12-2012

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Recommended Citation
Loehr, N. A., & Niese, E. (2012). A bijective proof of a factorization formula for specialized Macdonald polynomials. Annals of Combinatorics, 16(4), 815-828.

A bijective proof of a factorization formula for specialized Macdonald polynomials

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October 5, 2011

Abstract

Let μ and $\nu = (\nu_1, \ldots, \nu_k)$ be partitions such that μ is obtained from ν by adding m parts of size r. Descouens and Morita proved algebraically that the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ satisfy the identity $\tilde{H}_\mu = \tilde{H}_\nu \tilde{H}_{(r^m)}$ when the parameter t is specialized to an mth root of unity. Descouens, Morita, and Numata proved this formula bijectively when $r \leq \nu_k$ and $r \in \{1, 2\}$. This note gives a bijective proof of the formula for all $r \leq \nu_k$.

AMS Subject Classification Numbers: 05A19, 05E05, 05A05.

Keywords: Macdonald polynomials, inversions, q-multinomial coefficients, bijections.

1 Introduction

Macdonald polynomials [9, 10] have become central objects of study in the theory of symmetric functions. The modified Macdonald polynomials $\tilde{H}_\mu$ are symmetric polynomials in variables $x_1, \ldots, x_N$ with coefficients in $\mathbb{N}[q, t]$. This paper deals with a factorization formula satisfied by the modified Macdonald polynomials when t is specialized to a root of unity. More specifically, fix $m \in \mathbb{N}^+$, and let $\zeta = e^{2\pi i/m}$ be a primitive mth root of unity. Suppose $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$ is an integer partition, $r \in \mathbb{N}^+$, and μ is the partition obtained from ν by adding m new parts of size r. In [1], Descouens and Morita used symmetric function identities to give an algebraic proof of the following formula:

$$\tilde{H}_\mu(x_1, \ldots, x_N; q, \zeta) = \tilde{H}_\nu(x_1, \ldots, x_N; q, \zeta) \cdot \tilde{H}_{(r^m)}(x_1, \ldots, x_N; q, \zeta). \tag{1}$$

In [2], Descouens, Morita, and Numata pose the problem of proving (1) *bijectively* using the combinatorial formula for modified Macdonald polynomials found by Haglund [4] and proved by Haglund, Haiman, and Loehr [5, 6]. Descouens et al. are able to solve this problem in the case where $r \leq \nu_k$ and $r \in \{1, 2\}$. In this paper, we give a bijective proof of (1) for any choice of $r \leq \nu_k$. Our bijections restrict to the ones given in [2] when $r = 1$ or $r = 2$. We stress that formula (1) holds in full generality, without the assumption that $r \leq \nu_k$. But the bijective approaches studied here and in [2] cannot yet handle the cases where $r > \nu_k$.

The rest of the paper is organized as follows. Section 2 reviews the combinatorial definition of modified Macdonald polynomials from [4, 5, 6]. Section 3 describes the strategy we use to develop our bijection. This strategy is implemented in Sections 4 and 5. Final comments appear in Section 6, including a discussion of the combinatorial difficulties that arise when our hypothesis $r \leq \nu_k$ is dropped.

2 Combinatorial Definition of $\tilde{H}_\mu$

This section reviews the combinatorial formula for $\tilde{H}_\mu$, which was conjectured in [4] and proved in [5, 6]. Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l)$ be a fixed integer partition, and let $[N]$ denote the set $\{1, 2, \ldots, N\}$. The *Ferrers diagram* of μ is

$$\mathrm{dg}(\mu) = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \leq i \leq l, 1 \leq j \leq \mu_i\}.$$

A *filling* of μ is a function $T : \mathrm{dg}(\mu) \to [N]$. (Some authors refer to a filling as a *tableau*, whereas other authors use the word "tableau" as an abbreviation for "semistandard Young tableau." To avoid potential confusion, we use the term "filling" throughout.) Let $\mathcal{F}_\mu$ be the set of all fillings of μ. We often visualize $\mathrm{dg}(\mu)$ as a collection of unit squares, and we visualize T by placing the number $T(c)$ in the square representing $c \in \mathrm{dg}(\mu)$. For example, the Ferrers diagram of $\mu = (5, 3, 3, 3)$ is pictured (using the French convention) as

$$\mathrm{dg}(\mu) = \begin{array}{|c|c|c|cc}
\cline{1-3}
 & & & & \\
\cline{1-3}
 & & & & \\
\cline{1-3}
 & & & & \\
\hline
 & & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\
\hline
\end{array}\ ,$$

and a typical element of $\mathcal{F}_\mu$ is

$$T = \begin{array}{|c|c|c|cc}
\cline{1-3}
3 & 5 & 4 & & \\
\cline{1-3}
3 & 3 & 2 & & \\
\cline{1-3}
4 & 1 & 4 & & \\
\hline
2 & 5 & 3 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\
\hline
\end{array}\ . \tag{2}$$

Given $T \in \mathcal{F}_\mu$, the *content monomial* of T is $x^T = \prod_{c \in \mathrm{dg}(\mu)} x_{T(c)}$. For instance, the filling T pictured in (2) has content monomial $x_1^2 x_2^3 x_3^4 x_4^3 x_5^2$.

For any word $w = w_1 w_2 \cdots w_k$ with each $w_i \in [N]$, the *major index* of w, denoted $\mathrm{maj}(w)$, is the sum of all $i < k$ such that $w_i > w_{i+1}$. Given $T \in \mathcal{F}_\mu$, we obtain μ_1 *column words of* T by reading each column of T from top to bottom. We define $\mathrm{maj}_\mu(T)$ to be the sum of the major indices of these column words. For the example filling shown in (2), we compute

$$\mathrm{maj}_\mu(T) = \mathrm{maj}(3\,3\,4\,2) + \mathrm{maj}(5\,3\,1\,5) + \mathrm{maj}(4\,2\,4\,3) + \mathrm{maj}(2) + \mathrm{maj}(1) = 3 + 3 + 4 + 0 + 0 = 10.$$

Next, suppose $T \in \mathcal{F}_\mu$ and we have a triple of cells in T positioned as shown:

$$\begin{array}{|c|ccc|c|}
\cline{1-1}\cline{5-5}
y & & & & x \\
\cline{1-1}\cline{5-5}
z & & & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array} \tag{3}$$

Formally, we have two cells c_1 and c_2 in the same row of $\mathrm{dg}(\mu)$, with c_1 somewhere to the right of c_2, and a third cell c_3 just below c_2; and $x = T(c_1)$, $y = T(c_2)$, and $z = T(c_3)$. We allow c_1 and c_2 to be in the lowest row of $\mathrm{dg}(\mu)$; in this case, c_3 is outside the Ferrers diagram, and we set $z = \infty$. This triple of cells is called an *inversion triple of* T iff $x < y \leq z$ or $y \leq z < x$ or $z < x < y$. We define $\mathrm{inv}_\mu(T)$ to be the number of inversion triples of T. For the example filling shown in (2), $\mathrm{inv}_\mu(T) = 12$.

Haglund's combinatorial formula for the modified Macdonald polynomials is

$$\tilde{H}_\mu(x_1,\ldots,x_N;q,t) = \sum_{T\in\mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)} x^T. \tag{4}$$

For instance, the filling T pictured in (2) contributes the term $q^{12}t^{10}x_1^2x_2^3x_3^4x_4^3x_5^2$ to $\tilde{H}_{(5,3,3,3)}$.

For later work, it will be helpful to define the *inversion table* of a filling $T : \mathrm{dg}(\mu) \to [N]$. The inversion table of T is a function $I_T : \mathrm{dg}(\mu) \to \mathbb{N}$ that tells us how many inversion triples are "caused" by each cell $c \in \mathrm{dg}(\mu)$. More precisely, define $I_T(c)$ to be the number of inversion triples of T in which c is the upper-left cell of the triple (the cell containing y in (3)). For the example filling T shown in (2), we compute

$$I_T = \begin{array}{|c|c|c|cc}\hline 2&1&0\\\hline 1&1&0\\\hline 0&0&0\\\hline 1&3&2&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{0}\\\hline\end{array}\;.$$

Similarly, let $\mathrm{Des}(T)$ be the set of cells $(i,j) \in \mathrm{dg}(\mu)$ such that $(i-1,j) \in \mathrm{dg}(\mu)$ and $T((i,j)) > T((i-1,j))$. One may check that $\mathrm{maj}_\mu(T) = \sum_{c\in\mathrm{Des}(T)}(\mathrm{leg}_\mu(c)+1)$, where $\mathrm{leg}_\mu(c)$ is the number of squares above cell c in its column. In our example, $\mathrm{Des}(T) = \{(2,1),(2,3),(3,2),(4,2),(4,3)\}$.

3 Overall Strategy

Fix a partition ν of length k, fix $r \le \nu_k$, and let μ be obtained from ν by adding m parts of size r. Recall $\zeta = e^{2\pi i/m}$ satisfies $\zeta^m = 1$. The effect of setting $t = \zeta$ in (4) is to reduce the exponent $\mathrm{maj}_\mu(T)$ modulo m. To prove (1) bijectively (in the case $r \le \nu_k$), it will therefore suffice to find a bijection $G : \mathcal{F}_\mu \to \mathcal{F}_\nu \times \mathcal{F}_{(r^m)}$ with the following properties. Setting $G(T) = (T_1,T_2)$, we must have:

(i) $x^T = x^{T_1}x^{T_2}$;

(ii) $\mathrm{inv}_\mu(T) = \mathrm{inv}_\nu(T_1) + \mathrm{inv}_{(r^m)}(T_2)$;

(iii) $\mathrm{maj}_\mu(T) \equiv \mathrm{maj}_\mu(T_1) + \mathrm{maj}_{(r^m)}(T_2) \pmod{m}$.

Our goal in this paper is to construct an explicit bijection G satisfying (i), (ii), and (iii).

To motivate the construction of G, first consider the map $H : \mathcal{F}_\mu \to \mathcal{F}_\nu \times \mathcal{F}_{(r^m)}$ defined as follows. Given $T \in \mathcal{F}_\mu$, let $H(T) = (T_1,T_2)$ where T_1 is the restriction of T to $\mathrm{dg}(\nu)$, and $T_2((i,j)) = T((i+k,j))$ for $1 \le i \le m$ and $1 \le j \le r$. Informally, we get T_1 and T_2 from T by splitting off the last m rows of the filling. For example, suppose $\nu = (5)$, $r = 3$, $m = 3$, and T is the filling shown in (2). Then $H(T) = (T_1,T_2)$, where

$$T_1 = \begin{array}{|c|c|c|c|c|}\hline 2&5&3&2&1\\\hline\end{array}\,, \qquad T_2 = \begin{array}{|c|c|c|}\hline 3&5&4\\\hline 3&3&2\\\hline 4&1&4\\\hline\end{array}\,.$$

It is evident that H is a bijection satisfying condition (i) above. Moreover, H satisfies (iii). To see why, note that descents in the top $m-1$ rows of T become descents in the top $m-1$ rows of T_2 that give the same contribution to $\mathrm{maj}_\mu(T)$ and $\mathrm{maj}_{(r^m)}(T_2)$, respectively. Descents in the rows of T belonging to ν become descents in T_1. Since the legs of cells in ν differ from the legs of the corresponding cells in μ by 0 or m, we see that the descents in this part of the shape give the same contribution (modulo m) to $\mathrm{maj}_\mu(T)$ and $\mathrm{maj}_\nu(T_1)$, respectively. Finally, any descents in the mth row from the top of T do not appear in T_1 or T_2, but these descents contribute a multiple of m to $\mathrm{maj}_\mu(T)$ and are therefore irrelevant modulo m.

Unfortunately, the map H does not satisfy condition (ii) when $r > 1$. One readily checks that the inversion table I_{T_1} is the restriction of the inversion table I_T to $\mathrm{dg}(\nu)$. Similarly, after an appropriate shift of the indexing of the rows, the top $m-1$ rows of the inversion table I_{T_2} match the top $m-1$ rows of the inversion table I_T. However, the lowest row of I_{T_2} will not always agree with the corresponding row of I_T (which records inversions in the mth row from the top of T). In our example, we find that

$$I_{T_1} = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 2 & 1 & 0 \\ \hline\end{array}, \qquad I_{T_2} = \begin{array}{|c|c|c|}\hline 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ \hline\end{array}.$$

To fix this discrepancy, we modify the output of H in two stages. In the first stage (§4), we calculate (bijectively) a rearrangement of the letters in the mth row from the top of T that will become the new bottom row of T_2. This rearrangement is designed to make the inversion count in the bottom row of T_2 match the inversion count in the corresponding row of T. However, simply replacing the old bottom row of T_2 with this rearrangement might disturb inversion triples in the second row of T_2. So, in the second stage (§5), we implement the desired rearrangement of the bottom row of T_2 by performing a specified sequence of "column-switching" moves. Each such move interchanges two consecutive symbols in the bottom row of T_2 and (sometimes) modifies further entries in these columns to ensure the preservation of inversion triples and descents above the bottom row. The pair consisting of T_1 and the modified T_2 will be $G(T)$. We will see that this map G is a bijection satisfying conditions (i), (ii), and (iii) above.

4 Inversion Analysis

4.1 u-Inversions

Given a word $w = w_1 w_2 \cdots w_n$ with each $w_i \in [N]$, the classical *inversion number* of w, denoted $\mathrm{inv}(w)$, is the number of pairs $i < j$ with $w_i > w_j$. Dually, $\mathrm{coinv}(w)$ is the number of pairs $i < j$ with $w_i < w_j$. More generally, given any total ordering $\prec$ on the set $[N]$, we can define inversions relative to this ordering by letting $\prec$-$\mathrm{inv}(w)$ be the number of pairs $i < j$ with $w_i \succ w_j$. When studying inversion triples of fillings, we will only need to consider total orderings that are cyclic shifts of the usual ordering on $[N]$. More precisely, for each $u \in [N]$, let $\overset{u}{<}$ be the total ordering on $[N]$ given by

$$(u+1) \overset{u}{<} (u+2) \overset{u}{<} \cdots \overset{u}{<} N \overset{u}{<} 1 \overset{u}{<} 2 \overset{u}{<} \cdots \overset{u}{<} (u-1) \overset{u}{<} u.$$

(The superscript u reminds us that u is the greatest symbol relative to this ordering.) Let u-$\mathrm{inv}(w)$ be the number of pairs $i < j$ with $w_i \overset{u}{>} w_j$. Note that N-$\mathrm{inv}(w) = \mathrm{inv}(w)$.

Let $W = \mathcal{R}(1^{a_1} 2^{a_2} \cdots N^{a_N})$ be the set of all words that are rearrangements of a_1 copies of 1, a_2 copies of 2, etc. It can be shown that, for any total ordering $\prec$ of $[N]$,

$$\sum_{w \in W} q^{\prec\text{-inv}(w)} = \begin{bmatrix} a_1 + a_2 + \cdots + a_N \\ a_1, a_2, \ldots, a_N \end{bmatrix}_q, \tag{5}$$

where the right side is a q-multinomial coefficient [7]. Thus, the N statistics u-inv (for $u = 1, 2, \ldots, N$) are all equidistributed on W. We will require an explicit bijective proof of this fact.

Lemma 1. *Let $u \leq s \in [N]$, and let $W = \mathcal{R}(1^{a_1} 2^{a_2} \cdots N^{a_N})$. There is a bijection $f = f_{u,s} = f_{s,u} : W \to W$ (defined below) such that for all $w \in W$, $s\text{-inv}(f(w)) = u\text{-inv}(w)$ and $u\text{-inv}(f(w)) = s\text{-inv}(w)$. In fact, f is an involution.*

Proof. For fixed $u \leq s$, we define $f : W \to W$ as follows. Given $w = w_1 w_2 \cdots w_m$, first define $x = x_1 x_2 \cdots x_m$ by setting $x_i = 0$ if $u < w_i \leq s$ and $x_i = 1$ otherwise. (Intuitively, x relabels "small" letters relative to $\overset{u}{<}$ by 0 and "large" letters relative to $\overset{u}{<}$ by 1. Passage to $\overset{s}{<}$ will interchange the roles of small and large letters.) Let y (resp. z) be the subword of w consisting of symbols in the positions where $x_i = 0$ (resp. $x_i = 1$). Let x' be the reversal of x. Use x' to form $w' = f(w)$ by replacing the zeroes by the symbols in y and replacing the ones by the symbols in z (scanning y, z, and x' from left to right). For example, take $u = 3$, $s = 6$, $N = 7$, and $w = 25431167521745$. Then $x = 10011101011100$, $y = 546545$, $z = 23117217$, $x' = 00111010111001$, so $w' = f_{3,6}(w) = 54231615721457$. Since the reversal $x \mapsto x'$ is an involution, it readily follows that f is also an involution, hence a bijection. It now suffices to show that $s\text{-inv}(w') = u\text{-inv}(w)$. First observe that, by the definitions of x and $\overset{u}{<}$, we have

$$u\text{-inv}(w) = u\text{-inv}(y) + u\text{-inv}(z) + \text{inv}(x).$$

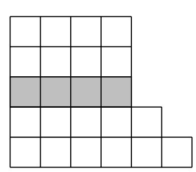
Similarly,

$$s\text{-inv}(w') = s\text{-inv}(y) + s\text{-inv}(z) + \text{coinv}(x').$$

We complete the proof by noting that $\text{inv}(x) = \text{coinv}(x')$, $u\text{-inv}(y) = s\text{-inv}(y)$, and $u\text{-inv}(z) = s\text{-inv}(z)$. The last two equalities follow since the orderings $\overset{u}{<}$ and $\overset{s}{<}$ agree on the set of u-small letters (resp. the set of u-large letters). □

4.2 Rearrangement of the Critical Row

For any filling $Z \in \mathcal{F}_{(r^m)}$ with bottom row consisting of the word $z = z_1 z_2 \cdots z_r$, the number of inversion triples in the bottom row of z is precisely $\text{inv}(z)$, as one readily checks. Now, suppose $T \in \mathcal{F}_\mu$ and $H(T) = (T_1, T_2)$ as in §3, where $\mu = (\nu_1, \ldots, \nu_k, r, \ldots, r)$. Suppose row k of T consists of the word $u = u_1 u_2 \cdots u_r \cdots$ and row $k+1$ of T consists of the word $w = w_1 w_2 \cdots w_r$. Call row $k+1$ of T (which is the mth row from the top) the *critical row* of T (or of μ). For example, suppose $\nu = (6, 5)$, $r = 4$, and $m = 3$, so that $\mu = (6, 5, 4, 4, 4)$. The critical row of μ is shaded in the following picture:



Define $n_i = I_T((k+1, i))$ for $1 \leq i \leq r$, so $n_1, \ldots, n_r$ count the inversion triples caused by the symbols $w_1, \ldots, w_r$ (respectively) in T. To accomplish stage 1 of our strategy in §3, we will use the bijection g appearing in the following lemma.

Lemma 2. *For fixed $u = u_1 u_2 \cdots u_r \cdots$, there is a bijection $g = g_u : [N]^r \to [N]^r$ such that, for all $w \in [N]^r$: (i) $g(w)$ is a rearrangement of w; (ii) $\text{inv}(g(w)) = n_1 + \cdots + n_r$.*

Proof. We give an algorithm defining g. Start with the input word $w = w_1 w_2 \cdots w_r$. For $j = r-1$ down to $j = 1$, take the current subword $w_{j+1}w_{j+2}\cdots w_r$ and replace it by $f_{u_j,u_{j+1}}(w_{j+1}\cdots w_r)$. At the end, replace the current word $w_1 \cdots w_r$ by $f_{N,u_1}(w_1 \cdots w_r)$. Output this word.

It is evident that g satisfies (i), since each $f_{u,s}$ satisfies (i). Since u is fixed and known, we can invert g by performing the steps in the algorithm backwards. So g is a bijection. To prove (ii), let w^r be the input word w, let w^i be the value of the word just after the $j = i$ iteration of the for-loop (for $r-1 \geq i \geq 1$), and let w^0 be the output word. We will show, by backwards induction, that $u_i\text{-inv}(w_i^i \cdots w_r^i) = n_i + \cdots + n_r$ for $r \geq i \geq 1$. This holds for the base case $i = r$, since $u_r\text{-inv}(w_r) = 0 = n_r$. Now assume $r > i \geq 1$ and

$$u_{i+1}\text{-inv}(w_{i+1}^{i+1} \cdots w_r^{i+1}) = n_{i+1} + \cdots + n_r.$$

The $j = i$ iteration of the for-loop applies $f_{u_i,u_{i+1}}$ to the subword $w_{i+1}^{i+1} \cdots w_r^{i+1}$ to produce the subword $w_{i+1}^i \cdots w_r^i$. By Theorem 1 and the induction hypothesis, we must have

$$u_i\text{-inv}(w_{i+1}^i \cdots w_r^i) = n_{i+1} + \cdots + n_r.$$

Consider the effect on u_i-inv when we add the next symbol $w_i^i = w_i$ to the left end of the displayed subword. This symbol will cause new u_i-inversions with each symbol to its right that is strictly less than w_i relative to $\stackrel{u_i}{<}$. On the other hand, the definition of $\text{inv}_\mu(T)$ shows that the symbol w_i in the filling T (and u_i directly below it) will cause an inversion triple for each symbol to the right of w_i in the critical row that is strictly less than w_i relative to $\stackrel{u_i}{<}$. The definition of inversion tables tells us that there are n_i such symbols. Since $w_{i+1}^i \cdots w_r^i$ is a rearrangement of $w_{i+1} \cdots w_r$, we must also get n_i new u_i-inversions when adding w_i to the left end of the subword above. We therefore have

$$u_i\text{-inv}(w_i^i w_{i+1}^i \cdots w_r^i) = n_i + u_i\text{-inv}(w_{i+1}^i \cdots w_r^i) = n_i + n_{i+1} + \cdots + n_r,$$

completing the induction step. Finally, since $g(w) = w^0 = f_{N,u_1}(w^1)$, we get

$$\text{inv}(g(w)) = N\text{-inv}(w^0) = u_1\text{-inv}(w^1) = n_1 + \cdots + n_r,$$

so (ii) is indeed true. $\square$

Example 3. Suppose $r = 7$, $N = 4$, $u = 2344312$, and $w = 1211332$. We compute $(n_1, \ldots, n_7) = (2, 2, 0, 0, 1, 1, 0)$. The following table shows the words w^i computed by the algorithm and the u_i-inversion count of each underlined subword $w_i^i \cdots w_r^i$. The answer in each case is $n_i + \cdots + n_r$, in accordance with the lemma. (By convention, $u_0 = N$ and $n_0 = 0$.) The output is $g_u(w) = 1213312$.

i	u_i	w^i	$u_i\text{-inv}(w_i^i \cdots w_r^i)$
7	2	$121133\underline{2}$	0
6	1	$12113\underline{32}$	1
5	3	$1211\underline{332}$	2
4	4	$121\underline{1332}$	2
3	4	$12\underline{11332}$	2
2	3	$1\underline{211332}$	4
1	2	$\underline{1233112}$	6
0	4	$\underline{1213312}$	6

Example 4. Suppose $r = 7$, $N = 4$, $u = 3442131$ and $v = 2311234$. The following table shows the steps in computing $g^{-1}(v) = w$. We use the notation $v^0 = v$, $v^1 = f_{N,u_1}(v^0)$, and $v^i = v_1^{i-1} \cdots v_{i-1}^{i-1} f_{u_{i-1},u_i}(v_i^{i-1} \cdots v_r^{i-1})$ for $2 \le i \le r$.

i	u_i	v^i	u_i-inv$(v_i^i \cdots v_r^i)$
0	4	$\underline{2311234}$	5
1	3	$\underline{4231123}$	5
2	4	$4\underline{231123}$	5
3	4	$42\underline{31123}$	3
4	2	$423\underline{3112}$	0
5	1	$4233\underline{211}$	0
6	3	$42332\underline{11}$	0
7	1	$423321\underline{1}$	0

The output is $w = 4233211$. Note that $(n_1, \ldots, n_7) = (0, 2, 3, 0, 0, 0, 0)$.

5 Column Switching

5.1 The Inversion Flip Operation

To implement stage 2 of our strategy, we require the following general combinatorial operation on fillings.

Definition 5. Suppose μ is a partition such that $\mu'_i = \mu'_{i+1}$ for some i (so columns i and $i+1$ of $\mathrm{dg}(\mu)$ have equal height). Define the *inversion flip* move $s_i : \mathcal{F}_\mu \to \mathcal{F}_\mu$ as follows:

- Given $T \in \mathcal{F}_\mu$, let a (resp. b) be the entry of T at the bottom of column i (resp. $i+1$).
- Switch entries a and b in the bottom row as shown here:

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline c & d \\ \hline a & b \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline c & d \\ \hline b & a \\ \hline \end{array} \ .$$

- If a, c, d and b, c, d are either both inversion triples or both not inversion triples, the move is complete. Otherwise, apply s_i recursively to the filling T' of $(\mu_2, \mu_3, \ldots)$ obtained by ignoring the bottom row of T.

A similar operation for standard fillings was studied in [8, 11]. For two-column rectangles, the inversion flip move s_1 is equivalent to an operation used in [2] to prove the $r = 2$ case of (1) (although those authors use different notation to describe the operation).

Lemma 6. *Given a partition μ and $i, j \in \mathbb{N}^+$ with $\mu'_i = \mu'_{i+1}$ and $\mu'_j = \mu'_{j+1}$,*

(a) $s_i^2 = s_i \circ s_i = \mathrm{id}_{\mathcal{F}_\mu}$*;*

(b) $s_i \circ s_j = s_j \circ s_i$ *when* $|i - j| \ge 2$.

Proof. Both properties follow directly from the definition of s_i. $\square$

Theorem 7. *Given a partition $\mu \in \mathrm{Par}(n)$ and $i \in \mathbb{N}^+$ with $\mu'_i = \mu'_{i+1}$, let $T \in \mathcal{F}_\mu$ have entries a and b in the bottom row of columns i and $i+1$, respectively. Then:*

(a) $\mathrm{maj}_\mu(s_i(T)) = \mathrm{maj}_\mu(T)$;

(b) $\mathrm{inv}_\mu(s_i(T)) = \mathrm{inv}_\mu(T) + \begin{cases} 1 & \text{if } a < b; \\ 0 & \text{if } a = b; \\ -1 & \text{if } b < a. \end{cases}$

Proof. Since every column contributes independently to maj_μ, to prove (a) it is sufficient to consider a filling T of shape $\mu = (2^n)$ and $i = 1$. By Lemma 6(a) we may also assume that $a \leq b$. Note that if $a = b$, $s_1(T) = T$, so we consider only $a < b$. This result is true when $n = 1$.

When $n = 2$, we can write

$$T = \begin{array}{|c|c|} \hline c & d \\ \hline a & b \\ \hline \end{array}\,.$$

If

$$s_1(T) = \begin{array}{|c|c|} \hline d & c \\ \hline b & a \\ \hline \end{array}\,,$$

column words are preserved, so $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$.

It is also possible that

$$s_1(T) = \begin{array}{|c|c|} \hline c & d \\ \hline b & a \\ \hline \end{array}.$$

If a, c, d formed an inversion triple in T, then b, c, d forms an inversion triple in $s_1(T)$. If $a < b < d < c$, $d < c \leq a < b$, or $c \leq a < b < d$, then the location and number of the descents are preserved, so $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$. If $a < d < c \leq b$, the number of descents is preserved, but the column in which the descent is located changes. However, since the columns are of equal height, $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$. Similarly, if a, c, d is not an inversion triple in T, then b, c, d is not an inversion triple in $s_1(T)$. If $a < b < c \leq d$, $c \leq d \leq a < b$, or $d \leq a < b < c$, the location and number of descents are preserved. If $a < c \leq d \leq b$, the number of descents are preserved, but the column in which the descent is located changes. As before, since the columns are of equal height, $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$. By induction, this result holds for any number of rows.

To show that $\mathrm{inv}_\mu(s_i(T)) = \mathrm{inv}_\mu(T) + 1$, we must show that the inversion flip does not affect the total number of inversion triples, excluding the triple a, b, ∞. By the definition of the inversion flip, it is sufficient to consider triples positioned as shown:

$$\begin{array}{|c|c|} \hline c & d \\ \hline a & b \\ \hline \end{array} \cdots \begin{array}{|c|} \hline z \\ \hline \\ \hline \end{array}$$

since all other triples in T will be preserved. Once again, there are two possibilities. First, if

$$s_i(T) = \begin{array}{|c|c|} \hline d & c \\ \hline b & a \\ \hline \end{array} \cdots \begin{array}{|c|} \hline z \\ \hline \\ \hline \end{array}$$

the inversion triples themselves are preserved. On the other hand, if

$$s_i(T) = \begin{array}{|c|c|} \hline c & d \\ \hline b & a \\ \hline \end{array} \cdots \begin{array}{|c|} \hline z \\ \hline \\ \hline \end{array}\,,$$

to show that the total number of inversion triples is preserved requires a tedious case analysis. We present several cases here and leave the remainder to the reader. First, suppose $z \leq a < b <$

$c < d$. Then none of $z \le a < c$, $z < b < c$, $z \le a < d$, and $z < b < d$ are inversion triples. Next, if $a < z \le b < c < d$, then $a < z < c$ is an inversion triple in T, and $z \le b < c$ is not an inversion triple in $s_i(T)$. On the other hand, $z \le b < d$ is not an inversion triple in T, but $a < z < d$ is an inversion triple in $s_i(T)$. Thus, the total number of inversion triples is preserved. The remaining cases are similar. $\square$

Example 8. The following picture illustrates successive applications of inversion flip moves on a filling in $\mathcal{F}_{(4^4)}$:

$$\begin{array}{|c|c|c|c|}\hline 3&4&2&2\\\hline 3&4&1&3\\\hline 4&5&2&3\\\hline 2&3&2&4\\\hline\end{array} \xrightarrow{s_3} \begin{array}{|c|c|c|c|}\hline 3&4&2&2\\\hline 3&4&3&1\\\hline 4&5&3&2\\\hline 2&3&4&2\\\hline\end{array} \xrightarrow{s_1} \begin{array}{|c|c|c|c|}\hline 3&4&2&2\\\hline 3&4&3&1\\\hline 4&5&3&2\\\hline 3&2&4&2\\\hline\end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|c|}\hline 3&2&4&2\\\hline 3&3&4&1\\\hline 4&3&5&2\\\hline 3&4&2&2\\\hline\end{array}\,.$$

$$q^{12}t^7 \qquad\qquad q^{13}t^7 \qquad\qquad q^{14}t^7 \qquad\qquad q^{15}t^7$$

In this case, each move increases the q-power by 1 while preserving the t-power.

5.2 The Map G

We are now ready to define the bijection $G : \mathcal{F}_\mu \to \mathcal{F}_\nu \times \mathcal{F}_{(r^m)}$ with properties (i), (ii), and (iii) from §3. Given $T \in \mathcal{F}_\mu$, let $H(T) = (T_1, T_2)$, where H is the "splitting map" discussed in §3. Let $u = u_1 \cdots u_r \cdots$ be the word in the top row of T_1 (the kth row of T), and let $w = w_1 \cdots w_r$ be the word in the bottom row of T_2 (the $(k+1)$th row of T). Let $v = v_1 \cdots v_r = g_u(w)$, where g_u is the map from §4.2.

For $i = 1$ to $r - 1$, do the following: Say the current bottom row of T_2 contains the word $w' = w'_1 \cdots w'_r$. By induction, we can assume that $w'_1 \cdots w'_{i-1} = v_1 \cdots v_{i-1}$. Find the least index $j \ge i$ such that $w'_j = v_i$. Perform inversion flip moves $s_{j-1}, s_{j-2}, \ldots, s_i$ (in this order) to bring the symbol w'_j to position i of the bottom row of T_2. After completing all loop iterations, the pair consisting of T_1 and the current value of T_2 is defined to be $G(T)$.

To see that G is a bijection, we describe how to compute $G^{-1}(T_1, T_2)$. Let u be the word in the top row of T_1, and let v be the word in the bottom row of T_2. Compute $w = g_u^{-1}(v)$. Knowing w and v, we can deduce which ordered sequence of inversion flip moves was used in the previous paragraph to convert w into v. Perform these same moves in the reverse order on the given filling T_2 to obtain a new filling T'_2. Finally, set $T = H^{-1}(T_1, T'_2)$. It is evident that this construction reverses the effect of G, so G is a bijection.

Since all constituent steps act by rearranging entries in fillings, we see that G satisfies condition (i). Similarly, since H satisfies (iii) and inversion flip moves preserve $\mathrm{maj}_{(r^m)}(T_2)$, G satisfies (iii). Finally, consider condition (ii). For the initial pair $(T_1, T_2) = H(T)$, we have already remarked that all inversion triples in T correspond to inversion triples in T_1 or T_2, except for the inversion triples in the critical row of T. However, the computation of $v = g_u(w)$ ensures that the number of inversion triples in T contributed by the word w in the critical row will equal the number of inversion triples in T_2 (i.e., ordinary inversions) contributed by the new word v in the bottom row. The loop above converts w to v by a sequence of inversion flips, and each of these moves does not affect the total number of inversion triples occurring in rows 2 and higher of T_2. So (ii) does indeed hold.

Example 9. Let $\mu = (6,6,5,5,5)$, $r = 5$, $m = 3$, and

$$T = \begin{array}{cccccc} 4 & 2 & 1 & 5 & 1 & \\ 2 & 4 & 5 & 2 & 3 & \\ \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{4} & \mathbf{1} & \\ 2 & 4 & 3 & 1 & 2 & 5 \\ 1 & 2 & 3 & 1 & 1 & 4 \end{array}.$$

(Entries in the critical row of T appear in bold type.) Then $u = 243125$, $w = 32241$, $v = g(w) = 22314$, and

$$H(T) = (T_1, T_2) = \left(\begin{array}{cccccc} 2 & 4 & 3 & 1 & 2 & 5 \\ 1 & 2 & 3 & 1 & 1 & 4 \end{array} \quad , \quad \begin{array}{ccccc} 4 & 2 & 1 & 5 & 1 \\ 2 & 4 & 5 & 2 & 3 \\ 3 & 2 & 2 & 4 & 1 \end{array} \right).$$

Set $U = s_4 \circ s_2 \circ s_1(T_2)$. Then

$$U = \begin{array}{ccccc} 4 & 2 & 1 & 5 & 1 \\ 2 & 4 & 5 & 2 & 3 \\ 2 & 2 & 3 & 1 & 4 \end{array},$$

and $G(T) = (T_1, U)$. Note that $\mathrm{inv}_\mu(T) = 20 = 10+10 = \mathrm{inv}_{(6^2)}(T_1) + \mathrm{inv}_{(5^3)}(U)$ and $\mathrm{maj}_\mu(T) = 27 \equiv 4 + 8 \pmod 3 = \mathrm{maj}_{(6^2)}(T_1) + \mathrm{maj}_{(5^3)}(U)$.

Example 10. Let $\nu = (6,6)$, $(r^m) = (5^3)$, $\mu = (6,6,5,5,5)$,

$$T = \begin{array}{cccccc} 3 & 2 & 1 & 1 & 5 & 4 \\ 1 & 3 & 3 & 4 & 2 & 5 \end{array} \qquad \text{and} \qquad U = \begin{array}{ccccc} 4 & 1 & 3 & 4 & 1 \\ 3 & 2 & 2 & 5 & 2 \\ 2 & 5 & 1 & 4 & 2 \end{array}.$$

Then

$$G^{-1}(T, U) = \begin{array}{cccccc} 4 & 1 & 3 & 4 & 1 & \\ 3 & 2 & 5 & 2 & 2 & \\ \mathbf{2} & \mathbf{5} & \mathbf{2} & \mathbf{1} & \mathbf{4} & \\ 3 & 2 & 1 & 1 & 5 & 4 \\ 1 & 3 & 3 & 4 & 2 & 5 \end{array}.$$

Note that $\mathrm{inv}_\mu(G^{-1}(T,U)) = 20 = 12 + 8 = \mathrm{inv}_\nu(T) + \mathrm{inv}_{(5^3)}(U)$ and $\mathrm{maj}_\mu(G^{-1}(T,U)) = 22 \equiv 2 + 8 \pmod 3 = \mathrm{maj}_\nu(T) + \mathrm{maj}_{(5^3)}(U)$.

When $r = 1$, the map G is the same as the map H. When $r \leq 2$, one may verify that our map G has the same effect on fillings as the bijections constructed by Descouens, Morita, and Numata in [2]. However, that paper uses the terminology of "attack inversions" instead of "inversion triples" when discussing the statistic $\mathrm{inv}_\mu(T)$.

6 Final Comments

6.1 Connection to q-Multinomial Coefficients

Suppose $m = 1 = \zeta$, so we are looking at the $t = 1$ specialization of $\tilde{H}_\mu$. Applying the map G repeatedly to remove the rows of μ one at a time, we obtain a bijective proof of the well-known formula

$$\tilde{H}_\mu(x_1, \ldots, x_N; q, 1) = \prod_{i=1}^{\ell(\mu)} \tilde{H}_{(\mu_i)}(x_1, \ldots, x_N; q, 1).$$

More generally, let $\mathcal{F}_\mu(a_{ij})$ be the set of fillings $T \in \mathcal{F}_\mu$ such that, for $1 \le i \le \ell(\mu)$, row i of T is in $\mathcal{R}(1^{a_{i1}}2^{a_{i2}}\cdots N^{a_{iN}})$. Repeated use of G and (5) proves that

$$\sum_{T\in\mathcal{F}_\mu(a_{ij})} q^{\mathrm{inv}_\mu(T)} = \prod_{i=1}^{\ell(\mu)} \begin{bmatrix} a_{i1}+\cdots+a_{iN} \\ a_{i1},\ldots,a_{iN}\end{bmatrix}_q. \tag{6}$$

(This formula was first observed in unpublished notes of Haglund, Haiman, Loehr, and Warrington; the special case of a two-row shape is stated without proof as Lemma 2 in [4].) Similarly, let $\mathcal{F}'_{\mu'}(a_{ij})$ be the set of fillings $T \in \mathcal{F}_{\mu'}$ such that, for $1 \le i \le \mu'_1 = \ell(\mu)$, the ith column word of T is in $\mathcal{R}(1^{a_{i1}}2^{a_{i2}}\cdots N^{a_{iN}})$. Since each column contributes independently to $\mathrm{maj}_\mu(T)$, we get

$$\sum_{T\in\mathcal{F}'_{\mu'}(a_{ij})} q^{\mathrm{maj}_\mu(T)} = \prod_{i=1}^{\ell(\mu)} \begin{bmatrix} a_{i1}+\cdots+a_{iN} \\ a_{i1},\ldots,a_{iN}\end{bmatrix}_q. \tag{7}$$

6.2 Symmetry

Adding (6) and (7) over all choices of a_{ij}, we obtain the *univariate symmetry*

$$\tilde{H}_\mu(X;q,1) = \tilde{H}_{\mu'}(X;1,q).$$

This proof can be made completely bijective using Foata's bijection on $\mathcal{R}(1^{a_1}\cdots N^{a_N})$ that sends inv to maj [3]. However, it is an open problem to give a bijective proof of the *joint symmetry* property

$$\tilde{H}_\mu(X;q,t) = \tilde{H}_{\mu'}(X;t,q),$$

which is known to be true by algebraic properties of Macdonald polynomials.

By combining joint symmetry with (1), we see that the modified Macdonald polynomials must also satisfy the identity

$$\tilde{H}_\mu(x_1,\ldots,x_N;\zeta,t) = \tilde{H}_\nu(x_1,\ldots,x_N;\zeta,t)\cdot \tilde{H}_{(m^r)}(x_1,\ldots,x_N;\zeta,t) \tag{8}$$

where $\zeta = e^{2\pi i/m}$ and μ,ν are integer partitions such that $\mathrm{dg}(\mu)$ is obtained from $\mathrm{dg}(\nu)$ by adding m new *columns* of height r. To shed some light on the combinatorial meaning of joint symmetry, it would be interesting to find a bijective proof of this identity. Such a bijection would likely be quite different from the maps considered here, since we are now reducing the inversion statistic mod m. Even in the case $r = 1$, constructing a bijection proving (8) appears to be a difficult problem.

6.3 The Case $r > \nu_k$

As remarked in the Introduction, the factorization formula (1) is valid for *all* r, but our bijective proof only works under the hypothesis $r \le \nu_k$. The following example illustrates the difficulties that arise in the case where $r > \nu_k$.

Example 11. Take $\nu = (5,2)$, $m = r = 3$, $\mu = (5,3,3,3,2)$ and let $T \in \mathcal{F}_\mu$ be the filling shown here.

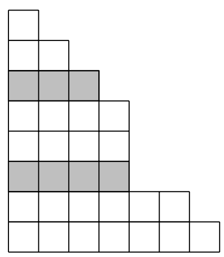
$$T = \begin{array}{|c|c|c|c|c|}
\hline 3 & 2 \\
\cline{1-3} 2 & 4 & 3 \\
\cline{1-3} 3 & 2 & 2 \\
\cline{1-3} 2 & 1 & 3 \\
\hline 4 & 2 & 3 & 1 & 4 \\
\hline
\end{array}$$

We compute $\mathrm{maj}_\mu(T) = 10$ and $\mathrm{inv}_\mu(T) = 9$. Again, starting with the naive step of removing the filling of (3^3) yields

$$T_1 = \begin{array}{|c|c|cccc} \cline{1-2} 3 & 2 \\ \hline 4 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \text{and} \qquad T_2 = \begin{array}{|c|c|c|} \hline 2 & \mathbf{4} & \mathit{3} \\ \hline 3 & \mathbf{2} & \mathit{2} \\ \hline 2 & 1 & 3 \\ \hline \end{array}.$$

Note that $\mathrm{maj}_{(5,2)}(T_1) = 0$ and $\mathrm{maj}_{(3^3)}(T_2) = 6$, but 10 is *not* congruent to $0 + 6 \pmod 3$. Observe that the descent in bold in T_2 contributes 1 less to the total major index compared to the corresponding descent in T. On the other hand, the italicized descent causes the same contribution to the major index in both T and T_2. Thus, one of the challenges when $r > \nu_k$ is how to maintain property (iii) from Section 3.

The other main challenge is that now there are really two critical rows, and adjacent cells in these rows may not have columns of equal height above them. An example of this appears in the partition diagram below, where $\nu = (7, 6, 3, 2, 1)$, $r = 4$, $m = 3$, $\mu = (7, 6, 4, 4, 4, 3, 2, 1)$, and the two critical rows are shaded. The unequal column heights prohibit the use of the inversion flip move to rearrange the critical rows, so it is no longer evident how to "fix" changes in the inversion table caused by excision of the m rows of size r.



References

[1] F. Descouens and H. Morita, "Factorization formulas for Macdonald polynomials," *European J. Combin.* **29** (2008), 395–410.

[2] F. Descouens, H. Morita, and Y. Numata, "A bijective proof of a factorization formula for Macdonald polynomials at roots of unity," FPSAC 2008 Proceedings, *Discrete Math. Theoret. Comput. Sci.* proc. **AJ** (2008), 471–482.

[3] D. Foata, "On the Netto inversion number of a sequence," *Proc. Amer. Math. Soc.* **19** (1968), 236–240.

[4] J. Haglund, "A combinatorial model for the Macdonald polynomials," *Proc. Natl. Acad. Sci. USA* **101** (2004), 16127–16131.

[5] J. Haglund, M. Haiman, and N. Loehr, "A combinatorial formula for Macdonald polynomials," *J. Amer. Math. Soc.* **18** (2005), 735–761.

[6] J. Haglund, M. Haiman, and N. Loehr, "Combinatorial theory of Macdonald polynomials I: proof of Haglund's formula," *Proc. Natl. Acad. Sci. USA* **102** #8 (2005), 2690–2696.

[7] N. Loehr, *Bijective Combinatorics*, Taylor and Francis/CRC Press (2011).

[8] N. Loehr and E. Niese, "Recursions and divisibility properties for combinatorial Macdonald polynomials," *Discrete Math. Theor. Comput. Sci.* **13** (2011), 21–43.

[9] I. Macdonald, "A new class of symmetric functions," *Actes du 20e Séminaire Lotharingien*, vol. 372/S-20, Publications I.R.M.A., Strasbourg (1988), 131–171.

[10] I. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford University Press (1995).

[11] E. Niese, *Combinatorial properties of the Hilbert series of Macdonald polynomials,* Virginia Tech doctoral dissertation, May 2010.