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Mathematical Discourse in Elementary Classrooms

by

Mary M. Shortino-Buck

A dissertation submitted in partial fulfillment of the
requirements for the degree of

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in
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Mathematical Discourse in Elementary Classrooms

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Mary Shortino-Buck

This dissertation is completed as a partial requirement for the Doctor of Education (EdD) degree at the University of Portland in Portland, Oregon.

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Abstract

Mathematic reasoning of elementary aged students in the United States is low compared to other like nations. For students living in poverty the disparity is even greater. Previous research has linked mathematical discourse with the improvement of mathematic conceptual understanding and reasoning. This research, which employed a qualitative case study methodology, examined six elementary classrooms, to investigate the complex nature of including mathematical discourse in instruction. The purpose of examining mathematical discourse in elementary classrooms was to provide contextual insight into teacher beliefs about mathematical discourse, how instruction was prepared and facilitated by the teacher, and how students responded through participation. Interview, observation, and artifact data were gathered, and cross analyzed. First, this study suggested that a combination of post-graduate coursework, adopted curriculum, and district professional development supported participating teachers to develop the content and pedagogical knowledge needed to include mathematical discourse in their instruction. Secondly, this study indicated that a teacher's personal experience with mathematic learning influenced his or her beliefs and impacted his or her instructional practices. Finally, a discrepancy in the cognitive level of discussions between high and low poverty 4^{th} and 6^{th} grade classrooms was noted, but no difference in participation was noted in 2^{nd} grade classrooms. This study supports current research by cross analyzing six in-depth case studies and providing insights into commonalities and differences in teacher beliefs, instructional preparation and facilitation, as well as, student participation.

Keywords: elementary, K-6, mathematical discourse, mathematics, math instruction, SES, student participation, teacher belief, poverty

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I would also like to express my gratitude for the TOSA, teachers, and students from my study district. Without your contributions, flexibility, and insight this study would not have been possible. I appreciate the openness of the teachers who shared their opinions and beliefs as well as opened their classrooms to an inquisitive stranger. Your commitment to improving elementary students’ mathematic instruction is inspiring.

I would like to thank the friends that were always there to provide encouragement. Laura and Judy had faith in my ability when I had none and they always listened to my thought process and lifted my spirits. Also, the friendships that were forged in my doctoral cohort were vital to my success during these intense years of study.

Finally, I would like to thank my family. They were there to encourage me and support me throughout this entire process. They listened, talked through my struggles, picked up the slack when I was not able, and let me know they were proud of my work. I do not have the words to express the admiration and love I have for them. They are what made this all possible.

Dedication

To Tom, Julia, and Angela, are and always will be my inspiration for being a better person. They make the world a better place and give me the confidence and support to achieve more than I think I am capable. Tom has gone above and beyond, taking on all aspects of our life so that I could concentrate on my work. He fed me literally and spiritually so I had the energy to keep going and succeed. My wish for my family is that they always see themselves through the eyes of those that love and admire them, of which I am merely one.

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Chapter 1: Introduction

The United States can no longer consider itself a leader in educating its population. Mann (1848, p.87) stated, “Education then, beyond all other devices of human origin, is the great equalizer of the conditions of men - the balance-wheel of the social machinery.” If education is the *balance-wheel* then our social machinery is out of alignment. This is evident in the mathematical achievement of U.S. students. For decades, the concern over how mathematics is taught in the U.S. has been prominent in analyses of instruction (Romberg, 1993; Smith, 1996; Yackel & Cobb, 1996). With the release of the Common Core State Standards (National Governors Association Center for Best Practices, Council of Chief State School Officers [CCSS]) in 2010 and international mathematics scores indicating a decline in U.S. mathematical understanding in 2015 (U.S. Department of Education Institute of Education Sciences National Center for Education Statistics [IES], 2015) there is greater urgency to implement curriculum and pedagogy that facilitates students’ conceptual understanding of mathematics.

According to the National Assessment of Educational Progress (NAEP) mathematics scores dropped in 2015 from the last administration in 2013 (IES, 2015). The results of the NAEP mathematics assessment provide a general overview of what U.S. students in fourth and eighth grades understand and can do in mathematics. The assessment measures both a student’s content knowledge and ability to apply mathematical reasoning. This assessment is given every two years to a cross section of U.S. students.

The 2015 administration of the test is the first time since 1990 that NAEP math scores have dropped in the U.S. Only 40% of 4th graders and 33% of 8th graders in the United States are considered proficient in mathematics an overall 2% drop since the previous year (IES, 2015). This 2% is reported as significantly different ($p < .05$) than 2015 (The Nations Report Card, 2017). Of greater interest for mathematics instruction is that U.S. students' achievement was higher when performing lower level mathematic skills such as handling data directly from tables or using simple formulas according to the Programme for International Student Assessment (PISA) (Organisation for Economic Co-operation and Development, 2013). Interpretations of the same data indicate that problems involving mathematical reasoning and strategies that ask students to think mathematically are a weakness for U.S. students with scores falling below like nations. Fewer than 9% of students in the U.S. achieved proficient or advanced in mathematical reasoning compared to 16% percent of students in Canada and 30% of students in Hong Kong, China, Korea, and Chinese Taipei.

The composite results are even more troublesome for some underserved children with 19% of 4th grade Black students and 26% of 4th grade Hispanic students scoring at or above proficiency compared to 51% of White students and 65% of Asian students.

For children from low-income households only 33% of 4th grade students reached basic achievement levels as defined by NAEP (U.S. Department of Education, 2015). The rate is two times higher for higher SES levels (U.S. Department of Education National Center for Education Statistics, 2011). In Oregon, 27% of fourth grade students who were eligible for free and reduced lunch scored proficient or above, compared to 54% of those who did not qualify. Reardon (2011) has found that the achievement gap between income groups in recent years has surpassed that of the black and white achievement gap. The achievement gap in mathematics is even greater for children whose mothers have not completed high school than that of overall family income (Reardon, 2011).

Carpenter, Franke, and Levi (2003) state,

> Students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical to their future success in mathematics and related fields. (p. 6)

This quote is pertinent to the state of mathematics education in the U.S. as reflected in the weak scores in mathematical reasoning on the 2015 NAEP. In reaction to the lower levels of mathematic achievement in the United States and ongoing research indicating that mathematical discourse is vital for student achievement, the National Council of Teachers of Mathematics (NCTM, 2014) provided eight teaching practices with the intent of strengthening student acquisition of mathematical concepts through improvement of instruction. Prominent in these eight teaching practices is including mathematical discourse as an important feature of instruction. "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (NCTM, 2014, p. 10).

Current mathematical achievement in the U.S. indicates a need to re-examine how and what is being taught in classrooms across the United States. Testing results like those reported above indicate an increase in a U.S. population unable to think mathematically which may have ramifications well beyond the classroom. The fact that the results indicate a large portion of students are unable to use mathematics for problem solving can have a negative impact on students' adult lives. Problem solving applications are used in the daily lives of adults for personal finances to careers that depend on being able to reason mathematically. Poor mathematical reasoning skills in the lives of adults, could result in loss of income and the inability to manage personal finances.

In response to the need for mathematics education to be more coherent in the United States, the National Governors' Association Center for Best Practices (2010) used state standards and international models for mathematical practices that have been shown to be effective to develop the Common Core State Standards (CCSS) for mathematical content and practices. The design of the CCSS was based on Schmidt, Houang, and Cogan's (2002) work that stressed conceptual understanding of key mathematical strands in addition to procedural skills.

Many teachers believe they should teach mathematics as they were taught through memorization of facts and repetitive practice of rote skills (Banilower, Boyd, Pasley, & Weiss, 2006; Boaler, 2016a; Weiss & Pasley, 2004). This discrepancy between everyday classroom instruction and research dates back decades. *The American Behavioral Scientist (pre-1986)* stated that, "The supposed agreement on pragmatic philosophy has not lead to agreement on the techniques most effective in furthering that pragmatism," (Chapter VIII Education, 1964, p. 93).

Despite evidence representing the acquisition of mathematics as a social activity, mathematic pedagogy has persisted in the United States as an unrelated series of procedures (Banilower, et. al., 2006). This view of mathematics is static, which leads to a belief that if a student is efficient at using formulas or procedures the student understands math. However, students can skillfully perform procedures without understanding the mathematical concepts that lead to the procedures. In this way, a student's mathematical ability remains at low cognitive levels of achievement (Thompson, 1992).

An aspect of why students are proficient at procedures and not mathematical thinking is the lack of instructional focus on mathematical concepts in elementary schools. Stipek, Givvin, Salmon, and MacGyvers' (2001) study examined the link between a teacher's beliefs and instructional practice finding that teacher beliefs about mathematical learning and ability, as well as a teacher's self-confidence and enjoyment of mathematics, are associated with instructional practices and their students' self-confidence. The more teachers' beliefs align with a student constructing his or her own understanding of mathematic concepts, the more instruction reflected an emphasis on understanding and the importance of mistakes in learning rather than speed and the right answer. This emphasis on conceptual understanding was found to improve student confidence and participation (Stipek, et. al., 2001).

Ball and Forzoni (2011) suggest there is a lack of agreement in schools and our greater society as to what constitutes good quality math instruction, with some suggesting speed and accuracy are most important and others suggesting conceptual mathematical thinking as the primary emphasis. This lack of agreement makes it difficult for school districts to establish coherent mathematics instruction because there is a disjointed implementation of pedagogy with teachers inconsistently applying instructional practices shown to increase student understanding along with pedagogy that research has shown (Jackson & Leffingwell, 1999; Swetman, 1994; Tankersley, 1993; Young, Wu, & Menon, 2012) to be detrimental to student learning, such as speed tests for basic facts.

Karp, Bush, and Dougherty (2014) have shown that, in an attempt to cover large quantities of content quickly or because of a general lack of content knowledge, teachers presented students with misrepresented generalizations that hold true for the moment but that do not broaden a student's understanding of mathematical concepts, cannot be used out of context, and should not be generalized outside of that context. The results are students that apply procedures incorrectly, with no means of understanding their error, because they do not understand the math behind the procedure. The authors explained this phenomenon as "always rules that are not so always" (p. 20). This is often expressed in terms of *math tricks* taught to students such as "when you multiply two numbers your answer is a bigger number." This *math trick* holds true until a student starts working with fractions, decimals, and negative numbers. This short-term fix in the classroom leads to long term misunderstanding of mathematic concepts that impede higher mathematic achievement.

Mathematical Discourse

A large part of encouraging students to develop reasoning skills and strategies in mathematics revolves around developing a classroom culture that emphasizes discourse to develop a shared understanding (Cobb & McClain, 2005). Several definitions of mathematical discourse exist but consistently include discussion, justification, argumentation, and negotiation as vital aspects at the center of mathematic pedagogy to improve student conceptual understanding (Cobb & McClain, 2005; Donovan & Bransford, 2005; NCTM, 2014; Thompson, 1994; Yackel & Cobb, 1996). Whitin and Whitin (2000) discussed the importance of recognizing children as the constructors of their own learning and encouraging them to express that learning through multiple avenues that both clarify their understanding and communicate it to others. The student interactions of discourse recognize the social aspects of conceptually understanding mathematic content.

Conceptual understanding in elementary school mathematics can be thought of as the ability to justify procedures through reasoning rather than to describe the computational procedures themselves (Kazemi, 2008). Mathematical discourse is an effective method for facilitating a child’s conceptual understanding and the acquisition of mathematical knowledge that allows for growth in achievement across student populations (Stylianou, Blanton, & Knuth, 2006). It is also the way in which knowledge is validated and organized providing a role in building new knowledge. When participating in mathematical discourse, students reason through their current thinking with input from peers and are asked to clarify their own, as well as peer understanding of the mathematical concept. This emphasis on understanding why, and the paths to the answer, as opposed to the correct answer, develops a deeper conceptual understanding for students. This process provides students time and feedback to understand how to amend their thinking to grasp concepts more precisely. Chapin, O'Connor, and Anderson (2009) found that mathematical discourse cleared up student misconceptions, improved students' ability to reason logically, gave students more opportunity to participate in their learning, and provided socially grounded motivation to learn.

As a social activity, mathematical discourse is dependent on language acquisition of the students participating in the discourse and the scaffolding that the teacher provides to bridge language gaps. While mathematical discourse encompasses more than linguistic patterns, it relies heavily on the ability of the student to explain or show his or her thinking to another participant. Johnstone (2002) defined discourse as communication using language. However, not all communication uses formal language. Communication occurs with symbols, physical gestures, visually, as well as orally, and with written language. When talking about mathematical discourse, these features of communication must be included to form a complete representation of the knowledge or understanding being passed from one person to the next.

Mathematical discourse is different than talk in other social contexts because in mathematical discourse one person, the teacher, controls the direction and topic of the discourse. The teacher's influence in student to student discourse can be seen when the teacher prepares students for mathematical interactions within the classroom and how discussion between students will be carried out. Not only is the direction and topic of the discourse guided by the teacher but the teacher also directs how student interactions will proceed; the accepted norm for classroom discussions. If there is a mismatch between the language norms of the teacher and the language norms the student produces and understands, the student's ability to fully participate can be negatively impacted which will interfere with the student's academic achievement (Herbel-Eisenmann, Choppin, Wagner, & Pimm, 2012). Taking this into consideration when analyzing levels of student discourse provides another important aspect of the teacher's influence in facilitating the student's conceptual understanding of mathematics.

Pedagogy of Mathematical Discourse

Along with believing that discourse is important to mathematical achievement, teachers need to understand the pedagogy of mathematical discourse. Smith and Stein (2011) developed five practices to be used in classrooms that implement mathematical discourse:

1. *Anticipating* student responses prior to the lesson
2. *Monitoring* students' work on and engagement with the task
3. *Selecting* particular students to present their mathematical work
4. *Sequencing* students' responses in a specific order for discussion
5. *Connecting* different students' responses and connecting the responses to key mathematical ideas.

A teacher's pedagogical skills are central to mathematical discourse because skilled questioning leads to productive discourse. By modeling a high cognitive level of questioning teachers show students how to interact with each other with rich discussion that leads to conceptual understanding (Akkuss & Hand, 2010). When a teacher can engage students in this manner as active participants in their own mathematical learning, the teacher creates an equitable environment for all students (Croom, 1997; Dale & Cuevas, 1992). An abundance of pedagogical strategies beyond questioning are needed to engage students in mathematical discourse to move students to higher cognitive understanding while simultaneously supporting the understanding of the variety of students in the classroom (Cazden, 2001).

Purpose

The purpose of this study was to examine the processes teachers with post graduate mathematic coursework took to implement mathematical discourse. This was done through analyzing how teachers believed mathematical discourse fit into the learning process, how lessons were prepared, how teachers facilitated mathematical discourse, and how students participated in it.

This study examined mathematical discourse in six second, fourth, and sixth grade classrooms. Data collection was focused with the domains of the Instructional Quality Assessment (IQA) (Boston, 2012) (Appendix A). The qualitative data were parsed into the four categories being examined; teacher belief, teacher planning, teacher facilitation, and student participation. The collected qualitative data was then used to analyze discourse.

Data were triangulated through individual teacher interview, classroom observation, in which students participated and teachers facilitated, and collection of student artifacts. These data were pertinent not only to measure discourse in classrooms but also to provide insight into teacher beliefs and practices that helped guide instruction in elementary mathematics.

This study considered mathematical discourse the dialogic communication whether oral, written, gestural, or graphic that purposefully reaches toward a shared mathematical understanding of specific mathematical content initiated by the teacher.

Specifically, this study will answer:

1. How do teachers think about mathematical discourse in the learning process?
2. How are lessons prepared to include mathematical discourse?
3. How do teachers facilitate mathematical discourse?
4. How do students participate in mathematical discourse?

Conceptually this research recognized four factors: the mathematical discourse used by students, the mathematical discourse facilitated by the teacher, the task chosen by the teacher, and the attitude and beliefs about math instruction held by the teacher.

Study Significance

Research has provided evidence to include mathematical discourse as a central tenet to develop conceptual understanding and improve mathematical achievement (Ball & Forzoni, 2011; Cobb & McClain, 2005; Donovan & Bransford, 2005; Kazemi, 2008; NCTM, 2014; Romberg, 1993; Schmidt, et al., 2002; Yackel & Cobb, 1996), however, not all talk is equal, indicating a need for teachers to engage students in higher cognitive levels of discourse that develop reasoning. The cognitive rigor of student participation in mathematical discourse has the potential to broaden a student's mathematical understanding. The process, knowledge, and skills a teacher must possess to implement mathematical discourse at a high cognitive level are great. This study focused on how teachers developed the knowledge and skills needed to implement mathematical discourse in their classrooms and how that knowledge was implemented. Starting from the individual teacher's personal experiences with mathematics, how those experiences impacted their belief, planning, and implementation of mathematical discourse to how students participated in mathematical discourse to understand content was examined.

Schools in economically diverse communities were considered to cross-reference discourse differences in various contexts. The results provided insights into instructional beliefs that impacted practices and student participation in these economically diverse communities.

Dissertation Overview

Divided into five chapters this study provided an investigation into current mathematic instructional practices in elementary classrooms in the Pacific Northwest. Chapter One provided a clarification of this study's focus and provided a rationale for the study. Chapter Two reviews relevant literature and provides a theoretical base for the study. Socially constructed understanding, teacher belief about mathematic instruction, and the impact of mathematical discourse on student understanding are the foci of Chapter Two.

Chapter Three describes methods used in the study. Collecting data that impacted various dimensions of mathematical instruction was triangulated to provide a full analysis of mathematical discourse that occurred in the classroom.

Chapter Four provides results of the study. The narrative includes qualitative data. The qualitative data includes quoted language used by students and teachers to give the classroom a voice in this study. The data in this chapter is organized to support the four study questions covering teacher beliefs, preparation, facilitation, and student participation. SES of the community is also provided in order to more clearly understand how these aspects interacted and impacted mathematic instruction as a whole.

Chapter Five provides an analysis of the data and meaningful conclusions based on the research questions presented above. It views the qualitative data both globally and in its parts noting similarities, patterns, relationships, and themes in an effort to shed light to various aspects that influenced mathematical discourse in these six elementary classrooms. Implications of the findings are discussed as they related to current literature on the importance of mathematical discourse in a child's understanding of mathematical concepts. The implications were then applied to the impact on teacher instruction and the development of best practices for mathematic pedagogy. The outcome provided an insight into current practices in mathematical discourse and from that insight possible avenues elementary teachers should consider to improve student outcomes through mathematical discourse.

Chapter 2: Review of Literature

Organized to address the vital aspects that impact mathematical discourse this review of literature builds on historical perspectives of mathematic instruction which is the basis for continued pedagogical influences in the classroom. Within this history various definitions of mathematical discourse have arisen from fields inside and outside mathematic education which changes the lens through which research on mathematical discourse has been conducted. One perspective addressed is how mathematical discourse promotes conceptual understanding of mathematics which goes beyond memorization of *math facts* and formulas (Kazemi, 1998; National Council of Teachers of Mathematics, 2014). While this perspective continues to gain momentum, it is not the only view of best practices in mathematics.

Within classrooms that conduct mathematical discourse, learning happens in a social context that provides support for students' shared understanding of concepts which can present opportunities for students who are situated socially within the academic culture of the classroom. An exploration of inequitable practices will be addressed when classroom culture is not aligned with a student's culture (Bishop, 2008; D'Ambrosio, 2008; Herbel-Eisenmann, Choppin, Wagner, & Primm, 2012; Forgasz & Rivera, 2012). Language levels of students play a large role in mathematical discourse, for this reason the literature review will cover equity issues in the mathematics classroom focusing on language norms used in mathematical discourse and teacher bias that goes into opportunities presented to students to expand their understanding.

This research adds to literature by looking at the intersection of the observed mathematical discourse in elementary classrooms, the belief system and planning process of the teacher, and any variance that occurs between classroom communities. Highlighting how mathematical discourse is facilitated in elementary classrooms illuminates an understanding of the various aspects that go into classroom instruction using mathematical discourse to improve student understanding of mathematical concepts. This conceptual understanding developed through mathematical discourse can provide the foundation to improve mathematic performance at higher cognitive levels beyond rote memorization of basic calculation (Boston, 2012).

Context of Mathematical Discourse

It is important to look at how mathematical instruction has been conceived in the past because it is still impacting how mathematics is being taught today. For decades mathematics reform in the United States has suggested a move away from rote memorization of rules to understanding mathematical concepts. In 1957 Gibbs and Van Engen wrote about the increase in the “discovery method” of pedagogy to learn mathematical concepts (Dawson & Ruddell, 1955) and described how the “meaning method” was shown to be more effective than the “rule method” (Miller, 1957). Both of which indicated a need to develop conceptual understanding over rote memorization, yet instructional methods focusing on memorization of rules and formulas continued to be prevalent in elementary mathematics instruction (Banilower, et. al., 2006).

One illustration of this comes from the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) Project (Silver &

Stein, 1996) which provided professional development and instructional materials for teachers in order to implement mathematical discourse in the classroom as a tool for improving student understanding of content. The study, led by a team of researchers at the Learning Research and Development Center (LRDC) at the University of Pittsburgh, began in 1990 and ran for 5 years in inner city middle schools with diverse populations across the United States. Research focused on issues of equity through the lenses of gender, race, ethnicity, and poverty.

Teacher focus included instructional strategies that developed discourse communities and cooperative strategies that encouraged students to develop their own thinking while teachers supported student collaborative reasoning through discourse and questioning (Silver, Smith, & Nelson, 1995). This structured discourse supported students in communicating their own thinking and reasoning as well interpreting another student's mathematical methods and reasoning.

While student participants in the QUASAR project showed improvement in mathematical performance over time (Silver & Stein, 1996) not all teacher participants implemented instruction in the same manner which demonstrates the difficulty faced with putting research into practice. A teacher participant in this study went to every professional development on the importance of allowing students to discuss mathematical tasks and build their own understanding of the mathematic concepts through mathematical discourse. She was provided with all the material support to instruct using this method yet she continued, throughout the year, to use the mathematic tasks to teach in a directive manner; providing students with the procedure they should use to solve the task and requiring students to use the procedure given by her to complete the task. It was not till she was videotaped and her colleagues, upon viewing the videotape together, discussed the pros and cons of her teaching style, did she understand the impact of allowing students to formulate their own understanding through talk.

In another study of over 350 elementary, middle, and high school lessons (Weiss, Pasley, Smith, Banilower, & Heck, 2003) the research team found that fewer than one in five lessons emphasized making sense of the mathematic content. Like the teacher participant in the QUASAR study, teacher practice did not move away from rule based memorization to promote conceptual understanding. These studies demonstrate that while the benefits to students of emphasizing mathematical understanding over the solitary use of rules based instruction is known, there is a reluctance to change instructional practice.

Historical perspectives about the purpose of education, social interaction, language use and development, role of teacher and student, are all vital to a complete view of mathematical discourse and how it impacts student understanding.

Problem solving has long been identified as a favorable method of teaching mathematics (Freudenthal, 1973; Polya, 1954). Traditionally students passively solved the problem posed by the teacher, in the way the teacher imparted to them (Silver, 1995). Students independently repeated the steps the teacher gave to get the answer the teacher anticipated. Students became skilled number crunchers but may not have understood why the procedures worked. For that matter, the teacher may not have been able to articulate or understand the math behind the procedure. In fact, Hungerford (1994) suggests that one of the weakest links in elementary education is elementary teacher's preparation to teach mathematics. Discourse in an academic community lead by a teacher who does not understand mathematic concepts or may understand but does not promote students arriving at their own understanding may accept mathematical rules as explanation. In these cases, the student is often not aware of the plausibility of their reasoning in real life because repeating procedure through memorization was the discursive norm that has been encouraged by the teacher.

These cultural beliefs about teaching and learning mathematics present a roadblock to student mathematic acquisition (NCTM, 2014). The prevalence of implementing the belief that supports memorizing procedures through teacher demonstration and repetitive duplication of the demonstration indicate a reluctance to stray from the cultural tradition and mathematic practices and the lack of belief that discourse and conceptual understanding are beneficial to students (Barkatsas & Malone, 2005, Wilkins, 2008).

Sfard (2008) takes the stance that all human intellectual activities are driven by communication. Since discourse can only be taken in the context of the community in which the communication is taking place the social context of that mathematical community must be examined. The way in which teachers and students interact with the content is influenced by the culture of their individual contexts and its relationship to the academic community which the mathematical discourse is taking place. What may seem obvious but deserves to be said is that the way students think about mathematics is influenced by the way they talk about mathematics. And the way they talk about mathematics is directly influenced by the teacher and the way the teacher thinks about mathematics.

Discourse in the classroom does not form arbitrarily, it is greatly influenced by the culture of the teacher and the classroom culture that the teacher cultivates (Sfard, 2008). In elementary classrooms there are rules of discourse that are explicitly taught but other discursive rules that are culturally based and instinctively followed and promoted as the expected norm in academic situations. “The way we speak and communicate with others conveys [these] unwritten regulations” (Sfard, 2008, p. 168). Lampert (1990) explains that students do not learn rules “simply by being told what to do anymore than one learns how to dance by being told what to do” (p. 58).

Bateson (1973) discussed the conflict that often occurs between discursive rules that are explicitly taught and those that are implied through action. He explains that this conflict creates a “double bind” in which the rules learned through action will be prominent over those taught explicitly. This conflict between actual practice and explicit instruction has the potential to impede learning mathematical content.

Another set of roadblocks to students’ mathematic education take the form of cultural bias that is embedded and expressed in our educational system, in sometimes subtle ways. As previously discussed the implied discursive norms of a mathematics classroom are greatly influenced by the cultural norms of the teacher and what has been accepted by the teacher as preferred mathematical discourse (Sfard, 2008). Components of language and its biases are considered by Orfield (2013) who discusses the issue of societal inequalities that are reinforced by the very system that is touted to be “the great equalizer” by Mann (1848). While saying education helps level the playing field educators continue to maintain the status quo of the dominant culture through the way mathematics is taught. Secada (1992) has accounted for ways in which multilingual students who are learning English are marginalized even when exceptional teachers are instructing. Often students of color, girls, and those in poverty do not see themselves as part of, or leaders in, the education system (Wagner, Herbel-Eisenmann, Choppin, 2012). These groups are not deemed acceptable until they adopt the dominant culture's manner and language. Yackel and Cobb (1996) used the term "sociomathematical norms" to describe the idea that each classroom has a set of routines or patterns of discussion that participants follow in order to relate understanding and participate in learning. These norms can be simultaneously supportive and disruptive to student understanding dependent on the cultural match between the norms and the student.

Vygotsky (1978) studies focused on the potential of students by building on what the students already know and do, their background, to be used as a point of entry to understanding new concepts. He recognized that students come to school as people who have life experiences in which new learning can be supported for a fuller understanding. However, historically, the background that Vygotsky spoke of was based on western white male culture. When starting from that perspective there is a possibility to skew what is useful background knowledge in which to build upon, and creates the perspective that non-dominant culture students are entering the education system with deficits. Anyon (1995) acknowledges these cultural biases in mathematic instruction by stating "Educational reforms cannot compensate for the ravages of society" (p. 88) however attention to educational history, classroom discursive norms, and how students build their own knowledge takes steps to addressing discursive bias.

Definitions of Mathematical Discourse

Ryve's (2011) review of 108 articles, addressed the topic of mathematical discourse, and found that only 20 articles contained a definition of mathematical discourse. Within those articles the definition varied and was often not clear. In some instances, discourse referred to talk only, while in others the greater human interaction concerning dialogue that includes symbolic representation was referenced. With the various definitions of mathematical discourse come different positions about mathematics in classroom discourse and mathematics as discourse. Discourse in mathematics is the action of having conversation about mathematic content. Mathematics as discourse refers to the nature of mathematics that does not exist without the language of mathematics to talk about the content. Sfard (2008) refers to mathematics as discourse not as the acquisition of knowledge but as the participation of creating that knowledge through the social activity of discourse.

Most definitions of mathematical discourse involve more than classroom talk. Visual representations are important to mathematical discourse because they give students a discussion point in which to make sense of problems, as well as support students with developing academic language skills needed to talk about mathematics (Arcavi, 2003; Fuson & Murata, 2007; Stylianou & Silver, 2004).

Johnstone (2002) defines discourse as communication using language. However, not all communication uses formal language. Communication occurs with symbols, physical gestures, visually, as well as, orally and with written language. When talking about mathematical discourse these features of communication must be included to form a complete representation of the knowledge or understanding being passed from one person to the next. Johnstone (2002) goes on to discuss the idea that discourse is both the source and the result of knowledge. The discourse people participate in together creates knowledge while the discourse used to explain what has been conceptualized interprets knowledge.

O'Halloran (2005) drawing on Halliday's (1978) social semiotic theory of discourse develops the idea that sign systems are used to impose order through meaning. These sign systems are composed of language, visual imagery, music, gesture, action and stance, and three-dimensional objects. When applied to mathematical discourse, connections can be made between Halliday's social semiotic theory and what takes place in the classroom. Through discourse students make sense and put order to mathematical concepts by using oral and written symbols, both linguistic and specialized, as well as using visuals, and possibly gestures by "acting out" mathematical situations. Three-dimensional objects that Halliday (1978) suggest can be equated to mathematical manipulatives. While Halliday's theory is a general language theory it does apply very directly to mathematical discourse which further emphasizes the social and linguistic nature of mathematic learning. Halliday focusses not only on the practice of discourse but the context in which such practices are conducted and informed. Based on this theory meaning is derived from a set of choices made through the intention of the signs and the social context of the problem (O'Halloran, 2005). While O'Halloran's work is useful, there is still a need for theoretical connections in research to be more clearly defined to analyze mathematics as a discourse (Ryve, 2011).

The idea of mathematics as discourse, theorized by Sfard (2008), suggested that mathematics makes great use of visual components that are developed primarily for the sake of being able to talk about math. In this way, mathematical discourse is circular. To communicate a collective understanding of math, a symbolic language was created in which math is represented, and that symbolic language becomes the math in which students communicate to understand. Whereas, zoology and history are discourse around animals and past societies, mathematical discourse is discourse around numbers, functions, sets, and geometric shapes. But unlike zoology and history where the objects being talked about exist as physical objects whether you discuss them or not, the objects of mathematics are abstract and were created for the sole purpose of discussing mathematics. Hence, Sfard's (2008) definition of mathematics as discourse is self-generating. If you are to participate in mathematical discourse there needs to be an understanding of mathematics, however, to understand mathematics there is a need to participate in the discourse of mathematics.

Theory

Social semiotics addresses the social aspect of developing meaning. In mathematics classrooms students use mathematical discourse as a method to develop this shared meaning of mathematic content that is facilitated by the classroom teacher. Halliday's social semiotic theory (1978) developed the concept of the use of language and interpersonal interactions to arrive at a common understanding. When applied to mathematic classrooms students and teachers develop a math culture where discourse - aural, verbal, visual – is the vehicle in which conceptual understanding is communicated and developed to a greater degree than previously attained. Within mathematical discourse lies a variance of cognitive levels that produce a variety of understanding outcomes, within the same classroom.

The way the teacher facilitates student conversation, and the agency that the student has during discourse, impact a student's ability to develop shared meaning with other participants that use the language of mathematics. Theorists have suggested that only in true dialogic exchanges can learning take place (Alexander, 2005; Freire, 1993; Mead, 1962; Nystrand, Wu, Gamoran, Zeiser, & Long, 2003). Bakhtin (1984) distinguishes true dialogic exchange from an exchange where one participant possesses the, "ready made truth" (p. 110) and the receptor of that truth passively accepts the truth without question or understanding. When this occurs, the student mimics the procedure of the teacher, but cannot be said to have learnt the mathematical concept.

Conceptual Understanding

According to NAEP, mathematics scores dropped in 2015 from the last administration in 2013 (IES, 2015). This is the first time since 1990 that NAEP math scores have dropped in the United States. Only 40% of 4^{th} graders and 33% of 8^{th} graders in the United States are considered proficient in mathematics (IES, 2015). According to the Programme for International Student Assessment (PISA) (Organisation for Economic Co-operation and Development, 2013) it appeared that U.S. students performed better when executing lower level mathematic skills such as handling data directly from tables or using simple formulas. The same data indicated that problems involving mathematical reasoning and strategies that ask students to think mathematically are a weakness for U.S. students with scores falling well below like nations. Fewer than 9% of students in the U.S. achieved the highest two mathematic levels compared to 16% percent of students in Canada and 30% of students in Hong Kong, China, Korea, and Chinese Taipei.

Based on the NAEP (U.S. Department of Education, IES, 2015) findings that presented a picture of low mathematical achievement among U.S. students, there is a need to understand how to teach mathematics so students of all backgrounds understand concepts and are not applying procedures to numbers without the conceptual understanding of those procedures. When procedures are applied without conceptual understanding students find it difficult to apply those procedures to new contexts or recognize when their recall of the procedure is flawed (NCTM, 2014).

The National Council of Teachers of Mathematics (NCTM, 2014) highlighted mathematical discourse as one of the primary instructional practices a teacher can facilitate to develop a student's conceptual understanding. NCTM (2014) promoted a greater emphasis on developing the ability to explain mathematical thinking in a manner that others can understand which challenges how math instruction has been traditionally conducted as well as basic beliefs around what effective mathematical instruction is. The idea of math reform has been suggested for decades but has been slow to have large scale adoption in classroom instruction (Jacobs, Lamb, & Phillip, 2010; Stigler & Hiebert, 2004; Weiss, Pasley, Smith, Banilower, & Heck, 2003).

Conceptual understanding of mathematic content has been shown to be developed through mathematical discourse (Lack, Swars, & Meyers, 2014). Development of expressive language both in an academic setting and at home not only influence students' ability to express their understanding but influence students' ability to participate in discussion that improves their understanding of mathematical concepts (Moschkovich, 2012). It follows that low levels of mathematical discourse decreases mathematical understanding and it is on that premise that this study is based. Lack, et. al. (2014) found that when the teacher was not present to facilitate discussion low-status students' quality of involvement decreased and was overshadowed by high-status students. It was postulated that these differences in discussion participation would be exacerbated over time without teacher facilitation indicating the need for teacher expertise in not only mathematical content but mathematical discourse pedagogy.

Research conducted by Donovan and Bransford (2005) as well as Lester (2007) suggest the foundation of effective mathematical teaching includes constructing "knowledge socially, through discourse, activity, and interaction related to meaningful problems," (NCTM, 2014, p. 9). A large part of encouraging students to develop reasoning skills and strategies in mathematics revolves around developing a classroom culture that emphasizes discourse as a means to develop a shared understanding. Discussion, justification, argumentation, and negotiation are all vital aspects of mathematical discourse (Cobb & McClain, 2005; Yackel & Cobb, 1996). These researchers believe, with the purpose to improve mathematical understanding, discourse should be at the center of student learning. As the complexity of mathematical discourse increases students' understanding of mathematical concepts deepens (National Research Council, 2001). In the transverse if a student has been relying on memorized algorithms and formulas early in their mathematic learning, they may struggle when math complexity increases, and requires an understanding of basic concepts for application to new learning.

Learning Mathematical Concepts Through Discourse

Theorist and researchers suggest that the pedagogy of mathematical discourse involves active participation of students collaboratively constructing meaning by equally controlling the conversation, and sharing in developing a conclusion based on the exchange of thinking, questioning, and critiquing (Alexander, 2008; Burbules, 1993; Freire, 1993; Reznitskaya & Gregory, 2013; Webb, Franke, Ing, Chan, Battey, Freund, & Shein, 2007). Discourse is thought of as both cognitive and social because discourse does not only involve use of language but also using and developing conceptual knowledge (Moschkovich, 2007).

It has been theorized that all learning takes place in a community and the main vehicle for communal participation and understanding is language (Davydov & Radzikhovski, 1985; Lave & Wenger, 1991). Steinberg, Empson, and Carpenter (2004) based their study on a community of learners and reported that classroom discussion was key to a student's conceptual understanding. This connection between conceptual understanding and student contribution to class discussion does not stand alone. Teachers must explicitly communicate participation procedures, and establish a classroom culture in which students are encouraged to participate even when they do not fully understand (Walshaw & Anthony, 2008). An important part of teacher communication to increase student participation and acquisition is making clear to students that contributions, whether correct or not, enhance the conceptual understanding of the student contributing and his or her fellow students.

One of the barriers that arises with implementing mathematical discourse in the elementary classroom is the lack of teacher training in mathematical concepts, and the unfamiliarity with the pedagogy that supports mathematical discourse (Hungerford, 1994). With poor teacher preparation to teach conceptual understanding of mathematics, implementing effective mathematical discourse can be very challenging and often not sustainable (Ball, 1991; Battista, 1993).

Equity in Mathematical Discourse

Mathematics is not usually thought of as a content that could be biased because it pertains to numbers which are thought of as culturally neutral. However, if we consider how vital language is to understand mathematical concepts, we must also acknowledge the possibility of bias built into that academic language. As with any society that uses language to realize a shared understanding there are inequities in the mathematical society of the classroom (Bishop, 2008).

Since mathematical discourse cannot happen in isolation, theories of community and social dynamics come into play (Burke & Stets, 2009). The context of the society in which the mathematical language is taking place, allows the student to go beyond stimulus-response patterns, of non-human animals, by participating in the interaction with another to arrive at a common understanding through shared language (Burke & Stets, 2009). Burke and Strets (2009) suggest that based on this interaction a student's agent identity is negotiated through the language between the student and the greater mathematical society. They go on to postulate that a student's agency in mathematical discourse impacts developing further insight of mathematical content, or it creates an atmosphere of preventing the student from developing a deeper understanding and marginalizing the student in the mathematical community. This marginalization further diminishes the student's ability to participate and learn more mathematical content.

Mathematic and social language play a role in Moschkovich and Nelson-Barber's (2009) study in which they consider how students must sort out the differences in meanings of terms, as well as phrases, that are used in both mathematical discourse and everyday communication. A person may "reduce" their calorie intake which results in a lessoning of the value of calories, whereas, "reducing" a fraction does not change the value of the fraction. These differences in meaning require students to learn the language of academic math which itself varies across different communities and across its different uses in the mathematics classroom. Within the same mathematics task, a student may have to linguistically navigate a situation in which there is "more" debt, which results in less money and "more" salary which results in an increase in money. Moschkovich and Nelson-Barber (2009) postulate that because of the influence of academic and everyday language in mathematical discourse it is not always possible to tell if a student's ability to participate and learn through discussion is initiated from school or outside school. The influences of both impact the effectiveness of classroom discourse. Students could be using colloquial meanings, while others in the classroom are using mathematical meanings, which can cause confusion or an incorrect analysis by the teacher of the student's understanding (Moschkovich & Nelson-Barber, 2009). Depending on the alignment of school language and outside of school language this negotiation of meaning is either impeded or benefitted.

When looking at vocabulary development to bridge the gap between neighborhood and school Moschkovich and Nelson-Barber (2009) point out that vocabulary development is not sufficient to mathematic content learning. Mathematical learning is most successful when it is in the context of a language rich lesson that requires receptive as well as expressive understanding, and when students actively participate in mathematical discourse. This active participation is highly influenced by teacher facilitation and development of mathematic culture (Walshaw & Anthony, 2008) within the classroom.

The importance of knowledge attainment, learning from and explaining through discourse, is central to the widening achievement gaps of various populations of students within the classroom (O'Connor, Hill, & Robinson, 2009). As student's progress through the school system, when the starting point is equalized, students from marginalized groups, especially those growing up in poverty, lose ground to white economically stable students and the achievement gap increases. O'Connor, et al.'s (2009) data shows that not only are students who live in poverty starting behind their white economically stable counterparts, but they are losing more ground as the years progress indicating an instructional system that benefits economically stable students to those living in poverty. This growing achievement gap has the possibility of being reduced with the facilitation of mathematical discourse that is aware of the discourse differences of students. Language choices made by teachers and students reflect what they value and what their culture values (Bishop, 2008). Bishop (2008) indicates that a teacher's known and unrecognized values pertaining to math and mathematical achievement directly influence a student's values of mathematics. These values are conveyed through the language that is used in the classroom.

The broad range of communicative practices, reading, writing, speaking, listening, is considered by Herbel-Eisenmann, Choppin, Wagner, and Pimm (2012), as well as prosodic features of communications which include gestures. Continued variances in student achievement levels emphasize the need for further research on mathematical discourse practices that have the potential to include or exclude various groups of children because classroom discourse is matched or mismatched to home discourse.

D'Ambrosia (2008) uses the term *ethnomathematics* to refer to the connection of mathematics to culture. For mathematics to be meaningful to students there needs to be a connection to that student's culture. One way to achieve this connection is through mathematical discourse when the discourse is made available to students in their cultural language. D'Ambrosia (2008) concludes that recognizing the culture of students influences how students think about and learn math. This indicates that how teachers choose their mathematical language is important. By examining the language teachers and students use in mathematical discourse, students can be encouraged to construct personal mathematical understanding and express that understanding through mathematical discourse which engages their cultural characteristics. Supportive classroom communities in which the teacher creates a safe classroom environment that values and encourages all student's participation through asking questions and sharing ideas creates opportunities for marginalized students to learn (Boaler, 2016b).

Embedded within discourse are attitudes and values along with social identities as to the student's place in the math community that become an important part of successful academic acquisition of math (Ball, 1991; Cobb, Wood, & Yackel, 1993; O'Connor, 1998; Sfard, 2000). With students coming from a multitude of backgrounds, some students come to school with the linguistic culture of the typical U.S. classroom while others may have conflicting home and community norms (Michaels, O'Connor, & Resnick, 2007). This variance in the teacher and other students' views of an individual's place in the math community impacts how students participate in mathematical discourse which in turn impacts the student's mathematic understanding.

The Pedagogy of Mathematical Discourse

Making discourse available to students happens in classroom decisions made daily by elementary teachers. Teachers decide routinely about how children will participate in mathematical discourse, what language is appropriate, and what language needs to be explicitly taught. An elementary school teacher would not expect students to conduct mathematical discourse as a mathematician but content appropriate vocabulary would be expected (NCTM, 2014). This is where mathematical discourse varies from general discourse.

Chapin, O'Connor, and Anderson (2009) suggest five teaching practices that convey to students the importance of their thinking and participation in discourse:

1. Revoicing – both teacher and students restate a previous speakers statement asking whether they understood correctly

2. Teacher initiated request that a student repeat a previous contribution by another student
3. Teacher's elicitation of a student's reasoning
4. Teacher's request for students to add on
5. Teacher wait time

Mathematical discourse is not only essential in acquiring mathematical understanding it is also the medium in which equity or inequality of that understanding is developed (Herbel-Eisnmann et al., 2011). Language choices made by the teacher and students in mathematical discourse impact the way understanding of concepts can be interpreted and expressed. A teacher's language can be privileging or limiting to students whose home language aligns with or differs from the language of discourse in the mathematics' classroom (Barton, 2008). There is a need to continue to analyze how students from various demographic populations use mathematical discourse, and the way teachers facilitate discussions on mathematical concepts in respect to those groups within the mathematics classroom.

What and how students learn, through the medium of mathematical discourse, is influenced through the discourse structures teachers put into place (Rigelman, 2009). The decisions teachers make on how to structure instruction projects to students what is valued and what is not. The teacher's questioning techniques, press for justifications, tools available to students, and the very structure of how students and teachers interact reflect a teacher's pedagogical beliefs and impact student learning.

Mathematical Instruction: Teacher Beliefs

While research points to mathematical discourse as a way to promote conceptual understanding in mathematics, the realization of this depends on teacher beliefs (Cross, 2009). Instructional decisions are filtered through teacher beliefs which are impacted by teachers' personal experiences as a teacher and as a learner (Hofer, 2001; Muis, Bendixen, & Haerle, 2006; Pajares, 1992). There is a large body of research that illustrates how teachers' understanding about what math is and how it should be taught is often based on their beliefs (Boaler, 2008; Laurenson, 1995).

Cooney (1985) found a disconnect between teachers' stated beliefs and practices, however, Ernest (1991) accounted for this discrepancy by taking into consideration the context of teaching with outside influences that impacted a teacher's ability to implement their stated beliefs. Anderson, Sullivan, and White (2005) took this component into consideration with their study and presented a model (Figure 1) in which the factors that influence a teacher's beliefs as well as their instruction are considered.

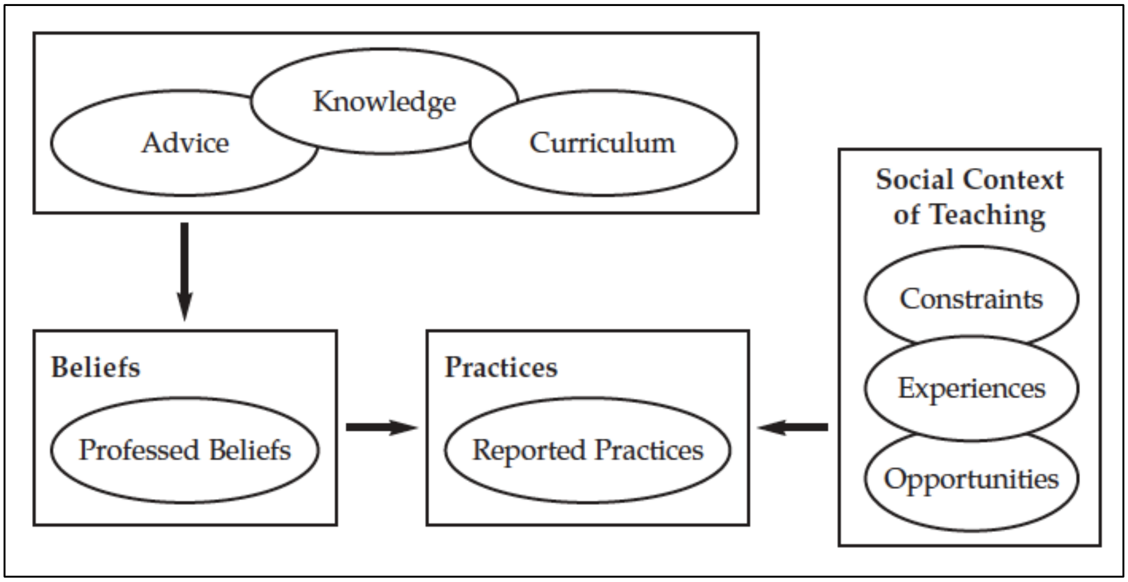

Figure 1. Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1).

Casa, McGivney-Burelle, and DeFranco (2007) developed a theory that explored the themes that make up preservice teachers' beliefs regarding discourse. This initial theory was analyzed through the development of an instrument, Preservice Teachers' Attitudes About Discourse in the Mathematics Classroom (PADM). From this instrument three reliable factors emerged: promoting mathematical reasoning, examining complex mathematical concepts, and valuing students' mathematical ideas. This instrument measures the attitudes of teachers about mathematical discourse but does not look at the relationship between those attitudes and instruction. This research also found that the questioning initiated by the teacher was limiting student cognitive discourse levels by only expecting short recall answers which draws a connection to the intricacies of belief in mathematical discourse to improve student outcome and execution of that belief to effectively instruct through mathematical discourse.

An earlier study conducted by Walsh and Sattes (2005) arrived at similar results. It found the more questions a teacher asked during a 30-minute period, the lower the cognitive level of student thinking. As in Casa, et. al. (2007) the presence of teacher questioning does not indicate the quality of the questions. Wash and Sattes (2005) study found an increase in number of questions often indicated that students were only asked to recall or perform rote tasks in a short amount of time. Walsh and Sattes' (2005) findings suggested that to develop higher cognitive demand in student discourse the questions or tasks implemented by the teacher needed to be thought provoking and engaging. These types of tasks and questions require students to spend a longer amount of time analyzing and formulating their thoughts through mathematical discourse. The latter instructional method conveys a teacher belief of math as discourse (Sfard, 2008) and a means to conceptual understanding.

The type of teacher belief, about mathematic content and pedagogical knowledge, needed to support higher cognitive demand in mathematical discourse requires teachers to personally experience and build confidence in their own mathematical ability. Polly, Neale, and Pugalee (2014) found that 84 hours of professional development which addressed content, pedagogy, and student learning produced a statistically significant gain in teachers' mathematical knowledge of mathematics and teaching mathematics. As a result of a change in teacher belief, pedagogical change was also noted. The results noted in Polly, et al.'s (2014) study support the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) which demonstrates the need for belief change, in response to new knowledge, prior to practice change. This model forms the theoretical foundation of this study.

Chapter 3: Methodology

This chapter describes the methods in which data were gathered through observation, artifact, and interview. These data shed light on teacher instructional beliefs, planning, and implementation of mathematical discourse, and student participation in mathematical discourse to provide further understanding of the instructional practice of mathematical discourse in elementary instruction.

Research Questions

The purpose of this study was to examine the processes teachers with post graduate mathematic coursework took to implement mathematical discourse. This was done through analyzing how teachers believed mathematical discourse fit into the learning process, how lessons were prepared, how teachers facilitated mathematical discourse, and how students participated in it.

This qualitative study took a post-positivist interpretive approach through the triangulation of data from multiple sources and followed a process of data distillation during analysis while still using interpretive data collection methods of observation and interview. The post-positivist researcher does, "not assume that their methods ensure certainty and universally generalizable results, or even take this as their goal" (Charney, 1996, p. 579). The post-positivist approach makes efforts toward context based generalizations.

Six elementary classrooms in one Oregon school district were studied, three in each of the top 20% and bottom 25% SES levels, to provide a comparative analysis of four factors: the mathematical discourse used by students, the mathematical discourse facilitated by the teacher, materials and planning of the instruction by the teacher, and the beliefs about math instruction held by the teacher. Specifically, this study will answer:

1. How do teachers think about mathematical discourse in the learning process?
2. How are lessons prepared to include mathematical discourse?
3. How do teachers facilitate mathematical discourse?
4. How do students participate in mathematical discourse?

Rationale for Methodology

A case study framework in a naturalistic setting best met the need of this research to provide observational data, self-reported data through interview, and artifact review. The descriptive focus of the data was designed to create a more complete picture of the setting in which the mathematical discourse takes place along with its observable influences on student and teacher interactions (Miles, Huberman, & Saldana, 2013). The Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) was used as the theoretical model for analysis of data. This model was used as the theoretical basis for discussion that addressed this study's questions and to develop conclusions.

Boston's (2012) IQA rubrics (Appendix A) were used to organize and interpret the cognitive level of discourse, question types, accountable talk, which includes linking and pressing as defined by Boston, and academic rigor of instruction. The IQA has been validated through multiple studies as a means to evaluate academic rigor in mathematics instruction (Boston, 2012; Matsumura, Garnier, Slater, & Boston, 2008; Wilhelm & Kim, 2015). The IQA (Boston, 2012) contains categories that focus on the task potential and task implementation that were found to be critical in the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) Project (Silver & Stein, 1996). The QUASAR Project was a five-year study that focused on analyzing reform efforts that attempted to provide opportunities for students to talk and reason through tasks, in an effort to improve student learning outcomes in diverse urban areas.

This study did not use the IQA (Boston, 2012) quantitatively but the domains were used to categorize observational and artifact data. The IQA (Boston, 2012) domains contained linking — teacher or students connect various mathematic methods that have similarities or supportive mathematical characteristics, and pressing — student or teacher questioning that caused students to think more deeply about their mathematical reasoning, providing — student responses, potential of the task, implementation of the task, student discussion, and mathematical residue — how the discussion extends or solidifies student understanding of the concept. This structure framed the qualitative theme analysis supported by the triangulation of data to create a deeper understanding of the intricacies and variables that influence mathematical discourse in elementary classrooms.

Three data sources, instructional observation, teacher interview, and artifact analysis, gave a richer perspective of mathematical discourse in elementary classrooms and provided insight into mathematics teaching and learning through discourse.

Participants and Setting

Observations of mathematical discourse were conducted in six elementary classrooms, in three schools, in the same district (Table 1) encompassing 125 students over a four-month span, October 2016 through January 2017. Criterion sampling (Patton, 1990) was used to develop an understanding of mathematical discourse from teachers who have access to the same district supports and trainings but have student populations on opposite ends of the economic spectrum. The following conditions were used to select teachers and classrooms for participation:

- Teacher completion of at least one course in mathematics, beyond typical elementary education certification, that include the practice of mathematical discourse. Graduate student rosters and recommendations from district staff were used to obtain possible participants.
- Six classrooms in the same Oregon school district representing the three of the highest and three of the lowest SES communities of the district.
- Classrooms at the same grade level, paired between schools with less than 25% economically disadvantaged and more than 75% economically disadvantaged. Building SES levels were obtained through district free and reduced lunch counts.
- Students representation at each SES level provided subjects with age comparable language levels so that observations were not influenced by age disparity.

Formally adopted mathematic materials within the guidelines of the state of Oregon were being used in 100% of the classrooms observed with individual classrooms using varying degrees of district created supplemental materials. Formally adopted materials give consistency throughout individual classrooms and schools within the same district but may not be adequate for implementing effective mathematical discourse to increase conceptual understanding. While teacher created or obtained materials can benefit students by being specifically developed or chosen to reflect the population in order to utilize students' current knowledge, the materials created or obtained by the teacher are dependent on the individual teacher's ability to analyze the context, content, and have enough content and pedagogical knowledge to develop or choose materials that address the needs of the class and challenge students to develop a deeper understanding of mathematics. Minimal teacher created materials were observed being used during instruction and through artifact analysis.

The six case studies are bounded by three characteristics to create a depth and breadth of data for analysis. The first characteristic was classrooms which were matched by grade level. Matching classrooms by grade level allowed for analysis of mathematical discourse across two classrooms with students at the same age range. This decision was made to maximize possible commonalities and differences in individual grade levels. This created a richness of data that allowed comparative analyses at the same grade level. If there were only one classroom at each grade this analysis could not be made. Comparing the mathematical discourse of second grade students to the mathematical

Table 1

School Demographics

Participating School	#1	#2	#3
SES			
Free Lunch	15.3%	75.1%	21.7%
Reduced Lunch	4.7%	0%	2.2%
Ethnicity			
Black/African American	2.69%	1.31%	0.66%
American Indian/Alaskan Native	0.54%	0.44%	0%
Asian	6.46%	0.66%	3.52%
Latinx	14%	64.85%	17.62%
Multiple	9.16%	4.80%	8.15%
Hawaiian/Other Pacific Islander	0.90%	0%	0.22%
White	66.25%	27.95%	69.82%
English Language Learners			
Active	7.90%	45.63%	5.95%
Monitored	0%	3.28%	1.76%
TAG	10.77%	2.62%	9.25%
SpEd	10.95%	8.95%	10.13%
Enrollment			
Total Students	557	458	454
2nd grade	78	62	64
4th grade	81	72	62
6th grade	82	68	58

Note: Free and reduced lunch percentages as reported 11-3-16; Ethnicity, enrollment, and ELL as reported 11-28-16 discourse of sixth grade students would not allow direct comparison because different levels of content, teaching materials, and developmentally appropriate language were used in second grade compared to that used in sixth grade. In addition, teacher content focus and pedagogical strategies would have a greater possibility of being different because of the developmental and content differences among grade levels. Secondly, grade levels for inclusion purposely spanned second through sixth grades at two year intervals. Second, fourth, and sixth grade classrooms were recruited in order to analyze developmental differences between discourse of students at different age levels. While pairing of the same grade levels allowed for a horizontal comparison of teacher belief, pedagogical practice, and student discourse, providing paired samples of participants across three grade levels allowed for vertical analysis in the progression of mathematical discourse as participants' linguistic skills and mathematic content being taught developed. Thirdly, reputational case selection was used. Teachers and their classrooms were chosen based on the recommendation of their college professor and/or district's Math Teacher on Special Assignment (TOSA) as teachers who actively planned and used mathematical discourse in their instruction. All six teachers (Table 2) were seen by district personnel as leaders in their buildings at implementing successful mathematical strategies to increase student acquisition. This allowed for an abundance of data in how these six individuals perceived and interacted with mathematical discourse from their beliefs about it, through planning, to their implementation decisions. While all six teachers received extended training involving mathematical discourse, from the same sources, the amount of training and their personal mathematic backgrounds varied which are influences on instruction based on the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1).

Table 2

Participating Teacher Demographics

Teacher	School SES	Grade Level Taught	Years' Experience Teaching	Post graduate mathematic coursework (number of courses completed)	
				Leadership & Coaching	Content & Pedagogy
Angela	high	second	17	0	1
Judy	high	fourth	7	1	3
Juan	high	sixth	11	1	3 (+ math endorsed)
Julia	low	second	28	1	2 (+ BS in math)
Tom	low	fourth	4	1	0
Laura	low	sixth	10	1	2

Note: self-reported during interview. Pseudonyms were used.

These six cases studies added confidence to the findings by interpreting a range of contrasting and similar cases in which mathematical discourse took place. Use of the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) provided the basis for the underlying theory of this study and the six case studies provided a continuum array in which to compare.

Design and Procedures

Six elementary school teachers and their students, in Oregon were selected to participate in this study based on interest generated through taking post graduate mathematic pedagogy courses that lead to Oregon's Elementary Math Instructional Leader specialization. An open invitation was given during class to teachers currently participating in post graduate coursework in addition, teachers were recruited by the district's Math TOSA through the district math team. Parameters for participation of matching grade levels in high and low SES level schools resulted in the participation of two second, two fourth, and two sixth grade teachers and their students, one from each high and low SES level at each grade.

Two 45-minute observations using the IQA (Boston, 2012) framework were conducted, in accordance with previous research using this tool (Matsumura, Garnier, Slater, & Boston, 2008; Wilhelm & Kim, 2015). Each teacher was observed during their normal workday instructional time. The observations spanned October 2016 through January 2017. A calendar was developed with the classroom teachers, prior to the observations, to give teachers enough time to plan instruction for the day of the observation. In addition to using the IQA framework during observations, the observations were videotaped to selectively transcribe and qualitatively review transcripts during analysis of data. Initial coding of observation was done using the 11 domains of the IQA (Boston, 2012) however quantitative scoring of these domains was not used in data analysis. The domains were used as an organizational tool for the qualitative data collection. Quotes and anecdotal notes were grouped by IQA domain, then a secondary distillation of data was conducted under the categories of teacher planning, teacher facilitation, and student participation. The fourth category of teacher belief was organized by interview questions.

The descriptive nature of the data was used to expose the reader to what was happening in the classroom during mathematical discourse. This method allows the perspective of the teacher and the student to surface to create a more encompassing perspective (Altrichter, 1993).

Individual interviews (Appendix B) were conducted with each teacher in order to give the teacher opportunity to expand on facets of mathematical discourse, teacher planning procedures, pedagogical and content beliefs, as well as, what the teacher valued in mathematic instruction for the grade level they were teaching. Each interview was audiotaped, transcribed, and took 30 to 60 minutes as necessitated by the teachers to explain their thinking about mathematical discourse more completely. The interviews were taken after at least one observation. In some cases, scheduling necessitated the interview taking place after both observations. All participants were asked the same predetermined questions and teachers were prompted, on an individual basis, to expand on their explanations to provide further insight into and clarification of their thinking about mathematical discourse.

Instruments

IQA. The IQA (Boston, 2012) provided a manner to report first-hand accounts of teaching and learning that went on in the mathematics' classrooms through direct classroom observation and analysis of artifacts. Prior to classroom observations Dr. Boston was contacted and training materials were obtained and used in accordance with Dr. Boston's direction. Training materials consisted of written guidelines, scored student samples, and videotaped observations as well as annotated use of rubrics. After completing all training procedures, made available by Dr. Boston, a pilot was conducted in two non-participating classrooms; one third grade and one sixth grade, prior to starting observations with participating classrooms.

Two observations were conducted in accordance with the reliability requirements of the IQA (Boston, 2012). Teachers were asked to engage students in a problem-solving task followed by whole class discussion as done in previous reliability studies (Matsumura, Garnier, Slater, & Boston, 2008; Wilhelm & Kim, 2015). Five point descriptive rubrics were used to score each of the 11 dimensions of the IQA and implementation checklists (See Appendix A for the complete classroom observation rubrics). The observations were videotaped to provide the opportunity to transcribe exact discourse exchanges that were analyzed to supplement and support notes taken and rubrics used during the observation. While rubrics were completed during all observations the themes that emerged during analysis did not rely on the rubrics and the quantitative data was not incorporated into this study. Video transcripts were also used to accurately include narratives in descriptions of classroom discourse.

Qualitative observational data were initially organized with the IQA into all 11 dimensions in the two areas of academic rigor and accountable talk. Accordingly, detail and rigor of expectations, and potential of task were grouped in teacher planning; teacher linking, teacher press, implementation of the task, questioning rigor, and mathematical residue were included in teacher facilitation; student discussion following the task, student questioning, student responses, student participation, and student linking were included in student participation (See Appendix A for domain descriptions). As overall categories emerged in the data analysis, interview responses were grouped, as appropriate, into each of the four concepts — student participation, teacher facilitation, teacher planning, and teacher beliefs.

Boston's IQA also addressed the thinking processes that a task has the potential to elicit, which was referred to as the cognitive demands of an instructional task by Stein, Grover, & Henningsen (1996). When student participation is analyzed the potential of the task becomes important. Tasks used in mathematical discourse with high-level cognitive demands have the potential to engage students with high-level thinking processes, such as problem-solving, conjecturing, justifying, generalizing, or proving (Van de Walle, 2004). The opposite of which is also true. If the students do not have a high cognitive task to talk about the result is mathematical discourse may not have the potential to promote mathematical reasoning.

Teacher interview. Thirteen questions were asked of teachers to gain clarity on the teacher's planning, instructional background, and beliefs about mathematics instruction and student learning. (See Appendix B for interview questions.) These interview questions were developed to clarify teacher planning procedures, and instructional beliefs. The questions were divided into four sections: demographic information, views on and experience with mathematic student learning, teacher facilitation of instruction, and beliefs around discourse in mathematics and SES of students. Questions were written and then reviewed by 14 professional colleagues, after which they were piloted with non-participating teachers on four occasions to clarify wording of the question, assess responses to provide data sought, and to measure time of response so as not to exceed one hour.

Interviews were conducted in a private room where only the teacher and researcher were present and would not be interrupted. Notes were taken during the interview and interviews were audio recorded for transcription and later analysis.

Problem solving belief and instruction model. The Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) was used as the theoretical model for analysis of data. This lens was used to synthesize data collected through interview, observation, and student work to provide commentary for this study's questions.

Ethical Considerations

The Institutional Review Board of the University of Portland, in Portland, Oregon granted permission to conduct this research on July 30, 2016. Teacher pseudonyms were given and all data were reported in the aggregate or in a manner that no personally identifiable data were recognizable. Individual schools and the names of the school district are not included in this study. All versions of the data were kept electronically under password protection on all devices. Within each password protected device an added level of security was taken with the individual raw data documents locked with a secondary passcode. Signed consent forms from all subjects and district representatives were kept electronically under above security. Any signed paper forms and student artifacts were shredded after being electronically uploaded.

Role of the Researcher

I am a focused participant observer in this research. I do not work in the same building as any of the participating teachers and have no pre-existing relationship with any of the participants or their students. As a Mathematics Instructional Leader for my school district I have participated in five years of coursework that promotes students socially constructing understanding of mathematics content through discourse. I have participated in my district's math grant for the last two years and am currently in the process of applying for an Oregon state Elementary Math Instructional Leader specialization. Participation in the coursework to obtain my Elementary Math Instructional Leader specialization has developed a bias toward the importance of conceptual understanding in mathematics and the role that mathematical discourse plays in that understanding. This background and belief is contrasted by some colleagues' who have differing views on the importance of mathematical discourse. The contrast of belief in the value of mathematical discourse along with the low math achievement and cultural gap in the school in which I am employed, is what prompted me to make mathematical discourse and its implementation in elementary classrooms the focus of my study.

Over the last eight years I have been an instructional leader at the school and district level, provided professional development as an Instructional Coach, which included math professional development. My role for the 2016 – 2017 school year has changed and in my current position I am not in a role that includes math instruction or mathematic leadership in the form of professional development. I had no influence and provided no mathematics instruction to the teachers or classroom children participating in this study. Prior to this study I had no knowledge of and never entered any of the participating schools.

My background in English Language Development, as a teacher of English Language Learners with an Oregon English to Speakers of Other Languages (ESOL) endorsement, as well as my experience as an endorsed Reading Specialist, and my participation in the Elementary Mathematics Instructional Leader program provided insight into language levels of students as well as mathematical practices and concepts that promote achievement. As a classroom teacher, discussion was always prominent in my classroom and I continue to promote collaboration in learning among students using cooperative learning strategies daily which may create a bias toward these types of instructional strategies.

My upbringing has influenced the pedagogical strategies I deem productive in the classroom. As a child, I talked through or drew out problems to understand the math. I value discourse as a means to develop, not only academic knowledge, but also to build upon student cultural backgrounds which differs one student to the next.

I grew up middle-class in a Central East Coast state outside two large metropolitan cities. The neighborhood in which I grew up was diverse with African, Cuban, European, Polynesian, white, and black families all living on the same block. The greater community in which I was raised was approximately 45% white, 40% black, 10% Latino, and 5% other races. Economically the community in which I was raised ranged from working-class through upper-middle-class, creating multiple economic cultures socially interacting. This exposure to cultures, outside my own, provided a comfort level with the different cultural ways of expressing one's self found in mathematical discourse.

One of my parents was born outside the United States and the other was born in the United States but did not speak English until entering school at age six. Italian was spoken in my household prior to the age of eight but I am a native English only speaker. I was raised with a severely developmentally disabled sibling which gave me insight into some of the stressors students encounter in their lives that impact a student's ability to participate in classroom discussions. While some cultural aspects match student populations being observed, being raised in a white middle-class home did privilege me in academic settings.

While growing up with both parents in a middle-class family, the immigrant status of my parents and the disability of my sibling made my upbringing closer to a working-class family with culture and stressors different than stereotypical white middle-class families. My personal background may bias me toward academic language and discourse but it has also given me insight into immigrant families, and those communities that are not part of the dominant academic culture, as well as home stressors that may impact student participation in mathematical discourse.

Data Analysis

Data were obtained through observation, student artifacts, and teacher interview. Observational and student artifact data were used qualitatively based on the IQA framework (Boston, 2012) grouping qualitative observational data and artifact data into 11 distinct domains: Language from the mathematic lessons was bundled into discrete language samples that represent student language on one topic within the content: explaining procedure, explaining reasoning, questioning others, and clarifying thinking; as well as teacher language: giving directions, explaining process to solve a problem, asking clarifying questions about process, asking open ended questions that press students to think more deeply about mathematical concepts, and asking funneling questions to lead students to a specific mathematical procedure. This procedure of grouping was done for whole group discussions. This data were organized into a two-column table with observational notes and times recorded in the left column and video selective transcripts and common characteristics of the conversations based on video review in the right column.

Interviews consisted of 13 questions allowing teachers to expand on their thinking about mathematical discourse in the classroom. The interviews were audiotaped and transcribed in full. The questions were developed to provide insight and clarification of areas of beliefs and implementation. In addition, these questions were linked to the first study question to pull out more information about instructional beliefs and implementation, as well as teacher beliefs about mathematical discourse and teaching student populations of varying cultures. Interview transcripts were grouped into four coding categories; teacher beliefs, teacher planning, teacher facilitation, and student participation. Contrasts and comparisons were made among the data sets looking for logical chains of evidence to support conclusions on the influence of teacher belief through planning and implementation on student participation in mathematical discourse (Miles, et al., 2013).

Through the detailed descriptive analysis of observation, interviews, and student artifacts, concepts emerged to support study questions (Creswell, 2013). These concepts concern the specific impact of teacher beliefs on instruction which impacts how and what students learn about mathematic concepts (Anderson, et. al., 2005), such as the focus of quickness of computation versus the ability to discuss mathematically, and how/if these beliefs and language intermingle with beliefs about student cultures and ability to use language to explain thinking.

Each of the six case studies was analyzed to look for patterns among each individual case study in a cross-case synthesis (Yin, 2009). Naturalistic generalizations were then inductively developed through conceptually clustered matrices (Miles, et al., 2013) the concepts that emerged derived from the 11 domains of the IQA (Boston, 2012) and the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1).

Data were initially parsed into the 11 IQA domains and the 13 survey questions, which were then grouped into four concepts; teacher beliefs, teacher planning of lessons to include mathematical discourse, teacher facilitation of mathematical discourse, and student participation in mathematical discourse. The data in these four concepts were cross referenced for common and outstanding characteristics. The concepts were then analyzed and four themes emerged; the importance of confidence and persistence, the influence of teacher math experiences, differences in learning between SES levels, and supports available to teachers to facilitate mathematical discourse.

Limitations

While the limited number of classrooms observed make generalizations difficult, the richness of data obtained from these six elementary classrooms contributes to an understanding of other more wide-scale data collection (Wilhelm & Kim, 2015). The qualitative nature of data that were collected provides insights into the specific situations of each classroom participating; however, the limited number of classrooms at each grade level makes it impossible to generalize to the greater community.

All attempts were made to make sure participating teachers had a background in and belief of the benefits of mathematical discourse that went beyond initial teacher preparatory courses. However, since this research focused on how teachers think about and implement mathematical discourse and how their students respond to that implementation, other factors such as teacher content knowledge, previous teaching experiences, and outside factors of the community in which they teach was not fully known prior to the study and could potentially impact the teacher's facilitation of mathematical discourse.

While observed, cultural match or mismatch of teacher and students in instructional methods and language was not a focus of this study. An in-depth study of equity in mathematical discourse would necessitate a larger sampling over an extended period, to produce data that would be useful in generalizing the influence of teacher cultural attitudes and practices on student acquisition of mathematic content. While this was not the focus of this study, general comparisons were made among SES demographics.

The data collected covered one moment in time, of a limited number of classrooms at the beginning of the school year. Results may have varied if data were collected at multiple times throughout the year. A longitudinal study following the same cohort of children over several years would add to the development of academic language and the influence of individual teachers and their belief about mathematical instruction and discourse.

Summary

This study compared mathematical discourse as measured by patterns that emerged in teacher beliefs, planning, and facilitation of mathematical discourse and student participation in classroom instruction. This analysis was based on the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1). Initial analysis was grouped by the pre-existing domains of the IQA (Boston, 2012) then parsed into the four concepts that emerged from the interview, observations, and artifact analysis. A tertiary grouping of four themes — the importance of confidence and persistence, teacher math experiences, differences in learning between SES levels, and supports available to teachers to facilitate mathematical discourse — cut across concepts was then analyzed. This narrowing process allowed patterns to emerge among each of the six case studies.

Chapter 4: Results

The results of data collection are reported in this chapter. The data were organized into four sections to reflect four different aspects that are associated with the research questions; teacher beliefs, teacher planning of lessons to include mathematical discourse, teacher facilitation of mathematical discourse, and student participation in mathematical discourse. Using the structure of the IQA (Boston, 2012) domains, interview questions, and Anderson, et. al.'s model (2005) as a guide, data were collected and parsed into four categories to support the following research questions.

1. How do teachers think about mathematical discourse in the learning process?
2. How are lessons prepared to include mathematical discourse?
3. How do teachers facilitate mathematical discourse?
4. How do students participate in mathematical discourse?

The first section focuses on teacher beliefs. This section includes teacher beliefs and experiences with mathematical discourse as self-reported through interview. The second section focuses on teacher preparation of lessons to include mathematical discourse. This section delves into the teacher's planning process; how they think about mathematical content and what materials they use to facilitate discourse in their math classroom. The third section focuses on teacher implementation of mathematical discourse. Examples of observational data and student work are included to demonstrate how each teacher implements mathematical discourse in the classroom. The fourth section describes student participation in mathematical discourse through observed student discussion and teacher reflection on student participation.

Beliefs

The results in this section pertain to how teachers, specifically those who have had post graduate mathematical coursework, think about math instruction and how their personal experiences influenced their beliefs about mathematical discourse and content. Interview questions were designed to support the first research question:

> How do teachers think about mathematical discourse in the learning process?

What is important for students to learn? When asked about the most important thing for students to learn, all teachers focused on qualities of learning over content. As Juan stated, "If students can leave my classroom with the ability to figure out math it doesn't matter what I don't teach them because they'll figure it out." This sentiment was reflected in all teacher responses. Participating teachers reported that the qualities they valued were influenced by CCSS Mathematical Practices (2010):

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Math confidence. A common thread through all teacher beliefs about the most important thing for students to learn was students' ability to be confident in their math capabilities, however, the three teachers who taught in the lower SES school, Julia, Tom, and Laura, highlighted this quality more so than those who taught at higher SES schools. The importance of feeling confident "in manipulating" values were expressed by Julia, while Tom expressed that confidence, "is something they can take with them for a lot longer than… content," and Laura felt knowing, "that math is accessible to them," is the most important thing for students to learn. In higher SES schools, teachers responded that the prominent characteristics for their students to learn were confidence with being able to discuss mathematically using logical reasoning, working together, celebrating challenge, being persistent, and confidence in taking risks.

The values that teachers held were often tied to their own personal experiences. Angela's childhood experience of receiving instruction while remaining silent was highlighted as why she felt mathematical discourse was vital. She felt she was at a disadvantage when she started college because her own personal experience with K-12 education was to sit quietly in class, listen to the teacher, take notes, and memorize what the teacher told her. When she got to college she did not know how to have an academic conversation and was uncomfortable sharing her thinking. She reflected that her students knowing how to have an academic discussion was vital because "reasoning, supporting your thinking, and questioning another's thinking was important in every aspect of life, not just in math."

Juan's perspective was based on his personal experience of always enjoying the challenge of math. He equated enjoying a challenge to not giving up. He wanted to instill persistence with his students by taking time during math to explicitly talk about what it feels like to persist, even when math is challenging. He wanted his students to learn that, "success is persisting and looking in different ways for something that makes sense; not waiting for someone to give you the answer."

Judy talked about her own experience in high school when she felt she "was not able to do math" and that she "should have had some sort of intervention." She believed if the math content was presented to her differently and if she was taught to talk through her thinking she would have had a positive relationship with math and would have been successful in high school math. Because of these experiences, Judy intervened when she saw her students struggle and helped them understand they may not currently be successful with math but they could develop the qualities to be successful. She wanted students to believe in themselves, so even when they did not understand math content they felt they had the skill to figure it out and persist through the struggle.

Persistence*.* The theme of students learning persistence came out in all interviews when asked about student struggle in the classroom. This belief draws directly from the first CCSS Math Practice; Make sense of problems and persevere in solving them. Angela introduced the concept of productive struggle early in the school year because she believed persistence is important to success in mathematics as well as "a life skill that is important for children to learn early and continue practicing because it applies to everything" in life. This belief came in part from her personal experience raising her own children and helping them persist when they struggled to complete something. She told her own children and her students, "This is the time to create those mistakes because you learn from it [*sic*] and that's why I'm here and it's my job to help you through it."

Julia had many years of experience with students coming to her class at the beginning of the year knowing procedures to complete a task without understanding the mathematic concept of the same task. An example she gave was she could ask students, "What is area?" and they would answer, "length times width," but they did not really understand what area was, they could only recite the memorized formula. Because of this, she focused on helping students persist to understand the concept. When her students struggled with understanding concepts of math, they wanted to stop as soon as they produced an answer. She believed students focusing on getting an answer created a situation in which students lacked the skills to persist to understand the mathematical concept. Lacking these skills in turn produced incorrect answers and lack of understanding, on her students' part, to know the answer was incorrect. She believed that focus on the correct answer produced students who were good number crunchers, who did not understand the math, and were unable apply the procedure outside the original context to a new situation.

Judy looked at persistence through mistakes. She believed that "being able to problem solve, and struggle through something, and understand that just because you get a wrong answer [*sic*], mistakes aren't bad, mistakes are when we learn." She, like Julia, believed she is undoing several years of training that focused children on getting the answer and not necessarily understanding math. She found that students that usually got correct answers, by following memorized procedures, struggled to persist when their first attempt did not arrive at the *correct answer*. These students did not always understand math, how to look at a task in different ways, and persist and learn through their mistakes. They tended to give up quickly and want to be told, *how to do it,* as opposed to understanding the math. She had a philosophy of complimenting student work ethic and persistence over correct answer. She believed that not being afraid of working hard and resilience in the face of failure is what will help students in life.

Student struggle. Both Juan and Laura talked about student struggle and persistence through the importance of working with others. Laura saw some students try to work individually and memorize everything instead of working with their team to understand mathematical concepts. She noticed that when the school's math materials changed from material that focused on memorized procedures, that were assessed by repeatedly "doing the same thing," to materials that required students to apply mathematical concepts to new situations, students that previously received high scores struggled because they did not understand the mathematical concepts. These students, she reported, also did not like working with others because they valued rapidly getting answers more than talking through and understanding the math more deeply with their classmates. Juan also related achievement in math to the student's personal experiences with math. He believed student ability to persist to understand a concept promoted a higher collaborative inclination and a higher rate of catching errors in calculations.

The importance of discourse in learning mathematics. When asked about math instruction, teachers expressed an evolving belief system. Julia believed, "that the way teachers teach comes directly from their preservice and inservice experiences and training." This idea of the influence of experience and coursework was echoed by the other teachers interviewed. Their approach to math instruction has changed since starting post graduate coursework, which focused on increasing teacher content knowledge and pedagogical skills. These teachers acquired a stronger belief in mathematical discourse as a means to promote student conceptual understanding. They expressed the importance of students explaining their thinking, critiquing others, being a critical listener, and thinking through someone else's thinking all of which is done through dialogue, writing, and drawing models which communicate understanding. However, the idea of communicating understanding was only one aspect expressed as why mathematical discourse is vital to student learning. Four of the six teachers brought up the aspect of talking through your thinking to learn. Angela expressed that "being told and doing is very different," students need to talk to learn. Teachers believed that mathematical discourse empowered students because as Angela stated, "the more they are able to talk about it the more they grasp" the mathematical concepts.

Juan observed that he knows "there is something connected to [the] verbal process with math that brings new understanding. It's never listening, it's never writing, it's never doing the math problem, it's when they're talking about what they're doing, when they're talking about the concept" that is when they learn. Laura supported the idea of learning through mathematical discourse when she discussed the concept that learning is a communal experience and cannot be done in isolation, therefore, discussion is necessary for any learning. Juan found that if he "slows down" and gives students ample time to discuss mathematical concepts he sees and hears them forming the concept in a concrete way through their words.

While all teachers expressed the importance of mathematical discourse, teachers at the school with more English language learners also expressed the need to increase and support student vocabulary through discourse. There were different feelings about using mathematical discourse with a large percentage of English language learners. Julia believed that since her students had more explicit language instruction, because they are English language learners, they are more confident in their speech and their discourse skills are stronger than students who speak English only, even when the English learners have less academic English vocabulary. She stated that, “over the years, because of the large ELL population, students have been given a lot of opportunity and structure to help practice and support language use in the classroom.” Whereas, Tom expressed concerns that his students were working from a deficit because they had the added hurdle of learning and talking about content in a language they were still learning. Laura expressed that, “every kid has the potential to do something great” and even though she acknowledged the achievement gap between her students that spoke English only and her multilingual students she believed, if all teachers were to “focus on discourse of content rather than just the content” student math knowledge would improve.

Pedagogical beliefs. Research conducted by Donovan and Bransford (2005) as well as Lester (2007) suggested the foundation of effective mathematical teaching includes constructing “knowledge socially, through discourse, activity, and interaction related to meaningful problems" (NCTM, 2014, p. 9). This constructivist view of learning as a social behavior is the foundation of mathematical discourse.

Julia summed up all six teachers' beliefs around pedagogy, "[I am a] two thirds believer in the constructivist view and that they really do have to make meaning of it on their own, so you want to provide all those opportunities…. Just that idea of students making their own meaning and coming up with their own equations… but at the same time there's this other third of me that knows there are certain things that they need to learn from me." This push and pull of teaching students to memorize components of math but still provide opportunities for children to think mathematically and experience math on their own terms is a struggle that all six teachers expressed.

Angela expressed that, after taking post graduate coursework centered on pedagogy and content, she was, "more open minded about what math can be or how math can work instead of it" being a series of equations and procedures to arrive at a predetermined answer. She believed that the "children should be the thinkers of their own learning and the teacher is more of a facilitator of the discussion." When reflecting on pedagogy Judy believed that, "not everyone can get what they need from a worksheet or the teacher talking to them;" students need "exploration of the core concepts [which are] crucial to [student] understanding." Judy went on to say, "we are doing a disservice to the students in current mathematical classrooms… they need that exploration time more than anything. [Not allowing exploration is] like taking the base out of the structure. Things fall apart." Laura also discussed her instructional belief change after taking post-graduate coursework. She confessed, "I was very worksheet-based because I didn't understand the curriculum." The coursework helped her understand the content better and why the curriculum was set up to promote discussion.

Laura's reflection on pedagogy focused on developing lessons that helped children understand math concepts through discourse and hands on experiences. She felt that, "getting the right answer is not as important as understanding how to do something." Juan agreed with this sentiment and connected it to the push and pull teachers expressed about memorization versus understanding concepts, "you might be good at calculating and memorizing things but you don't understand the concept or you might understand the concept but you don't have your facts down, so until you have all of those things in place you are not a mathematician."

Five teachers in this study expressed that the post graduate courses they took in teaching math content changed how they viewed math instruction. Angela said that her, "eyes have been open to find out how much children bring into their own learning," and that teachers should allow, "children to be the thinkers and process their and other students' ways of thinking."

Juan contrasted his post graduate math coursework to his own pre-graduate teacher education experience where, "everything [he] learned was a very traditional way to [teach] and [he] was not successful in that traditional [way], [he] always felt there was a different way to teach and reach students who struggle and don't come in already understanding math content." He explained that his post graduate coursework, as well as his work with the company that published the curriculum materials he is using, helped him find a way of teaching through mathematical discourse structures to promote conceptual understanding.

Tom was the one teacher that did not mention a change in belief based on post-graduate coursework. He is also the only teacher who has yet to take a mathematic content and pedagogy course. At the time of this research he had completed one course in mathematical leadership which focused on a teacher's leadership role. This course did not develop mathematic content knowledge or pedagogy.

Beliefs about mathematical discourse in high and low SES schools. When reflecting on how her instruction might be different in a school with a higher SES and lower ELL populations Julia responded, "My guess would be that the high kids are going to do all the talking at [higher SES schools] and the kids that aren't as high, [who] are not confident, will let them do the talking." She contrasted this to her students who all attempted to participate in class discussions and felt comfortable making mistakes in front of their peers. She attributed this to explicit academic conversation supports given to students from when they first enter her school. With the substantial ELL population at Julia's school, she said, "language supports are put into place early in all aspects of student education, and children become comfortable expressing their understanding in academic situations."

In analysis, the strongest theme that emerged as an influence of SES level was a parent's beliefs about education and math instruction, and the family's ability to access learning opportunities in their communities.

Family and Community. When discussing differences between high and low SES schools. Five of the six teachers pointed to the outside influence of family and community on a student's ability to achieve in math. Laura found strength in students who were first generation American stating that with, "first generation families, school is really important and a lot of our second and third generation families, it slowly gets less important." She believed this created a positive atmosphere for first generation students to learn in school.

Angela, Judy, and Juan, who teach at higher SES schools, expressed that their students come in understanding academic vocabulary, behavior expectations, along with having families that have the finances to expose their children to many opportunities that support their understanding of content. Teachers working in higher SES schools commented that all this benefits instruction allowing teachers to go more in depth with math concepts more quickly. Angela explained that at one lower SES school, where she taught, many families did not have cars, making it difficult for families to take advantage of community resources, like libraries, in the same way that wealthier children were able. She believed that the ability to access learning opportunities in the community advantaged students who accessed those resources on a regular basis.

While Angela could not find any negative aspects about teaching at a higher SES school both Judy and Juan pointed out that parents tend to push back strongly when students are taught differently than when the parent was in grade school. They believed the push back is probably not as strong in lower SES schools. Judy explained that parents at higher SES schools sometimes felt their child was not being challenged because they "don't value how we're challenging their child to understand the concept and not only get the answer by following a procedure they don't understand." She believed that this parental influence on students “leads kids to rush,” to get the right answer. Students think, “Why do I need to understand the math because I can just use a calculator?” She continued, “A calculator is not always going to save your bacon. I mean if you don’t know what’s going on in the math and you type in a [wrong] number you’re going to have a problem, and this can affect you financially, it can affect your job… it can affect their daily lives.”

On the other hand, Juan also pointed out that students are, “heavy in positive mathematical resources” as compared to his experience teaching in a low SES school. He felt that teaching at a low SES school was more challenging to build the structure needed to teach math. He went on to say that both types of schools can have roadblocks to student learning just in different ways.

Lesson Preparation

The results in this section pertain to how teachers, who have had post-graduate mathematical coursework, plan math instruction. Julia expressed the sentiments of every teacher in this study concisely, “I did not go into teaching to design curriculum. I take what I have and make it better by the questions I ask the students and [my expectations] of them.” While all teachers initially expressed the sentiment of following the curriculum’s scope and sequence provided by the district, they realized, upon further reflection, that they were not following the curriculum blindly. They put thought into how the content would be presented to facilitate a greater amount of student participation and discourse.

This section focusses on how teachers used district approved materials and incorporated mathematical discourse into their lessons. Interview questions, classroom observations, and student work samples support the second research question:

How are lessons prepared to include mathematical discourse?

Several themes emerged that provided support to teachers when they planned instruction. The following section discusses district developed resources, district adopted materials, and how these resources influenced mathematical discourse, as well as, how teachers prepared lessons to include mathematical discourse and lastly, instructional strategies used to support mathematical discourse for students.

Resources. All teachers who participated in this study taught in the same school district but did not all use the same instructional materials. The two sixth grade teachers, though housed in elementary schools, used the materials that were adopted by the middle schools, which were different than kindergarten through fifth grade materials. Three of the four second and fourth grade teachers used the most recent version of the district adopted materials while one of the fourth-grade teachers used an earlier version of the district adopted materials which has been adapted with supplements by the district to address Common Core State Standards.

All teachers expressed that they followed the curriculum, but when they described their process more deeply they realized they followed the structure of the curriculum but adapted it as their experience dictated to fit the needs of their students. All teachers felt the materials adopted by their district provided good opportunities for students to talk and make meaning of math content. In addition, the districts math leader team has created and gathered lessons and activities to supplement the purchased materials. The supplemental materials include:

- *Low floor/high ceiling tasks:* Tasks (math problems or situations) that have easy entry points for all students, and have the potential to go to a deeper level of math concept.
- *Which one does not belong questions:* Students are asked to analyze which problem does not belong in a four-square grid with math content in each square. There is no correct answer. This is used to promote mathematical reasoning skills, discourse, and support practice in defending current understanding while critiquing other students' understanding.
- *Three reads:* A math story problem is posted without the quantities and without the question. The first read is to make sense of the situation. Students brainstorm questions they have and what could possibly go in the blanks. Values are then put into the problem and it is read again. Students discuss what they can figure out about the situation with the values. Then for the third read the question is added and students discuss what is being asked before answering the question.
- *Number strings*: A type of task to promote mathematical discourse and reasoning, in which students start with a simple equation and then build on that equation looking for patterns to solve the subsequent equations and link to previous equations in the string (example: 2x50, 4x100, 100÷2, 100÷4, 200÷4, 400÷8, 800÷16, 800/16)
- *Three act tasks*: Act one is an attention getter based on the content, usually a video. Act two, students can ask about any information they need to solve the problem. Act three, students solve the problem and discuss their thinking about

the procedure and how it is related to other students' methods and to the original problem.

These materials, that sit apart from adopted curriculum, while created to supplement the older version of the K-5 adoption are available for all teachers, K-6, to use in their planning.

In addition to material supports, this district provided opportunities for teachers to participate in professional development with a mathematics focus. Teachers participating in this study have received district professional development focusing on Boaler's (2016a) work on mathematical mindset, as well as resources from Stanford University's *Youcubed* website. As a result, mindset philosophy and instructional methods are prominent in lesson preparation.

One of the professional development opportunities taught teachers about the benefits of using compendia. Laura, consistently planned and made use of compendia during instruction. The compendium is a chart on content created with student input, displayed throughout the lesson, and is often referenced in subsequent lessons. Laura planned the content of the compendium to include numeric, visual, and text information on the math topic and thought through how she would guide students in completing the chart as an entire class. During instruction, student input dictated what went on the compendium which often differed from what she planned, but she felt it was important to state content in student language and not her language. This responsiveness to students during instruction was referenced by all teachers. While their plans were created in advance the specifics of how the plan got carried out often changed in response to student needs at the time of the lesson.

While Angela and Tom used various online resources as supplements in addition to what the district provided the other four teachers did not use additional online resources. In addition to the online resources provided by the district, five of the participating teachers indicated that they used tasks and philosophies they learned during their post-graduate coursework to replace or supplement lessons they deemed weak. Teachers used a task they saw taught in their post-graduate course or a case study lesson found in the course text. The post graduate coursework, are a combination of pedagogy and math content knowledge. All teachers have participated in this classwork (Table 2), however, the one course that Tom completed was on the topic of professional leadership in mathematics not content and pedagogy.

Both Julia and Juan have degrees in mathematics in addition to their elementary education degrees. They both agreed that this background knowledge of “where the math is going,” as Julia put it, facilitated their decision making in adapting lessons on the spot during instruction. They both have modified curriculum lessons without using any resources outside their own knowledge learned through experience and their own education.

Material influence on mathematical discourse. No matter which adopted material was used, all teachers agreed that their materials supported mathematical discourse. When asked about the material’s influence on discourse in the classroom Angela highlighted the key questions in the teacher’s manual that translated into classroom discussions on specific concepts. Julia also pointed out that, “the amount of discourse is increased even more by using the supplemental materials in the district created planner.”

While the materials supported discourse, Judy reflected that this often caused teachers, "to stray from a lesson as necessary to address student misunderstanding. Lessons that were planned for one day can take three." All teachers expressed that often lessons designed by the textbook to be completed in a single day suggested timeframe took more than one day. In the eyes of these teachers, this aspect of lesson planning was neither a positive nor a negative, but more of a frustration. Extending the lesson was stated as a necessity, to conduct the lesson in a way that students have the time and opportunity to talk about the mathematical concepts, and build an understanding of those concepts.

Laura and Juan shared that their materials are cyclical and according to Laura, "support discourse because everything is based on real world examples and the Math Practices (CCSS, 2010). Students complete two problems a day on average because of all the discussion that goes into each problem." Laura went on to share that training for their math materials concentrated on, "how to walk around the classroom and involve students in discourse about the concept. Moving away from worksheets and memorization and replacing instruction with deep thinking and relying on each other's thought processes." During Juan's interview he mirrored Laura's statement, that the materials are set up so that, "every student has a role/job. If one person doesn't do their job the group breaks down." He believed that this, "forced cooperativeness, increases discourse." In addition to material in the lesson that promoted discourse Juan also pointed out that there are, "study team teaching strategies that are based on SIOP [Sheltered Instruction Observation Protocol] strategies like numbered heads, huddle, …" Juan also stated that while their materials encouraged the use of discourse as a central instructional method there is a lack of buy-in amongst the seventh and eighth grade teachers. This lack of buy-in often resulted in the adopted materials being used as worksheets or not teaching from the current adoption in favor of a previous adoption that does not promote mathematical discourse.

Preparation process. All teachers started by reading the lesson in the teacher manual and made decisions of whether the lesson addressed the needs of their students. Laura kept binders from year to year of modifications she made to lessons and how her students responded to the modifications to make best use of the information based on past lessons for the following school year.

Juan strategically planned lesson presentation methods that would get students working cooperatively and talking about math content. These strategies are explicitly taught to his students and provided a structure to increase engagement and academic discourse. In addition, he prepared a set of questions for students to help facilitate mathematical discourse in their small groups and when they are leading the class in a discussion. The lesson structure that he prepared is very intentional and math tasks were purposefully chosen to support making connections among the equations which support multiple methods of mathematical reasoning.

Mathematical discourse integration into lesson. All teachers who participated in this study integrated math discussion protocols and explicit math mindset discussions into their lessons early in the school year and provided refreshers throughout the year. Lessons that focused on mathematical mindsets were created or adapted from Boaler's (2016a) *Mathematical mindset: Unleashing Students' potential through creative math, inspiring messages and innovative teaching,* which was the focus of a book study in this district. Boaler's (2016a) strategies were meant to develop positive mathematic thinking processes within students based on Dweck's (1999, 2017) growth mindset concepts.

Two themes dominated when teachers explained how they integrated mathematical discourse into their lessons; the use of sentence frames and mathematical tasks. The two strategies came up in reflections of all six teachers as a way to promote and support mathematical discourse in their classrooms.

Sentence frames. Part of teacher lesson preparation were sentence frames to support children with verbal interaction. In addition to Talk Moves (Chapin, O'Connor, & Anderson, 2009) which incorporates the five main strategies of teacher prompting, wait time, revoicing, restating, and students applying reasoning, students are taught, through teacher prepared sentence frames, to support their answers as well as critique and add on to other students' reasoning. While there are premade sentence frames at the district level, teachers modified or created their own sentence frames as needed for their students.

Juan also provided question stems for students to use when they were presenting their math reasoning in front of the class. He prompted students to use these question stems to promote discussion when there was a lull in the discourse. Part of Juan's planning was to predict possible situations when students may struggle. He then connected these situations to the proper sentence frame or question stem. Students used these frames to support their further understanding or get the information they needed from peers to proceed forward. The protocols that Juan used in his class for discussion are specific and explicitly taught. He stated that he developed the strategies students need, to "hold them accountable not only for their but also their partner's understanding."

Julia used sentence frames to increase math vocabulary. She believed that, "it's important to make students use math vocabulary. Sentence frames are used extensively to facilitate this." While Julia had confidence in her content knowledge and ability to think on the spot she thought through and prepared sentence frames in advance of instruction.

Angela also used sentence frames to facilitate discourse. She shared that many of her students started out the year knowing sentence frames such as, "I respectfully disagree because …, I'd like to add …, I did it differently because…." This was the first year students started the year already knowing some sentence frames used in mathematical discourse, which she believed indicated that teachers are starting to have more math discussions in lower grades. Angela, like the other teachers, prepared sentence frames in advance, in anticipation of what students might need. In addition, she also prepared speech bubbles to help students with discourse vocabulary.

Tasks. While all teachers instructed using materials adopted, provided, and supported by the district, planning and implementation decisions about those materials were prevalent with all teachers. All teachers made task decisions which provided students with tasks that were open ended so that students could represent their thinking in multiple ways to develop perspectives that aid in conceptual understanding. In some cases, teachers chose a three reads task or another supplemental strategy that better supported discourse and conceptual understanding than the adopted curriculum materials. These decisions were made lesson-by-lesson and based on the teacher's perceived student need.

When analyzing the potential of student tasks, they fell into two categories. The tasks either (1) engaged students in using procedures; giving students little possibility of demonstrating understanding beyond procedural competence, or (2) they had the potential to develop conceptual understanding through mathematical reasoning and relationships. While procedural competence involved the ability to follow steps as with formulas and algorithms, developing conceptual understanding involved applying mathematical reasoning and analysis to situations that have no dictated procedure. An example of the latter could be a task that asked students to consider *What is happening?* or *What is the relationship?* in a series of equations; 5+5, 2X5, 4X5, 2X10, 4X10. This type of task required students to apply mathematic knowledge and reasoning to the numerical relationship and does not have a singular manner which to do so. There is no one right answer because the questions are open ended enough that the series of equations can be seen to relate to each other in various ways all mathematically sound.

Task analysis showed that independent student tasks tended to engage students in items that required procedural competence while in class tasks had greater potential to engage students in mathematical reasoning that developed a deeper understanding of mathematical concepts.

Lesson Facilitation

The results in this section pertain to how teachers, who have had post graduate mathematical coursework, facilitated mathematical discourse. Interview questions addressing implementation, two observations of math instruction in each classroom, along with analysis of 121 pieces of student work from 17 different assignments, were used to support the third research question:

How do teachers facilitate mathematical discourse?

Teacher reflections on implementation. When asked how content and strategies are facilitated, teachers responded with specific teaching strategies they used during math instruction to support students in realizing their mathematical potential.

Juan's main goal was to, "facilitate discussion, differentiate content, and provide structure for the students to carry the conversation." Juan has advanced his focus from primary teachers who taught students how to talk about math to teaching students to drive their own conversations, which he feels sixth grade students should learn. He conveyed that his goal is to guide the conversation but by the end of the school year he wanted his students to maintain academic conversations about mathematics on their own for the majority of the class.

While Laura had similar desires for her students' independence in mathematical discourse, she supported that independence through compendia. The compendia, "reflect student need, the content on the same topic changes from year-to-year but always includes: visuals, vocabulary, doesn't show steps, as in, first do this, then do that, but it shows different student thinking on the concept." This is a scaffold so students had information to refer, supporting their independence. Laura constantly went back to the compendium through subsequent lessons and highlighted areas or had students highlight the area they were using. She saw students on a regular basis use the compendium when they worked in groups or independently.

Laura, like Angela, facilitated “the importance of cooperation and working together through having team tasks every day as the central part of [her] lessons.” The compendia were always present to support mathematical discourse deemed necessary for successful team tasks.

Tom, like Laura, believed that working with others is vital to learning mathematical concepts. He changed lessons that were set up in the adopted materials as whole class discussions. He turned them into partner or team talk discussions so more students could participate in the discussion. He also believed that this lowered student apprehension for students about talking in front of the class, by allowing them to talk in a small group or with partners first (Krashen, 1982). This small group practice increased student participation in whole class discussions, it gave students time to process their understanding with a small group.

Judy felt it was “important to impart to [her] students that math may not come easily but the pride you feel when you are successful in understanding is important.” She facilitated this persistence by calling on every student two times each math class which helped them get, “comfortable talking about their thinking.” She lead students in conversations about what worked well and what didn’t work, in order to understand “where students are in their thinking,” so she could support them in finding strategies that would work for them.

One strategy Judy used was giving students her microphone to share their process. She wanted to promote the importance of mathematical thinking by focusing on the process and not the correct answer. When the student used her microphone, she promoted the idea that the student was currently teaching through sharing his or her process. The microphone also supported students that spoke quietly. She celebrated, "mistakes as an opportunity for the entire class to make sense of the mistake and learn." She echoed the sentiments of Angela in that it was important for students to learn how to struggle through problems while they still have teachers to support them, "so they are comfortable with the struggle and have options when they get older."

Julia facilitated vocabulary development by using sentence frames "extensively" and expecting mathematical vocabulary when students discussed mathematic topics. If students did not use mathematic terminology she pressed for the math term in the form of a question like, "What do we call triangles with three equal sides?" If the student speaking could not answer the question, she opened it up to the entire class, then went back to the original student and had him or her restate his or her thoughts using academic vocabulary. In her school students learned academic conversation procedures, like turn and talk, starting in kindergarten. This allowed her to focus on the content of what is being discussed as well as the nuance of student understanding when providing support for student answers, instead of describing the procedure.

Teacher task expectations. Teachers were clear in stating expectations of what students should complete and, with the case of younger students, teachers walked students through the first few steps of the task. The expectation for the quality of completed student work was not explicitly stated during the lessons observed or in written work. This is not to say that teachers did not discuss quality at other times outside of the observation period. Teachers clearly stated expectations concerning protocols and how student answers should look, as far as what should be included, but were not as explicit about what quality mathematical explanations or proofs looked like. Many student answers, oral and written, pertained to the procedures they used.

An example—typical of all the participating teachers—of how expectations were presented to students can be seen during the first observation of Judy's classroom. She was conducting a lesson on measuring time and explained her expectations to her students:

> You should have an elbow partner you are sitting next to. I want you to turn to an elbow partner and talk to them please, about what you know about these three ways to measure time. [Students partner talk.] What do we know about the relationship of these three things? What do we know about the measurement of these three things? [Partner talk, then students shared observations whole class.] I'm going to ask you a question and I'm going to want you to think first, then you're going to share your thinking with that elbow partner you're sitting with. Remembering that our good partner sharing means turning to someone, and you're looking at them, and you're answering the question fully, and you're taking turns on who starts, right? Take turns on who starts, let's make sure we're not leaving anyone out. For those of you who love to start talking first, for this next question I want you to wait and let the other person go first. It will be ok, you can still share after they're done. Ok, first question, you're going to answer it with your partner then I'm going to give you the next instruction so hold on for me, alright?

Less explicitly she emphasized quality of what the student's answer looked like by stating, so the entire class could hear, "I like the way you gave me sentences.

… Nice sentence frame," to individual students that had written complete sentences on their white boards. Later in the lesson she described her expectations for the quality of the content:

> I want to see proof on your board of how you came up with that answer. How many seconds in an hour? Proof on your board about how you came up with that answer. I need you to show me proof. How do you know? Can you prove it to me? The answer doesn't do me any good. [to individuals] That doesn't make sense to me. …Interesting proof. Here's my problem: I have people who are just writing answers on their boards. The answer is not important to me here. You have to be confident in your answer and prove that it's the correct one. So, if you just write a number down, that does not give me what I need. I need proof to me that that is the answer. How you know that that is the correct answer? Show me how you know. So, work with your elbow partner to come up with a proof on every board.

For all teachers, expectations were made more explicit during in class work than student independent work. Judy's expectations were typical of what was seen in all twelve observations.

Whereas, with independent work teachers relied on previous class instruction to frame expectations. Of the 17 assignments collected teachers reported on all 17 that expectations of what to do were given but only 3 assignments had clear expectations for quality of work. A typical example of expectations about what to do were, "Work by yourself," "Show your work," "Write your answer on the line," whereas an expectation of quality was, "Show your mathematical thinking on each step using pictures, numbers, and words. Use the various math tools around our room to show your thinking."

Implementation of task. Julia supported her students through protocols that were put into place but she did not lead her students to any one way of solving problems. She guided students through their thinking with a series of questions and visuals to help them understand the concept being presented. She developed this through repetitive hand motions to indicate values and math processes. In one instance when Julia talked about a pattern she consistently gave wait time for independent thinking and always pointed to what she was talking about so students could follow visually as well as aurally.

Angela was precise with her directions to students prior to starting a Three Reads task. She told students that she was expecting them to talk about how they were thinking about the task, not about what the answer is. She also let them know they will, "not talk about what the answer is until tomorrow."

When the opportunity arose she quickly related language aspects in math to other content areas. She consistently brought out the language of math with her students as with the following exchange.

> Angela: What [are] snowballs here? In this math problem? We all know what snowballs are. We know that 14 and 25 are the values, or the amount, or the quantity. Snowballs is [*sic*] the [Angela pauses]. When we did the question for [Jon's], books were this, when we did [Efran's] question, it was apples.
>
> Student: It's the thing
>
> Angela: You're right the thing. There's another math word, do you remember?

Student: It's the unit

Angela: The unit, you're right it is the thing in this situation but sometimes we have people, sometimes we have places, right? The thing is the noun, the unit.

Angela highlighted students who were precise in their language and consistently explained why she asked students to understand their thinking and explain themselves clearly. In this excerpt, she used the time it took a student to come to the front of the room and put the microphone on to talk to the students about precision.

Angela: In the second box it was a different kind of question. Now when we did that we left off with [Sean]. And he was actually thinking about, can you write 14 then 25 (directed to the student teacher who was documenting student thought). [Sean] can you come up really quickly and tell us what you said? I'm going to give you the microphone so we can hear you. So, think about, hey you guys, I was having a conversation with Mrs. [Hightower] this morning, and I like how you revise your thinking or you say, oh I respectfully disagree with myself. However, think before you say something, try this today, think, ok? How am I not going to try not to have to revise my answer? Not that I don't want you to revise your answer, but I want you to think carefully before you say something, so you know exactly what you want to say without having to revise later, possibly. Because sometimes I think what we do is, oh, I'm just going to say an answer because I think I know it. I haven't really thought all the way through but I'm just going to say it and later when you think it through yourself, or someone else gives you an idea, and you say, 'Oh that's not exactly what I was thinking.' That's actually a good thing to do but some of us

> do it a lot. Which means maybe, some of us need to think a little bit longer before we share something. Does that make sense to you? I'm not saying you can't change your answer. That's not what I'm saying at all, but instead of saying, '2 plus 5 is 8. Oh, oh, oh, wait, I want to revise my answer.' Instead of just shouting out, think first '2 plus 5 [*pause*] oh it's 7. Then I don't need to revise my answer all the time. Right? OK, so [Sean]…

Judy also used time efficiently and kept students engaged constantly for 90 minutes. Students were challenged as exhibited by having to rethink their understanding, then go back to the task, and try again with a different understanding. The lesson flowed organically, the teacher constantly checked in with students, provided support, and made modification of time to work independently, with a partner, and with the whole group as needed. The series of tasks had entry points for all students and allowed students to investigate the topic, which was defined by the teacher, at their individual level of understanding.

Laura's students struggled with the topic of greatest common factor and many were attempting to follow procedures without understanding why the procedures worked. Laura stepped back from the team work and walked students through the task explicitly explaining her thinking. She then released the students back to teams. During a second observation Laura was using realia, a jar containing raisins and nuts that duplicated the situation in the math task, to explicitly show students the situation in the task they were working on in their teams. In her interview she stated that she, "strives to show everything visually," because she believed it helped students make connections to their lives. This belief stemmed from her personal experience learning math. Her instructions were clear but the mathematical concepts were not clear to the students. She moved around the classroom, facilitated discussion in teams, and directed student attention to the information around the classroom that could help them understand. She also worked with individual teams to help students see connections to other tasks they had completed. Students continued to struggle with the concept but were progressing in their understanding.

Juan started his lesson with a focus on mathematical procedures. He focused his students on brainstorming various interactions that might happen when working together, then students problem solved how to respond to the instance.

> Teacher: I want you to put that hat back on when we're talking about how is this room a safe place to come up here and present something when you're (a) nervous, (b) not a hundred percent confident, and (c) maybe not even right. What you guys are doing out here to make it safe for those people. What do you need to do?
>
> Student 1: When they're up there you don't really giggle because then they think they got the wrong answer.
>
> Teacher: You maintain your self-control, right? What else do you do?
>
> Student 2: Like if they say something, and they get the wrong answer. Um, like don't make a weird face.
>
> Teacher: Control your voice and control your facial expressions. What else?
>
> Student 3: Um, just give them all your attention, I mean, you're not like messing around.

Teacher: So, should you have your pencil in your hand, or be playing with it, or be writing? [Students shake their heads.] No, you should be giving them 100%. What else?

Student 4: You should listen to them so you can give feedback.

Teacher: What do I ask from you? What do I expect when there is a comment and it's like crickets in the room? What do I say? What do I usually say?

Students: Um…

Teacher: Do you have any feedback for me? Do you agree with me? Do you disagree? Do you think that if there is a student up here talking do you think you can give them feedback? What are three ways you can give feedback to people without even using your voice? [Mark] has one he did this [rolling hand in a circular motion in front of his chest]. What does that mean?

Student: I think you're on the right track.

Teacher: Hey, I think you're on the right track. I think there's a little bit more. Maybe we can get at it. What does this mean? [Hands rocking back and forth on either side of his head, by his ears.]

Students: I agree.

Teacher: What does this mean? [Hands crossing back and forth over one another, with palms down, in front of chest.]

Students: I disagree.

Teacher: So, it's important to give feedback from the audience to know what is going on. Not overwhelming feedback but a little feedback is good because otherwise you don't know what's happening inside other people's heads.

In the second observation Juan moved students from making isolated observations of four equations, to comparing the equations. Juan used the *Which One Doesn't Belong* task to intentionally move students from discrete individual observations, to comparison among the four equations, then to working as a team—sharing individual thinking and listening to teammates to support their conjecture, then finally students created an argument for why the equation they chose does not belong to convince their classmates of their thinking. The purpose of this lesson is not to learn specific content but to use previously learned information to think mathematically and form an argument based on mathematical understanding.

In contrast Tom's students struggled with conceptual understanding. In both observations, Tom led students through a task scaffolding student thinking. Students were released to carry out instructions but did not contribute any mathematical thinking different than Tom's process. Students duplicated what was done or instructed by Tom. The following discourse took place during whole class discussion that followed team work.

Tom: What would that skeleton look like, what would the red part look like?

Student 1: 3 across and 5 down

Tom: [Jerry] what do you think it looks like?

Jerry: 4 going down and 13 across

Tom: [Annie] what did you share with your partner? What did it look like?

Annie: umm…

Tom: What do you think that skeleton would look like?

Annie: 10 going down and 13 going the other way

Tom: Now work with your partner to build the skeleton for 4 X 13. How many going down for 4 X 13?

Choral Response: 4

Tom: How many across?

Choral Response: 13

Tom: Then you're going to fill it in with base ten pieces. You are going to use this to build the outside. How many down?

Choral Response: 4

Tom: How many across?

Choral Response: 13

Tom: Then you and your partner are going to fill it in using your base ten pieces.

At this point Tom showed the class the blue skeleton pieces and then instructed students to build it with their partner. Two students built the array on the document camera and their work was projected so all students could see. When the students at the document camera stopped in confusion Tom arranged the skeleton pieces and told the students to fill in the center with yellow pieces, at which time the students finished the visual.

As part of the teacher led discussion teachers linked concepts, representations, and processes to encourage students to evaluate their own and other student's thinking about the concept presented.

Teacher linking, press, and questioning. Teacher linking is the process of drawing connections between student procedures and thinking to the original task, past tasks, and differing methods to develop a deeper understanding of the underlying concept of the task. This requires teachers to understand the underlying mathematic concepts of the task and the many ways in which students may approach or think about the task (NCTM, 2014). In addition to linking student thinking to the task, teachers also press students to consider critical aspects of the task by asking purposeful questions (NCTM, 2014).

Julia consistently linked student thinking methods together and actively looked for representations that could be linked to deepen student understanding. Her questioning pressed students to make connections, “Does that look like what you did when you…,” “Can you explain what [Jim] did and how it’s like yours?” “Where is the four in your number sentence? Where is the four in your picture? Where is the four in [Nicole’s] picture?”

Julia also connected the current activity to a previous activity where students used beans as counters. She asked students to get a mental image of what 20 beans looked like so they could estimate a scoop of 50 beans. By addressing students in a very explicit way she provided a new strategy of using a previous experience for a present solution. She verbalized her thinking, which does not come naturally to many of her students, as a framework they could use in verbalizing their thinking.

Julia’s questions supported pressing students to consider their process of doing the task and the concept of numbers and their values. Some of the questions she asked were, “Where did you see that?”, “Why do you think that?”, “What patterns do you see?”, “What are you thinking about this strategy?”, “What would happen if__?”, “Explain to your partner what [Enrique] did,” “How can you show your thinking?”, “That is after, how do we know what’s before?”

Angela included explanation, vocabulary, and language that she expected her students to use in her instruction but she did not dwell on pre-teaching vocabulary explicitly. She focused on using vocabulary in context as it became necessary to express math in clear terms. Angela pressed students through highlighting student thinking, questioning, and encouraging risk taking. An example of risk taking encouragement was evident in the first observation where she told students, “There is no really wrong way of doing box number two. … Some of you are wondering and some of you are jumping in. Just go ahead and take a risk.” Angela gave examples of how various students approached the task. “Come up with your own way of showing those two numbers.” She followed this statement up by asking, “What do you know about those two numbers?” to focus student attention to the math and precision of student representations.

Judy provided quick response time to students asking students to provide evidence to their thinking visually and orally. She connected student thinking to previous content learned, and highlighted different ways students thought about the task consistently during the first observation. Judy complimented students on taking risks and shared their thinking focusing on making sense of a task and not racing to the answer. “Remember that it’s important to listen to each other. It’s important to hear the different methods that we’re using to find the ones that make sense with your brains. Nobody’s brains are exactly the same. Nobody makes the exact same connection. I’m really glad you shared that with us [Nicole.]”

In addition to her quick response time, Judy's questions challenged students but never frustrated them. She was observed using questioning for three reasons: to clarify student thinking, improve the precision of student language, and press students to a deeper understanding.

Questioning observed in Tom's instruction elicited a response concerning following procedure and the teacher's press for explanation tended to encourage students to duplicate the teacher's thinking.

This is highlighted in the previous excerpt from Tom's instruction that started with, "Tom: What would that skeleton look like, what would the red part look like? Student 1: 3 across and 5 down."

This structure of questioning funneled student thinking into duplicating what the teacher expected to see in a *correct* answer and did not allow students to think through the task independently.

Laura pressed students to make a connection between the visual of the equation and the numerical representation of the equation. She did this to help students see the connection between the values in the equation, how they related within the equation, and how they corresponded to the pictorial representation of the equation. In the second observation, she pressed students to link their explanations by adding on to other students' thinking. She guided students as opposed to explaining the link outright.

In both observations, Laura pressed students to be persistent in their thinking about math tasks, and precise in their language when talking about math tasks. She explained that sometimes you need to, "step back and rethink your answer, talk it through with the class or student team which allows you to clarify and process your thinking."

Juan's goal was to have his students independently carry out mathematic discourse by the Spring. To this end he often put the questioning and thinking back on the students to compare their processes and not just their answers. "Did you or your table group agree at any portion of the problem? … Did your table group agree with you on that?" However, even with the students' advanced understanding of mathematical concepts there was a need for the teacher to more explicitly make connections for the students, in order to expand student understanding and strategic thinking. Juan chose to make these connections through questions, "Can anyone tell me what the advantage of 60 is? Or the advantage of 300 when doing this problem? Which would be more mathematically efficient and why?"

Juan's technique did not go without effort. In both observations, there were times when he needed to pull information from his students and at those times the instruction concentrated on procedure. In these cases, the teacher asked students to restate another student's thinking to include mathematical vocabulary, or explain what the directions meant. "So, in this case when it says to simplify the following expression what does that mean? What part of the expression do they want you to simplify?" Juan pressed students to go beyond description of the procedure, beyond the steps they took to solve the task, but he started with the procedural description, highlighted the students' ability to use various methods to reach the same outcome, then guided, through questioning, deeper analysis of why all the different procedures worked.

The residue. Davis (1992) explains mathematical residue as:

> Instead of starting with *mathematical* ideas, and then *applying* them, we should start with *problems* or *tasks*, and as a result of working on these problems the children would be left with a residue of *mathematics*—we would argue that mathematics is what you have left over after you have worked on problems. We reject the notion of *applying* mathematics, because of the suggestion that you start with mathematics and then look around for ways to use it (p. 237).

As Davis emphasizes mathematical residue goes beyond memorizing steps and formulas and applying them in known situations, to being able to manipulate the underlying mathematical concepts in future unknown situations and arrive at a logical conclusion. Reflection on the presence of mathematical residue lies in the possibility of students being able to apply understanding beyond the current task. Building this possibility into instruction involved choosing tasks that promote residue, and facilitating conversation that went beyond recounting the steps taken to solve a specific math task. Julia facilitated this by eliciting strategies students could use in multiple situations.

An example of this, in Julia's instruction, involved students who struggled to see a pattern on their calendar. Julia simply moved the calendar piece over to the right of the previous row, as an eighth day, so the students could see the pattern. When the calendar piece was the first piece on the left of the calendar the diagonal pattern was not obvious. Julia analyzed student confusion, made an adjustment that resonated with the students, and in doing so provided an example of a strategy that can be used in many situations. Later in the lesson it was observed that several students were using this new strategy.

Judy on the other hand developed mathematical residue through vocabulary and linking concepts. Students were struggling with a task that involved the passage of time and she reminded students of a past lesson where they learned base ten. “Ah, it becomes base 60. We talked about that before, right? Time kinda messes with us a little bit because we change our counting just a little bit. We go from 11:59 to 12:00, not to 11:60 as we would in base ten, right?”

Both Angela and Juan embeded protocols into their instructional strategies to guide students in how to work through any mathematical task. These teachers, directed students to use their background knowledge to answer unknowns. Both teachers developed purposeful application of mathematical logic with their students. Through the simple process of expecting students to rethink and restate their and other student thinking while linking it to what they have learned in the past these teachers developed strategies that will help students conjecture and analyze future unknowns.

Both Tom and Laura were not observed developing mathematical residue because their children were struggling to understand the task. Because students did not understand what to do and struggled with the math content, much of the conversations revolved around specific procedures for the specific task.

Student Participation

The results in this section pertain to how students participated in mathematical discourse. This section is broken down into overall student participation in the lesson through completing tasks, student participation in mathematical discourse through explaining their thinking, linking their thinking to other students' thinking, and answering questions from classmates and teacher. In addition, teachers' views on how students participated with classroom instruction will be included. Observation, interview, and analysis of student work were used to support the fourth research question:

How do students participate in mathematical discourse?

Teacher views. Cooperative learning techniques and student exploration of mathematical concepts were mentioned by all teachers as a method to encourage maximum participation and discussion among students. As the grade level increased teachers talked more about creating a classroom environment that put students in charge of *figuring out the math*, leading their own conversations, and working with fellow students to answer content questions. Judy talked about, "treating her class as a community that is responsible for each other as well as theirselves [*sic*]."

The first thing that both Julia and Juan mentioned was the quick pacing of their instruction to keep students engaged. Juan expressed that he, "expects students to talk five times more than [he] does." While Judy did not mention a fast pace, she did emphasize that students contributed more than she does to mathematical discourse. She built her classroom's mathematical community to "instill an independence that students know how to figure problems out by using appropriate tools, talking to other students, and adults in the room, trying a different approach to the problem. Basically, when things get hard [students] have options to persist through it." She tied this into the idea that as students learned they were bound to make mistakes so they needed to understand what to do with those mistakes to progress their learning.

All teachers mentioned developing a classroom culture where students felt safe to make mistakes. Angela celebrated mistakes to create an "environment where [students] can talk and think and try new things." She encouraged students to talk directly to each other and like Juan she stated they have, "more student discourse than [the teacher] talking." Like Angela, Tom encouraged students to respond to each other instead of using him as an intermediary. "Students come to me wanting to be guided through every step as a whole class and I help them be more independent." While student independence was what teachers strived for, observations were conducted early in the school year and teacher guided lessons were still in place to varying degrees. As mentioned the most independent student participation occurred in both second-grade classrooms and the higher SES fourth and sixth grades.

Two student observations were conducted in each classroom during the first half of the school year; October through January. The categories for focused observation were guided by the domains of the IQA (Boston, 2012). The following descriptions of student participation are addressed by student grade level for comparison and not by teacher pseudonym.

Student participation. In all classrooms student participation in completing assignments was over 90% throughout the lessons observed. The degree of engagement ranged from procedural, following steps the teacher gave students, to independent discussion of mathematical concepts.

The engagement in five of the six classes showed students actively participating in partner, team, and whole class discussions while processing mathematical concepts. This was exemplified through students engaged in discussion about their understanding, acting out the word problem, showing their understanding through pictures and manipulatives, and showing excitement about sharing their understanding with the entire class.

One classroom stood out in that during both observations students were *doing* math, in that they were following the steps the teacher laid out, but they did not engage in trying to make sense of the math. A typical example in this classroom was a student observed conducting a non-math related conversation, turning around to copy what the teacher wrote on the document camera, returning to his conversation, then repeating the behavior.

Student linking. Student linking did not develop in a linear fashion as the students aged. In the second-grade classrooms students linked their thinking with teacher prompting but also independently linked their thinking to other student thinking. Students interacted with each other directly about the connections they drew from each other's mathematic representations and thinking. The interactions were student directed, then revoiced by the teacher. All second grade students used the same sentence frames, "I respectfully disagree__," "I'd like to add__," "Mine is different because__ ." Students did not need to be prompted during this section of the lesson to add more to their explanations.

In both second-grade classrooms visuals and ample manipulative choice helped students make connections between their thinking, other student's thinking, and various strategies used to solve the task. Students were not restricted to specific manipulatives, they were able to choose any manipulative available in the classroom. The freedom to use any manipulative in the classroom to solve a task produced various methods of solving the task that could be linked. This freedom of choice was only apparent in second grade classrooms.

The fourth-grade classrooms needed more teacher prompting for students to link their strategy or thinking to another student's thinking or a previously learnt strategy. A reduction in manipulative support and representational graphics was observed in fourth grade classrooms. In one second grade class the students were guided through a story line visualization of the mathematic task prior to considering the mathematic concept. This scaffolding created a common visualization of the task situation because students acted it out, which facilitated student linking since all students were working from a common understanding.

Students in the lower SES fourth-grade classroom did not link their thinking to others. The tasks were guided by the teacher step by step and student thinking did not venture outside of the structure initiated by the teacher. The higher SES 4th grade student linking was generally initiated by the teacher but then students developed their own strategies for thinking about the task. Teacher prompting was needed to link student thinking to another student’s thinking.

In both sixth-grade classrooms students were able to link their thinking to their classmate’s thinking, frequently drawing connections to procedures. The students in the lower SES sixth grade class connected their procedure for solving a task to a classmate who used a different procedure but arrived at the same answer. The students in the higher SES classroom made connections between procedures and the concepts behind the procedures.

In one exchange in the lower SES sixth grade classroom a student wrote an equation for the task and showed where the values in his equation could be found in a previous student's visual representation of the same task. Earlier in the lesson, to aid in the student's understanding, the teacher provided raisins and peanuts in a mason jar which duplicated the task situation. This visual appeared to assist the student in making the connection.

In the higher SES school, sixth grade students linked their thinking to other student's thinking with and without prompting from the teacher. Student linking took on the form of defending their own thinking and challenging classmate's thinking. In

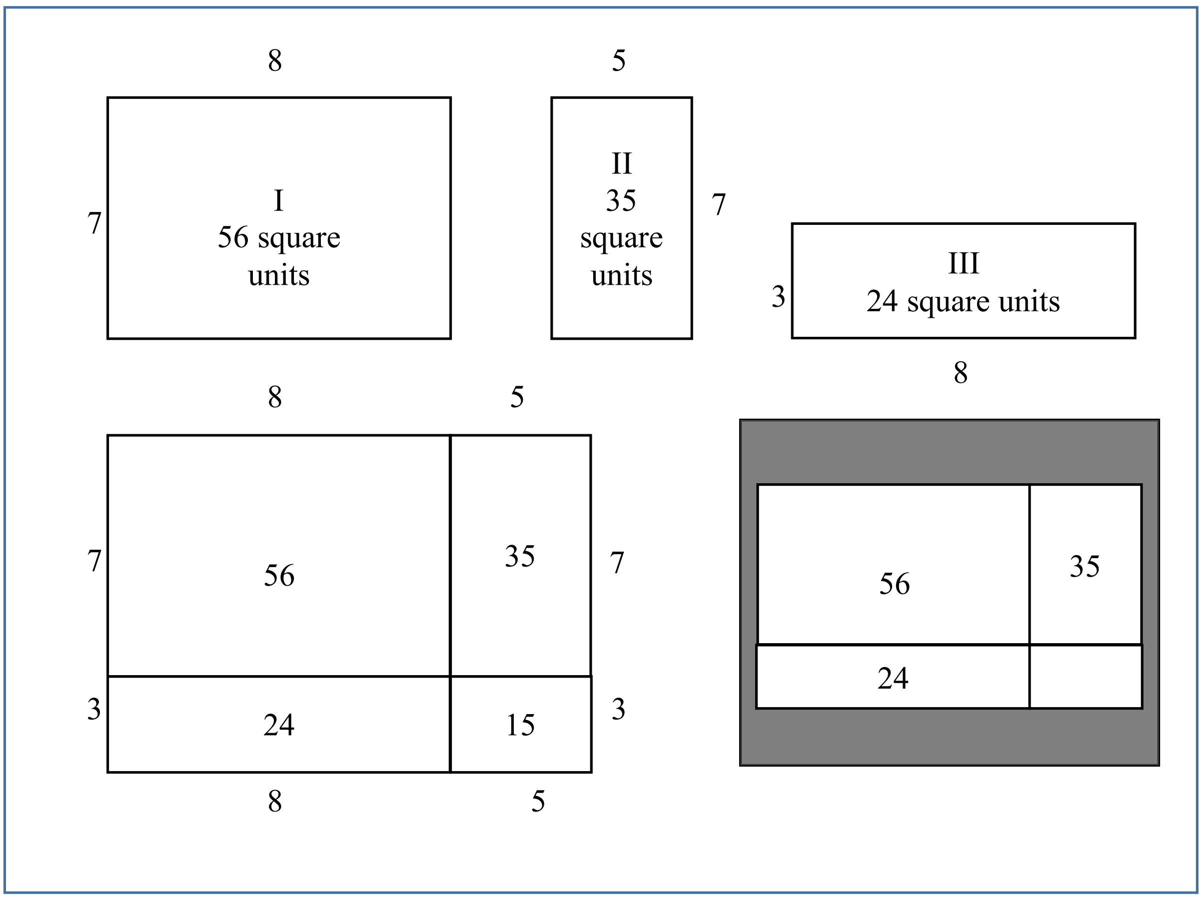

Figure 2. Student 1, sixth grade example. Graphic of Student 1's work which was projected when this exchange occurred. The shaded inset was the original task.

other instances, students presented how their thinking is the same as another student's thinking but the procedure is different, as in the following example (Figure 2).

> Student 1: So, the first thing that I did is I was trying to get the missing number for that (points to an area array on her paper). That's how I got the numbers on the outside of those (points to I, II, and III). I got them by like thinking, what times what equals 56 and what times what equals 35 and what times what equals 24. I got 8 and 7, 5 and 7, and an 8. Then what I did was that I put all those together (points to I and II) and got this (points to where she redrew the pieces of the array into one larger array). I got this answer by, I know you have to find one of the numbers like these, so if 5 is right here then 5 is over there (points to the fives on the opposite sides of her array then does the same for the remaining values). Seven is right here so 7 has to be here. And if 8 is right here then 8 is here and here and 3 is here so 3 goes right there. And then when I was done doing that I got 3 into 15 which 3 time 5 equals 15. That's how I got my answer.
>
> (Students agree with student 1's answer through hand signals.)
>
> Teacher: What's a question you can ask your audience? (um) Look on the right side of the paper. Remember I have those question stems there.
>
> Student 1: So, what do you guys think?
>
> Student 2: My answer is similar to student 1 because I got the same answer of 15 but I think I got it a slightly different way because when I did my work I, I kind of. I wrote down the multiples of 24 and 35 next to each other and I got. I just got a bunch of different multiples. And I took those multiples and I tried to

figure out which combination between them by multiplying them together could I get 56. And I got 7 times 8 equals 56, because 8 times 3 equals 24. So, I agree I just got to it in a little bit of a different way.

Student 3: Can you explain how you got the 8 and the 7 and all those number and where you put them again?

Student 1: All I did was I just, thought in my head, um. Could you say that again?

Student 3: How did you know what number to put where? I mean like why did you put the 8 at the top and the 3 at the bottom?

Student 1: Um, I did that because, um, I realized that you have to like line up the numbers (she points to the same number on opposite sides of her area array). So. if I put 8 right here then it will go on top too.

Teacher: If you are up at the Elmo and you say I did something similar to yours, is it all right to go up and show your work next to that persons? Absolutely.

Student responses and discussion. It was noted during the two observations in each classroom that students in lower SES fourth and sixth grades required more teacher prompting than in the higher SES schools. When an explanation is provided by students in lower SES schools it tends to be computational or procedural relating to the steps the student took to achieve their answer.

While student responses in higher SES classrooms explained procedures also, these students used more mathematical vocabulary, and their responses were not prompted by the teacher as often as they were in lower SES schools.

The following exchange is a typical discussion in the higher SES school.

Teacher: I heard one excellent partnership over here that said I think blah, blah, blah. Ok, so what's your idea? So, they passed it on to their neighbor. Like a game, like my turn your turn. I love that. I want you to think about one thing that your partner told you. What's one observation that your partner shared with you? What were you just thinking that first time?

Student 1: My partner said that he noticed on the first that there was half a square, on the second there were two full squares, on the third there was half a square, and on the fourth there were two full squares.

Teacher: Ok, so that was a lot of observations. That was like four observations.

Student 2: My partner noticed that the colors are red, red, green, green, then redder.

Teacher: Ok she noticed the colors that we see.

Student 3: My partner noticed that there was half of a square on the first and then, kind of like what Student 1 said, on the second one there are two full squares, the third has half a square, and the fourth one is technically two squares.

Teacher: Technically, can you explain that a little?

Student 3: Both of them have pieces on two of the sides which makes four of the little squares and if you put them together it makes two full squares.

Teacher: Put them together, is that something we can do?

Student 3: Yes.

Teacher: And when we're talking about two squares are we talking about perimeter? What are we talking about there? Lines of symmetry? What is that concept we're thinking about called? Talk it over with your partner. Share it with your partner then put your hands on your shoulders if you think you know. I'm thinking back to last month's calendar. [short partner discussion time] Everyone whisper to me what they think it's called.

Students whisper choral response: Area.

While this typical exchange in a lower SES school was guided by the teacher.

Teacher: Did anyone find the area of one group then the second group? Raise your hand if you did it that way. Did anyone count every individual square? Why wouldn't you want to do it that way?

Student 1: It takes too long.

Teacher: Someone that didn't do the two groups how did you do it? Someone that didn't do it?

Student 2: I counted 10, 20, 30, 40, then we counted the little ones.

Teacher: (Teacher revoices what the student explained, pointing to the pieces.) Anyone do it that way?

Student 3: Kinda both.

Teacher: So, similar. Let's see what answer everyone got.

Five sets of students all responded 52.

Teacher: Do you think that's the answer?

Choral response: Yes

All four second grade observations yielded similar student response rates and types of discussions. In all four observations discussions took on a visual nature as students uniformly contributed to discussions about patterns, task situations, and possible strategies, which included use of manipulatives, that could be useful to solve the task and explain mathematical reasoning. One second grade classroom conducted a task where students estimated the number of beans needed and scooped the beans based on previous knowledge of what 20 beans looked like. Students then went back to their partners and counted the beans putting ten beans in each Dixie cup to see how close they were. This lesson also gave students the opportunity to practice counting by tens and develop place value number sense. Two typical discussions in second grade on this task are as follows:

Example 1:

Student 1: Because it's like you have that line then you put it together [the student is referring to how the number looks on paper when the teacher wrote 11 cups|9 extras. The student put the two numbers together, which may indicate that the student understood how to count by tens or may show the student is pulling the numbers, 11 and 9, then putting them next to each other in that order.]

Student 2: Oh! I know [unsolicited]. I have a different strategy. So, if you take one out, well first you count 11 then take one out and you count to 100, like 10, 20, and then you count the left-over ones and then you have — 110, 101, 102, 103, 104, 105, 106, 107, 108, 109. [The student said 101… but wrote 111…]

Example 2:

Student 1: Wait. I only have, I think I over shooted a little bit. Yeah, I definitely over shooted.

Teacher: How do you know that you overshot?

Student 1: Because I have these ones left.

Teacher: Why does having those ones left mean you overshot?

Student 1: Well, I have another ten. Oh, I got really close. I need another cup.

Student 2: I need another cup too. And I overshot too.

Student 1: So, this is ten but I got so close though.

In addition, mathematical vocabulary, such as “digits,” “analogue,” “diagonal,” “pattern,” and “odd and even,” were used by students in both second-grade classrooms. This mathematical language was appropriate to student grade level and task. Both teachers also reinforced mathematical language by revoicing the correct vocabulary, as can be seen in example two and highlighting key vocabulary through their speech and written word.

Another element that was evident in second grade classrooms was students comfortably amending their thinking or changing their understanding all together when presented with new evidence. Fourth and sixth grade students fell into three categories; students reluctant to change their thinking even in the face of evidence that suggested they should, students that changed their thinking when presented with evidence to the contrary of their current understanding, and those that changed their thinking immediately without question when their thinking was challenged or contradicted in any manner.

Observation of individual teams in fourth and sixth grades at the lower income school gave insight into student struggle. In both classrooms while students discussed the task in small groups students did not understand the mathematical concepts and attempted to apply procedures in which they had little understanding. Students understood what they were being asked to do, the values involved, and had a general understanding of the situation, but they did not have the mathematical conceptual knowledge to apply the understanding they did have in a systematic way. This difference was observed in fourth and sixth grades but was not observed in second grade. The following student conversations from one lower SES classroom highlights the student struggle to make sense of concepts and apply procedure.

Student 1: It's 40 right here, it's 40 times, 40 and 24.

Student 2 & Student 4: Wrote down what Student 1 said; Student 3 looked at his textbook.

Teacher: So, it's really helpful if we attend to precision by using the lines and columns with the rectangles. (Teacher observes for a moment, comments, then moves on to next group of students.)

Student 2: What goes here?

Student 1: 2

Student 2: How do you do this one?

Student 1: 24 and actually

Student 3: 20 times 2 is 40

Student 1: 40 times 10 is 40 and 4, no it wouldn't be 4

Student 2: looks to Student 3 for answer

Student 4: 4 times 8 is, uh, 32.

Student 1: Oh, I got this one. 32? I did this then I did this (pointing to work)

Student 4: I did that too.

Student 1: Did you get this number? Yeah.

Student 2 to Student 3: That's too small see? That would be 2.

Student 3: Wait, wait, wait (wrote in journal)

Wait, wait, hold on, hold on

(Students wrote in their journals, then there was a long pause while trying to figure out what to do next.)

Student 2: It's 40.

Student 1: No, it's "a" we need to do it as much as possible.

Student 3: I found 70.

Student 2: I found 4.

Student 1 to Student 3: 5 times 4, 5 times 4

Student 2 to Student 3: It's 40

Student 3: Oh yeah, it's 5 times 4

Student 1: No, no, no it's 5 times 8, sorry.

Student 2: It's 13

Student 1: Ok, 8 and 5 and

Student 4: 4, 4

Student 2: 48?

Student 4: No, 32.

Student 1: Is this the number? Oh, I know, we did it wrong, we did it wrong, oh you guys, we need to put a zero in one of these. Next to the 4, you guys.

At this point the teacher pulls the class back to whole group discussion and Student 1 is trying to change her work quickly. Others in the group do not.

While engagement in completing the task was high, there was general confusion about the task itself. From this exchange students were focused on getting the answer and not attending to the concept. Students seemed to be applying multiplication to the situation without a real understanding of the values or the situation the task represented. The students treated the values in the task as disembodied numbers that did not represent anything in reality.

Student artifacts. Four different assignments were collected from three teachers and five assignments from one teacher for a total of 17 assignments. Both high and low SES second grade classrooms were represented, as well as, fourth and sixth grade low SES classrooms. Assignments were not obtained from the fourth and sixth grade high SES classrooms. Each assignment for each classroom was represented by two samples of high achieving work, two samples of medium achieving work, and two samples of interesting work as identified by the classroom teacher, some teachers provided more samples than requested, for a total of 121 pieces of student work. The teacher provided a narrative of any oral instructions that were given to the students, teacher expectations given to students for quality work, and any information about grading criteria that was shared with students for each of the 17 assignments.

A common characteristic of the assignments was that students tended to engage with tasks at the procedural level. Student work was numerical and formulaic as opposed to offering an insight into the student's understanding of the mathematical concept. In six cases, this was because the task was a procedural or algorithmic task (Table 3).

Table 3

Distribution of Student Artifacts

Task type	Assessment	In Class Individual Assignment	In Class Partner or Group Assignment
Procedural Task (ex. algorithm)	1	0	0
Conceptual Task (ex. word problem requiring application to new situation)	0	4	6
Mixture of Procedural and Conceptual Tasks	4	1	0

Tasks that were started with classroom discussion then proceeded to individual, partner, or group written work displayed more student engagement in the concepts behind the procedures. Engagement in concepts was defined by the potential of the task to engage students in exploration and understanding of mathematical concepts as well as students engaging in creating meaning of the concepts with non-algorithmic thinking (Boston, 2012).

Second grade students had more pictorial representations than older students which is reflective of the tasks requested of them. A typical second grade task presented a story problem that related to the student, such as making snow balls, then asked students to; “Draw what you know about the story,” “Show how the numbers go together,” “What math questions can you come up with for this story?” and “Solve the problem. Show your thinking using words, numbers, and pictures with labels.” A typical task that was collected from older students was more removed from the student and directions for completing the task lent itself to algorithmic knowledge. One example asks,

> Maned wolves are a threatened species that live in South America. People estimate that there are about 24,000 of them living in the wild. The dhole is an endangered species that lives in Asia. People estimate there are ten times as many maned wolves as dholes living in the wild. About how many dholes are there living in the wild? Your third-grade cousin doesn’t understand how to figure out the answer. Use numbers, words, or pictures to show your work and explain your reasoning so they can understand.

All samples showed either answers alone or an algorithm showing how the student got their answer. When the student did include written explanation, the written explanations were the steps to the algorithm written out. This type of answer was typical with students older than second-grade.

Summary

In this chapter, I presented the findings of this study. These findings are based on analysis of interview transcripts, classroom observations, and student assignment samples. These findings are supported by collection and review of these qualitative data sets in each of six classrooms. Findings were discussed in four parts in correspondence with the major concepts presented by the data. Data of the first section focused on teacher beliefs and how they thought about mathematic instruction especially mathematical discourse. The data in this first section described participant beliefs about what is important for students to learn, the role of discourse in mathematic learning, pedagogical beliefs, and beliefs around teaching in a high and low SES school.

The second section focused on teacher planning lessons to include mathematical discourse. This section was grouped into four sections based on interview responses. Participants described the resources they used, how those materials influenced mathematical discourse, how they prepared math lessons and integrated mathematical discourse into their lessons.

The third section used lesson observations to focus on how teachers facilitated mathematical discourse. This section was broken into five themes that emerged based on the IQA (Boston, 2012) framework. The five themes of the third section are teacher reflection on implementation, teacher task expectations, teacher implementation of the task, how teachers linked, pressed, and questioned students, as well as mathematical residue as a result of teacher instruction.

The fourth and final section focused on how students participated in mathematical discourse. Through interview, observation, and artifact data, using the structure of the IQA (Boston, 2012) domains, five themes emerged. Data was presented through teacher views on student participation, observed student participation, student linking, student response and discussion, and student assignments.

Chapter 5: Discussion, Recommendations, and Conclusion

The purpose of this study was to examine the processes teachers with post graduate mathematic coursework took to implement mathematical discourse. This was done through analyzing how teachers believed mathematical discourse fit into the learning process, how lessons were prepared, how teachers facilitated mathematical discourse, and how students participated in it.

Research was conducted through individual face-to-face interviews with six teachers, 12 instructional observations, and analyses of 17 assignments that comprised 121 pieces of student work. This chapter reviews, analyzes, and discusses the findings of this study, provides suggestions for future research, and outlines the implication for mathematics instruction in elementary schools.

Discussion

Four questions framed this research:

1. How do teachers think about mathematical discourse in the learning process?
2. How are lessons prepared to include mathematical discourse?
3. How do teachers facilitate mathematical discourse?
4. How do students participate in mathematical discourse?

The Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) was used as the theoretical framework on which this study was conceptualized. The research questions were answered by four emergent themes developed through initial organization of data collected based on the IQA (Boston, 2012) framework. Data for this research were obtained through interview, classroom observation, and artifacts. The 11 domains of the IQA (Boston, 2012, Appendix A) provided the structure of three initial concepts; teacher planning, teacher implementation, and student participation. The data for the final concept of teacher beliefs came from interview questions (Appendix B), as reported in Chapter 4. From these initial concepts, four themes emerged that cut across these concepts and fell within the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1):

1. the importance of developing the qualities of a mathematical thinker
2. teacher math experiences shaped instruction and student experience
3. variables that influenced student participation in mathematical discourse
4. outside influences on facilitating mathematical discourse

The following section is organized by the four research questions and within each section the cross cutting themes are addressed as they relate to the research question.

Beliefs

The first research question addresses teacher beliefs which directly impact teacher practice (Anderson, et. al., 2005).

> How do teachers think about mathematical discourse in the learning process?

A large body of research suggests that teachers are the pivotal component to change the direction of mathematics education, and that change in teacher belief proceeds change in practice (Boaler, 2008; Ernest, 1991; Fang, 1996; Stipek, et. al, 2001; Thompson, 1992). This is supported by the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) which shows the influence of teacher beliefs on teacher practice.

The personal mathematic experience of teachers by participating in post graduate coursework, was reported to change their beliefs around math. Not only did their belief about teaching mathematics change but also, they reported that their confidence in their own math skills improved. Teachers not only reported directly through interview that their beliefs around mathematical discourse changed, as a result of increased content and pedagogical knowledge, but this belief could be seen in how teachers prepared and facilitated instruction as well as in the artifact analysis.

Teacher belief in mathematical discourse, as a means to promote conceptual understanding was reportedly initiated by post-graduate coursework, then it was supported by the school district through text adoptions and supplemental materials which was evident in interview and observations of preparation and facilitation of instruction. This in turn, laid the foundation for mathematical discourse in the classroom, with the continued support of professional development opportunities that broadened the teacher's content and pedagogical understanding further.

Teacher participants unanimously believed that facilitating students' development of math confidence and persistence were the core qualities of a mathematical thinker. When asked, "What is the most important thing for students to learn?" teachers initially responded with qualities of a learner and not a content objective. The qualities of math confidence and persistence where characteristics that all teachers mentioned, showed evidence in their planning and instruction, as well as, was evidenced in student talk during lessons. Not only did teachers believe in the importance of mathematic confidence and persistence, but they acted on that belief, and students responded to the instruction demonstrating independence in the way they spoke about mathematics.

There was an interplay among mathematical discourse, mathematical confidence, and persistence. Teachers held the belief that to develop mathematical confidence and persistence students needed to participate in mathematical discourse. This is exampled in Laura's reflection that students who did not participate in mathematical discourse did not grow as math thinkers and learners. However, it was also acknowledged that to participate in mathematical discourse students needed to be confident and persistent. This came up in teacher beliefs and in planning for the supports needed to provide a culture where students felt confident to take mathematical risks and persist. In addition to relating these qualities to their students, teachers reflected on their own learning process and how their perception of what math is and how it should be taught changed with their increased confidence, as a result of learning mathematic content as an adult.

The influence the teacher's own personal experiences had on their beliefs about how math should be taught, and in their confidence in teaching math, could be seen in their instructional practices. Stipek, Givvin, Salmon, and MacGyvers' (2001) study examined the link between a teacher's beliefs and instructional practice finding that teacher beliefs about mathematical learning and ability, as well as a teacher's self-confidence and enjoyment of mathematics, were associated with instructional practices and their students' self-confidence. In reflecting on their own mathematic self-confidence several teachers shared that it was not until having the experience of participating in math content through discourse themselves that they felt confident in their mathematic abilities.

The more teachers' beliefs aligned with a student constructing their own understanding of mathematic concepts, the more instruction reflected an emphasis on understanding and the importance of mistakes in learning rather than speed and the right answer (Stipek, et. al., 2001). In addition to self-confidence and persistence, teachers all had a belief that a student should construct his or her own understanding of mathematic concepts using mathematical discourse as the vehicle. It was sometimes a struggle to implement this belief. Teachers struggled with also believing that they should conduct some instruction that included rote memorization of algorithms and quick recall of math facts. They tried to balance this with their views that students needed to develop a collective understanding of mathematic concepts which takes more instructional time than memorization.

All teachers connected their beliefs about math instruction to their own experiences learning math, good and bad. Two teachers not only related their instructional beliefs to their own experiences but also to experiences of their children. These personal experiences were often talked about with passion and created a connection between the teacher as a learner and the teacher as an instructor of learners which provided insights that they acted upon.

Teachers in the low SES school community felt vocabulary needed to be explicitly taught to students due to the large number of English language learners

(ELL). Whereas, in the high SES schools vocabulary was implicitly taught or highlighted during mathematical discourse when the situation presented itself.

Julia believed students benefited attending a school with a high ELL population because academic conversation procedures were a focus of instruction from kindergarten. The benefit to her students was that she could focus on the content being taught instead of procedures. She, however, was the only teacher in the low SES school that related this sentiment.

In addition to having a belief in the importance of mathematical discourse to support instruction, it was unanimous that teachers believed adopted curriculum, supported mathematical discourse to promote confidence, persistence, and a students' conceptual understanding development.

Lesson Preparation

The second research question addresses teacher lesson preparation which is driven by the teacher's beliefs, knowledge, materials, and outside experiences and opportunities (Anderson, et. al., 2005).

How are lessons prepared to include mathematical discourse?

This study indicates a need for content and pedagogical teacher development beyond initial teacher preparation courses in order for teachers to have the skills and knowledge necessary to implement mathematical discourse. Professional development is part of the advice, knowledge, and curriculum section of the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1). According to this model teacher beliefs are influenced by professional development which in turn influences preparation of lessons and instruction.

To prepare instruction that included mathematical discourse teachers used adopted curriculum, supplemental district materials, as well as content and pedagogical knowledge they gained through district professional development and post graduate coursework. The interview revealed that teachers relied heavily on the adopted and other supportive materials. As one teacher put it, “I didn’t go into teaching to write curriculum.”

Content knowledge and pedagogical skills, which teachers learned through professional development and coursework, could be seen during observations throughout teacher facilitation of instruction. A teacher’s pedagogical skills are central to mathematical discourse because skilled questioning and task selection leads to productive discourse (Akkuss & Hand, 2010). To this end all teachers felt the district’s adopted materials supported mathematical discourse by providing task oriented lessons. In addition, the district provided supplemental material support that all teachers accessed. The supplemental materials were separate from the adopted materials but supported the development of lessons that promote mathematical discourse. This electronic material bank provided quick access for teacher planning and supported the philosophy of developing conceptual understanding through mathematical discourse. Previous research has found that curriculum used by teachers influence discourse in the mathematics classroom by what decisions teachers make in preparing lessons (Drake & Sherin, 2006; Remillard, 1999; Rigelman, 2009).

The initial response from many teachers was that they did not plan, they followed the curriculum, but were accustomed to responding *in the moment* of the lesson. While *in the moment*, instructional adaptations did occur, teachers, upon further reflection, realized that over the years they have developed responses for instructional situations when students struggle with mathematical content. One teacher kept binders of lessons from year to year adding notes each year of what went well and where students struggled to be proactive in future instruction, while other teachers developed a culture and a language that celebrated mistakes as a positive contribution to student understanding.

Initial teacher language, to encourage persistence, was developed by teachers through district professional development, book studies, post graduate coursework, and independent reading of professional texts. While teachers expressed they did not, "go into teaching to write curriculum," they did devote time and effort into expanding their knowledge base about classroom math culture so they would have the tools and language at their disposal to respond to students *in the moment*. It was also through the extended professional learning that teachers developed the skills and content knowledge required to make informed decisions about tasks used in the classroom and questions to promote student thinking. Some teachers adapted the curriculum tasks so they more closely aligned to student lives while other teachers brought in realia or acted out tasks in the classroom to support understanding of the mathematical situation.

In addition to, *in the moment*, adaptations that encouraged mathematical understanding, all teachers started out the year building the math community in their classrooms by explicitly planning lessons that taught protocols for discussion and "the power of yet," which is a growth mindset (Dweck, 2016) philosophy that is supported by the district through professional development and videos used during instruction.

Teachers spoke openly about having to let go of completing instruction in a short time frame if they were to act on their belief that students needed to build their own meaning of mathematic concepts through discourse. They struggled in their planning between allowing the time needed for mathematical discourse that promoted mathematical understanding and the more traditional skills needed by students to quickly calculate. This struggle bore out in comparison of observed lessons, that focused on students building their understanding of concepts, versus artifacts that promoted algorithmic and formulaic calculations more often than in class tasks.

The district's math team structure was yet another layer of district teacher training and support that provided opportunities for teachers to expand their content and pedagogical knowledge. All teachers mentioned the structure their district provided of academic communication and professional development as a strong component to their continued development as a math instructor and learner. Having opportunities to participate in book studies and professional development added to the strategies to which they had access when they planned lessons.

Teachers reported that the strongest influence for developing their ability to conduct mathematical discourse confidently was post graduate coursework provided through a grant in conjunction with a local university. Evidence could be seen in teacher instruction and in their reflection of the benefits of in depth mathematic education, beyond initial teacher training coursework. The post graduate courses that teachers participated in influenced their belief about math instruction and what constitutes learning math. Their new beliefs more clearly aligned to research that promotes conceptual learning through mathematical discourse which in turn impacted how teachers planned instruction. (Alexander, 2008; Burbles, 1993; Cobb & McClain, 2005; Donovan & Bransford, 2005; Lester, 2007; Moschkovich, 2012; NCTM, 2014; Reznitskaya & Gregory, 2013; Yackel & Cobb, 1996).

Teachers shared their views of differences in low and high SES schools. In addition, to the need to explicitly teach vocabulary in the lower SES schools, access to opportunities in the community was suggested to be a difference that benefited higher SES students. While this was not independently confirmed, teachers believed this to be a factor for mathematic understanding with higher SES students advantaged because of their greater access to academic supports outside of school.

Lesson Facilitation

The third research question addresses teacher lesson facilitation which is impacted by the teacher's beliefs, knowledge, materials, and outside experiences and opportunities (Anderson, et. al., 2005).

How do teachers facilitate mathematical discourse?

Participating teachers had a belief in developing confident students who could think mathematically, which was reflected in their interviews and seen in their lesson facilitation. They put into practice their constructivist beliefs, by facilitating academic conversations where students shared their thinking and mistakes were celebrated as contributions to learning.

Hungerford (1994) suggests that one of the barriers that arises with implementing mathematical discourse in the elementary classroom is the lack of teacher training in mathematical concepts, and the pedagogy that supports mathematical discourse. Ball (1991) and Battista (1993) found that implementing effective mathematical discourse can be very challenging and often not sustainable because of poor teacher preparation in mathematics. This research suggests that, while challenging, teachers with strong mathematical content and pedagogical knowledge can facilitate mathematical discourse where students reason, examine, and think mathematically.

The idea of developing student confidence expressed by teacher participant beliefs appeared in their instruction as structures put in place and strategies used by teachers when conducting lessons featuring mathematical discourse. All teachers used strategies such as sentence frames to support students who struggled with confidence or did not have the language to participate in mathematic discussions, and all teachers used cooperative learning strategies, such as think-pair-share and small group work to encourage student participation. While all teachers included supportive strategies in their lessons the flexibility of those strategies varied. Whereas some instances showed teachers not straying from their plans, other instances displayed teachers adjusting on the go.

Celebrating mistakes, as supported by the research of Stipek, et. al. (2001), was facilitated in class by teacher lead discussions highlighting student or teacher mistakes. Recognizing mistakes as a learning tool was also dominant in the classroom culture. To build a math community that was positive, and promoted students feeling comfortable sharing their thinking, teachers spoke directly with students about the benefit of making mistakes. They encouraged students to share ideas that were not completely formed or correct because they valued student thinking. They made efforts to make sure students understood that mistakes can add to student understanding when trying to reason through a task.

Participant teachers had personal experiences of math as memorizing, successfully and unsuccessfully, that impacted their own math confidence and developed a negative impression of mistakes. During instruction teachers made efforts to make sure students did not have a negative experience of math or making mistakes. They did not want students to experience math as unrelated memorized equations, so they facilitated math tasks that were experienced by students in their daily lives or brought the experience to the students through realia and manipulatives.

One common factor that emerges in research about teacher beliefs is the relationship between beliefs, concerning best instructional practices for mathematics, with the mathematic content and pedagogical knowledge of the teacher (Anderson, et al., 2005; Ernest, 1991; Fennema, Carpenter, & Peterson, 1989; Raymond, 1997; Romberg, 1993). These studies support the theory of Casa, et al. (2007) that highlighted the three reliable factors of successful mathematic teachers; promoting mathematical reasoning, examining complex mathematical concepts, and valuing students' mathematical ideas. To develop these three factors of instruction, teachers need sufficient content and pedagogical knowledge on which to build. This idea of building content and pedagogical knowledge was emphasized by all participants. Five of the six participants reflected on how their views of mathematic instruction, in general, and mathematical discourse, in particular, changed as a result of taking post graduate coursework. The one participant that did not relay this sentiment had yet to take a course that focused on mathematical pedagogy and teacher math content knowledge development. The coursework provided teachers with personal experiences which developed their content knowledge and expanded their pedagogical thinking and available strategies. This seems to have impacted teacher beliefs about what and how students learn math as evidenced in participants directly stating the coursework made a difference in their belief or was included when they talked about what is important for students to learn.

Laura stated, "Everyone should take the [classes]. I think even if you took just one class your teaching is totally changed." Whereas Juan emphasized the value that he developed because of his experience in post graduate math education concerning working to support mathematical thinkers and not only teach discrete facts, "[Students have] learned to challenge each other, they've learned to disagree with each other. … if they can leave my classroom with the ability to figure out math, it doesn't matter what I don't teach them because they'll figure it out."

Teachers' ability to articulate and implement his or her philosophy about mathematical discourse appeared to increase with the amount of additional professional development and post graduate coursework focused on math content and pedagogy. Teachers with extensive mathematic content and pedagogical knowledge could articulate their philosophy and how it fit into their instruction more clearly. Teachers with less content or pedagogical experience struggled more with articulating their philosophy and how they developed instruction that supported their constructivist views.

Mathematical discourse was the primary vehicle used to support teacher constructivist belief of students developing their own understanding. To facilitate the instructional strategy of mathematical discourse student language was supported in the classroom. Language was emphasized in Laura's instruction using compendia that students could access for reference during lessons and that the teacher referred to on a regular basis. Student developed compendia supported language in classroom discussions by providing easy access to vocabulary, thinking strategies, and visuals of specific content. In all classrooms visuals, manipulative, and realia, were evident to support student discourse.

The amount of content and pedagogical knowledge teachers must have in order to successfully develop student conceptual understanding through discourse is great and does not happen overnight (Ball, 1991; Battista, 1993; Hungerford, 1994; Moschkovich, 2007; Polly, Neale, & Pugalee, 2014). All the supports the teachers received work in collaboration to help provide a foundation in which teachers can build and implement lessons that provide students with opportunities to think mathematically. Teachers demonstrated this orchestration of support in their instruction and students responded to that instruction by participating in mathematical discourse in some instances taking control of the lesson and talking directly to other students without the teacher as a mediator.

Student Participation

The fourth research question addresses student participation which is directly impacted by the teacher's content and pedagogical knowledge as well as materials used (Alexander, 2008; Burbles, 1993; Battista, 1993; Cobb & McClain, 2005; Hungerford, 1994; Polly, Neale, & Pugalee, 2014).

How do students participate in mathematical discourse?

A reflection of student participation in the Teacher Mathematic Practice model (Anderson, et. al., 2005, Figure 1) is seen though the teacher's professed beliefs versus their actual beliefs and the influence of the social context of teaching on actual practice versus reported practice which impacts student participation.

Teacher language was specific and consistent in developing the philosophy of math confidence and persistence to support students to participate fully in mathematical discourse. Teachers explicitly taught students coping strategies when they got frustrated, such as taking a walk to the water fountain and coming back to class with a clear head and ready to work. Students responded in their own language during instruction with the willingness and confidence to change their thinking in front of their peers, "Oh, wait a second, I'd like to change my thinking." Students also were seen confidently carrying on mathematical discussions, challenging each other's thinking, and persisting through a task by talking through their thinking with the whole class or with partners.

Teacher beliefs about the importance of student mathematic confidence and persistence influenced their instructional practice and how the students responded to mathematic tasks. This is reflected in many studies that support the idea that instructional change is dependent on what teachers believe about instruction (see for instance: Anderson, et al., 2005; Hofer, 2001; Muis, Bendixen, & Haerle, 2006; Pajares, 1992; Sarason, 1982). While during interviews teachers discussed student participation in mathematical discourse much in the same way, observation data yielded a difference in the content of discourse between high and low SES fourth and sixth grade classrooms.

Even with the differences in fourth and sixth grade classrooms student participation in completing tasks was high in all classrooms. Analyses of student discourse showed that in the two lower SES classrooms students were guided more directly by the teacher than in the higher SES schools. In the higher SES schools, students conducted mathematical discourse without a large amount of teacher guidance, whereas, students in lower SES schools appeared to need more teacher prompting. In addition, student conversation in lower SES schools more often focused on steps they took to solve a task and less about their understanding of the task. Students were observed comparing answers in lower SES fourth and sixth grade classrooms whereas in the higher SES classrooms students more often linked their thinking to arrive at the answer with other students thinking about process. A limitation of this study is while a difference in participation of low and high SES students was observed, there was not enough data to make any clear deductions about why this may have occurred.

Limitations

While the limited number of classroom observations make generalization difficult, the richness of data obtained from the 12 observations contribute to an understanding of other more wide-scale data collections (Wilhelm & Kim, 2015). The qualitative nature of data that was collected provided insights into the specific situations of each classroom participating; however, the limited number of classrooms in each SES grouping makes it impossible to generalize to the greater community.

All attempts were made to make sure participating teachers had a background in and belief of the benefits of mathematical discourse that went beyond initial teacher preparation courses. However, since this research focused only on how teachers think about and implement mathematical discourse and how their students respond to that implementation, other factors such as teacher content knowledge, previous teaching experiences, and outside factors of the community in which they teach was not fully known prior to the study and could potentially impact the teacher's facilitation of mathematical discourse.

The data collected covered one moment in time, of a limited number of classrooms at the beginning of the school year. Results may have varied if data was collected at multiple times throughout the year. Because of this limited timeframe in which to collect data patterns of student participation did not fully evolve.

While not the focus of this study, general comparisons were made among SES demographics. Cultural match or mismatch of teacher and students, based on economic demographics, in instructional methods and language was not a focus of this study. An in-depth study of equity in mathematical discourse would necessitate a larger sampling over an extended period, to produce data that would be useful in generalizing the influence of teacher cultural attitudes and practices on student acquisition of mathematic content.

Major Findings

The purpose of this study was to examine the processes teachers with post graduate mathematic coursework took to implement mathematical discourse. This was done through analyzing how teachers believed mathematical discourse fit into the learning process, how lessons were prepared, how teachers facilitated mathematical discourse, and how students participated in it.

The data collected supported previous studies such as Anderson, et al. (2005) that suggested prior to instructional change a belief change is needed which is prompted by teacher knowledge, materials, and outside supports ("advice" in Anderson, et al., 2005, Figure 1). This study seems to suggest that it was a combination of post graduate coursework, adopted curriculum, and district professional development support that provided the foundation for teachers to change their belief system to include mathematical discourse in their instruction. This in turn provided the opportunities, through instruction, for students to develop an understanding of mathematical concepts at a deeper level than if not supported by mathematical discourse (Cobb & McClain, 2005; Donovan & Bransford, 2005; NCTM, 2014; Thompson, 1994; Whitin & Whitin, 2000; Yackel & Cobb, 1996).

In addition to the many opportunities to advance content and pedagogical knowledge, the teachers personal experiences seemed to be part of their desire to create a safe mathematical environment where students were confident learners. Teachers voiced their belief in the potential of their students and in some cases related that they did not want their students to experience math as they did. These experiences, good and bad, with math content and pedagogy provided a foundation for the way in which these teachers thought about and taught math.

Another finding was the difference in second and fourth grade mathematical discourse. While there were not enough data to reach any conclusion, the differences were enough to warrant future research to investigate why.

Finally, the in-depth look at the entire process of teacher belief through student participation in mathematical discourse adds to the body of research. If we are to accept the research that supports mathematical discourse as a factor in developing student conceptual knowledge (Chapin, O'Connor, and Anderson, 2009; Cobb &

McClain, 2005; Donovan & Bransford, 2005; Kazemi, 2008; NCTM, 2014) then research that illuminates the process of developing mathematical discourse in elementary classrooms is needed. This research attempted to create a complete picture of the setting in which the mathematical discourse takes place along with its observable influences on student and teacher interactions (Miles, Huberman, & Saldana, 2013) in order to expand understanding of the factors that contribute to mathematical discourse.

Future Research

This study attempted to increase understanding regarding the impact of teacher belief about mathematical discourse to instruction, how mathematical discourse is planned and implemented in elementary classrooms, and how students participate in that discourse. This qualitative study offered a detailed examination, through triangulation of data, of mathematical discourse in six elementary classrooms, focusing on teacher beliefs through student participation.

Although this study represents a start for developing a larger body of research on the relationship between mathematical discourse and various student populations, further research is needed. First, a future study should include gaining student perspectives on mathematical discourse as well as quantitative outcomes for students. While this study was primarily teacher focused the component of student perspective would add to the richness of understanding the impacts of mathematical discourse in elementary classrooms. This would also allow for an investigation of cultural differences in how students think about, respond to, and participate in mathematical discourse.

Second, it would be prudent to examine a greater variety and quantity of elementary classrooms to broaden the research to participants who have experienced various types of training and supports. Broadening the pool of participant classrooms would add to the generalizability of the data and help indicate conditions that promote success in implementing mathematical discourse in elementary schools. This study focused on a small subsection of teachers to paint a clear picture of how they thought about and implemented mathematical discourse. The focus on one school district allowed elimination of variables, such as curriculum adoption and supports offered teachers, to examine how this one segment of the teaching population addresses mathematical discourse. The narrow focus of one school district's efforts, however, is also a limitation because of the lack of variables. Adding more participants would allow greater confidence in comparisons and conclusion.

Third, there was indication in this study that a student's ability to participate in mathematical discourse was not a linear growth pattern or that there is some factor that caused the participation gap between SES levels to grow, as evidenced through the different type of participation in second and fourth grades in the high and low SES schools. However, there were not enough data to confirm why there was a difference. Tracking paired high/low comparison schools' discourse patterns over time would be beneficial. A longitudinal study could add to this component by following student cohorts over several years to document their participation in mathematical discourse as mathematic content becomes more abstract.

Finally, an in-depth analysis of instructional mismatch between cultures is needed. A thorough analysis of tasks used to promote mathematical discourse would help understand more clearly if the tasks themselves may be a roadblock to students' participation in mathematical discourse or whether the academic language in the discourse poses a roadblock. Along with task analysis for cultural bias a more directed look at participation in mathematical discourse from various cultural groups within the classroom is needed. This examination would necessitate considering the intersectional characteristics of overlapping cultures based on ethnicity, race, gender, and SES.

Conclusion

Research agrees that mathematical discourse supports student conceptual understanding; accessing mathematical understanding through questioning, challenging thinking, and analyzing their and other student thinking, to arrive at communal knowledge of the content (Donovan & Bransford, 2005; Kazemi, 2008; Krummheuer, 1995; Lester, 2007; Moschkovich, 2012; NCTM, 2014; Wood, Cobb, & Yackel, 1991). A large part of developing reasoning skills in mathematics revolves around developing a classroom culture that emphasizes discourse as the path to shared understanding (Cobb & McClain, 2005; Yackel & Cobb, 1996). In order to support this, classroom structures must be put in place to create a safe classroom community which is vital to student mathematic development.

To create this environment, teachers need to possess both strong pedagogical skills and an understanding of math content. However, Hungerford (1994) suggests that teacher preparation in mathematics is perhaps the weakest link in elementary education. This roadblock to a quality math education is compounded by the possibility that the classroom culture may not be aligned with the student's culture, ethnically, as well as, economically (Bishop, 2008; D'Ambrosio, 2008; Forgasz & Rivera, 2012; Herbel-Eisenmann, et. al, 2012).

With the weak preparation of elementary teachers in mathematics, and the possible gap between the teacher and student culture, developing a mathematic community of learners in the classroom, facilitated by mathematical discourse, is challenging (Ball, 1991; Battista 1993). A large body of research suggests that teachers are the pivotal component to change the direction of mathematics education, and change in teacher belief proceeds change in practice (Boaler, 2008; Ernest, 1989; Fang, 1996; Stipek, et. al, 2001; Thompson, 1992).

This research looked in depth, through triangulation of data, into the processes of six elementary teachers to implement mathematical discourse into their classroom routines and how their students responded to that implementation. The personal mathematic experience of teachers by participating in post graduate university coursework, changed their beliefs around math. Not only did their belief about teaching mathematics change, but also they reported that their confidence in their own math skills improved.

During interviews teacher participants shared that they can only plan lessons to a certain extent because they must respond to students, *in the moment*, depending on student need. To do this effectively a teacher must have a depth of conceptual understanding, which includes the foundation knowledge students need for the current concept and where the concept goes next in the math continuum, to support the students' understanding. Teachers reported that they did not learn these types of mathematical concepts fully until participating in post graduate coursework. Teachers who have not had the opportunity to participate in mathematic coursework beyond their initial teacher preparation courses may find it challenging to support students to conceptual understanding by using mathematical discourse.

The belief change, initiated by post-graduate coursework, was then supported by the school district through text adoptions and supplemental materials that laid the foundation for mathematical discourse in the classroom, as well as professional development opportunities that broadened the teacher's content and pedagogical understanding. A structured districtwide hierarchy of professional development gave teachers the opportunity to work with their colleagues and continue their discussions about mathematical discourse, and its implementation, which provided another layer of teacher support. Through teacher interview and classroom observation it was observed that all layers of support seemed to be needed to sustain instructional change that supported mathematical discourse.

Data collected indicated differences in mathematical discourse between high and low SES fourth and sixth grade classrooms. The reason for this difference is unknown based on the data collected because there were too many compounding variables (SES, language level, ELL, classroom culture, teacher delivery, gender). An interesting component was that the second-grade classrooms did not show differences in student participation based on observations and collection of artifacts. These results prompt a continued in-depth analysis of this phenomena from multiple perspectives to narrow down possible implications.

Deepening mathematic content knowledge while simultaneously learning to think about mathematic instruction in new ways is not something that happens overnight. The results of this study suggest that to improve student mathematic understanding we must begin with teacher beliefs. Not only beliefs about mathematical discourse and instruction but beliefs about what math is.

Developing a teacher's content knowledge while simultaneously developing an understanding of mathematic pedagogy is at the core of a sustainable change in mathematic instruction in elementary schools. These data suggest that elementary teachers would benefit from opportunities to participate in math education as a learner, so they can develop their own conceptual understanding of math, through personal experiences, which could expand their belief system about mathematic instruction to include the use of mathematical discourse.

This research adds to the body of research by examining the process of implementing mathematical discourse in elementary schools, by teachers who have had post graduate coursework that develops mathematic content knowledge and mathematic pedagogy that includes discourse. Research agrees that mathematical discourse promotes conceptual understanding in mathematics yet it has not been implemented across the U.S. in large scale (Ball & Forzoni, 2011; Cobb & McClain, 2005; Cooney, 1985; Donovan & Bransford, 2005; Kazemi, 2008; Lampert, 1990; Miller, 1957; NCTM, 2014; Romberg, 1993; Schmidt, et al., 2002; Yackel & Cobb, 1996). This evidence shows that there is promising progress in developing elementary math instruction that includes mathematical discourse but there is still more data that needs to be collected to develop a clearer understanding of what is needed to increase U.S. students' ability to reason mathematically.

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Appendix A

Instructional Quality Assessment
Classroom Observation Tool

COVER PAGE – COMPLETE FOR EACH LESSON AND ATTACH TO FIELD NOTES, COPY OF INSTRUCTIONAL TASK, AND SCORE SHEET

Background Information

Date of observation:____________________________ Observer:
____

Start Time:___________________________________ End Time:
____

District:______________________________________ School:
____

Grade:______________________________ Day 1 or Day 2
____

Classroom Context

Total number of students in the classroom:
____
_____

Boys

_____Girls

Sketch of

seating

arrangement(

s):

Mathematical Topic of the Lesson:

Field Notes (attach).

Part 1: Documents Needed During the Observation

Accountable Talk Function Reference List

Most of these moves will be made by the teacher, but in some cases, students might make them. In recording the actual moves, note T for Teacher move, S for Student move.

1. Accountability to the Learning Community

Keeping everyone together so they can follow complex thinking

"What did she just say?"

"Can you repeat what Juan said in your own words?"

Getting students to relate to one another's ideas

"Jay just said…and Susan, you're saying…"

"Who wants to add on to what Ana just said?"

"Who agrees and who disagrees with what Ana just said?"

"How does what you're saying relate to what Juan just said?"

"I agree with Sue, but I disagree with you, because…"

"I agree with Fulano because…"

Revoicing/Recapping

"Can you repeat what Juan said in your own words?"

"So, what I'm hearing you say is…"

Marking

"That's a really important point."

"Jenna said something really interesting. We need to think about that."

2. Accountability to Knowledge and Rigorous Thinking

Pressing for accuracy

“Where could we find more information about that?”

“Are we sure about that? How can we know for sure?”

“What evidence is there?”

“How do you know?”

“How did you get 50?”

Building on prior knowledge / recalling prior knowledge

“How does this connect with what we did last week?”

“Do you remember when we talked about slope?”

Pressing for reasoning

“What made you say that?”
“Why do you think that?”
“Can you explain that?”
“Why do you disagree?”
“Say more about that.”
“What do you mean?”

Academic Rigor 2: Implementation Lesson Checklist:

A	The Lesson provided opportunities for students to engage in high-level thinking:	B	The Lesson DID NOT provide opportunities for students to engage in high-level thinking:
Students ○ engaged with the task in a way that addressed the teacher's goals for high-level thinking and reasoning. ○ communicated mathematically with peers. ○ had appropriate prior knowledge to engage with the task. ○ had opportunities to serve as mathematical authority in classroom ○ had access to resources that supported their engagement with the task. Teacher ○ supported students to engage with the high- level demands of the task while maintaining the challenge of the task ○ provided sufficient time to grapple with the demanding aspects of the task and for expanded thinking and reasoning. ○ held students accountable for high-level products and processes. ○ provided consistent presses for explanation and meaning. ○ provided students with sufficient modeling of high-level performance on the task. ○ provided encouragement for students to make conceptual connections.		The task ○ expectations were not clear enough to promote students' engagement with the high-level demands of the task. ○ was not complex enough to sustain student engagement in high-level thinking. ○ was too complex to sustain student engagement in high-level thinking (i.e., students did not have the prior knowledge necessary to engage with the task at a high level). The teacher ○ Allowed classroom management problems to interfere with students' opportunities to engage in high-level thinking. ○ provided a set procedure for solving the task ○ shifted the focus to procedural aspects of the task or on correctness of the answer rather than on meaning and understanding. ○ Gave feedback, modeling, or examples that were too directive or did not leave any complex thinking for the student. ○ Did not press students or hold them accountable for high-level products and processes or for explanations and meaning. ○ Did not give students enough time to deeply engage with the task or to complete the task to the extent that was expected. ○ Did not provide students access to resources necessary to engage with the task at a high level.	

C	The **Discussion** provides opportunities for students to engage with the high-level demands of the task. Students:

- use multiple strategies and make explicit connections or comparisons between these strategies, or explain why they choose one strategy over another.
- use or discuss multiple representations and make connections between different representations or between the representation and their strategy, underlying mathematical ideas, and/or the context of the problem
- identify patterns or make conjectures, predictions, or estimates that are well grounded in underlying mathematical concepts or evidence.
- generate evidence to test their conjectures. Students use this evidence to generalize mathematical relationships, properties, formulas, or procedures.
- (rather than the teacher) determine the validity of answers, strategies or ideas.

Academic Rigor Q: Questioning Types:

Question Type	Description	Examples
Probing	• Clarifies student thinking • Enables students to elaborate their own thinking for their own benefit and for the class	• *"How did you get that answer?"* • *"Why did you use that scale for your graph?"* • *"Why did you use that formula to solve the problem?"* • *"Explain to me how you got that expression."*
Exploring mathematica l meanings and relationships	• Points to underlying mathematical relationships and meanings • Makes links between mathematical ideas	• *"What does 'n' represent in terms of the diagram?"* • *"How does the 'x' in your table related to the 'x' in your graph?"* • *"How would your expression work for any "function?"* • *"What is staying the same in your equation? Why is it staying the same?"*
Generating discussion	• Enables other members of class to contribute and comment on ideas under discussion	• *"Explain to me what John was saying."* • *"What else did you notice about the graph of the parabola?"* • *"Who agrees with what Sue said? Why do you agree?"*
Procedural or factual	• Elicits a mathematical fact or procedure • Requires a yes/no or single response answer. • Requires the recall of a memorized fact or procedure	• *"What is the square root of 4?"* • *"What is a co-efficient?"* • *"What is 3 x 5?* • *"Does this picture show ½ or ¼ ?"*
Other mathematica l	• Related to teaching and learning mathematics but do not request mathematical procedures or factual	• *"How could you use this in the real world?"* • *'Which problem was the most difficult?"*

	knowledge, probe students' thinking, press for explanations, or generate discussion.	
Non-mathematical	• Does not relate to teaching and learning mathematics	• *"Why didn't you use graph paper?"* • *"Who has ever seen a caterpillar?"*

Adapted from Boaler & Humphries (2005).

Part 2: IQA Mathematics Rubrics

Accountable Talk

Consider talk from the whole-group discussion only.

I. How effectively did the lesson-talk build Accountability to the Learning Community?

Participation in the Learning Community
Was there widespread participation in teacher-facilitated discussion?

Rubric 1: Participation	
4	Over 75% of the students participated throughout the discussion.
3	50-75% of the students participated in the discussion.
2	25-50% of the students participated in the discussion.
1	Less than 25% of the students participated in the discussion.
0	None of the students participated in the discussion.
N/A	Reason:

_______ **Number of students in class**

_______ **Number of students who participated**

Teacher's Linking Contributions: Does the teacher support students in connecting ideas and positions to build coherence in the discussion?

Rubric 2: Teacher's Linking	
4	The teacher consistently (at least 3 times) explicitly connects (or provides opportunities for students to connect) speakers' contributions to each other and describes (or provides opportunities for students to describe) how ideas/positions shared during the discussion relate to each other.
3	At least twice during the lesson, the teacher explicitly connects (or provides opportunities for students to connect) speakers' contributions to each other and describes (or provides opportunities for students to describe) how ideas/positions relate to each other.
2	At one or more points during the discussion, the teacher links speakers' contributions to each other, but does not show how ideas/positions relate to each other (weak links -- e.g., local coherence; implicit building on ideas; noting that ideas/strategies are different but not describing how). OR teacher revoices or recaps only, but does not describe how ideas/positions relate to each other OR only one strong effort is made to connect speakers' contributions to each other (1 strong link).
1	Teacher does not make any effort to link or revoice speakers' contributions.
0	No class discussion OR Class discussion was not related to mathematics.
N/A	Reason:

Students' Linking Contributions: Do student's contributions link to and build on each other?

Rubric 3: Students' Linking	
4	The students consistently explicitly connect their contributions to each other and describe how ideas/positions shared during the discussion relate to each other. (e.g. I agree with Jay because…")
3	At least twice during the lesson, students explicitly connect their contributions to each other and describe ideas/positions shared during the discussion relate to each other. (e.g. I agree with Jay because…")
2	At one or more points during the discussion, the students link students' contributions to each other, but do not describe how ideas/positions relate to each other. (e.g., e.g., local coherence; implicit building on ideas; "I disagree with Ana.") OR students make only one strong effort to connect their contributions with each other.
1	Students do not make any effort to link or revoice students' contributions.
0	No class discussion OR Class discussion was not related to mathematics.
N/A	Reason:

II. How effectively did the lesson-talk build <u>Accountability to Knowledge and Rigorous Thinking</u>?

Asking: Were students pressed to support their contributions with evidence and/or reasoning?

Rubric 4: Asking (Teachers' Press)	
4	The teacher consistently (almost always) asks students to provide evidence for their contributions (i.e., press for conceptual explanations) or to explain their reasoning. (There are few, if any instances of missed press, where the teacher needed to press and did not.)
3	Once or twice during the lesson the teacher asks students to provide evidence for their contributions (i.e., press for conceptual explanations) or to explain their reasoning. (The teacher sometimes presses for explanations, but there are instances of missed press.)
2	Most of the press is for computational or procedural explanations or memorized knowledge OR There are one or more superficial, trivial efforts, or formulaic efforts to ask students to provide evidence for their contributions or to explain their reasoning (i.e., asking everyone, "How did you get that?").
1	There are no efforts to ask students to provide evidence for their contributions AND there are no efforts to ask students to explain their thinking.
0	Class discussion was not related to mathematics OR No class discussion
N/A	Reason:

Providing: Did students support their contributions with evidence and/or reasoning? (This evidence must be appropriate to the content area—i.e., evidence from the text; citing an example, referring to prior classroom experience.)

Rubric 5: Providing (Students' Responses)	
4	Students consistently provide evidence for their claims, OR students explain their thinking using reasoning in ways appropriate to the discipline (i.e. conceptual explanations).
3	Once or twice during the lesson students provide evidence for their claims, OR students explain their thinking, using reasoning in ways appropriate to the discipline (i.e. conceptual explanations).
2	Students provide explanations that are computational, procedural or memorized knowledge, OR What little evidence or reasoning students provide is inaccurate, incomplete, or vague.
1	Speakers do not back up their claims, OR do not explain the reasoning behind their claims.
0	Class discussion was not related to mathematics OR No class discussion
N/A	Reason:

Academic Rigor

RUBRIC 1: Potential of the Task

Did the task have potential to engage students in rigorous thinking about challenging content?

4	The task has the potential to engage students in exploring and understanding the nature of mathematical concepts, procedures, and/or relationships, such as: • Doing mathematics: using complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example); OR • Procedures with connections: applying a broad general procedure that remains closely connected to mathematical concepts. The task must explicitly prompt for evidence of students' reasoning and understanding. For example, the task MAY require students to: • solve a genuine, challenging problem for which students' reasoning is evident in their work on the task; • develop an explanation for why formulas or procedures work; • identify patterns and form and justify generalizations based on these patterns; • make conjectures and support conclusions with mathematical evidence; • make explicit connections between representations, strategies, or mathematical concepts and procedures. • follow a prescribed procedure in order to explain/illustrate a mathematical concept, process, or relationship.
3	The task has the potential to engage students in complex thinking or in creating meaning for mathematical concepts, procedures, and/or relationships. However, the task does not warrant a "4" because: • the task does not explicitly prompt for evidence of students' reasoning and understanding. • students may be asked to engage in doing mathematics or procedures with connections, but the underlying mathematics in the task is not appropriate for the specific group of students (i.e., too easy <u>or</u> too hard to promote engagement with high-level cognitive demands); • students may need to identify patterns but are not pressed for generalizations or justification; • students may be asked to use multiple strategies or representations but the task does not explicitly prompt students to develop connections between them; • students may be asked to make conjectures but are not asked to provide mathematical evidence or explanations to support conclusions

2	The potential of the task is limited to engaging students in using a procedure that is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • There is little ambiguity about what needs to be done and how to do it. • The task does not require students to make connections to the concepts or meaning underlying the procedure being used. • Focus of the task appears to be on producing correct answers rather than developing mathematical understanding (e.g., applying a specific problem solving strategy, practicing a computational algorithm). OR There is evidence that the mathematical content of the task is at least 2 grade-levels below the grade of the students in the class.
1	The potential of the task is limited to engaging students in memorizing or reproducing facts, rules, formulae, or definitions. The task does not require students to make connections to the concepts or meaning that underlie the facts, rules, formulae, or definitions being memorized or reproduced.
0	The task requires no mathematical activity.
N/A	Students did not engage in a task.

ATTACH OR DESCRIBE THE TASK.

RUBRIC 2: Implementation of the Task

At what level did the teacher guide students to engage with the task in implementation?

4	Students engaged in exploring and understanding the nature of mathematical concepts, procedures, and/or relationships, such as: • Doing mathematics: using complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example); OR • Procedures with connections: applying a broad general procedure that remains closely connected to mathematical concepts. There is explicit evidence of students' reasoning and understanding. For example, students may have: • solved a genuine, challenging problem for which students' reasoning is evident in their work on the task; • developed an explanation for why formulas or procedures work; • identified patterns, formed and justified generalizations based on these patterns; • made conjectures and supported conclusions with mathematical evidence; • made explicit connections between representations, strategies, or mathematical concepts and procedures. • followed a prescribed procedure in order to explain/illustrate a mathematical concept, process, or relationship.
3	Students engaged in complex thinking or in creating meaning for mathematical concepts, procedures, and/or relationships. However, the implementation does not warrant a "4" because: • there is no explicit evidence of students' reasoning and understanding. • students engaged in doing mathematics or procedures with connections, but the underlying mathematics in the task was not appropriate for the specific group of students (i.e., too easy or too hard to sustain engagement with high-level cognitive demands); • students identified patterns but did not form or justify generalizations; • students used multiple strategies or representations but connections between different strategies/representations were not explicitly evident; • students made conjectures but did not provide mathematical

	evidence or explanations to support conclusions
2	Students engaged in using a procedure that was either specifically called for or its use was evident based on prior instruction, experience, or placement of the task. • There was little ambiguity about what needed to be done and how to do it. • Students did not make connections to the concepts or meaning underlying the procedure being used. • Implementation focused on producing correct answers rather than developing mathematical understanding (e.g., applying a specific problem solving strategy, practicing a computational algorithm). OR There is evidence that the mathematical content of the task is at least 2 grade-levels below the grade of the students in the class.
1	Students engage in memorizing or reproducing facts, rules, formulae, or definitions. Students do not make connections to the concepts or meaning that underlie the facts, rules, formulae, or definitions being memorized or reproduced.
0	The students did not engage in mathematical activity.
N/A	The students did not engage with a mathematical task.

RUBRIC 3: Student Discussion Following Task
To what extent did students show their work and explain their thinking about the important mathematical content?

4	Students present their mathematical work and thinking for solving a task and/or engage in a discussion (teacher- guided or student-led) of the important mathematical ideas in the task. During this discussion: •students provide complete and thorough explanations of their strategy, idea, or procedure. •students make connections to the underlying mathematical ideas (e.g., "I divided because we needed equal groups"). •students provide reasoning and justification for their mathematical work and thinking. OR •students present and/or discuss more than one strategy or representation for solving the task, and a) provide explanations, comparisons, etc., of why/how the different strategies/representations were used to solve the task, and/or b) make explicit connections between strategies or representations; •there is *thorough presentation and discussion across strategies or representations*
3	Students present their mathematical work and thinking for solving a task and/or engage in a discussion (teacher- guided or student-led) of the important mathematical ideas in the task. During this discussion: •students *attempt to* provide explanations of why their strategy, idea, or procedure is valid and/or students *begin to* make connections. The justifications, explanations and connections are conceptually-based (and on the right track), *but are not complete and thorough* (e.g., student responses often require extended press from the teacher, are incomplete, lack precision, or fall short of making explicit connections). OR •students present and/or discuss more than one strategy or representation for solving the task, and provide explanations of how the individual strategies/representations were used to solve the task *but do not make connections between different strategies or representations.* •*there are thorough presentation and/or discussion of individual strategies or representations, but there is not discussion, comparison, connections, etc., across strategies/representations.*

2	Students show/describe/discuss procedural work for solving the task. During this discussion: • connections are **not** made with mathematical concepts and the discussion focuses solely on procedures (e.g., the steps for a multiplication problem, finding an average, or solving an equation; what they did first, second, etc.), OR • students make presentations of their work, and questioning or prompting from the teacher is for procedural explanations only, OR • students show/discuss only one strategy/representation for solving the task, OR • students present their work with no questioning or prompting from the teacher (to the presenters or to the class) to explain the mathematical work, make connections, etc. [*Presentations with no discussion.*]
1	• Students provide brief or one-word answers, fill in blanks, or IRE pattern (e.g., T: What is the answer to Question 5? S: 4.5 T: Correct!), OR • Students' responses are vague, unclear, or contain several misconceptions regarding the overall concept or procedure. [Student responses are incorrect or do not make sense mathematically.]
0	There was no mathematical discussion of the task: a) no discussion occurred following students' work on the task; or b) teacher's questions and/or student's responses are non-mathematical.
N/A	Reason:

AR-Q: Rigor of Teachers' Questions

Rubric AR-Q: Questioning	
4	**The teacher consistently asks** academically relevant questions that provide opportunities for students to elaborate and explain their mathematical work and thinking (probing, generating discussion), identify and describe the important mathematical ideas in the lesson, or make connections between ideas, representations, or strategies (exploring mathematical meanings and relationships).
3	**At least 3 times during the lesson, the teacher asks** academically relevant questions that provide opportunities for students to elaborate and explain their mathematical work and thinking (probing, generating discussion), identify and describe the important mathematical ideas in the lesson, or make connections between ideas, representations, or strategies (exploring mathematical meanings and relationships).
2	**There are one or more superficial, trivial, or formulaic efforts** to ask academically relevant questions probing, generating discussion, exploring mathematical meanings and relationships) (i.e., every student is asked the same question or set of questions) or to ask students to explain their reasoning; OR **Only one (1) effort** is made to ask an academically relevant question (e.g., one instance of a strong question, or the same strong question is asked multiple times)
1	The teacher asks procedural or factual questions that elicit mathematical facts or procedure or require brief, single word responses.
0	The teacher did not ask questions during the lesson, or the teacher's questions were not relevant to the mathematics in the lesson.
N/A	Reason:

AR-X: Mathematical Residue Rubric

Rubric AR-X: Mathematical Residue	
4	The discussion following students' work on the task surfaces the important mathematical ideas, concepts, or connections embedded in the task and serves to extend or solidify students' understanding of the main mathematical goals/ideas/concepts of the lesson. The discussion leaves behind important mathematical residue.
3	During the discussion following students' work on the task, the important mathematical ideas, concepts, or connections begin to surface, are wrestled with by students, but are not pursued in depth or have not materialized/solidified by the close of the lesson. The lesson is beginning to amount to something mathematically but the mathematics is only partially developed; perhaps due to time or student readiness.
2	During the discussion following students' work on the task, the important mathematical ideas, concepts, or connections in the task are explained or made explicit by the teacher primarily (i.e., the teacher is telling students what connections should have been made; students take notes or provide brief answers but do not make meaningful mathematical contributions to the discussion, students make superficial contributions that are taken over by the teacher). The discussion is mathematical, but does not address the concepts, ideas, or connections embedded in the task (random or not consistent with the mathematical goal) OR the discussion is about mathematics that is not relevant/important for the group of students.
1	Important mathematical ideas do not surface during the discussion following students' work on the task. The discussion is mathematical, but there is no apparent mathematical goal; the discussion does not focus on developing (or building up) students' understanding of the important mathematical ideas.
0	There was no discussion following the task. OR The discussion was about non-mathematical aspects of the task and did not leave behind mathematical residue.

Appendix B

Interview questions

1. How many years have you been teaching?
2. What grades have you taught?
3. How many courses have you taken through the Elementary Mathematics Instructional Leader series?
4. Why did you decide to take these courses?

This first set of questions are about mathematical learning in your classroom.

5. Tell me about what it's like to learn math in your classroom.
6. *(If not addressed in #4)* Tell me about your beliefs concerning math instruction. Please describe one belief and give an example of how that belief shows up in your teaching.
7. What's the most important thing(s) for your students to learn at this grade? How do you facilitate that learning?

This next set of questions are about teaching math.

8. I am interested in how you prepare math lessons especially the instructional resources you use. Could you tell me about that? (What instructional materials or resources are used in your lesson preparation?)

The final set of questions are specifically about mathematical discourse.

9. How do you see the relationship between mathematical discourse and student learning in elementary grades? Why do you think this?

10. Do your students struggle with the math content and/or discourse procedures? (if not addressed in answer ask: Can you tell me about one student and how you helped him/her?)
11. Do the district adopted materials influence the amount of discourse and kind of discourse opportunities offered in your classroom instruction? If so, in what way? (Follow-up if appropriate: How do you remedy this?)
12. Are there any characteristics of your school's population that might impact mathematical discourse, positively or negatively? How? Explain why you think this. How would it be different with a different student population? (follow up with SES if this is not mentioned)
13. Is there anything you'd like to add to help me understand your thinking on the topic of mathematical discourse?