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Model for chaotic dynamics of the perpendicular-pumping spin-wave instability

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We propose a model for the dynamics of spin-wave instabilities driven by a rf field perpendicular to the dc magnetic field in the second-order Suhl process. We show that a self-oscillation arises from the dynamic nonlinear interaction between the $k=0$ mode driven by the pumping field and a degenerate $k\neq 0$ magnon, with frequency that depends on the dissipation rates and the nonlinear interaction parameters and not on the sample dimensions. For certain parameter values, as the driving field increases we find a period-doubling route to chaos and odd-period bifurcation windows consistent with recent experiments in yttrium iron garnet.

In a very elegant microwave experiment Gibson and Jeffries[1] (GJ) recently observed chaotic behavior in spin-wave instabilities driven by the perpendicular-pumping Suhl process.[2] The experiments utilized a ferromagnetic resonance configuration in which a sphere of Ga-doped yttrium iron garnet (YIG) is surrounded by driving and pick-up coils at right angles. At certain crystal orientations and sufficient pumping intensity GJ observed strong low-frequency (~ 16 kHz) self-oscillations in the amplitude of the transmitted microwave signal (frequency 1.3 GHz) when the static magnetic field H_0 is close to the resonance value. This low-frequency oscillation displays a period-doubling bifurcation route to chaos and periodic windows as the driving field H_1 increases above the threshold for the Suhl instability. GJ have qualitatively interpreted their observations as arising from the nonlinear behavior of a dimensional resonance of the sphere pumped by the uniform mode $k=0$, which is driven by the microwave field. The prevailing idea is that the standing spin-wave modes of the dimensional resonance generate the self-oscillations and the chaotic attractors in the manner predicted by Nakamura, Ohta, and Kawasaki.[3] However, there are several problems with this interpretation. First, the theoretical studies of Nakamura *et al.*[3] are valid for the parallel-pumping instabilities[4] which are described by equations quite different than those for the transverse pumping. Secondly, in order to explain the observed self-oscillation frequency, the spin-wave modes must propagate[1,5] at an angle $\theta_k=60.4°$ with respect to the static field $\mathbf{H}_0$, which is surprisingly large considering that the mode with the lowest threshold[2] in the second-order Suhl process has $\theta_k=0$. Finally, the fact that GJ did not observe the low-frequency oscillation in pure YIG seems an indication that this oscillation depends more on the material parameters than on sample boundary conditions.

In this paper we propose a model that explains the essential features of the spin-wave chaotic dynamics observed by GJ. We show that a self-oscillation can arise from the dynamic nonlinear competition between the $k=0$ mode and a degenerate $k,-k$ magnon-pair mode, which may display chaotic behavior as the driving field increases. Consider a ferromagnetic spin-wave system driven by a rf field H_1 of frequency ω applied transversely to the static field $\mathbf{H}_0$, described by the following Hamiltonian:

$$H=\sum_k \hbar\omega_k c_k^\dagger c_k+\hbar\sum_{k,k'}\left(\tfrac{1}{2}S_{kk'}c_k^\dagger c_{-k}^\dagger c_{k'}c_{-k'}+T_{kk'}c_k^\dagger c_{k'}^\dagger c_k c_{k'}\right)+\hbar\gamma(SN/2)^{1/2}H_1(c_0^\dagger e^{-i\omega t}+\text{H.c.})\ , \quad (1)$$

where $c_k^\dagger$ and c_k are the creation and destruction operators of magnons with energy $\hbar\omega_k$, $\gamma=g\mu_B/\hbar$ is the gyromagnetic ratio, N is the number of spins S in the system, and $S_{kk'}$ and $T_{kk'}$ denote the interaction parameters between magnons with wave vectors k and k'. The justification for using spin-wave formalism is that above threshold only a few modes are drive parametrically, with population of order $\gamma_k/S_{kk}\sim 10^{17} << NS$ (γ_k is the thermal relaxation rate). The use of the restricted four-magnon interaction Hamiltonian in (1) has been justified in detail.[6] In simple ferromagnets the magnon-magnon interaction arises from the exchange, dipolar, and anisotropy interactions between the spins. For small values of k ($\leq 10^5$ cm^{-1}), such as in YIG pumped by microwave fields, the contribution from the exchange energy is negligible. The vertex of the magnetic dipolar contribution is[7]

$$V_{1234}^{\text{dip}}\simeq\frac{\pi(g\mu_B)^2}{2V}(4\cos^2\theta_{k_3-k_1}-\sin^2\theta_{k_1}-\sin^2\theta_{k_3})\ , \quad (2)$$

where V is the volume of the sample. This expression is valid for $10R^{-1}\leq k << a^{-1}$, where R is the sample dimension and a is the lattice parameter. For $k_1=0$ or $k_3=0$, the corresponding $\sin^2\theta_k$ should be replaced by N_x+N_y and for $k_1-k_3=0$, $\cos^2\theta_{k_3-k_1}$ is replaced by N_z, where N_x, N_y, and N_z are the sample demagnetizing coefficients. The anisotropy contribution depends on the crystal symmetry and on the sample orientation with respect to $\mathbf{H}_0$. For cubic symmetry and for $\mathbf{H}_0$ along the [100] or [111] crystal axes only, the four-magnon anisotropy interaction coefficient has the simple form

$$V_{1234}^{\text{anis}}\simeq -2g\mu_B H_A/SN\ , \quad (3)$$

where H_A is the effective anisotropy field in the direction of the magnetization. For YIG, $H_A(100)=-80$ Oe and $H_A(111)=+60$ Oe.

Based on the idea of the two-mode model of Nakamura *et al.*[3] we assume that above threshold only one $k,-k$ magnon pair is parametrically pumped by the $k=0$ mode. Introducing relaxation phenomenologically in the usual manner and using approximations similar to those of Ref. 6, we obtain the equations of motion for the two modes:

$$\begin{aligned}\frac{d\tilde{c}_0}{dt}&=[-i\Delta\omega_0-\gamma_0-i2(T_{00}n_0+2T_{0k}n_k)]\tilde{c}_0\\&\quad -i(S_{00}\sigma_0+2S_{0k}\sigma_k)\tilde{c}_0^*-i\gamma(SN/2)^{1/2}H_1\ ,\\ \frac{1}{2}\frac{d\sigma_k}{dt}&=[-i\Delta\omega_k-\gamma_k-i2T_{0k}n_0\\&\quad -i4(S_{kk}+T_{kk})n_k]\sigma_k-iS_{0k}\sigma_0 n_k\ ,\end{aligned} \quad (4)$$

where $\tilde{c}_0 = \langle c_0 \rangle \exp(i\omega t)$ denotes a slowly varying amplitude, $\sigma_k = \langle c_k c_{-k} \rangle \exp(i2\omega t)$ is a Cooper-pair density, $n_k = \langle c_k^\dagger c_k \rangle$ is the magnon population, which[6] for $\gamma_k t >> 1$ is $n_k \simeq |\sigma_k|$, $\Delta\omega_k = \omega_k - \omega$ is the detuning, and γ_k is the relaxation rate due to various dissipation processes. Since $\tilde{c}_0$ and σ_k are complex, Eqs. (4) describe a nonlinear dynamic system in a four-dimensional space where the external field drives the $k=0$ mode, and this in turn pumps the $k \neq 0$ magnon. For sufficiently small driving fields, all $k \neq 0$ magnons are essentially at the thermal level. When H_1 exceeds the Suhl threshold $H_c = (2\hbar \gamma_k \gamma_0^2 / \gamma M V S_{0k})^{1/2}$, the $\theta_k = 0$ magnon degenerate with the uniform mode grows exponentially in time until the nonlinear interactions become important. The behavior of n_0 and n_k in the steady state then depends on the set of interaction parameters and dissipation rates. Numerical studies of the evolution of Eqs. (4) reveal that in general n_0 and n_k are attracted to fixed points. However, for some sets of parameter values they are attracted to periodic trajectories corresponding to an oscillating dynamic interplay of energy between the two modes. Since the variation of n_0 modulates the absorbed microwave signal, this leads to a self-oscillation with frequency which unlike the dimensional resonance does not depend on the sample dimensions. We have found periodic trajectories for several sets of parameter values. Since GJ observe the lower-frequency (16 kHz) oscillation well above the minimum threshold, it is not possible from the experiments to determine which $k \neq 0$ mode is involved in the process; it is certainly the degenerate magnon that has the "right" dissipation and nonlinear interaction with the $k=0$ mode to generate the oscillation first.

Since the $\theta_k = 0$ degenerate mode is the one of lowest threshold, we have considered it as a likely candidate. From (2) we determine for this mode the parameters arising from the dipolar interaction: $S_{00} = T_{00} = 0$, $T_{0k} = 2G$, $S_{0k} = 10G/3$, $S_{kk} = -4G/3$, $T_{kk} = -G/3$, where $G = \pi (g\mu_B)^2 / \hbar V$ ($\sim 10^{-11}$ sec^{-1} in Ga:YIG with $4\pi M \sim 300$ Oe). In YIG the anisotropy interaction (3) is of the same order as the dipolar and its sign depends on the direction of the applied field, so that actually the nonlinear parameters vary substantially with the sample orientation. Unfortunately, the orientation of the sample in the GJ experiments is not known with precision.[8] Guided by these values we have found a set of parameters that gives results qualitatively similar to the experimental observation of GJ: $\Delta\omega_0 = \Delta\omega_k = 0$, $S_{00}/\gamma_k F = T_{00}/\gamma_k F = 0.1$, $S_{0k}/\gamma_k F = 3.0$, $T_{0k}/\gamma_k F = 2.9$, $S_{kk}/\gamma_k F = -0.5$, $T_{kk}/\gamma_k F = -0.2$, and $\gamma_0/\gamma_k = 5.0$, where F is a normalization factor of order G/γ_k ($\sim 10^{-17}$ in YIG) used to make the normalized quantities of order unity.

Figure 1 shows the self-oscillations of the (normalized) uniform-mode magnon population n_0 in the steady-state regime, for several values of the pumping intensity $R \equiv H_1/H_c$ (note that the actual population is n_0/F). The oscillation is first observed at $R \simeq 5.5$. As R increases a cascade of period-doubling bifurcations is observed in Figs. 1(a)–1(d) leading to chaos at $R \simeq 7.820$. At higher R a

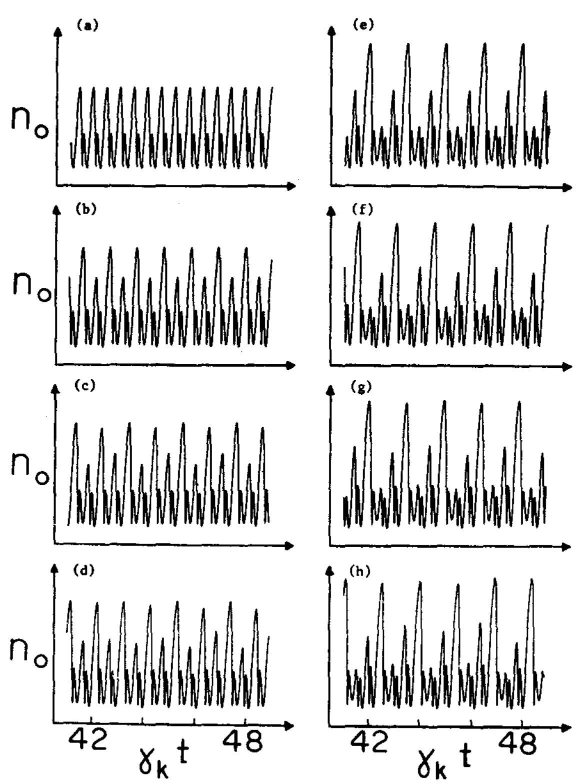

FIG. 1. Bifurcations in the auto-oscillations of the (normalized) uniform-mode magnon population, n_0 vs $\gamma_k t$: (a) period 1 at $R = 7.360$, (b) period 2 at $R = 7.640$, (c) period 4 at $R = 7.720$, (d) onset of chaos at $R = 7.770$, (e) period 3 at $R = 8.197$, (f) period 6 at $R = 8.210$, (g) period 12 at $R = 8.220$, (h) chaos at $R = 8.240$.

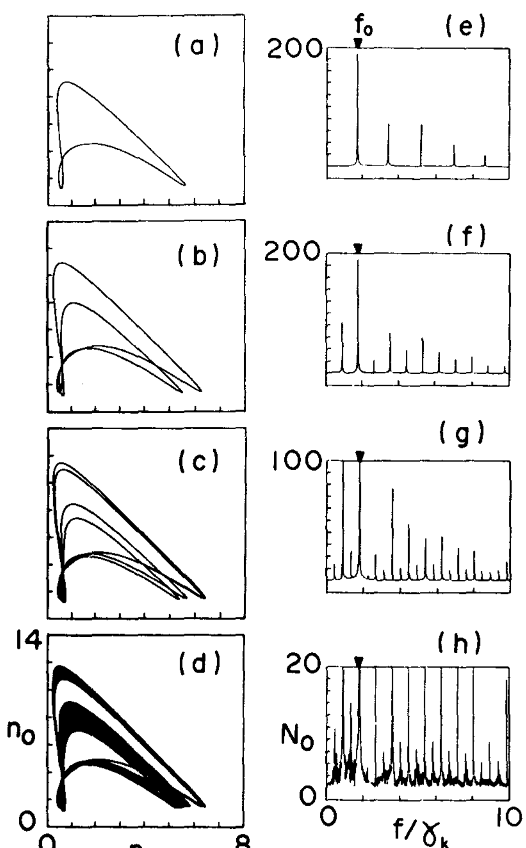

FIG. 2. Phase-space trajectories n_0 vs n_k for period-2^n self-oscillations and corresponding power spectra of the n_0 mode amplitude: (a) and (e) $n=0$ at $R = 7.360$, (b) and (f) $n=1$ at $R = 7.640$, (c) and (g) $n=2$ at $R = 7.720$, (d) and (h) onset of chaos at $R = 7.770$.

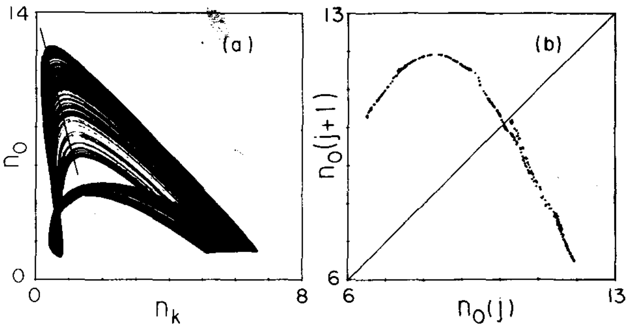

FIG. 3. (a) Fully developed strange attractor at $R=7.820$, (b) Lorenz plot for n_0 at the same value of R, constructed from the Poincaré section shown in (a).

narrow window of period 5×2^n develops (not shown) and for still larger R a narrow 3×2^n window is seen in Figs. 1(e)–1(h), in good qualitative agreement with the observations of GJ. The period-doubling bifurcation route can be further analyzed with the phase-space $n_0\times n_k$ trajectories and the Fourier transform of n_0 shown in Fig. 2. The subharmonic components $f_0/2$, $f_0/4$, $f_0/8$, and $f_0/16$ emerge at $R_2\simeq 7.398$, $R_4\simeq 7.6626$, $R_8\simeq 7.73291$, and $R_{16}\simeq 7.74879$, respectively. These values give a convergence scaling parameter $\delta\simeq(R_8-R_4)/(R_{16}-R_8)\simeq 4.43$. This is close to the Feigenbaum[9] universal value 4.669 . . . , indicating that the system behaves like a one-dimensional map. The scaling parameter $\alpha\simeq 2.6$ is also close to the universal value 2.502 . . . . Figure 3 shows a fully developed strange attractor and the corresponding Lorenz plot, this being quite similar to the return map (Fig. 8) of GJ. Finally, in Fig. 4 we show spectra and trajectories characteristic of a narrow $(f_0/3)\times 2^n$ window obtained at larger R, consistent with the observations of GJ.

In regard to the numerical value of the self-oscillation frequency f_0, we note that the calculations give $f_0\simeq 2\gamma_k$. Since the actual relaxation rate has not been measured in the experiments, we used microscopic spin-wave theory to calculate it, assuming that its source is the three-magnon interaction process.[10,11] For the $\theta_k=0$ magnon degenerate with the $k=0$ mode, we obtain for $\omega/2\pi=1.3$ GHz, $4\pi M_s=300$ Oe, at room temperature $\gamma_k=4.4\times 10^5$ sec^{-1}. Since the uniform-mode linewidth is $\Delta H_0\sim 0.2$–0.4 Oe, $\gamma_0=\gamma\Delta H_0/2\sim(1.7-\!-3.4)\times 10^6$ sec^{-1}, this is consistent with the ratio $\gamma_0/\gamma_k=5$ used in the calculation. However, this value of γ_k gives an oscillation frequency f_0 more than an order of magnitude higher than the observed one (16 kHz). A similar discrepancy has been found in studies of the parallel-pumping spin-wave instability.[12] This suggests that the actual relaxation rates γ_0 and γ_k at high pumping levels is smaller than at low power. This is consistent with the remarkably slow decay of the low-frequency oscillations reported by GJ. Our calculation of γ_k assumes that the magnons involved in the relaxation of the k mode are in thermal equilibrium with the lattice. Probably, due to a bottleneck effect at high power, this is not true and the actual value of γ_k is smaller than the calculated one.

In conclusion, we believe that the 250-kHz oscillation observed by GJ is a dimensional resonance, since it scales

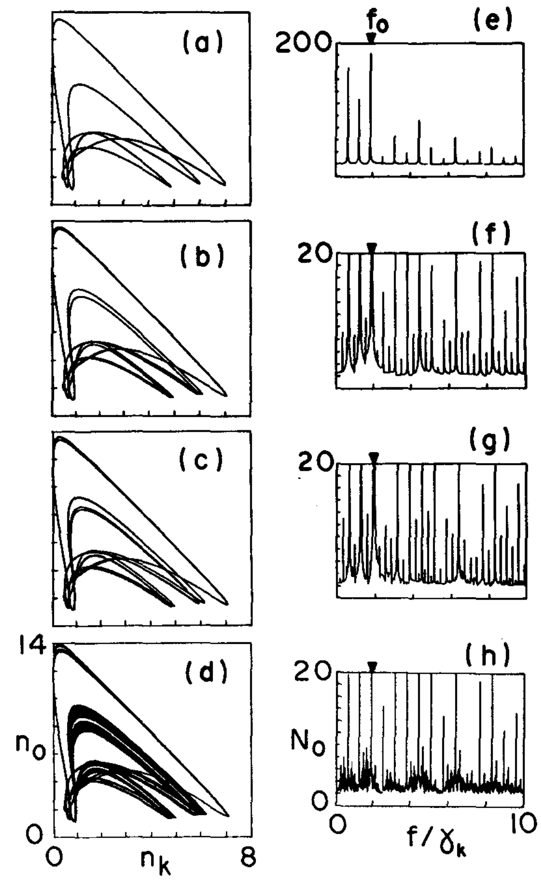

FIG. 4. Same as Fig. 2 for the period-3×2^n window: (a) and (e) $n=0$ at $R=8.197$, (b) and (f) $n=1$ at $R=8.210$, (c) and (g) $n=2$ at $R=8.220$, (d) and (h) chaos at $R=8.240$.

properly with sample diameter and corresponds to $\theta_k=0$ standing waves. However, the low-frequency (16 kHz) oscillation which displays the interesting nonlinear dynamics is associated with an oscillating interplay of energy between the $k=0$ magnon and a degenerate $k\neq 0$ pair mode. The similarity between all the results above and the observation of GJ demonstrate that the two-mode mechanism proposed here can explain the low-frequency (16 kHz) self-oscillations and the chaotic dynamics observed in Ref. 1. It remains to be determined which $k\neq 0$ magnon oscillates with the $k=0$ mode and its relaxation rate at high power. This would presumably allow a quantitative comparison of the calculated oscillation frequency and pumping intensities with the measured ones. More combined experimental and theoretical work is needed in this direction.

Since this paper was originally submitted for publication we learned that Zhang and Suhl[13] have proposed a model similar to ours to explain the GJ experiments.

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