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Finding Positive Solutions of Boundary Value Dynamic Equations on Time Scale

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FINDING POSITIVE SOLUTIONS OF BOUNDARY VALUE DYNAMIC EQUATIONS ON TIME SCALE

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ABSTRACT

This thesis is on the study of dynamic equations on time scale. Most often, the derivatives and anti-derivatives of functions are taken on the domain of real numbers, which cannot be used to solve some models like insect populations that are continuous while in season and then follow a difference scheme with variable step-size. They die out in winter, while the eggs are incubating or dormant; and then they hatch in a new season, giving rise to a non overlapping population. The general idea of my thesis is to find the conditions for having a positive solution of any boundary value problem for a dynamic equation where the domain of the unknown function is a so called time scale, an arbitrary nonempty closed subset of the reals.

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CONTENTS

1. Introduction

This paper focuses on determining eigenvalues λ, for which there exist positive solutions with respect to a cone, of the nonlinear eigenvalue dynamic equation

$$y^{\Delta\Delta} + \lambda f(t, y^{\sigma}) = 0, \quad t \in \mathbb{T}$$

with boundary conditions

$$\alpha_{11} y(t_1) + \alpha_{12} y^{\Delta}(t_1) = 0$$

$$\alpha_{21} y(\sigma(t_2)) + \alpha_{22} y^{\Delta}(\sigma(t_2)) = 0.$$

In this equation, we consider the case where the solution is defined on any closed subset of real numbers (called a time scale) denoted by $\mathbb{T}$, initiated by Stephan Hilger in order to create a theory that unifies discrete and continuous analysis in calculus. This is fully defined in Section 5.

For the special case where the time scale is the real numbers, the equation takes the form

$$y^{''} + \lambda f(t, y) = 0, \quad t \in [t_1, t_2],$$

subject to the two-point boundary conditions

$$\alpha_{11} y(t_1) + \alpha_{12} y^{'}(t_1) = 0,$$

$$\alpha_{21} y(t_2) + \alpha_{22} y^{'}(t_2) = 0.$$

Also, we consider the 3rd-order eigenvalue problem

$$y^{\Delta\Delta\Delta} - \lambda f(t, y^{\sigma}) = 0, \quad t \in \mathbb{T},$$

with boundary conditions

$$y(t_1) = \beta_1,$$

$$y(\sigma(t_2)) = \beta_2,$$

$$y(\sigma^2(t_3)) = \beta_3$$

on a general time scale $\mathbb{T}$.

In the case where the time scale is real numbers, the equation is of the form

$$y^{'''} = \lambda f(t, y) \quad t \in [t_1, t_3],$$

subject to the three-point boundary conditions

$$y(t_1) = \beta_1,$$

$$y^{'}(t_2) = \beta_2,$$

$$y^{''}(t_3) = \beta_3.$$

Boundary value problems for higher order differential equations play a role in both theory and applications. The existence of positive solutions for two-point eigenvalue problems, has been studied by many researchers by using the Guo-Krasnosel'skii fixed point theorem. We refer readers to [11, 12, 13] for some recent results. However, few papers can be found in the literature for third order three-point boundary value problems (BVPs), most papers deal with existence of positive solutions when the nonlinear term f is nonnegative [1]. In this paper, we deal with the existence of a positive solution for the 2nd and 3rd order BVPs on the general time scale, when the nonlinear term f is nonnegative, by first defining their respective Green's function. This Green's function is then used to derive the Green's function for the $2n^{th}$ order BVP. The Green's function is also used to derive the condition for which a positive solution of the $2n^{th}$ order eigenvalue differential equation can be derived.

The rest of this paper is organised as follows. In Section 2, we compute the Green's function for the two-point boundary value problem on $\mathbb{R}$ and also find the condition under which a positive solution will exist for the two-point problem. In Section 3, we derive the Green's functions for even order BVPs, compute the bounds which are finally used to proof the existence of positive solution(s) for $2n^{th}$ order BVPs. In Section 4, we find the conditions in which positive solution(s) will exist for the three-point problem. Sections 5 and 6 offer some background on time scales and we derive similar existence results for even order problems and third order problem. Conclusions and future work are discussed in the last section.

2. Second Order Boundary Value Problem on $\mathbb{R}$

2.1. *Solution to the Second Order Differential Equation*

For this section, we are going to consider the second order boundary value eigenvalue problem on the time scale $\mathbb{T} = \mathbb{R}$.
Consider the second order eigenvalue BVP

$$y^{''}(t) + \lambda f(t, y(t)) = 0, \quad t \in [t_1, t_2] \tag{1}$$

with boundary conditions

$$\begin{cases} \alpha_{11} y(t_1) + \alpha_{12} y^{'}(t_1) &= 0 \\ \alpha_{21} y(t_2) + \alpha_{22} y^{'}(t_2) &= 0 \end{cases} \tag{2}$$

where $f : [t_1, t_2] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and $\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}$ are real constants.

We will assume the following condition: $A_1 : f : [t_1, t_2] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

We define the nonnegative numbers f_0, f^0, f_∞, and f^∞ by

$$f_0 = \lim_{y \to 0^+} \min_{t \in [t_1, t_2]} \frac{f(t, y)}{y}$$

$$f^0 = \lim_{y \to 0^+} \max_{t \in [t_1, t_2]} \frac{f(t, y)}{y}$$

$$f_\infty = \lim_{y \to \infty} \min_{t \in [t_1, t_2]} \frac{f(t, y)}{y}$$

$$f^\infty = \lim_{y \to \infty} \max_{t \in [t_1, t_2]} \frac{f(t, y)}{y}$$

and assume that they all exist in the extended reals.

Now we are going to find the solution of the second order problem. We shall show that the solution $y(t)$ is of the form $y(t) = \int_{t_1}^{t_2} G(t,s)g(s)\,ds$ where $G(t,s)$ is defined below.

Writing $y^{''}(t) = -g(t,y(t))$ where $g(t,y(t)) = \lambda f(t,y(t))$ and solving the differential equation (1) using Laplace transform, we have $L(y^{''}(t)) = -L(g(t))$ which implies $s^2L(y(t)) - sy(0) - y^{'}(0) = -L(g(t))$. That is $L(y(t)) = \frac{1}{s}y(0) + \frac{1}{s^2}y^{'}(0) - \frac{1}{s^2}L(g(t))$.

Taking the inverse Laplace of both side, we have

$$\begin{aligned} y(t) &= y(0) + ty^{'}(0) - \int_{t_1}^{t} (t-s)g(s)\,ds, and \\ y^{'}(t) &= y^{'}(0) - \int_{t_1}^{t} g(s)\,ds. \end{aligned}$$

Using the boundary conditions, we have

$$\alpha_{11}y(t_1) + \alpha_{12}y^{'}(t_1) = \alpha_{11}(y(0) + t_1y^{'}(0)) + \alpha_{12}y^{'}(0) = 0,$$

which implies,

$$\alpha_{11}y(0) + (\alpha_{11}t_1 + \alpha_{12})y^{'}(0) = 0.$$

Likewise,

$$\alpha_{21}y(t_2) + \alpha_{22}y^{'}(t_2) = 0$$

implies

$$\alpha_{21}\left(y(0) + t_2y^{'}(0) - \int_{t_1}^{t_2} (t_2-s)g(s)\,ds\right) + \alpha_{22}\left(y^{'}(0) - \int_{t_1}^{t_2} g(s)\,ds\right) = 0,$$

which implies that

$$\alpha_{21}y(0) + (\alpha_{21}t_2 + \alpha_{22})y^{'}(0) = \int_{t_1}^{t_2} (\alpha_{21}(t_2 - s) + \alpha_{22})g(s)\, ds.$$

Solving for $y(0)$ and $y^{'}(0)$, we have

$$\begin{aligned} y(0) &= \frac{-\beta_1 A}{D} \\ y^{'}(0) &= \frac{\alpha_{11} A}{D} \end{aligned}$$

where

$$(3) \qquad \begin{cases} \beta_i = \beta(t_i) = \alpha_{i1}t_i + \alpha_{i2}, \quad i = 1, 2 \\ A = \int_{t_1}^{t_2} (\beta_2 - \alpha_{21}s)g(s)ds \\ D = \alpha_{11}\beta_2 - \alpha_{21}\beta_1, \end{cases}$$

So,

$$\begin{aligned} y(t) &= \frac{-\beta_1 A}{D} + \frac{\alpha_{11}}{D}At - \int_{t_1}^{t} (t - s)g(s)\, ds \\ &= \int_{t_1}^{t_2} \frac{-\beta_1}{D}(\beta_2 - \alpha_{21}s)g(s)\, ds + \int_{t_1}^{t_2} \frac{\alpha_{11}}{D}t(\beta_2 - \alpha_{21}s)g(s)\, ds \\ &\quad - \int_{t_1}^{t} (t - s)g(s)\, ds \\ &= \frac{1}{D}\int_{t_1}^{t_2} (\beta_2 - \alpha_{21}s)(\alpha_{11}t - \beta_1)g(s)\, ds - \int_{t_1}^{t} (t - s)g(s)\, ds. \end{aligned}$$

Therefore,

$$y(t) = \int_{t_1}^{t_2} G(t, s)g(s)\, ds$$

where

$$(4) \qquad G(t, s) = \begin{cases} \frac{1}{D}(\beta_1 - \alpha_{11}s)(\alpha_{21}t - \beta_2) & \text{if } t_1 \leq s \leq t \leq t_2; \\ \frac{1}{D}(\beta_2 - \alpha_{21}s)(\alpha_{11}t - \beta_1) & \text{if } t_1 \leq t \leq s \leq t_2. \end{cases}$$

Throughout this section, we will require the following conditions:

$$\begin{aligned} A_2 &: \alpha_{11} > 0, \alpha_{21} > 0; \\ A_3 &: m_1 \le t_1 \le t_2 \le m_2, \quad \text{where } \ m_i = \frac{\beta(t_i)}{\alpha_{i1}} = \frac{\beta_i}{\alpha_{i1}}, i = 1, 2 \end{aligned}$$

Note: $\frac{\beta_1}{\alpha_{11}} \le t_1$ implies that $\beta_1 - \alpha_{11} t_1 \le 0$ which implies that $\alpha_{12} \le 0$ and $\frac{\beta_2}{\alpha_{21}} \ge t_2$ implies that $\beta_2 - \alpha_{21} t_1 \le 0$ which implies that $\alpha_{22} \ge 0$. Now, we establish some preliminary results that will be used later.

2.2. *Properties of the Function G(t,s)*

We give some Lemmas on the above function G(t,s) .

Lemma 2.1. $G(t,s) > 0 \ \ for \ \ (t,s) \in [t_1, t_2] \times [t_1, t_2]$.

Proof. For $t_1 \le s \le t \le t_2$ using conditions A_1 and A_2, we have

$$\begin{aligned} \frac{\beta_1}{\alpha_{11}} &\le s \le t \le \frac{\beta_2}{\alpha_{21}} \quad \text{so that} \quad D = \alpha_{11}\beta_2 - \alpha_{21}\beta_1 > 0 \ \text{ and} \\ G(t,s) &= \frac{1}{D}(\alpha_{21} t - \beta_2)(\beta_1 - \alpha_{11} s) > 0. \end{aligned}$$

Also, for $t_1 \le t \le s \le t_2$, we have $\frac{\beta_1}{\alpha_{11}} \le t \le s \le \frac{\beta_2}{\alpha_{21}}$ so that $G(t,s) > 0$. Therefore, $G(t,s) > 0$ for $(t,s) \in [t_1, t_2] \times [t_1, t_2]$. □

Lemma 2.2. *The function $G(t,s)$ satisfies the homogeneous differential equation $-y^{''} = 0$ and the boundary conditions (2) for fixed s.*

Proof. Since $G(t,s)$ is a polynomial of degree one, then it satisfies $\frac{d^2}{t^2} G(t,s) = 0$ for all $(t,s) \in [t_1, t_2] \times [t_1, t_2]$.

Note that differentiation is with respect to t.

For $t_1 \le t \le s \le t_2$, $\frac{d}{dt} G(t,s) = \frac{1}{D}\alpha_{11}(\beta_2 - \alpha_{21} s)$ so that

$$\alpha_{11} G(t_1, s) + \alpha_{12} G^{'}(t_1, s) = 0.$$

Also for $t_1 \leq s \leq t \leq t_2$, $G^{'}(t,s) = \frac{1}{D}\alpha_{21}(\beta_1 - \alpha_{11}s)$ so that

$$\alpha_{21}G(t_2,s) + \alpha_{22}G^{'}(t_2,s) = 0.$$

□

Lemma 2.3. *For any fixed $s \in [t_1,t_2]$, the function $G(t,s)$ is continuous for every $t \in [t_1,t_2]$.*

Proof. Clearly, $G(t,s)$ is continuous everywhere on $[t_1,t_2]\times[t_1,t_2]$ since it is continuous at the point $t = s$. Hence, the proof is complete. □

Lemma 2.4. *$\frac{d}{dt}G(t,s)$ has a jump discontinuity with a jump of factor -1 at the point $t = s$.*

Proof. Here, we show that the limit of $\frac{d}{dt}G(t,s)$ as t approaches s from above differ from its limit as t approaches s from below by -1.

$$\begin{aligned} G^{'}(s^+,s) - G^{'}(s^-,s) &= \lim_{t\to s^+} G^{'}(t,s) - \lim_{t\to s^-} G^{'}(t,s) \\ &= \frac{1}{D}(\alpha_{21}\beta_1 - \alpha_{21}\alpha_{11}s - \alpha_{11}\beta_2 + \alpha_{11}\alpha_{21}s) \\ &= \frac{1}{D}(\alpha_{21}\beta_1 - \alpha_{11}\beta_2) = -1. \end{aligned}$$

□

Lemma 2.5. *Define*

$$\gamma = \min\left\{\frac{G(t_1,s)}{G(s,s)}, \frac{G(t_2,s)}{G(s,s)}\right\}, \tag{5}$$

then $0 < \gamma < 1$.

Proof. Since $G(t,s) > 0$ for all $(t,s) \in [t_1,t_2] \times [t_1,t_2]$, $\gamma > 0$.
Case (i) If $s = t_1$, $\gamma = \min\left\{1, \frac{G(t_2,t_1)}{G(t_1,t_1)}\right\}$, which implies

$$\gamma = \frac{G(t_2,t_1)}{G(t_1,t_1)} = \frac{\alpha_{21}t_2 - \beta_2}{\alpha_{11}t_1 - \beta_1} < 1.$$

Case (ii) If $s = t_2$, then $\gamma = \min\left\{1, \frac{G(t_1,t_2)}{G(t_2,t_2}\right\}$ which implies

$$\gamma = \frac{G(t_1,t_2)}{G(t_2,t_2)} < 1.$$

Hence, the proof is complete. □

Theorem 2.6. *Assume that conditions $A_1 - A_3$ holds then,*

$$\gamma G(s,s) \leq G(t,s) \leq G(s,s)$$

where

$$0 < \gamma = \min\{\frac{G(t_1,s)}{G(s,s)}, \frac{G(t_2,s)}{G(s,s)}\} < 1.$$

Proof. Case (i) For $t_1 \leq s \leq t \leq t_2$, $G^{'}(t,s) = \frac{\alpha_{21}}{D}(\beta_1 - \alpha_{11}s) < 0$, which implies that $G(t,s)$ is a decreasing function of t so that $G(t,s) \leq G(s,s)$ and also for $t \leq t_2$, $\frac{G(t,s)}{G(s,s)} \geq \frac{G(t_2,s)}{G(s,s)} \geq \gamma$ which implies

$$\gamma G(s,s) \leq G(t,s).$$

Case (ii) For $t_1 \leq t \leq s \leq t_2$, $G^{'}(t,s) = \frac{1}{D}\alpha_{11}(\beta_2 - \alpha_{21}s) > 0$
implies that $G(t,s)$ is an increasing function of t so that

$$G(t,s) \leq G(s,s)$$

and also for $t \geq t_1$,

$$\frac{G(t,s)}{G(s,s)} \geq \frac{G(t_1,s)}{G(s,s)} \geq \gamma$$

and so we have

$$\gamma G(s,s) \leq G(t,s).$$

Therefore, $\gamma G(s,s) \leq G(t,s) \leq G(s,s)$ for $t_1 \leq t, s \leq t_2$. □

2.3. *Definition of Green's Function*

Consider the linear homogeneous differential equation

$$\sum_{i=0}^{n} a_i y^{(i)} = 0, \quad t \in [t_1, t_n] \tag{6}$$

subject to the homogeneous boundary conditions

$$\alpha_{i1} y(t_i) + \alpha_{i2} y^{'}(t_i) + \cdots + \alpha_{in} y^{n-1}(t_i) = 0, \ i = 1, 2..., n. \tag{7}$$

For each fixed $s \in [t_1, t_2]$, a function H(t,s) with the property that

(i) H(t,s) satisfies the differential equation

(ii) H(t,s) satisfies the homogeneous boundary conditions

(iii) $H^{(i)}(t,s)$, i = 0,1,2,. . ., n-2 is a continuous function of t on $[t_1, t_n]$

(iv) $H^{(n-1)}(s^+, s) - H^{(n-1)}(s^-, s) = \frac{1}{a_n(s)}$ at t = s

is called the Green's function of (6) satisfying (7).

So, from this definition, we can conclude from Lemmas 2.2, 2.3 and 2.4 that the function $G(t,s)$ is the Green's function for the equation

$$-y^{''}(t) = 0, \quad t \in [t_1, t_2]$$

with boundary conditions

$$\alpha_{11} y(t_1) + \alpha_{12} y^{'}(t_1) = 0$$

$$\alpha_{21} y(t_2) + \alpha_{22} y^{'}(t_2) = 0.$$

2.4. *Existence of Positive Solutions*

In this section,we find the range of λ for which there exist a positive solution for (1) satisfying (2).

Definition 2.7. Let X be a Banach space. A non empty closed convex set κ is called a **cone** of X, if it satisfies the following conditions:
(i) $\alpha_1 u + \alpha_2 v \in \kappa, \forall\, u, v \in \kappa$ and $\alpha_1, \alpha_2 \geq 0$,
(ii) $u \in \kappa$ and $-u \in \kappa$, implies $u = 0$.

Let y(t) be the solution of the BVP (1) satisfying (2), given by

$$y(t) = \lambda \int_{t_1}^{t_2} G(t,s) f(s, y(s))\, ds. \tag{8}$$

Define

$$X = \{u | u \in C[t_1, t_2]\},$$

with norm

$$\|u\| = \max_{t \in [t_1, t_2]} |u(t)|.$$

Then, $(X, \|.\|)$ is a Banach space. Define a set κ by

$$\kappa = \{u \in X : u(t) \geq 0 \;\text{ on }\; [t_1, t_2] \text{ and } \min_{t \in [t_1, t_2]} u(t) \geq \; \gamma \|u\|\} \tag{9}$$

where γ is defined in (5).

In the next Lemma, we show that κ defined above is a cone.

Lemma 2.8. *The set κ is a cone in X, where κ is defined in (9).*

Proof. Let $\{u_n\} \in \kappa$,$n \in \mathbb{N}$, be such that $\|u_n - u_0\| \to 0$ as $n \to \infty$, where $u_0 \in X$. Then $u_n(t) \geq 0$ on $[t_1, t_2]$, and $\min\{u_n(t)\} \geq$

$\gamma\|u_n\|, \forall \quad n \in \mathbb{N}$. Thus, given $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $-\epsilon < u_n(t) - u_0(t) < \epsilon$, $t \in [t_1, t_2]$, $n \geq N$ and so $0 \leq u_n(t) \leq u_0(t) + \epsilon$, $t \in [t_1, t_2], n \geq N$. Hence $u_0(t) \geq 0$ on $[t_1, t_2]$, then $\lim_{n\to\infty} \min\{u_n(t)\} \geq \gamma \lim_{n\to\infty} \|u_n\|$ and $\min_{t\in[t_1,t_2]} \{u_n(t)\} \geq \gamma\|u_0\|$, $t \in [t_1, t_2]$, implies $u_0 \in \kappa$ and so κ is closed.

Now let u,$v \in \kappa$ and $\alpha_1, \alpha_2 \geq 0$. Then $\alpha_1 u(t) + \alpha_2 v(t) \geq 0$, $t \in [t_1, t_2]$, and

$$\begin{aligned}
\min_{t\in[t_1,t_2]} \{\alpha_1 u(t) + \alpha_2 v(t)\} &\geq \alpha_1 \min_{t\in[t_1,t_2]} \{u(t)\} + \alpha_2 \min_{t\in[t_1,t_2]} \{v(t)\} \\
&\geq \alpha_1 \gamma \|u\| + \alpha_2 \gamma \|v\| \\
&\geq \gamma \|\alpha_1 u + \alpha_2 v\|.
\end{aligned}$$

Therefore, $\alpha_1 u + \alpha_2 v \in \kappa$. Hence the proof is complete. □

Define the operator $T : \kappa \to X$ by

$$(10) \qquad (Ty)(t) = \lambda \int_{t_1}^{t_2} G(t,s) f(s, y(s)) ds, \quad \text{for all} \quad t \in [t_1, t_2].$$

If $y \in \kappa$ is a fixed point of T , then y satisfies (8); hence y is a positive solution of the BVP (1) - (2). We seek a fixed point of the operator T in the cone κ.

Now, we show that the operator defined in (10) preserves the cone.

Lemma 2.9. *The operator T, as defined in (10), preserves κ. That is, $T : \kappa \to \kappa$.*

Proof. Let y $\in \kappa$. Since $G(t, s) > 0$ for all $t \in [t_1, t_2]$, we have $(Ty)(t) \geq 0$ *for all* $t \in [t_1, t_2]$. Then from Lemma 2.6,

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s,y(s))\, ds \\
&\geq \lambda \int_{t_1}^{t_2} \gamma G(s,s) f(s,y(s))\, ds \\
&\geq \lambda\gamma \int_{t_1}^{t_2} \max_{t\in[t_1,t_2]} G(t,s) f(s,y(s))\, ds, \\
&\geq \gamma\lambda \max_{t\in[t_1,t_2]} \int_{t_1}^{t_2} G(t,s) f(s,y(s))\, ds, \\
&= \gamma \| Ty \|.
\end{aligned}$$

Therefore,

$$\min_{t\in[t_1,t_2]} (Ty)(t) \geq \gamma \| Ty \|.$$

So, $(Ty)(t) \in \kappa$. Hence $T : \kappa \to \kappa$. □

Now we need to show that the operator T is completely continuous on the cone κ.

Lemma 2.10. *The operator T is completely continuous, where T is define in (10).*

Proof. Let y $\in \kappa$ and $\epsilon > 0$ be given. By the continuity of f, there exists $\delta > 0$ such that for any $u \in [0,\infty)$ with $|y(t) - u| < \delta$, $t \in [t_1, t_2]$, then $|f(t,y(t)) - f(t,u)| < \epsilon$. Let $w \in \kappa$ with $\|y - w\| < \delta$, then $|w(t) - y(t)| < \delta$, for all t $\in [t_1, t_2]$. So we have,

$$\begin{aligned}
|(Ty)(t) - (Tw)(t)| &= \lambda \int_{t_1}^{t_2} G(t,s) |f(s,y(s)) - f(s,w(s))|\, ds \\
&\leq \epsilon\lambda \int_{t_1}^{t_2} G(t,s)\, ds
\end{aligned}$$

Thus, $\|(Ty)(t)-(Tw)(t)\| \leq \epsilon\lambda \int_{t_1}^{t_2} G(t,s)\,ds$ and T is continuous. Now, let $\{y_n\}$ be a bounded sequence in κ. Since f is continuous, there exists $N > 0$ such that $|f(t, y(t))| \leq N$ for all n where t $\in [t_1, t_2]$. For each n,

$$
\begin{aligned}
|(Ty_n)(t)| &= |\lambda \int_{t_1}^{t_2} G(t,s) f(s, y_n(s)) ds| \\
&\leq \lambda \int_{t_1}^{t_2} |G(s,s)||f(s, y_n(s))| ds \\
&\leq N\lambda \int_{t_1}^{t_2} G(s,s) ds.
\end{aligned}
$$

By choosing successive subsequences, there exists a subsequence $\{Ty_{n_j}\}$ which converges uniformly on $[t_1, t_2]$. Hence T is completely continuous.

□

To establish the eigenvalue intervals where a fixed point exists, we will employ the following Fixed Point Theorem due to Guo and Krasnosel'skii.

Theorem 2.11 (Guo-Krasnosel'skii Fixed Point Theorem)**.** *Let X be a Banach space, $\kappa \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \Omega_2$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : \kappa \cap (\overline{\Omega_2} \backslash \Omega_1) \to \kappa$ is completely continuous operator such that either*
(i) $\|Tu\| \leq \|u\|, u \in \kappa \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in \kappa \cap \partial\Omega_2$, or
(ii) $\|Tu\| \geq \|u\|, u \in \kappa \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in \kappa \cap \partial\Omega_2$,
holds. Then T has a fixed point in $\kappa \cap (\overline{\Omega_2} \backslash \Omega_1)$.

We are going to present our first existence result.

Theorem 2.12. *Assume that conditions (A_1)-(A_3) are satisfied. Then, for each λ satisfying*

$$\frac{1}{[\gamma^2 \int_{t_1}^{t_2} G(s,s)\,ds] f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{t_2} G(s,s)\,ds] f^0}, \tag{11}$$

there exist at least one positive solution of the BVP (1)-(2) in κ, where f_∞ and f^0 are as define in Section 2.1.

Proof. Let λ be given as in (11). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{[\gamma^2 \int_{t_1}^{t_2} G(s,s)\,ds](f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{t_2} G(s,s)\,ds](f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (10). By definition of f^0, there exists $H_1 > 0$ such that

$$\max_{t \in [t_1, t_2]} \frac{f(t,y)}{y} \leq (f^0 + \epsilon), \text{ for } 0 < y \leq H_1.$$

It follows that, $f(t,y) \leq (f^0 + \epsilon)y$, for $0 < y \leq H_1$. Choose $y_1 \in \kappa$ with $\|y_1\| = H_1$. Then, we have from the boundedness of $G(t,s)$ and the nature of λ, that

$$\begin{aligned}
(Ty_1)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_1(s))\,ds \\
&\leq \lambda \int_{t_1}^{t_2} G(s,s) f(s, y_1(s))\,ds \\
&\leq \lambda \int_{t_1}^{t_2} G(s,s)(f^o + \epsilon) y_1(s)\,ds \\
&\leq \lambda \int_{t_1}^{t_2} G(s,s)(f^o + \epsilon)\|y_1\|\,ds \\
&\leq \|y_1\|.
\end{aligned}$$

Consequently, $\|Ty_1\| \leq \|y_1\|$. So, if we define

$$\Omega_1 = \{u \in X : \|u\| < H_1\},$$

then,

$$\|Ty\| \leq \|y\|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{12}$$

By definition of f_∞, there exists $\overline{H_2} > 0$ such that

$$\min_{t \in [t_1, t_2]} \frac{f(t,y)}{y} \geq (f_\infty - \epsilon), \text{ for } y \geq \overline{H_2}.$$

It follows that

$$f(t,y) \geq (f_\infty - \epsilon)y, \text{ for } y \geq \overline{H_2}.$$

Let

$$H_2 = \max\{2H_1, \frac{1}{\gamma}\overline{H_2}\},$$

and let

$$\Omega_2 = \{u \in X : \|u\| < H_2\}.$$

Now, choose $y_2 \in \kappa \cap \partial\Omega_2$ with $\|y_2\| = H_2$, so that

$$\min_{t \in [t_1, t_2]} y_2(t) \geq \gamma \|y_2\| \geq \overline{H_2}.$$

Then,

$$\begin{aligned}
(Ty_2)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_2(s))\, ds \\
&\geq \lambda \int_{t_1}^{t_2} \gamma G(s,s) f(s, y_2(s))\, ds \\
&\geq \lambda\gamma \int_{t_1}^{t_2} G(s,s)(f^\infty - \epsilon) y_2(s)\, ds
\end{aligned}$$

$$\geq \gamma^2\lambda\int_{t_1}^{t_2} G(s,s)(f^\infty-\epsilon)\|y_2\|\,ds$$
$$\geq \|y_2\|.$$

Thus,

(13) $$\|Ty\| \geq \|y\|, \quad \text{for y } \in \kappa\cap\partial\Omega_2$$

Applying Theorem 2.11(i), from (12) and (13) we have that T has a fixed point $y(t)\in\kappa\cap(\overline{\Omega_2}\backslash\Omega_1)$. This fixed point is the positive solution of the BVP (1)-(2) for the given λ. □

Another existence result applying Theorem 2.11(ii) is as follow:

Theorem 2.13. *Assume that conditions (A_1)-(A_3) are satisfied. Then, for each λ satisfying*

(14) $$\frac{1}{[\gamma^2\int_{t_1}^{t_2} G(s,s)\,ds]f_0} < \lambda < \frac{1}{[\int_{t_1}^{t_2} G(s,s)\,ds]f^\infty}$$

there exist at least one positive solution of the BVP (1)-(2) in κ .

Proof. Let λ be given as in (14). Now, let $\epsilon>0$ be chosen such that

$$\frac{1}{[\gamma^2\int_{t_1}^{t_2} G(s,s)\,ds](f_0-\epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{t_2} G(s,s)\,ds]u(f^\infty+\epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (10). By definition of f_0, there exists $J_1>0$ such that

$$\min_{t\in[t_1,t_2]}\frac{f(t,y)}{y} \geq (f_0-\epsilon), \text{ for } 0<y\leq J_1.$$

It follows that, $f(t,y) \geq (f_0 - \epsilon)y$, for $0 < y \leq J_1$. Choose y $\in \kappa$ with $\|y\| = J_1$. Then, we have from the boundedness of G(t,s) that

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s,y(s)) ds \\
&\geq \lambda \int_{t_1}^{t_2} \gamma G(s,s) f(s,y(s)) ds \\
&\geq \gamma\lambda \int_{t_1}^{t_2} G(s,s)(f_o - \epsilon) y(s) ds \\
&\geq \gamma^2\lambda \int_{t_1}^{t_2} G(s,s)(f_o - \epsilon)\|y\| ds \\
&\geq \|y\|.
\end{aligned}$$

Consequently, $\|Ty\| \geq \|y\|$. So, if we define

$$\Omega_1 = \{u \in X : \|u\| < J_1\},$$

then

$$\|Ty\| \geq \|y\|, \quad for \quad y \in \kappa \cap \partial\Omega_1. \tag{15}$$

It remains for us to consider f^∞. By the definition of f^∞, there exists an $\overline{J_2} > 0$ such that

$$\max_{t \in [t_1, t_2]} \tfrac{f(t,y)}{y} \leq (f^\infty + \epsilon), \text{ for y} \geq \overline{J_2}.$$

It follows that

$$f(t,y) \leq (f^\infty + \epsilon)y,$$

for y $\geq \overline{J_2}$.

There are two cases to consider.

Case (i). The function f is bounded. Suppose L >0 is such that $f(t,y) \leq$ L, for all $0 < y < \infty$.

Let

$$J_2 = \max\{2J_1, L\lambda \int_{t_1}^{t_2} G(s,s) \ \ ds\},$$

Then, for $y_2 \in \kappa$ with $\|y_2\| = J_2$, we have

$$\begin{aligned} (Ty_2)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_2(s))\, ds \\ &\leq \lambda \int_{t_1}^{t_2} G(s,s) f(s, y_2(s))\, ds \\ &\leq \lambda L \int_{t_1}^{t_2} G(s,s)\, ds \\ &\leq \|y_2\|. \end{aligned}$$

Thus $\|Ty\| \leq \|y\|$. So, if we define

$$\Omega_2 = \{u \in X : \|u\| < J_2\},$$

then

$$\|Ty\| \leq \|y\|, \quad for \quad y \in \kappa \cap \partial\Omega_2. \tag{16}$$

Case (ii). The function f is unbounded. Let $J_2 > \max\{2J_1, \overline{J_2}\}$ be such that $f(t,y) \leq f(t, J_2)$, for $0 < y \leq J_2$. Let $y_2 \in \kappa$ with $\|y_2\| = J_2$. Then,

$$\begin{aligned} (Ty_2)(t) &= \lambda \int_{t_1}^{t_2} G(t,s) f(s, y_2(s))\, ds \\ &\leq \lambda \int_{t_1}^{t_2} G(s,s) f(s, y_2(s))\, ds \\ &\leq \lambda \int_{t_1}^{t_2} G(s,s) f(s, J_2)\, ds \end{aligned}$$

which implies

$$
\begin{aligned}
(Ty_2)(t) &\leq \lambda \int_{t_1}^{t_2} G(s,s)(f^{\infty}+\epsilon) J_2 \, ds \\
&\leq J_2 = \|y_2\|.
\end{aligned}
$$

Thus , $\|Ty\| \leq \|y\|$. For this case, if we define

$$\Omega_2 = \{u \in X : \|u\| < J_2\},$$

Then

$$\|Ty\| \leq \|y\|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{17}$$

Thus, in either of the cases, Theorem 2.11 in light of (15),(16) and (17) yields that T has fixed point $y(t) \in \kappa \cap (\overline{\Omega_2} \backslash \Omega_1)$. This fixed point is the solution of the BVP (1)-(2) for the given λ. □

2.5. *Example*

Let's consider the example

$$y^{''}(t) + \lambda \frac{y(1+200y)}{1+y} = 0, \quad t \in [0,1]$$

with boundary conditions

$$
\begin{aligned}
y(0) - y^{'}(0) &= 0 \\
2y(1) + 3y^{'}(1) &= 0
\end{aligned}
$$

The green's function is given by

$$
G(t,s) = \begin{cases} \frac{1}{7}(-1-s)(-5+2t) & \text{if } 0 \leq s \leq t; \\ \frac{1}{7}(5-2s)(1+t) & \text{if } 0 \leq t \leq s. \end{cases}
$$

We found that $\gamma = \frac{1}{2}$, $f_\infty = 200$, and $f^0 = 1$. So, employing (11), there is a positive solution for all λ in the range $(\frac{3}{125}, \frac{6}{5})$.

3. Green's Function and Bounds for the $2n^{(th)}$ Order Boundary Value Differential Equation

Our interest in this section is finding positive solutions to all differential equation of the form

$$(-1)^{\frac{n}{2}} y^{(n)} + \lambda f(t, y(t)) = 0 \tag{18}$$

for even n, with boundary conditions

$$\begin{cases} \alpha_{11} y^{(2k)}(t_1) + \alpha_{12} y^{(2k+1)}(t_1) &= 0 \\ \alpha_{21} y^{(2k)}(t_2) + \alpha_{22} y^{(2k+1)}(t_2) &= 0, \quad k = 0, 1, 2, \ldots \frac{n}{2} - 1. \end{cases} \tag{19}$$

Before we can do this, we need to be able to generate the Green's function of the homogeneous boundary value problem which we do in the following subsection.

3.1. *Finding the Green's Function for the $2n^{(th)}$ Order DE*

In this section, we will derive Green's function for 2nth order homogeneous differential equation (21) satisfying (22).

Theorem 3.1. *Suppose that $G_2(t, s)$ is the Green's function satisfying*

$$-y''(t) = 0$$

with boundary conditions

$$\begin{aligned} \alpha_{11} y(t_1) + \alpha_{12} y'(t_1) &= 0 \\ \alpha_{21} y(t_2) + \alpha_{22} y'(t_2) &= 0 \end{aligned}$$

Then ,

$$G_n(t,s) = \int_{t_1}^{t_2} G_2(t,w)G_{n-2}(w,s)\,dw \quad n \in \{2k+2 : k \in \mathbb{N}\} \tag{20}$$

is the Green's function for

$$(-1)^{\frac{n}{2}} y^n(t) = 0, \quad n \in \{2k+2 : k \in \mathbb{N}\}, \tag{21}$$

with boundary conditions

$$\begin{cases} \alpha_{11} y^(2k)(t_1) + \alpha_{12} y^(2k+1)(t_1) &= 0 \\ \alpha_{21} y^(2k)(t_2) + \alpha_{22} y^(2k+1)(t_2) &= 0, \;\; k = 0,1,2,\ldots \frac{n}{2}-1. \end{cases} \tag{22}$$

Proof. Suppose $G_2(t,s)$ is the Green's function satisfying $-y^{''}(t) = 0$, then

$$-y^{''}(t) = g \quad \Rightarrow \quad y(t) = \int_{t_1}^{t_2} G_2(t,s)g(s)\,ds$$

so that

$$y^{''''}(t) = g \Rightarrow (y^{''})^{''} = g$$

which implies $$y^{''}(t) = -\int_{t_1}^{t_2} G_2(t,s)g(s)\,ds = -H(t).$$

Then,
$$\begin{aligned} y(t) &= \int_{t_1}^{t_2} G_2(t,w)H(w)\,dw \\ &= \int_{t_1}^{t_2} G_2(t,w)\left\{\int_{t_1}^{t_2} G_2(w,s)g(s)\,ds\right\} dw \\ &= \int_{t_1}^{t_2} \left\{\int_{t_1}^{t_2} G_2(t,w)G_2(w,s)g(s)\,ds\right\} dw \\ &= \int_{t_1}^{t_2} \left\{\int_{t_1}^{t_2} G_2(t,w)G_2(w,s)\,dw\right\} g(s)\,ds \\ &= \int_{t_1}^{t_2} G_4(t,s)g(s)\,ds \end{aligned}$$

where

$$G_4(t,s) = \int_{t_1}^{t_2} G_2(t,w)G_2(w,s)\,dw.$$

From definition of $G_2(t,s)$, $G_4(t,s) = \int_{t_1}^{t_2} G_2(t,w)G_2(w,s)\,dw$ implies $G_4^{''}(t,s) = -G_2(t,s)$ which in turn implies that $y^{''}$ satisfies the boundary conditions (2).

That is,

$$\begin{aligned} \alpha_{11}y^{''}(t_1) + \alpha_{12}y^{'''}(t_1) &= 0, \\ \alpha_{21}y^{''}(t_2) + \alpha_{22}y^{'''}(t_2) &= 0. \end{aligned}$$

Likewise, $G_4(t,s)$ satisfies boundary conditions (2) so that $y(t)$ satisfies the BC

$$\begin{aligned} \alpha_{11}y(t_1) + \alpha_{12}y^{'}(t_1) &= 0 \\ \alpha_{21}y(t_2) + \alpha_{22}y^{'}(t_2) &= 0 \\ \alpha_{11}y^{''}(t_1) + \alpha_{12}y^{'''}(t_1) &= 0 \\ \alpha_{21}y^{''}(t_2) + \alpha_{22}y^{'''}(t_2) &= 0. \end{aligned}$$

So, $G_4(t,s)$ is the Green's function for the equation

$$y^{''''}(t) = 0,$$

satisfying the BCs

$$\begin{aligned} \alpha_{11}y(t_1) + \alpha_{12}y^{'}(t_1) &= 0 \\ \alpha_{21}y(t_2) + \alpha_{22}y^{'}(t_2) &= 0 \end{aligned}$$

$$
\begin{aligned}
\alpha_{11} y^{''}(t_1) + \alpha_{12} y^{'''}(t_1) &= 0 \\
\alpha_{21} y^{''}(t_2) + \alpha_{22} y^{'''}(t_2) &= 0.
\end{aligned}
$$

For n=6, we have $-y^{(6)}(t) = g(t)$ or, equivalently, $-(y^{('')})^{('''')}(t) = g(t)$

$$
\begin{aligned}
\text{which implies} \qquad -y^{''}(t) &= \int_{t_1}^{t_2} G_4(t,s) g(s)\, ds = H(t) \\
\text{so that} \qquad y(t) &= \int_{t_1}^{t_2} G_2(t,w) H(w)\, dw \\
&= \int_{t_1}^{t_2} G_2(t,w) \left\{ \int_{t_1}^{t_2} G_4(w,s) g(s)\, ds \right\} dw \\
&= \int_{t_1}^{t_2} \left\{ \int_{t_1}^{t_2} G_2(t,w) G_4(w,s) g(s)\, dw \right\} ds \\
&= \int_{t_1}^{t_2} G_6(t,s) g(s)\, ds,
\end{aligned}
$$

where

$$
G_6(t,s) = \int_{t_1}^{t_2} G_2(t,w) G_4(w,s) g(s)\, dw. \tag{23}
$$

By definition of $G_2(w,s)$, (23) implies

$$
G_6^{''}(t,s) = -G_4(t,s)
$$

which means that $y^{''}$ satisfies the boundary conditions above. That is, we have

$$(24)\qquad \begin{cases} \alpha_{11}y^{''}(t_1)+\alpha_{12}y^{'''}(t_1) & =0\\ \alpha_{21}y^{''}(t_2)+\alpha_{22}y^{'''}(t_2) & =0\\ \alpha_{11}y^{''''}(t_1)+\alpha_{12}y^{'''''}(t_1) & =0\\ \alpha_{21}y^{''''}(t_2)+\alpha_{22}y^{'''''}(t_2) & =0 \end{cases}$$

Also, $G_6^{''}(t,s)$ satisfies the boundary conditions (2) so that $y(t)$ satisfies the BC

$$\begin{aligned} \alpha_{11}y(t_1)+\alpha_{12}y^{'}(t_1) &= 0\\ \alpha_{21}y(t_2)+\alpha_{22}y^{'}(t_2) &= 0\\ \alpha_{11}y^{''}(t_1)+\alpha_{12}y^{'''}(t_1) &= 0\\ \alpha_{21}y^{''}(t_2)+\alpha_{22}y^{'''}(t_2) &= 0\\ \alpha_{11}y^{''''}(t_1)+\alpha_{12}y^{'''''}(t_1) &= 0\\ \alpha_{21}y^{''''}(t_2)+\alpha_{22}y^{'''''}(t_2) &= 0. \end{aligned}$$

Continuing in this way, we find out that

$$G_n(t,s)=\int_{t_1}^{t_2} G_2(t,w)G_{n-2}(w,s)\,dw \quad n\in\{2k+2; k\in\mathbb{N}\}$$

is the Green's function for

$$(-1)^{\frac{n}{2}}y^n(t)=0, \quad n\in\{2k+2; k\in\mathbb{N}\}$$

with boundary conditions

$\alpha_{11}y^{(2k)}(t_1)+\alpha_{12}y^{(2k+1)}(t_1)=0$

$\alpha_{21}y^{(2k)}(t_2)+\alpha_{22}y^{(2k+1)}(t_2)=0, \quad k=0,1,2,\ldots \frac{n}{2}-1.$ □

3.2. *Bounds for the Green's Function*

Here, we find bound for the Green's function for 2nth order problem.

Theorem 3.2. *Assuming conditions (A1)-(A3),then*

$$\gamma^{\frac{n}{2}} G_n(s,s) \leq G_n(t,s) \leq G_n(s,s) \quad \textit{for } n \quad \in \{2k; k \in \mathbb{N}\}$$

Proof. From previous theorem, $\gamma G_2(s,s) \leq G_2(t,s) \leq G_2(s,s) \, \forall \; (t,s) \in [t_1,t_2]$.

So,

$$\begin{aligned} G_4(t,s) &= \int_{t_1}^{t_2} G_2(t,x) G_2(x,s)\, dx \\ &\leq \int_{t_1}^{t_2} G_2(s,x) G_2(x,s)\, dx = G_4(s,s). \end{aligned}$$

Therefore,

$$G_4(t,s) \leq G_4(s,s).$$

Also

$$\begin{aligned} G_4(t,s) &= \int_{t_1}^{t_2} G_2(t,x) G_2(x,s)\, dx \\ &\geq \int_{t_1}^{t_2} \gamma G_2(s,x) G_2(x,s)\, dx \quad \textit{and} \quad \textit{since} \quad 0 < \gamma < 1, \quad \textit{we} \quad \textit{have} \\ &\geq \int_{t_1}^{t_2} \gamma G_2(s,x) \gamma G_2(x,s)\, dx = \gamma^2 G_4(s,s). \end{aligned}$$

Therefore,

$$G_4(t,s) \geq \gamma^2 G_4(s,s),$$

so that

$$\gamma^2 G_4(s,s) \leq G_4(t,s) \leq G_4(s,s).$$

Similarly,

$$\begin{aligned} G_6(t,s) &= \int_{t_1}^{t_2} G_2(t,x)G_4(x,s)\,dx \\ &\le \int_{t_1}^{t_2} G_2(s,x)G_4(x,s)\,dx = G_6(s,s). \end{aligned}$$

Therefore,

$$G_6(t,s) \le G_6(s,s).$$

Also,

$$\begin{aligned} G_6(t,s) &= \int_{t_1}^{t_2} G_2(t,x)G_4(x,s)\,dx \\ &\ge \int_{t_1}^{t_2} \gamma G_2(s,x)\gamma^2 G_4(s,s)\,dx \quad \text{and since} \quad G_4(t,s) \le G_4(s,s), \quad \text{we have} \\ &\ge \int_{t_1}^{t_2} \gamma G_2(s,x)\gamma^2 G_4(x,s)\,dx = \gamma^3 G_6(s,s). \end{aligned}$$

Therefore $G_6(t,s) \ge \gamma^3 G_6(s,s)$ so that

$$\gamma^3 G_6(s,s) \le G_6(t,s) \le G_6(s,s).$$

Continuing in this, we have that

$$\gamma^{\frac{n}{2}} G_n(s,s) \le G_n(t,s) \le G_n(s,s) \text{ for } n \in \{2k+2; k \in \mathbb{N}\}$$

□

The following theorem gives us the eigenvalue interval for which there exists positive solution(s) for even order problems.

Theorem 3.3. *For $n \in \{2k; k \in \mathbb{N}\}$, assuming that conditions (A_1)-(A_3) is satisfied, then for each λ satisfying*

$$(25) \qquad \frac{1}{[\gamma^n \int_{t_1}^{t_2} G_n(s,s)ds]f_0} < \lambda < \frac{1}{[\int_{t_1}^{t_2} G_n(s,s)ds]f^\infty},$$

there exist at least one positive solution of the BVP

$$(26) \qquad (-1)^{\frac{n}{2}} y^n(t) = \lambda f(t, y(t)).$$

with boundary conditions

$$\begin{aligned} \alpha_{11} y^{(2k)}(t_1) + \alpha_{12} y^{(2k+1)}(t_1) &= 0 \\ \alpha_{21} y^{(2k)}(t_2) + \alpha_{22} y^{(2k+1)}(t_2) &= 0, \quad k = 0, 1, 2, \dots \frac{n}{2} - 1. \end{aligned}$$

Proof. The proof follows by using Theorem 2.11 and changing γ to be $\gamma^{\frac{n}{2}}$ in (11) and (14). Doing this, we have

$$\frac{1}{[\gamma^n \int_{t_1}^{t_2} G_n(s,s) \ ds]f_o} < \lambda < \frac{1}{[\int_{t_1}^{t_2} G_n(s,s) \ ds]f^\infty}.$$

□

3.3. *Example*

Using equation (21), we can easily generate the Green's function for the case where n = 4, 6, 8, 10, and so on. Below is one of such computed Green's function using mathematica.
For the case where n=4,

$$G_4(t,s) = \begin{cases} -\frac{(\beta_1-\alpha_{11}s)(s-t)(-\beta_2+\alpha_{21}t)(3\beta_1(-2\beta_2+\alpha_{21}(s+t))+\alpha_{11}(3\beta_2(s+t)-2\alpha_{21}(s^2+st+t^2)))}{6D^2} & \\ +\frac{(\beta_1-\alpha_{11}s)(\beta_1-\alpha_{11}t)((\beta_2-\alpha_{21}t)^3+(-\beta_2+\alpha_{21}t_2)^3)}{3\alpha_{21}D^2} & \\ +\frac{(\beta_2-\alpha_{21}s)(\beta_2-\alpha_{21}t)((-\beta_1+\alpha_{11}s)^3+(\beta_1-\alpha_{11}t_1)^3)}{3\alpha_{11}D^2} & \text{if } t_1 \le s \le t \le t_2; \\ \frac{(\beta_2-\alpha_{21}s)(s-t)(\beta_1-\alpha_{11}t)(-3\beta_1(-2\beta_2+\alpha_{21}(s+t))+\alpha_{11}(-3\beta_2(s+t)+2\alpha_{21}(s^2+st+t^2)))}{6D^2} & \\ +\frac{(\beta_1-\alpha_{11}s)(\beta_1-\alpha_{11}t)((\beta_2-\alpha_{21}s)^3+(-\beta_2+\alpha_{21}t_2)^3)}{3\alpha_{21}D^2} & \\ +\frac{(\beta_2-\alpha_{21}s)(\beta_2-\alpha_{21}t)((-\beta_1+\alpha_{11}t)^3+(\beta_1-\alpha_{11}t_1)^3)}{3\alpha_{11}D^2} & \text{if } t_1 \le t \le s \le t_2. \end{cases}$$

is the Green's function for

$$-y^{(4)} = 0$$

with boundary conditions

$$(27) \qquad \begin{cases} \alpha_{11}y(t_1) + \alpha_{12}y^{'}(t_1) & = 0 \\ \alpha_{21}y(t_2) + \alpha_{22}y^{'}(t_2) & = 0 \\ \alpha_{11}y^{''}(t_1) + \alpha_{12}y^{'''}(t_1) & = 0 \\ \alpha_{21}y^{''}(t_2) + \alpha_{22}y^{'''}(t_2) & = 0. \end{cases}$$

Considering the equation

$$y^{(4)}(t) + \lambda \frac{y(1+200y)}{1+y} = 0, \quad t \in [0,1],$$

with boundary conditions

$$\begin{aligned} y(0) - y^{'}(0) &= 0 \\ 2y(1) + 3y^{'}(1) &= 0 \\ y''(0) - y^{'''}(0) &= 0 \\ 2y^{''}(1) + 3y^{'''}(1) &= 0 \end{aligned}$$

the Green's function is

$$G_4(t,s) = \begin{cases} \frac{1}{147}(8+(-1-s)^3)(5-2s)(5-2t) \\ +\frac{1}{294}(-1-s)(-1-t)(125+(-5+2t)^3) \\ +\frac{1}{294}(-1-s)(5-2t)(s-t) \\ (-15(s+t)+4(s^2+st+t^2)+3(-10+2(s+t))) & \text{if } t_1 \le s \le t \le t_2; \\ \frac{1}{147}(5-2s)(8+(-1-t)^3)(5-2t) \\ +\frac{1}{294}(-1-s)(125+(-5+2s)^3)(-1-t) \\ -\frac{1}{294}(5-2s)(-1-t)(s-t) \\ (-15(s+t)+4(s^2+st+t^2)-3(10-2(s+t))) & \text{if } t_1 \le t \le s \le t_2. \end{cases}$$

We found that $\gamma = \frac{1}{2}$, $f_\infty = 200$, and $f^0 = 1$. Employing (11), we get the eigenvalue interval $0.0748886 < \lambda < \frac{630}{673}$, for which there exists a positive solution.

4. Third-Order Boundary Value Problem on $\mathbb{R}$ with Green's Function and Bound

For this section, we are going to consider the third order eigenvalue problem on $\mathbb{R}$. We are going to consider nonhomogeneous boundary conditions. In this section, we assume $f(t, y(t))$ to be as defined in Section 2.

4.1. *Solving the Third Order Equation*

Consider the boundary value problem

$$y^{'''}(t) = \lambda f(t, y(t)), \quad t \in [t_1, t_3] \tag{28}$$

with boundary conditions

$$\begin{cases} y(t_1) &= \rho_1 \\ y^{'}(t_2) &= \rho_2 \\ y^{''}(t_3) &= \rho_3 \end{cases} \tag{29}$$

Defining $g(t) \equiv \lambda f(t, y(t))$ and taking the Laplace transform of 28, we have

$$\begin{aligned} L(y^{'''}(t)) &= L(g) \text{ which implies} \\ s^3 L(y) - s^2 y(0) - s y^{'}(0) - y^{''}(0) &= L(g). \end{aligned}$$

This implies that

$$L(y) = \frac{1}{s} y(0) + \frac{1}{s^2} y^{'}(0) + \frac{1}{s^3} y^{''}(0) + \frac{1}{s^3} L(g).$$

Taking the inverse Laplace gives

$$
\begin{aligned}
y(t) &= y(0) + ty^{'}(0) + \frac{1}{2}t^2y^{''}(0) + L^{-1}\{\frac{1}{s^3}L(g)\} \\
&= y(0) + ty^{'}(0) + \frac{1}{2}t^2y^{''}(0) + \frac{1}{2}\int_{t_1}^{t}(t-s)^2g(s)\,ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
y^{'}(t) &= y^{'}(0) + ty^{''}(0) + \int_{t_1}^{t}(t-s)g(s)\,ds, \text{ and} \\
y^{''}(t) &= y^{''}(0) + \int_{t_1}^{t}g(s)\,ds
\end{aligned}
$$

Using the boundary conditions, we have that

$$
\begin{aligned}
y(0) &= \rho_1 - t_1\rho_2 - \frac{1}{2}(t_1^2 - 2t_1t_2)\rho_3 + \frac{1}{2}\int_{t_1}^{t_3}(t_1^2 - 2t_1t_2)g(s)\,ds \\
&\quad + \int_{t_1}^{t_2} t_1(t_2 - s)g(s)\,ds, \\
y^{'}(0) &= \rho_2 - t_2\rho_3 + \int_{t_1}^{t_3} t_2g(s)\,ds - \int_{t_1}^{t_2}(t_2 - s)g(s)\,ds, \\
y^{''}(0) &= \rho_3 - \int_{t_1}^{t_3} g(s)\,ds,
\end{aligned}
$$

so that

$$
\begin{aligned}
y(t) &= y(0) + ty^{'}(0) + \frac{1}{2}t^2y^{''}(0) + \frac{1}{2}\int_{t_1}^{t}(t-s)^2g(s)\,ds \\
&= \rho_1 + (t - t_1)\rho_2 + \frac{1}{2}(t^2 - 2tt_2 - t_1^2 + 2t_1t_2)\rho_3 \\
&\quad -\frac{1}{2}\int_{t_1}^{t_3}(t^2 - 2tt_2 - t_1^2 + 2t_1t_2)g(s)\,ds \\
&\quad + \int_{t_1}^{t_2}(t_2 - s)(t_1 - t)g(s)\,ds + \frac{1}{2}\int_{t_1}^{t}(t-s)^2g(s)\,ds
\end{aligned}
$$

That is,

$$\begin{aligned} y(t) &= \rho_1 + (t-t_1)\rho_2 + \frac{1}{2}\left((t-t_2)^2 - (t_2-t_1)^2\right)\rho_3 \\ &\quad -\frac{1}{2}\int_{t_1}^{t_3}((t-t_2)^2-(t_2-t_1)^2)g(s)\,ds \\ &\quad -\int_{t_1}^{t_2}(t_2-s)(t-t_1)g(s)\,ds + \frac{1}{2}\int_{t_1}^{t}(t-s)^2 g(s)\,ds. \end{aligned}$$

Defining

$$z(t) \equiv \rho_1 + (t-t_1)\rho_2 + \frac{1}{2}((t-t_2)^2-(t_2-t_1)^2)\rho_3, \tag{30}$$

we have

$$\begin{aligned} y(t) &= z(t) - \frac{1}{2}\int_{t_1}^{t_3}\left((t-t_2)^2-(t_2-t_1)^2\right)g(s)\,ds \\ &\quad -\int_{t_1}^{t_2}(t_2-s)(t-t_1)g(s)\,ds + \frac{1}{2}\int_{t_1}^{t}(t-s)^2 g(s)\,ds, \end{aligned}$$

where $z(t)$ is the solution of the homogeneous boundary value differential equation

$$y'''(t) = 0,$$

with boundary conditions

$$\begin{cases} y(t_1) &= \rho_1 \\ y'(t_2) &= \rho_2 \\ y''(t_3) &= \rho_3. \end{cases}$$

Also,

(31)

$$G(t,s) = \begin{cases} \frac{1}{2}(s-t_1)^2 & \text{if } t_1 \le s \le t \le t_2 < t_3; \\ \frac{1}{2}\left[(s-t_1)^2 - (s-t)^2\right] & \text{if } t_1 \le t \le s \le t_2 < t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t_2-t)^2\right] & \text{if } t_1 \le t \le t_2 \le s \le t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t-t_2)^2 + (t-s)^2\right] & \text{if } t_1 < t_2 \le s \le t \le t_3; \\ \frac{1}{2}\left[(t_2-t_1)^2 - (t-t_2)^2\right] & \text{if } t_1 < t_2 \le t \le s \le t_3; \\ \frac{1}{2}(s-t_1)^2 & \text{if } t_1 \le s \le t_2 \le t \le t_3. \end{cases}$$

is the Green's function for the equation

(32)
$$y^{'''}(t) = 0,$$

with boundary conditions

(33)
$$\begin{cases} y(t_1) & = 0 \\ y^{'}(t_2) & = 0 \\ y^{''}(t_3) & = 0. \end{cases}$$

From above, $z(t)$ as defined in (30) has zeroes $t^{'}$ and $t^{''}$ where

(34)
$$\begin{cases} t^{'} & = \frac{(\rho_3 t_2 - b_2)+\sqrt{A}}{\rho_3}, \\ t^{''} & = \frac{(\rho_3 t_2 - b_2)-\sqrt{A}}{\rho_3}, \text{ and} \\ A & = [\rho_3(t_1-t_2)+\rho_2]^2 - 2\rho_1\rho_3. \end{cases}$$

We assume the following conditions on $t_1,\ t_2,\ t_3$ and $\rho_1,\ \rho_2, \rho_3$ throughout this section:

$B1:\ t_2 > \frac{t_1+t_3}{2},\ \ t_3 < t^{''}$

$B2: \ \rho_1 > 0, \quad \rho_3 < 0, \quad \rho_3(t_2 - t_1) < \rho_2 < \rho_3(t_2 - t_3).$

Note: $B1$ is derived from the fact that $G(t_3, s)$ must be nonnegative on the interval $t_1 < t_2 \le t \le s \le t_3$ and we choose $t_3 < t''$ so that $(t_1, t_3) \subset (t', t'')$. $B2$ is derived such that $t_1 < t_2 - \frac{\rho_2}{\rho_3} < t_3$ where $t_2 - \frac{\rho_2}{\rho_3}$ is the maximum point of $z(t)$. Also, we make $\rho_3 < 0$ because we want $z(t)$ to be concave down and $\rho_1 > 0$ since we want a positive solution for $y(t)$.

4.2. *Bounds for the Green's Function*

In this section, we find the bounds for the Green's function (31).

Theorem 4.1. *Given that condition $B1$ holds, $G(t,s) > 0$ for (t,s) $\in$* $(t_1, t_3] \times (t_1, t_3]$.

Proof. For $t_1 \le s \le t \le t_2 < t_3$, $G(t, s) > 0$ since $s \ne t_1$.
For $t_1 < t < s \le t_2 < t_3$, since $t_1 < t < s$, we have $s - t_1 > s - t > 0$ and so $G(t, s) = \frac{1}{2}\left[(s - t_1)^2 - (s - t)^2\right] > 0$. Also, if $t = s$, then
$G(t, s) = \frac{1}{2}(s - t_1)^2 > 0$. Therefore $G(t, s) > 0$.
For $t_1 < t < t_2 \le s \le t_3$, since $t_1 < t < t_2$, we have $t_2 - t_1 > t - t_1$ and so $G(t, s) = \frac{1}{2}\left[(t_2 - t_1)^2 - (t_2 - t)^2\right] > 0$. Also, if $t = t_2$, then
$G(t, s) = \frac{1}{2}(t_2 - t_1)^2 > 0$. Therefore $G(t, s) > 0$.
For $t_1 < t_2 \le s \le t \le t_3$, since $t_2 > \frac{t_1+t_3}{2}$, we have
$t_2 - t_1 > t_3 - t_2 > t - t_2$. So, $G(t, s) = \frac{1}{2}\left[(t_2 - t_1)^2 - (t - t_2)^2 + (t - s)^2\right] > 0$.
For $t_1 < t_2 \le t \le s \le t_3$, $G(t, s) > 0$ since $t_2 > \frac{t_1+t_3}{2}$.
Lastly, for $t_1 < s \le t_2 \le t < t_3$, $G(t, s) > 0$ since $s \ne t_1$. □

In the next theorem, we find the bounds for the Green's function 31. This bound is later used to find the range of λ values for which (28) satisfying (29) has a positive solution.

Theorem 4.2. *For a fixed s,* $G(t,s) \le \frac{1}{2}(s-t_1)^2$ *for all (t,s)* $\in (t_1,t_3] \times (t_1,t_3]$.

$G(t,s) \ge \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)$ *for all (t,s)* $\in [t_2,t_3] \times [t_2,t_3]$.

Proof. For $t_1 \le t < s < t_2 < t_3$, $G^{'}(t,s) = s-t > 0$
which implies that $G(t,s)$ is an increasing function of t. So, $G(t,s) < G(s,s)$ for $t < s$.

For $t_1 \le t \le t_2 \le s \le t_3$, $G^{'}(t,s) = t_2 - t \ge 0$ which implies that $G(t,s)$ is a nondecreasing function of t. So, $G(t,s) \le G(t_2,s) = \frac{1}{2}(t_2-t_1)^2 \le \frac{1}{2}(s-t_1)^2$ for $t \le t_2 \le s$.

Likewise, for $t_1 < t_2 \le s \le t \le t_3$, $G^{'}(t,s) = t_2 - s \le 0$, so $G(t,s)$ is a nonincreasing function of t so that $G(t,s) \le G(s,s) = \frac{1}{2}\left[(t_2-t_1)^2-(s-t_2)^2\right] \le \frac{1}{2}(t_2-t_1)^2 \le \frac{1}{2}(s-t_1)^2$.

For $t_1 < t_2 \le t \le s \le t_3$, $\quad t < t_3$ so that $\quad -(t-t_2)^2 > -(t_3-t_2)^2$ which implies $G(t,s) = \frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2\right) \ge \frac{1}{2}\left((t_2-t_1)^2-(t_3-t_2)^2\right)$.

Likewise on $t_1 < t_2 \le s \le t \le t_3$, from above,

$$\begin{aligned} G(t,s) &= \frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2+(t-s)^2\right) \ge \frac{1}{2}\left((t_2-t_1)^2-(t-t_2)^2\right) \\ &\ge \frac{1}{2}\left((t_2-t_1)^2-(t_3-t_2)^2\right) \end{aligned}$$

□

4.3. *Existence of Positive Solution(s)*

In this subsection, we find the range of λ for which (28) satisfying (29) has positive solution. Let $y(t)$ be the solution of the BVP (28)- (29), given by

$$y(t) = z(t) + \lambda \int_{t_1}^{t_3} G(t,s) f(s, y(s))\, ds \tag{35}$$

Defining

$$v(t) \equiv y(t) - z(t),$$

equation (35) can be re-written as

$$v(t) = \lambda \int_{t_1}^{t_3} G(t,s) f(s, v(s))\, ds, \tag{36}$$

which is the solution of the homogeneous boundary value differential equation

$$v^{'''}(t) = \lambda f(t, v(t)), \quad t \in [t_1, t_3], \tag{37}$$

with boundary conditions

$$\begin{cases} v(t_1) &= 0 \\ v^{'}(t_2) &= 0 \\ v^{''}(t_3) &= 0. \end{cases} \tag{38}$$

Also $G(t,s)$ is the Green's function for the differential equation

$$v^{'''}(t) = 0, \quad t \in [t_1, t_3]$$

with boundary conditions

$$\begin{cases} v(t_1) &= 0 \\ v^{'}(t_2) &= 0 \\ v^{''}(t_3) &= 0. \end{cases}$$

Define a set, X , by

$$X = \{u | u \in C[t_1, t_3]\}$$

with norm

$$\|u\| = \max_{t \in [t_1, t_3]} |u(t)|,$$

Then $(X, \|.\|)$ is a Banach space.
Let

$$m = \min\left\{ \min_{t_2 \le s \le t} \left\{ \frac{(t_2-t_1)^2 - (t_3-t_2)^2 + (t_3-s)^2}{(t_2-t_1)^2 + (t_2-s)^2} \right\}, \frac{(t_2-t_1)^2 - (t_3-t_2)^2}{(t_2-t_1)^2} \right\}. \tag{39}$$

We first show that $0 < m < 1$. Since for $t_1 < t_2 \le s \le t \le t_3$, we have $G^{'}(t,s) = t_2 - s \le 0$, so $G(t,s)$ is a decreasing function of t and $G(t_3, s) < G(t_2, s)$. Also, for $t_1 < t_2 \le t \le s \le t_3$, we have $G^{'}(t,s) = t_2 - t \le 0$, so $G(t,s)$ is a decreasing function of t and $G(t_3,s) < G(t_2,s)$.

Define a set κ by

$$\kappa = \{u \in X : u(t) \ge 0 \text{ on } [t_1, t_2] \text{ and } \min_{t \in [t_2, t_3]} u(t) \; \ge \; m\|u\|\}.$$

Then by Lemma 2.8, κ is a cone. Using condition B2,

$$z(t) > 0 \text{ for } t \in (t^{'}, t^{''}),$$

where $t^{'}$ and $t^{''}$ are as define in (34).

From the fact that $z(t^{'}) = 0$ and $z(t_1) = \rho_1 > 0$, we conclude that $t^{'} < t_1$ since z(t) is concave down. Also, since $t_3 < t^{''}$ then $(t_1, t_3) \subseteq (t^{'}, t^{''})$. So, we conclude that

$$z(t) \ge 0 \text{ for } t \in [t_1, t_3].$$

Define the operator T : $\kappa \to X$ by

$$(Tv)(t) = \lambda \int_{t_1}^{t_3} G(t,s) f(s, v(s)) ds, \quad \forall \;\; t \in [t_1, t_3] \tag{40}$$

From Lemma 2.9, T preserves κ. If $v \in \kappa$ is a fixed point of T , then v satisfies (37) and hence v is a positive solution of the BVP (37) - (38). We seek a fixed point of the operator ,T, in the cone κ.
From Lemma 2.9, the operator T as defined in (40) preserves κ.

Now, we find the range of λ that gives a positive solution for (36)

Theorem 4.3. *Assume that conditions (B1),(B2) is satisfied. Then, for each λ satisfying*

$$(41) \qquad \frac{1}{[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\,ds]f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2\,ds]f^0},$$

there exist at least one positive solution of the BVP (37)-(38) in κ where m is defined in (39).

Proof. Let λ be given as in (41). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\;\;ds](f_\infty-\epsilon)} \le \lambda \le \frac{1}{[\int_{t_1}^{t_3}G(s,s)\,ds](f^0+\epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (40). By definition of f^0, there exist $H_1 > 0$ such that

$$\max_{t\in[t_1,t_3]}\frac{f(t,v)}{v} \le (f^0+\epsilon), \quad \text{for} \;\; 0 < v \le H_1.$$

It follows that, $f(t,v) \le (f^0+\epsilon)v$, for $0 < v \le H_1$. So choosing $v_1 \in \kappa$ with $\|v_1\| = H_1$. Then, we have from the boundedness of $G(t,s)$ that

$$\begin{aligned}
(Tv_1)(t) &= \lambda\int_{t_1}^{t_3} G(t,s)f(s,v_1(s))\,ds \\
&\le \lambda\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2 f(s,v_1(s))\,ds \\
&\le \lambda\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2(f^0+\epsilon)v_1(s)\,ds \\
&\le \lambda\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2(f^0+\epsilon)\|v_1\|\,ds \\
&\le \|v_1\|.
\end{aligned}$$

Consequently, $\|Tv\| \le \|v\|$. So, if we define

$$\Omega_1 = \{u \in X : \|u\| < H_1\},$$

Then

$$\|Tv\| \leq \|v\|, \quad \text{for} \;\; v \in \kappa \cap \partial\Omega_1. \tag{42}$$

By definition of f_∞, there exists an $\overline{H_2} > 0$ such that

$$\min_{t \in [t_1, t_3]} \tfrac{f(t,v)}{v} \geq (f_\infty - \epsilon), \text{ for } v \geq \overline{H_2}.$$

It follows that $f(t,v) \geq (f_\infty - \epsilon)v$, for $v \geq \overline{H_2}$.

Let

$$H_2 = \max\{2H_1, \frac{1}{m}\overline{H_2}\},$$

and let

$$\Omega_2 = \{u \in X : \|u\| < H_2\}.$$

Now choose $v_2 \in \kappa \cap \partial\Omega_2$ with $\|v_2\| = H_2$, so that

$$\min_{t \in [t_1, t_2]} v_2(t) \geq m\|v_2\| \geq \overline{H_2}.$$

Consider,

$$\begin{aligned}
T(v_2)(t) &= \lambda \int_{t_1}^{t_3} G(t,s) f(s, v_2(s)) \; ds \\
&\geq \lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2) f(s, v_2(s)) \; ds \\
&\geq \lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2)(f^\infty - \epsilon) v_2(s) \; ds \\
&\geq m\lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2)(f^\infty - \epsilon)\|v_2\| \; ds \\
&\geq \|v_2\|.
\end{aligned}$$

Thus,

$$\|Tv\| \geq \|v\|, \text{ for } \;\; v \; \in \kappa \cap \partial\Omega_2 \tag{43}$$

Applying Theorem 2.11 to (42) and (43) yields a fixed point for $Tv(t) \in \kappa \cap (\overline{\Omega_2} \backslash \Omega_1)$. This fixed point is the positive solution of the BVP (37)- (38) for the given λ. □

Next, we prove the other range for λ for which a positive solution exists.

Theorem 4.4. *Assume that conditions (B1)-(B2) is satisfied. Then, for each λ satisfying*

$$\frac{1}{[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t-t_2)^2)\,ds]f_0} < \lambda < \frac{1}{[\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2\ \ ds]f^\infty} \tag{44}$$

there exist at least one positive solution of the BVP (37)-(38) in κ .

Proof. Let λ be given as in (44). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{[m\int_{t_2}^{t_3}\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\ \ ds](f_0-\epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{t_3}\frac{1}{2}(s-t_1)^2\ \ ds]u(f^\infty+\epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (2.9). By definition of f_0, there exist $J_1 > 0$ such that

$$\min_{t\in[t_1,t_3]}\frac{f(t,v)}{v} \geq (f_0-\epsilon),\ \text{for}\ \ 0 < v \leq J_1.$$

It follows that, $f(t,v) \geq (f_0 - \epsilon)v$, for $0 < v \leq J_1$. So choosing $v_1 \in \kappa$ with $\|v_1\| = J_1$. we have from the boundedness of G(t,s) that

$$\begin{aligned}
(Tv_1)(t) &= \lambda \int_{t_1}^{t_3} G(t,s) f(s, v_1(s))\ ds \\
&\geq \lambda \int_{t_2}^{t_3} G(t,s) f(s, v_1(s))\ ds \\
&\geq \lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2) f(s, v_1(s))\ ds \\
&\geq \lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2)(f_o - \epsilon) v_1(s)\ ds \\
&\geq m\lambda \int_{t_2}^{t_3} \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2)(f_o - \epsilon)\|v_1\|\ ds \\
&\geq \|v_1\|.
\end{aligned}$$

Consequently, $\|Tv\| \geq \|v\|$. So, if we define

$$\Omega_1 = \{u \in X : \|u\| < J_1\},$$

Then

$$\|Tv\| \geq \|v\|, \quad \text{for} \quad v \in \kappa \cap \partial\Omega_1. \tag{45}$$

It remains for us to consider f^∞. By the definition of f^∞, there exists an $\overline{J_2} > 0$ such that

$$\max_{t \in [t_1, t_3]} \frac{f(t,v)}{v} \leq (f^\infty + \epsilon), \quad \text{for} \quad v \geq \overline{J_2}.$$

It follows that

$$f(t,v) \leq (f^\infty + \epsilon)v, \quad \text{for } v \geq \overline{J_2}.$$

There are two cases.

Case (i): The function f is bounded. Suppose L >0 is such that $f(t,v) \leq$ L, for all $0 < v < \infty$.

Let

$$J_2 = \max\{2J_1, \lambda L \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 ds\},$$

Then for $v_2 \in \kappa$ with $\|v_2\| = J_2$,

$$\begin{aligned} (Tv_2)(t) &= \lambda \int_{t_1}^{t_3} G(t,s) f(s, v_2(s)) \ ds \\ &\leq \lambda \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 f(s, v_2(s)) \ ds \\ &\leq \lambda L \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 \ ds \\ &\leq J_2 = \|v_2\|. \end{aligned}$$

Thus, $\|Tv\| \leq \|v\|$. So, if we define

$$\Omega_2 = \{u \in X : \|u\| < J_2\},$$

then

$$\|Tv\| \leq \|v\|, \quad \text{for} \quad v \in \kappa \cap \partial\Omega_2. \tag{46}$$

Case (ii): The function f is unbounded. Let $J_2 > \max\{2J_1, \overline{J_2}\}$ be such that $f(t,v) \leq f(t, J_2)$, for $0 < v \leq J_2$. Let $v_2 \in \kappa$ with $\|v_2\| = J_2$. Then

$$\begin{aligned} (Tv_2)(t) &= \lambda \int_{t_1}^{t_3} G(t,s) f(s, v_2(s))\, ds \\ &\leq \lambda \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 f(s, v_2(s))\, ds \\ &\leq \lambda \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 f(s, J_2)\, ds \\ &\leq \lambda \int_{t_1}^{t_3} \frac{1}{2}(s-t_1)^2 (f^\infty + \epsilon) J_2\, ds \\ &\leq J_2 = \|v_2\|. \end{aligned}$$

Thus , $\|Tv\| \leq \|v\|$. For this case, if we define

$$\Omega_2 = \{u \in X : \|u\| < J_2\},$$

then

$$\|Tv\| \leq \|v\|, \quad for \;\; v \in \kappa \cap \partial\Omega_2. \tag{47}$$

In either of the cases, an application of part (ii) of Theorem 2.11 to (45), (46) and (47) yields a fixed point for $Tv(t) \in \kappa \cap (\overline{\Omega_2} \backslash \Omega_1)$. This fixed point is the solution of the BVP (37)-(38) for the given λ. □

4.4. *Green's Function and Bound for the 3n$^{(th)}$ Order BVP*

Our interest in this section is to find positive solutions to all differential equations of the form

$$y^{(n)} + \lambda f(t, y(t)) = 0 \tag{48}$$

subject to some boundary conditions

$$\begin{cases} y^{(3k)}(t_1) & = \rho_1 \\ y^{(3k+1)}(t_2) & = \rho_2 \\ y^{(3k+2)}(t_3) & = \rho_3, \;\; k = 0, 1, 2, \ldots, \frac{n}{2} - 1. \end{cases} \tag{49}$$

Before we can do this, we need to be able to generate the Green's function of the homogeneous boundary value problem. The following theorem offers us a method.

Theorem 4.5. *Suppose that $G_3(t, s)$ is the Green's function of*

$$y^{'''}(t) = 0$$

satisfying the boundary conditions

$$(50) \qquad \begin{cases} y(t_1) &= 0 \\ y^{'}(t_2) &= 0 \\ y^{''}(t_3) &= 0 \end{cases}$$

then ,

$$(51) \qquad G_n(t,s) = \int_{t_1}^{t_3} G_3(t,w)G_{n-3}(w,s)\,dw \;\; n \in \{3k+3 : k \in \mathbb{N}\}$$

is the Green's function for

$$(52) \qquad y^n(t) = 0, \;\; n \in \{3k+3 : k \in \mathbb{N}\},$$

with boundary conditions

$$(53) \qquad \begin{cases} y^{(3k)}(t_1) &= 0 \\ y^{(3k+1)}(t_2) &= 0 \\ y^{(3k+2)}(t_3) &= 0, \;\; k = 0,1,2,\ldots,\frac{n}{2}-1. \end{cases}$$

Proof. Suppose $G_3(t,s)$ is the Green's function of $y^{'''}(t) = 0$, satisfying the boundary conditions $y(t_1) = 0,\; y^{'}(t_2) = 0,\; y^{''}(t_3) = 0$ then, the solution of

$$y^{'''}(t) = g$$

satisfying the above BCs is

$$y(t) = \int_{t_1}^{t_3} G_3(t,s)g(s)\,ds.$$

So, if $y^{(6)}(t) = g$, that is, $(y^{(3)})^{(3)} = g$, then,

$$y^{(3)}(t) = \int_{t_1}^{t_3} G_3(t,s)g(s)\,ds \equiv H(t)$$

which implies

$$\begin{aligned} y(t) &= \int_{t_1}^{t_3} G_3(t,w)H(w)\,dw \\ &= \int_{t_1}^{t_3} G_3(t,w)\left\{\int_{t_1}^{t_3} G_3(w,s)g(s)\,ds\right\} dw \\ &= \int_{t_1}^{t_3}\int_{t_1}^{t_3} G_3(t,w)G_3(w,s)g(s)\,ds\,dw \\ &= \int_{t_1}^{t_3}\left\{\int_{t_1}^{t_3} G_3(t,w)G_3(w,s)\,dw\right\} g(s)\,ds \\ &= \int_{t_1}^{t_3} G_6(t,s)g(s)\,ds, \end{aligned}$$

where

$$G_6(t,s) = \int_{t_1}^{t_3} G_3(t,w)G_3(w,s)\,dw.$$

By definition of $G_3(t,s)$,

$$G_6(t,s) = \int_{t_1}^{t_3} G_3(t,w)G_3(w,s)\,dw$$

implies that

$$G_6^{'''}(t,s) = G_3(t,s)$$

which implies that $y^{'''}$ satisfies the boundary conditions for the equation (50), that is,

$$\begin{cases} (y^{'''})(t_1) &= 0 \\ (y^{'''})'(t_2) &= 0 \\ (y^{'''})''(t_3) &= 0. \end{cases}$$

Likewise, $G_6(t,s)$ satisfies the boundary conditions (50) so that $y(t)$ satisfies the boundary conditions

$$\begin{cases} y(t_1) &= 0 \\ y^{'}(t_2) &= 0 \\ y^{''}(t_3) &= 0. \end{cases}$$

So, $G_6(t,s)$ is the Green's function for the equation

$$y^{(6)}(t) = 0,$$

with boundary conditions

$$\begin{cases} y(t_1) &= 0 \\ y^{'}(t_2) &= 0 \\ y^{''}(t_3) &= 0 y^{'''}(t_1) \quad = 0 \\ y^{(4)}(t_2) &= 0 \\ y^{(5)}(t_3) &= 0. \end{cases} \tag{54}$$

Similarly,

$$y^{(9)}(t) = g(t)$$

implies that

$$(y^{(3)})^{(6)}(t) = g(t)$$

which gives us

$$y^{(3)}(t) = \int_{t_1}^{t_3} G_6(t,s)g(s)\,ds = H(t)$$

so that

$$
\begin{aligned}
y(t) &= \int_{t_1}^{t_3} G_3(t,w)H(w)\,dw \\
&= \int_{t_1}^{t_3} G_3(t,w)\left\{\int_{t_1}^{t_3} G_6(w,s)g(s)\,ds\right\}\,dw \\
&= \int_{t_1}^{t_3} \left\{\int_{t_1}^{t_3} G_3(t,w)G_6(w,s)g(s)\,dw\right\}\,ds \\
&= \int_{t_1}^{t_3} G_9(t,s)g(s)ds,
\end{aligned}
$$

where

$$G_9(t,s) = \int_{t_1}^{t_3} G_3(t,w)G_6(w,s)g(s)\,dw.$$

By definition of $G_3(w,s)$

$$G_9^{'''}(t,s) = G_6(t,s)$$

which means that $y^{'''}$ satisfies the boundary conditions (54), that is

$$
\begin{aligned}
y^{(3)}(t_1) &= 0 \\
y^{(4)}(t_2) &= 0 \\
y^{(5)}(t_3) &= 0 \\
y^{(6)}(t_1) &= 0 \\
y^{(7)}(t_2) &= 0 \\
y^{(8)}(t_3) &= 0.
\end{aligned}
$$

Also, $G_9(t,s)$ satisfies the boundary conditions (38) and y(t) satisfies the boundary conditions

$$\begin{cases} y(t_1) & = 0 \\ y^{'}(t_2) & = 0 \\ y^{(2)}(t_3) & = 0 \\ y^{(3)}(t_1) & = 0 \\ y^{(4)}(t_2) & = 0 \\ y^{(5)}(t_3) & = 0 \\ y(6)(t_1) & = 0 \\ y^{(7)}(t_2) & = 0 \\ y^{(8)}(t_3) & = 0. \end{cases}$$

Continuing in this way, we find that

$$G_n(t,s) = \int_{t_1}^{t_3} G_3(t,w) G_{n-3}(w,s) dw, \quad n \in \{3k+3; k \in \mathbb{N}\},$$

is the Green's function for

$$y^n(t) = 0, \quad n \in \{3k+3; k \in \mathbb{N}\},$$

with boundary conditions

$$\begin{cases} y^{(3k)}(t_1) & = & 0 \\ y^{(3k+1)}(t_2) & = & 0 \\ y^{(3k+2)}(t_3) & = & 0, \ k = 0, 1, 2, \ldots \frac{n}{2} - 1. \end{cases}$$

□

4.5. *Bounds for the Green's Function*

In this section, we find the bounds for Green's function, $G_n(t,s)$.

Theorem 4.6. *Assuming conditions (B1),(B2),then for* $n \in \{3k; k \in \mathbb{N}\}$,

$$\left(\frac{t_3 - t_1}{2}\right)^{\frac{n}{3}-1} \left((t_2 - t_1)^2 - (t_3 - t_2)^2\right)^{\frac{n}{3}} \leq G_n(t, s) \text{ for all } (t, s) \in [t_2, t_3] \times [t_2, t_3].$$

$$G_n(t, s) \leq 3 \left(\frac{1}{6}\right)^{\frac{n}{3}} (t_3 - t_1)^{n-3}(s - t_1)^2 \text{ for all } (t, s) \in [t_1, t_3] \times [t_1, t_3].$$

Proof. From Theorem (4.2),

$$G_3(t, s) \leq \frac{1}{2}(s - t_1)^2 \text{ for all } (t, s) \in [t_1, t_3] \times [t_1, t_3], \text{and}$$

$$G_3(t, s) \geq \frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2) \text{ for all } (t, s) \in [t_2, t_3] \times [t_2, t_3].$$

So,

$$\begin{aligned} G_6(t, s) &= \int_{t_1}^{t_3} G_3(t, x)G_3(x, s)\, dx \\ &\leq \int_{t_1}^{t_3} \frac{1}{2}(x - t_1)^2 \frac{1}{2}(s - t_1)^2\, dx = \frac{1}{3}\left(\frac{1}{2}\right)^2 (s - t_1)^2(t_3 - t_1)^3. \end{aligned}$$

Therefore,

$$G_6(t, s) \leq \frac{1}{3}\left(\frac{1}{2}\right)^2 (s - t_1)^2(t_3 - t_1)^3.$$

Also for all $(t, s) \in [t_2, t_3] \times [t_2, t_3]$,

$$\begin{aligned} G_6(t, s) &= \int_{t_1}^{t_3} G_3(t, x)G_3(x, s)\, dx \\ &\geq \int_{t_1}^{t_3} \left(\frac{1}{2}\left\{(t_2 - t_1)^2 - (t_3 - t_2)^2\right\}\right)^2 dx \\ &\geq \left(\frac{1}{2}\left\{(t_2 - t_1)^2 - (t_3 - t_2)^2\right\}\right)^2 (t_3 - t_1). \end{aligned}$$

Therefore,

$$G_6(t, s) \geq \left(\frac{1}{2}((t_2 - t_1)^2 - (t_3 - t_2)^2)\right)^2 (t_3 - t_1).$$

Similarly,

$$\begin{aligned}
G_9(t,s) &= \int_{t_1}^{t_3} G_3(t,x)G_6(x,s)\,dx \\
&\le \int_{t_1}^{t_3} \frac{1}{2}(x-t_1)^2\frac{1}{3}\left(\frac{1}{2}\right)^2(s-t_1)^2(t_3-t_1)^3\,dx \\
&= \left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3(s-t_1)^2(t_3-t_1)^6.
\end{aligned}$$

Also,

$$\begin{aligned}
G_9(t,s) &= \int_{t_1}^{t_3} G_3(t,x)G_6(x,s)\,dx \\
&\ge \int_{t_1}^{t_3} \frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\left[\frac{1}{2}\left\{(t_2-t_1)^2-(t_3-t_2)^2\right\}\right]^2(t_3-t_1)\,dx \\
&= \left(\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\right)^3(t_3-t_1)^2.
\end{aligned}$$

Therefore,

$$G_9(t,s) \ge \left(\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\right)^3(t_3-t_1)^2.$$

Continuing in this sense, we have that

$$G_n(t,s) \le \left(\frac{1}{3}\right)^{\frac{n-3}{3}}\left(\frac{1}{2}\right)^{\frac{n}{3}}(t_3-t_1)^{n-3}(s-t_1)^2 \text{ for all } (t,s)\in[t_1,t_3]\times[t_1,t_3],$$

$$G_n(t,s) \ge \left(\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\right)^{\frac{n}{3}}(t_3-t_1)^{\frac{n-3}{3}} \text{ for all } (t,s)\in[t_2,t_3]\times[t_2,t_3],$$

□

By defining the two functions

$$F_n(s,s) = \left(\frac{1}{3}\right)^{\frac{n-3}{3}}\left(\frac{1}{2}\right)^{\frac{n}{3}}(t_3-t_1)^{n-3}(s-t_1)^2,$$

$$E_n(s,s) = \left(\frac{1}{2}((t_2-t_1)^2-(t_3-t_2)^2)\right)^{\frac{n}{3}}(t_3-t_1)^{\frac{n-3}{3}},$$

going by (41) and (44), we can state the following theorems:

Theorem 4.7. *Assume that conditions (B1),(B2) are satisfied. Then, for each λ satisfying*

$$\frac{1}{[m\int_{t_2}^{t_3} E_n(s,s)\,ds]f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{t_3} F_n(s,s)\,ds]f^0},$$

there exist at least one positive solution of the BVP (48)-(49) in κ .

Proof. The proof is similar to that of Theorem 4.3. □

Theorem 4.8. *Assume that conditions (B1),(B2) are satisfied. Then, for each λ satisfying*

$$\frac{1}{[m\int_{t_2}^{t_3} E_n(s,s)\,ds]f_0} < \lambda < \frac{1}{[\int_{t_1}^{t_3} F_n(s,s)\,ds]f^\infty}$$

there exist at least one positive solution of the BVP (48)-(49) in κ .

Proof. The proof is similar to that of Theorem 4.4. □

4.6. *Example*

Consider the third order boundary value problem

$$y^{'''}(t) + \lambda y(200 - 199.5e^{-7y}) = 0, \quad t \in [0,1],$$

with boundary conditions,

$$\begin{cases} y(1) & = 1 \\ y'(2.6) & = 0 \\ y''(4) & = -1 \end{cases}$$

The Green's function is given by

$$G(t,s)=\begin{cases}\frac{1}{2}(s-1)^2 & \text{if } 1\le s\le t\le 2.6<4;\\ \frac{1}{2}(-1+2s-t)(-1+t) & \text{if } 1\le t\le s\le 2.6<4;\\ \frac{1}{2}(4.2-t)(t-1) & \text{if } 1\le t\le 2.6\le s\le 4;\\ \frac{1}{2}(-4.2+s^2+5.2t-2st) & \text{if } 1<2.6\le s\le t\le 4;\\ \frac{1}{2}(4.2-t)(-1.+t) & \text{if } 1<2.6\le t\le s\le 4;\\ \frac{1}{2}(s-1)^2 & \text{if } 1\le s\le 2.6\le t\le 4.\end{cases}$$

For this particular example,

$$z(t)=1+\frac{1}{2}(2.56-(t-2.6)^2),$$

$m=0.132743,\ f_\infty=200,\ f^0=\frac{1}{2}.$

Using (41), positive solution exists for all λ in the interval (0.0896828,0.222222).

5. Second Order Boundary Value Problems on a Time Scale

In this section, we will find positive solutions for the two boundary value problem discussed in Section 2 and Section 3, on a general time scales. First, we will discuss what a time scale is.

5.1. *Time Scales*

The calculus of time scales was introduced by Stefan Hilger in his Ph.D. thesis (Universität Würzburz,1988) in order to unify the discrete and continuous analysis. The definitions and the theorems in this subsection are from [2].
A time scale is an arbitrary non-empty closed subset of the real numbers. It is usually denoted by $\mathbb{T}$. Thus $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ are some examples of time scales. But $\mathbb{Q}, \mathbb{R} - \mathbb{Q}\left\{irrationals\right\}, \mathbb{C}$ and $(0,1)$,i.e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. We move through the time scale using forward and backward jump operators. The gaps in the time scale are measured by a function μ, defined in terms of forward jump operator, σ.

Definition 5.1. *Forward jump operator* Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, by

$$\sigma(t) := \inf\left\{s \in \mathbb{T} : s > t\right\}$$

Backward jump operator : An operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) := \sup\left\{s \in \mathbb{T} : s < t\right\}$$

Note 1: If $\sigma(t) > t$, we say that t is $right-scattered$, while if $\rho(t) < t$ we say that t is $left-scattered$. The points which are both right-scattered and left-scattered are called isolated.

Note 2 : If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense.

Note 3 : If $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

The forward jump operator defined on t, $\sigma(t)$, is not always equal to t. The difference between $\sigma(t)$ and t is called graininess.

Graininess Function: The Graininess of a time scale ,$\mathbb{T}$, $\mu :\rightarrow [0,\infty)$ is defined by

$\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

Note 4: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$. That is,

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup\ \mathbb{T}), sup\mathbb{T}] & \text{if } \sup T < \infty \\ \mathbb{T} & \text{if } \sup\ \mathbb{T} = \infty. \end{cases}$$

Note 5: Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function, then we define the function, $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$, by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$ $\quad$ *i.e.*, $f^\sigma = f \circ \sigma$.

Using σ we define the delta derivative of a function f in a natural way.

Definition 5.2. Differentiation: Assume $f : \mathbb{T}^k \rightarrow \Re$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\triangle(t)$ to be the number(provided it exists) with the property that given any $\epsilon > 0$ there exists a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that

$|[f(\sigma(t)) - f(s)] - f^\triangle(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$ for all $s \in U$

where $f^\triangle(t)$ is called delta derivative of f at t.

Using the limit definition,

Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous and let $t \in \mathbb{T}^k$. Then we define

$$f^\triangle(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

provided the limits exist.

We will introduce the delta derivative $f^\triangle$ for a function f defined on $\mathbb{T}$. It is expressed as

(i) $f^{\triangle} = f'$ (is the usual derivative) if $\mathbb{T} = \mathbb{R}$ and

(ii) $f^{\triangle} = \triangle f$ (is the forward difference operator) if $\mathbb{T} = \mathbb{Z}$.

Theorem 5.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(iii) If t is right- dense, then f is differentiable at t iff the limit

$$\lim_{s\to\ t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\triangle}(t) = \lim_{s\to\ t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t,then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t).$$

Now we introduce the most powerful fundamentals of derivatives: the sum rule, product rule, quotient rule and the transformation of the sigma function in terms of original function and its derivative.

Theorem 5.4. *Assume f, g: $\mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

(ii) For any constant $\alpha,\ \alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) \\ &= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$*, then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left\{\frac{1}{f}\right\}^{\triangle}(t) = \frac{-f^{\triangle}(t)}{f(t)f(\sigma(t))}$$

(v) If $g(t)g(\sigma(t)) \neq 0$ *then* $\frac{f}{g}$ *is differentiable at* t *and*

$$\left\{\frac{f}{g}\right\}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}$$

In addition to the differentiability we need couple of more conditions for integrability of the function.

Definition 5.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated*, provided its right-sided limits exists(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exists(finite) at all left-dense points in $\mathbb{T}$. The set of such function is denoted by R.

A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-continuous$ provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. It is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable in the region of differentiation D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k - D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each each $t \in$

D. Assume $f : \mathbb{T} \to \mathbb{T}$ is regulated function. Any function F is called a $pre-antiderivative$ of f if $F^{\triangle}(t) = f(t)$.

Theorem 5.6. *Existence of Pre-Antiderivative Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^{\triangle}(t) = f(t)$ holds for all $t \in D$.*

The indefinite integral of a regulated function f is given by

$$\int f(t)\triangle t = F(t) + C$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by:

$$\int_r^s f(t)\triangle t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

TABLE: Time scale derivative and Antiderivative for $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{T} = \mathbb{Z}$

Time $\mathbb{T}$	symbol	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator	$\rho(t)$	t	$t-1$
Forward jump operator	$\sigma(t)$	t	$t+1$
Graininess	$\mu(t)$	0	1
Derivative	$f^{\triangle}(t)$	$f'(t)$	$\triangle f(t)$
Integral	$\int_a^b f(t)\triangle t$	$\int_a^b f(t)dt$	$\sum_{t=a}^{b-1} f(t)(\text{if } \quad a < b)$
Rd-continuous	f	continuous f	any f

Theorem 5.7. *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t).$$

Some fundamental laws of integration are summarized in the following theorem including two laws of integration by parts.

Theorem 5.8. *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*
(i) $\int_a^b (f(t) + g(t))\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$;
(ii) $\int_a^b (\alpha f)(t)\triangle t = \alpha \int_a^b f(t)\triangle t$;
(iii) $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t$;
(iv) $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$
(v) $\int_a^b f(\sigma(t))g(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(t)\triangle t$;
(vi) $\int_a^b f(t)g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t$;
(vii) $\int_a^a f(t)\triangle t = 0$
(viii) If $|f(t)| \leq g(t)$ *on* $[a, b)$, *then* $|\int_a^b f(t)\triangle t| \leq \int_a^b g(t)\triangle t$;
(ix) If $f(t) \geq 0$ *for all* $a \leq t \leq b$, *then* $\int_a^b f(t)\triangle t \geq 0$.

The interesting part of time scale calculus is that the integration can also be performed if the domain of the function is a subset of the integers. Thus integration of any function depends upon the domain of the function.

Theorem 5.9. *Let $a, b, c \in \mathbb{T}$ and $f \in C_{rd}$*

(i) If $\mathbb{T} = \mathbb{R}$,

$$\int_a^b f(t)\triangle t = \int_a^b f(t)dt$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If [a,b] consists of only isolated points, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a<b \\ 0 & \text{if } a=b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a>b \end{cases}$$

(iii) if $\mathbb{T}=\mathbb{Z}$*, then*

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a<b \\ 0 & \text{if } a=b \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a>b \end{cases}$$

Now we move to the dynamic equation with the delta derivative.

Definition 5.10. For $h>0$, we define the Hilger complex numbers , the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle as

$$\begin{aligned} \mathbb{C}_h &:= \left\{ z\in\mathbb{C} : z \neq -\frac{1}{h} \right\}, \\ \mathbb{R}_h &:= \left\{ z\in\mathbb{C}_h : z\in\mathbb{R} \ \text{ and } \ z > -\frac{1}{h} \right\}, \\ \mathbb{R}_h &:= \left\{ z\in\mathbb{C}_h : z\in\mathbb{R} \ \text{ and } \ z < -\frac{1}{h} \right\}, \\ \mathbb{I}_h &:= \left\{ z\in\mathbb{C}_h : \left| z+\frac{1}{h} \right| = \frac{1}{h} \right\}, \end{aligned}$$

respectively. For $h=0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\mathbb{R}$ and $\mathbb{A}_0 := \emptyset$.

The generalized exponential function is denoted by $e_p(t,t_0)$. The exponential function is defined as follows;

$$e_p(t,t_0) = \int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau \tag{55}$$

where the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$, $h > 0$ is defined as

$$\xi_h(z) = \frac{1}{h}\log(1 + zh)$$

where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

It is used to solve the differential equation equations. The basic properties of exponential function can be summarized as follows:

If p, $q \in R$, then $e_0(t, s) = 1$ and $e_p(t, t) = 1$, $\frac{1}{e_p(t,s)} = e_{\ominus p}(s, t)$, $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$ and $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$.

Definition 5.11. We say that the function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

The set of all regressive and rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by R or $R(\mathbb{T})$ or $R(\mathbb{T}, \mathbb{R})$.

Definition 5.12. Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^{\triangle} = f(t, y, y^{\sigma}) \tag{56}$$

is called a first order dynamic equation, sometimes called a differential equation. If

$$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t) \text{ or } f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t)$$

for the rd continuous functions f_1 and f_2, then (56) is called a linear equation. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of (56) if $y^{\triangle}(t) = f(t, y(t), y(\sigma(t)))$ is satisfied for for all $t \in \mathbb{T}^k$.

The general solution of (56) is defined to be the set of all solutions of (56). Now we move to the definition of regressive function which is different for first and second order differential equations.

The general form of second order linear dynamic equation is written as:

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t) \tag{57}$$

where $p, q, f \in C_{rd}$. Let us consider an operator $L_2 : C^2_{rd} \to C_{rd}$ by

$$L_2y(t) = y^{\triangle\triangle}(t) + p(t)y^{\triangle}(t) + q(t)y(t) \quad \text{for } t \in \mathbb{T}^k.$$

Thus the general form of second order equation can be written as

$$L_2y = f.$$

where $L_2y = 0$ is called homogenous dynamic equation.

Theorem 5.13. *The operator $L_2 : C^2_{rd} \to C_{rd}$ is a linear operator, i.e, $L_2(\alpha y_1 + \beta y_2) = \alpha L_2(y_1) + \beta L_2(y_2)$ for all α, $\beta \in \mathbb{R}$ and $y_1, y_2 \in C^2_{rd}$.*

The second order linear dynamic equation $y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t)$ is called regressive provided $p, q, f \in C_{rd}$ such that

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^k.$$

Theorem 5.14. *Existence and Uniqueness:*
Assume that the second order linear dynamic equation

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t)$$

is regressive. If $t_0 \in \mathbb{T}^k$, then the initial value problem

$$L_2y = f(t),\ y(t_0) = y_0,\ y^{\triangle}(t_0) = y_0^{\triangle},$$

where y_0 *and* $y_0^{\triangle}$ *are given constants, has a unique solution, and this solution is defined on the whole time scale* $\mathbb{T}$.

5.2. *Solution to the Second Order Differential Equation*

In this section, we will consider the second order boundary value problem on the general time scale $\mathbb{T}$. We define $\beta(t_i)$ as

$$\beta(t_i) = \alpha_{i1}t_i + \alpha_{i2}, \quad i = 1, 2.$$

Consider the boundary value problem

$$y^{\Delta\Delta}(t) + \lambda f(t, y(\sigma(t))) = 0, \quad t \in [t_1, \sigma(t_2)]_{\mathbb{T}}, \tag{58}$$

with boundary conditions

$$\begin{cases} \alpha_{11}y(t_1) + \alpha_{12}y^{\Delta}(t_1) = 0 \\ \alpha_{21}y(\sigma(t_2)) + \alpha_{22}y^{\Delta}(\sigma(t_2)) = 0 \end{cases} \tag{59}$$

where we assume $f\colon [t_1, \sigma(t_2)] \times \mathbb{R}^+ \to \mathbb{R}^+$ is rd continuous.
The general solution to this differential equation is

$$y(t) = y_c(t) + y_p(t)$$

where $y_c(t)$ is the complementary solution to the homogeneous equation $y^{\Delta\Delta}(t) = 0$ and $y_p(t)$ is a particular solution of (58).
The fundamental solution of

$$y^{\Delta\Delta}(t) = 0 \tag{60}$$

is made up of

$$\begin{aligned} y_1(t) &= e_0(t, t_1) = 1, \\ y_2(t) &= v(t)y_1(t) = v(t) \end{aligned}$$

such that

$$y_2^{\Delta}(t) = v^{\Delta}(t) \ \text{ and } y_2^{\Delta\Delta}(t) = 0 = v^{\Delta\Delta}(t).$$

Putting $v^{\Delta}(t) = 1$, we have

$$v(t) = y_2(t) = \int_{t_1}^{t} \Delta s = t - t_1.$$

So, the complementary solution $y_c(t)$ is

$$y_c(t) = A + B(t - t_1),$$

where A and B are real constants.

The particular solution $y_p(t)$ of the equation

$$y_p^{\Delta\Delta}(t) = -g$$

is of the form

$$y_p(t) = \alpha(t)y_1(t) + \beta(t)y_2(t)$$

$$= \alpha(t) + \beta(t)(t - t_1)$$

where $\alpha(t)$ and $\beta(t)$ are functions of t to be found. The first derivative of $y(t)$ is given as

$$y_p^{\Delta}(t) = \alpha^{\Delta} + \beta^{\Delta}(\sigma(t) - t_1) + \beta(t).$$

Assuming

(61) $$\alpha^{\Delta} + \beta^{\Delta}(\sigma(t) - t_1) = 0,$$

then

$$y_p^{\Delta}(t) = \beta(t)$$

and

$$y_p^{\Delta\Delta}(t) = \beta^{\Delta}(t) = -g(t).$$

From this, we have

$$\beta(t) = -\int_{t_1}^{t} g(s)\,\Delta s.$$

From (61), $\alpha^{\Delta} + \beta^{\Delta}(\sigma(t) - t_1) = 0$ implies $\alpha^{\Delta}(t) = g(t)(\sigma(t) - t_1)$ and therefore,

$$\alpha(t) = \int_{t_1}^{t} (\sigma(s) - t_1) g(s)\,\Delta s.$$

So,

$$y_p(t) = \int_{t_1}^{t} (\sigma(s) - t_1) g(s)\,\Delta s + \int_{t_1}^{t} (t_1 - t) g(s)\,\Delta s$$

so that the general solution becomes

$$\begin{aligned} y(t) &= A + (t - t_1)B + \int_{t_1}^{t} (\sigma(s) - t_1) g(s)\,\Delta s + \int_{t_1}^{t} (t_1 - t) g(s)\,\Delta s \\ &= A + (t - t_1)B + \int_{t_1}^{t} (\sigma(s) - t) g(s)\,\Delta s, \end{aligned}$$

and

$$y^{\Delta}(t) = B - \int_{t_1}^{t} g(s)\,\Delta s,$$

where A and B are constants.

Using the boundary conditions, (59),

$$\begin{aligned} A &= -\frac{1}{d} \int_{t_1}^{\sigma(t_2)} \alpha_{12}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s)) g(s)\,\Delta s \\ B &= \frac{1}{d} \int_{t_1}^{t_2} \alpha_{11}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s)) g(s)\,\Delta s \end{aligned}$$

where

$$d = \alpha_{11}\beta(\sigma(t_2)) - \alpha_{21}\beta(t_1)$$

we have

$$y(t) = \frac{1}{d} \int_{t_1}^{t_2} (\beta_2 - \alpha_{21}\sigma(s))(\alpha_{11}t - \beta_1) g(s)\,\Delta s - \int_{t_1}^{t} (t - \sigma(s)) g(s)\,\Delta s,$$

where β_1, β_2 are defined in (3). The Green's function for $y^{\Delta\Delta} = 0$ satisfying (59) is

$$(62)\quad G(t,s) = \begin{cases} \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}t)(\alpha_{11}\sigma(s) - \beta_1) & \text{if } t_1 \le s \le \sigma(s) \le t; \\ \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}t - \beta_1) & \text{if } t_1 \le t \le s \le \sigma(s). \end{cases}$$

We will assume the following condition throughout this section:

$$A_4 : m(t_1) \le t_1 \le \sigma(t_2) \le m(\sigma(t_2)), \quad \textit{where} \;\; m(t_i) = \frac{\beta(t_i)}{\alpha_{i1}}, i = 1, 2$$

Throughout this section, we assume conditions A_2 and A_4.

5.3. *Bounds for the Green's Function*

In this subsection, we find the bounds for the Green's function (62).

Theorem 5.15. *Assuming conditions A_2 and A_4, $G(t,s) \le G(\sigma(s), s)$ for $(t,s) \in [t_1, \sigma(t_2)] \times [t_1, t_2]$*
$G(t,s) \ge \gamma G(\sigma(s), s)$ for $(t,s) \in [\frac{\sigma(t_2)}{4}, \frac{3\sigma(t_2)}{4}] \times [t_1, t_2]$.

where

$$\gamma = \min\left\{ \frac{\alpha_{11}\sigma(t_2) - 4\beta_1}{4\alpha_{11}\sigma(t_2) - 4\beta_1}, \frac{\alpha_{21}\sigma(t_2) + 4\alpha_{22}}{4\beta(\sigma(t_2)) - 4\alpha_{21}\sigma(t_1)} \right\}$$

Proof. On the interval $t \le s \le \sigma(s)$,

$$\begin{aligned} G(t,s) &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}t - \beta_1) \\ &\le \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}\sigma(s) - \beta_1) \\ &= G(\sigma(s), s). \end{aligned}$$

also on $\sigma(s) \leq t$,

$$\begin{aligned} G(t,s) &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}t)(\alpha_{11}\sigma(s) - \beta_1) \\ &\leq \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}\sigma(s) - \beta_1) \\ &= G(\sigma(s),s) \end{aligned}$$

So, $G(t,s) \leq G(\sigma(s),s)$ for $(t,s) \in [t_1,\sigma(t_2)] \times [t_1,t_2]$. Also for $(t,s) \in [\frac{\sigma(t_2)}{4}, \frac{3\sigma(t_2)}{4}] \times [t_1,t_2]$, and $t \leq s$,

$$\begin{aligned} G(t,s) &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}t - \beta_1) \\ &\geq \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))\left(\frac{\alpha_{11}\sigma(t_2)}{4} - \beta_1\right) \\ &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}\sigma(s) - \beta_1)\left(\frac{\alpha_{11}\sigma(t_2) - 4\beta_1}{4\alpha_{11}\sigma(s) - 4\beta_1}\right) \\ &\geq \gamma G(\sigma(s),s). \end{aligned}$$

Likewise, on $\sigma(s) \leq t$,

$$\begin{aligned} G(t,s) &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}t)(\alpha_{11}\sigma(s) - \beta_1) \\ &\geq \frac{1}{d}\left(\beta(\sigma(t_2)) - \frac{3\alpha_{21}\sigma(t_2)}{4}\right)(\alpha_{11}\sigma(s) - \beta_1) \\ &= \frac{1}{d}\left(\alpha_{21}\sigma(t_2) + \alpha_{22} - \frac{3\alpha_{21}\sigma(t_2)}{4}\right)(\alpha_{11}\sigma(s) - \beta_1) \\ &= \frac{1}{d}\left(\frac{\alpha_{21}\sigma(t_2) + 4\alpha_{22}}{4}\right)(\alpha_{11}\sigma(s) - \beta_1) \\ &= \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11}\sigma(s) - \beta_1)\left\{\frac{\alpha_{21}\sigma(t_2) + 4\alpha_{22}}{4\beta(\sigma(t_2)) - 4\alpha_{21}\sigma(s)}\right\} \\ &\geq G(\sigma(s),s)\left\{\frac{\alpha_{21}\sigma(t_2) + 4\alpha_{22}}{4\beta(\sigma(t_2)) - 4\alpha_{21}\sigma(t_1)}\right\} \\ &\geq \gamma G(\sigma(s),s). \end{aligned}$$

□

We will need the following condition:

(B) $f_0 = \lim_{x\to 0^+} \frac{f(x)}{x}$ and $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$ both exist on the extended real line.

5.4. *Existence of Positive Solution*

In this section, we will be discussing conditions in which there exist a positive solution for the boundary value problem. We assume $\sigma(t_2)$ is right-dense so that G(t,s) ≥ 0 for t $\in [t_1, \sigma(t_2)]$, s $\in [t_1, \sigma(t_2)]$.
Assume that $[t_1, \sigma(t_2)]$ is such that

$$\xi = \min\{t \in \mathbb{T} : \ t \geq \frac{\sigma(t_2)}{4}\}, \quad \omega = \max\{t \in \mathbb{T} : \ t \leq \frac{3\sigma(t_2)}{4}\},$$

both exist and satisfy

$$\frac{\sigma(t_2)}{4} \leq \xi < \omega \leq \frac{3\sigma(t_2)}{4}.$$

And if $\sigma(\omega) = t_2$, also assume $\sigma(\omega) < \sigma(t_2)$. Next, let $\tau \in [\xi, \omega]$ be defined by

$$\int_\xi^\omega G(\tau, s)\Delta s = \max_{t\in[\xi,\omega]_{\mathbb{T}}} \int_\xi^\omega G(t,s)\,\Delta s \tag{63}$$

For any interval S, we will denote $S \bigcap \mathbb{T}$ by $S_{\mathbb{T}}$.
Finally, we define

$$l = \min_{s\in[t_1,\sigma^2(t_2)]} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)} \tag{64}$$

and set

$$r = \min\{\gamma, l\}. \tag{65}$$

On the set

X $= \{x : [t_1, \sigma^2(t_2)] \to \mathbb{R}\}$ define the cone $\kappa \subset X$ by

(66) $\kappa = \{x \in X : x(t) \geq 0 \text{ on } [t_1, \sigma^2(t_2)]_{\mathbb{T}}, \text{ and } x(t) \geq r\|x\| \text{ for } t \in [\xi, \sigma(\omega)]_{\mathbb{T}}\}$.

Define an integral operator T : $\kappa \to X$ by

$$(Tu)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\, \Delta s, \quad u \in \kappa, \tag{67}$$

for t $\in [t_1, \sigma(t_2)]_{\mathbb{T}}$. From the nonnegativity of $G(t,s)$ and from assumption (B), if u $\in \kappa$, then Tu(t) ≥ 0 on $[t_1, \sigma(t_2)]$.

We now show that T : $\kappa \to \kappa$ and that T is completely continuous since for $u \in \kappa$.

From (67) and Theorem 5.15, for $t \in [t_1, \sigma(t_2)]$,

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\, \Delta s \\ &\leq \lambda\gamma \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) f(u(\sigma(s)))\, \Delta s \end{aligned}$$

and so,

$$\|Tu\| \leq \lambda\gamma \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) f(u(\sigma(s)))\, \Delta s. \tag{68}$$

Also, from Theorem 5.15 ,

$$\begin{aligned}
\min_{t\in[\xi,\omega]} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s)f(u(\sigma(s)))\,\Delta s \\
&\geq \lambda \int_{t_1}^{\sigma(t_2)} \gamma G(\sigma(s),s)f(u(\sigma(s)))\,\Delta s \\
&\geq \gamma\|Tu\| \text{ from } (66) \\
&\geq r\|Tu\| \text{ from } (65)
\end{aligned}$$

and also

$$\begin{aligned}
Tu(\sigma(\omega)) &= \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(\omega),s)f(u(\sigma(s)))\,\Delta s \\
&\geq \lambda \int_{t_1}^{\sigma(t_2)} lG(\sigma(s),s)f(u(\sigma(s)))\,\Delta s \text{ from } (64) \\
&\geq r \int_{t_1}^{\sigma(t_2)} lG(\sigma(s),s)f(u(\sigma(s)))\,\Delta s \text{ from } (65) \\
&\geq r\|Tu\|.
\end{aligned}$$

Thus, $Tu \in \kappa$, and we conclude that $T : \kappa \to \kappa$. which shows that T is completely continuous.

Theorem 5.16. *Assume that conditions A_2, A_4, and B is satisfied. Then, for each λ satisfying*

(69)
$$\frac{1}{[r\int_\xi^\omega G(\tau,s)\,\Delta s]f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)\,\Delta s]f_0},$$

there exists at least one positive solution of the BVP (58)-(59) in κ .

Proof. Let λ be as in (69) and choose ϵ such that

$$\frac{1}{[r\int_\xi^\omega G(\tau,s)\,\Delta s](f_\infty-\epsilon)} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)\,\Delta s](f_0+\epsilon)}.$$

We seek a fixed point of T which belongs to κ.

According to the definition of f_0, there exist $H_1 > 0$ such that

$$f(x) \le (f_0 + \epsilon)x \quad \text{for } 0 < x \le H_1.$$

Let

$$\Omega_1 = \{x \in X : \|x\| < H_1\},$$

and choose $u \in \kappa$ with $\|u\| = H_1$. Then from Theorem 5.15 and assuming right-density of $\sigma(t_2)$, for $t \in [t_1, \sigma^2(t_2)]_{\mathbb{T}}$,

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\,\Delta s \\ &\le \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)(f_0+\epsilon)u(\sigma(s))\,\Delta s \\ &\le \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)(f_0+\epsilon)\|u\|\,\Delta s \\ &\le \|u\|. \end{aligned}$$

Thus, $\|Tu\| \le \|u\|$, and in particular,

$$\|Tu\| \le \|u\| \text{ for u} \in \kappa \cap \partial\Omega_1. \tag{70}$$

Considering next f_∞, there exist $J_1 > 0$ such that $f(x) \ge (f_\infty - \epsilon)x$ for $x \ge J_1$. Let $J_2 = \max\{2H_1, \frac{1}{r}J_1\}$ and let

$$\Omega_2 = \{x \in X : \|x\| < J_2\}.$$

If $u \in \kappa$ with $\|u\| = J_2$, then $\min_{t\in[\xi,\omega]_{\mathbb{T}}} u(t) \geq r\|u\| \geq J_1$ and

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s)f(u(\sigma(s)))\,\Delta s \\ &\geq \lambda \int_{\xi}^{\omega} G(t,s)f(u(\sigma(s)))\,\Delta s \\ &\geq \lambda \int_{\xi}^{\omega} G(t,s)(f_\infty - \epsilon)u(\sigma(s))\,\Delta s \\ &\geq r\lambda \int_{\xi}^{\omega} G(t,s)(f_\infty - \epsilon)\|u\|\,\Delta s \\ &\geq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \leq \|u\|$ and, in particular,

(71) $$\|Tu\| \leq \|u\| \text{ for u} \in \kappa \cap \partial\Omega_2.$$

An application of Theorem 2.11 to (70) and (71) yields that T has a fixed point $u \in \kappa \cap (\overline{\Omega_2} \setminus \Omega_1)$. Such a fixed point is a desired solution for the given λ. □

We will show another range of λ for positive solution.

Theorem 5.17. *Assume that conditions A_2, A_4, and B are satisfied. Then, for each λ satisfying*

(72) $$\frac{1}{[r\int_{\xi}^{\omega} G(z,s)\,\Delta s]f_0} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)\,\Delta s]f_\infty},$$

there exists at least one positive solution of the BVP (58)-(59) in κ .

Proof. From definition of f_0, there exist an $H_1 > 0$ such that $f(x) \geq (f_0 - \epsilon)x$ for $0 < x \leq H_1$. Let

$$\Omega_1 = \{x \in X : \|x\| < H_1\}.$$

Then, for $u \in \kappa$ with $\|u\| = H_1$, we have

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\,\Delta s \\ &\geq \lambda \int_{\xi}^{\omega} G(t,s) f(u(\sigma(s)))\,\Delta s \\ &\geq \lambda \int_{\xi}^{\omega} G(t,s)(f_0 - \epsilon) u(\sigma(s))\,\Delta s \\ &\geq r\lambda \int_{\xi}^{\omega} G(t,s)(f_0 - \epsilon)\|u\|\,\Delta s \\ &\geq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$ and in particular,

$$\|Tu\| \geq \|u\| \text{ for } u \in \kappa \cap \partial\Omega_1. \tag{73}$$

As we turn to f_∞, there exists a $J_1 > 0$ such that $f(x) \leq (f_\infty + \epsilon)x$ for all $x \geq J_1$. We consider two cases: (a) f is bounded and (b) f is unbounded. For case (a), suppose N > 0 is such that $f(x) \leq N$ for all $0 < x < \infty$. Let

$$H_2 = \max\{2H_1, N\lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s)\Delta s\}, \tag{74}$$

and define

$$\Omega_2 = \{x \in X : \|x\| < H_2\}.$$

Then, for $u \in \kappa$ with $\|u\| = H_2$,

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\,\Delta s \\ &\leq \lambda N \int_{t_1}^{\sigma(t_2)} G(t,s)\,\Delta s \\ &\leq \|u\|, \text{ from } (74) \end{aligned}$$

and so

$$\|Tu\| \leq \|u\| \text{ for } u \in \kappa \cap \partial\Omega_2. \tag{75}$$

For case (b), let $H_2 > \max\{2H_1, J_1\}$ be such that $f(x) \leq f(H_2)$ for $0 < x \leq H_2$.

Defining

$$\Omega_2 = \{x \in X : \|x\| < H_2\},$$

we choose $u \in \kappa$ with $\|u\| = H_2$. We have

$$\begin{aligned} Tu(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) f(u(\sigma(s)))\, \Delta s \\ &\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) f(H_2)\, \Delta s \\ &\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)(f_\infty + \epsilon) H_2\, \Delta s \\ &\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)(f_\infty + \epsilon)\|u\|\, \Delta s \\ &\leq \|u\|. \end{aligned}$$

Again

$$\|Tu\| \leq \|u\| \text{ for u } \in \kappa \cap \partial\Omega_2. \tag{76}$$

An application of Theorem 2.11 to (73) ,(75) and (76) yields that T has a fixed point $u \in \kappa \cap (\overline{\Omega_2} \backslash \Omega_1)$. Such a fixed point is a desired solution for the given λ. □

5.5. *Green's Function for $2n^{th}$ Order BVP on Time Scale*

The proof we will state in this section is similar to that of Theorem 3.1 which gives the Green's function of $-y''(t) = 0$ satisfying (2).

Theorem 5.18. *Suppose that $G_2(t,s)$ is the Green's function satisfying*

$$-y^{\Delta\Delta}(t) = 0$$

with boundary conditions

$$\begin{aligned}\alpha_{11}y(t_1)+\alpha_{12}y^{\Delta}(t_1) &= 0\\ \alpha_{21}y(t_2)+\alpha_{22}y^{\Delta}(t_2) &= 0\end{aligned}$$

then ,

$$G_n(t,s)=\int_{t_1}^{t_2}G_2(t,w)G_{n-2}(w,s)\,\Delta w \quad n\in\{2k+2:k\in\mathbb{N}\} \tag{77}$$

is the Green's function for

$$(-1)^{\frac{n}{2}}y^{\Delta^{(n)}}(t)=0,\quad n\in\{2k+2:k\in\mathbb{N}\}, \tag{78}$$

with boundary conditions

$$\begin{cases}\alpha_{11}y^{(2k)}(t_1)+\alpha_{12}y^{\Delta^{(2k+1)}}(t_1)=0\\ \alpha_{21}y^{\Delta(2k)}(\sigma(t_2))+\alpha_{22}y^{\Delta^{(2k+1)}}(\sigma(t_2))=0,\ k=0,1,2,\dots\frac{n}{2}-1\end{cases} \tag{79}$$

Proof. The proof is similar to Theorem 3.1. □

Theorem 5.19. *Assuming conditions (A2) and (A4),then for* $n\in\{2k;k\in\mathbb{N}\}$,

$$\gamma^{\frac{n}{2}}G_n(\sigma(s),s)\le G_n(t,s) \text{ for } (t,s)\in[\frac{\sigma(t_2)}{4},\frac{3\sigma(t_2)}{4}]\times[t_1,t_2],$$

and

$$G_n(t,s)\le G_n(\sigma(s),s) \text{ for } (t,s)\in[t_1,\sigma(t_2)]\times[t_1,t_2].$$

Proof. The proof is similar to Theorem 3.2. □

5.6. *Example*

In this section, we will solve some dynamic equations and find the range of λ for which a positive solution can be obtained.

We consider the general boundary value problem,

$$y^{\Delta\Delta}(t) + \lambda f(t, y(\sigma(t))) = 0, \quad t \in \mathbb{T}$$

with boundary conditions

$$\begin{aligned} \alpha_{11} y(t_1) + \alpha_{12} y^{\Delta}(t_1) &= 0 \\ \alpha_{21} y(\sigma(t_2)) + \alpha_{22} y^{\Delta}(\sigma(t_2)) &= 0 \end{aligned}$$

The Green's function is

$$G(t,s) = \begin{cases} \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21} t)(\alpha_{11}\sigma(s) - \beta_1) & \text{if } t_1 \le s \le \sigma(s) \le t; \\ \frac{1}{d}(\beta(\sigma(t_2)) - \alpha_{21}\sigma(s))(\alpha_{11} t - \beta_1) & \text{if } t_1 \le t \le s \le \sigma(s). \end{cases}$$

Case (1):

If $\mathbb{T} = [t_1, t_2] \in \mathbb{R}$, then

$$\sigma(t) = t.$$

Considering again the conditions, let $t_1 = 0,\ t_2 = 1,\ \alpha_{11} = 1,\ \alpha_{12} = -1,$ $\alpha_{21} = 2,\ \alpha_{22} = 3,\ f(t, y) = y + 1$, then

This turns out to be the same example as in Section 2.

Let's solve the same differential equation with different time scales.

Case (2):

If $\mathbb{T} = \mathbb{N}_0$, then

$$\sigma(t) = t + 1.$$

and the equation becomes

$$y(\sigma^2(t)) - 2y(\sigma(t)) + y(t) + \lambda f(y(t)) = 0$$

with boundary conditions

$$a_{11} y(t_1) + a_{12}(y(\sigma(t_1)) - y(t_1)) = 0$$

$$a_{21}y(\sigma(t_2)) + a_{12}(y(\sigma^2(t_2)) - y(\sigma(t_2))) = 0$$

Considering the previous condition, $t_1 = 0,\ t_2 = 1,\ \alpha_{11} = 1,\ \alpha_{12} = -1,$ $\alpha_{21} = 2,\ \alpha_{22} = 3,\ f(t, y) = y + 1$, then

$$G(t,s) = \begin{cases} \frac{1}{9}(-2-s)(-7+2t) & \text{if } t_1 \le s \le \sigma(s) \le t; \\ \frac{1}{9}(5-2s)(1+t) & \text{if } t_1 \le t \le s \le \sigma(s). \end{cases}$$

We have that $\xi = \frac{1}{2},\ \omega = \frac{3}{4},\ \mathrm{l} = \frac{2}{5}$, $\gamma = \frac{1}{2}$, therefore $\mathrm{r} = \frac{2}{5}$.

$f_0 = \infty,\ f_\infty = 1.$

$$\int_{t_1}^{\sigma(t_2)} G(\sigma(t_2), s)\Delta s = \sum_{t=t_1}^{\sigma(t_2)-1} G(\sigma(t_2), s) = \sum_{t=t_1}^{\sigma(t_2)-1} G(t_2+1, s) = \frac{19}{9}$$

$$\int_{\xi}^{\omega} G(\tau, s)\Delta s = \max_{t\in[\xi,\omega]} \int_{\xi}^{\omega} G(t,s)\,\Delta s = \max_{t\in[\xi,\omega]} \sum_{t=\tau}^{\omega-1} G(t,s) = \frac{2}{3}$$

According to Theorem 5.17, there exist a positive solution for all λ in the interval $(0, \frac{9}{19})$.

6. Third Order Boundary Value Problem on Time Scale

In this section, we are going to consider the third order boundary value eigenvalue problem on a general time scale

6.1. *Solution to the Eigenvalue Boundary Value Problem on* $\mathbb{T}$

In this section, we will try to solve the third order BVP on a time scale. Consider the boundary value problem

$$y^{\Delta\Delta\Delta}(t) - \lambda f(t, y(\sigma(t))) = 0,\ \ t \in \mathbb{T} \tag{80}$$

with boundary conditions,

$$\begin{cases} y(t_1) & = \rho_1 \\ y^{\Delta}(\sigma(t_2)) & = \rho_2 \\ y^{\Delta\Delta}(\sigma^2(t_3)) & = \rho_3 \end{cases} \tag{81}$$

For simplicity, define $g(t) = \lambda f(t, y(\sigma(t)))$
The general solution to this differential equation is

$$y(t) = y_c(t) + y(t)_p(t)$$

where $y_c(t)$ is the complementary solution to the homogeneous equation $y^{\Delta\Delta\Delta}(t) = 0$ and $y_p(t)$ is the particular solution of (80) satisfying (81).

The equation

$$y^{\Delta\Delta\Delta}(t) = 0$$

has three equal auxiliary solutions, so the solutions are

$$\begin{aligned} y_1(t) &= e_0(t, t_1) = 1, \\ y_2(t) &= v(t)y_1(t) = v(t) = t - t_1 \\ y_3(t) &= w(t)y_1(t) \end{aligned}$$

such that $y_3^{\Delta}(t) = w^{\Delta}(t)$, $y_3^{\Delta\Delta}(t) = w^{\Delta\Delta}(t)$, $y_3^{\Delta\Delta\Delta}(t) = w^{\Delta\Delta\Delta}(t) = 0$ Setting $w^{\Delta\Delta}(t) = 1$, we have that $w^{\Delta} = t - t_1$ and

$$w(t) = \int_{t_1}^{t} (s - t_1)\,\Delta s$$

so that

$$y_3(t) = w(t) \quad = \quad \int_{t_1}^{t} (s - t_1)\,\Delta s.$$

So,

$$\begin{aligned} y_c(t) &= Ay_1 + By_2 + Cy_3, \\ &= A + B(t - t_1) + C\int_{t_1}^{t} (s - t_1)\,\Delta s \end{aligned}$$

where A, B and C are real constants, and

$$y_p(t) = \alpha(t)y_1(t) + \rho(t)y_2(t) + \gamma(t)y_3(t)$$

$$= \alpha(t) + \rho(t)(t - t_1) + \gamma(t)\int_{t_1}^{t} (s - t_1)\,\Delta s$$

so that

$$y^{\Delta}(t) \quad = \quad \alpha^{\Delta}(t) + \rho^{\Delta}(t)(\sigma(t) - t_1) + \rho(t) + \gamma^{\Delta}(t)\int_{t_1}^{\sigma(t)} (s - t_1)\,\Delta s + \gamma(t)(t - t_1).$$

Setting

$$\alpha^{\Delta}(t) + \rho^{\Delta}(t)(\sigma(t) - t_1) + \gamma^{\Delta}(t)\int_{t_1}^{\sigma(t)} (s - t_1)\,\Delta s = 0, \tag{82}$$

we have

$$\begin{aligned} y^{\Delta}(t) &= \rho(t) + \gamma(t)(t - t_1), \text{ and} \\ y^{\Delta\Delta}(t) &= \rho^{\Delta}(t) + \gamma^{\Delta}(t)(\sigma(t) - t_1) + \gamma(t). \end{aligned}$$

If

$$\rho^{\Delta}(t) + \gamma^{\Delta}(t)(\sigma(t) - t_1) = 0, \tag{83}$$

we have

$$y^{\Delta\Delta}(t) = \gamma(t) \;\; \text{yielding}$$

$$y^{\Delta\Delta\Delta}(t) = \gamma^{\Delta}(t) = g(t)$$

which implies

$$\gamma(t) = \int_{t_1}^{t} g(s)\,\Delta s.$$

From (83),

$$\rho(t) = \int_{t_1}^{t} (t_1 - \sigma(s))g(s)\,\Delta s. \tag{84}$$

From (82),(83) and (84), we have

$$\alpha(t) = \int_{t_1}^{t} (t_1 - \sigma(s))^2 g(s)\,\Delta s + \int_{t_1}^{t}\int_{t_1}^{\sigma(w)} (t_1 - s)g(w)\,\Delta s\,\Delta w \tag{85}$$

so that

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= A + (t - t_1)B + C\int_{t_1}^{t}(s - t_1)\,\Delta s + \int_{t_1}^{t}(t_1 - \sigma(s))^2 g(s)\,\Delta s \\ &+ \int_{t_1}^{t}\int_{t_1}^{\sigma(w)}(t_1 - s)g(w)\,\Delta s\,\Delta w + (t - t_1)\int_{t_1}^{t}(t_1 - \sigma(s))g(s)\,\Delta s \\ &+ \int_{t_1}^{t} g(s)\,\Delta s\int_{t_1}^{t}(s - t_1)\,\Delta s. \end{aligned}$$

Using the boundary conditions, we have

$$
\begin{aligned}
A &= \rho_1 \\
B &= \rho_2 - \int_{t_1}^{\sigma(t_2)} (\sigma(t_2) - \sigma(s))g(s)\,\Delta s - (\sigma(t_2) - t_1)\rho_3 + \int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - t_1)g(s)\,\Delta s, \text{and} \\
C &= \rho_3 - \int_{t_1}^{\sigma^2(t_3)} g(s)\,\Delta s,
\end{aligned}
$$

so that

$$
\begin{aligned}
y(t) &= \rho_1 + (t - t_1)\rho_2 + \rho_3 \left(\int_{t_1}^{t} (s - t_1)\,\Delta s - (t - t_1)(t_2 - t_1) \right) \\
&+ \int_{t_1}^{t} \left((t_1 - \sigma(s))(t - \sigma(s)) + \int_{t_1}^{t} (w - t_1)\Delta w - \int_{t_1}^{\sigma(s)} (w - t_1)\Delta w \right) g(s)\Delta s \\
&- \int_{t_1}^{\sigma(t_2)} (t - t_1)(\sigma(t_2) - \sigma(s))g(s)\Delta s \\
&+ \int_{t_1}^{\sigma^2(t_3)} \left((t - t_1)(\sigma(t_2) - t_1) - \int_{t_1}^{t} (w - t_1)\Delta w \right) g(s)\Delta s
\end{aligned}
$$

where

$$
z(t) = \rho_1 + (t - t_1)\rho_2 + \rho_3 \left(\int_{t_1}^{t} (s - t_1)\,\Delta s - (t - t_1)(t_2 - t_1) \right)
$$

is the solution to the homogeneous equation

$$
y^{\Delta\Delta\Delta}(t) = 0
$$

with boundary conditions

$$
\begin{cases}
y(t_1) = \rho_1 \\
y(\sigma(t_2)) = \rho_2 \\
y(\sigma^2(t_3)) = \rho_3.
\end{cases}
$$

Also,

(86)

$$G(t,s) = \begin{cases} \int_{t_1}^{\sigma(s)} (\sigma(s) - w)\,\Delta w & \text{if } t_1 \le \sigma(s) \le t \le \sigma(t_2) < \sigma^2(t_3); \\ \int_{t_1}^{t} (\sigma(s) - w)\,\Delta w & \text{if } t_1 \le t \le \sigma(s) \le \sigma(t_2) < \sigma^2(t_3); \\ \int_{t_1}^{t} (\sigma(t_2) - w)\,\Delta w & \text{if } t_1 \le t \le \sigma(t_2) \le \sigma(s) \le \sigma^2(t_3); \\ (t - t_1)(\sigma(t_2) - t_1) & \\ + \int_{t_1}^{\sigma(s)} (\sigma(s) - t + t_1 - w)\,\Delta w & \text{if } t_1 < \sigma(t_2) \le \sigma(s) \le t \le \sigma^2(t_3); \\ \int_{t_1}^{t} (\sigma(t_2) - w)\,\Delta w & \text{if } t_1 < \sigma(t_2) \le t \le \sigma(s) \le \sigma^2(t_3); \\ \int_{t_1}^{\sigma(s)} (\sigma(s) - w)\,\Delta w & \text{if } t_1 \le \sigma(s) \le \sigma(t_2) \le t \le \sigma^2(t_3). \end{cases}$$

is the Green's function satisfying

$$y^{\Delta\Delta\Delta}(t) = 0$$

with boundary conditions

$$\begin{aligned} y(t_1) &= 0 \\ y(\sigma(t_2)) &= 0 \\ y(\sigma^2(t_3)) &= 0. \end{aligned}$$

Condition:

C1 : $\int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\Delta w > 0$

6.2. *Existence of Positive Solution*

The following Lemma and Theorems will be stated without proof.

Lemma 6.1. *Assuming condition C1 holds, the Green's function* $G(t,s) >$ *0 for* $(t,s) \in (t_1, \sigma^2(t_3)] \times (t_1, \sigma^2(t_3)]$.

Proof. The proof is similar to Theorem 4.1. □

Theorem 6.2. *$G(t,s) \leq G(\sigma(s),s)$ for $(t,s) \in [t_1, \sigma^2(t_3)] \times [t_1, \sigma^2(t_3)]$ and $G(t,s) \geq \int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\Delta w$ for $(t, \sigma(s)) \in [\sigma(t_2), \sigma^2(t_3)] \times [\sigma(t_2), \sigma^2(t_3)]$*

Proof. The proof is similar to Theorem 4.2. □

We will now show the interval for λ for which there exist a positive solution. All the symbols used here are as define in Section 4.

Define

$$r = \min_{\sigma(t_2) \leq \sigma(s) \leq t} \left\{ \frac{(t_3 - t_1)(\sigma(t_2) - t_1) + \int_{t_1}^{\sigma(s)} (\sigma(s) - t_3 + t_1 - w)\Delta w}{(t_2 - t_1)(\sigma(t_2) - t_1) + \int_{t_1}^{\sigma(s)} (\sigma(s) - t_2 + t_1 - w)\Delta w}, \frac{\int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\,\Delta w}{\int_{t_1}^{\sigma(t_2)} (\sigma(t_2) - w)\,\Delta w} \right\}.$$

Now, we state without proof the range of the values of λ for which there exist a positive solution.

Theorem 6.3. *For each λ satisfying*

(87)
$$\frac{1}{[r \int_{\sigma(t_2)}^{\sigma^2(t_3)} \left(\int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\Delta w \right) \Delta s] f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{\sigma^2(t_3)} \int_{t_1}^{\sigma(s)} (\sigma(s) - w)\Delta w \Delta s] f_0},$$

there exists at least one positive solution of the BVP (80) satisfying (81) in κ .

Proof. The proof is similar to Theorem 4.3 and 4.4. □

Theorem 6.4. *For each λ satisfying*

(88)
$$\frac{1}{[r \int_{\sigma(t_2)}^{\sigma^2(t_3)} \left(\int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\Delta w \right) \Delta s] f_0} < \lambda < \frac{1}{[\int_{t_1}^{\sigma^2(t_3)} \int_{t_1}^{\sigma(s)} (\sigma(s) - w)\,\Delta s] f_\infty},$$

there exist at least one positive solution of the BVP (80) satisfying (81) in κ .

Proof. The proof is similar to Theorem 4.3 and 4.4. □

Next, we find the Green's function of the $3n^{th}$ order BVP defined below on time scale.

Theorem 6.5. *Suppose that $G_3(t,s)$ is the Green's function satisfying*

$$y^{\Delta\Delta\Delta}(t) = 0$$

with boundary conditions

$$y(t_1) = 0$$

$$y^{\Delta}(t_2) = 0$$

$$y^{\Delta\Delta}(t_3) = 0$$

then ,

$$G_n(t,s) = \int_{t_1}^{t_3} G_3(t,w)G_{n-3}(w,s)\,\Delta w \quad n \in \{3k+3 : k \in \mathbb{N}\} \tag{89}$$

is the Green's function for

$$y^{\Delta^{(n)}}(t) = 0, \quad n \in \{3k+3 : k \in \mathbb{N}\}, \tag{90}$$

with boundary conditions

$$\begin{cases} y^{\Delta^{(3k)}}(t_1) &= 0 \\ y^{\Delta^{(3k+1)}}(t_2) &= 0 \\ y^{\Delta^{(3k+2)}}(t_3) &= 0, \;\; k = 0, 1, 2, \ldots \frac{n}{2} - 1. \end{cases} \tag{91}$$

Proof. The proof is similar to Theorem 4.5. □

6.3. *Example*

In this section, we will find the solutions and interval for λ for which the third order boundary value problem has a positive solution.

Case 1 : $\mathbb{T} = [t_1, t_3]$

In this case, $\sigma(t) = t$ so that the differential equation

$$y^{\Delta\Delta\Delta}(t) - \lambda(y+1) = 0,\ t \in \mathbb{T}$$

with boundary conditions

$$\begin{cases} y(t_1) = \rho_1 \\ y^{\Delta}(\sigma(t_2)) = \rho_2 \\ y^{\Delta\Delta}(\sigma^2(t_3)) = \rho_3 \end{cases}$$

becomes that of Section 1.

Case 2 : $\mathbb{T} = \mathbb{N}_0$

For this,

$$\sigma(t) = t + 1,$$

such that the equation becomes

$$y(\sigma^3(t)) - 3y(\sigma^2(t)) + 3y(\sigma(t)) - y(t) - \lambda(y(t) + 1) = 0,\ t \in \mathbb{T}$$

with boundary conditions

$$\begin{cases} y(t_1) = \rho_1 \\ y(\sigma^2(t_2)) - y(\sigma(t_2)) = \rho_2 \\ y(\sigma^4(t_3)) - 2y(\sigma^3(t_3)) + y(\sigma^2(t_3)) = \rho_3. \end{cases}$$

Let $t_1 = 1,\ t_2 = 2.6,\ t_3 = 4,\ \rho_1 = 1,\ \rho_2 = 0,\ \rho_3 = -1$. Then the Green's function is given as

$$G(t,s) = \begin{cases} \frac{1}{2}s(s+1) & \text{if } t_1 \le \sigma(s) \le t \le \sigma(t_2) < \sigma^2(t_3); \\ \frac{1}{2}(2s-t+2)(t-1) & \text{if } t_1 \le t \le \sigma(s) \le \sigma(t_2) < \sigma^2(t_3); \\ \frac{1}{2}(1-t)(t-7.2) & \text{if } t_1 \le t \le \sigma(t_2) \le \sigma(s) \le \sigma^2(t_3); \\ -2.6+0.5s^2+s(1.5-1.t)+2.6t & \text{if } t_1 < \sigma(t_2) \le \sigma(s) \le t \le \sigma^2(t_3); \\ \frac{1}{2}(1-t)(t-7.2) & \text{if } t_1 < \sigma(t_2) \le t \le \sigma(s) \le \sigma^2(t_3); \\ \frac{1}{2}s(s+1) & \text{if } t_1 \le \sigma(s) \le \sigma(t_2) \le t \le \sigma^2(t_3). \end{cases}$$

So,

$$\int_{\sigma(t_2)}^{\sigma^2(t_3)} \left(\int_{t_1}^{\sigma^2(t_3)} (\sigma(t_2) - w)\Delta w \right) \Delta s = \sum_{s=\sigma(t_2)}^{\sigma^2(t_3)-1} \sum_{w=t_1}^{\sigma^2(t_3)-1} (\sigma(t_2) - w) = 6$$

$$\int_{t_1}^{\sigma^2(t_3)} \left(\int_{t_1}^{\sigma(s)} (\sigma(s) - w)\Delta w \right) \Delta s = \sum_{s=t_1}^{\sigma^2(t_3)-1} \sum_{w=t_1}^{\sigma(s)-1} (\sigma(s) - w) = 35$$

and $r = 0.692943,\ f_0 = \infty,\quad f_\infty = 1$. We have that there exist a positive solution of $y(t)$ for all λ in the interval $\left(0, \frac{1}{35}\right)$.

7. Conclusion

In this work, the use of Guo-Krasnosel'skii fixed point theorem for solving for positive solution of the second and third order BVP dynamical equations on time scale is established. Theorem 3.1 and 4.5 helps in extending our work to solve the $2n^{(th)}$ and $3n^{(th)}$ order BVP dynamical equation respectively. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale (also known as a time-set). In this way, results apply not only to the set of real numbers or set of integers but to more general time scales such as a Cantor set.

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