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Weihrauch Reducibility and Finite-Dimensional Subspaces

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WEIHRAUCH REDUCIBILITY AND FINITE-DIMENSIONAL SUBSPACES

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APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Sean Sovine, affirm that the thesis, Weihrauch Reducibility and Finite-Dimensional Subspaces, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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CONTENTS

LIST OF FIGURES

ABSTRACT

In this thesis we study several principles involving subspaces and decompositions of vector spaces, matroids, and graphs from the perspective of Weihrauch reducibility. We study the problem of decomposing a countable vector space or countable matroid into 1-dimensional subspaces. We also study the problem of producing a finite-dimensional or 1-dimensional subspace of a countable vector space, and related problems for producing finite-dimensional subspaces of a countable matroid. This extends work in the reverse mathematics setting by Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán (2007) and recent work of Hirst and Mummert (2017). Finally, we study the problem of producing a nonempty subset of a countable graph that is equal to a finite union of connected components and the problem of producing a nonempty subset of a countable graph that is equal to a union of connected components that omits at least one connected component. This extends work of Gura, Hirst, and Mummert (2015). We briefly investigate some of these problems in the reverse mathematics setting.

CHAPTER 1

INTRODUCTION

Weihrauch reducibility and reverse mathematics are two frameworks for classifying the logical strengths of mathematical principles. Weihrauch reducibility, based on computability theory, involves formalizing mathematical principles as mappings from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$, while reverse mathematics involves formalizing principles within second-order arithmetic. Our work deals with the Weihrauch reducibility and reverse mathematics classifications of some mathematical principles related to dependence. More specifically, we investigate the logical strengths in the two settings of principles involving the existence of subspaces and decompositions of vector spaces, graphs, and matroids.

In the following three subsections, we provide brief introductions to computability theory, Weihrauch reducibility, and reverse mathematics. We then give a brief overview of the main problems we address in this thesis.

Computability Theory

In this section we will outline some of the basic concepts of computability theory and some results that are important for the work in this thesis. We state most results without proof. Proofs of all of the results stated here can be found in a standard reference on computability, such as Rogers [12]. Computability Theory is the subfield of mathematical logic that studies the properties of functions that can be computed algorithmically. The first step in this study is to formally define what it means for a function to be "algorithmically computable". Informally, an algorithmically computable function is one whose output on a given input can be determined by a human or a machine using a finite number of steps – where there is a clearly-defined set of rules that determines what action is taken at each step, and each rule and each action is mechanical in nature – using a finite amount of memory or scratch paper, and taking a finite amount of time to complete. During the 1930's, several formal models of computation were introduced which were intended provide a mathematical characterization of those functions that can be algorithmically computed. The most important of these early models were the λ-calculus introduced by Alonzo

Church, the Turing machine introduced by Alan Turing, and the theory of μ-recursive functions introduced by Stephen Cole Kleene. In particular, a Turing machine is a mathematical description of a simple device that can be physically constructed, making it clear that functions computable by Turing machine can be mechanically computed by a machine.

It was proved during the 1930s and 1940s that these three and other models of computation were equivalent, in the sense that a function is computable by one of these models if and only if it is computable by each of the others. The resulting set of functions characterized by these models is referred to as the set of *computable* (or sometimes Turing computable) *functions*. The equivalence of these formal models lends support for the Church–Turing Thesis, which says that any function that is "algorithmically computable" in the informal sense is a computable function, and vice-versa.

As would be expected from the informal definition of an algorithm, in each model of computation the formal algorithm that specifies how a function is computed can be described by a finite string in a finite language. In fact, modern programming languages are "Turing equivalent", meaning that the set of functions they can compute is exactly the set of computable functions. Hence, when thinking of an algorithm for a computable function, one may think of an implementation of that algorithm in a modern programming language, such as C++, Java, or Python. Beware that our concept of algorithm assumes that the input and output of an algorithmically computable function both consist of a finite amount of information. It is for this reason that computable functions are required to have domain and codomain as the set $\mathbb{N}$ of natural numbers. However, we can also represent functions from and to other countable sets as computable functions, as long as appropriate codings for the domain and codomain sets are available.

Because the set of finite strings in a finite language is countable, it follows that there are only countably many programs. This suggests that each possible program could be encoded by a single number, and that there could be another program that takes as input a number and computes the function whose program is specified by the given number. Alan Turing proved the existence of such a function, which is referred to as a *universal function*. Suppose that Φ represents the two-place universal function. Then $\Phi(e, n)$ is equal to evaluating the function f that is computed by the eth program in the list of all possible programs with natural number n as input.

The reader who has some experience with computer programming may notice that there is an issue we have overlooked in our description of programs and the universal function: A given program run with a given input may never produce any output. This situation is often referred to in computer programming as "entering an infinite loop". Hence, we say that some computable functions are *partial.* To say that a computable function $f = \Phi(e, \cdot)$ is partial means that the set of input numbers for which program e produces an output is a proper subset of the natural numbers. We use the notation $f(n)\downarrow$ to indicate that f halts on input n and $f(n)\uparrow$ to indicate that f fails to halt on input n. To say that two computable functions f and g are equal means that, for each n, $f(n)\downarrow$ if and only if $g(n)\downarrow$, and if $f(n)\downarrow$, then $f(n) = g(n)$. If f halts on all inputs, we say that f is *total.*

Computable and C.E. Sets

In addition to the computability of functions, computability theory also studies the computability of sets of natural numbers. We say that a set $S \subseteq \mathbb{N}$ is computable if and only if its characteristic function, i.e., the function χ_S with $\chi_S(n) = 1$ if $n \in S$ and $\chi_S(n) = 0$ otherwise, is computable. There are also sets S which are only half-computable, in the sense that there is a computable function f which will halt on input n if and only if $n \in S$. These sets are called *computably enumerable sets*, and they play an important role in computability theory. We often abbreviate computably enumerable as c.e. The following theorem justifies the name "computably enumerable" for these sets:

Theorem 1.1.1 (See [12], Sec. 5.2, Thm. V)**.** *A set $S \subseteq \mathbb{N}$ is computably enumerable if and only if there is a total computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that $S = \mathrm{range}(f)$.*

We also have the following important characterization of computable sets in terms of c.e. sets:

Theorem 1.1.2 (See [12], Sec. 5.1, Thm. II)**.** *A set $S \subseteq \mathbb{N}$ is computable if and only if both S and $S^c = \mathbb{N} \setminus S$ are c.e.*

Given the definitions of computable and computably enumerable sets, the natural question to ask is whether there exists a set that is of one type but not the other. It is straightforward to show that any computable set is computably enumerable. Here is an example of a set that is computably enumerable but not computable, which was introduced by Alan Turing in 1936 [15]. Let

$$H := \{e : \Phi(e, e) \downarrow\},$$

where Φ is the two-place universal function described above. It is straightforward to show that H is computably enumerable. If we define $h(e) := \Phi(e, e)$, then h halts on input n if and only if $n \in H$. We will show that H is not a computable set.

Theorem 1.1.3. *The set $H = \{e : \Phi(e, e) \downarrow\}$ is c.e., but not computable.*

Proof. Suppose to the contrary that H is computable. Then, $H^c = \mathbb{N} \setminus H$ is c.e., so there is some index e_0 such that $\Phi(e_0, \cdot)$ halts on n if and only if $n \in H^c$. Now consider the result of running $\Phi(e_0, e_0)$. If $\Phi(e_0, e_0) \downarrow$, then this implies that $e_0 \in H$. However, by our choice of the index e_0, this also implies that $e_0 \in H^c$, a contradiction. On the other hand, if $\Phi(e_0, e_0) \uparrow$, then $e_0 \in H^c$, by the definition of H. But, $\Phi(e_0, e_0) \uparrow$ also implies that $e_0 \notin H^c$, by our choice of e_0. Hence, there can be no such e_0, and therefore H^c is not c.e. Hence, H is not computable by Theorem 1.1.2. ■

Relative Computability and Turing Degrees

We can also define the concept of *relative computability.* Suppose that A is any subset of $\mathbb{N}$. (Notice in particular that we do not require that A is computable.) Assume that we are working with the Turing machine formalism, so that a given function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if there is a Turing machine program that computes f. We can extend this formalism by adding another operation to the set of operations that a Turing machine can carry out. This additional operation, in one step of computation, can answer the question "is n in A?". In referring to this extra capability we say that we are given an "oracle" for the set A. We thus get an extended set of possible programs, which can again be indexed by the natural numbers. If a function f can be computed in this extended formalism, then we say that f is *computable in A* or that f is *computable relative to A.* This same extension can be carried out in any of the other formal models of computation.

The set of all possible oracle programs, i.e., programs in the extended formalism, can again be enumerated. Hence, there is also a universal function for this extended formalism, denoted $\Phi^A(\cdot,\cdot)$, where $\Phi^A(e,x)$ indicates the output of oracle program e with input x and oracle set A, if that output exists. Notice that $\Phi^A(e,x)$ depends on the three inputs A, e, and x. If we fix only the program e, then $\Phi^A(e,\cdot)$ can be seen as a function mapping each set $A \subseteq \mathbb{N}$ to a function $\Phi^A(e,\cdot)$ from $\mathbb{N}$ to $\mathbb{N}$. When taking this perspective, we refer to $\Phi^A(e,\cdot)$ as a *Turing functional.*

We say that a set $A \subseteq \mathbb{N}$ is *Turing reducible* to a set $B \subseteq \mathbb{N}$ if there is an index e such that the Turing functional $\Phi^B(e,\cdot)$ with oracle set B computes the characteristic function of A. We denote this relationship by $A \leq_T B$, and we also indicate this relationship by saying that A *is computable from* B. It is straightforward to verify that $\leq_T$ is a pre-order, i.e., that it is reflexive and transitive. By taking equivalence classes of subsets of $\mathbb{N}$, where C is equivalent to D if and only if both $C \leq_T D$ and $D \leq_T C$, we obtain a partial order of equivalence classes. These equivalence classes of sets are referred to as the *Turing degrees.* We say that a Turing degree $\mathcal{A}$ is *reducible* to another Turing degree $\mathcal{B}$ if and only if $A \leq_T B$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The Turing degree of all computable sets is denoted $\emptyset$.

By relativizing the construction of the set H defined above, we define the set

$$A' = \{x : \Phi^A(x,x)\downarrow\}.$$

This set is called the *Turing jump* of the set A. It can be shown that A is computable from A' and that A' is c.e. relative to A, but $A' \not\leq_T A$. We can iterate the Turing jump of A to obtain the n-th jump $A^{(n)}$ of A. The iterated Turing jumps $\emptyset^{(n)}$ play a fundamental role in the relationship between computability theory and formal arithmetic, as we will see below.

The Arithmetical Hierarchy and Post's Theorem

The last concepts we will discuss in this section are the *arithmetical hierarchy* and its relation to computability theory, which is established by Post's Theorem. The arithmetical hierarchy is a system of organizing formulas in first-order arithmetic according to the number of quantifiers and types of quantifiers they contain, modulo logical equivalence with formulas expressed in a simple normal form.

Definition 1.1.1. We define the following classifications of formulas in the arithmetical hierarchy:

1. A formula ϕ which is logically equivalent to a formula with only bounded quantifiers is defined to be both Σ^0_0 and Π^0_0.

2. A formula ϕ which is logically equivalent to a formula of the form $\exists n\psi$, where ψ is Π^0_n, is defined to be Σ^0_{n+1}.

3. A formula ϕ which is logically equivalent to a formula of the form $\forall n\psi$, where ψ is Σ^0_n, is defined to be Π^0_{n+1}.

Post's Theorem shows that there is a close connection between sets which are computable relative to a jump of the degree $\emptyset$ and classifications of sets in the arithmetical hierarchy. Many-one reduction is a form of reduction among subsets of $\mathbb{N}$ that is stronger than Turing reduction.

Theorem 1.1.4 (Post's Theorem; See [12], Sec. 14.5)**.** *The following relationships hold:*

1. *A set B is Σ^0_{n+1} if and only if it is c.e. relative to $\emptyset^{(n)}$.*

2. *Every Σ^0_n set is many-one reducible (and hence Turing reducible) to $\emptyset^{(n)}$.*

Post's Theorem is particularly important in reverse mathematics. As will be described below, reverse mathematics involves classifying theorems by formalizing them within second-order arithmetic. The relationship between computability theory and arithmetic that is established by Post's Theorem allows one to use computability-theoretic methods in reverse mathematics.

Weihrauch Reducibility

Weihrauch reducibility is a framework for comparing theorems based on computability. In this framework, mathematical objects are represented by elements in Baire space, i.e., by functions from $\mathbb{N}$ to $\mathbb{N}$. A theorem is formalized as a mapping from a domain set to a codomain set, referred to as a Weihrauch principle, and these domain and codomain sets are represented as subsets of Baire space. Work in Weihrauch reducibility typically deals with relationships between principles by quantifying over all representations of a given problem. A more detailed description of the methodology of Weihrauch reducibility is given by Brattka and Gherardi [1] and by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [3]. In this work we follow the approach of Hirst and Mummert [8] and do not work directly with representations. We instead identify mathematical objects with elements of $\mathbb{N}^{\mathbb{N}}$. In this way, we assume that each object has been encoded as a function in $\mathbb{N}^{\mathbb{N}}$ and make comparisons between principles involving such functions. This approach is in line with the approach of reverse mathematics, in which we assume that the mathematical objects under consideration have been encoded using natural numbers and sets of natural numbers.

For our purposes, principles in Weihrauch reducibility are given by sets of ordered pairs (A, B), where $A \in \mathbb{N}^{\mathbb{N}}$ is an *instance* of a problem and $B \in \mathbb{N}^{\mathbb{N}}$ a *solution* to the instance A.

Definition 1.2.1. A principle P is said to be *Weihrauch reducible* to a principle Q if there are computable functionals Φ and Ψ such that:

1. for each instance A of P, Φ^A is an instance of Q, and
2. given a solution B to the instance Φ^A of Q, $\Psi^{A,B}$ is a solution the instance A of P.

In this case, we write $P \leq_W Q$. If there exists a functional Ψ satisfying (2) that is independent of the input A, then we say that P is *strongly Weihrauch reducible* to Q, and we write $P \leq_{sW} Q$.

The diagram in Figure 1.1 illustrates the relationships of the various parts of a Weihrauch reduction of a principle P to another principle Q.

$$
\begin{tikzcd}
I_P \arrow[r, "P"] \arrow[d, "\Phi(I_p)"'] \arrow[dr] & S_P \\
I_Q \arrow[r, "Q"'] & S_Q \arrow[u, "\Psi(I_P{,}S_Q)"']
\end{tikzcd}
$$

Figure 1.1: Diagram of the Weihrauch reduction of P to Q.

Intuitively, if P is Weihrauch reducible to Q, then there is a computer program that converts an instance of P into an instance of Q, and another computer program that converts a solution of Q into a solution of P.

If P and Q are Weihrauch principles such that each is (strongly) Weihrauch reducible to the other, then we say that P and Q are (strongly) Weihrauch equivalent.

Definition 1.2.2. Suppose P and Q are Weihrauch principles. If P is Weihrauch reducible to Q and Q is Weihrauch reducible to P, then we say that P and Q are *Weihrauch equivalent*, and we write $P \equiv_W Q$. If P is strongly Weihrauch reducible to Q and Q is strongly Weihrauch reducible to P, then we say that P and Q are *strongly Weihrauch equivalent*, and we write $P \equiv_{sW} Q$.

Weihrauch equivalence and strong Weihrauch equivalence are equivalence relations. The equivalence classes of principles under (strong) Weihrauch equivalence are referred to as the *Weihrauch degrees* (respectively, *strong Weihrauch degrees*). The relation of (strong) Weihrauch reducibility induces a partial order on the (strong) Weihrauch degrees.

Weihrauch reducibility results in a relatively fine-grained classification of the relationships between theorems because it requires the existence of functions Φ and Ψ that provide conversions for *all* instances A of P and *all* solutions to Φ^A. This independence of Φ and Ψ from particular instances and solutions is an instance of the general phenomenon of uniformity, which plays an important role in the study of computability.

In this work we will often speak of a Weihrauch problem as a mapping that takes an element of $\mathbb{N}^\mathbb{N}$ as input and returns an element of $\mathbb{N}^\mathbb{N}$ as output. Notice that in our definition of a Weihrauch principle as a set of ordered pairs, we do not require that this set of ordered pairs defines a function – it may contain pairs (A, B), (A, C) with $B \neq C$. This will not cause any problems because all that is required in our use of Weihrauch principles is that the sets B, C satisfy some particular conditions in relation to A, which depend on the theorem formalized by a particular Weihrauch principle. In the more formal Weihrauch reducibility setting, this issue is resolved using *realizers*, which we will not discuss in depth here.

We state the following definition of the Weihrauch principle LLPO, as given by Brattka and Gherardi [1]:

Definition 1.2.3. We define $\mathrm{LLPO} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ to be the Weihrauch principle such that if $p \in \mathbb{N}^\mathbb{N}$ with $p(n) \neq 0$ for at most one n, then

$$\mathrm{LLPO}(p) \ni \begin{cases} 0 & \text{if } (\forall n \in \mathbb{N}) p(2n) = 0 \\ 1 & \text{if } (\forall n \in \mathbb{N}) p(2n+1) = 0. \end{cases}$$

In particular, notice that $\mathrm{LLPO}(0^{\mathbb{N}}) = \{0,1\}$.

In this definition the expression $\mathrm{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ indicates that LLPO takes as input a subset of all functions in $\mathbb{N}^{\mathbb{N}}$ and that for a given input $p \in \mathbb{N}^{\mathbb{N}}$ there is actually a set of equally valid outputs corresponding to the input p. In more formal treatments of Weihrauch reducibility, a *realizer* of the problem LLPO would be a function $f\colon \mathrm{dom}(\mathrm{LLPO}) \to \mathbb{N}$ such that $f(p) \in \mathrm{LLPO}(p)$ for all $p \in \mathrm{dom}(\mathrm{LLPO})$. In other words, a realizer is a *section* of the relation given by the set of ordered pairs that defines LLPO. In our simplified presentation of Weihrauch reducibility we consider LLPO to be the set of all pairs (p, f) where p is in the domain of LLPO, as defined above, and f is a solution of the input p, also as defined above. However, notice that the definition above says that a realizer of LLPO gives no information about the input $0^{\mathbb{N}}$, since a given realizer may return either 0 or 1 on this input. This fact affects the strength of LLPO in a nontrivial way. In our version of LLPO, as a set of ordered pairs, we have that $(0^{\mathbb{N}}, 0)$ and $(0^{\mathbb{N}}, 1)$ are both in LLPO.

We define several more Weihrauch principles which serve as important landmarks for classifying theorems in the Weihrauch reducibility setting. The principles $\mathrm{C}_{\mathbb{N}}$ and WKL_W will be particularly important in the work presented in this thesis.

Definition 1.2.4. Define $\mathrm{C}_{\mathbb{N}}$ to be the Weihrauch principle that takes as input a nonsurjective function $f\colon \mathbb{N} \to \mathbb{N}$ and returns an element $n \in \mathbb{N} \setminus \mathrm{range}(f)$.

Definition 1.2.5. Define WKL_W to be the principle that takes as input a function $f \in \mathbb{N}^{\mathbb{N}}$ that encodes an infinite binary tree T (a subset $\{0,1\}^{<\mathbb{N}}$ such that (1) if $\sigma \in T$ then each initial segment of σ is also in T, and (2) there is a function $f \in \{0,1\}^{\mathbb{N}}$ such that each finite restriction of f is in T) and returns a function function $f \in \{0,1\}^{\mathbb{N}}$ such that any finite restriction of f is in T.

Definition 1.2.6. Define C to be the Weihrauch principle that takes as input a function $f\colon \mathbb{N} \to \mathbb{N}$ and returns the characteristic function $\chi_{\mathrm{range}(f)}$.

Definition 1.2.7. Define LPO to be the Weihrauch principle that takes as input a function $f\colon \mathbb{N} \to \mathbb{N}$ and returns the characteristic function of a set $G \subseteq \{0,1\}$, such that $G = \{0\}$ if $0 \in \mathrm{range}(f)$ and $G = \{1\}$ otherwise.

The following theorem is proved by Brattka and Gherardi [1].

Theorem 1.2.1. WKL_W *is Weihrauch equivalent to* $\widehat{\mathrm{LLPO}}$.

Next, we define the parallelization of a Weihrauch principle.

Definition 1.2.8. Given a Weihrauch principle P, the *parallelization of* P, denoted $\widehat{P}$ is the set of all sequences $\{(f_n, g_n)\}_{n\in\mathbb{N}}$ of ordered pairs, where f_n is an instance of P and g_n is a solution of f_n, for each n.

Brattka and Gherardi [1] mention the following properties of the parallelization operation.

Theorem 1.2.2. *Let f and g be multi-valued functions on represented spaces. Then*

1. $f \leq_W \widehat{f}$,
2. $f \leq_W g \Longrightarrow \widehat{f} \leq_W \widehat{g}$,
3. *and* $\widehat{f} \equiv_W \widehat{\widehat{f}}$.

Analogous results hold for strong Weihrauch reducibility.

We obtain the following corollary to Theorems 1.2.1 and 1.2.2.

Corollary 1.2.3. WKL_W *is Weihrauch equivalent to* $\widehat{\mathrm{WKL}_W}$.

Proof. We have $\mathrm{WKL}_W \equiv_W \widehat{\mathrm{LLPO}} \equiv_W \widehat{\widehat{\mathrm{LLPO}}} \equiv_W \widehat{\mathrm{WKL}_W}$. ■

On the other hand, we have:

Theorem 1.2.4. $\widehat{\mathrm{C}_\mathbb{N}}$, C, *and* $\widehat{\mathrm{LPO}}$ *are strongly Weihrauch equivalent.*

Proof. First we show that $\mathrm{C} \leq_{sW} \widehat{\mathrm{LPO}}$. Suppose we are given a function $f\colon \mathbb{N} \to \mathbb{N}$. For each n define $g_n(m) = 0$ if $f(m) = n$, and $g_n(m) = 1$ otherwise. Then $0 \in \mathrm{range}(g_n)$ if and only if $n \in \mathrm{range}(f)$, and (g_n) is an instance of $\widehat{\mathrm{LPO}}$. Hence, we can apply $\widehat{\mathrm{LPO}}$ to obtain a solution h to (g_n), and from h we can compute $\chi_{\mathrm{range}(f)}$, thus solving the instance f of C.

Now we show that $\widehat{\mathrm{C}_\mathbb{N}} \leq_{sW} \mathrm{C}$. Suppose we are given an instance (f_n) of $\widehat{\mathrm{C}_\mathbb{N}}$, i.e., that each f_n is a nonsurjective function from $\mathbb{N}$ to $\mathbb{N}$. Then, we can use a standard computable bijective encoding of pairs to enumerate the set S of pairs of the form $(n, f_n(m))$, where n and m range over all natural numbers. Then, applying C gives us the characteristic function χ_S. From χ_S, for each n we can compute the least k such that $(n,k) \notin S$, thus solving the instance (f_n) of $\mathrm{C}_{\mathbb{N}}$.

Finally, we show that $\widehat{\mathrm{LPO}} \leq_{sW} \widehat{\mathrm{C}_{\mathbb{N}}}$. Suppose we are given an instance (f_n) of $\widehat{\mathrm{LPO}}$. From (f_n) we can uniformly compute a sequence of functions (g_n) such that $g_n(m) = 0$ if $f_n(m) = 0$ and $g_n(m) = m + 1$ otherwise. Then, $\mathbb{N} \setminus \mathrm{range}(g_n) = \{0\}$ if $0 \notin \mathrm{range}(f_n)$ and $0 \notin \mathbb{N} \setminus \mathrm{range}(g_n)$ if $0 \in \mathrm{range}(f_n)$. Hence, we can apply $\widehat{\mathrm{C}_{\mathbb{N}}}$ to (g_n) to obtain a solution h, and from h we can compute a solution to the instance (f_n) of $\widehat{\mathrm{LPO}}$. ■

The following result is mentioned in Brattka and Gherardi [1]:

Theorem 1.2.5. *The following relationships hold*

$$\mathrm{LLPO} <_W \mathrm{LPO} \,|_W\, \widehat{\mathrm{LLPO}} <_W \widehat{\mathrm{LPO}},$$

where $|_W$ *denotes incomparability in the Weihrauch sense.*

By combining the results of Theorems 1.2.1, 1.2.2, 1.2.4, and 1.2.5, we obtain the following result.

Theorem 1.2.6. $\mathrm{C}_{\mathbb{N}}$ *is not Weihrauch reducible to* WKL_W*.*

Proof. Suppose that $\mathrm{C}_{\mathbb{N}} \leq_W \mathrm{WKL}_W$ held. Then by Theorems 1.2.1, 1.2.2, and 1.2.4 we would have

$$\widehat{\mathrm{LPO}} \equiv_{sW} \widehat{\mathrm{C}_{\mathbb{N}}} \leq_W \widehat{\mathrm{WKL}_W} \equiv \widehat{\mathrm{LLPO}},$$

contradicting the result of Theorem 1.2.5. ■

In this section, we have presented several principles that serve as important landmarks in Weihrauch reducibility, namely the princples $\mathrm{C}_{\mathbb{N}}$, LPO, $\widehat{\mathrm{LPO}}$, and WKL_W. In establishing relationships between these principles we have demonstrated some basic techniques for proving relationships in the Weihrauch reducibility setting. The result of Theorem 1.2.6 will be particularly relevant to our work on classifying the Weihrauch proper subspace principle.

Reverse Mathematics

In reverse mathematics, we formalize mathematical theorems in the language of second-order arithmetic, which is a theory in two-sorted first-order logic, with one sort of variables intended to represent natural numbers and the other sort intended to represent sets of natural numbers. This two-sorted logic can be interpreted within the usual, one-sorted version of first-order logic, and we have at our disposal all of the fundamental theorems concerning first-order logic, including Gödel's Completeness Theorem, which ensures that a proposition which is true in all models of a theory is syntactically provable. When working in reverse mathematics, we work with subsystems of second-order arithmetic, which are subsets of the full set of axioms of second-order arithmetic along with weakened versions of some axioms of second-order arithmetic and possibly some additional axioms, and principles, which are represented by additional axioms in the language of second-order arithmetic. A standard and comprehensive reference for reverse mathematics is Stephen Simpson's text *Subsystems of Second-Order Arithmetic* [13].

Because the only objects in the language of second-order arithmetic are natural numbers and sets of natural numbers, other mathematical objects that appear in theorems studied in reverse mathematics must be coded either as natural numbers or as sets of natural numbers. As a corollary to this fact, in reverse mathematics we may only deal with objects that are countable or that can be represented by sets of countable objects. For example, any complete, separable metric space can be represented by a set of countable objects, since each point in such a space can be represented by a member of an equivalence class of Cauchy sequences with terms in a countable, dense subset. In contrast, an uncountable, discrete topological space cannot be represented in second-order arithemtic.

In order to ensure the availability of enough logical tools to complete mathematical proofs, in reverse mathematics we work over a base system of relatively weak axioms. By this we mean that there is a particular subsystem – called the *base system* – that is assumed along with the other principles and subsystems under consideration, so that implications among subsystems and principles are obtained relative to this base system. The most common base system for reverse mathematics is RCA_0, which consists of basic arithmetical axioms plus weakened induction and comprehension axioms. Most results in reverse mathematics have the form "A implies C over B", where A, B, and C are subsystems of second-order arithmetic. The meaning of this statement is that every model of both A and B is also a model of C. Here the B is the base system. For example, one result of reverse mathematics says that, assuming RCA_0, the principle that "every vector space has a basis" is equivalent to the subsystem ACA_0. It is important that the base system has the right amount of logical strength. If the base system implies too many nontrivial results, then the distinctions that can be made between theorems over this base system may be too coarse. On the other hand, if the base system is too weak, then proofs may become very laborious or impossible to carry out, or the distinctions made over that system may be too fine.

Many results in reverse mathematics compare mathematical theorems or principles to a set of subsystems known as the "Big Five" subsystems. These are, in increasing order of strength, RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. It has been discovered that a significant number of the fundamental results of mathematics are equivalent to one of these five principles. There are, however, some interesting examples in the reverse mathematics literature of principles that are not equivalent to any of the "Big Five" subsystems. See, for example, the classification of Ramsey's Theorem for pairs by Cholak, Jockusch, and Slaman [11]. In this work, all reverse mathematical results involve comparisons to WKL_0 or ACA_0, and we always work over the base system RCA_0.

Principles Related to Dependence

Most of the principles we study involve the existence of subspaces or decompositions into subspaces, where the meaning of 'subspace' depends on which mathematical dependence structure the principle refers to. The structures we work with are matroids, graphs, and vector spaces. Throughout this work we assume that all graphs are simple, meaning that in any graph discussed here there is at most one edge between any two distinct vertices and there are no self-loops. Matroids axiomatize a form of dependence between objects that generalizes forms of dependence that arise in several settings in mathematics, including linear dependence within vector spaces and connectedness within graphs. With respect to a matroid, a graph, or a vector space one can define a *subspace* to be a set that is saturated under the corresponding dependence relation. Within the graph, vector space, and matroid structures one can also define the concept of a basis, and then one can define the dimension of a subspace to be the cardinality of any basis.

Our notion of dependence within a graph is based on connectedness, and as a result a subspace of a graph in this setting is a union of connected components. There is a large body of work on graphical matroids, in which a set of vertices is considered to be dependent if it contains a cycle. The reader may refer to the text by Oxley [10] for an introduction to the study of graphical matroids. We follow Gura, Hirst, and Mummert [6] in studying dependence within graphs in terms of connectedness. This study is motivated by the fact that problems pertaining to connected components in graphs have interesting logical properties, particularly in relation to computability.

The principles studied in this work are related to those studied by Hirst and Mummert [8] and Gura, Hirst, and Mummert [6]. We are interested in the following general principles, where M represents a graph, vector space, or matroid:

1. *Decomposition into subspaces*: Given an object M equipped with a notion of dependence, there is a decomposition of M into 1-dimensional subspaces.

2. *Existence of subspaces*: Given an object M equipped with a notion of dependence and with dimension greater than 1, there exists a nontrivial proper subspace S of M.

3. *Existence of finite-dimensional subspaces*: Given an object M equipped with a notion of dependence and with dimension greater than 1, there exists a finite-dimensional nontrivial proper subspace S of M.

4. *Existence of 1-dimensional subspaces*: Given an object M equipped with a notion of dependence and with dimension greater than 1, there exists a 1-dimensional subspace S of M.

As can be readily seen, a given principle of type (4) will imply the analogous principles of type (2) and type (3), since within a matroid, a vector space, or a graph with dimension greater than 1, a 1-dimensional subspace will always be nontrivial and proper. We are interested in whether the additional specificity in (3) and (4) makes a principle of type (3) or (4) logically stronger than the analogous principle of type (2) or (3).

The diagrams in Figures 1.2 and 1.3 show the known relationships between the principles studied in this work, which include specific instances of the general principles (1)-(4). In these diagrams, an arrow from a principle A to a principle B indicates that B is reducible to A in the Weihrauch or reverse mathematics sense, respectively.

$$
\begin{tikzcd}
 & \mathrm{PSG}_W^{<\mathbb{N}} \arrow[r, leftrightarrow] & \widehat{\mathrm{LPO}} \arrow[r] \arrow[dll, leftrightarrow] \arrow[dl] \arrow[d] \arrow[dr, leftrightarrow] \arrow[drr, leftrightarrow] & \mathrm{LPO} & \\
\mathrm{P}_W & \mathrm{L}_W \arrow[d] & \mathrm{D}_{\mathrm{V}} \arrow[l] & \mathrm{DEP}_W & \mathrm{D}_{\mathrm{M}} \arrow[d, leftrightarrow] \\
 & \mathrm{PS}_W \arrow[r] & \mathrm{SEP} \arrow[r, leftrightarrow] & \mathrm{WKL}_W & \mathrm{WD}_{\mathrm{M}} \\
 & \mathrm{WKL}_W \star \mathrm{C}_{\mathbb{N}} \arrow[u] \arrow[r] & \mathrm{PSG}_W \arrow[u] \arrow[r] & \mathrm{C}_{\mathbb{N}} &
\end{tikzcd}
$$

Figure 1.2: Weihrauch reducibility relationships between principles studied here.

$$
\begin{tikzcd}
\mathrm{D}^{\mathrm{V}} \arrow[r, leftrightarrow] & \mathrm{ACA}_0 \arrow[r] \arrow[dl] \arrow[d, leftrightarrow] & \mathrm{WKL}_0 \arrow[r] \arrow[d, leftrightarrow] & \mathrm{RCA}_0 \\
\mathrm{D}^{\mathrm{M}} \arrow[d, leftrightarrow] & \mathrm{L} \arrow[d, leftrightarrow] & \mathrm{PS} & \\
\mathrm{WD}^{\mathrm{M}} & \mathrm{PS}_W^{<\mathbb{N}} & &
\end{tikzcd}
$$

Figure 1.3: Reverse mathematics relationships between principles studied here.

CHAPTER 2

DECOMPOSITION INTO SUBSPACES

In this section we study strength of the problem of decomposing a matroid into a collection of 1-dimensional subspaces from the Weihrauch reducibility and reverse mathematics perspectives.

Matroid Decomposition

Matroids can be axiomatically defined in several apparently different but equivalent ways. We use the following definition, which is used by Hirst and Mummert [8] as the basis of their definition of an e-matroid. In this definition and in the rest of this thesis, given a set S, $[S]^{<\mathbb{N}}$ denotes the collection of all finite subsets of S.

Definition 2.1.1. A matroid is a pair (M, D), where M is a set called the *ground set* and $D \subseteq [M]^{<\mathbb{N}}$ is called the *set of finite dependent subsets of* M, which satisfies the following axioms.

1. The empty set is not dependent: $\emptyset \notin D$.
2. Finite supersets of dependent sets are dependent: If A is dependent and $B \supseteq A$, then B is also dependent.
3. Independent sets have the *exchange property*: Suppose that A and B are finite independent subsets of M and $|A| > |B|$. Then there is an element $x \in A \setminus B$ such that $B \cup \{x\}$ is also independent.

For infinite sets $A \subseteq M$ we say that A is dependent if it contains a finite dependent subset, and that A is independent otherwise.

We define the notions of subspace, span, basis, and dimension for matroids, as well as the sets of zero and nonzero elements of a matroid.

Definition 2.1.2. If (M, D) is a matroid, we define

$$Z(M, D) := \{x \in M : \{x\} \in D\}$$

and refer to $Z(M, D)$ as the *set of zero elements of* (M, D). We define $N(M, D) := M \setminus Z(M, D)$, and refer to $N(M, D)$ as the *set of nonzero elements of* (M, D).

Whenever the matroid (M, D) that is being considered is clear from context, we will refer to $Z(M, D)$ using Z and $N(M, D)$ using N.

Definition 2.1.3. If (M, D) is a matroid, we say that a nonempty set $S \subseteq M$ is a *subspace* of (M, D) if, whenever $m \in M$ and $N \cup \{m\} \in D$ for a finite subset $N \subseteq S$ with $N \notin D$, then $m \in S$. In this case we say that S is *saturated* under the dependence relation D.

Definition 2.1.4. Suppose (M, D) is a matroid, $S \subseteq M$, and $T \subseteq S$. We say that T *spans* S if for each $s \in S$ there is a finite subset $N \subseteq T$ such that $N \cup \{s\} \in D$.

Definition 2.1.5. Suppose that (M, D) is a matroid and $S \subseteq M$ is a subspace of (M, D). Then $T \subseteq S$ is a *basis* for S if T is independent and T spans S.

Definition 2.1.6. If (M, D) is a matroid and $S \subseteq M$ is a subspace, then the *dimension* of S is the cardinality of any basis for S.

It is straightforward to show that if (M, D) is a matroid and $S \subseteq M$ is a subspace, then the pair (S, E), where E is the collection of sets in D that are subsets of S, is again a matroid. It is also straightforward to show using the exchange property that all bases for S have the same cardinality.

The following basic theorem shows that for any matroid (M, D) a decomposition of (M, D) into 1-dimensional subspaces always exists.

Theorem 2.1.1. *Every matroid M can be decomposed into a family of 1-dimensional subspaces such that the intersection of any two of these subspaces is exactly the set Z of zero elements of (M, D). If M is the matroid obtained from the linear dependence relation of a vector space V, then the decomposition obtained is exactly the set of lines through the origin in V.*

Proof. Let (M, D) be a matroid. For each $x \in N$, define

$$U_x := \{x\} \cup \{y \in M : \{x, y\} \text{ is dependent}\}.$$

We claim that for any distinct $y, z \in U_x$ the set $\{y, z\}$ is dependent. Suppose to the contrary that $\{y, z\}$ is independent. From the definition of U_x it follows that $x \neq y$ and $x \neq z$. Hence, $\{y, z\}$ and $\{x\}$ are independent sets with $|\{x\}| < |\{y, z\}|$, so by the matroid exchange axiom one of $\{x, y\}$ or $\{x, z\}$ is independent. This contradicts our definition of U_x, so $\{y, z\}$ must be dependent.

Observe that, because $\{x, z\}$ is dependent for each $z \in Z$, it follows that $Z \subseteq U_x$ for each $x \in N$. Now suppose that $U_x \neq U_y.$, and further suppose that $\{a, b\}$ is dependent for all $a \in U_x$ and $b \in U_y$; then it follows from the definitions of U_x and U_y that $U_x = U_y$, contradicting our assumption. Hence, there must be an $a_0 \in U_x$ and a $b_0 \in U_y$ such that $\{a_0, b_0\}$ is independent. We now will show that if $z \in U_x \cap U_y$, then $z \in Z$. Suppose that this is not the case, so that there is an element $z_0 \in U_x \cap U_y$ with $z_0 \in N$. Now consider the independent sets $\{a_0, b_0\}$ and $\{z_0\}$. We have $|\{a_0, b_0\}| > |\{z_0\}|$, but $\{z_0\} \cup \{t\}$ is dependent for all $t \in \{a_0, b_0\}$, contradicting the matroid exchange axiom. Hence, $z_0 \in Z$. Therefore, $U_x \cap U_y \subseteq Z$; by our remark above we thus have $U_x \cap U_y = Z$.

Hence, we have shown that $M = \bigcup_{x \in N} U_x$, and $U_x \cap U_y = Z$ for $U_x \neq U_y$. By definition, each U_x is 1-dimensional. Hence, $\{U_x\}_{x \in N}$ gives a decomposition of (M, D) into 1-dimensional subspaces. It is straightforward to verify that if (M, D) encodes the relation of linear dependence in a vector space V, then each U_x is exactly the line through the origin in V in the direction of x. ■

The following corollary will be useful when proving results about matroids.

Corollary 2.1.2. *If (M, D) is a matroid, $m, n, k \in N$, and $\{m, k\}$ and $\{n, k\}$ are both dependent, then $\{m, n\}$ is also dependent.*

Proof. Let $\{U_x : x \in M\}$ be a decomposition of (M, D), as in the proof of Theorem 2.1.1. Then, we have $U_m \cap U_k \neq Z$ and $U_n \cap U_k \neq Z$, so $U_m = U_n = U_k$. Hence, $\{m, n\}$ is dependent, by the definition of U_m. ■

Our goal is to formalize the result of Theorem 2.1.1 and to study its relative strength in the contexts of Weihrauch reducibility and reverse mathematics. We formalize these results using a more general structure, called an *e-matroid.* The dependence structure axiomatized by an e-matroid generalizes the dependence structures that arise in computable vector spaces and graphs, since in each of these cases the collection of finite dependent subsets can be enumerated, but may not be computable, from the vector space operations or the set of graph edges.

Definition 2.1.7. An *e-matroid* is a pair (M, D), where $M \subseteq \mathbb{N}$ and D is a function from $\mathbb{N}$ to $[\mathbb{N}]^{<\mathbb{N}}$, such that the pair $(M, \mathrm{range}(D))$ is a matroid.

It is relatively straightforward to show that, from a countable vector space or a graph with an enumerated edge set, we can uniformly compute an e-matroid (M, D) that encodes the dependence structure of that vector space or graph. We include several lemmas that formalize this statement. To say that we are given a graph $G = (V, E)$ with an enumerated edge set means that we are given the vertex set V and a function $f : \mathbb{N} \to E$ that enumerates the set E of edges of G.

If the matroid (M, D) is computed from a vector space or graph in the way we describe, then there is a one-to-one correspondence between subspaces of (M, D) and subspaces of the original vector space or graph, and a decomposition of (M, D) into 1-dimensional subspaces will give a decomposition of the initial vector space or graph into 1-dimensional subspaces. Hence, in obtaining classification results about decompositions of e-matroids we obtain as corollaries upper bounds for the strenghts of related principles involving countable vector spaces and graphs with enumerated edge sets.

The following lemma shows that we can computably turn a graph with an enumerated edge set into an e-matroid. We require that the graph (V, E) in this lemma has at least one edge, else there would be no dependent sets to enumerate.

Definition 2.1.8. Suppose that $G = (V, E)$ is a graph. Then, we define a finite set $A \subseteq V$ to be *dependent* if A contains two distinct vertices that lie in the same connected component of G.

Lemma 2.1.3. *Given a simple graph $G = (V, E)$, where $V \subseteq \mathbb{N}$ and E is nonempty and given as an enumeration, there exists an enumeration of the finite dependent subsets of G which is uniformly computable from G.*

Proof. Let $f : \mathbb{N} \to E$ be an enumeration of E, and let g be an enumeration of $[\mathbb{N}]^{<\mathbb{N}}$. Suppose that $\{m_0, n_0\} = f(0)$. A pair of vertices (s, t) is in the same connected component iff there is a finite path connecting s to t consisting of vertices in E. Hence, we can enumerate such pairs by the following procedure: Suppose $g(n)$ codes the set $\{x_0, \ldots, x_k\}$. Check if the set of edges $\{f(x_0), \ldots, f(x_k)\}$ is exactly the set of edges in some (non-closed) finite path. If so, then let $h(n)$ equal the code for $\{s, t\}$, where s is initial vertex and t is the final vertex in this path. If not, then let $h(n)$ equal the code for $\{m_0, n_0\}$. It is straightforward to show that h is computable.

Now let φ be a computable bijective pairing function. We can use the following procedure to enumerate all of the finite dependent sets of vertices in V. Suppose $\varphi(n) = (s, t)$. Let A_s be the intersection of V with the set coded by $h(s)$. Let $\psi(n)$ be the (code of the) union A_s and the set coded by $g(t)$. It is straightforward to show that each set in this enumeration contains a pair of connected vertices, and if a finite set contains a pair of connected vertices it is included in this enumeration. ■

The collection of dependent sets in the above lemma satisfies the matroid axioms. Further, the subspaces of the matroid obtained from a graph G using Lemma 2.1.3 are exactly the unions of connected components of G. Hence, we have the following lemma.

Lemma 2.1.4. *If $G = (V, E)$ is a graph, where $V \subseteq \mathbb{N}$ and E is nonempty and given as an enumeration, then from G we can uniformly compute an e-matroid (M, D) such that $M = V$ and, for $A \in [M]^{<\mathbb{N}}$, $A \in D$ if and only if A contains two distinct path-connected vertices of G. Further, a subset $S \subseteq V$ is a subspace of (M, D) if and only if it is equal to a union of connected components in G.*

Proof. It is straightforward to verify that, if ψ is the enumeration of dependent sets of the graph $G = (V, E)$ given in the proof of Lemma 2.1.3, then (V, ψ) is an e-matroid. We verify the third matroid axiom here. Assume that A, B are subsets of V with $|A| > |B|$ and that neither A nor B contains a pair of vertices connected by a path in G. Suppose that each $a \in A$ is connected by a path to some element of B. Then by the Pigeonhole Principle there must be some vertex b_0 in B that is connected by a path to two distinct vertices in A. This contradicts the fact that A is independent, and hence there must be an $a_0 \in A$ such that $B \cup \{a_0\}$ is dependent.

To verify the second claim we only need to apply the definitions of connected component and matroid subspace. Observe that $S \subseteq V$ is a subspace of (M, D) if and only if whenever $v \in V$ and $N \cup \{v\}$ is dependent for some finite $N \subseteq S$ then $v \in S$. This says exactly that whenever v is connected by a path in G to a vertex in $N \subseteq S$ then v is already in S, which is equivalent to saying that S is union of connected components. ■

Weihrauch Perspective

The following Weihrauch principle formalizes the notion of a decomposition of an e-matroid into 1-dimensional subspaces. The definition of an e-matroid decomposition given here generalizes the definition of a decomposition of a graph into connected components given by Hirst [7].

Definition 2.2.1. We define $\mathrm{D_M}$ to be the principle that, given an e-matroid (M, D), produces a *decomposition of* (M, D) *into* 1-*dimensional subspaces*, which is a function $f\colon M \to \mathbb{N}$ such that

1. if $n \in Z$ then $f(n) = 0$,
2. if $n \in M \setminus Z$, then $f(n) > 0$,
3. if $n, m \in M \setminus Z$ and $m \neq n$, then $f(n) = f(m)$ if and only if $\{n, m\}$ is dependent.

We define $\mathrm{WD_M}$ to be the principle that, given an e-matroid (M, D), produces a *weak decomposition of* (M, D) *into* 1-*dimensional subspaces*, which is a function $f\colon M \to \mathbb{N}$ that satisfies condition (3).

We now show that $\mathrm{WD_M} \equiv_{sW} \mathrm{D_M} \equiv_{sW} \widehat{\mathrm{LPO}}$. To do so, we first show that $\widehat{\mathrm{LPO}}$ is equivalent to the Weihrauch principle DEP_W, which takes as input an e-matroid and returns as output the characteristic function for its set of finite dependent sets. This is the Weihrauch equivalent of Theorem 7 of Hirst and Mummert [8].

Definition 2.2.2. We define DEP_W to be the Weihrauch principle given by the set of pairs $((M, D), \chi_{\mathrm{range}(D)})$, were (M, D) is an e-matroid, and $\chi_{\mathrm{range}(D)}$ is the characteristic function for range(D).

Theorem 2.2.1. DEP_W *is strongly Weihrauch equivalent to* $\widehat{\mathrm{LPO}}$.

Proof. ($\widehat{\mathrm{LPO}} \leq_{sW} \mathrm{DEP}_W$.) Suppose the sequence of functions $g_n\colon \mathbb{N} \to \mathbb{N}$ constitutes an instance of $\widehat{\mathrm{LPO}}$. Now let E be the collection of 2-element sets of ordered pairs of the form $\{(n, k), (n, k+1)\}$ for each $n, k \in \mathbb{N}$ and $\{(0, 0), (j+1, k)\}$ for each j, k such that $g_j(k) = 0$. It is straightforward to show that E is computable from the sequence (g_n). Let $V = \mathbb{N} \times \mathbb{N}$, and observe that the pair (V, E) is a computable graph.

Now apply Lemma 2.1.4 to obtain an e-matroid (M, D) with $M = V$ and D corresponding to finite subsets of V containing pairs of vertices that are connected in (V, E). Apply DEP_W to (M, D) to obtain the characteristic function $\chi_{\mathrm{range}(D)}$. Observe that, in the graph (V, E), $(0, 0)$ and $(n + 1, 0)$ are in the same connected component if and only if $(\exists k)[g_n(k) = 0]$. Hence, we can compute a solution ψ to the instance (g_n) of $\widehat{\mathrm{LPO}}$ from $\chi_{\mathrm{range}(D)}$ as follows. To compute $\psi(n)$, first compute $\chi_{\mathrm{range}(D)}((0, 0), (n + 1, 0))$, then subtract the result from one.

($\mathrm{DEP}_W \leq_{sW} \widehat{\mathrm{LPO}}$.) Let (M, D) be an e-matroid. Let g be an enumeration of $[\mathbb{N}]^{<\mathbb{N}}$. For each $m \in \mathbb{N}$ define the function $f_m \colon \mathbb{N} \to \{0, 1\}$ by

$$f_m(n) := \begin{cases} 0, & \text{if } D(n) = g(m), \\ 1, & \text{otherwise.} \end{cases}$$

Observe that (f_m) constitutes an instance of $\widehat{\mathrm{LPO}}$ and that $0 \in \operatorname{range} f_m$ if and only if the finite set $g(m)$ is a dependent subset of M. Hence, if h is a solution to the instance (f_m) of $\widehat{\mathrm{LPO}}$, then $\chi_{\mathrm{range}(D)} := 1 - h$. ■

We now apply Theorem 2.2.1 to show that $\mathrm{WD_M} \equiv_{sW} \mathrm{D_M} \equiv_{sW} \widehat{\mathrm{LPO}}$. Notice that in the proof that $\widehat{\mathrm{LPO}} \leq_{sW} \mathrm{WD_M}$ we make use of the fact that a graph has no "zero" elements according to our definition of dependence. This fact allows us to obtain a connected component from a weak decomposition.

Theorem 2.2.2. *The principles* WD_M, D_M, *and* $\widehat{\mathrm{LPO}}$ *are strongly Weihrauch equivalent.*

Proof. ($\mathrm{D_M} \leq_{sW} \widehat{\mathrm{LPO}}$) Suppose we are given an e-matroid (M, D) with $M = \mathbb{N}$. Apply $\mathrm{DEP}_W \equiv_{sW} \widehat{\mathrm{LPO}}$ to obtain the characteristic function $\chi_{\mathrm{range}(D)}$. Now, define a function $f \colon M \to \mathbb{N}$ by

- $f(m) = 0$ if $m \in Z$;
- if $m \in M \setminus Z$ and $\{m, n\}$ is independent for all $n < m$ with $n \in M \setminus Z$, then $f(m) = m + 1$;

- if $m \in M \setminus Z$ and there exists a least $n_0 < m$ such that $n_0 \in M \setminus Z$ and $\{m, n_0\}$ is dependent, then $f(m) = n_0 + 1$.

Observe that f is computable from $\chi_{\mathrm{range}(D)}$. It is straightforward to verify that f satisfies conditions (1) and (2) in Definition 2.2.1.

We show that f also satisfies (3). Suppose that $m, n \in M \setminus Z$ with $m < n$, and $f(n) = f(m) = k + 1 \leq m + 1$. If $k = m$, then by the definition of f it must be the case that $\{m, n\}$ is dependent. If $k < m$, then by the definition of f both $\{m, k\}$ and $\{n, k\}$ are dependent, and so by an application of Corollary 2.1.2 to the e-matroid $(M, \mathrm{range}(D))$ we see that $\{m, n\}$ is dependent in this case as well.

Now suppose that $m, n \notin Z$ with $m < n$, and that $\{m, n\}$ is dependent. It follows from Corollary 2.1.2 that, for $k \notin \{m, n\}$ and $k \notin Z$, $\{n, k\}$ is dependent iff $\{m, k\}$ is dependent. From the definition of f we see that if $f(m) = m + 1$, then m is the least nonzero $t < n$ with $\{t, n\}$ dependent, so $f(n) = m + 1 = f(m)$. On the other hand, if $f(m) = k + 1 < m + 1$, then k is the least nonzero $t < m < n$ such that $\{m, t\}$ is dependent. Then, $\{n, t\}$ will also be dependent, so $f(n) = k + 1$ also. Hence, f satisfies condition (3), so f is a decomposition of (M, D) into 1-dimensional subspaces.

If $M \neq \mathbb{N}$, then we can utilize the above construction by first encoding the membership relation for M into the dependence relation D. One way of doing this is to encode the elements of M using the odd natural numbers, and then to use the even natural numbers to encode another matroid structure in which $\{0, 2k\}$ is dependent if and only if $k \in M$. This will result in a new e-matroid (M', D'). Then, it is straightforward to compute the functions χ_M and $\chi_{\mathrm{range}(D)}$ from $\chi_{\mathrm{range}(D')}$.

($\mathrm{WD_M} \leq_{sW} \mathrm{D_M}$) This follows directly from the definitions of $\mathrm{D_M}$ and $\mathrm{WD_M}$.

($\widehat{\mathrm{LPO}} \leq_{sW} \mathrm{WD_M}$) By Theorem 6.4 of Gura, Hirst, and Mummert [6], the principle D_G saying that a countable graph can be decomposed into its connected components is strongly Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. If we are given a countable graph $G = (V, E)$, as in Lemma 2.1.4 we can compute an e-matroid (M, D) from G, in which the finite dependent sets are exactly the finite subsets $S \subseteq V$ such that there are distinct elements $u, v \in S$ that are connected by a path in G. Hence, a decomposition of (M, D) into 1-dimensional subspaces is exactly a decomposition of G into connected components. Moreover, the weak decomposition of Definition 2.2.1 specializes to the decomposition defined by Gura, Hirst, and Mummert [6] whenever (M, D) is obtained from the path-connectedness relation on a graph. Hence, D_G is strongly Weihrauch reducible to WD_{M}, so $\widehat{\mathrm{LPO}}$ is also strongly Weihrauch reducible to WD_{M}. ■

Reverse Mathematics Perspective

In this section we consider e-matroid decompositions from the reverse mathematics perspective. We formalize an e-matroid in second-order arithmetic as a pair (M, D), where M is a subset of $\mathbb{N}$ and D is a function from $\mathbb{N}$ to $[M]^{<\mathbb{N}}$ whose range is the collection of finite dependent subsets of M.

Definition 2.3.1. We define D^{M} to be the second-order arithmetic formalization of the principle "every e-matroid has a decomposition into 1-dimensional subspaces". Formally, we define a 1-dimensional decomposition of an e-matroid (M, D) to be a function $f \colon M \to \mathbb{N}$ such that

1. $n \in Z \Leftrightarrow f(n) = 0$,
2. and $n, m \notin Z \wedge m \neq n \Rightarrow (f(n) = f(m) \Leftrightarrow (\exists k) D(k) = \{n, m\})$,

where $Z = \{m \in M : (\exists n)[D(n) = \{m\}]\}$.

We define WD^{M} to be the formalization of the principle "every e-matroid has a weak decomposition into 1-dimensional subspaces", where a weak decomposition of an e-matroid (M, D) into 1-dimensional subspaces is a function $f \colon M \to \mathbb{N}$ that satisfies condition (2).

The following theorem gives the reverse mathematics classifications of WD^{M} and D^{M}.

Theorem 2.3.1. *The subsystems* WD^{M} *and* D^{M} *are equivalent to* ACA_0 *over* RCA_0*.*

Proof. It is straightforward to show that ACA_0 implies D^{M}, and it follows from the definitions that D^{M} implies WD^{M}. We show that WD^{M} implies the principle that every graph has a connected component. If we are given a graph $G = (V, E)$, then application of an instance of Δ^0_1-comprehension with parameter G shows that there exists an e-matroid (M, D), where $M = V$ and D is an enumeration of the set of all finite subsets of G containing a pair of distinct, path-connected vertices. Now apply WD^{M} to obtain a weak decomposition f of (M, D) into

1-dimensional subspaces. Since (M, D) has no zero elements, f is a decomposition of (M, D) into 1-dimensional subspaces. Hence, a 1-dimensional subspace of V is given by the set

$$S := \{v \in V : f(v) = i\},$$

where i is in the range of f. The set S exists by Δ^0_1 comprehension with parameter f. It is straightforward to verify that S is a connected component of G. In Theorem 2.1, Gura, Hirst, and Mummert [6] show that the principle "every graph has a connected component" is equivalent to ACA_0 over RCA_0. Hence, WD^{M} implies ACA_0 over RCA_0. ■

CHAPTER 3

EXISTENCE OF SUBSPACES OF VECTOR SPACES

In this section we examine the strengths of principles involving the existence of subspaces of countable vector spaces.

Countable Vector Spaces

We begin with a definition of a countable vector space, which is based on the definition given by Metakides and Nerode [9].

Definition 3.1.1. A countable vector space V over an infinite computable field $\mathbb{F}$ is a tuple $(|V|, \equiv_V, +, \cdot, 0^V)$ consisting of

1. a set of vectors $|V| = \mathbb{N}$,
2. a function $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which represents vector space addition,
3. a function $\cdot\colon \mathbb{F} \times \mathbb{N} \to \mathbb{N}$ which represents scalar multiplication,
4. an element $0_V \in |V|$ that is an identity for $+$,
5. and an equivalence relation $\equiv_V \subseteq \mathbb{N}^2$, which represents equality in V,

such that $\left(|V| / \equiv_V, +, \cdot, 0^V\right)$ is a vector space over $\mathbb{F}$.

In this definition we could have allowed $|V|$ to be any countably infinite set. However, in this case there would be a uniformly computable bijection between $\mathbb{N}$ and $|V|$, and we would gain no additional generality. Specifically, if Φ is a computable functional such that $\Phi^A(k)$ is the k-th smallest element of A, if such an element exists, then $\Phi^{|V|}$ is a bijective enumeration of $|V|$ whenever $|V| \subseteq \mathbb{N}$ is an infinite set. Hence, we assume that $|V| = \mathbb{N}$. If $|V|$ and $\mathbb{F}$ were finite, then all questions about V would be answerable by exhaustive computation. Also notice that we can take $\equiv_V$ to be normal equality with no loss of generality by representing each vector by the least element of its equivalence class under $\equiv_V$ and requiring that each operation returns the least representative of its result. There is no loss of generality in this case because finding a least representative is uniformly computable from $\equiv_V$.

We define the following Weihrauch and reverse mathematics principles which assert the existence of subspaces of countable vector spaces of dimension greater than 1.

Definition 3.1.2. Let PS denote the formalization in second-order arithmetic of the principle "every vector space of dimension greater than 1 has a nontrivial proper subspace", where "dimension greater than 1" is formalized as "there exist two linearly independent vectors".

Let PS_W be the Weihrauch principle that takes as input a countable vector space of dimension greater than 1 and returns a nontrivial proper subspace.

Definition 3.1.3. Let L denote the formalization in second-order arithmetic of the principle "every vector space of dimension greater than 1 has a 1-dimensional subspace", where "dimension greater than 1" is formalized as "there exist two linearly independent vectors".

Let L_W be the Weihrauch principle that takes as input a countable vector space of dimension greater than 1 and returns a 1-dimensional subspace.

The next theorem follows directly from the definitions.

Theorem 3.1.1. L *implies* PS *over* RCA_0. PS_W *is strongly Weihrauch reducible to* L_W.

Weihrauch Perspective

The following lemma is the vector space equivalent of Lemma 2.1.4, and can be proven by a similar procedure in which finite subsets of $|V|$ and linear combinations of vectors are simultaneously enumerated.

Lemma 3.2.1. *Given a computable vector space V, we can uniformly compute from V an e-matroid (M, D) with $M = |V|$ and D the collection of finite linearly dependent subsets of V. Further, the subspaces of (M, D) are exactly the subspaces of V.*

We can apply this lemma to obtain the following upper bound for L_W.

Theorem 3.2.2. L_W *is strongly Weihrauch reducible to* $\widehat{\mathrm{LPO}}$.

Proof. Suppose we are given a countable vector space V of dimension greater than 1. Apply Lemma 3.2.1 to obtain an e-matroid (M, D) computable from V. Apply $\mathrm{D_M} \equiv_{sW} \widehat{\mathrm{LPO}}$ to obtain a decomposition f of (M, D) into 1-dimensional subspaces, and suppose $t \in \mathrm{range}(f)$. Now define $v \in S$ if and only if $f(v) \in \{0, t\}$. Observe that S is a 1-dimensional subspace of V that is computable from f. ∎

We now consider the Weihrauch classification of PS_W. In the proof of Theorem 3.2.6 we adapt a construction given by Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] in the context of computability to produce a Weihrauch reduction. This proof makes use of several lemmas from linear algebra. We adapt the statements and proofs of these lemmas from those given by Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4]. Theorem 3.2.6 shows that we can reduce the principle SEP, which produces a separating set for two disjoint enumerated sets, to the principle PS_W.

Definition 3.2.1. We define SEP to be the Weihrauch principle that takes as input two functions $f, g : \mathbb{N} \to \mathbb{N}$ with $\mathrm{range}(f) \cap \mathrm{range}(g) = \emptyset$ and returns as output a set S such that $\mathrm{range}(f) \subseteq S$ and $\mathrm{range}(g) \cap S = \emptyset$. The set S is referred to as a *separating set for* $\mathrm{range}(f), \mathrm{range}(g)$.

Definition 3.2.2. Let V be a vector space and let $X \subseteq V$. We let $\langle X \rangle$ denote the *span of X in V*. Given $v \in V$ and a basis B for V, let $\mathrm{supp}(v)$ denote the *support of v with respect to the basis B*, which is set of vectors that appear with nonzero coefficients in the expansion of v as a linear combination of vectors in B.

Suppose that V is a vector space and that $X \subseteq V$. Let $[v]$ denote the equivalence class of $v \in V$ in the quotient space $V/\langle X \rangle$. A subset $B \subseteq V$ spans V over $\langle X \rangle$ if the set $\{[b] : b \in B\}$ spans $V/\langle X \rangle$. It is straightforward to show that this is equivalent to the condition that each $v \in V$ is equal to a sum of the form $\alpha_1 b_1 + \ldots + \alpha_n b_n + x$, where $b_1, \ldots, b_n \in B$, $\alpha_1, \ldots, \alpha_n$ are scalars, and $x \in \langle X \rangle$. By definition, the subset $B \subseteq V$ is linearly independent if $\alpha_1[b_1] + \ldots + \alpha_n[b_n] = [0]$ implies that $\alpha_1 = 0, \ldots, \alpha_n = 0$. An equivalent condition is that $\alpha_1 b_1 + \ldots + \alpha_n b_n \in \langle X \rangle$ implies that $\alpha_1 = 0 \ldots, \alpha_n = 0$. In the following proofs it will be more convenient to use these alternative conditions for linear independence and span in $V/\langle X \rangle$.

Lemma 3.2.3. *Suppose that V is a vector space, that $X \subseteq V$, that $\{v, w\}$ is linearly independent over $\langle X\rangle$, and that $u \notin \langle X\rangle$. Then there exists at most one λ such that $u \in \langle X \cup \{v - \lambda w\}\rangle$.*

Proof. Suppose that $u \in \langle X \cup \{v - \lambda_1 w\}\rangle$ and $u \in \langle X \cup \{v - \lambda_2 w\}\rangle$. Fix scalars μ_1, μ_2 and vectors $x_1, x_2 \in \langle X\rangle$ such that $u = \mu_1(v - \lambda_1 w) + x_1$ and $u = \mu_2(v - \lambda_2 w) + x_2$. Notice that $\mu_1, \mu_2 \neq 0$, since $u \notin \langle X\rangle$. Then, we have

$$(\mu_1 - \mu_2)v + (\mu_2\lambda_2 - \mu_1\lambda_1)w = x_2 - x_1 \in \langle X\rangle.$$

Hence, since $\{v, w\}$ is linearly independent over $\langle X\rangle$, it follows that $\mu_1 - \mu_2 = 0$ and $\mu_2\lambda_2 - \mu_1\lambda_1 = 0$. Since $\mu_1 = \mu_2 \neq 0$, it follows that $\lambda_1 = \lambda_2$. ■

Lemma 3.2.4. *Suppose that V is a vector space, that $X \subseteq V$, and that B is a basis for V over $\langle X\rangle$ that is linearly ordered by $\prec$. Suppose that*

1. *$v \in V$,*
2. *$e \in B$,*
3. *λ is a nonzero scalar,*
4. *$e \succ \max(\mathrm{supp}(v))$.*

Then $B \setminus \{e\}$ is a basis for V over $\langle X \cup \{e - \lambda v\}\rangle$ and, for all $w \in V$,

$$\max(\mathrm{supp}_{B\setminus\{e\}}(w + \langle X \cup \{e - \lambda v\}\rangle)) \preceq \max(\mathrm{supp}_B(w)).$$

Proof. Notice that $e \in \langle (B \setminus \{e\}) \cup X \cup \{e - \lambda v\}\rangle$, because $e \notin \mathrm{supp}(v)$ and B is a basis for V over $\langle X\rangle$. Hence, $B \setminus \{e\}$ spans V over $\langle X \cup \{e - \lambda v\}\rangle$. Now suppose that $e_1, e_2, \ldots, e_n \in B \setminus \{e\}$ are distinct and $\mu_1, \mu_2, \ldots, \mu_n$ are scalars such that

$$\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_n e_n \in \langle X \cup \{e - \lambda v\}\rangle.$$

Fix a scalar μ and a vector $x \in \langle X \rangle$ such that

$$\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_n e_n - \mu(e - \lambda v) = x \in \langle X \rangle.$$

Notice that $\mu = 0$ must hold, because the coefficient of e on the left-hand side is μ, and the vectors $e_1, \ldots, e_n, e$ are linearly independent over X, by assumption. Thus, the μ_i's must also each be 0, because B is a basis over $\langle X \rangle$. Therefore $B \setminus \{e\}$ is a basis for V over $\langle X \cup \{e - \lambda v\} \rangle$. The last line of the theorem now follows by hypothesis 4. ■

Lemma 3.2.5. *Suppose that V is a vector space, that $X \subseteq V$, and that B is a basis for V over $\langle X \rangle$ that is linearly ordered by $\prec$. Suppose that*

1. $v_1, v_2 \in V$,
2. $e_1, e_2 \in B$ *with* $e_1 \neq e_2$,
3. λ *is a scalar,*
4. $e_1 \succ \max(\mathrm{supp}(v_1) \cup \mathrm{supp}(v_2))$,
5. $\{v_1, e_1\}$ *is linearly independent over* $\langle X \rangle$,
6. $v_1 \notin \langle X \cup \{e_2 - \lambda v_2\} \rangle$.

Then $\{v_1, e_1\}$ is linearly independent over $\langle X \cup \{e_2 - \lambda v_2\} \rangle$.

Proof. Suppose that $\mu_1 v_1 + \mu_2 e_1 \in \langle X \cup \{e_2 - \lambda v_2\} \rangle$, so that

$$\mu_1 v_1 + \mu_2 e_1 - \mu_3(e_2 - \lambda v_2) = x \in \langle X \rangle,$$

for some scalar μ_3 and $x \in \langle X \rangle$. We need to show that $\mu_1 = \mu_2 = 0$.

Case 1: $e_1 \prec e_2$. In this case the coefficient of e_2 on the left-hand side is μ_3, so we must have $\mu_3 = 0$. Hence, $\mu_1 v_1 + \mu_2 e_1 \in \langle X \rangle$, and $\mu_1 = \mu_2 = 0$ since $\{v_1, e_1\}$ is linearly independent over $\langle X \rangle$.

Case 2: $e_1 \succ e_2$. In this case, the coefficient of e_1 on the left-hand side is μ_2, so $\mu_2 = 0$. Then, $\mu_1 v_1 - \mu_3(e_2 - \lambda v_2) \in \langle X \rangle$. Since $v_1 \notin \langle X \cup \{e_2 - \lambda v_2\} \rangle$, $\mu_1 = 0$. Hence $\{v_1, e_1\}$ is linearly independent over $\langle X \cup \{e_2 - \lambda v_2\} \rangle$. ■

With these lemmas established, we give the following proof, which is adapted directly from Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4]. We make no significant changes to the proof as it is presented by the original authors, adding only some additional commentary to verify that the constructions given in this proof are uniformly computable from the input functions f and g.

Theorem 3.2.6. SEP *is Weihrauch reducible to* PS_W*.*

Proof. Suppose we are given an instance of SEP, which is a pair of functions f, g from $\mathbb{N}$ to $\mathbb{N}$ such that $\mathrm{range}(f) \cap \mathrm{range}(g) = \emptyset$. To solve this instance of SEP, we need to produce a set S such that $\mathrm{range}(f) \subseteq S$ and $\mathrm{range}(g) \cap S = \emptyset$. We will reduce this problem to that of producing a nontrivial proper subspace of a countable vector space in two steps. First, we describe a procedure that takes f and g as input parameters and constructs a vector space V. Then, we describe a procedure that, given f, g, and a proper subspace W of V as parameters, computes from W a separating set S for $\mathrm{range}(f)$ and $\mathrm{range}(g)$.

To begin, let V^∞ be the vector space over $\mathbb{Q}$ generated by the countable basis $e_0, e_1, \ldots$ with the ordering $e_i \prec e_j$ if and only if $i < j$. Here e_i is the vector that has a 1 in its ith component and a 0 in all other components. Let $v_0, v_1, \ldots$ be a computable enumeration of V^∞, and assume that v_0 is the zero vector of V^∞. Using this enumeration, we can computably go back and forth between v_i and a code for its expansion in terms of the basis $\{e_i\}_{i \in \mathbb{N}}$. Let $\mathrm{supp}(v_i)$ be the set of basis elements that have nonzero coefficients when v_i is expressed as a linear combination of basis elements e_j. Now let $\phi : \mathbb{N}^3 \to \mathbb{N}$ be a computable injective function such that $e_{\phi(i,j,n)} \succ \max\{\mathrm{supp}(v_i) \cup \mathrm{supp}(v_j)\}$ for each i, j, n. It is straightforward to show that such a function ϕ exists.

Part 1, constructing the vector space V:

Construction:

We construct a subspace U of V^∞ by computing an increasing sequence $U_2, U_3, U_4, \ldots$ of finite subsets of V and setting $U = \bigcup_{n \geq 2} U_n$. We define a set of requirements $R_{i,j,n}$ to be satisfied by all pairs $v_i, v_j \notin U$. The requirement $R_{i,j,n}$ says that the following hold, where $v_i, v_j \notin U$:

Requirement $R_{i,j,n}$:

1. If $n \notin \text{range}(f) \cup \text{range}(g)$, then each of $\{v_i, e_{\phi(i,j,n)}\}$ and $\{v_j, e_{\phi(i,j,n)}\}$ are linearly independent over U.

2. If $n \in \text{range}(f)$, then $e_{\phi(i,j,n)} - \lambda v_i \in U$ for some nonzero $\lambda \in \mathbb{Q}$.

3. If $n \in \text{range}(g)$, then $e_{\phi(i,j,n)} - \lambda v_j \in U$ for some nonzero $\lambda \in \mathbb{Q}$.

Define $h\colon \mathbb{N}^4 \to \{0,1\}$ to be such that $h(i,j,n,s) = 1$ if and only if we have acted to satisfy requirement $R_{i,j,n}$ at some stage $t \leq s$, during which we constructed U_t. This construction will ensure that $v_k \in U$ if and only if $v_k \in U_k$, for each $k \geq 2$. Since each U_k is computable by our procedure, this will ensure that U is also computable.

Set $U_2 = \{v_0\}$ and $h(i,j,n,s) = 0$ for all i,j,n,s with $s \leq 2$. Now suppose that $s \geq 2$ and that we have defined U_s and $h(i,j,n,s)$ for all i,j,n. Suppose that for any i,j,n such that $v_i, v_j \notin \langle U_s \rangle$ the following are satisfied:

1. If $h(i,j,n,s) = 0$, then each of $\{v_i, e_{\phi(i,j,n)}\}$ and $\{v_j, e_{\phi(i,j,n)}\}$ is linearly independent over $\langle U_s \rangle$.

2. If $h(i,j,n,s) = 1$:

   (a) If $n \in f(\{1, \ldots, s\})$, then $e_{\phi(i,j,n)} - \lambda v_i \in U_s$ for a nonzero $\lambda \in \mathbb{Q}$.

   (b) If $n \in g(\{1, \ldots, s\})$, then $e_{\phi(i,j,n)} - \lambda v_j \in U_s$ for a nonzero $\lambda \in \mathbb{Q}$.

Notice that, because U_s is finite and we can compute the coefficients of the expansion of each v_k in terms of basis elements, from the index k, then we can compute the characteristic function of $\langle U_s \rangle$, using Gaussian elimination, for example. Assume throughout the rest of the proof that we have a fixed, effective, bijective coding $c\colon \mathbb{N}^3 \to \mathbb{N}$ of triples such that $m \leq c(i,j,k)$ for each $m \in \{i,j,k\}$. To compute U_{s+1}, first check whether there exists a triple (i,j,n) with code less than s such that the following condition holds:

Condition A:

1. $v_i, v_j \notin \langle U_s \rangle$.

2. $n \in f(\{1, \ldots, s\}) \cup g(\{1, \ldots, s\})$.

3. $h(i, j, n, s) = 0$.

First suppose that no such triple (i, j, n) exists. If $v_{s+1} \in \langle U_s \rangle$, then set $U_{s+1} = U_s \cup \{v_{s+1}\}$; otherwise let $U_{s+1} = U_s$. Also let $h(i, j, n, s+1) = h(i, j, n, s)$ for all i, j, n.

Now suppose that there is a least triple $(i, j, n) < s$ satisfying requirements (1) through (3) of condition A. Next:

- If $n \in f(\{1, \ldots, s\})$, then search for the least $\lambda \in \mathbb{Q}$ (we assume a fixed effective encoding of $\mathbb{Q}$) such that $v_k \notin \langle U_s \cup \{e_{\phi(i,j,n)} - \lambda v_i\} \rangle$ for all $k \leq s$ with $v_k \notin \langle U_s \rangle$. Such a $\lambda \in \mathbb{Q}$ is guaranteed to exist by Lemma 3.2.3 and the fact that $\mathbb{Q}$ is infinite. Let $U'_s = U_s \cup \{e_{\phi(i,j,n)} - \lambda v_i\}$ and let $h(i, j, n, s+1) = 1$. If $v_{s+1} \in \langle U'_s \rangle$, then set $U_{s+1} = U'_s \cup \{v_{s+1}\}$; otherwise set $U_{s+1} = U'_s$.

- If $n \in g(\{1, \ldots, s\})$, then search for the least $\lambda \in \mathbb{Q}$ such that $v_k \notin \langle U_s \cup \{e_{\phi(i,j,n)} - \lambda v_j\} \rangle$ for all $k \leq s$ with $v_k \notin \langle U_s \rangle$. Let $U'_s = U_s \cup \{e_{\phi(i,j,n)} - \lambda v_j\}$ and let $h(i, j, n, s+1) = 1$. If $v_{s+1} \in \langle U'_s \rangle$, then set $U_{s+1} = U'_s \cup \{v_{s+1}\}$; otherwise set $U_{s+1} = U'_s$.

Verification:

It follows directly from our construction that part (2) of the induction hypothesis is maintained. To verify that part (1) of the induction hypothesis is maintained, suppose that $U_{s+1} = U_s \cup \{e_{\phi(i,j,n)} - \lambda v_i\}$. Suppose that $v_k \notin \langle U_{s+1} \rangle$, that $(k, l, m) \neq (i, j, n)$, and that $\{v_k, e_{\phi(k,l,m)}\}$ is linearly independent over $\langle U_s \rangle$. We will show that $\{v_k, e_{\phi(k,l,m)}\}$ is also linearly independent over $\langle U_{s+1} \rangle$. Suppose that $U_s = \{v_0,\, v_{j_1}, \ldots, v_{j_m},\, e_{i_1} - \lambda_1 v_{k_1},\, \ldots,\, e_{i_n} - \lambda_n v_{k_n}\}$. By our construction, the vectors $T = \{v_0,\, v_{j_1},\, \ldots,\, v_{j_m}\}$ are all in the span of $U_s \setminus T$, so we can ignore these vectors. Hence, by repeated application of Lemma 3.2.4, $B := \{e_1, e_2, \ldots\} \setminus \{e_{i_1}, \ldots, e_{i_n}\}$ is a basis for V over $\langle U_s \rangle$. Now, if $\{v_k, e_{\phi(k,l,m)}\}$ is linearly dependent over $\langle U_{s+1} \rangle$, then there exist coefficents α, β, δ with $\delta \neq 0$ and α and β both not zero such that

$$(\alpha e_{\phi(k,l,m)} + \beta v_k) + \delta(e_{\phi(i,j,n)} - \lambda v_i) \in \langle U_s \rangle.$$

Suppose that $e_{\phi(k,l,m)} \prec e_{\phi(i,j,n)}$. Then, by the definition of ϕ we have

$e_{\phi(i,j,n)} \succ \max(\mathrm{supp}(v_k) \cup \mathrm{supp}(v_i))$. Hence, if we expand $(\alpha e_{\phi(k,l,m)} + \beta v_k) + \delta(e_{\phi(i,j,n)} - \lambda v_i)$ in terms of the basis B, then the coefficient of $e_{\phi(i,j,n)}$ in this expansion must be zero. But, this coefficient is δ, which we assumed to be nonzero.

Now suppose that $e_{\phi(k,l,m)} \succ e_{\phi(i,j,n)}$. In this case all of the hypotheses of Lemma 3.2.5 are met; hence, we can apply Lemma 3.2.5 to show that $\{v_k, e_{\phi(k,l,m)}\}$ is linearly independent over $\langle U_{s+1}\rangle$, contradicting our assumption. Hence, $\{v_k, e_{\phi(k,l,m)}\}$ must be linearly independent over $\langle U_{s+1}\rangle$. The proof will be nearly identical if instead $U_{s+1} = U_s \cup \{e_{\phi(i,j,n)} - \lambda v_j\}$. Hence, part (1) of our induction hypothesis is maintained.

In the construction of U_{s+1} we have ensured that, if $k \leq s$ and $v_k \notin \langle U_s\rangle$, then $v_k \notin \langle U_{s+1}\rangle$ also. This ensures that, if v_k enters the span of U, then it does so at a stage $s \leq k$. The construction also ensures that, if v_k enters the span of U at a stage $s \leq k$, then v_k will be added to U_{k-1} in the construction of U_k during stage k. Hence, the set $U = \bigcup_s U_s$ is closed under taking linear combinations, so it is a subspace of V^∞. As described above, U is computable.

It remains to verify that the requirements $R_{i,j,n}$ are satisfied for each (i,j,n) with $v_i, v_j \notin U$. To verify this, choose (i,j,n) such that $v_i, v_j \notin U$. Notice that this implies that $v_i, v_j \notin U_s$ for each s. If $n \notin \mathrm{range}(f) \cup \mathrm{range}(g)$, then our induction hypotheses ensure that $\{v_i, e_{\phi(i,j,n)}\}$ and $\{v_j, e_{\phi(i,j,n)}\}$ are linearly independent over $\langle U_s\rangle$ for each s, so each of these sets is also linearly independent over U. Hence, $R_{i,j,n}$ is satisfied in this case. Now suppose that $f(i_0) = n$. Then, at some stage $s \geq i_0$ the triple (i,j,n) will satisfy condition A. The triple (i,j,n) will then continue to satisfy condition A for $t \geq s$ until at some stage $s_0 \geq i_0$, (i,j,n) is the least triple that satsifies condition A. At stage s_0 we will act to satisfy requirement $R_{i,j,n}$. Because our induction hypotheses are maintained, this ensures that $R_{i,j,n}$ is satisfied for U. The case is similar if $g(i_0) = n$ for some i_0. Hence, the requirements $R_{i,j,n}$ are satisfied.

Now define $V = V^\infty/U$. V is a computable vector space over $\mathbb{Q}$: the equivalence relation $\equiv_V$ defined by $x \equiv_V y$ iff $x - y \in U$ is computable because V^∞ and U are both computable. We can represent each equivalence class $[x]$ under $\equiv_V$ by the least element of $[x]$ according to the enumeration of vectors in V^∞. This representation is computable, and the modifications of the operations on V^∞ so that they return least representatives of equivalence classes under $\equiv_V$ are also computable. By our construction $\{v_1, e_{\phi(1,2,n)}\}$ is linearly independent over U for any

$n \notin \mathrm{range}(f) \cup \mathrm{range}(g)$, so V is has dimension at least two. Hence, V is an instance of PS_W.

Notice that throughout this construction so far, the input functions f and g have been treated as oracles, with each step of the construction accessing finitely many output values for f or g. Hence, this construction is uniform in the input functions f and g, and the construction constitutes the first half of a Weihrauch reduction of SEP to PS_W.

Part 2, computing a separating set from a subspace of V:

Now suppose we apply PS_W to V to obtain a nontrivial proper subspace W of V. Then, from W we can compute a subspace W_0 of V^∞ such that $U \subset W_0 \subset V^\infty$ and $W = W_0/U$. Suppose we have access to the original instance of SEP, which is to say the functions f and g. From f and g we can compute U and $V = V^\infty/U$, just as in the first part of the reduction. Recall that elements of V are coded by least representatives of equivalence classes of V^∞ over U. Hence, to check if $x \in V^\infty$ is in W_0, we only have to compute the least representative of the equivalence class $[x]$ and determine whether or not it is in W. Hence, W_0 is computable from f, g, and W.

Now fix $v_i, v_j \in V^\infty$ such that $v_i \in W_0$ and $v_j \notin W_0$. Such vectors exist, because $U \subset W_0 \subset V^\infty$ and both inclusions are proper, and can be obtained from W_0 by enumerating and checking finitely many vectors in V^∞. Now define $S := \{n : e_{\phi(i,j,n)} \in W_0\}$. Observe that S is computable from W_0, because each element of V^∞ is represented by the finite set of nonzero coefficients in its representation as a linear combination of basis elements e_k. Further, since $v_i \in W_0$ and $v_j \notin W_0$, $n \in \mathrm{range}(f)$ implies that $n \in S$ and $n \in \mathrm{range}(g)$ implies that $n \notin S$, by requirement $R_{i,j,n}$. Specifically, if $n \in \mathrm{range}(f)$, then there is a nonzero $\lambda \in \mathbb{Q}$ such that $e_{\phi(i,j,n)} - \lambda v_i \in U \subset W_0$. Since $v_i \in W_0$, it follows that $e_{\phi(i,j,n)} \in W_0$ also in this case, and therefore $n \in S$. If $n \in \mathrm{range}(g)$, then there is a nonzero $\lambda \in \mathbb{Q}$ such that $e_{\phi(i,j,n)} - \lambda v_j \in U \subset W_0$. If $n \in \mathrm{range}(g)$ in this case, then $e_{\phi(i,j,n)} \in W_0$ implies that $v_j \in W_0$, a contradiction. Hence, $n \notin S$ in this case. Therefore, S is a separating set for $\mathrm{range}(f)$ and $\mathrm{range}(g)$.

In part 2 of this proof, we have defined a procedure that takes as input the functions f and g and a proper subspace of W of $V = V^\infty/U$, and from W, f, and g computes a separating set for $\mathrm{range}(f)$ and $\mathrm{range}(g)$. Further, the given procedure doesn't depend on the particular functions f and g, which are each accessed only finitely many times by each computation using this procedure, so this procedure is uniform in f and g. Hence, we conclude that the construction in part 2 of this proof completes the (non-strong) Weihrauch reduction SEP $\leq_W$ PS_W. ∎

We have the following equivalence of Weihrauch principles, which we will use to obtain a Weihrauch parallel to Theorem 1.5 of Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4]. For more information on results related to the following lemma, see Simpson [14, pg. 30]. The proof of this lemma makes use of standard techniques from computability theory.

Definition 3.2.3. We define WKL_W to be the Weihrauch principle that takes as input an infinite binary tree T (encoded as a subset of $\{0,1\}^{<\mathbb{N}}$ that is closed under taking initial segments) and returns a path through T (encoded as a function $f : \mathbb{N} \to \{0,1\}$).

Lemma 3.2.7. SEP *is strongly Weihrauch equivalent to* WKL_W*.*

Proof. (SEP $\leq_{sW}$ WKL_W) Suppose we are given two r.e. sets A and B with $A \cap B = \emptyset$. Our goal is to produce a set $C \supseteq A$ with $C \cap B = \emptyset$. We will build a subtree T of $\{0,1\}^{<\mathbb{N}}$ in which each node represents a finite initial segment of a candidate separating set. A path through this tree will represent a set C of the desired type. For each string $\sigma \in \{0,1\}^{<\mathbb{N}}$, define $\sigma \in T$ if and only if each element of $A_{|\sigma|+1} \cap \{0,\ldots,|\sigma|\}$ is in σ and no element of $B_{|\sigma|+1} \cap \{0,\ldots,|\sigma|\}$ is in σ. We claim that T has the desired properties.

Suppose that h is a path through T, and suppose there is an element n of A not in h (here we identify h with $\{n : h(n) = 1\}$). There is some stage s at which n is enumerated into A; let $s_0 := \max\{s, n\}$. Now, the restriction $h \upharpoonright (s_0+1)$ with length s_0+1 is in T, but $n \in (A_{s_0+2} \cap \{0,\ldots,s_0+1\}) \setminus h \upharpoonright (s_0+1)$, contradicting the condition for membership in T. Hence, h contains A. Now suppose that there is an element $m \in B \cap h$; suppose m is enumerated into B at stage t. Let $t_0 := \max\{m, t\}$. We have $h \upharpoonright (t_0+1) \in T$, but $m \in B_{t_0+2} \cap \{0,\ldots,t_0+1\}$, contradicting the condition for membership in T. Hence, h contains no element of B, so $C := \operatorname{range} h$ is a set of the desired form.

We have shown thus far that, given disjoint r.e. sets A, B, we can compute from these sets a subtree T of $\{0,1\}^{<\mathbb{N}}$ such that any path through T is the characteristic function for a separating set for A and B. This constitutes the first half of a Weihrauch reduction of the problem of separation to that of finding a path through an infinite binary tree. Now, we can use WKL_W to obtain a path h through T, and from our construction of T the path h is exactly the characteristic function of a computable separation of A and B. So the second half of the reduction is simply the identity map.

($\mathrm{WKL}_W \leq_{sW}$ SEP) Suppose we are given an infinite tree $T \subseteq \{0,1\}^{<\mathbb{N}}$. We define A to be the set of nodes $\sigma \in T$ such that $\sigma ^\frown (0)$ is not extendible, and we define B to be the set of nodes $\sigma \in T$ such that $\sigma ^\frown (1)$ is not extendible, with the additional specification that if both $\sigma ^\frown (0)$ and $\sigma ^\frown (1)$ are not extendible, then σ is put in B if $\sigma ^\frown (1)$ ceases to be extendible at an earlier level than or at the same level as $\sigma ^\frown (0)$, and σ is put into A otherwise.

It is straightforward to show that both A and B are c.e. in T: a procedure to compute whether a node is in A or B involves iterating over t and checking whether the finite number of possible paths of the form $\sigma ^\frown (n_1 = 0, n_2, \ldots, n_t)$ and $\sigma ^\frown (m_1 = 1, m_2, \ldots, m_t)$ are in T, then applying the appropriate rule to determine if σ is added to A or B at stage t. Moreover, our definition of A and B guarantees that A and B are disjoint.

Hence, we can apply SEP to obtain an oracle for a set C such that $A \subseteq C$ and $C \cap B = \emptyset$. Now, from C we can compute a path through T as follows. Define p_1 to be the root of T. Suppose we have computed the extendible subpath $(p_1, \ldots, p_k) \in T$. We can computably choose p_{k+1} using C by choosing $p_{k+1} = 1$ if $(p_1, \ldots, p_k) \in C$ and $p_{k+1} = 0$ otherwise. We claim that this choice guarantees that the node p_{k+1} is extendible. Suppose that p_{k+1} is not extendible. Then, because $(p_1, \ldots, p_k)$ is extendible, at most one of $(p_1, \ldots, p_k, 0)$, $(p_1, \ldots, p_k, 1)$ is not extendible. If $(p_1, \ldots, p_k, 0)$ is not extendible, then $(p_1, \ldots, p_k) \in A \subseteq C$, so we have chosen $(p_1, \ldots, p_k, 1)$. On the other hand, if $(p_1, \ldots, p_k, 1)$ is not extendible, then $(p_1, \ldots, p_k) \in B$, hence $(p_1, \ldots, p_k) \notin C$, so we have chosen $(p_1, \ldots, p_k, 0)$. It is straightforward to see that this procedure is uniformly computable in C. Hence, at each kth stage we will have chosen an extendible node p_k, so the infinite path $(p_1, p_2, \ldots)$ is contained in T, and we have solved the instance T of WKL_W. ■

We introduce the compositional product of two Weihrauch principles, and use that operation to establish an upper bound for PS_W. Brattka and Pauly [2] define the compositional product, and then provide an equivalent characterization. We take that equivalent characterization as our definition, because it is easier to use in establishing classifications of Weihrauch principles.

Definition 3.2.4. The Weihrauch degree of the compositional product $f \star g$ of two Weihrauch principles is defined as

$$f \star g \equiv_W \max_{\leq_W}\{f' \circ g' : f' \leq_W f \wedge g' \leq_W g\},$$

where $f' \circ g'$ is the set of all pairs (I, S) such that there exists a $J \in \mathbb{N}^{\mathbb{N}}$ with $(I, J) \in g'$ and $(J, S) \in f'$.

We now show that an upper bound for PS_W is given by the compositional product of $\mathrm{C}_{\mathbb{N}}$ with WKL_W.

Theorem 3.2.8. *The principle* PS_W *is Weihrauch reducible to* $\mathrm{WKL}_W \star \mathrm{C}_{\mathbb{N}}$.

Proof. Suppose we are given a countable vector space V. We first apply $\mathrm{C}_{\mathbb{N}}$ to locate two linearly independent vectors u, v in V. Then, we define a binary tree using u, v, and V, as follows. Let $v_0, v_1, v_2, \ldots$ be an enumeration of the vectors in V and $q_0, q_1, q_2, \ldots$ be an enumeration of the scalars in $\mathbb{Q}$. Interpret each $\sigma \in \{0,1\}^{<\mathbb{N}}$ as the characteristic function for a subset of $\{v_0, \ldots, v_{|\sigma|-1}\}$. Define $\sigma \in T$ if and only if the subset S with characteristic function σ satisfies the conditions for a subspace of V containing u and not containing v, with consideration restricted to the vectors $\{v_0, \ldots, v_{|\sigma|-1}\}$ and the scalars $\{q_0, \ldots, q_{|\sigma|-1}\}$. This means that, if $u \in \{v_0, \ldots, v_{|\sigma|-1}\}$ then $u \in S$, if $v \in \{v_0, \ldots, v_{|\sigma|-1}\}$ then $v \notin S$, and S is closed under taking linear combinations of vectors from $\{v_0, \ldots, v_{|\sigma|-1}\}$ with scalars from $\{q_0, \ldots, q_{|\sigma|-1}\}$, whenever the resulting vector is also in $\{v_0, \ldots, v_{|\sigma|-1}\}$. Then, T is computable from u, v, and V, and any infinite path through T will represent the characteristic function of a nontrivial proper subspace of V. We know classically that T is infinite, because classically PS_W is true. Hence, the application of WKL_W to the tree T yields the desired subspace. ■

Combining Theorems 3.2.6 and 3.2.8 and Lemma 3.2.7 yields the following corollary, which is an approximate Weihrauch reducibility parallel to part of the reverse mathematics result of Theorem 1.5 of Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4]. We show in a later section that the upper and lower bounds given in this corollary are not Weihrauch equivalent.

Corollary 3.2.9. *We have the following bounds for the Weihrauch strength of* PS_W*:*

$$\mathrm{WKL}_W \leq_W \mathrm{PS}_W \leq_W \mathrm{WKL}_W \star \mathrm{C}_{\mathbb{N}}.$$

Reverse Mathematics Perspective

The following theorems give the reverse mathematics classifications of the subsystems L and PS.

Theorem 3.3.1. L *is equivalent to* ACA_0 *over* RCA_0*.*

Proof. Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] show that ACA_0 is equivalent over RCA_0 to the principle "every vector space of dimension greater than 1 has a finite-dimensional nontrivial proper subspace." L is at least as strong as this principle, and hence is at least as strong as ACA_0. It is straightforward to show that ACA_0 implies L over RCA_0. ■

The following classification of PS is obtained by Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4].

Theorem 3.3.2. PS *is equivalent to* WKL_0 *over* RCA_0*.*

CHAPTER 4

EXISTENCE OF SATURATED SUBGRAPHS

Finite-Dimensional Saturated Subgraphs

In this section we consider principles asserting the existence of subspaces of graphs. These principles are related to the principle 'P' of Gura, Hirst, and Mummert [6], which formalizes the statement that "every graph has a connected component". In the following we refer to this principle of Gura, Hirst, and Mummert as P_W to emphasize that it is a Weihrauch principle.

Definition 4.1.1. If $G = (V, E)$ is a graph, then we say that $S \subseteq V$ is a *saturated subgraph* if for each $v \in V$, if v is path connected to a vertex in S, then $v \in S$.

The following straightforward lemma characterizes saturated subgraphs.

Lemma 4.1.1. *If $G = (V, E)$ is a graph, then $S \subseteq V$ is a saturated subgraph of G if and only if S is equal to a union of connected components of G.*

We define the finite-dimensional saturated subgraph principle, $\mathrm{PSG}_W^{<\mathbb{N}}$.

Definition 4.1.2. Define $\mathrm{PSG}_W^{<\mathbb{N}}$ to be the principle that takes as input a graph $G = (V, E)$ and returns a nonempty set S that is equal to the union of finitely many connected components of G. We refer to such a set S as a *nonempty finite-dimensional saturated subgraph* of the graph G.

The following proof is an adaptation and slight extension of the proof given by Gura, Hirst, and Mummert of Theorem 2.1 in the paper titled "On the existence of a connected component of a graph" [6], which says that $\mathrm{P}_W \equiv_{sW} \widehat{\mathrm{LPO}}$.

Theorem 4.1.2. $\widehat{\mathrm{LPO}}$ *is strongly Weihrauch equivalent to* $\mathrm{PSG}_W^{<\mathbb{N}}$*.*

Proof. ($\widehat{\mathrm{LPO}} \leq_{sW} \mathrm{PSG}_W^{<\mathbb{N}}$) Suppose that (f_n) is an instance of $\widehat{\mathrm{LPO}}$. We will construct a graph $G = (V, E)$ from (f_n), such that from any set $C \subseteq V$ that is the union of finitely many connected components of G we can compute a solution to the instance (f_n). Let V be the set of vertices of the form $v_k^{\sigma,n}$, where $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $n, k \in \mathbb{N}$. Let E contain the following types of edges:

(i) $(v_k^{\sigma,n}, v_{k+1}^{\sigma,n})$ for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $n, k \in \mathbb{N}$;

(ii) $(v_k^{\sigma,n}, v_0^{\sigma^\frown\langle j\rangle,m})$ for all $m, n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, if $f_j(k) = 0$.

Now suppose that $C \subseteq V$ is equal to the union of finitely many connected components of G, and suppose that $v_0^{\sigma,n} \in C$. It can be verified that $v_0^{\sigma,n}$ and $v_0^{\sigma^\frown\langle j\rangle,m}$ lie in the same connected component of G if and only if $f_j(k) = 0$ for some k. Hence, if there is no k such that $f_j(k) = 0$, then for each m the vertex $v_0^{\sigma^\frown\langle j\rangle,m}$ will lie in a different connected component from $v_0^{\sigma,n}$. Since C is the union of finitely many connected components, $v_0^{\sigma^\frown\langle j\rangle,m} \notin C$ for some m in this case. Hence, by checking sequentially if $v_0^{\sigma^\frown\langle j\rangle,m}$ is in C, we will eventually detect that there is no k such that $f_j(k) = 0$, if this is the case. We can simultaneously evaluate $f_j(k)$ for each k, halting if we discover a k such that $f_j(k) = 0$. Since there either is or is not an k such that $f_j(k) = 0$, our procedure will eventually halt, determing whether or not such an k exists. It is straightforward to show that this procedure is computable from C, and hence from C we can compute a solution to the instance (f_n).

($\mathrm{PSG}_W^{<\mathbb{N}} \leq_{sW} \widehat{\mathrm{LPO}}$) There are numerous ways to establish this reduction. Here we note that the principle P_W of Gura, Hirst, and Mummert [6] implies $\mathrm{PSG}_W^{<\mathbb{N}}$ by definition, and $\mathrm{P}_W \equiv_{sW} \widehat{\mathrm{LPO}}$, as is proved by Gura, Hirst, and Mummert. ∎

Because we can computably turn a graph into an e-matroid, using Lemma 2.1.4, we obtain as a corollary that the principle that "every e-matroid has a finite-dimensional subspace" is strongly Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. However Lemma 2.1.4 requires that the graph under consideration have at least one edge. Because the graph G constructed in the first part of the proof of Theorem 4.1.2 has infinitely many edges, by the same proof we obtain the following corollary.

Corollary 4.1.3. *The principle* $\mathrm{PSG}_W^{<\mathbb{N}*}$ *that formalizes "every graph with at least one edge has a nonempty finite-dimensional saturated subgraph" is strongly Weihrauch equivalent to* $\widehat{\mathrm{LPO}}$.

When referring to an e-matroid, *nontrivial* means that the given e-matroid contains at least one nonzero element.

Definition 4.1.3. Suppose that (M, D) is an e-matroid and that S is a subspace of (M, D), as defined in Definition 2.1.3. If there exists an element $m \in S \setminus Z$, then we say that S is *nontrivial*;

otherwise we say that S is *trivial.* We say that S is a *finite-dimensional subspace* of (M, D) if S is trivial or if S is nontrivial and there is a finite basis $B \subseteq S$ for S.

We now show that $\mathrm{PSM}_W \equiv_{sW} \widehat{\mathrm{LPO}}$.

Corollary 4.1.4. *The principle* PSM_W *that takes as input a nontrivial e-matroid* (M, D) *and returns a nontrivial finite-dimensional subspace of* (M, D) *is strongly Weihrauch equivalent to* $\widehat{\mathrm{LPO}}$*.*

Proof. ($\mathrm{PSM}_W \leq_{sW} \widehat{\mathrm{LPO}}$): Assume that (M, D) is a nontrivial e-matroid. We define several instances of $\widehat{\mathrm{LPO}}$: First, define an instance (f_n) where $0 \in \mathrm{range}(f_n)$ if and only if $n \in M$. Second, define an instance (g_n) with $0 \in \mathrm{range}(g_n)$ if and only if $n \in M$ and there is an $m \neq n$ in M with $\{m, n\} \notin \mathrm{range}(D)$. Finally, define an instance (h_n) as in the proof of Theorem 2.2.1 such that a solution to (h_n) computes $\chi_{\mathrm{range}(D)}$. These three instances of $\widehat{\mathrm{LPO}}$ can be encoded into a single instance of $\widehat{\mathrm{LPO}}$. The entire procedure described so far is uniformly computable from (M, D).

Now suppose we apply $\widehat{\mathrm{LPO}}$ to obtain solutions to (f_n), (g_n), and (h_n). We can use the solution to (f_n) compute χ_M, and we can use the solution to (g_n) to locate a nonzero element m of (M, D). Then we can use the solution to (h_n) to compute $\chi_{\mathrm{range}(D)}$, and $\chi_{\mathrm{range}(D)}$ can be used to compute the 1-dimensional subspace of all elements $n \in M$ such that $\{m, n\} \in D$. This subspace is a solution to PSM_W. Since this computation can be carried out from any set of solutions to (f_n), (g_n), and (h_n), it constitutes the second half of the desired strong Weihrauch reduction.

($\widehat{\mathrm{LPO}} \leq_{sW} \mathrm{PSM}_W$): We know that $\mathrm{PSG}_W^{<\mathbb{N}*} \equiv_{sW} \widehat{\mathrm{LPO}}$. Suppose we are given a countable graph $G = (V, E)$ with at least one edge. From G we can compute a nontrivial e-matroid (M, D) where subspaces of (M, D) are exactly unions of connected components in G. Then, a nontrivial finite-dimensional subspace of (M, D) is also a nontrivial finite-dimensional saturated subgraph of G. Hence, $\mathrm{PSG}_W^{<\mathbb{N}*} \leq_{sW} \mathrm{PSM}_W$. ■

Nontrivial Proper Saturated Subgraphs

Above we considered the principle $\mathrm{PSG}_W^{<\mathbb{N}}$ that says informally that "every graph has a nonempty finite-dimensional saturated subgraph". Here we consider the principle that says that "every graph with more than one connected component has a nontrivial proper saturated subgraph".

Definition 4.2.1. Define PSG_W to be the Weihrauch principle given by the set of pairs (G, S), where G is a countable graph with at least two connected components and S is a *nontrivial proper subgraph* of G, i.e., S is a nonempty union of connected components of G that is not all of G.

We show that $\mathrm{C}_{\mathbb{N}}$ is reducible to PSG_W.

Theorem 4.2.1. $\mathrm{C}_{\mathbb{N}}$ *is strongly Weihrauch reducible to* PSG_W.

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be nonsurjective. We define a graph $G = (V, E)$. Let $V = \mathbb{N} \times \mathbb{N}$, and let E consist of edges of the form $\{(k, n), (k, 0)\}$ for $k, n \in \mathbb{N}$ and $\{(0, t), (n + 1, t)\}$ where $f(t) = n$. Suppose that we are given a nontrivial poper subgraph $S \subseteq V$. Then, we can check if $(0, 0) \in S$.

Case 1, $(0, 0) \in S$*:* In this case, we know that S contains $(n + 1, 0)$ for all $n \in \mathrm{range}(f)$. Hence, we can search until we find an element $(k + 1, 0)$ that is not in S, and we will have $k \notin \mathrm{range}(f)$.

Case 2, $(0, 0) \notin S$*:* In this case, we know that S does not contain any vertices of the form $(k + 1, 0)$, where $k \in \mathrm{range}(f)$. Hence, we only need to search until we find a vertex $(k + 1, 0) \in S$, and we will know that $k \notin \mathrm{range}(f)$.

In either case the search is guaranteed to succeed because the subgraph S is nonempty and proper. ■

We can leverage the vector space construction from Theorem 3.2.6 to show that WKL_W is Weirhauch reducible to PSG_W.

Theorem 4.2.2. WKL_W *is Weihrauch reducible to* PSG_W.

Proof. Build the vector space V: Suppose we are given two functions $f, g \in \mathbb{N}^{\mathbb{N}}$ with $\mathrm{range}(f) \cap \mathrm{range}(g) = \emptyset$. Then, with these functions as input, we can use the construction from Theorem 3.2.6 to obtain a subspace $U \subseteq V^{\infty}$ such that any nontrivial proper subspace of $V := V^{\infty}/U$ uniformly computes a separating set S for f and g. In that construction, each vector in V is an equivalence class $[v_i]$ with representative $v_i \in V^{\infty}$. We can identify each vector $[v_i] \in V$ with its least representative and computably modify the vector space operations to operate on least representatives. And, given any vector in V^{∞}, we can compute the least representative of its equivalence class in V, which may be the zero vector v_0. Let $L \subseteq V^{\infty}$ be the set of least representatives of equivalence classes in U, and observe that L is computable from U. Denote the least representative of $[v_j] \in V$ by $\mathrm{Lr}([v_j])$.

Build the graph G: Now let $N = (L \setminus \{v_0\}) \times \mathbb{N}$. We will define a countable graph $G = (N, E)$ that is computable from V. Let E consist of all pairs of the form $\{(v_n, t), (v_n, t+1)\}$ such that $v_n \in L \setminus \{v_0\}$ and $t \in \mathbb{N}$ and of the form $\{(v_n, t), (v_m, t)\}$ such that $[v_n] = \lambda_t [v_m]$ for vectors $v_n, v_m \in L \setminus \{v_0\}$ and scalar λ_t. Then, it follows that $(v_n, 0)$ and $(v_m, 0)$ lie in the same connected component of G if and only if $[v_n]$ and $[v_m]$ lie on the same line in V. The functional that computes this graph G can be composed with the functional that computes the vector space V to obtain a single functional that computes G uniformly from an oracle for f and g.

Apply PSG_W*:* Now suppose we apply PSG_W to G to obtain a nontrivial, proper saturated subgraph H, i.e., a nonempty subset of N that is equal to a union of connected components in G, which leaves out at least one connected component of G. We intend to use H to compute a separating set for f and g.

There is a bijection between connected components in H and lines through the origin in V, since $(v_t, 0)$ and $(v_s, 0)$ lie in the same connected component in $G \supseteq H$ if and only if $[v_t]$ and $[v_s]$ lie on the same line through the origin in V, and vertices (v_n, t) can be identified with $(v_n, 0)$. Since H is nonempty, we can search to find a (nonzero) $v_i \in L$ such that the line through $[v_i]$ in V (minus $[v_0]$) is contained in H, which happens if and only if $(v_i, 0) \in H$. Since H omits at least one connected component of G we can also search to find a nonzero $v_j \in L$ such that the line through $[v_j]$ in V is not contained in H, i.e., such that $(v_j, 0) \notin H$.

Compute separating set S from H: Now define $K = \{[v_r] \in V : (\mathrm{Lr}(v_r), 0) \in H\}$, and observe that K is computable from H. Now define the set $S = \{n : [e_{\phi(i,j,n)}] \in K\}$, where $\{e_r\}_{r \in \mathbb{N}}$ are the standard basis vectors for V^∞, and $\phi : \mathbb{N}^3 \to \mathbb{N}$ is the injective function defined in the construction in the proof of Theorem 3.2.6. Observe that S is computable from K and U, and hence from H and U. Specifically, we can compute the t such that $v_t = e_{\phi(a,b,n)}$, then check if $[v_t] \in K$.

Since $[v_i], [v_j]$ are both $\neq [0] = [v_0]$, it follows that $v_i, v_j \notin U$. Hence, by the construction of U it follows that $n \in \mathrm{range}(f)$ implies that there is a nonzero $\lambda \in \mathbb{Q}$ such that $e_{\phi(i,j,n)} - \lambda v_i \in U$, which implies that $[e_{\phi(i,j,n)}] = [\lambda v_i]$. Since $[v_i] \in K$, K is closed under nonzero scalar multiplication, and $\lambda[v_i] = [\lambda v_i]$ (true in any quotient vector space), it follows that $[\lambda v_i] = [e_{\phi(i,j,n)}] \in K$, and hence that $n \in S$. It follows that range$(f) \subseteq S$.

If $n \in$ range(g) then there is a nonzero $\lambda \in \mathbb{Q}$ such that $e_{\phi(i,j,n)} - \lambda v_j \in U$, which implies that $[e_{\phi(i,j,n)}] = [\lambda v_j]$. Then, if $n \in S$, we have $[\lambda v_j] = [e_{\phi(i,j,n)}] \in K$. Since K closed under nonzero scalar multiplication, this implies that $[v_j] \in K$, contradicting our assumption that $[v_j] \notin K$. It follows that range$(g) \cap S = \emptyset$. Hence, S is a separating set for f, g. We can verify that the second half of this proof defines a functional that computes S from f, g, and H. Hence, we have shown that SEP $\equiv_W$ WKL_W is Weihrauch reducible to PSG_W. ■

Hence, as a consequence of Theorem 1.2.6 we have:

Corollary 4.2.3. PSG_W *is not Weihrauch reducible to* WKL_W*.*

We showed above that the composition $\mathrm{WKL}_W \star \mathrm{C}_\mathbb{N}$ is an upper bound for PS_W. Here we show that it is also an upper bound for PSG_W.

Theorem 4.2.4. PSG_W *is Weihrauch reducible to* $\mathrm{WKL}_W \star \mathrm{C}_\mathbb{N}$*.*

Proof. This proof is essentially the same as the proof of $\mathrm{PS}_W \leq_W \mathrm{WKL}_0 \star \mathrm{C}_\mathbb{N}$. Let $G = (V, E)$ be a countable graph with at least two connected components. We first apply $\mathrm{C}_\mathbb{N}$ to find a pair (v_1, v_2) of non–path-connected vertices in G. Then we define a tree $T \subseteq \{0,1\}^{<\mathbb{N}}$ such that $\sigma \in T$ if and only if σ is the characteristic function of a subset $S \subseteq V$ that is consistent with the following requirements, only considering vertices with index at most $|\sigma|$ and paths of length at most $|\sigma|$:

1. $v_1 \in S$ and $v_2 \notin S$.

2. If $u \in S$ and u, w are path-connected, then $w \in S$.

Then a path through T will be the characteristic function for a connected component of G, which is certainly a proper subgraph of G. We can apply WKL_W to obtain such a path. ■

Combining the results of Corollary 4.2.3 and Theorem 4.2.4 gives us the following result.

Theorem 4.2.5. WKL_W *is not Weihrauch equivalent to* $\mathrm{WKL}_W \star \mathrm{C}_\mathbb{N}$*.*

Proof. It is straightforward to show that $\mathrm{WKL}_W \leq_W \mathrm{WKL}_W \star \mathrm{C}_\mathbb{N}$. If the opposite relation held, then it would follow from Theorem 4.2.4 that $\mathrm{PSG}_W \leq_W \mathrm{WKL}_W$, contradicting
Corollary 4.2.3. ■

CHAPTER 5

ADDITIONAL DEPENDENCE RESULTS

Equivalence of Vector Space Basis and Decomposition Principles

Combining Theorems 2.2.2 and 2.3.1 in this work with Theorems 3 and 12 of Hirst and Mummert [8] shows that finding a basis for an e-matroid and decomposing an e-matroid into 1-dimensional subspaces are equivalent problems from both the reverse mathematics and the Weihrauch reducibility perspectives. It thus seems plausible that the same equivalence holds when "e-matroid" is replaced with "countable vector space".

Friedman, Simpson, and Smith [5] prove that the principle "every vector space has a basis" is equivalent to ACA_0 over RCA_0. Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] prove that the principle "every vector space of dimension greater than one has a finite-dimensional nontrivial proper subspace" is equivalent to ACA_0 over RCA_0. Here we use the latter result to show that the principle "every vector space has a decomposition into 1-dimensional subspaces" is also equivalent to ACA_0 over RCA_0. The definition of a decomposition of a computable vector space into 1-dimensional subspaces is analogous to Definition 2.3.1.

Theorem 5.1.1. *The principle* D^V *that formalizes the statement "every vector space has a decomposition into 1-dimensional subspaces" is equivalent to* ACA_0 *over* RCA_0*.*

Proof. It is straightforward to show that ACA_0 implies D^{V}. To show that D^{V} implies ACA_0, we reason as follows. Let V be a countable vector space. Apply D^{V} to obtain a decomposition f of V into 1-dimensional subspaces. A 1-dimensional subspace S of V can now be obtained from f using Δ^0_1 comprehension by defining S to be the set of all $v \in V$ such that $f(v) = 0$ or $f(v) = 1$. As mentioned above, obtaining a 1-dimensional subspace from an arbitrary countable vector space requires ACA_0. Hence, D^{V} implies ACA_0. ■

We would like to obtain the parallel result to Theorem 5.1.1 in the Weihrauch reducibility setting, which would be that D_{V} is Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. To discuss one possible approach for obtaining this result, we define the Weihrauch reducibility version of the finite-dimensional subspace principle for vector spaces.

Definition 5.1.1. Let $\mathrm{PS}_W^{<\mathbb{N}}$ be the Weihrauch principle defined by the set of pairs (V, S), where V is a countable vector space of dimension greater than 1 and S is a finite-dimensional nontrivial proper subspace of V.

It follows from Theorem 2.1.1 that $\mathrm{D_V} \leq_W \widehat{\mathrm{LPO}}$. If we could show that $\widehat{\mathrm{LPO}} \leq_W \mathrm{PS}_W^{<\mathbb{N}}$, this would be one way to establish the lower bound $\widehat{\mathrm{LPO}} \leq_W \mathrm{D_V}$, since one can compute a 1-dimensional subspace from a 1-dimensional decomposition. We discuss the plausibility of the relationship $\widehat{\mathrm{LPO}} \leq_W \mathrm{PS}_W^{<\mathbb{N}}$ as a parallel to a computability result of Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] in Chapter 6.

CHAPTER 6

OPEN PROBLEMS AND FUTURE WORK

In this section we describe a few open problems related to our work, as well as an extension of some of the classification problems we have considered.

Classification of $\mathrm{PS}_W^{<\mathbb{N}}$ and L_W

In Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] it is shown that there is a computable vector space V of dimension greater than 1 such that the Turing degree of each finite-dimensional nontrivial proper subspace of V is at least $\emptyset'$. It is also shown that the reverse mathematics version of $\mathrm{PS}_W^{<\mathbb{N}}$, which says that "every countable vector space of dimension greater than 1 has a finite-dimensional nontrivial proper subspace", is equivalent to ACA_0 over RCA_0. It thus seems plausible that $\mathrm{PS}_W^{<\mathbb{N}}$ is Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. However, the proof given by Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [4] for the finite-dimensional case makes use of a nonuniform argument, and thus does not translate directly to the Weihrauch reducibility setting.

It follows by definition that $\mathrm{PS}_W^{<\mathbb{N}} \leq_{sW} \mathrm{L}_W$. Hence, if we established $\widehat{\mathrm{LPO}} \leq_W \mathrm{PS}_W^{<\mathbb{N}}$, then we would also obtain $\mathrm{L}_W \equiv_{sW} \mathrm{PS}_W^{<\mathbb{N}}$. One argument for the plausibility of this result is that the analogous result holds in the graph setting: the principle that takes a countable graph and produces a connected component and the principle that takes a countable graph and produces a finite-dimensional saturated subgraph are both Weihrauch equivalent to $\widehat{\mathrm{LPO}}$.

Classification of PS_W and PSG_W

We have established that both of PS_W and PSG_W are bounded above by $\mathrm{WKL}_W \star \mathrm{C}_\mathbb{N}$ and below by WKL_W. It remains to find an exact Weihrauch reducibility classification of these two principles. In particular, it remains to determine whether these principles are Weihrauch equivalent.

Subspaces and Decompositions of Structures with Bounded Dimension

Gura, Hirst, and Mummert [6] define the principle P_k, which takes as input a graph with exactly k connected components and returns a connected component, and the principle D_k, which decomposes a graph with exactly k connected components into its connected components. They show that these principles are equivalent to $\mathrm{C}_{\mathbb{N}}$. We are interested in the analogous principles in the vector space and matroid settings, which involve producing a subspace or a decomposition of a vector space or matroid with dimension k.

Hirst and Mummert [8] define principles that take as input a countable graph, matroid, or vector space and produce a basis. They show that each of these principles is strongly Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. They also define versions of these principles in which the input object comes with a finite upper bound on its dimension. They show that these bounded principles are each equivalent to a Weihrauch principle $\mathrm{C}^{\#}_{\max}$, and they show that the formalization of $\mathrm{C}^{\#}_{\max}$ is equivalent to Σ^0_2 induction in the reverse mathematics setting. In the same spirit, we are interested in versions of the subspace and decomposition principles for graphs, vector spaces, and matroids in which the input object comes with a finite upper bound on its dimension.

APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

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Office of Research Integrity

February 16, 2017

Sean Sovine
5271 W. Pea Ridge Road, Apt 17
Huntington, WV 25705

Dear Mr. Sovine:

This letter is in response to the submitted thesis abstract entitled *"Weihrauch Reducibility and Finite-Dimensional Subspaces."* After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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