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The combinatorics of modified Macdonald polynomials

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THE COMBINATORICS OF MODIFIED MACDONALD POLYNOMIALS

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APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Jacob Rodeheffer, affirm that the thesis, *The Combinatorics of Modified Macdonald Polynomials*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the editorial standards of our discipline and the Graduate College of Marshall University. With our signatures, we approve the manuscript for publication.

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABSTRACT

This paper is concerned with finding a combinatorial Schur expansion of the modified Macdonald polynomials. We will use the Robinson–Schensted–Knuth (RSK) algorithm to obtain the result for a limited set of partition shapes, and we will use Foata's bijection to extend the result to conjugate shapes. We will explore possibilities for modifying the RSK algorithm in such a way that it could be applied to obtain the result for more general shapes.

CHAPTER 1

INTRODUCTION

In algebraic combinatorics, we are concerned with the interplay between algebraic structures and combinatorics, where the combinatorics often involve permutations and tableaux. This paper focuses on proving a special case of Macdonald's conjecture, which deals with the Schur function expansion of the modified Macdonald polynomials. Macdonald's conjecture predicts that the coefficients in the Schur function expansion, which comes from the algebraic study of vector spaces and their bases, are the combinatorial generating function for certain statistics on tableaux.

The vector space under consideration in relation to the modified Macdonald polynomials is Λ^n, the set of symmetric polynomials that are homogeneous of degree n. This vector space has a number of known bases, including the Schur polynomials s_μ, the elementary symmetric polynomials e_μ, the monomial symmetric polynomials m_μ, the Hall–Littlewood polynomials, the Jack symmetric polynomials, and the zonal symmetrical polynomials [3]. Of particular interest is the Schur basis, which has both algebraic and combinatorial definitions, and which features prominently in the representation theory of the symmetric group [2].

The original Macdonald polynomials $P_\mu(X; q, t)$, for $X = \{x_1, x_2, \ldots\}$ and certain values of parameters q and t, specialize to several of these bases [3]. For instance: $P_\mu(X; q, q) = s_\mu$, $P_\mu(X; 1, t) = e_\mu$, and $P_\mu(X; q, 1) = m_\mu$. However, the original Macdonald polynomials are defined in a way that makes them difficult to interpret combinatorially, and this is the motivation behind the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ (see [9]). The modified Macdonald polynomials have a clear combinatorial definition, and can be transformed to the original Macdonald polynomials $P_\mu(X; q, t)$ by an operation only involving the indeterminates q and t [3]. Furthermore, the modified Macdonald polynmials are a basis for the vector space $\Lambda^n(q, t)$, the symmetric and degree-n-homogeneous subset of $\mathbb{Q}(q, t)[X]$, the ring of formal power series in $X = \{x_1, x_2, \ldots\}$ with coefficients in the field $\mathbb{Q}(q, t)$ (see [8]).

As it turns out, the modified Macdonald polynomials are themselves symmetric in X and homogeneous [8]; that is, $\tilde{H}_\mu(X; q, t) \in \Lambda^n$. This implies that the modified Macdonald polynomials have an expansion in the Schur polynomial basis. Macdonald's conjecture [9] states that the coefficient of the Schur polynomial s_λ in the Schur expansion of $\tilde{H}_\mu(X;q,t)$ is $\sum_{T\in\mathrm{SYT}(\lambda)} q^{\tilde{a}_\mu(T)}t^{\tilde{b}_\mu(T)}$, where $\mathrm{SYT}(\lambda)$ is a set of tableaux and $\tilde{a}_\mu$ and $\tilde{b}_\mu$ are statistics on tableaux. Macdonald's conjecture has been proven for particular cases of partition shape μ, including, for example, hook shapes and two-column shapes [3].

Loehr [7] has a proof for Macdonald's conjecture when μ is a vertical strip, and in Chapter 6 of this paper we adapt Loehr's method to prove the conjecture for horizontal strips. The horizontal strip case was also recently proven by Assaf using a different method [1]. Essential to our proof method is the bijection provided by the Robinson–Schensted–Knuth (RSK) algorithm, the properties of which we define and prove in Chapter 4, and also Foata's bijection, which is explored in Chapter 5.

Discovering a proof of Macdonald's conjecture for general μ remains as a goal for further research. In Chapter 7 we will examine certain approaches to this goal, including that of designing new tableau statistics and designing a modified RSK algorithm.

CHAPTER 2

BACKGROUND

Permutations can be viewed as words composed of unique letters, or alternatively as bijections mapping from a set to itself. They are an important building block of combinatorics and are fundamental to the definitions of symmetric functions and the RSK algorithm. Certain permutation statistics have useful properties in algebraic combinatorics. In particular, the inversion statistic and the major index statistic will be used in our proofs of Macdonald's conjecture in Chapter 6.

2.1 Permutations

Definition 2.1. A *word* of length n is an n-tuple $\pi = \pi_1\pi_2\ldots\pi_n$ where $\pi_1, \pi_2, \ldots, \pi_n \in \mathbb{Z}^+$. We call each π_i a *letter.*

Definition 2.2. A *permutation* of length n is a bijection $\pi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. A permutation of length n may be expressed as a word of length n, $\pi_1\pi_2\ldots\pi_n$, where $\pi(i) = \pi_i$. The set of all permutations of length n is denoted S_n.

Example 2.3. The word $\pi = 3241$ is a permutation $\pi \in S_4$ where $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 4$, and $\pi(4) = 1$.

Definition 2.4. An *inversion* in a word $\pi = \pi_1\pi_2\ldots\pi_n$ is an ordered pair (π_i, π_j) such that $i < j$ and $\pi_i > \pi_j$. The *inversion statistic* of a word π is defined

$$\operatorname{inv}(\pi) = |\{(\pi_i, \pi_j) : i < j, \pi_i > \pi_j\}|.$$

In other words, $\operatorname{inv}(\pi)$ is the number of inversions in π.

Example 2.5. In the permutation 235164, the inversions are $(2, 1)$, $(3, 1)$, $(5, 1)$, $(5, 4)$, and $(6, 4)$, and $\operatorname{inv}(235164) = 5$.

Definition 2.6. A *descent* of a word $\pi = \pi_1\pi_2\ldots\pi_n$ is an index $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$. The notation $\operatorname{Des}(\pi)$ denotes the descent set of π, that is, the set of all descents of π.

Definition 2.7. For $w \in S_n$, the *inverse descent set* IDes(w) is defined

$$\mathrm{IDes}(w) = \mathrm{Des}(w^{-1}).$$

Remark 2.8. Note that, because $x \in \mathrm{Des}(w^{-1})$ if and only if x appears later than $x+1$ in w, an equivalent expression for IDes(w) is

$$\mathrm{IDes}(w) = \{x : w^{-1}(x) > w^{-1}(x+1)\}.$$

Definition 2.9. The *major index statistic* of a word π is defined

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

In other words, maj(π) is the sum of the descents of π.

Example 2.10. For $\pi = 235164 \in S_6$, $\mathrm{Des}(\pi) = \{3, 5\}$ and $\mathrm{maj}(\pi) = 3 + 5 = 8$.

The following definition and theorem concern the formula for the generating function of the inversion statistic.

Definition 2.11. The notation $[n]_q$ denotes $1 + q + q^2 + \cdots + q^{n-1}$. The notation $[n]_q!$ denotes $[n]_q[n-1]_q[n-2]_q \ldots [1]_q$, and $[0]_q!$ is considered to be $[0]_q! = 1$.

Theorem 2.12 ([6])**.** *For any positive integer n,*

$$\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} = [n]_q!.$$

Proof. We proceed by induction. For $n = 1$ we have $S_n = \{1\}$, and note that $\mathrm{inv}(1) = 0$. So

$$\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} = q^0 = 1 = [1]_q!$$

and the base case holds. Next, assume $\sum_{\omega \in S_{n-1}} q^{\mathrm{inv}(\omega)} = [n-1]_q!$ for some n. For any $i \in \{1, 2, \ldots, n\}$, define $S_{n,i} = \{w \in S_n : w_i = n\}$. Note that S_n equals the disjoint union of

$S_{n,1}, S_{n,2}, \dots, S_{n,n}$, so

$$\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} = \sum_{\omega \in S_{n,1}} q^{\mathrm{inv}(\omega)} + \sum_{\omega \in S_{n,2}} q^{\mathrm{inv}(\omega)} + \cdots + \sum_{\omega \in S_{n,n}} q^{\mathrm{inv}(\omega)}.$$

For each i there exists a bijection $f_i \colon S_{n-1} \to S_{n,i}$ defined by inserting the label n in between π_i and π_{i+1} of $\pi \in S_{n-1}$. Therefore $\sum_{\omega \in S_{n,i}} q^{\mathrm{inv}(\omega)} = \sum_{\pi \in S_{n-1}} q^{\mathrm{inv}(f_i(\pi))}$. Since f_i preserves inversions of the form (π_j, π_k) and adds $n-i$ inversions of the form (n, π_k) for any $j < k$, we have

$$\mathrm{inv}(f_i(\pi)) = \mathrm{inv}(\pi) + n - i,$$

and

$$\sum_{\omega \in S_{n,i}} q^{\mathrm{inv}(\omega)} = \sum_{\pi \in S_{n-1}} q^{\mathrm{inv}(f_i(\pi))} = \sum_{\pi \in S_{n-1}} q^{\mathrm{inv}(\pi)} q^{n-i} = q^{n-i}[n-1]_q!.$$

Therefore,

$$\begin{aligned}\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} &= \sum_{\omega \in S_{n,1}} q^{\mathrm{inv}(\omega)} + \sum_{\omega \in S_{n,2}} q^{\mathrm{inv}(\omega)} + \cdots + \sum_{\omega \in S_{n,n}} q^{\mathrm{inv}(\omega)} \\ &= q^{n-1}[n-1]_q! + q^{n-2}[n-1]_q! + \cdots + q^{n-n}[n-1]_q! \\ &= [n]_q[n-1]_q! \\ &= [n]_q!.\end{aligned}$$

□

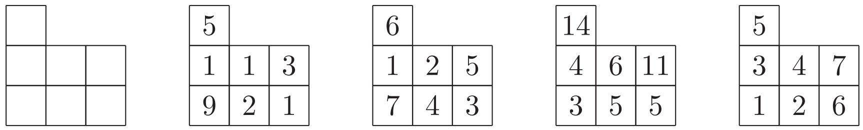
2.2 Fillings and Tableaux

We extend the idea of permutations and words to two-dimensional arrays known as fillings.

Definition 2.13. A *partition* λ of a positive integer n, denoted $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$.

Example 2.14. The partitions of 4 are $(4), (3,1), (2,2), (2,1,1)$, and $(1,1,1,1)$.

Definition 2.15. A *Ferrers diagram* of a partition $(\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ is an arrangement of n cells on a row-column grid such that, for all $i \in \{1, 2, \dots, k\}$, the i^{th} row has λ_i cells. The rows are left-justified, contain no gaps, and follow the French convention, that is, they are numbered



(a) (b) (c) (d) (e)

Figure 2.1: Ferrers Diagrams and Types of Fillings
For the partition $\lambda = (3, 3, 1)$, (a) is the Ferrers diagram of λ, (b) is a filling with shape λ, (c) is a standard filling with shape λ, (d) is a semistandard tableau with shape λ, and (e) is a standard tableau with shape λ.

from bottom to top, as seen in Fig. 2.1(a). A cell in a Ferrers diagram at row r and column c is specified (r, c).

Definition 2.16. A *filling* T of a partition λ is an assignment of a positive integer to each cell in the Ferrers diagram of λ, as seen in Fig. 2.1(b). The partition λ is called the *shape* of T. For a filling T, the notation $T(r, c)$ denotes the content of the cell (r, c) in T. The set of all fillings of shape λ is denoted $\mathcal{F}_\lambda$.

Definition 2.17. A *standard filling* of shape $\lambda \vdash n$ is a filling of shape λ whose entries are precisely $1, 2, \ldots, n$ in some order, as seen in Fig. 2.1(c). The set of all standard fillings of shape λ is denoted $\mathcal{SF}_\lambda$.

Definition 2.18. Given $\lambda \vdash n$, a *semistandard tableau* is a filling of shape λ such that up each column (from bottom to top) entries strictly increase, and along each row (from left to right) entries weakly increase, as seen in figure Fig. 2.1(d). The set of all semistandard tableaux of a partition λ is denoted $\mathrm{SSYT}(\lambda)$; a partial example of $\mathrm{SSYT}((2, 1))$ is seen in Fig. 2.2.

Definition 2.19. Given $\lambda \vdash n$, a *standard tableau* of shape λ is a standard filling of shape λ such that up each column and along each row, entries strictly increase, as seen in Fig. 2.1(e). The set of all standard tableaux of a partition λ is denoted $\mathrm{SYT}(\lambda)$.

Definition 2.20. The *content* of a filling T is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where k is the value of the greatest entry in T and, for all $i \in \{1, 2, \ldots, k\}$, α_i is the number of entries of the integer i in T. For example, the filling in Fig. 2.1(b) has content $(3, 1, 1, 0, 1, 0, 0, 0, 1)$.

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} 3 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 3 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 3 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} 3 \\ 2 & 2 \end{ytableau} \quad \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}$$

Figure 2.2: Semistandard tableaux of shape (2,1)
The eight semistandard tableaux above are the subset of SSYT((2,1)) with entries restricted to 1, 2, and 3. (The entire set SSYT((2,1)) has tableaux with entries from all the positive integer entries and is infinite.)

Definition 2.21. The *content monomial* x^T of a filling T with content $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is $x^T = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$. For example, the content monomial of the filling in Fig. 2.1(b) is $x_1^3 x_2 x_3 x_5 x_9$.

The following definitions and theorem allow us to partition sets of fillings based on underlying structural similarities.

Definition 2.22. For a filling T, the reading word, denoted $\mathrm{rw}(T)$, is the word formed by listing the entries of T in reading order (left to right across each row, starting with the top row).

Remark 2.23. Note that a filling is uniquely determined by its shape and reading word.

Definition 2.24. We define the process of *standardizing* a filling by a function stdz, which maps the set of all fillings of a partition λ onto the set of standard fillings of λ. For filling T, $\mathrm{stdz}(T)$ is defined by the following. Begin by giving $\mathrm{stdz}(T)$ the same Ferrers diagram as T. Next, scan the entries of T in reading order, looking for entries of 1. Then repeat looking for entries of 2, and then entries of 3, and so on, until all entries have been found. The order of finding the entries of T determines the entries of $\mathrm{stdz}(T)$: for the i^{th} entry found in the scan, assign the entry at the same cell in $\mathrm{stdz}(T)$ to be i.

For example, the standard filling in Fig. 2.1(c) is the standardization of the filling in Fig. 2.1(b).

Theorem 2.25. *If T is a semistandard tableau, then* $\mathrm{stdz}(T)$ *is a standard tableau.*

Proof. Let T be a semistandard tableau of shape $\lambda \vdash n$. Note that none of the entries in $\mathrm{stdz}(T)$ exceeds n.

Let (i, k) and (j, k) be two cells in column k in the diagram of λ with $i < j$, and let a, b be the entries of these cells respectively in T, and a', b' be the entries of these cells in $\mathrm{stdz}(T)$. Since T

has strictly increasing columns, $a < b$, and thus by definition of standardization, $a' < b'$ and $\mathrm{stdz}(T)$ has strictly increasing columns.

Let (l, m) and (l, n) be two cells in row l in the diagram of λ with $m < n$, and let c, d be the entries of these cells respectively in T, and c', d' be the entries of these cells in $\mathrm{stdz}(T)$. Since T has weakly increasing rows, $c \leq d$. If $c < d$, then $c' < d'$. If $c = d$, then $c' < d'$ since the scanning order of standardization proceeds from left to right along each row. Thus $c' < d'$, and $\mathrm{stdz}(T)$ has strictly increasing rows. Therefore, $\mathrm{stdz}(T)$ is a standard tableau. $\square$

Definition 2.26. For a standard filling T, the *inverse descent set* $\mathrm{IDes}(T)$ is the set of entries a in T that appear later in the reading word of T than $a + 1$. Notationally,

$$\mathrm{IDes}(T) = \{a \in \mathrm{rw}(T) : (\mathrm{rw}(T))^{-1}(a) > (\mathrm{rw}(T))^{-1}(a+1)\}.$$

Definition 2.27. The *descent set* of a standard tableau T is defined

$$\mathrm{Des}(T) = \{i : i \text{ appears in a lower row of } T \text{ than } i+1\}.$$

Note that for standard tableau T, $\mathrm{IDes}(T) = \mathrm{Des}(T)$.

Example 2.28. Let T be the standard filling in Fig. 2.1(c) and U be the standard tableau in Fig. 2.1(e). Then $\mathrm{IDes}(T) = \{3, 4, 5\}$ and $\mathrm{Des}(U) = \mathrm{IDes}(U) = \{2, 4, 6\}$.

2.3 Schur Functions

In this section we introduce the vector space of symmetric functions, and one of its bases, the Schur functions. Macdonald's conjecture concerns the Schur expansion of the modified Macdonald polynomials. Schur functions are defined using tableau content monomials.

Definition 2.29. A *symmetric function* is a function $f(x_1, x_2, \ldots)$ such that for all $n \in \mathbb{Z}^+$ and all $\sigma \in S_n$,

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots),$$

where we consider $\sigma(i) = i$ for $i > n$.

Example 2.30. The polynomial $f(x_1, x_2, x_3) = x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2$ is symmetric. For example, for the permutation $\sigma = 312$,

$$\begin{aligned} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) &= f(x_3, x_1, x_2) \\ &= x_3x_1^2 + x_3x_2^2 + x_1x_3^2 + x_1x_2^2 + x_2x_3^2 + x_2x_1^2 \\ &= x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2 \\ &= f(x_1, x_2, x_3). \end{aligned}$$

Definition 2.31. The *Schur function* of a partition $\lambda \vdash n$ is defined by

$$s_\lambda(x_1, x_2, \ldots) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

Example 2.32. Observing the subset of SSYT((2, 1)) in Fig. 2.2, we compute the first eight terms of $s_{(2,1)}(x_1, x_2, \ldots)$ to be

$$x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Theorem 2.33 ([6])**.** *All Schur functions are symmetric.*

Proof. Let $\lambda \vdash n$. To show $s_\lambda(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = s_\lambda(x_1, x_2, \ldots)$ for every $\sigma \in S_n$ it is sufficient to show that

$$s_\lambda(x_{\pi(1)}, x_{\pi(2)}, \ldots) = s_\lambda(x_1, x_2, \ldots) \tag{2.1}$$

for every adjacent transposition $\pi = (i, i + 1)$, since every permutation may be written as a composition of adjacent transpositions.

Let $\pi = (i, i + 1)$ be an adjacent transposition with $1 \le i \le n - 1$. Since $s_\lambda(x_1, x_2, \ldots)$ is the sum of all the content monomials of SSYT(λ), to show (2.1) it is sufficient to show that every content monomial $x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_i^{\alpha_i}x_{i+1}^{\alpha_{i+1}} \ldots = x^T$ with $T \in \mathrm{SSYT}(\lambda)$ is uniquely paired with another monomial $x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_{i+1}^{\alpha_i}x_i^{\alpha_{i+1}} \ldots = x^U$, $U \in \mathrm{SSYT}(\lambda)$. We will establish this with an involution

$f\colon \mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda)$.

Let $T \in \mathrm{SSYT}(\lambda)$. Define a "fixed pair" of T to be a pair of cells (j,k), $(j+1,k)$ with respective entries i and $i+1$. Any is or $i+1$s that do not make up a fixed pair are "free". Let r be the index of a row in T. Let a be the number of free is in row r of T and b be the number of free $i+1$s in row r of T. Then row r of $f(T)$ is the same as row r of T but with the a free is and b free $(i+1)$s replaced by b is and a $(i+1)$s, in increasing order. Note that this will produce the desired relationship between the content monomials of T and $f(T)$. An example computation of f is seen in Fig. 2.3.

Observe that in row r of T the is and $(i+1)$s follow the relative left-to-right order of: fixed is, free is, free $(i+1)$s, fixed $(i+1)$s. A fixed i cannot occur after a free i since there is an $i+1$ in row $r+1$ above the fixed i and the row $r+1$ is weakly increasing, and a fixed $i+1$ cannot occur before a free $i+1$ since there is an i in row $r-1$ below the fixed $i+1$ and row $r-1$ is weakly increasing. This order of is and $(i+1)$s in row r of T implies the same order in row r of $f(T)$, and thus $f(T)$ has weakly increasing rows.

Also, $f(T)$ has strictly increasing columns: let e be an entry in T. Suppose $e = i$. If the entry above e is $i+1$ in T, then e is part of a fixed pair and its neighbors above and below do not change in $f(T)$. If the entry above e is greater than $i+1$, then e might or might not change to $i+1$ in $f(T)$ but regardless its neighbors above and below will not change and will not break the strict column increasing rule with e. Similar reasoning applies if $e = i+1$; and if $e \neq i, e \neq i+1$ then either e has neighbors above and below that do not change in $f(T)$ or else they change in such a way so as to still maintain the strict column increasing rule with e. Thus, $f(T) \in \mathrm{SSYT}(\lambda)$.

Finally, observe that f is its own inverse. Thus f is an involution $f\colon \mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda)$ with the property that if $T \in \mathrm{SSYT}(\lambda)$ has $x^T = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \dots$ then $x^{f(T)} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{i+1}^{\alpha_i} x_i^{\alpha_{i+1}} \dots$. Thus we have shown (1), and $s_\lambda(x_1, x_2, \dots)$ is symmetric. $\square$

Definition 2.34. A formal power series $f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} c_i x_{i,1}^{p_{i,1}} x_{i,2}^{p_{i,2}} \dots x_{i,k_i}^{p_{i,k_i}}$ is *homogeneous* with degree n if for all $i \in \{1, 2, \dots\}$, $\sum_{j=1}^{k_i} p_{i,j} = n$.

Definition 2.35. The set of all symmetric, homogeneous of degree n formal power series in $X = \{x_1, x_2, \dots\}$ is denoted Λ^n.

T

4	4	5									
2	3	3	3	4	4						
1	1	2	2	2	3	3	4	4	4	4	6

$f(T)$

3	4	5									
2	3	3	4	4	4						
1	1	2	2	2	3	3	3	3	3	4	6

Figure 2.3: Computation of f for proving symmetry of s_λ
Let $i = 3$. For computing f, we consider all entries of 3 and 4 in the semistandard tableau T. There is a fixed pair at $(2,2),(3,2)$ and another at $(1,6),(2,6)$. In row 3 of T, there are no free 3s and one free 4; thus row 3 of $f(T)$ has one free 3 and no free 4s. Row 2 of T has two free 3s and one free 4, so row 2 of $f(T)$ has one free 3 and two free 4s. Row 1 of T has one free 3 and four free 4s, so row 1 of $f(T)$ has four free 3s and one free 4.

Example 2.36. For $n = 2$,

$$x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_2x_3 + x_2x_4 + x_2x_5 + \cdots + \cdots \in \Lambda^2$$

and

$$x_1^2 + x_2^2 + x_3^2 + \cdots \in \Lambda^2.$$

Theorem 2.37 ([6])**.** *For any positive integer n, Λ^n is a vector space with $\{s_\lambda : \lambda \vdash n\}$ as a basis.*

The vector space of symmetric functions is a subspace of the vector space of quasisymmetric functions. Consequently, any symmetric function can be written as a linear combination of the elements of a quasisymmetric function basis. We give such an expansion of the Schur functions.

Definition 2.38. A *quasisymmetric* function is a formal power series $p(x_1, x_2, \ldots)$ such that for all compositions $\alpha = (\alpha_1, \ldots, \alpha_s)$ and for all integers $1 \leq i_1 < i_2 < \ldots < i_s$, the coefficient of $\prod_{j=1}^s x_{i_j}^{\alpha_j}$ in p equals the coefficient of $\prod_{j=1}^s x_j^{\alpha_j}$ in p.

Example 2.39. The polynomial $p(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3$ is quasisymmetric. For example, choosing $\alpha = (2,1)$ and $i_1 = 2$, $i_2 = 3$, the coefficient of $\prod_{j=1}^s x_{i_j}^{\alpha_j} = x_2^2x_3$ is 1 and the coefficient of $\prod_{j=1}^s x_j^{\alpha_j} = x_1^2x_2$ is 1.

Definition 2.40 ([3])**.** For a positive integer n and set $I \subseteq \{1, 2, \ldots, n-1\}$, the *fundamental* *quasisymmetric function* $F_{n,I}(x_1, x_2, \ldots)$ is defined by

$$F_{n,I}(x_1, x_2, \ldots) = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n, \\ \text{if } j \in I \text{ then } i_j \neq i_{j+1}}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

Example 2.41. Let $n = 3$ and $I = \{1\}$. Then the first ten terms of $F_{n,I}(x_1, x_2, \ldots)$ are

$$x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3^2 + x_1 x_3 x_4 + x_1 x_4^2 + x_2 x_3^2 + x_2 x_3 x_4 + x_2 x_4^2 + x_3 x_4^2.$$

Theorem 2.42 ([3])**.** *For any partition $\lambda \vdash n$ for any positive integer n,*

$$s_\lambda = \sum_{U \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(U)}.$$

Proof. Let $\lambda \vdash n$. Observe that $\mathrm{SSYT}(\lambda)$ may be partitioned by standardization class,

$$s_\lambda = \sum_{U \in \mathrm{SYT}(\lambda)} A_U,$$

where

$$A_U = \sum_{\substack{T \in \mathrm{SSYT}(\lambda) \\ \mathrm{stdz}(T) = U}} x^T.$$

Let $U \in \mathrm{SYT}(\lambda)$. Note that from Definition 2.40,

$$F_{n,\mathrm{Des}(U)} = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ \text{if } j \in \mathrm{Des}(U) \text{ then } i_j \neq i_{j+1}}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

We will show that

$$A_U = F_{n,\mathrm{Des}(U)}$$

by showing the two polynomials to have the same set of terms. Note that neither polynomial has any repeat summands: for A_U, two distinct tableaux that standardize to the same tableau cannot have the same content, and for $F_{n,\mathrm{Des}(U)}$, distinct choices of $i_1 \le i_2 \le \ldots \le i_n$ will always result in distinct monomials.

Let $x_{i_1}x_{i_2}\ldots x_{i_n}$ be a term in A_U. Let $j \in \mathrm{Des}(U)$. Then $j+1$ is in a row above j in U. This implies $i_{j+1} \neq i_j$, since the only case in which the scanning order of standardization skips to a higher row is when a new letter is being scanned for. Thus $x_{i_1}x_{i_2}\ldots x_{i_n}$ is a term in $F_{n,\mathrm{Des}(U)}$.

Let $x_{i_1}x_{i_2}\ldots x_{i_n}$ be a term in $F_{n,\mathrm{Des}(U)}$. Construct a new tableau T by replacing each entry j in U with i_j. Observe that $x^T = x_{i_1}x_{i_2}\ldots x_{i_n}$ and $\mathrm{stdz}(T) = U$. Also, T has weakly increasing rows: for i_j and i_k in the same row of T with i_j to the left of i_k, U's strictly increasing rows guarantee that $j < k$, and thus $i_j \leq i_k$. Finally, we will show that T has strictly increasing columns. Let i_j, i_k be in the same column of T with $\mathrm{row}(i_j) < \mathrm{row}(i_k)$ (implying $j < k$ and $\mathrm{row}(j) < \mathrm{row}(k)$ in U). Since $i_j \leq i_{j+1} \leq \ldots \leq i_{k-1} \leq i_k$, to show $i_j < i_k$ it is sufficient to show that some $j, j+1, \ldots, k-1$ is an element of $\mathrm{Des}(U)$. Observe that $j \notin \mathrm{Des}(U)$ would imply $\mathrm{row}(j) \geq \mathrm{row}(j+1)$ in U. Similarly, $j, j+1, \ldots, k-1 \notin \mathrm{Des}(U)$ would imply $\mathrm{row}(j) \geq \mathrm{row}(j+1) \geq \ldots \geq \mathrm{row}(k)$ in U, which contradicts $\mathrm{row}(j) < \mathrm{row}(k)$. Thus $i_j < i_k$ and T has strictly increasing columns. Thus, $T \in \mathrm{SSYT}(\lambda)$ and $x_{i_1}x_{i_2}\ldots x_{i_n} = x^T$ is a term in A_U.

We have shown

$$A_U = F_{n,\mathrm{Des}(U)},$$

and thus

$$s_\lambda = \sum_{U \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(U)}. \qquad \square$$

CHAPTER 3

MACDONALD POLYNOMIALS

At this point we have sufficient foundation to define the modified Macdonald polynomials and state the Macdonald Conjecture. The modified Macdonald polynomials, generalizing Schur polynomials, are defined using the content monomials of fillings as well as two statistics on tableau. The Macdonald Conjecture states that the modified Macdonald polynomials have an expansion in Schur polynomials such that the coefficients of the expansion are determined by particular tableau statistics.

Definition 3.1. An *inversion triple* in a standard filling T is a triple

$$((r_1, c_1), (r_1 + 1, c_1), (r_1 + 1, c_2))$$

of cells in T such that $c_2 > c_1$ and the entries of the cells are counterclockwise increasing: for

$$x = T(r_1, c_1)$$
$$y = T(r_1 + 1, c_1)$$
$$z = T(r_1 + 1, c_2)$$

it is the case that

$$z < y < x, \text{ or } x < z < y, \text{ or } y < x < z.$$

Note that we allow the case $r_1 = 0$; by convention, $T(0, c_1) = \infty$ for all c_1. We also define the set of inversion triples of a filling of shape μ,

$$\operatorname{Inv}_\mu(T) = \{(x, y, z) : (x, y, z) \text{ is an inversion triple in } \operatorname{stdz}(T)\}.$$

Definition 3.2. For $\mu \vdash n$, inv_μ is a statistic on fillings of μ defined by

$$\operatorname{inv}_\mu(T) = |\operatorname{Inv}_\mu(T)|.$$

Note that T is standardized in the computation of $\mathrm{Inv}_\mu(T)$. For $w \in S_n$, we also define the usage

$$\mathrm{inv}_\mu(w) = \mathrm{inv}_\mu(U)$$

where U is the standard filling with shape μ and reading word w.

Definition 3.3. The *column word* of a column c of a filling is the word made by listing the entries of c in order from top to bottom.

Definition 3.4. For $\mu \vdash n$, maj_μ is a statistic on fillings of μ defined by

$$\mathrm{maj}_\mu(T) = \sum_{\omega \in C} \mathrm{maj}(\omega),$$

where C is the set of column words of $\mathrm{stdz}(T)$ and $\mathrm{maj}(\omega)$ is the major index permutation statistic as in Definition 2.9. For $w \in S_n$, we also define the usage

$$\mathrm{maj}_\mu(w) = \mathrm{maj}_\mu(U),$$

where U is the standard filling with shape μ and reading word w.

Definition 3.5 ([3])**.** For $\mu \vdash n$, the *modified Macdonald polynomial* $\tilde{H}_\mu$ is defined

$$\tilde{H}_\mu(X; q, t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} x^T,$$

where $X = \{x_1, x_2, \ldots\}$ and x^T is the content monomial of filling T (recall Definition 2.21).

Theorem 3.6 ([3])**.** *For all $\mu \vdash n$, the modified Macdonald polynomial $\tilde{H}_\mu(X; q, t)$ is symmetric in X.*

Corollary 3.7 ([3])**.** *For all $\mu \vdash n$, the modified Macdonald polynomial $\tilde{H}_\mu(X; q, t)$ has an expansion in Schur polynomials,*

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda \vdash n} K_\lambda(q, t) s_\lambda$$

for some q, t-dependent coefficients $K_\lambda(q, t)$.

Proof. The result follows immediately from Theorem 3.6 and Theorem 2.37. $\square$

Macdonald's conjecture states that the coefficients $K_\lambda(q, t)$ of the Schur expansion of the modified Macdonald polynomials are generating functions for some standard tableau statistics $\tilde{a}_\mu$ and $\tilde{b}_\mu$.

Conjecture 3.8 ([9])**.** *The Macdonald Conjecture for the Schur expansion of modified Macdonald polynomials states that, for all $\mu \vdash n$, for some tableau statistics $\tilde{a}_\mu$ and $\tilde{b}_\mu$,*

$$K_\lambda(q, t) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_\mu(T)} t^{\tilde{b}_\mu(T)}$$

where $K_\lambda(q, t)$ is the $K_\lambda(q, t)$ in

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda \vdash n} K_\lambda(q, t) s_\lambda$$

as obtained from Corollary 3.7.

In the remainder of this thesis we lay the foundation for proving limited cases of the Macdonald Conjecture in Chapter 6: Chapter 4 is devoted to the RSK algorithm and Chapter 5 contains Foata's bijection.

CHAPTER 4

THE RSK ALGORITHM

The Robinson-Schensted-Knuth (RSK) algorithm establishes a bijection between the set of permutations of length n and the set of all same-shape pairs of standard tableaux with n cells. This bijection provides us with Theorem 4.9, a combinatorial result concerning pairs of standard tableaux. The RSK algorithm gives equivalence classes on permutations known as the Knuth relations [5], and the algorithm is used in the proof of the Littlewood-Richardson Rule [11], which deals with the products of Schur polynomials. It also has been used in computing the length of longest increasing subsequences [12] and in representations of the symmetric group [10].

In the computation of the algorithm, the letters of the input permutation are considered one at a time in left-to-right order. The output pair of tableaux are iteratively constructed one cell at a time, depending on the current letter of the permutation. The bijection provided by the RSK algorithm is used in the proof of the Macdonald Conjecture in Chapter 6.

Definition 4.1. A *partial tableau* is a filling with distinct entries in which the entries within each row (strictly) increase from left to right and the entries within each column (strictly) increase from bottom to top.

Our definitions of row insertion and the RSK algorithm follow Loehr [6].

Definition 4.2. We will define *row insertion*, a function denoted by r. Let T be a partial tableau. Let x be a positive integer not in T. Then $r(T, x)$ is an ordered pair $r(T, x) = (\tilde{T}, \tilde{x})$ where $\tilde{T}$ is a partial tableau and $\tilde{x}$ is a positive integer. The tableau $\tilde{T}$ is formed by starting with T and performing an iterative row operation, defined as follows.

1. Set $y = x$ and $i = 1$.

2. If y is greater than every entry in row i, append y to the end of row i and end the algorithm. The resulting tableau is $\tilde{T}$, and $\tilde{x} = y$.

3. Otherwise, let z be the least entry in row i greater than y. Replace z with y in the tableau.

$$
\begin{array}{ccccc}
T & \text{iteration } 0 & \text{iteration } 1 & \text{iteration } 2 & \tilde{T} = \text{iteration } 3 \\
\begin{array}{|c|c|c|}\hline 9 \\ \cline{1-1} 8 \\ \hline 3 & 10 & 12 \\ \hline 1 & 4 & 5 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 9 \\ \cline{1-1} 8 \\ \hline 3 & 10 & 12 \\ \hline 1 & 4 & 5 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 9 \\ \cline{1-1} 8 \\ \hline 3 & 10 & 12 \\ \hline 1 & \underline{2} & 5 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 9 \\ \cline{1-1} 8 \\ \hline 3 & \underline{4} & 12 \\ \hline 1 & \underline{2} & 5 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 9 \\ \cline{1-1} 8 & \underline{10} \\ \hline 3 & \underline{4} & 12 \\ \hline 1 & \underline{2} & 5 \\ \hline \end{array} \\
x = 2 & y = 2 & y = 4 & y = 10 & \tilde{x} = 10
\end{array}
$$

Figure 4.1: Row insertion of 2 into a partial tableau T
Row insertion operates sequentially on the rows of a partial tableau, starting from the bottom. Each iteration takes the y from the previous row, finds the least $z > y$ in the current row, and swaps y and z.

4. Reset y to $y = z$, increment $i = i + 1$, and go to step 2.

Example 4.3. The computation of $r(T, 2)$ for partial tableau T is represented in Fig. 4.1. Beginning with $y = 2$ we update row 1 and obtain iteration 1, resetting $y = 4$. Then we update row 2 to obtain iteration 2, resetting $y = 10$. Finally, in row 3 we reach the terminal iteration 3, y is appended to the end of the row and we have $r(T, 2) = (\tilde{T}, 10)$.

Remark 4.4. Note that the row insertion function r has an inverse r^{-1}, performed by unraveling the iterative row-wise operations used to compute r. Let $\tilde{T}$ be a partial tableau and $\tilde{x}$ an entry of T. Then $(T, x) = r^{-1}(\tilde{T}, \tilde{x})$ is computed by the process:

1. Let $i + 1$ be the row of $\tilde{x}$ in $\tilde{T}$. Remove $\tilde{x}$ from $\tilde{T}$. Set $y = \tilde{x}$.

2. If $i = 0$ then terminate the process, T is complete and $x = y$.

3. Otherwise, let z be the greatest entry in row i less than y, and replace z by y in the tableau.

4. Reset $y = z$ and $i = i - 1$, and go to step 2.

The following theorem lays the foundation for computing the RSK algorithm, which relies on repeated applications of row insertion.

Theorem 4.5 ([6]). *Given T, a partial tableau, and x, a positive integer not in T, and $(\tilde{T}, \tilde{x}) = r(T, x)$, it follows that $\tilde{T}$ is a partial tableau.*

Proof. Let T be a partial tableau and x a positive integer not in T. Let $(\tilde{T}, \tilde{x}) = r(T, x)$. Observe that r does not change the content of T other than adding x. Thus $\tilde{T}$ has distinct entries. $\tilde{T}$ has strictly increasing rows because $\tilde{T}$ differs from T only by some sequence of row-wise operations, each of which by definition preserves the strictly-increasing-row property of the row it acts on.

To prove that $\tilde{T}$ has strictly increasing columns, let b and a be any two entries in the same column of $\tilde{T}$ where b is in the row above a. Let d and c be the original entries (in T) of the cells now (in $\tilde{T}$) occupied by b and a, respectively. Suppose that $b = d$. Then $b > a$, since $d > c$ and $c \geq a$.

Suppose that $b \neq d$. So $r(T, x)$ replaced d in its row with b, meaning b had come from the same row as c and had itself been replaced by some $e < b$. If $e = a$, then $b > a$. If $e \neq a$, then $e > a$, since $e < a$ would imply b had started to the left of c and thus would have replaced an entry further to the left than d. In every case, $b > a$. Thus $\tilde{T}$ has strictly increasing columns.

Proving that $\tilde{T}$ is a filling requires showing that the rows of $\tilde{T}$ follow the shape of a partition, that is, their sequence of lengths is weakly decreasing. Suppose for contradiction that the sequence of lengths of the rows of $\tilde{T}$ is not weakly decreasing. Then there exists a row $\tilde{t}$ of $\tilde{T}$ that is longer than its previous row, $\tilde{s}$, that is, $\text{length}(\tilde{t}) > \text{length}(\tilde{s})$. Let t and s be the rows in T corresponding to $\tilde{t}$ and $\tilde{s}$ in $\tilde{T}$. Since T is a filling, $\text{length}(t) \leq \text{length}(s)$, and since the only way that $r(T, x)$ can change the lengths of t or s is by an increase of 1, it must be that $\text{length}(s) = \text{length}(t) = \text{length}(\tilde{s}) = \text{length}(\tilde{t}) - 1$. In other words, there must have been an entry of s appended to t to form $\tilde{t}$. Let z be this entry of s. Since z was added to the end of t, it must have been greater than every element in t. But this is impossible: since s and t have the same length, z must have been in the same column as one of t's elements, and since T is a filling it follows the strictly increasing column rule, so z is less than the element in t located in the same column. Thus the sequence of lengths of rows in $\tilde{T}$ is weakly decreasing and $\tilde{T}$ is a filling. We have shown that $\tilde{T}$ is a partial tableau. □

Definition 4.6. The *RSK algorithm* is a function $\text{RSK}\colon S_n \to \{(P, Q) : P, Q \in \text{SYT}(\lambda), \lambda \vdash n\}$. Given input $\pi = \pi_1\pi_2 \ldots \pi_n \in S_n$ then $\text{RSK}(\pi) = (P, Q)$, defined by the following.

1. Let P_1 be the filling that is composed of a single entry π_1. Note that P_1 is a partial tableau.

i	π_i	P_i	Q_i
1	2	$\begin{array}{\|c\|} \hline 2 \\ \hline \end{array}$	$\begin{array}{\|c\|} \hline 1 \\ \hline \end{array}$
2	5	$\begin{array}{\|c\|c\|} \hline 2 & 5 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|} \hline 1 & 2 \\ \hline \end{array}$
3	4	$\begin{array}{ll} 5 & \\ 2 & 4 \end{array}$	$\begin{array}{ll} 3 & \\ 1 & 2 \end{array}$
4	1	$\begin{array}{ll} 5 & \\ 2 & \\ 1 & 4 \end{array}$	$\begin{array}{ll} 4 & \\ 3 & \\ 1 & 2 \end{array}$
5	3	$\begin{array}{ll} 5 & \\ 2 & 4 \\ 1 & 3 \end{array}$	$\begin{array}{ll} 4 & \\ 3 & 5 \\ 1 & 2 \end{array}$

Figure 4.2: Computation of RSK(25413)
To obtain P_i, perform the row insertion $(P_i, \tilde{x}) = r(P_{i-1}, \pi_i)$. To obtain Q_i, add a cell containing i to Q_{i-1} in same cell location as $\tilde{x}$ in P_i. Then $\mathrm{RSK}(\pi) = (P_n, Q_n)$.

2. Let Q_1 be the filling that is composed of a single entry 1.

3. Iterating from $i = 2$ to $i = n$, define P_i by $(P_i, \tilde{x}_i) = r(P_{i-1}, \pi_i)$, and define Q_i as a copy of Q_{i-1} but with the additional placement of a new entry i in the same cell location as π_i in P_i.

4. Set $P = P_n$ and $Q = Q_n$.

For example, Fig. 4.2 displays the iterative computation of $(P, Q) = \mathrm{RSK}(\pi)$ when $\pi = 25413$.

The results in the remainder of this chapter establish properties of the RSK algorithm that will be used in our proof of Macdonald's conjecture in Chapter 6.

Lemma 4.7 ([6])**.** *If* $(P, Q) = \mathrm{RSK}(\pi)$ *for some* $\pi \in S_n$*, then* $P \in \mathrm{SYT}(\lambda)$ *and* $Q \in \mathrm{SYT}(\lambda)$ *for some* $\lambda \vdash n$*; in other words,*

$$\mathrm{RSK}\colon S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}.$$

Proof. Observe that since $\pi_1 \pi_2 \dots \pi_n$ is a rearrangement of $(1, 2, \dots, n)$ and since the function r_{π_i}

has the effect of adding π_i as an entry to P_{i-1}, it follows that the entries of P are precisely $1, 2, \ldots, n$ in some order; and, by definition of r_{π_i}, each P_i is a filling with strictly increasing rows and columns. Thus, $P \in \mathrm{SYT}(\lambda)$ for some $\lambda \vdash n$.

We will prove that $Q \in \mathrm{SYT}(\lambda)$. Note that the entries of Q are placed in counting order: $(1, 2, \ldots, n)$. Thus Q has distinct entries. Since Q has the same shape as P by construction, Q is a (standard) filling. Q has strictly increasing rows because the method of constructing Q's shape is the same as that of P, meaning that no row entry gets placed unless it is the start of a row or it follows an earlier (and thus, for Q, lesser) entry. Similarly, Q has strictly increasing columns because every column entry either is the start of a column or follows an earlier (and thus lesser) entry. Thus, $Q \in \mathrm{SYT}(\lambda)$. $\square$

Theorem 4.8 ([6])**.** *The RSK algorithm* $\mathrm{RSK}\colon S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}$ *is a bijection.*

Proof. RSK has an inverse $\mathrm{RSK}^{-1}\colon \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\} \to S_n$ obtained by iteratively applying r^{-1}. Let $\pi \in S_n$ and $(P, Q) = \mathrm{RSK}(\pi)$. Set $P_n = P$, $Q_n = Q$. Iterating from $i = n$ to $i = 1$: let x_i be the entry of P_i at the same cell location as i in Q_i; then perform the computation $(P_{i-1}, \pi_i) = r^{-1}(P_i, x_i)$ and make Q_{i-1} by removing i from Q_i. Thus we have that the function $\mathrm{RSK}\colon S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}$ is a bijection. For example, Fig. 4.3 shows that for the given standard tableaux P, Q, $\mathrm{RSK}^{-1}((P, Q)) = 23514$. $\square$

The following theorem is a combinatorial result of the RSK bijection concerning pairs of standard tableaux of the same shape.

Theorem 4.9 ([6])**.** *For any positive integer n,*

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n!,$$

where $f^{\lambda} = |\,\mathrm{SYT}(\lambda)|$.

i	P_{i-1}	Q_{i-1}	π_i
6	$\begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}$	$\begin{array}{ccc} 4 & 5 & \\ 1 & 2 & 3 \end{array}$	-
5	$\begin{array}{ccc} 2 & & \\ 1 & 3 & 5 \end{array}$	$\begin{array}{ccc} 4 & & \\ 1 & 2 & 3 \end{array}$	4
4	$\begin{array}{ccc} 2 & 3 & 5 \end{array}$	$\begin{array}{ccc} 1 & 2 & 3 \end{array}$	1
3	$\begin{array}{cc} 2 & 3 \end{array}$	$\begin{array}{cc} 1 & 2 \end{array}$	5
2	2	1	3
1	$\emptyset$	$\emptyset$	2

Figure 4.3: Computation of $\mathrm{RSK}^{-1}((P,Q))$
The top row of the figure (corresponding to $i = 6$) contains the input P and Q tableaux. Given P_i for some i, to obtain P_{i-1} and π_i perform $(P_{i-1}, \pi_i) = r^{-1}(P_i, \tilde{x})$ where $\tilde{x}$ is the entry in P_i at the same location as i in Q_i. To obtain Q_{i-1}, remove the cell containing i from Q_i.

Proof. Since RSK is a bijection, we have that $|S_n| = |\{(P,Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}|$. Thus

$$\begin{aligned} n! &= \sum_{\lambda \vdash n} |\{(P,Q) : P, Q \in \mathrm{SYT}(\lambda)\}| \\ n! &= \sum_{\lambda \vdash n} |\{P : P \in \mathrm{SYT}(\lambda)\}|^2 \\ n! &= \sum_{\lambda \vdash n} (f^\lambda)^2. \end{aligned}$$ □

The following results concern properties of the RSK bijection that will be needed in our proof of the Macdonald Conjecture in Chapter 6.

Definition 4.10. Let T be a partial tableau and x a positive integer not in T. Then the *bump sequence* of x in T is the maximal sequence $(y_1, \ldots, y_k)$ such that $y_1 = x$ and for $i \geq 2$, y_i is the entry in row $i-1$ of T that gets replaced by the row insertion $r(T, x)$. For example, in Fig. 4.1, the bump sequence of 2 in the tableau T is $(2, 4, 10)$.

The RSK algorithm relates the tableau descent set with the permutation descent set and inverse descent set (recall Definitions 2.27, 2.6, and 2.7).

Lemma 4.11 ([3]). *If* $(P, Q) = \mathrm{RSK}(w)$ *for some* $w \in S_n$*, then* $\mathrm{Des}(Q) = \mathrm{Des}(w)$.

Proof. Let $w \in S_n$ and $(P, Q) = \mathrm{RSK}(w)$. Let $x \in \mathrm{Des}(w)$. Let $(y_1, \ldots, y_k)$ be the bump sequence of w_x in P_{x-1}. Note that for all i, $y_i < y_{i+1}$. From $x \in \mathrm{Des}(w)$ we have that $w_{x+1} < w_x$, and thus it can be shown by induction along the rows of P_x that the bump sequence of w_{x+1} in P_x has length $l > k$. Observe that k is the row of the entry x in Q and l is the row of the entry $x+1$ in Q. Thus $x \in \mathrm{Des}(Q)$ and we have $\mathrm{Des}(w) \subseteq \mathrm{Des}(Q)$.

Let $x \in \mathrm{Des}(Q)$. Then $k < l$, where k is the length of the bump sequence $(a_1, \ldots, a_k)$ of w_x in P_{x-1} and l is the length of the bump sequence $(b_1, \ldots, b_l)$ of w_{x+1} in P_x. Since a_k is at the end of its row in P_x and $l > k$, it follows that $b_k < a_k$. By induction it can be shown that $b_1 < a_1$. Since $b_1 = w_{x+1}$ and $a_1 = w_x$, we have that $x \in \mathrm{Des}(w)$ and $\mathrm{Des}(Q) \subseteq \mathrm{Des}(w)$. Thus
$\mathrm{Des}(Q) = \mathrm{Des}(w)$. □

Lemma 4.12 ([11]). *For all* $w \in S_n$*, if* $\mathrm{RSK}(w) = (P, Q)$ *and* $\mathrm{RSK}(w^{-1}) = (\tilde{P}, \tilde{Q})$*, then* $P = \tilde{Q}$.

Corollary 4.13. *For all* $w \in S_n$*, if* $\mathrm{RSK}(w) = (P, Q)$*, then* $\mathrm{Des}(P) = \mathrm{IDes}(w)$.

Proof. Let $w \in S_n$, $(P, Q) = \mathrm{RSK}(w)$, and $(\tilde{P}, \tilde{Q}) = \mathrm{RSK}(w^{-1})$. From Lemma 4.11 we know $\mathrm{Des}(\tilde{Q}) = \mathrm{Des}(w^{-1})$. By Definition 2.7 we have $\mathrm{Des}(w^{-1}) = \mathrm{IDes}(w)$ and from Lemma 4.12 we have $P = \tilde{Q}$. Thus $\mathrm{Des}(P) = \mathrm{IDes}(w)$. □

CHAPTER 5

FOATA'S BIJECTION

In this chapter we define Foata's map and show it is a bijection on S_n. Foata's bijection has an important weight-preserving property linking the inversion and major index permutation statistics. Foata's bijection also has the property of preserving the inverse descent set of a permutation π, $\mathrm{IDes}(\pi)$. In Chapter 6 we will use the weight-preserving and inverse-descent-set-preserving properties of Foata's bijection to extend our proof of the Macdonald Conjecture to a broader class of partition shapes. We follow Loehr's presentation [6] by defining a helper function h_x.

Definition 5.1 ([6])**.** Let σ be a word of length k of distinct letters and let $x \in \mathbb{Z}^+$ be not in σ. Then h_x is a function where $h_x(\sigma)$ is defined by the following. We will partition σ into "runs". If $\sigma_k > x$ (resp. $\sigma_k < x$), then define $i_1 < \ldots < i_m$ to index the letters in σ such that $\sigma_{i_j} > x$ (resp. $\sigma_{i_j} < x$). Then there are m runs in σ, where the j^{th} run is

$$\sigma_{i_{j-1}+1}\sigma_{i_{j-1}+2}\ldots\sigma_{i_j}$$

where i_0 is understood to be $i_0 = 0$. Note that σ can be written as partitioned into consecutive runs (we use vertical bars to distinguish the consecutive runs):

$$\sigma = \sigma_1\sigma_2\ldots\sigma_{i_1}|\sigma_{i_1+1}\sigma_{i_1+2}\ldots\sigma_{i_2}|\ldots|\sigma_{i_{m-1}+1}\sigma_{i_{m-1}+2}\ldots\sigma_{i_m}.$$

Then $h_x(\sigma)$ is the rearrangement of σ where the letters within each run are right-cyclic-shifted:

$$\begin{aligned} h_x(\sigma) &= \sigma_{i_1}\sigma_1\ldots\sigma_{i_1-1}|\sigma_{i_2}\sigma_{i_1+1}\ldots\sigma_{i_2-1}|\ldots|\sigma_{i_m}\sigma_{i_{m-1}+1}\ldots\sigma_{i_m-1} \\ &= \sigma_{i_1}\sigma_1\ldots\sigma_{i_1-1}\sigma_{i_2}\sigma_{i_1+1}\ldots\sigma_{i_2-1}\ldots\sigma_{i_m}\sigma_{i_{m-1}+1}\ldots\sigma_{i_m-1}. \end{aligned}$$

Definition 5.2 ([6])**.** We define *Foata's map* ϕ by the following. Let $\pi = \pi_1\ldots\pi_n$ be a word with

k	$\phi(\pi_1 \dots \pi_k)$	π_{k+1}	$h_{\pi_{k+1}}(\phi(\pi_1 \dots \pi_k))$	$\phi(\pi_1 \dots \pi_{k+1})$
1	6	2	$\vert 6 \vert$	62
2	62	5	$\vert 26 \vert$	265
3	265	3	$\vert 62 \vert 5 \vert$	6253
4	6253	4	$\vert 26 \vert 35 \vert$	26354
5	26354	1	$\vert 2 \vert 6 \vert 3 \vert 5 \vert 4 \vert$	263541

Table 5.1: Computation of $\phi(625341)$
Foata's map is computed using the recursion $\phi(\pi_1 \dots \pi_k) = h_{\pi_k}(\phi(\pi_1 \dots \pi_{k-1}))\pi_k$. For example, in the row above where $k = 4$, we start with $\phi(6253) = 6253$. Because $(\phi(6253))_4 = 3$ is less than $\pi_5 = 4$, we start a new run in 6253 with every letter less than 4, obtaining the partitioning $|62|53|$. Finally, we right-cyclic-shift the letters within each run to obtain $h_4(\phi(6253)) = 2635$ and append $\pi_5 = 4$ to obtain $\phi(62534) = 26354$. Also note that $\mathrm{inv}(\phi(625341)) = 9 = \mathrm{maj}(625341)$.

n distinct letters. Then $\phi(\pi)$ is computed by the recursion

$$\phi(\pi_1 \dots \pi_k) = h_{\pi_k}(\phi(\pi_1 \dots \pi_{k-1}))\pi_k$$

for all $2 \le k \le n$ with initial condition $\phi(\pi_1) = \pi_1$. In other words, $\phi(\pi_1 \dots \pi_k)$ is computed by appending π_k to the end of the word $h_{\pi_k}(\phi(\pi_1 \dots \pi_{k-1}))$.

Remark 5.3. Note that $h_x(\sigma)$ is a rearrangement of the letters of σ. It can be shown by induction on k that $\phi(\pi)$ has precisely the letters $\pi_1, \dots, \pi_n$ in some order, so $\phi\colon S_n \to S_n$ (Definition 2.2).

Example 5.4. For $\pi = 625341$, the computation of $\phi(\pi)$ is seen in Table 5.1. As an example iteration of the computation, take the row in the table beginning with $k = 3$. Since $\pi_4 = 3$ and $(\phi(625))_3 = 5 > 3$, we define $i_1, \dots, i_m$ to be the indices of the letters in $\phi(\pi_1\pi_2\pi_3) = \phi(265)$ which are greater than 3. These indices are $i_1 = 2$ and $i_2 = 3$. Thus we partition $\phi(\pi_1\pi_2\pi_3)$ into two runs, 26 and 5, and each run gets right-cyclic-shifted, resulting in $h_3(265) = 625$. Finally, appending π_{k+1} yields

$$\phi(6253) = 6253.$$

Theorem 5.5 ([6])**.** *Foata's map, $\phi : S_n \to S_n$, is a bijection.*

Proof. Let Foata's map $\phi : S_n \to S_n$ and h_x be defined as above. Since ϕ is computed by iteratively applying h_x and appending a letter, identifying an inverse for h_x is sufficient to show that ϕ has an inverse. Let τ be a word of length $k-1$ with distinct letters from the set $\{1, \ldots, n\}$, and let x be a letter in $\{1, \ldots, n\}$ not in τ. We will again partition τ into runs. If $\tau_1 > x$, then define $i_1 < \ldots < i_m$ to index the letters in τ such that $\tau_{i_j} > x$. If $\tau_1 < x$, then define $i_1 < \ldots < i_m$ to index the letters in τ such that $\tau_{i_j} < x$. Then there are m runs, where the j^{th} run is identified

$$\tau_{i_j}\tau_{i_j+1}\ldots\tau_{i_{j+1}-1}$$

where i_{m+1} is understood to be $i_{m+1} = k$. So τ can be written as partitioned into consecutive runs,

$$\tau = \tau_1\tau_2\ldots\tau_{i_2-1}|\tau_{i_2}\tau_{i_2+1}\ldots\tau_{i_3-1}|\ldots|\tau_{i_m}\tau_{i_m+1}\ldots\tau_{k-1}.$$

Then $h_x^{-1}(\tau)$ left-cyclic-shifts each run in τ,

$$h_x^{-1}(\tau) = \tau_2\ldots\tau_{i_2-1}\tau_1|\tau_{i_2+1}\ldots\tau_{i_3-1}\tau_{i_2}|\ldots|\tau_{i_m+1}\ldots\tau_{k-1}\tau_{i_m}.$$

We argue h_x^{-1} is the inverse of h_x for the reason that, if $\tau = h_x(\sigma)$, the letters $\tau_{i_1}, \ldots, \tau_{i_m}$ identified in computing $h_x^{-1}(\tau)$ are the same as the letters $\sigma_{i_1}, \ldots, \sigma_{i_m}$ identified in computing $\tau = h_x(\sigma)$. These letters begin the runs in τ; thus h_x^{-1} identifies the same runs in its input as h_x had produced in its output. Since h_x^{-1} cycles each group left where h_x had cycled each group right, $h_x^{-1}(\tau)$ ends up $h_x^{-1}(\tau) = \sigma$. Now that we have an inverse for h_x, we can define an inverse for ϕ by an iterative procedure that "peels off" one letter at a time from the input to create the output. Suppose $\pi \in S_n$. Start the procedure by setting $k = n$ and $\tau = \pi$. Then for iteration k, set $(\phi^{-1}(\pi))_k = \tau_k$ and then reset $\tau = h^{-1}(\tau_1, \ldots, \tau_{k-1})$, then proceed with iteration $k-1$. Thus the output $\phi^{-1}(\pi)$ is determined in reverse order, $(\phi^{-1}(\pi))_n$, $(\phi^{-1}(\pi))_{n-1}$, $\ldots$, $(\phi^{-1}(\pi))_1$. Since h_x^{-1} performs only a rearrangement of its input word, we see that the letters peeled off of all the τs to create $\phi^{-1}(\pi)$ are precisely the same letters as those in π; thus $\phi^{-1}(\pi) \in S_n$. It may be verified that one iteration of the procedure in computing ϕ^{-1} is the inverse of one recursion in the formula for computing ϕ. Thus ϕ has an inverse and is a bijection $\phi : S_n \to S_n$. □

Example 5.6. Let $\sigma = 263541$ and compute $\phi^{-1}(\sigma)$, as in Table 5.2. Note that assembling the letters of $\phi^{-1}(\sigma)$ in order from the column $(\phi^{-1}(\sigma))_k = \tau_k$ yields $\phi^{-1}(\sigma) = 625341$. As an example

k	$\tau = \tau_1 \dots \tau_k$	$(\phi^{-1}(\sigma))_k = \tau_k$	$h_{\tau_k}^{-1}(\tau_1 \dots \tau_{k-1})$
6	263541	1	$\mid 2 \mid 6 \mid 3 \mid 5 \mid 4 \mid$
5	26354	4	$\mid 62 \mid 53 \mid$
4	6253	3	$\mid 26 \mid 5 \mid$
3	265	5	$\mid 62 \mid$
2	62	2	$\mid 6 \mid$
1	6	6	–

Table 5.2: Computation of $\phi^{-1}(263541)$
Where Foata's map had been computed by recursive applications of h_x and appending the next letter of π, the inverse of Foata's map is computed by recursions of the reverse procedure: removing the last letter of τ and applying h_x^{-1} to the remainder. For example, in the row where $k = 5$ we start with $\tau = 26354$, remove the 4, and reset τ to $\tau = h_4^{-1}(2635) = 6253$. The removed letter 4 is recorded as the k^{th} letter of the output word, $(\phi^{-1}(263541))_5 = 4$.

iteration of the computation, take the row in the table beginning with $k = 5$. The last letter of the current τ, $\tau_5 = 4$, becomes the 5^{th} letter of $\phi^{-1}(\sigma)$. This letter is removed, and the remaining letters are divided into runs. Since $\tau_1 = 2 < 4 = \tau_k$, we define $i_1, \dots, i_m$ to be the indices of the letters in τ that are less than 4. These indices are $i_1 = 1$, $i_2 = 3$. Since each index i_j begins a run, we obtain the runs 26 and 35, and each is left-cyclic-shifted to obtain

$$h_4^{-1}(2635) = 6253.$$

The following theorem provides the weight-preserving property of Foata's bijection. The two weights are the inversion and major index statistics on permutations (recall Definitions 2.4 and 2.9).

Theorem 5.7 ([6])**.** *For any* $\pi \in S_n$*,*

$$\mathrm{inv}(\phi(\pi)) = \mathrm{maj}(\pi).$$

Proof. Let $\pi \in S_n$. We will prove the above result by induction on the k^{th} recursion of the computation of $\phi(\pi)$. Observe that for $k = 0$, $\mathrm{maj}(\pi_1) = 0$ and $\mathrm{inv}(\phi(\pi_1)) = \mathrm{inv}(\pi_1) = 0$, so the base case holds. Suppose that for some $k \in \{1, \dots, n\}$, $\mathrm{inv}(\phi(\pi_1 \dots \pi_{k-1})) = \mathrm{maj}(\pi_1 \dots \pi_{k-1})$.

Suppose $\pi_{k-1} > \pi_k$. Then the descents in $\pi_1 \dots \pi_k$ are the same as those in $\pi_1 \dots \pi_{k-1}$ but with the addition of $k-1$. Thus,

$$\text{maj}(\pi_1 \dots \pi_k) = \text{maj}(\pi_1 \dots \pi_{k-1}) + k - 1.$$

Also consider $\text{inv}(\phi(\pi_1 \dots \pi_k)) = \text{inv}(h_{\pi_k}(\phi(\pi_1 \dots \pi_{k-1}))\pi_k)$. Let $(\tau_1 \dots \tau_{k-1}) = \phi(\pi_1 \dots \pi_{k-1})$. In $h_{\pi_k}(\tau_1 \dots \tau_{k-1})$, due to the cyclic nature of the map, the only relative letter positions that change are those involving the letters $\tau_{i_1}, \dots, \tau_{i_m}$, each of which moves to the left end of all members of its run where before it had been on the right end. No relative positions change between members of different runs. Thus the change in inv caused by h_{π_k} is entirely due to the change in number of inversions involving τ_{i_j} within the j^{th} run for all $j \in \{1, \dots, m\}$. Since, as per our supposition, $\tau_{i_j} > \pi_k$ and all other members of the run are less than π_k, τ_{i_j} is greater than all other members of its run, of which there are $i_j - i_{j-1} - 1$. Thus (recall $i_m = k-1$ and $i_0 = 0$),

$$\text{inv}(h_{\pi_k}(\tau_1 \dots \tau_{k-1})) = \text{inv}(\tau_1 \dots \tau_{k-1}) + \sum_{j=1}^{m}(i_j - i_{j-1} - 1)$$

$$\text{inv}(h_{\pi_k}(\tau_1 \dots \tau_{k-1})) = \text{inv}(\phi(\pi_1 \dots \pi_{k-1})) + k - 1 - m.$$

Then, appending π_k to the end of $h_{\pi_k}(\tau_1 \dots \tau_{k-1})$ to form $\phi(\pi_1 \dots \pi_k)$ has the effect of adding a further m inversions, since the letters in $h_{\pi_k}(\tau_1 \dots \tau_{k-1})$ greater than π_k are precisely $\tau_{i_1}, \dots, \tau_{i_m}$. In other words,

$$\text{inv}(\phi(\pi_1 \dots \pi_k)) = \text{inv}(\phi(\pi_1 \dots \pi_{k-1})) + k - 1 - m + m$$

$$= \text{maj}(\pi_1 \dots \pi_{k-1}) + k - 1$$

$$= \text{maj}(\pi_1 \dots \pi_k).$$

Suppose instead that $\pi_{k-1} < \pi_k$. Then $\text{maj}(\pi_1 \dots \pi_k) = \text{maj}(\pi_1 \dots \pi_{k-1})$, since no descents are added or removed. Also, by similar logic as above, applying h_{π_k} to $\pi_1 \dots \pi_k$ *subtracts* $k - 1 - m$ from inv, since this time each π_{i_j} is less than all its other run members and so loses $i_j - i_{j-1} - 1$ inversions when it moves to the left end of its run. And, the letters in $h_{\pi_k}(\pi_1 \dots \pi_{k-1})$ greater than π_k are all those letters in $\pi_1 \dots \pi_{k-1}$ except $\pi_{i_1}, \dots, \pi_{i_m}$, which number $k - 1 - m$. Thus appending π_k to $h_{\pi_k}(\pi_1 \ldots \pi_{k-1})$ increases the number of inversions by $k-1-m$. Thus we have

$$\begin{aligned}\operatorname{inv}(\phi(\pi_1 \ldots \pi_k)) &= \operatorname{inv}(\phi(\pi_1 \ldots \pi_{k-1})) - (k-1-m) + (k-1-m) \\ &= \operatorname{inv}(\phi(\pi_1 \ldots \pi_{k-1})) \\ &= \operatorname{maj}(\pi_1 \ldots \pi_{k-1}) \\ &= \operatorname{maj}(\pi_1 \ldots \pi_k).\end{aligned}$$

□

Recall from Definition 2.11 that $[n]_q!$ denotes

$$\begin{aligned}[n]_q! &= [n]_q[n-1]_q[n-2]_q \ldots [1]_q \\ &= (1+q+q^2+\cdots+q^{n-1})(1+q+q^2+\cdots+q^{n-2})\ldots(1).\end{aligned}$$

Combining Theorems 5.7 and 2.12 we obtain the generating function for the major index statistic.

Theorem 5.8 ([6]). *For any positive integer n,*

$$\sum_{\omega \in S_n} q^{\operatorname{maj}(\omega)} = [n]_q!.$$

Proof. Let n be a positive integer. From Theorem 5.7 we know

$$\sum_{\omega \in S_n} q^{\operatorname{maj}(\omega)} = \sum_{\omega \in S_n} q^{\operatorname{inv}(\phi(\omega))}.$$

By Theorem 5.5, Foata's map is a bijection $\phi\colon S_n \to S_n$, and so $\sum_{\omega \in S_n} q^{\operatorname{inv}(\phi(\omega))}$ is an equivalent rearrangement of the terms of $\sum_{\omega \in S_n} q^{\operatorname{inv}(\omega)}$,

$$\sum_{\omega \in S_n} q^{\operatorname{inv}(\phi(\omega))} = \sum_{\omega \in S_n} q^{\operatorname{inv}(\omega)}$$

and by Theorem 2.12,

$$\sum_{\omega \in S_n} q^{\operatorname{inv}(\omega)} = [n]_q!.$$

Thus

$$\sum_{\omega \in S_n} q^{\operatorname{maj}(\omega)} = [n]_q!. \qquad \square$$

In addition to sending major index statistic to inversion statistic, ϕ preserves the inverse descent set of the permutation (see Definition 2.7).

Lemma 5.9 ([6])**.** *For any $w \in S_n$,*

$$\operatorname{IDes}(\phi(w)) = \operatorname{IDes}(w),$$

where ϕ is Foata's bijection.

Proof. Let $w \in S_n$. Let $x \in \operatorname{IDes}(w)$. Then $w^{-1}(x) > w^{-1}(x+1)$. Define $z_i = \phi_i(w_1 w_2 \dots w_i)$, the word which is the i^{th} iteration in the computation of $\phi(w)$ and which is some arrangement of the letters $w_1, w_2, \dots, w_i$. Thus $z_{w^{-1}(x)}$ contains both the letters x and $x+1$, and note that $z_j^{-1}(x) > z_j^{-1}(x+1)$, where $j = w^{-1}(x)$. We will show that the application of the helper function of Foata's map, $h_{w_{j+1}}(z_j)$, does not change the relative ordering of x and $x+1$. Recall that the computation of $h_{w_{j+1}}(z_j)$ involves partitioning z_j into consecutive runs where the last letters of the runs are the letters which have the same order-relationship to w_{j+1} as w_j does. Observe $x > w_{i+1}$ if and only if $x + 1 > w_{i+1}$, and $x < w_{i+1}$ if and only if $x + 1 < w_{i+1}$. Thus, either x and $x+1$ are both at the ends of their runs, or neither is. If x and $x+1$ are both at the ends of their runs then they are in separate runs and their relative order does not change. If neither is at the end of its run, then the index of each in the word will increase by 1, according to the right-cyclic-shifting of the runs; as a result, their relative order remains the same. Since z_{j+1} is, by definition, the word $h_{w_{j+1}}(z_j)$ appended by the letter w_{j+1}, we conclude that the relative order of x and $x+1$ in z_{j+1} is the same as in z_j, in other words $z_{j+1}^{-1}(x) > z_{j+1}^{-1}(x+1)$. The same reasoning can be applied inductively to obtain $z_n^{-1}(x) > z_n^{-1}(x+1)$, and so $(\phi(w))^{-1}(x) > (\phi(w))^{-1}(x+1)$, giving $x \in \operatorname{IDes}(\phi(w))$, and thus

$$\operatorname{IDes}(w) \subseteq \operatorname{IDes}(\phi(w)).$$

Let $x \in \operatorname{IDes}(\phi(w))$. Then $(\phi(w))^{-1}(x) > (\phi(w))^{-1}(x+1)$. By similar induction as above, considering the runs used in computing the inverse helper function h^{-1}, it can be shown that $w^{-1}(x) > w^{-1}(x+1)$. Thus $x \in \mathrm{IDes}(w)$, $\mathrm{IDes}(\phi(w)) \subseteq \mathrm{IDes}(w)$, and

$$\mathrm{IDes}(\phi(w)) = \mathrm{IDes}(w). \qquad \square$$

CHAPTER 6

PROVING MACDONALD'S CONJECTURE IN CERTAIN CASES

In this chapter we prove the Macdonald Conjecture for two classes of partition shapes, $\mu = (1^n)$ (Theorem 6.3) and $\mu = (n)$ (Theorem 6.4). The proof of Theorem 6.3 follows Loehr [7], and the proof of Theorem 6.4 takes a similar approach but with added use of Foata's map. This method used to prove Theorem 6.4 differs from the that used in the recent proof by Assaf [1]. We start by rewriting the modified Macdonald polynomials in terms of fundamental quasisymmetric polynomials (see Definition 2.40), mirroring the expansion of the Schur functions.

Lemma 6.1 ([3])**.** *For any $\mu \vdash n$,*

$$\tilde{H}_\mu(X;q,t) = \sum_{T \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} F_{n,\mathrm{IDes}(T)}.$$

Proof. We present a similar argument as in the proof of Theorem 2.42. Let $\mu \vdash n$. Beginning from the definition of the modified Macdonald polynomial (Definition 3.5),

$$\tilde{H}_\mu(X;q,t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} x^T$$

observe that $\mathcal{F}_\mu$ may be partitioned by standardization class,

$$\tilde{H}_\mu(X;q,t) = \sum_{U \in \mathcal{SF}_\mu} \sum_{\substack{T \in \mathcal{F}_\mu \\ \mathrm{stdz}(T)=U}} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} x^T$$

$$\tilde{H}_\mu(X;q,t) = \sum_{U \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(U)} t^{\mathrm{maj}_\mu(U)} \sum_{\substack{T \in \mathcal{F}_\mu \\ \mathrm{stdz}(T)=U}} x^T$$

$$\tilde{H}_\mu(X;q,t) = \sum_{U \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(U)} t^{\mathrm{maj}_\mu(U)} B_U,$$

where

$$B_U = \sum_{\substack{T \in \mathcal{F}_\mu \\ \mathrm{stdz}(T)=U}} x^T.$$

Let $U \in \mathcal{SF}_\mu$. Note that from Definition 2.40,

$$F_{n,\mathrm{IDes}(U)} = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n, \\ \text{if } j \in \mathrm{IDes}(U) \text{ then } i_j \neq i_{j+1}}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

We will show that

$$B_U = F_{n,\mathrm{IDes}(U)}$$

by showing the two polynomials to have the same set of terms. Note that neither polynomial has any repeated summands: for B_U, two distinct fillings that standardize to the same filling cannot have the same content, and for $F_{n,\mathrm{IDes}(U)}$ distinct choices of $i_1 \le i_2 \le \ldots \le i_n$ will always result in distinct monomials.

Let $x_{i_1} x_{i_2} \ldots x_{i_n}$ be a term in B_U, taking $i_1 \le i_2 \le \ldots \le i_n$. Let $j \in \mathrm{IDes}(U)$. Then $j+1$ appears before j in $\mathrm{rw}(U)$. This implies $i_{j+1} \neq i_j$, since the only case in which the scanning order of standardization goes against reading order is when a new letter is being scanned for. Thus $x_{i_1} x_{i_2} \ldots x_{i_n}$ is a term in $F_{n,\mathrm{IDes}(U)}$.

Let $x_{i_1} x_{i_2} \ldots x_{i_n}$ be a term in $F_{n,\mathrm{IDes}(U)}$. Construct a new filling T by replacing each entry j in U with i_j. Observe that $x^T = x_{i_1} x_{i_2} \ldots x_{i_n}$, $\mathrm{stdz}(T) = U$, and the entries of T are drawn from $\{1, 2, \ldots, n\}$. Thus, $x_{i_1} x_{i_2} \ldots x_{i_n}$ is a term in B_U.

We have shown

$$B_U = F_{n,\mathrm{IDes}(U)},$$

and thus

$$\tilde{H}_\mu(X; q, t) = \sum_{T \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} F_{n,\mathrm{IDes}(T)}. \qquad \square$$

Since we can now generate $\tilde{H}_\mu(X; q, t)$ by standard fillings, we can use the reading word (Definition 2.22) to restate $\tilde{H}_\mu(X; q, t)$ in terms of permutations.

Lemma 6.2 ([7]). *For any $\mu \vdash n$,*

$$\tilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} F_{n,\mathrm{IDes}(w)}.$$

Proof. Let $\mu \vdash n$. From Lemma 6.1 we know

$$\tilde{H}_\mu(X;q,t) = \sum_{T \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} F_{n,\mathrm{IDes}(T)}.$$

Recall from Definitions 3.2 and 3.4 that for a standard filling T,

$$\mathrm{inv}_\mu(\mathrm{rw}(T)) = \mathrm{inv}_\mu(T)$$

and

$$\mathrm{maj}_\mu(\mathrm{rw}(T)) = \mathrm{maj}_\mu(T).$$

Also, from Definitions 2.7 and 2.26 we have that $\mathrm{IDes}(\mathrm{rw}(T)) = \mathrm{IDes}(T)$. Thus

$$\tilde{H}_\mu(X;q,t) = \sum_{T \in \mathcal{SF}_\mu} q^{\mathrm{inv}_\mu(\mathrm{rw}(T))} t^{\mathrm{maj}_\mu(\mathrm{rw}(T))} F_{n,\mathrm{IDes}(\mathrm{rw}(T))}.$$

Observe that, for a given μ, $\mathrm{rw}\colon \mathcal{SF}_\mu \to S_n$ is a bijection. (A filling is uniquely determined by its shape and reading word.) Thus we obtain

$$\tilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} F_{n,\mathrm{IDes}(w)}. \qquad \square$$

Following the method of Loehr, we use the RSK bijection and its properties, as well as the fundamental quasisymmetric expansions of the modified Macdonald polynomials and Schur functions, to prove Macdonald's conjecture for the case $\mu = (1^n)$.

Theorem 6.3 ([7]). *Macdonald's Conjecture holds for $\mu = (1^n) = (1, 1, \ldots, 1)$.*

Proof. Let $\mu = (1^n) \vdash n$. Because μ consists of a single column, $\mathrm{inv}_\mu(w) = 0$ and

$\mathrm{maj}_\mu(w) = \mathrm{maj}(w)$ for all $w \in S_n$. From Lemma 6.2 we know

$$\tilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} F_{n,\mathrm{IDes}(w)}.$$

Thus

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{w \in S_n} q^0 t^{\mathrm{maj}(w)} F_{n,\mathrm{IDes}(w)}.$$

We define the tableau statistics $\tilde{a}_{(1^n)}$, $\tilde{b}_{(1^n)}$ by

$$\tilde{a}_{(1^n)}(T) = 0$$

and

$$\tilde{b}_{(1^n)}(T) = \sum_{i \in \mathrm{Des}(T)} i.$$

Let $w \in S_n$ and $(P,Q) = \mathrm{RSK}(w)$. From Lemma 4.11 we know $\mathrm{Des}(Q) = \mathrm{Des}(w)$, and thus by Definition 2.9 we have

$$\mathrm{maj}(w) = \tilde{b}_{(1^n)}(Q).$$

Also, from Corollary 4.13 we have

$$\mathrm{IDes}(w) = \mathrm{Des}(P).$$

Thus

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{w \in S_n,\ (P,Q)=\mathrm{RSK}(w)} q^{\tilde{a}_{(1^n)}(Q)} t^{\tilde{b}_{(1^n)}(Q)} F_{n,\mathrm{Des}(P)}.$$

And since $\mathrm{RSK}\colon S_n \to \{(P,Q) : P,Q \in \mathrm{SYT}(\lambda),\ \lambda \vdash n\}$ is a bijection,

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{(P,Q)\,:\, P,Q \in \mathrm{SYT}(\lambda),\ \lambda \vdash n} q^{\tilde{a}_{(1^n)}(Q)} t^{\tilde{b}_{(1^n)}(Q)} F_{n,\mathrm{Des}(P)}$$

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{\lambda \vdash n} \sum_{P \in \mathrm{SYT}(\lambda)} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(1^n)}(Q)} t^{\tilde{b}_{(1^n)}(Q)} F_{n,\mathrm{Des}(P)}$$

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{\lambda \vdash n} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(1^n)}(Q)} t^{\tilde{b}_{(1^n)}(Q)} \sum_{P \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(P)}$$

and by Theorem 2.42,

$$\tilde{H}_{(1^n)}(X;q,t) = \sum_{\lambda \vdash n} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(1^n)}(Q)} t^{\tilde{b}_{(1^n)}(Q)} s_\lambda. \qquad \square$$

The following proof of the Macdonald Conjecture for the shape $\mu = (n)$ uses the same technique as the proof for $\mu = (1^n)$ but first implements the weight-preserving property of Foata's bijection to change between the inversion statistic and the major index statistic.

Theorem 6.4. *Macdonald's Conjecture holds for $\mu = (n)$.*

Proof. Let $\mu = (n) \vdash n$. Because μ consists of a single row, $\mathrm{inv}_\mu(w) = \mathrm{inv}(w)$ and $\mathrm{maj}_\mu(w) = 0$ for all $w \in S_n$. From Lemma 6.2 we know

$$\tilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} F_{n,\mathrm{IDes}(w)}.$$

Thus

$$\tilde{H}_{(n)}(X;q,t) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} t^0 F_{n,\mathrm{IDes}(w)}.$$

By Theorem 5.7 and Lemma 5.9 we obtain

$$\tilde{H}_{(n)}(X;q,t) = \sum_{w \in S_n} q^{\mathrm{maj}(\phi(w))} t^0 F_{n,\mathrm{IDes}(\phi(w))}.$$

By Theorem 5.5 Foata's map $\phi\colon S_n \to S_n$ is a bijection, and so

$$\tilde{H}_{(n)}(X;q,t) = \sum_{w \in S_n} q^{\mathrm{maj}(w)} t^0 F_{n,\mathrm{IDes}(w)}.$$

We now follow the same argument as in Theorem 6.3. We define the tableau statistics $\tilde{a}_{(n)}$, $\tilde{b}_{(n)}$ by

$$\tilde{a}_{(n)}(T) = \sum_{i \in \mathrm{Des}(T)} i$$

and

$$\tilde{b}_{(n)}(T) = 0.$$

Let $w \in S_n$ and $(P, Q) = \mathrm{RSK}(w)$. From Lemma 4.11 we know $\mathrm{Des}(Q) = \mathrm{Des}(w)$, and thus by Definition 2.9 we have

$$\mathrm{maj}(w) = \tilde{a}_{(n)}(Q).$$

Also, from Lemma 4.13 we have

$$\mathrm{IDes}(w) = \mathrm{Des}(P).$$

Thus

$$\tilde{H}_{(n)}(X; q, t) = \sum_{w \in S_n,\ (P,Q)=\mathrm{RSK}(w)} q^{\tilde{a}_{(n)}(Q)} t^{\tilde{b}_{(n)}(Q)} F_{n,\mathrm{Des}(P)}.$$

And since $\mathrm{RSK}\colon S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda),\ \lambda \vdash n\}$ is a bijection,

$$\tilde{H}_{(n)}(X; q, t) = \sum_{(P,Q)\,:\ P,Q \in \mathrm{SYT}(\lambda),\ \lambda \vdash n} q^{\tilde{a}_{(n)}(Q)} t^{\tilde{b}_{(n)}(Q)} F_{n,\mathrm{Des}(P)}$$

$$\tilde{H}_{(n)}(X; q, t) = \sum_{\lambda \vdash n} \sum_{P \in \mathrm{SYT}(\lambda)} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(n)}(Q)} t^{\tilde{b}_{(n)}(Q)} F_{n,\mathrm{Des}(P)}$$

$$\tilde{H}_{(n)}(X; q, t) = \sum_{\lambda \vdash n} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(n)}(Q)} t^{\tilde{b}_{(n)}(Q)} \sum_{P \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(P)}$$

and by Theorem 2.42,

$$\tilde{H}_{(n)}(X; q, t) = \sum_{\lambda \vdash n} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\tilde{a}_{(n)}(Q)} t^{\tilde{b}_{(n)}(Q)} s_\lambda. \qquad \square$$

CHAPTER 7

MODIFYING THE RSK ALGORITHM

In Chapter 6 we used the RSK algorithm to prove the Macdonald Conjecture for the case $\mu = (1^n)$, and also for the case $\mu = (n)$ with the help of Foata's bijection. The RSK algorithm has the following properties:

1. RSK: $S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}$ is a bijection (see Theorem 4.8).

2. For all $w \in S_n$, if $(P, Q) = \mathrm{RSK}(w)$, then $\mathrm{IDes}(w) = \mathrm{Des}(P)$ (see Corollary 4.13).

3. For all $w \in S_n$, if $(P, Q) = \mathrm{RSK}(w)$, then we have tableau statistics $\tilde{a}_\mu(Q) = \mathrm{inv}_\mu(w)$ and $\tilde{b}_\mu(Q) = \mathrm{maj}_\mu(w)$ for $\mu = (1^n)$ (see proof of Theorem 6.3).

4. Every filling P_i in the computation of P must be a partial tableau.

Note that properties (1), (2), and (3) are essential to the proof of Theorem 6.3. We would like to define an algorithm RSK$'$ similar to RSK that could prove the Macdonald Conjecture for $\mu = (n)$ (and for even more general shapes) without relying on Foata's bijection. To do so, the new algorithm would have the first three properties above, extending the third property to apply to the shape $\mu = (n)$ (and more general shapes). It should also possess the fourth property to be recognizable as an iterative insertion algorithm. Loehr [7] has given an RSK' that satisfies the four properties for shapes μ such that $\mu_1 \leq 3$ and $\mu_2 \leq 2$, though not for shapes μ with $\mu_1 > 3$. The most direct way of achieving the extension of the third property to the shape $\mu = (n)$ would be to define RSK$'$ in such a way that $\mathrm{RSK}'(w) = \mathrm{RSK}(\phi(w))$. We have found no obvious way to do so, however. Furthermore, we will show that any RSK$'$ algorithm that possesses the four properties must include possible movement of tableau entries into lower rows during row insertion, which is a significant departure from the original definition of row insertion. This result holds regardless of the precise definitions of the statistics $\tilde{a}_{(n)}$ and $\tilde{b}_{(n)}$.

Theorem 7.1. *Suppose $\tilde{a}_\mu$ is a statistic on standard tableaux of shape μ and suppose* RSK$'$: $S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}$ *is a bijection computed by iterative partial tableaux* *such that for every* $w \in S_n$ *and* $(P, Q) = \mathrm{RSK}'(w)$, $\mathrm{Des}(P) = \mathrm{IDes}(w)$ *and* $\tilde{a}_{(n)}(Q) = \mathrm{inv}_{(n)}(w) = \mathrm{inv}(w)$. *Then for some* n *and some* $w \in S_n$, *some iteration of the computation of* $\mathrm{RSK}'(w)$ *requires moving a tableau entry down a row.*

Proof. Let $\tilde{a}_{(n)}$ and $\mathrm{RSK}' \colon S_n \to \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda), \lambda \vdash n\}$ be as above. Let $n = 4$ and $\lambda = (3, 1) \vdash 4$. Let $P, Q \in \mathrm{SYT}(\lambda)$ where Q has reading word 2134. Let $w = (\mathrm{RSK}')^{-1}((P, Q))$. Since $\tilde{a}_{(n)}(Q) = \mathrm{inv}(w)$ and the range of inv on S_4 is $\{0, 1, 2, 3, 4, 5, 6\}$ (see Table 7.1), the possible values of $\tilde{a}_{(n)}(Q)$ are $\{0, 1, 2, 3, 4, 5, 6\}$. Consider each case in turn; use Table 7.1 for reference.

- $\underline{\tilde{a}_{(n)}(Q) = 0}$: Impossible, since $\mathrm{inv}(w) = 0$ implies $w = 1234$ which has $\mathrm{IDes}(w) = \{\}$ and there are no $P \in \mathrm{SYT}((3, 1))$ which have $\mathrm{Des}(P) = \{\}$.

- $\underline{\tilde{a}_{(n)}(Q) = 1}$: Consider when P has reading word 4123, so that $\mathrm{Des}(P) = \{3\}$. Since $\mathrm{inv}(w) = 1$ and $\mathrm{IDes}(w) = \{3\}$, we infer $w = 1243$. We observe that all possible ways of computing $\mathrm{RSK}'(w)$ involve moving an entry down a row (see Fig. 7.1).

- $\underline{\tilde{a}_{(n)}(Q) = 2}$: Consider when P has reading word 2134, so that $\mathrm{Des}(P) = \{1\}$. Since $\mathrm{inv}(w) = 2$ and $\mathrm{IDes}(w) = \{1\}$, we infer $w = 2314$. We observe that all possible ways of computing $\mathrm{RSK}'(w)$ involve moving an entry down a row (see Fig. 7.1).

- $\underline{\tilde{a}_{(n)}(Q) = 3}$: Consider when P has reading word 2134, so that $\mathrm{Des}(P) = \{1\}$. Since $\mathrm{inv}(w) = 3$ and $\mathrm{IDes}(w) = \{1\}$, we infer $w = 2341$. We observe that all possible ways of computing $\mathrm{RSK}'(w)$ involve moving an entry down a row (see Fig. 7.1).

- $\underline{\tilde{a}_{(n)}(Q) = 4}$: Consider when P has reading word 3124, so that $\mathrm{Des}(P) = \{2\}$. Since $\mathrm{inv}(w) = 4$ and $\mathrm{IDes}(w) = \{2\}$, we infer $w = 3412$. We observe that all possible ways of computing $\mathrm{RSK}'(w)$ involve moving an entry down a row (see Fig. 7.1).table

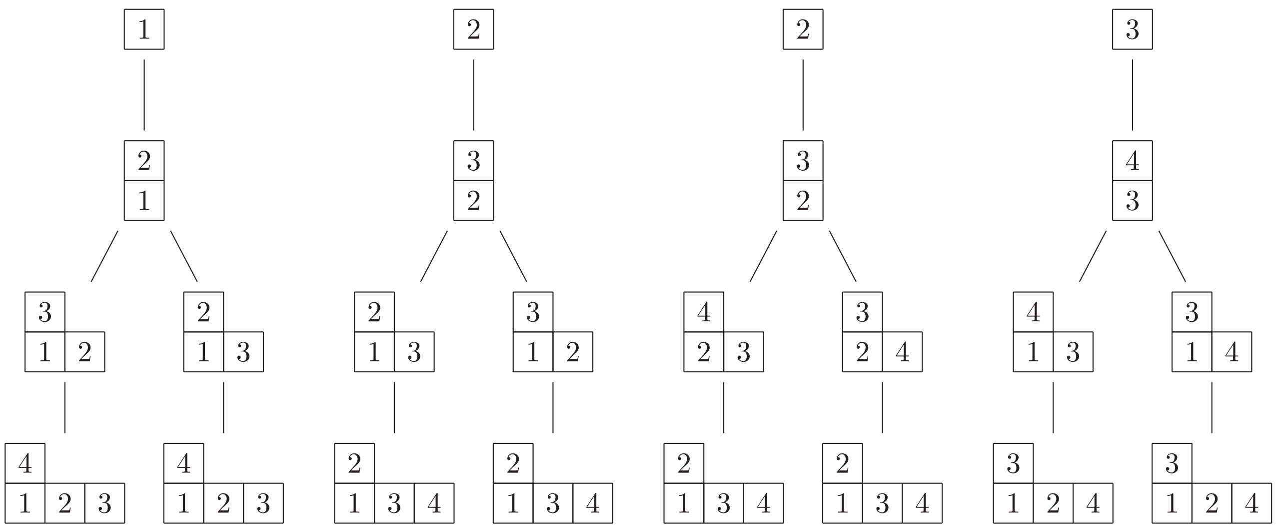
- $\underline{\tilde{a}_{(n)}(Q) = 5}$: Impossible, since $\mathrm{inv}(w) = 5$ implies either $w = 4231$ or $w = 4312$, and neither possibility has $\mathrm{IDes}(w)$ such that there exists $P \in \mathrm{SYT}((3, 1))$ with $\mathrm{Des}(P) = \mathrm{IDes}(w)$.

- $\underline{\tilde{a}_{(n)}(Q) = 6}$: Impossible, since $\mathrm{inv}(w) = 6$ implies $w = 4321$ which has $\mathrm{IDes}(w) = \{1, 2, 3, 4\}$ and there are no $P \in \mathrm{SYT}((3, 1))$ which have $\mathrm{Des}(P) = \{1, 2, 3, 4\}$.

Thus, no RSK$'$ algorithm can avoid moving an entry down a row in at least one of its iterations. $\square$

$w \in S_4$	$\mathrm{inv}(w)$	$\mathrm{maj}(w)$	$\mathrm{IDes}(w)$
1234	0	0	$\{\}$
1243	1	3	$\{3\}$
1324	1	2	$\{2\}$
1342	2	3	$\{2\}$
1423	2	2	$\{3\}$
1432	3	5	$\{2,3\}$
2134	1	1	$\{1\}$
2143	2	4	$\{1,3\}$
2314	2	2	$\{1\}$
2341	3	3	$\{1\}$
2413	3	2	$\{1,3\}$
2431	4	5	$\{1,3\}$
3124	2	1	$\{2\}$
3142	3	4	$\{2\}$
3214	3	3	$\{1,2\}$
3241	4	4	$\{1,2\}$
3412	4	2	$\{2\}$
3421	5	5	$\{1,2\}$
4123	3	1	$\{3\}$
4132	4	4	$\{2,3\}$
4213	4	3	$\{1,3\}$
4231	5	4	$\{1,3\}$
4312	5	3	$\{2,3\}$
4321	6	6	$\{1,2,3\}$

Table 7.1: Inversion, major index, and inverse descent set on S_4
Note the joint distribution (discussed in Chapter 8) of the inversion and major index statistics on S_4: for every $w \in S_4$, there exists $u \in S_4$ such that $\mathrm{inv}(u) = \mathrm{maj}(w)$ and $\mathrm{maj}(u) = \mathrm{inv}(w)$.



$\tilde{a}_{(n)}(Q) = 1$, $\mathrm{rw}(P) = 4123$, so $w = 1243$

$\tilde{a}_{(n)}(Q) = 2$, $\mathrm{rw}(P) = 2134$, so $w = 2314$

$\tilde{a}_{(n)}(Q) = 3$, $\mathrm{rw}(P) = 2134$, so $w = 2341$

$\tilde{a}_{(n)}(Q) = 4$, $\mathrm{rw}(P) = 3124$, so $w = 3412$

Figure 7.1: Possible RSK' **progressions when** $\lambda = (3,1)$ **and** $\mathrm{rw}(Q) = 2134$
The above trees, which are read top to bottom, display the P_1, P_2, P_3, P_4 of possible RSK' algorithms for specific cases considered in Theorem 7.1: given $\lambda = (3,1)$ and $\mathrm{rw}(Q) = 2134$, we consider all possibilities for $\tilde{a}_{(n)}(Q)$. For each possible value for $\tilde{a}_{(n)}(Q)$, we focus on a particular value for $\mathrm{rw}(P)$ which forces the RSK' progression to include at least one iteration where a tableau entry is moved down a row.

CHAPTER 8

TABLEAU STATISTICS

In Chapter 7 we considered the requirements for a modified RSK-like algorithm that could be used to extend the proof of Macdonald's conjecture to more general shapes. One of those requirements was the existence of tableau statistics $\tilde{a}_\mu$ and $\tilde{b}_\mu$ such that $\tilde{a}_\mu(Q) = \mathrm{inv}_\mu(w)$ and $\tilde{b}_\mu(Q) = \mathrm{maj}_\mu(w)$ for $(P, Q) = \mathrm{RSK}'(w)$. In this chapter we discuss candidates for these statistics that are extensions of the inversion and major index permutation statistics defined by Haglund and Stevens [4]. A cause for special interest in these statistics is that there exists a weight-preserving bijection Ψ that is an extension of Foata's map, sending the tableau major index statistic to the tableau inversion statistic. In addition to behaving in a parallel manner to the inversion and major index permutation statistics in this way, the Haglund–Stevens tableau statistics have a certain invariance property concerning tableau conjugates, which we explore in Theorems 8.4 and 8.19. However, we also show that these tableau statistics are not jointly distributed, an important property that the inversion and major index statistics have on permutations.

Note that we will use the notation (r, c) to denote the cell of a filling in row r and column c.

Definition 8.1 ([4])**.** Let T be a standard tableau. Then we define

$$\mathrm{maj}(T) = \sum_{e \in \mathrm{Des}(T)} e.$$

For example, for the tableau T in Fig. 8.1, $\mathrm{maj}(T) = 2 + 6 + 9 + 3 + 7 + 12 = 39$.

Definition 8.2. Let T be a standard tableau and k an entry in T. Define $\pi_k(T)$ as the path in the diagram of T constructed in the following way:

1. Start at the lower-left corner of the cell in T containing the entry k.

2. Let (a, b) be the row-column coordinates of the cell most recently added to the path. If $(a, b) = (1, 1)$, the path is complete; if $b = 1$ then the next cell is $(a - 1, b)$, and if $a = 1$ then the next cell is $(a, b - 1)$. If $a \neq 1, b \neq 1$ and $T(a, b - 1) > T(a - 1, b)$ then $(a, b - 1)$ is the

T

10	13		
4	8	11	
3	5	7	12
1	2	6	9

$\Psi_{11}(T)$

9	13		
3	7	11	
2	5	8	12
1	4	6	10

Figure 8.1: Computation of $\Psi_{11}(T)$
We find the path $\pi_{11}(T) = ((3,3),(3,2),(2,2),(2,1),(1,1))$ by starting at the cell of the entry 11 and repeatedly stepping left or down, depending on which cell contains the greater entry. We identify the cycle blocks in T according to $\pi_{11}(T)$ as (1), $(2,3,4)$, (5), (6), $(7,8)$, and $(9,10)$, by forming lists of consecutive integer entries whose first element is the only entry on the same side of $\pi_{11}(T)$ as the entry 1. We compute $\Psi_{11}(T)$ by right-cyclic-shifting each cycle block.

T

5		
3	4	7
1	2	6

T'

6	7	
2	4	
1	3	5

Figure 8.2: Conjugate of a tableau T
For tableau T, the conjugate T' is the tableau produced by inverting the row and column coordinates of each cell in T.

next cell. Otherwise, $(a-1,b)$ is the next cell.

3. Repeat step 2 until the path is complete.

For example, for tableau T in Fig. 8.1, the path $\pi_{11}(T) = ((3,3),(3,2),(2,2),(2,1),(1,1))$.

Definition 8.3. The *conjugate*, T', of a standard tableau T is the tableau formed by reflecting T over the line $y = x$. Formally, for each cell (r,c) in T, T' has a cell (c,r) such that $T'(c,r) = T(r,c)$. We define conjugation on a tableau cell by $(a,b)' = (b,a)$. Similarly, the conjugate π' of a path π is the path formed by taking the conjugate of each cell in π: if $\pi = ((r_1,c_1),\ldots,(r_m,c_m))$ then $\pi' = ((c_1,r_1),\ldots,(c_m,r_m))$. Fig. 8.2 demonstrates an example of a tableau T and its conjugate T'.

The following result gives a sort of complementary relationship between the Haglund–Stevens major index of a standard tableau and the major index of its conjugate, in that the sum of the major indices depends only on the size of the tableau, not its shape or reading word.

Theorem 8.4. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Then*

$$\operatorname{maj}(T) + \operatorname{maj}(T') = \frac{n(n-1)}{2}.$$

Proof. Let T be as above. Let $[n-1]$ denote $\{1, 2, \ldots, n-1\}$. Let $\overline{\mathrm{Des}(T)}$ represent $[n-1] \setminus \mathrm{Des}(T)$. Let $i \in [n-1]$. Let (r, c) be the cell containing i in T and (s, d) the cell containing $i+1$ in T.

Suppose $i \in \mathrm{Des}(T)$. Then $r < s$. Suppose for contradiction that $c < d$. Consider that the cell (s, c) is above (r, c) in the same column and to the left of (s, d) in the same row. But this implies the contradiction $i < T(s, c) < i + 1$ by increasing row and column rules. Thus $c \geq d$. This implies $i \notin \mathrm{Des}(T')$ since c is the row of i in T' and d is the row of $i+1$ in T'. Thus $\mathrm{Des}(T) \subseteq \overline{\mathrm{Des}(T')}$.

Suppose $i \in \overline{\mathrm{Des}(T)}$. Then $r \geq s$. Suppose for contradiction that $c \geq d$. It cannot be that $r = s$, because that would imply $c < d$ by the increasing row rule. Thus $r > s$. Similarly, it cannot be that $c = d$, so it must be that $c > d$. Consider that the cell (r, d) is above (s, d) in the same column and to the left of (r, c) in the same row. But this implies the contradiction $i + 1 < T(r, d) < i$ by the increasing row and column rules. Thus $c < d$, implying $i \in \mathrm{Des}(T')$. By contraposition, if $i \in \overline{\mathrm{Des}(T')}$ then $i \in \mathrm{Des}(T)$. Thus $\overline{\mathrm{Des}(T')} \subseteq \mathrm{Des}(T)$.

We have shown $\mathrm{Des}(T) = \overline{\mathrm{Des}(T')}$. Thus $[n-1] = \mathrm{Des}(T) \sqcup \mathrm{Des}(T')$, so

$$\begin{aligned}
\sum_{i \in [n-1]} i &= \sum_{i \in \mathrm{Des}(T) \sqcup \mathrm{Des}(T')} i \\
\sum_{i=1}^{n-1} i &= \sum_{i \in \mathrm{Des}(T)} i + \sum_{i \in \mathrm{Des}(T')} i \\
\frac{n(n-1)}{2} &= \operatorname{maj}(T) + \operatorname{maj}(T').
\end{aligned}$$

□

To define the tableau inv statistic as in [4], we first define cycle blocks, the function Ψ_k, tableau inversion path, and tableau inversion pair.

Definition 8.5. Let T be a standard tableau and k an entry in T. Suppose $a < k$ is an entry of T located on the same side of $\pi_k(T)$ as the entry 1 in T (above $\pi_k(T)$ if 1 is above $\pi_k(T)$ and below $\pi_k(T)$ if 1 is below $\pi_k(T)$; consider the beginning of the path to have a directly rightward extension until the rightmost extremity of the diagram). Suppose $a+1, a+2, \ldots, a+m$ is a maximal-length sequence of consecutive-integer entries in T such that $a+1, a+2, \ldots, a+m$ are on the opposite side of $\pi_k(T)$ as a and $a+m < k$. Then $(a, a+1, a+2, \ldots, a+m)$ is a *cycle block* in T according to the path $\pi_k(T)$.

Definition 8.6. Let T be a standard tableau and k an entry in T. Define $\Psi_k(T)$ to be the standard tableau formed by performing the following modification to T: for every cycle block $(a_1, \ldots, a_m)$ in T according to $\pi_k(T)$, perform a right-cyclic-shift of the cycle block members in their respective cells in T.

Example 8.7. For the standard tableau T in Fig. 8.1, the cells in T below the path $\pi_{11}(T)$ are those containing 5, 7, 12, 1, 2, 6, and 9, and the cycle blocks in T according to $\pi_{11}(T)$ are (1), $(2, 3, 4)$, (5), (6), $(7, 8)$, and $(9, 10)$. We perform a right-cyclic shifting of each cycle block $(a_1, a_2, \ldots, a_{m-1}, a_m) \to (a_m, a_1, \ldots, a_{m-2}, a_{m-1})$ to obtain $\Psi_{11}(T)$ as shown.

The following definition and theorem establish a weight-preserving bijection on tableaux analogous to Foata's map for permutations.

Definition 8.8. Let the function $\Psi = \Psi_3 \circ \Psi_4 \circ \cdots \circ \Psi_{n-1} \circ \Psi_n$.

Theorem 8.9 ([4])**.** *The function* $\Psi\colon \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$ *is a bijection such that for all* $T \in \mathrm{SYT}(\lambda)$, $\mathrm{inv}(T) = \mathrm{maj}(\Psi(T))$.

Definition 8.10. Let T be a standard tableau with shape $\lambda \vdash n$. The *inversion paths* of T are the paths

$$\begin{aligned}
&\pi_n(T),\\
&\pi_{n-1}(\Psi_n(T)),\\
&\pi_{n-2}(\Psi_{n-1} \circ \Psi_n(T)),\\
&\vdots\\
&\pi_2(\Psi_3 \circ \cdots \circ \Psi_{n-1} \circ \Psi_n(T)).
\end{aligned}$$

Definition 8.11. Let T be a standard tableau and let π be an inversion path of T. Let b be the entry of the first cell of π. Suppose a is an entry in a cell below π such that $a < b$. Then (b, a) is an *inversion pair* in T.

Definition 8.12 ([4]). Let T be a standard tableau. Then we define $\mathrm{inv}(T)$ to be the number of inversion pairs in T.

Example 8.13. Table 8.1 presents an example computation of $\mathrm{inv}(T)$, where T is the first tableau in the column marked $\Psi_{k+1} \circ \cdots \circ \Psi_9(T)$. To follow the first two iterations: the path $\pi_9(T) = ((3,3),(3,2),(2,2),(2,1),(1,1))$ is an inversion path of T. Thus

$$(9,8),(9,5),(9,3),(9,2),(9,1)$$

are inversion pairs in T. The cycle blocks of T according to $\pi_9(T)$ are (1), $(2,3,4)$, (5), (6), and $(7,8)$. We thus compute $\Psi_9(T)$ as shown. Then we find the path

$$\pi_8(\Psi_9(T)) = ((2,3),(1,3),(1,2),(1,1)),$$

which is an inversion path of T. Considering this inversion path of T, we see that

$$(7,6)$$

is an inversion pair in T. The cycle blocks of $\Psi_9(T)$ according to $\pi_8(\Psi_9(T))$ are (1), (2), (3), (4), and $(5,6)$. We then compute $\Psi_8 \circ \Psi_9(T)$ as shown. From counting the number of inversion pairs found in each iteration, we conclude that $\mathrm{inv}(T) = 5+1+4+1+3+0+2+0 = 16$.

The following four lemmas give some results dealing with the relationship between conjugation, π_k paths, and the function Ψ_k, laying the foundation for Theorem 8.19.

Lemma 8.14. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Let $k \in \{2,3,\ldots,n\}$. Then*

$$\pi_k(T') = \pi_k(T)'.$$

k	$\Psi_{k+1} \circ \cdots \circ \Psi_9(T)$	$\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_9(T))$	inversion pairs identified
9	$\begin{array}{\|c\|c\|c\|}\hline 4&8&9\\\hline 3&5&7\\\hline 1&2&6\\\hline\end{array}$	$((3,3),(3,2),(2,2),(2,1),(1,1))$	$(9,7),(9,6),(9,5),(9,2),(9,1)$
8	$\begin{array}{\|c\|c\|c\|}\hline 3&7&9\\\hline 2&5&8\\\hline 1&4&6\\\hline\end{array}$	$((2,3),(1,3),(1,2),(1,1))$	$(7,6)$
7	$\begin{array}{\|c\|c\|c\|}\hline 3&7&9\\\hline 2&6&8\\\hline 1&4&5\\\hline\end{array}$	$((3,2),(2,2),(1,2),(1,1))$	$(8,7),(8,6),(8,5),(8,2)$
6	$\begin{array}{\|c\|c\|c\|}\hline 4&7&9\\\hline 2&6&8\\\hline 1&3&5\\\hline\end{array}$	$((2,2),(1,2),(1,1))$	$(5,2)$
5	$\begin{array}{\|c\|c\|c\|}\hline 5&7&9\\\hline 3&6&8\\\hline 1&2&4\\\hline\end{array}$	$((3,1),(2,1),(1,1))$	$(4,3),(4,2),(4,1)$
4	$\begin{array}{\|c\|c\|c\|}\hline 5&7&9\\\hline 3&6&8\\\hline 1&2&4\\\hline\end{array}$	$((1,3),(1,2),(1,1))$	none
3	$\begin{array}{\|c\|c\|c\|}\hline 5&7&9\\\hline 3&6&8\\\hline 1&2&4\\\hline\end{array}$	$((2,1),(1,1))$	$(3,2),(3,1)$
2	$\begin{array}{\|c\|c\|c\|}\hline 5&7&9\\\hline 3&6&8\\\hline 1&2&4\\\hline\end{array}$	$((1,2),(1,1))$	none

Table 8.1: Computation of $\mathrm{inv}(T)$
We compute $\mathrm{inv}(T)$ iteratively. For each iteration $k = n$ to $k = 2$ find the path π_k in the working tableau $\Psi_{k+1} \circ \cdots \circ \Psi_9(T)$ and use that path to identify inversion pairs in the original tableau T. Then apply Ψ_k to the working tableau to obtain the working tableau for the next iteration. Note that the tableau in row $k = 9$ is T.

Proof. Let T be as above. Let $k \in \{2, 3, \ldots, n\}$. Consider the paths $\pi_k(T)$, $\pi_k(T')$. For $1 \leq i \leq \text{length}(\pi_k(T))$, define

$$c_i = i^{\text{th}} \text{ cell of } \pi_k(T)$$

$$d_i = i^{\text{th}} \text{ cell of } \pi_k(T').$$

We will prove by induction that $d_i = c_i'$. Consider that $T'(d_1) = T(c_1) = k$, so it must be that $d_1 = c_1'$. Suppose that $d_{i-1} = c_{i-1}'$. Let $(a, b) = c_{i-1}$. Then $d_{i-1} = (a, b)' = (b, a)$. Note that the cell beneath c_{i-1} is $(a-1, b)$. Note that the cell to the left of d_{i-1} is $(b, a-1)$. Thus the cell beneath c_{i-1} is the conjugate of the cell to the left of d_{i-1}, so the content of these two cells is the same. Label this content as x. Similarly the cell to the left of c_{i-1} and the cell beneath d_{i-1} have the same content, and label this content y. If $x > y$ then, by the definition of π_k, $c_i = (a-1, b)$ and $d_i = (b, a-1)$. If $y > x$ then $c_i = (a, b-1)$ and $d_i = (b-1, a)$. Either way, $d_i = c_i'$. Since this is true for all $1 \leq i \leq \text{length}(\pi_k(T))$, we have shown

$$\pi_k(T') = \pi_k(T)'.$$

□

Lemma 8.15. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Let $k \in \{2, 3, \ldots, n\}$. Let x be an entry in T. Let π be a path in T. Then x is located above π in T if and only if x is located below π', the conjugate of π, in T'.*

Proof. Let T, k, x, and π be as above. Let (r, c) be the cell containing x in T and p be the uppermost row of π at the column c, $(p, c) \in \pi$. Suppose x is above π in T. Then $r \geq p$. Observe that (c, r) is the cell of x in T', and the cell in π' corresponding to (p, c) in π is (c, p). Since $r \geq p$, we can say that x is to the right of the path π in T'. Thus x is below π' in T'. By analogous logic it can be shown that if x is below π in T then x is above π' in T', which gives by contraposition that if x is below π' in T' then x is above π in T. Thus x is above π in T if and only if x is below π' in T'. □

Lemma 8.16. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Let $k \in \{2, 3, \ldots, n\}$. Then $(a_1, \ldots, a_m)$ is a cycle block in T according to the path $\pi_k(T)$ if and only if $(a_1, \ldots, a_m)$ is a cycle block in T' according to the path $\pi_k(T')$.*

Proof. Let T and k be as above. Let $(a_1, \ldots, a_m)$ be a cycle block in T according to the path $\pi_k(T)$.

Suppose a_1 is above $\pi_k(T)$ in T. Then $a_2, \ldots, a_m$ are below $\pi_k(T)$ in T. By Lemma 8.14, $\pi_k(T')$ is the conjugate of $\pi_k(T)$. Then by Lemma 8.15, a_1 is below $\pi_k(T')$ in T' and $a_2, \ldots, a_m$ are above $\pi_k(T')$ in T'. Thus $(a_1, \ldots, a_m)$ is a cycle block in T' according to $\pi_k(T')$. Similarly, if a_1 is below $\pi_k(T)$ in T, then a_1 is above $\pi_k(T')$ in T' and $a_2, \ldots, a_m$ are below $\pi_k(T')$ in T', and $(a_1, \ldots, a_m)$ is a cycle block in T' according to $\pi_k(T')$.

The same argument holds to show that if $(a_1, \ldots, a_m)$ is a cycle block in T' according to $\pi_k(T')$ then $(a_1, \ldots, a_m)$ is a cycle block in T according to $\pi_k(T)$. $\square$

Lemma 8.17. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Let $k \in \{2, 3, \ldots, n\}$. Then*

$$\Psi_k(T') = \Psi_k(T)'.$$

Proof. Let T and k be as above. Let x be an entry in T with cell c.

Suppose x is not in any cycle block of T according to $\pi_k(T)$. Then the cell of x in $\Psi_k(T)$ is c. Note that the cell of x in T' is c'. Also, by Lemma 8.16, x is not in any cycle block of T' according to $\pi_k(T')$. Thus the cell of x in $\Psi_k(T')$ is c'.

Suppose x is in the cycle block $(a_1, \ldots, a_m)$ of T according to $\pi_k(T)$, $x = a_l$ for some $l \in \{1, \ldots, m\}$. Then the cell of x in $\Psi_k(T)$ is the cell of a_{l+1} in T (or the cell of a_1 in T if $l = 1$). By Lemma 8.16, x is in the cycle block $(a_1, \ldots, a_m)$ of T' according to $\pi_k(T')$. Thus the cell of x in $\Psi_k(T')$ is the cell of a_{l+1} in T' (or the cell of a_1 in T' if $l = 1$). But we know that the cell of a_{l+1} in T' (or a_1 in T') is the conjugate of the cell of a_{l+1} in T (or a_1 in T). Thus the cell of x in $\Psi_k(T')$ is the conjugate of the cell of x in $\Psi_k(T)$.

We have shown that for all entries x in T, the cell of x in $\Psi_k(T')$ is the conjugate of the cell of x in $\Psi_k(T)$. Thus

$$\Psi_k(T') = \Psi_k(T)'.$$ $\square$

Corollary 8.18 is a re-expression of Lemma 8.17 in the form that will be needed to prove results about inversion paths in Theorem 8.19.

Corollary 8.18. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Let $k \in \{2, 3, \ldots, n\}$. Then*

$$\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T' = (\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)'.$$

Proof. Follows immediately from applying Lemma 8.17 inductively. $\square$

The following theorem shows the same complementarity of conjugate tableau pairs in Haglund–Stevens inversion statistic as for the Haglund–Stevens major index statistic in Theorem 8.4.

Theorem 8.19. *Let T be a standard tableau of shape λ where $\lambda \vdash n$. Then*

$$\mathrm{inv}(T) + \mathrm{inv}(T') = \frac{n(n-1)}{2}.$$

Proof. Let T be as above. Let I be the set of legal inversion pairs in $[n]^2$,

$$I = \{(x, y) : x, y \in \{1, \ldots, n\}, x > y\}.$$

Note $|I| = \frac{n(n-1)}{2}$. Let J be the set of inversion pairs in T and K the set of inversion pairs in T'. Note that J and K are subsets of I.

Let $(x, y) \in J$. Let $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)$ be the inversion path beginning at x in T. By Corollary 8.18 we have that

$$\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T' = (\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)',$$

so by Lemma 8.14 we have that

$$\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T') = \pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)'.$$

Consequently $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T')$ is the inversion path beginning at x in T'. Since $(x, y) \in J$, we know that y is below $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)$ in T. Thus, by Lemma 8.15, y is above $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T')$ in T', and so $(x, y) \notin K$. In other words, $J \subseteq \overline{K}$ where $\overline{K} = I \setminus K$. By similar reasoning, if $(x, y) \in \overline{K}$ then y is above $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T')$ in T' and y is below $\pi_k(\Psi_{k+1} \circ \cdots \circ \Psi_n \circ T)$ in T, so $(x, y) \in J$ which implies $\overline{K} \subseteq J$. Thus $J = \overline{K}$ and $I = J \sqcup K$. Thus

$$\begin{aligned} \operatorname{inv}(T) + \operatorname{inv}(T') &= |J| + |K| \\ &= |J \sqcup K| \\ &= |I| \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

□

Definition 8.20. Two statistics f and g are *jointly distributed* on a set X if for every $x \in X$, there exists $y \in X$ such that $f(y) = g(x)$ and $g(y) = f(x)$. For example, inversion statistic and major index statistic are jointly distributed on S_5, as seen in Table 7.1. Also, the tableau inversion statistic and major index statistic are not jointly distributed on $\mathrm{SYT}(\lambda \vdash n)$, as seen in Table 8.2.

The modified Foata's bijection F on S_n provides a joint distribution of inversion statistic and major index statistic on S_n.

Definition 8.21. Let ϕ be Foata's bijection. Then the modified Foata's bijection is

$$F = I \circ \phi \circ I \circ \phi^{-1} \circ I,$$

where I denotes taking the permutation inverse.

Theorem 8.22 ([6])**.** *For every* $w \in S_n$, $\operatorname{inv}(w) = \operatorname{maj}(F(w))$ *and* $\operatorname{maj}(w) = \operatorname{inv}(F(w))$.

Since the inversion statistic and major index statistic are jointly distributed on S_n, we would expect this property to be paralleled on the tableau side with the tableau statistics we use for the Macdonald Conjecture. Since the tableau inversion and major index statistics fail to be jointly distributed on $\mathrm{SYT}(\lambda \vdash n)$ (see Table 8.2), this decreases their suitability as candidates for $\tilde{a}_{(n)}$ and $\tilde{b}_{(n)}$.

Macdonald's conjecture remains unproven in the general case. The method of proof would involve developing a modified RSK algorithm, which, as discussed in Chapter 7, would need to allow for moving entries of partial tableaux to lower rows in some applications of the modified row

$T \in \mathrm{SYT}(\lambda \vdash 5)$	$\mathrm{inv}(T)$	$\mathrm{maj}(T)$	$\mathrm{Des}(T)$	$T \in \mathrm{SYT}(\lambda \vdash 5)$	$\mathrm{inv}(T)$	$\mathrm{maj}(T)$	$\mathrm{Des}(T)$
$\begin{array}{ccccc} 1&2&3&4&5 \end{array}$	0	0	$\{\}$	$\begin{array}{cccc} 5 \\ 1&2&3&4 \end{array}$	4	4	$\{4\}$
$\begin{array}{cccc} 4 \\ 1&2&3&5 \end{array}$	3	3	$\{3\}$	$\begin{array}{cccc} 3 \\ 1&2&4&5 \end{array}$	2	2	$\{2\}$
$\begin{array}{cccc} 2 \\ 1&3&4&5 \end{array}$	1	1	$\{1\}$	$\begin{array}{ccc} 4&5 \\ 1&2&3 \end{array}$	6	3	$\{3\}$
$\begin{array}{ccc} 3&5 \\ 1&2&4 \end{array}$	5	6	$\{2,4\}$	$\begin{array}{ccc} 2&5 \\ 1&3&4 \end{array}$	3	5	$\{1,4\}$
$\begin{array}{ccc} 3&4 \\ 1&2&5 \end{array}$	4	2	$\{2\}$	$\begin{array}{ccc} 2&4 \\ 1&3&5 \end{array}$	2	4	$\{1,3\}$
$\begin{array}{cc} 5 \\ 3&4 \\ 1&2 \end{array}$	8	6	$\{2,4\}$	$\begin{array}{cc} 4 \\ 3&5 \\ 1&2 \end{array}$	7	5	$\{2,3\}$
$\begin{array}{cc} 5 \\ 2&4 \\ 1&3 \end{array}$	6	8	$\{1,3,4\}$	$\begin{array}{cc} 4 \\ 2&5 \\ 1&3 \end{array}$	5	4	$\{1,3\}$
$\begin{array}{cc} 3 \\ 2&5 \\ 1&4 \end{array}$	4	7	$\{1,2,4\}$	$\begin{array}{cc} 5 \\ 4 \\ 3 \\ 1&2 \end{array}$	9	9	$\{2,3,4\}$
$\begin{array}{cc} 5 \\ 4 \\ 2 \\ 1&3 \end{array}$	8	8	$\{1,3,4\}$	$\begin{array}{cc} 5 \\ 3 \\ 2 \\ 1&4 \end{array}$	7	7	$\{1,2,4\}$
$\begin{array}{cc} 4 \\ 3 \\ 2 \\ 1&5 \end{array}$	6	6	$\{1,2,3\}$	$\begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}$	10	10	$\{1,2,3,4\}$

Table 8.2: inv **and** maj **are not jointly distributed on** $\mathrm{SYT}(\lambda \vdash 5)$
For example, let U be the tableau of shape $(3,2)$ and reading word 25134. Observe $\mathrm{inv}(U) = 3$, $\mathrm{maj}(U) = 5$, but there is no $T \in \mathrm{SYT}(\lambda \vdash 5)$ such that $\mathrm{inv}(T) = 5$ and $\mathrm{maj}(T) = 3$.

insertion procedure. The proof would also involve defining tableau statistics $\tilde{a}_\mu$ and $\tilde{b}_\mu$ satisfying the condition $\tilde{a}_\mu(Q) = \mathrm{inv}_\mu(w)$ and $\tilde{b}_\mu(Q) = \mathrm{maj}_\mu(w)$ for $(P, Q) = \mathrm{RSK}'(w)$, and possibly satisfying the condition of being jointly distributed on $\{T \in \mathrm{SYT}(\lambda) : \lambda \vdash n\}$.

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APPENDIX A

LETTER FROM INSTITUTIONAL RESEARCH BOARD

MARSHALL UNIVERSITY®

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Office of Research Integrity

January 18, 2017

Jacob Rodeheffer
1456 4th Avenue, #303
Huntington, WV 25701

Dear Mr. Rodeheffer:

This letter is in response to the submitted thesis abstract entitled *"The Combinatorics of Modified Macdonald Polynomials."* After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director

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Jacob Rodeheffer

— Curriculum Vitae —

January 24, 2017

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Education

MA, Mathematics
College of Science, Marshall University
Expected May 2017
GPA: 4.0
Thesis advisor: Dr. Elizabeth Niese
Coursework includes 12 hours of probability and statistics.

BS, Electrical Engineering
Viterbi School of Engineering, University of Southern California
2014
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Teaching Certifications and Development

Certificate in College Teaching
Center for Teaching and Learning, Marshall University
Expected Spring 2017

Graduate Mathematics Seminar
Marshall University
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Weekly seminar concerning teaching theory and practice, professional development, and academic success.

Teaching Experience as Primary Instructor

Developmental Mathematics
Marshall University
Summer 2016, Fall 2016, and Spring 2017

Middle School

Christ the King Catholic School, Detroit

2014-2015 academic year

Taught two classes of math and three classes of religion for 6th, 7th, and 8th graders. All the responsibilities of a full-time teacher. Placed at school through Christ the King Service Corps (www.ckscdetroit.org).

Teaching Experience as Assistant

College Algebra

Marshall University

Fall 2016, Spring 2017

Developmental Mathematics

Marshall University

Fall 2015 (2 sections) and Spring 2016 (2 sections)

Tutoring Experience

Math Tutoring Lab

Marshall University

Fall 2015, Spring 2016, Fall 2016, Spring 2017

After School Assistance

Christ the King Catholic School

2014-2015 academic year

Homeboy Industries

Spring 2014

Tutored former gang members in math towards earning their GEDs.

Foshay Learning Center

Spring 2014

Tutored struggling high school math students.

Holy Name of Jesus Catholic School

Spring 2014

Tutored struggling middle school math students.

Navy Reserve Officers' Training Corps, University of Southern California

Spring 2013

Tutored undergraduate ROTC students in math, physics, and programming.

Computer Languages Experience

Projects in C++
- Thesis computation aid
- 3D ray tracer
- 3D roller coaster simulator
- 3D heightmap manipulator
- Neural network simulator with 2D interface
- Maze solver

Projects in Java
- 2D village simulator

Projects in Verilog
- Mastermind solver

Projects in Assembly Code
- CPU emulator

Projects in LaTeX
- Master's thesis

Other Experience

3D Printing Research Intern
- Jet Propulsion Laboratory
- Summer 2013

  Advised by engineers, used software to process digital models of Martian terrain in preparation for 3D printing. Gave presentation summarizing research.

Eagle Scout
- Boy Scout Troop 80, Mountain View, CA
- 2010

Publications

The Combinatorics of Modified Macdonald Polynomials
- Master's Thesis
- Advisor: Dr. Elizabeth Niese
- Expected Spring 2017