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The Rank of Recurrence Matrices

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When teaching a course in linear algebra, instructors may ask students to produce examples that illustrate their understanding of particular definitions or concepts and test their ability to apply them. For example, to elicit knowledge of the *rank* of a matrix (i.e., the number of pivots in the matrix once reduced to echelon form, or, equivalently, the size of the largest collection of linearly independent columns of the matrix), both authors have themselves asked students to give an example of a 3×3 matrix with *full* rank, meaning that it has as many pivots as possible (here, three). Savvy students often produce the identity matrix, its three pivots clearly displayed, but the following answer has also appeared many times in our various courses:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Of course, a quick check by row reduction reveals that A does *not* have full rank, rather it has rank 2: In reduced row echelon form,

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\sim$ denotes row equivalence. This is not obvious at first glance—there are no zero entries in A and certainly no row is a multiple of another—but it turns out that

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every $m \times n$ matrix (with $m, n \geq 2$) filled with consecutive terms of any arithmetic sequence has rank 2. (If you have not already, verify that the third row of A above is twice the second row minus the first.) This curiosity is the starting point for our investigation, which then broadens to consider the ranks of matrices filled with geometric sequences and more generally defined recursive sequences. Though most examples involve integers for simplicity, all of the following results hold for arbitrary real-valued entries.

Matrices of arithmetic and geometric sequences

The defining characteristic of the matrix A above is that its entries are consecutive elements in an arithmetic sequence.

Definition. An *arithmetic sequence* $\{a_k\}$ is defined by a *seed* (or *initial value*) a_1, a common difference x, and the equation $a_k = a_{k-1} + x$ for $k \geq 2$. We assume that $x \neq 0$ and, by re-indexing if necessary, we also assume that $a_1 \neq 0$. An *arithmetic matrix* is a matrix whose entries (read row-by-row) form an arithmetic sequence.

Proposition 1. *Every* $m \times n$ *arithmetic matrix* A *with* $m, n \geq 2$ *has rank* 2.

Proof. Note that A may be written as

$$A = \begin{bmatrix} a & a+x & \dots & a+(n-1)x \\ a+nx & a+(n+1)x & \dots & a+(2n-1)x \\ a+2nx & a+(2n+1)x & \dots & a+(3n-1)x \\ \vdots & \vdots & \ddots & \vdots \\ a+(m-1)nx & a+((m-1)n+1)x & \dots & a+(mn-1)x \end{bmatrix},$$

where a is the seed and x is the difference between successive terms of the sequence. Simply expressing the entries of A in this way makes the dependencies between rows much more apparent! We see that $2(\text{Row } 2) - (\text{Row } 1) = \text{Row } 3$, and that every row beyond the third can be expressed as a linear combination of the first two rows:

$$(i-1)(\text{Row } 2) - (i-2)(\text{Row } 1) = \text{Row } i$$

for $3 \leq i \leq m$. Thus, by applying the $m-2$ appropriate row operations, we cancel out Row 3 through Row m. Carrying out one last row operation, replacing Row 2 by $-\frac{a+nx}{a}(\text{Row } 1) + (\text{Row } 2)$, we have

$$A \sim \begin{bmatrix} a & a+x & \dots & a+(n-1)x \\ a+nx & a+(n+1)x & \dots & a+(2n-1)x \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} a & a+x & \dots & a+(n-1)x \\ 0 & \frac{-nx^2}{a} & \dots & \frac{-(n-1)x^2}{a} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

and, since $x \neq 0$, we see that A is row equivalent to a rank 2 matrix. ■

This is not a new result—it has likely been observed by many before us, was mentioned (without proof) in [**3**], was the starting point for a related question in [**8**] involving Vandermonde determinants, and has been rediscovered by every one of our students who mistakenly suggested an arithmetic matrix had full rank on one of our exams!—yet it provides an intuitive starting place for further investigation. A natural next question is, perhaps, "What happens if we replace an arithmetic sequence with a *geometric* sequence?" For those adept at mental row reduction, it will be unsurprising that these *geometric matrices* are all rank 1, but we include the result for completeness.

Definition. A *geometric sequence* takes the form $a_k = ap^{k-1}$, where a and p are nonzero and $k \geq 1$. This can also be written recursively as $a_k = pa_{k-1}$ for $k \geq 2$ with initial value $a_1 = a$ and common ratio p. A *geometric matrix* is a matrix whose entries (read row-by-row) form a geometric sequence.

Proposition 2. *Every $m \times n$ geometric matrix G has rank 1.*

Proof. Note that G takes the form

$$\begin{bmatrix} a & ap & \dots & ap^{n-1} \\ ap^{n} & ap^{n+1} & \dots & ap^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ ap^{(m-1)n} & ap^{(m-1)n+1} & \dots & ap^{mn-1} \end{bmatrix},$$

and that rows 2 through m are multiples of the first row, specifically, Row $i = p^{(i-1)m}$(Row 1) for $2 \leq i \leq m$. Therefore, G is row equivalent to a matrix with exactly one nonzero row, which must be a pivot row. Thus, $\text{rank}(G) = 1$. ■

Matrices of linear recurrence relations

The arithmetic and geometric examples we have just examined are two cases of sequences defined by *recurrence relations.* We turn now to a broader examination of sequences defined by recurrence relations of arbitrary order.

Definition. A *homogeneous linear recurrence sequence* is a sequence $\{a_k\}$ defined by a relation of the form $a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \cdots + \gamma_r a_{k-r}$, where each $\gamma_i \in \mathbb{R}$ with $\gamma_r \neq 0$ and the *initial values* (or *seeds*) $a_1, \dots, a_r \in \mathbb{R}$ are given. Adding a nonzero constant to a homogeneous linear recurrence relation results in an *inhomogeneous linear recurrence sequence*: $a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \cdots + \gamma_r a_{k-r} + b$, where γ_r and b are nonzero. (Since we consider only linear sequences, we will omit descriptor "linear" from here on.)

The *order* of a recurrence sequence is the largest index j for which γ_j is nonzero; we assume all recurrences have a finite order r. As before, by re-indexing if necessary, we assume $a_1 \neq 0$, though other seeds (and coefficients) may be zero. Finally, a *recurrence matrix* is a matrix whose entries (read row-by-row) are the consecutive terms in some recurrence sequence.

Any arithmetic sequence, such as $1, 2, 3, 4, 5, \dots$, is an example of a first-order inhomogeneous recurrence sequence. Geometric sequences form the class of first-order homogeneous recurrence sequences. The classical *Fibonacci numbers*, 1, 1, 2, 3, 5, 8, 13, …, are a homogeneous recurrence sequence of order 2 of the form $a_k = a_{k-1} + a_{k-2}$, likewise the *Lucas numbers*, 2, 1, 3, 4, 7, 11, …, which are defined by the same relation as the Fibonacci numbers but with different seeds ($a_1 = 2$, $a_2 = 1$). Examples of recurrence matrices for these sequences are

$$F = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 8 & 13 & 21 & 34 & 55 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 11 \\ 18 & 29 & 47 \end{bmatrix}.$$

Clearly F has rank 2; we encourage you to verify that L also has rank 2. The references [**2, 4, 7**] contain a wealth of information on Fibonacci and Lucas numbers.

The earlier cases of arithmetic and geometric matrices having ranks 2 and 1, respectively, can now be viewed as illustrating more general and somewhat surprising facts, which we prove next: Given a homogeneous recurrence sequence, the rank of the associated recurrence matrix is bounded above by the order r of the recurrence. For inhomogeneous sequences, the upper bound on matrix rank is $r + 1$.

Theorem 1. *Consider the order r homogeneous recurrence sequence $\{a_k\}$ defined by $a_k = \gamma_1 a_{k-1} + \cdots + \gamma_r a_{k-r}$. If R is the associated $m \times n$ recurrence matrix, then rank$(R) \leq r$.*

Proof. If either m or n is less than r, then the theorem is automatically true, since the rank of R is bounded above by $\min(m, n)$. We therefore assume that $m, n \geq r$; note that this implies all r of the seeds for the sequence fall in the first row of R.

For the sake of simplicity, we first consider the special case wherein all $\gamma_i = 1$. To this end, let $a_k = a_{k-1} + \cdots + a_{k-r}$. Writing $R_1, \ldots, R_n$ for the columns of the recurrence matrix R, the equations $R_k = R_{k-1} + \cdots + R_{k-r}$ for $r + 1 \leq k \leq n$ give linear dependencies among the columns of R. Therefore, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{v}_{n-r} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ -1 \end{bmatrix},$$

each with r ones, all belong to $\text{Null}(R) \subseteq \mathbb{R}^n$. Letting $\vec{e}_1, \ldots, \vec{e}_n$ denote the standard basis vectors in $\mathbb{R}^n$, we have

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_r - \vec{e}_{r+1}, \ldots, \vec{v}_{n-r} = \vec{e}_{n-r} + \vec{e}_{n-r+1} + \cdots + \vec{e}_{n-1} - \vec{e}_n.$$

Observe that the vectors $\{\vec{v}_i\}$ are linearly independent (from the placement of zeros and ones, for instance). There are $n - r$ of these vectors; thus, the dimension of the null space of R is at least $n - r$. The Rank-Nullity Theorem [**5**, ch. 4] states that $\text{rank}(R) + \dim(\text{Null}(R)) = n$, which implies $\text{rank}(R) \leq r$.

Generalizing, we now consider $a_k = \gamma_1 a_{k-1} + \cdots + \gamma_r a_{k-r}$ for arbitrary coefficients γ_i with $\gamma_r \neq 0$. The dependence relations among columns of R are now

$$R_k = \gamma_1 R_{k-1} + \gamma_2 R_{k-2} + \cdots + \gamma_r R_{k-r}$$

for $r + 1 \leq k \leq n$. Thus, the vectors

$$\vec{v}_1 = \gamma_r\vec{e}_1 + \gamma_{r-1}\vec{e}_2 + \cdots + \gamma_1\vec{e}_r - \vec{e}_{r+1}, \ldots$$
$$\vec{v}_{n-r} = \gamma_r\vec{e}_{n-r} + \gamma_{n-1}\vec{e}_{n-r+1} + \cdots + \gamma_1\vec{e}_{n-1} - \vec{e}_n$$

belong to Null(R). As before, the $\vec{v}_i$ are linearly independent: Each $\vec{v}_i$ is guaranteed to have a nonzero entry in Row i (from γ_r) and in Row $(r+i)$ (from $-\vec{e}_{r+i}$). The Rank-Nullity Theorem again implies rank$(R) \leq r$. ∎

Thus, in the case of a recurrence matrix for a linear homogeneous sequence of order r, "full rank" could be taken to mean "rank r" (the fullest possible rank).

Corollary. *Let $\{a_k\}$ be an order r inhomogeneous recurrence sequence defined by the relation $a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \cdots + \gamma_r a_{k-r} + b$ where $b \neq 0$. If R is the associated $m \times n$ recurrence matrix, then rank$(R) \leq r + 1$.*

Proof. Since we may write

$$a_{k+1} = \gamma_1 a_k + \gamma_2 a_{k-1} + \cdots + \gamma_r a_{k-r+1} + b,$$
$$a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \cdots + \gamma_r a_{k-r} + b,$$

the difference between successive terms is given by

$$a_{k+1} - a_k = \gamma_1 a_k + (\gamma_2 - \gamma_1)a_{k-1} + \cdots + (\gamma_r - \gamma_{r-1})a_{k-r+1} + \gamma_r a_{k-r}.$$

Adding a_k to each side, we have

$$a_{k+1} = (\gamma_1 + 1)a_k + (\gamma_2 - \gamma_1)a_{k-1} + \cdots + (\gamma_r - \gamma_{r-1})a_{k-r+1} + \gamma_r a_{k-r}.$$

By definition, $\gamma_r \neq 0$, hence an arbitrary term in the sequence can be expressed as an order $r + 1$ homogeneous relation and the result follows from Theorem 1. ∎

We note that this upper bound on rank is not necessarily tight: Experimenting with coefficients and initial conditions leads to examples of homogeneous recurrences of order r, whose matrices have rank substantially less than r, as we see next.

Consider the second-order recurrence $a_n = -3a_{n-1} - 2a_{n-2}$ with $a_1 = 1$, $a_2 = -2$. This relation produces the sequence $1, -2, 4, -8, 16, -32, 64, \ldots$, which can also be written as the geometric sequence $a_n = -2a_{n-1}$. As such, Proposition 2 implies that every recurrence matrix for this sequence will have rank 1.

As another example, the somewhat trivial fourth-order relation given by $b_n = b_{n-4}$ with seeds $b_1 = 1, b_2 = b_3 = b_4 = 0$ produces the sequence $1, 0, 0, 0, 1, 0, 0, 0, \ldots$. In this case, the rank of the associated recurrence matrix depends on the number of columns present: A 4×4 matrix clearly has rank 1, while a 6×6 matrix has rank 2, and a 5×5 has ("full") rank 4. In this fashion, we may construct a recurrence of arbitrarily large order where some associated recurrence matrices are rank 1.

When does the rank drop?

To explore precisely when and why a particular choice of coefficients or seeds leads to a drop in rank, we employ a strategy familiar to those who have studied differential equations. A homogeneous recurrence relation of order r may be written as a system of r homogeneous first-order equations. This is done by introducing $r - 1$ variables to take the place of $a_{k-1}, \ldots, a_{k-r+1}$ **[6]**.

Definition. The roots of the characteristic polynomial for the coefficient matrix describing this linear system are the *eigenvalues* of the recurrence relation.

We illustrate this in the order 2 case, which will be the focus of the classification to follow. The second-order recurrence relation $a_k = ca_{k-1} + da_{k-2}$ may be converted into a system of first-order equations by setting $b_k = a_{k-1}$, thereby implying $b_{k-1} = a_{k-2}$. Then, $a_k = ca_{k-1} + db_{k-1}$ and the recurrence relation may be written as a matrix equation

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} = \begin{bmatrix} c & d \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ b_{k-1} \end{bmatrix}.$$

The characteristic polynomial of the coefficient matrix is $P(\lambda) = \lambda^2 - c\lambda - d$ and the roots of $P(\lambda)$ are the eigenvalues, namely

$$\lambda = \frac{c \pm \sqrt{c^2 + 4d}}{2}. \tag{1}$$

The rank of a recurrence matrix for a second-order homogeneous relation depends on a particular relationship between the seeds and the eigenvalues of the relation. Before giving this classification of drops in rank in the order 2 homogeneous case, we present a technical lemma, which allows us to express a_k as a linear combination of the initial values a_1 and a_2.

Lemma. *Let $a_k = ca_{k-1} + da_{k-2}$ be a homogeneous recurrence relation of order* 2. *Let $\{s_k\}$ and $\{t_k\}$ be the sequences of coefficients of a_2 and a_1 in $\{a_k\}$, respectively, such that $a_k = s_k a_2 + t_k a_1$. Then, for all $k \geq 3$,*

(i) $s_k = cs_{k-1} + ds_{k-2}$ *and*

(ii) $a_k = s_k a_2 + ds_{k-1}a_1$.

Proof. To prove (i), note that

$$\begin{aligned} a_k &= ca_{k-1} + da_{k-2} \\ &= c(s_{k-1}a_2 + t_{k-1}a_1) + d(s_{k-2}a_2 + t_{k-2}a_1) \\ &= (cs_{k-1} + ds_{k-2})a_2 + (ct_{k-1} + dt_{k-2})a_1. \end{aligned}$$

Since $a_k = s_k a_2 + t_k a_1$, comparing the coefficients of a_2 yields the result.

The proof of (ii) proceeds by induction. Note that the first three terms of the sequence $\{a_k\}$ are $a_1, a_2, ca_2 + da_1$, and so the first three terms of $\{s_k\}$ are $0, 1, c$. The base case $k = 3$ is therefore true with $s_3 = c$ and $s_2 = 1$. Now, assume that (ii) is true for all $\ell < k$. Then, our inductive assumption implies that

$$\begin{aligned} a_k &= ca_{k-1} + da_{k-2} \\ &= c(s_{k-1}a_2 + ds_{k-2}a_1) + d(s_{k-2}a_2 + ds_{k-3}a_1) \\ &= (cs_{k-1} + ds_{k-2})a_2 + d(cs_{k-2} + ds_{k-3})a_1. \end{aligned}$$

By (i), it follows that $a_k = s_k a_2 + ds_{k-1}a_1$. ∎

We are now able to state precisely when a drop in rank occurs for a recurrence matrix of a homogeneous relation of order 2.

Theorem 2. *Consider the second-order homogeneous recurrence sequence* $a_k = ca_{k-1} + da_{k-2}$ *for nonzero* $c, d \in \mathbb{R}$. *If* R *is the associated* $m \times n$ *recurrence matrix, then* $rank(R) = 1$ *whenever* $\frac{a_2}{a_1}$ *is an eigenvalue of the recurrence relation for seeds* a_1 *and* a_2.

Proof. Without loss of generality, we compute the rank of the transpose, R^T [**5**]. By part (ii) of the lemma, the terms of the sequence $\{a_k\}$ can be written entirely in terms of the coefficient sequence $\{s_k\}$ and the seeds a_1 and a_2. Specifically, $a_k = s_k a_2 + ds_{k-1}a_1$. As a result, the transpose R^T may be written as

$$\begin{bmatrix} a_1 & s_{n+1}a_2 + ds_n a_1 & \dots & s_{(m-1)n+1}a_2 + ds_{(m-1)n}a_1 \\ a_2 & s_{n+2}a_2 + ds_{n+1}a_1 & \dots & s_{(m-1)n+2}a_2 + ds_{(m-1)n+1}a_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_n a_2 + ds_{n-1}a_1 & s_{2n}a_2 + ds_{2n-1}a_1 & \dots & s_{mn}a_2 + ds_{mn-1}a_1 \end{bmatrix}.$$

As detailed in Theorem 1, the columns of R, and hence the rows of R^T, are linearly dependent. In particular, when taken three at a time sequentially, the rows of R^T are dependent. This implies that R^T is row equivalent to

$$M = \begin{bmatrix} a_1 & s_{n+1}a_2 + ds_n a_1 & \dots & s_{(m-1)n+1}a_2 + ds_{(m-1)n}a_1 \\ a_2 & s_{n+2}a_2 + ds_{n+1}a_1 & \dots & s_{(m-1)n+2}a_2 + ds_{(m-1)n+1}a_1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now, for $1 \leq j \leq m-1$, consider the $(j+1)$st column in M,

$$M_{j+1} = \begin{bmatrix} s_{jn+1}a_2 + ds_{jn}a_1 \\ s_{jn+2}a_2 + ds_{jn+1}a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

After replacing Row 2 with $\frac{-a_2}{a_1}$(Row 1) + (Row 2), we have

$$\begin{bmatrix} s_{jn+1}a_2 + ds_{jn}a_1 \\ -\frac{1}{a_1}\left(s_{jn+1}a_2^2 + (ds_{jn} - s_{jn+2})a_1a_2 - ds_{jn+1}a_1^2\right) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since part (i) of the lemma implies that $ds_{jn} - s_{jn+2} = -cs_{jn+1}$, this reduced form of the $(j+1)$st column of M may be written as

$$\begin{bmatrix} s_{jn+1}a_2 + ds_{jn}a_1 \\ -\frac{s_{jn+1}}{a_1}\left(a_2^2 - ca_1a_2 - da_1^2\right) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It follows, then, that R^T is row equivalent to

$$\begin{bmatrix} a_1 & s_{n+1}a_2 + ds_n a_1 & \dots & s_{(m-1)n+1}a_2 d s_{(m-1)n} a_1 \\ 0 & -\frac{s_{n+1}}{a_1} q & \dots & -\frac{s_{(m-1)n+1}}{a_1} q \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \tag{2}$$

where $q = a_2^2 - ca_1a_2 - da_1^2$. Therefore, $\text{rank}(R^T) = 1$ whenever $q = 0$. Solving for a_2 in terms of a_1 in $q = 0$ yields

$$a_2 = a_1 \left(\frac{c \pm \sqrt{c^2 + 4d}}{2} \right),$$

showing by (1) that $\frac{a_2}{a_1}$ is an eigenvalue of the relation. ■

Note that the converse of the theorem fails: If R is any $m \times n$ recurrence matrix for a homogeneous relation of order 2 and $\text{rank}(R) = 1$, then $\frac{a_2}{a_1}$ need not be an eigenvalue of the relation. To see see this, note from the reduced form of R^T in (2) that $\text{rank}(R)$ is also 1 when the coefficients $s_{n+1}, \dots, s_{(m-1)n+1}$ are zero (i.e., when the a_2 component of a_k vanishes periodically with n). This is highlighted by the next example.

Define a sequence by $a_k = 2a_{k-1} - 2a_{k-2}$ with seeds $a_1 = 1, a_2 = 2$. The characteristic polynomial $\lambda^2 - 2\lambda + 2$ has complex roots, so the ratio of initial values is not an eigenvalue of the relation (this is true for any real seeds). Yet, writing out a few terms of this sequence, we have $1, 2, 2, 0, -4, -8, -8, 0, 16, 32, 32, 0$ and you can verify $a_{4k} = 0$ for all integers $k \geq 1$. Any recurrence matrix with $n = 4k$ columns, then, will have rank 1. If we write the sequence another way, as in part (ii) of the lemma, then we can easily see the entries in which the coefficients s_{kn+1} of a_2 vanish:

$$a_1, a_2, 2a_2 - 2a_1, 2a_2 - 4a_1, \underbrace{-4a_1}_{s_{n+1}=0}, -4a_2, -8a_2 + 8a_1, -8a_2 + 16a_1, \underbrace{16a_1}_{s_{2n+1}=0}, \dots.$$

Observe, however, that a 3×3 recurrence matrix for this sequence has rank 2. This indicates that, in some cases, the rank of a recurrence matrix depends on subtle relationships between the coefficients defining the relation, the initial values, and the number of columns in R.

When c, d, a_1, a_2 are all positive, the coefficients s_k are always nonzero in the second-order recurrence relation $a_k = ca_{k-1} + da_{k-2}$. Hence, we have the following corollary to Theorem 2.

Corollary. *For $m, n \geq 2$, the $m \times n$ recurrence matrix for any generalized Fibonacci sequence $\mathcal{F}_k = \mathcal{F}_{k-1} + \mathcal{F}_{k-2}$ with positive integer seeds $\mathcal{F}_1 = a_1$ and $\mathcal{F}_2 = a_2$ has rank 2. In particular, any $m \times n$ recurrence matrix for the standard Fibonacci sequence 1, 1, 2, 3, 5, ... or the Lucas sequence 2, 1, 4, 3, 7, 11, ... has rank 2.*

Proof. All terms in the sequence above are positive, so $s_k \neq 0$ for $k > 1$. The characteristic polynomial of the relation $a_k = a_{k-1} + a_{k-2}$ is $P(\lambda) = \lambda^2 - \lambda - 1$, hence the eigenvalues of the relation are $\frac{1 \pm \sqrt{5}}{2}$. Since the seeds a_1, a_2 are integers, their ratio cannot be either of these eigenvalues. The result follows from Theorem 2. ■

Suggestions for further study

By the proof of Theorem 2, the rank of a recurrence matrix for the relation $a_k = s_k a_2 + t_k a_1$ is linked to the dynamics of the sequence of coefficients $\{s_k\}$: If these vanish periodically, then the rank of the matrix depends on its width (i.e., number of columns), as seen in the last example. In attempting to examine this more deeply (as well as in extending to higher-order relations), we suspect that an alternate viewpoint will be useful. Specifically, rather than writing an arbitrary term a_k with respect to initial values a_1 and a_2 (and other seeds, in the case $r > 2$) and then examining how the sequence $\{s_k\}$ behaves, we may instead find a general formula for a_k as a linear combination of *fundamental solutions* for the recurrence. These fundamental solutions are defined with respect to eigenvalues of the relation and their multiplicities; the method for finding general solutions to recurrences is well-known [**1**]. Preliminary investigations using this perspective suggest a characterization of when an order r recurrence can actually be realized as an order q recurrence for some $q < r$. In this case, Theorem 1 would imply that the rank of the recurrence matrix is bounded above by q, necessarily indicating a drop in rank from r.

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Summary. A recurrence matrix is defined as a matrix whose entries (read left-to-right, row-by-row) are sequential elements generated by a linear recurrence relation. The maximal rank of this matrix is determined by the order of the corresponding recurrence. In the case of an order-two recurrence, the associated matrix fails to have full rank whenever the ratio of the two initial values of the sequence is an eigenvalue of the relation.

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