1-1-2011

Statistical Properties of a Convoluted Beta-Weibull Distribution

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STATISTICAL PROPERTIES OF
A CONVOLUTED BETA-WEIBULL DISTRIBUTION

A Thesis submitted to
the Graduate College of
Marshall University

in partial fulfillment of
the requirements for the degree of
Master of Arts

in

Mathematics

by
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May 2011

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ABSTRACT

A new class of distributions recently developed involves the logit of the beta distribution. Among this class of distributions are the *beta-normal* (Eugene et al. (2002)); *beta-Gumbel* (Nadarajah and Kotz (2004)); *beta-exponential* (Nadarajah and Kotz (2006)); *beta-Weibull* (Famoye et al. (2005)); *beta-Rayleigh* (Akinsete and Lowe (2008)); *beta-Laplace* (Kozubowski and Nadarajah (2008)); and *beta-Pareto* (Akinsete et al. (2008)), among a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in the literature. One approach by which a new statistical distribution is generated is by the transformation of random variables having known distribution function(s). The focus of this work is to investigate the statistical properties of the convoluted beta-Weibull distribution, defined and extensively studied by Famoye et al. (2005). That is, if X is a random variable having the beta-Weibull distribution with parameters α_1, β_1, c_1 and γ_1, i.e. $X \sim \mathrm{BW}(\alpha_1, \beta_1, c_1, \gamma_1)$, and Y has a beta-Weibull distribution expressed as $Y \sim \mathrm{BW}(\alpha_2, \beta_2, c_2, \gamma_2)$, what then is the distribution of the convolution of X and Y. That is, the distribution of the random variable $Z = X + Y$. We obtain the probability density function (pdf) and the cumulative distribution function (cdf) of the convoluted distribution. Various statistical properties of this distribution are obtained, including, for example, moment, moment and characteristic generating functions, hazard function, and the entropy. We propose the method of Maximum Likelihood Estimation (MLE) for estimating the parameters of the distribution. The open-source software R is used extensively in implementing our results.

Acknowledgment

I would like to thank all those who helped and supported me. My sincere appreciation goes to Dr. Alfred Akinsete. I would not have made it this far without his tremendous support. I would like to thank him for his advice, encouragement and patience with me throughout.

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1. INTRODUCTION

Distribution functions, their properties and interrelationships play significant roles in modeling naturally occurring phenomena. For this reason, a large number of distribution functions have been proposed and defined in the literature, which are found to be applicable to many events in real life. Various methods exist in defining statistical distributions. Many of these have risen from the need to model naturally occurring events. For example, the normal distribution addresses real-valued variables that tend to cluster at a single mean value, whereas the Poisson distribution models discrete rare events. Yet few other distributions are functions of one or more distributions. For example, a random variable T is said to have a t-distribution if $T=\frac{Z}{\sqrt{W/n}}$, where Z has the standard normal distribution, and W has the chi-squared distribution with n degrees of freedom.

A new class of distributions recently developed involves the logit of the beta distribution. Among this class of distributions are the *beta-normal* (Eugene et al. (2002)); *beta-Gumbel* (Nadarajah and Kotz (2004)); *beta-exponential* (Nadarajah and Kotz (2006)); *beta-Weibull* (Famoye et al. (2005)); *beta-Rayleigh* (Akinsete and Lowe (2008)); *beta-Laplace* (Kozubowski and Nadarajah (2008)); and *beta-Pareto* (Akinsete et al. (2008)), and a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in the literature. One approach by which a new statistical distribution is generated is by the transformation of random variables having known distribution function(s). Many useful properties of statistical distributions are revealed by transformations of random variables. For example, if X and Y are independent and identically distributed random variables having the gamma distribution with parameters (α, s), and (β, s) respectively, then a random variable G defined by G = $\frac{X}{X+Y}$ is known to have the beta distribution with parameter α and β.

The focus of this work is to investigate the statistical properties of the convoluted beta-Weibull distribution, defined and extensively studied by Famoye et al. (2005). That is, if X is a random variable having the beta-Weibull distribution with parameters α_1, β_1, c_1 and γ_1, i.e. $X\sim$ BW($\alpha_1, \beta_1, c_1, \gamma_1$), and Y has a beta-Weibull distribution expressed as $Y\sim$BW($\alpha_2, \beta_2, c_2, \gamma_2$), what then is the distribution of the convolution of X and Y. That is, the distribution of the random variable $Z = X + Y$. We obtain the probability density function (pdf) and the cumulative distribution function (cdf) of the convoluted distribution. Various statistical properties of this distribution are obtained, including for example, moment, moment and characteristic generating functions, hazard function, and the entropy. We propose the method of Maximum Likelihood Estimation (MLE) for estimating the parameters of the distribution. The open-source software R is used extensively in implementing our results.

2. The Literature Review

The beta distribution has been widely applied as a statistical distribution to address various kinds of problems in reliability. According to Nadarajah (2002), a generalized class of beta distribution has been introduced in recent years. Under this scheme, the cumulative distribution function (cdf) for the generalized class of distribution for the random variable X is generated by applying the inverse cdf of X to a beta distributed random variable to obtain,

$$G(\mathrm{x}) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt; \qquad 0<\alpha,\ 0<\beta.$$

The corresponding probability density function (pdf) from G(x) is given by

$$g(\mathrm{x})=\frac{1}{B(\alpha,\beta)}[F(x)]^{\alpha-1}[1-F(x)]^{\beta-1}F'(x),$$

where $F'(x)$=f(x) is the pdf of X.

We discuss, in what follows, summaries of some of the beta compounded distributions that have been defined and studied in literature.

2.1 The beta-exponential distribution (BED)

The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability. The beta-exponential distribution, defined and studied by Nadarajah and Kotz (2006), is generated from the logit of a beta random variable. In the paper, authors provide a comprehensive treatment of statistical properties of the beta-exponential distribution. The paper also discusses and derives expressions for the moment generating function, characteristic function, the first four moments, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, Renyi entropy, and the Shannon entropy.

The paper proposes a generalization of the exponential distribution with the hope that it would attract wider applications in reliability. The generalization is motivated by the following general class:

If G denotes the cdf of a random variable, then a generalized class of distribution can be defined by

$$F(x) = I_{G(x)}(a, b); \qquad a>0 \text{ and } b>0,$$

where,

$$I_y(a, b) = \frac{B_y(a, b)}{B(a, b)}$$

denotes the incomplete beta function ratio, and

$$B_y(a, b) = \int_0^y w^{a-1}(1-w)^{b-1} dw$$

denotes the incomplete beta function.

The author defined the beta-exponential distribution by taking G to be the cdf of an exponential distribution with parameterλ. The cdf of beta-exponential distribution then becomes,

$$F(x) = I_{1-exp(-\lambda x)}(a, b); \quad x>0, a>0, b>0, \lambda>0,$$

and the corresponding pdf as obtained by Nadarajah and Kotz (2006) is,

$$f(x) = \frac{\lambda}{B(a,b)} exp(-b\lambda x) \{1 - exp(-\lambda x)\}^{a-1}; \quad x>0, a>0, b>0, \lambda>0.$$

This distribution is the generalization of the exponentiated exponential distribution defined by Gupta and Kundu (2003) when b=1. The beta-exponential distribution reduces to the exponential distribution with parameter $b\lambda$ when a =1.

Besides its mathematical simplicity when compared to other beta compounded distributions, the beta-exponential distribution can be used as an improved model for failure time data. The distribution exhibits both increasing and decreasing failure rates, and the shape of the failure rate function depends on the parameter a.

2.2 The beta-Gumbel distribution (BGD)

The Gumbel distribution is perhaps the most widely applied statistical distribution for problems in engineering. The paper by Nadarajah and Kotz (2004) introduced and defined the beta-Gumbel distribution from the logit of a beta random variable. The paper provides a comprehensive treatment of the mathematical properties of the beta-Gumbel distribution and discusses the analytical shapes of the corresponding probability density function and the hazard rate function. Expressions for the moment generating function, variation of the skewness and kurtosis, asymptotic distribution of the extreme order statistics and estimation are also discussed in the paper.

In the essence of the logit of beta distribution, the cdf $G(x)$ has the Gumbel distribution defined by

$$G(x) = exp\left\{- exp\left(-\frac{x-\mu}{\sigma}\right)\right\}; \qquad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0.$$

Thus, the cdf of the BGD is given by $F(x)=I_{\exp(-u)}(a,b)$, where $u=\exp\{-(x-\mu)/\sigma\}$. The corresponding pdf is

$$f(x) = \frac{u\, exp(-au)\{1-exp(-u)\}^{b-1}}{\sigma B(a,b)}.$$

The above pdf has the equivalent form

$$f(x) = \frac{\Gamma(a+b)}{\sigma\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k\, u\, exp\{-(a+k)u\}}{k!\,\Gamma(b-k)}$$

The beta-Gumbel distribution allows for greater flexibility of its tail, which enables some real-life problems with tail features to be analyzed more accurately, leading to better estimation and prediction of parameters.

2.3 The beta-Rayleigh distribution (BRD)

According to Akinsete and Lowe (2008), the problem of estimating the reliability of components is of utmost importance in many areas of research, for example in medicine, engineering and control systems. If X represents a random strength capable of withstanding a random amount of stress Y in a component, the quantity $R=P(Y<X)$ measures the reliability of the component. In the paper, the authors defined and studied

the beta-Rayleigh distribution (BRD) and obtained a measure of reliability when both X and Y have the beta-Rayleigh distribution. Some properties of the BRD are discussed in the paper, including, for example, special cases of the distribution, moments, and parameter estimation.

By taking F(x) as the cdf of the Rayleigh distribution, the pdf for BRD can be written as

$$g(x) = \frac{x}{\sigma^2 B(\alpha,\beta)} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2 \beta} \left(1 - e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^{\alpha-1} \quad ; \quad x \geq 0.$$

Using the relationship between the incomplete beta function and the Gauss Hypergeometric function, the cdf for BRD can be expressed as

$$G(x) = 1 - \frac{e^{-\frac{\alpha}{2}\left(\frac{x}{\sigma}\right)^2}}{\alpha B(\alpha,\beta)}\, {}_2F_1(\alpha, 1-\beta; 1+\alpha; e^{-\frac{\alpha}{2}(\frac{x}{\sigma})^2}),$$

where ${}_2F_1(a, b; c; z)$ is a second order hypergeometric function.

The above distribution is used in calculating the measure of reliability, which is vital in many fields requiring safety. The reliability measure obtained from BRD is seen to generalize the known Rayleigh reliability measure and addresses more cases of reliability measures.

2.4 The beta-Weibull distribution (BWD)

The Weibull distribution has wide applications in many fields of studies. One generalization of the Weibull distribution is the beta-Weibull distribution, defined by Famoye et al. (2005).

The authors discussed some properties of a four-parameter beta-Weibull distribution. The distribution is shown to have bathtub, unimodal, increasing, and decreasing hazard functions. The distribution is applied to censored data sets on bus-motor failures, a censored data set on head-and-neck-cancer clinical trial, and also to survival data.

By taking F(x) to be the cdf of a Weibull random variable X, the corresponding pdf for the beta-Weibull random variable is expressed as:

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{c}{\gamma}(x/\gamma)^{c-1}[1-e^{-(x/\gamma)^c}]^{\alpha-1}e^{-\beta(x/\gamma)^c},$$

$$\mathrm{x} > 0, \alpha > 0, \beta > 0, c > 0, \gamma > 0.$$

2.5 The beta-Laplace distribution (BLD)

Motivated by the work of Eugene et al.(2002), Kozubowski and Nadarajah (2008) introduced the beta-Laplace distribution generated from the logit of a beta random variable. The basic theoretical properties of the distribution are discussed, including, for example, modality and concavity of the density, moments and related parameters, and stochastic representations that aid in random variate generation from the model.

By the usual method of the logit of the beta distribution, and using the cdf of the Laplace distribution given by

$$G(x) = \begin{cases} \frac{1}{2} exp\left(\frac{x-\theta}{\sigma}\right), \; if \; x < \theta, \\ 1 - \frac{1}{2} exp\left(\frac{x-\theta}{\sigma}\right), \; if \; x \geq \theta, \end{cases} \quad - \infty < \theta < \infty \, , \sigma > 0.$$

The corresponding pdf of the beta-Laplace distribution is expressed as

$$f_{a,b,\theta,\sigma}(x) = \frac{1}{2\sigma B(a,b)} exp\left(-\frac{|x-\theta|}{\sigma}\right) G^{a-1}(x)\{1 - G(x)\}^{b-1}.$$

We see from this function that θ and σ are location and scale parameters, respectively. In a particular case where $\theta = 0$ and $\sigma = 1$, the pdf becomes

$$f_{a,b}(x) = \left(\frac{1}{2}\right)^{a+b+1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} e^{ax}(2-e^{x})^{b-1} \, , \; if \; x \leq 0, \\ e^{-bx}(2-e^{-x})^{a-1} \, , if \; x > 0. \end{cases}$$

2.6 The beta-Pareto distribution (BPD)

According to Akinsete et al. (2008), the family of the Pareto distribution is well known in the literature for its capability in modeling the heavy- tail distribution, such as the data on income distribution, city population size, and size of companies. Some other quantities measured in the physical, biological, technological and social systems of various kinds have been found to follow the Pareto distribution.

Different types of the Pareto distributions and their generalizations exist in the literature. In the paper by Akinsete et al. (2008), a four-parameter beta-Pareto distribution is generated and studied. Some properties are discussed in the paper, including the unimodality of the distribution, the unimodal or decreasing hazard rate, the expressions for the mean, mean deviation, variance, skewness, kurtosis, Renyi and Shannon entropies, maximum likelihood estimates of the parameters and applications to real-life data.

A random variable Y is said to have the Pareto distribution if its pdf is given as

$$f(y) = \frac{k\theta^k}{y^{k+1}}\,; \qquad \mathrm{k} > 0, \theta > 0, y \geq \theta.$$

The Pareto distribution is skewed to the right and characterized by a shape parameter k and a scale parameter θ. The density function f(y) is a decreasing function of y and achieves its maximum when y is smallest.

The probability density function of the beta-Pareto distribution is given in Akinsete et al. (2008) as

$$g(x) = \frac{k}{\theta B(\alpha,\beta)}\left\{1 - \left(\frac{x}{\theta}\right)^{-k}\right\}^{\alpha-1}\left(\frac{x}{\theta}\right)^{-k\beta-1}; \quad x \geq \theta, \alpha, \beta, \theta, \mathrm{k} > 0.$$

In the following chapter, we define a convolution of two beta-Weibull distributions. Various properties of this distribution are obtained.

3. The Convoluted beta-Weibull Distribution

3.1 Definition, Density and Distribution Functions

According to Famoye et al. (2005), the pdf of a beta-Weibull random variable X is expressed as

$$f(x) = \frac{1}{B(\alpha,\beta)}\frac{c}{\gamma}(x/\gamma)^{c-1}[1-e^{-(x/\gamma)^c}]^{\alpha-1}e^{-\beta(x/\gamma)^c} \ ,$$

$$\alpha > 0, \beta > 0, c > 0, \gamma > 0, x > 0.$$

Assume X has the BWD with parameters α_1, β_1, c_1 and γ_1, and Y has the BWD with parameters α_2, β_2, c_2 and γ_2. That is, $X \sim \text{BW}(\alpha_1, \beta_1, c_1, \gamma_1)$ and $Y \sim \text{BW}(\alpha_2, \beta_2, c_2, \gamma_2)$. Let Z=$X+Y$ be a random variable. By using the concept of convolution of the two random variables, we may write the pdf of Z as,

$$f(z) = \int_0^z f_x(z-y)f_y(y)dy$$

$$= \frac{c_1c_2}{B(\alpha_1,\beta_1)\cdot B(\alpha_2,\beta_2)\cdot \gamma_1{}^{c_1}\cdot \gamma_2{}^{c_2}} \int_0^z (z-y)^{c_1-1}\,(y)^{c_2-1}\left[1-e^{-\left(\frac{z-y}{\gamma_1}\right)^{c_1}}\right]^{\alpha_1-1} \times$$

$$\left[1-e^{-\left(\frac{y}{\gamma_2}\right)^{c_2}}\right]^{\alpha_2-1} e^{-\beta_1\left(\frac{z-y}{\gamma_1}\right)^{c_1}}\, e^{-\beta_2\left(\frac{y}{\gamma_2}\right)^{c_2}}dy.$$

Let $\alpha_1 = \alpha_2 = 1$ for simplicity, so that

$$f(z) = \frac{\beta_1\beta_2c_1c_2}{\gamma_1{}^{c_1}\gamma_2{}^{c_2}} \int_0^z (z-y)^{c_1-1}\,(y)^{c_2-1}e^{-\beta_1\left(\frac{z-y}{\gamma_1}\right)^{c_1}}\, e^{-\beta_2\left(\frac{y}{\gamma_2}\right)^{c_2}}dy.$$

Again for computational simplicity, we set $c_1 = c_2 = 1$ to have,

$$f(z) = \frac{\beta_1\beta_2}{\gamma_1\,\gamma_2} \int_0^z e^{-\beta_1\left(\frac{z-y}{\gamma_1}\right)}\; e^{-\beta_2(\frac{y}{\gamma_2})^{c_2}} dy$$

$$= \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\; e^{-\frac{\beta_1}{\gamma_1}z} \int_0^z e^{(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2})y}\, dy$$

$$= \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\; e^{-\frac{\beta_1}{\gamma_1}z}\; \frac{\gamma_1\gamma_2}{\beta_1\gamma_2-\beta_2\gamma_1} \left[e^{\left(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2}\right)y}\right]_0^z$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\; e^{-\frac{\beta_1}{\gamma_1}z}\; [e^{\left(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2}\right)z} - 1]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1} \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right), \qquad (1)$$

where $\beta_1 > 0, \beta_2 > 0,\ \gamma_1 > 0, \gamma_2 > 0, z > 0, and\ \beta_1\gamma_2 \neq \beta_2\gamma_1$.

Equation (1) is the pdf of the random variable Z , a convolution of two independent and identically distributed beta-Weibull random variables. We say that Z has a Convoluted beta-Weibull Distribution (CBWD), and write for notational purpose;

$$Z \sim \text{CBWD}(\beta_1, \beta_2, \gamma_1, \gamma_2).$$

This is a special case of CBWD$(c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ with $c_1 = c_2 = \alpha_1 = \alpha_2 = 1$. In that case, $Z \sim \text{CBWD}(1,1,1,1, \beta_1, \beta_2, \gamma_1, \gamma_2)$.

To show that Equation (1) is indeed a pdf, we require that

$$\int_0^\infty f(z)dz = 1.$$

We show this as follows:

$$\int_0^{\infty} f(z)dz = \lim_{t\to\infty} \int_0^t \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ (e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}) \text{ dz}$$

$$= \lim_{t\to\infty} \left[\frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1} \ e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_2}{\gamma_2}z}\right]_0^t$$

$$= (0+0) - (\frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1} - \frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1})$$

$$= \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1}$$

= 1, as excepted.

The corresponding cdf of a convoluted beta-Weibull distribution is defined as $F(z) = P(Z \leq z)$. Using Equation (1), we have,

$$F(z) = \int_0^z f(t)dt$$

$$= \int_0^z \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ (e^{-\frac{\beta_2}{\gamma_2}t} - e^{-\frac{\beta_1}{\gamma_1}t}) \ dt$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ \int_0^z (e^{-\frac{\beta_2}{\gamma_2}t} - e^{-\frac{\beta_1}{\gamma_1}t}) \ dt$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ \left[\frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}t} - \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}t}\right]_0^z$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ [\frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}z} + \frac{\gamma_2}{\beta_2}]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \ [\frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}z} + \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1\beta_2}]$$

$$= \frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1}\, e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_2}{\gamma_2}z} + 1.$$

We see immediately that

$$\lim_{z\to\infty} F(z) = 1.$$

The graph of the cdf of the convoluted beta-Weibull distribution for $\beta_1 = 2, \beta_2 = 3,\ \gamma_1 = 2, \gamma_2 = 4$ is shown in Figure (1).

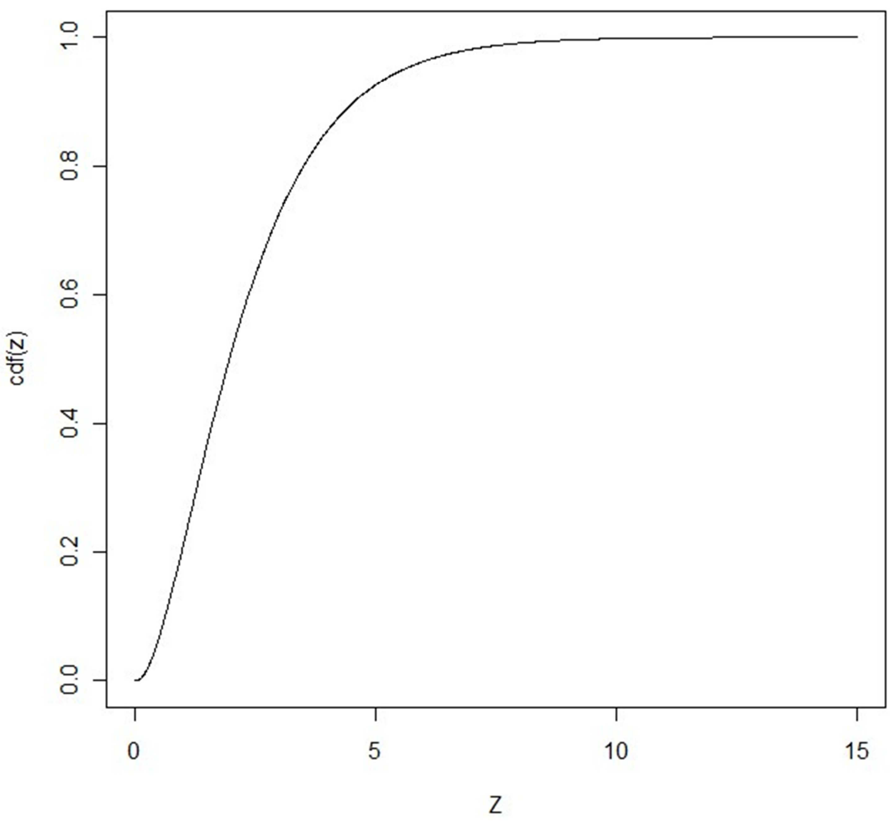

Fig. 1 . The graph of cdf ($\beta_1 = 2, \beta_2 = 3,\ \gamma_1 = 2, \gamma_2 = 4$)

From the graph, the cdf increases when z increases, and approaches 1 when z becomes large, as excepted.

3.2 Shape of the PDF

We investigate the shape of the CBWD in what follows:

Given,

$$f(z) = \frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1} \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right).$$

Differentiate the equation above with respect to z and set equal to zero to have,

$$f'(z) = \frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1} \left(\frac{\beta_1}{\gamma_1} e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\beta_2}{\gamma_2} e^{-\frac{\beta_2}{\gamma_2}z}\right) = 0.$$

The above may be written as

$$exp[\left(\frac{\beta_2}{\gamma_2} - \frac{\beta_1}{\gamma_1}\right) z] = \frac{\beta_2\gamma_1}{\beta_1\gamma_2}.$$

Solving for z finally gives,

$$z = \frac{\gamma_1\gamma_2}{\beta_2\gamma_1-\beta_1\gamma_2} \left[\ln(\beta_2\gamma_1) - \ln(\beta_1\gamma_2)\right] > 0 \text{, where } \beta_1\gamma_2 \neq \beta_2\gamma_1.$$

This value shows that the distribution is unimodal.

By choosing different values for parameters $\beta_1, \beta_2, \gamma_1$ and γ_2 in the distribution, corresponding shapes of the distribution are shown in the graph below:

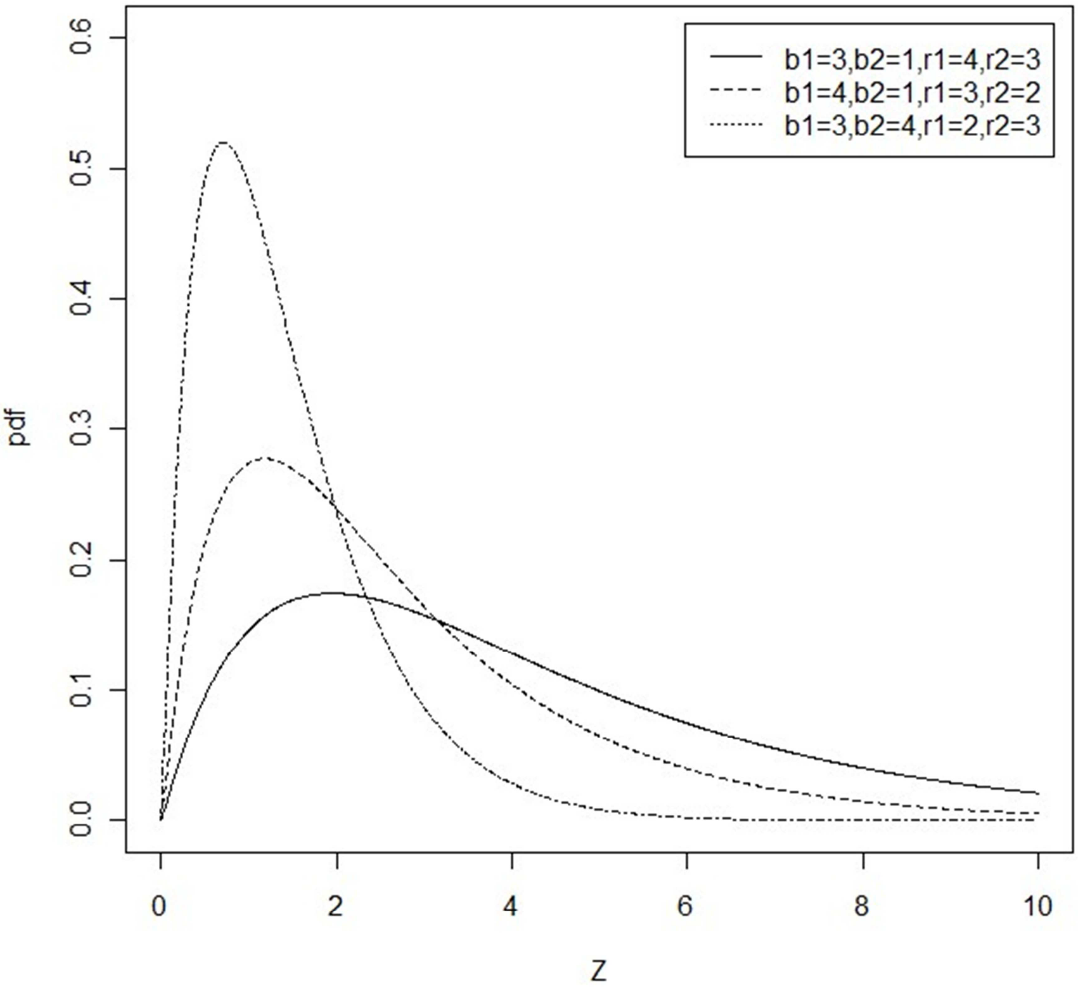

Fig. 2. The graph of pdf for different parameters values ($\mathrm{b} = \beta, \mathrm{r} = \gamma$)

It is interesting to note that all parameters in the CBWD are scale parameters. They affect the graph of pdf in different directions and different rates. Among the parameters, the bigger γ_1 and γ_2 are, the more spread out the graph is, but the bigger β_1 and β_2 are, the more concentrated the graph is.

4. The Hazard Function

The hazard function is a measure of the tendency of a component to fail. And the greater the value of the hazard function is, the greater the probability of impending failure is. Technically, the hazard function is the probability of failure in a very small time interval. Mathematically, the hazard function for random variable X is defined as

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

Hence the hazard rate function h(z) associated with Equation (1) is given as,

$$h(z) = \frac{f(z)}{1 - F(z)}$$

$$= \left(\frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\right)\left[\frac{e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}}{1 - \left(\frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1}e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1}e^{-\frac{\beta_2}{\gamma_2}z} + 1\right)}\right]$$

$$= (\beta_1\beta_2)\left(\frac{e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}}{\beta_1\gamma_2 e^{-\frac{\beta_2}{\gamma_2}z} - \beta_2\gamma_1 e^{-\frac{\beta_1}{\gamma_1}z}}\right). \qquad (2)$$

We consider the behavior of the hazard function as z approachs and as z approaches infinity as follows:

Taking the limit of Equation (2) as $z \to 0$, we have,

$$\lim_{z\to 0} h(z) = \beta_1\beta_2 \frac{e^{-\frac{\beta_2}{\gamma_2}0} - e^{-\frac{\beta_1}{\gamma_1}0}}{\beta_1\gamma_2 e^{-\frac{\beta_2}{\gamma_2}0} - \beta_2\gamma_1 e^{-\frac{\beta_1}{\gamma_1}0}}$$

$$= \beta_1\beta_2 \frac{1-1}{\beta_1\gamma_2 - \beta_2\gamma_1}$$

$$= 0.$$

Also,

$$\lim_{z\to\infty} h(z) = \lim_{z\to\infty}\beta_1\beta_2 \frac{e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}}{\beta_1\gamma_2 e^{-\frac{\beta_2}{\gamma_2}z} - \beta_2\gamma_1 e^{-\frac{\beta_1}{\gamma_1}z}}$$

$$= \lim_{z\to\infty}\beta_1\beta_2 \frac{1- e^{-(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2})z}}{\beta_1\gamma_2 - \beta_2\gamma_1 e^{-(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2})z}}$$

$$= \begin{cases} \frac{\beta_1}{\gamma_1} & if\ \frac{\beta_2}{\gamma_2} > \frac{\beta_1}{\gamma_1} \\ \frac{\beta_2}{\gamma_2} & if\ \frac{\beta_2}{\gamma_2} < \frac{\beta_1}{\gamma_1} \end{cases}$$

Figure (3) shows the graph of hazard rate function.

From the graph, we can see that the value of the hazard function increases when z increases. It comes close to a constant value as the value of z increases. The implication of this behavior explains that the convoluted beta-Weibull distribution may be appropriate in modeling age-dependent events, where risk or hazard increases with age. Many examples are found in systems of components that fail as a result of the age of those components.

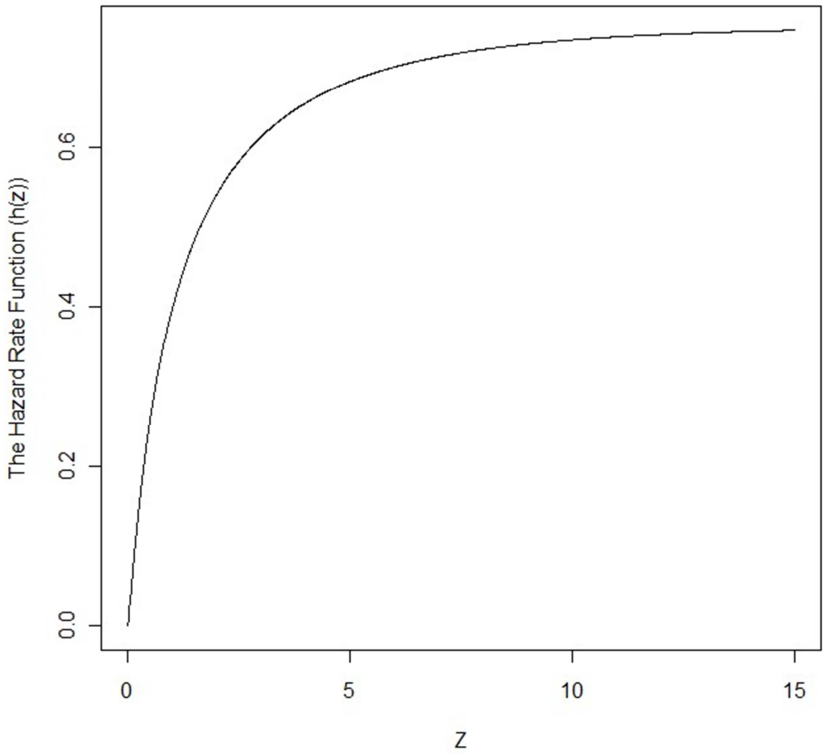

Fig. 3. The Graph of the Hazard Rate Function

5. Moments and Generating Functions

5.1 Generating Functions

We derive the moment generating function and characteristic function for a random variable Z having the CBWD density function given in Equation (1) as follows:

By definition, the moment generating function of a random variable Z is defined as

$$M_Z(t) = E[exp(tZ)], \text{ where } |t| < 1.$$

Using Equation (1), we have,

$$M_Z(t) = \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \int_0^{\infty} [e^{(t-\frac{\beta_2}{\gamma_2})z} - e^{(t-\frac{\beta_1}{\gamma_1})z}]\, dz$$

$$= \lim_{x\to\infty} \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_2}{t\gamma_2 - \beta_2} e^{\left(t-\frac{\beta_2}{\gamma_2}\right)z} - \frac{\gamma_1}{t\gamma_1 - \beta_1} e^{\left(t-\frac{\beta_1}{\gamma_1}\right)z}\right]_0^x$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(0 - 0 - \frac{\gamma_2}{t\gamma_2 - \beta_2} + \frac{\gamma_1}{t\gamma_1 - \beta_1}\right)$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(\frac{\beta_1\gamma_2 - t\gamma_1\gamma_2 - \beta_2\gamma_1 + t\gamma_1\gamma_2}{(t\gamma_1 - \beta_1)(t\gamma_2 - \beta_2)}\right)$$

$$= \frac{\beta_1\beta_2}{(\beta_1 - t\gamma_1)(\beta_2 - t\gamma_2)}$$

$$= [(1 - \frac{\gamma_1}{\beta_1}t)(1 - \frac{\gamma_2}{\beta_2}t)]^{-1}. \qquad (3)$$

It follows from the above that the characteristic function of Z defined by

$$\emptyset_Z(t) = E[ex\,p(itZ)],$$

may be written as

$$\emptyset_Z(\mathrm{t}) = [(1-\tfrac{\gamma_1}{\beta_1}it)(1-\tfrac{\gamma_2}{\beta_2}it)]^{-1}$$

From the above representation of the moment and characteristic generating functions, we may generalize that if X_j , j=1,2,…,k, are independent identically distributed random variables, each with density function given in Equation (1), the characteristic generating function $\Phi_R(t)$ of R=$X_1 + X_2 + \cdots + X_k$ may be expressed as,

$$\Phi_R(t) = \prod_{j=1}^{k}(1-\frac{\gamma_j}{\beta_j}it)^{-1}.$$

5.2 Moments

The moment generating function (3) can also be expressed as

$$M_Z(t) = \beta_1\beta_2\,(t\gamma_1-\beta_1)^{-1}(t\gamma_2-\beta_2)^{-1}. \quad (4)$$

By definition, the k th raw moment of the random variable Z is expressed as,

$$E(Z^k) = \frac{\partial^k}{\partial t^k}M_Z(t)|_{t=0}.$$

Taking the derivative of Equation (4) with respect to t, we have,

$$M_Z{}'(t) = [-\gamma_2(t\gamma_1-\beta_1)^{-1}(t\gamma_2-\beta_2)^{-2} - \gamma_1(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-1}]\,\beta_1\beta_2\,,$$

from where we obtain,

$$E(Z) = M_Z{}'(0) = \left[\frac{\gamma_2}{\beta_1{\beta_2}^2} + \frac{\gamma_1}{{\beta_1}^2\beta_2}\right]\beta_1\beta_2$$

$$= \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}. \quad (5)$$

The table below shows the mean of the CBWD with different parameters.

Mean(Z)										
	r2	b1=2			b1=3			b1=4		
		b2			b2			b2		
		1	2	3	1	2	3	1	2	3
r1=1	1	1.50	1.00	0.83	1.33	0.83	0.67	1.25	0.75	0.58
	2	2.50	1.50	1.17	2.33	1.33	1.00	2.25	1.25	0.92
	3	3.50	2.00	1.50	3.33	1.83	1.33	3.25	1.75	1.25
	4	4.50	2.50	1.83	4.33	2.33	1.67	4.25	2.25	1.58
r1=2	1	2.00	1.50	1.33	1.67	1.17	1.00	1.50	1.00	0.83
	2	3.00	2.00	1.67	2.67	1.67	1.33	2.50	1.50	1.17
	3	4.00	2.50	2.00	3.67	2.17	1.67	3.50	2.00	1.50
	4	5.00	3.00	2.33	4.67	2.67	2.00	4.50	2.50	1.83
r1=3	1	2.50	2.00	1.83	2.00	1.50	1.33	1.75	1.25	1.08
	2	3.50	2.50	2.17	3.00	2.00	1.67	2.75	1.75	1.42
	3	4.50	3.00	2.50	4.00	2.50	2.00	3.75	2.25	1.75
	4	5.50	3.50	2.83	5.00	3.00	2.33	4.75	2.75	2.08
r1=4	1	3.00	2.50	2.33	2.33	1.83	1.67	2.00	1.50	1.33
	2	4.00	3.00	2.67	3.33	2.33	2.00	3.00	2.00	1.67
	3	5.00	3.50	3.00	4.33	2.83	2.33	4.00	2.50	2.00
	4	6.00	4.00	3.33	5.33	3.33	2.67	5.00	3.00	2.33

Table 1. The Mean of the CBWD with different paramenters (r=γ, b=β)

From Table 1, the mean of the CBWD increases when γ_1 and γ_2 increase, while the mean of the CBWD decreases when β_1 and β_2 increase.

According to Famoye et al. (2005), the r^{th} raw moment of the beta-Weibull distribution is given by

$$E(X^{\mathrm{r}}) = \frac{\Gamma(\alpha+\beta)\Gamma(\frac{\mathrm{r}}{c}+1)\gamma^{\mathrm{r}}}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(-1)^k(\beta+k)^{-(\mathrm{r}+k)/c}}{k!\,\Gamma(\alpha-k)}.$$

Setting $\alpha = c = 1$ as we have in our model, the above can be shown to become, for $\mathrm{r} = 1$,

$$E(X) = \frac{\gamma}{\beta}.$$

This result coincides with Equation (5) for corresponding random variables X_1 and X_2. Further, if $\beta = 1$, we have the mean of the Weibull distribution.

The result shows that the mean of the sum of independent beta-Weibull distribution is the sum of individual means.

In that case, given that R=$X_1 + X_2 + \cdots + X_k$,

$$E(R) = \sum_{j=1}^{k} \frac{\gamma_j}{\beta_j}.$$

Expressions for other higher moments are calculated as follows:

$$M_Z''(t) = \beta_1\beta_2 \{-\gamma_2[-\gamma_1(t\gamma_1 - \beta_1)^{-2}(t\gamma_2 - \beta_2)^{-2} - 2\gamma_2(t\gamma_1 - \beta_1)^{-1}(t\gamma_2 - \beta_2)^{-3}]$$
$$- \gamma_1[-\gamma_2(t\gamma_1 - \beta_1)^{-2}(t\gamma_2 - \beta_2)^{-2} - 2\gamma_1(t\gamma_1 - \beta_1)^{-3}(t\gamma_2 - \beta_2)^{-1}]\}$$
$$= \beta_1\beta_2 \{[\gamma_1\gamma_2(t\gamma_1 - \beta_1)^{-2}(t\gamma_2 - \beta_2)^{-2} + 2\gamma_2{}^2(t\gamma_1 - \beta_1)^{-1}(t\gamma_2 - \beta_2)^{-3}] +$$
$$[\gamma_1\gamma_2(t\gamma_1 - \beta_1)^{-2}(t\gamma_2 - \beta_2)^{-2} + 2\gamma_1{}^2(t\gamma_1 - \beta_1)^{-3}(t\gamma_2 - \beta_2)^{-1}]\}$$
$$= \beta_1\beta_2[2\gamma_1\gamma_2(t\gamma_1 - \beta_1)^{-2}(t\gamma_2 - \beta_2)^{-2} + 2\gamma_2{}^2(t\gamma_1 - \beta_1)^{-1}(t\gamma_2 - \beta_2)^{-3} +$$
$$2\gamma_1{}^2(t\gamma_1 - \beta_1)^{-3}(t\gamma_2 - \beta_2)^{-1}]$$

$$E(Z^2) = M_Z''(0)$$
$$= \beta_1\beta_2 [\left(\gamma_1\gamma_2\beta_1{}^{-2}\beta_2{}^{-2} + 2\gamma_2{}^2\beta_1{}^{-1}\beta_2{}^{-3}\right) + (\gamma_1\gamma_2\beta_1{}^{-2}\beta_2{}^{-2} + 2\gamma_1{}^2\beta_1{}^{-3}\beta_2{}^{-1})]$$
$$= \gamma_1\gamma_2\beta_1{}^{-1}\beta_2{}^{-1} + 2\gamma_2{}^2\beta_2{}^{-2} + \gamma_1\gamma_2\beta_1{}^{-1}\beta_2{}^{-1} + 2\gamma_1{}^2\beta_1{}^{-2}$$
$$= 2\gamma_1\gamma_2\beta_1{}^{-1}\beta_2{}^{-1} + 2\gamma_2{}^2\beta_2{}^{-2} + 2\gamma_1{}^2\beta_1{}^{-2}$$

$$M_Z'''(t) = \beta_1\beta_2\,\{2\gamma_1\gamma_2[-2\gamma_1(t\gamma_1-\beta_1)^{-3}(t\gamma_2-\beta_2)^{-2} - 2\gamma_2(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-3}] +$$
$$2{\gamma_2}^2[-3\gamma_2(t\gamma_1-\beta_1)^{-1}(t\gamma_2-\beta_2)^{-4} - \gamma_1(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-3}] +$$
$$2{\gamma_1}^2[-3\gamma_1(t\gamma_1-\beta_1)^{-4}(t\gamma_2-\beta_2)^{-1} - \gamma_2(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-3}]\}$$

$$E(Z^3) = M_Z'''(0)$$
$$=\beta_1\beta_2\,(4{\gamma_1}^2\gamma_2\,{\beta_1}^{-3}{\beta_2}^{-2} + 4\gamma_1\,{\gamma_2}^2{\beta_1}^{-2}{\beta_2}^{-3} + 6{\gamma_2}^3{\beta_1}^{-1}{\beta_2}^{-4} + 2\gamma_1\,{\gamma_2}^2{\beta_1}^{-2}{\beta_2}^{-3} +$$
$$6{\gamma_1}^3{\beta_1}^{-4}{\beta_2}^{-1} + 2{\gamma_1}^2\gamma_2\,{\beta_1}^{-3}{\beta_2}^{-2})$$
$$=4{\gamma_1}^2\gamma_2\,{\beta_1}^{-2}{\beta_2}^{-1} + 4\gamma_1\,{\gamma_2}^2{\beta_1}^{-1}{\beta_2}^{-2} + 6{\gamma_2}^3{\beta_2}^{-3} + 2\gamma_1\,{\gamma_2}^2{\beta_1}^{-1}{\beta_2}^{-2} + 6{\gamma_1}^3{\beta_1}^{-3} +$$
$$2{\gamma_1}^2\gamma_2\,{\beta_1}^{-2}{\beta_2}^{-1}$$
$$=6{\gamma_1}^2\gamma_2\,{\beta_1}^{-2}{\beta_2}^{-1} + 6\gamma_1\,{\gamma_2}^2{\beta_1}^{-1}{\beta_2}^{-2} + 6{\gamma_2}^3{\beta_2}^{-3} + 6{\gamma_1}^3{\beta_1}^{-3}$$

$$M_Z^{(4)}(t) = 6\beta_1\beta_2\,[3{\gamma_1}^3\gamma_2(t\gamma_1-\beta_1)^{-4}(t\gamma_2-\beta_2)^{-2} + 2{\gamma_1}^2{\gamma_2}^2\,(t\gamma_1-\beta_1)^{-3}(t\gamma_2-\beta_2)^{-3}$$
$$+ 3\gamma_1{\gamma_2}^3(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-4} + 2{\gamma_1}^2{\gamma_2}^2\,(t\gamma_1-\beta_1)^{-3}(t\gamma_2-\beta_2)^{-3}$$
$$+ 4{\gamma_1}^4(t\gamma_1-\beta_1)^{-5}(t\gamma_2-\beta_2)^{-1} + {\gamma_1}^3\gamma_2\,(t\gamma_1-\beta_1)^{-4}(t\gamma_2-\beta_2)^{-2}$$
$$+ 4{\gamma_2}^4(t\gamma_1-\beta_1)^{-1}(t\gamma_2-\beta_2)^{-5} + \gamma_1\,{\gamma_2}^3\,(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-4}]$$
$$= [{\gamma_1}^3\gamma_2(t\gamma_1-\beta_1)^{-4}(t\gamma_2-\beta_2)^{-2} + \gamma_1{\gamma_2}^3(t\gamma_1-\beta_1)^{-2}(t\gamma_2-\beta_2)^{-4} +$$
$${\gamma_1}^2{\gamma_2}^2\,(t\gamma_1-\beta_1)^{-3}(t\gamma_2-\beta_2)^{-3} + {\gamma_1}^4(t\gamma_1-\beta_1)^{-5}(t\gamma_2-\beta_2)^{-1} +$$
$${\gamma_2}^4(t\gamma_1-\beta_1)^{-1}(t\gamma_2-\beta_2)^{-5}]$$

$E(Z^4) = M_Z{}^{(4)}(0)$

$= 24\beta_1\beta_2\,({\gamma_1}^3\gamma_2\,{\beta_1}^{-4}{\beta_2}^{-2} + \gamma_1\,{\gamma_2}^3{\beta_1}^{-2}{\beta_2}^{-4} + {\gamma_1}^2{\gamma_2}^2{\beta_1}^{-3}{\beta_2}^{-3} + {\gamma_1}^4{\beta_1}^{-5}{\beta_2}^{-1} +$

${\gamma_2}^4{\beta_1}^{-1}{\beta_2}^{-5})$

$=24\left({\gamma_1}^3\gamma_2\,{\beta_1}^{-3}{\beta_2}^{-1} + \gamma_1\,{\gamma_2}^3{\beta_1}^{-1}{\beta_2}^{-3} + {\gamma_1}^2{\gamma_2}^2{\beta_1}^{-2}{\beta_2}^{-2} + {\gamma_1}^4{\beta_2}^{-4} + {\gamma_2}^4{\beta_2}^{-4}\right).$

It is not difficult to see that the n-th moment of Z can be written in general as:

$$E(Z^n) = n!\sum_{i=0}^{n} {\gamma_1}^i{\gamma_2}^{n-i}\,{\beta_1}^{-i}{\beta_2}^{-(n-i)}.$$

Using the above results, we obtain expressions for the variance, the third and fourth central moments, from which expressions for the skewness and kurtosis are calculated. For the variance, we have,

$Var(Z) = E[Z - E(Z)]^2$

$= E(Z^2) - [E(Z)]^2$

$= 2\gamma_1\,\gamma_2\,{\beta_1}^{-1}{\beta_2}^{-1} + 2{\gamma_1}^2{\beta_1}^{-2} + 2{\gamma_2}^2{\beta_2}^{-2} - {\gamma_1}^2{\beta_1}^{-2} - {\gamma_2}^2{\beta_2}^{-2} -$

$2\gamma_1\,\gamma_2\,{\beta_1}^{-1}{\beta_2}^{-1}$

$$= \left(\frac{\gamma_1}{\beta_1}\right)^2 + \left(\frac{\gamma_2}{\beta_2}\right)^2. \qquad (5^*)$$

Table 2 shows the variance of the CBWD for various parameter values. It is clear from this table that the variance of the CBWD increases when γ_1 and γ_2 increase, and the variance of the CBWD decreases when β_1 and β_2 increases. The same behavior exists in the case of the mean.

Variance(Z)										
	r2	b1=2			b1=3			b1=4		
		b2			b2			b2		
		1	2	3	1	2	3	1	2	3
r1=1	1	1.25	0.50	0.36	1.11	0.36	0.22	1.06	0.31	0.17
	2	4.25	1.25	0.69	4.11	1.11	0.56	4.06	1.06	0.51
	3	9.25	2.50	1.25	9.11	2.36	1.11	9.06	2.31	1.06
	4	16.25	4.25	2.03	16.11	4.11	1.89	16.06	4.06	1.84
r1=2	1	2.00	1.25	1.11	1.44	0.69	0.56	1.25	0.50	0.36
	2	5.00	2.00	1.44	4.44	1.44	0.89	4.25	1.25	0.69
	3	10.00	3.25	2.00	9.44	2.69	1.44	9.25	2.50	1.25
	4	17.00	5.00	2.78	16.44	4.44	2.22	16.25	4.25	2.03
r1=3	1	3.25	2.50	2.36	2.00	1.25	1.11	1.56	0.81	0.67
	2	6.25	3.25	2.69	5.00	2.00	1.44	4.56	1.56	1.01
	3	11.25	4.50	3.25	10.00	3.25	2.00	9.56	2.81	1.56
	4	18.25	6.25	4.03	17.00	5.00	2.78	16.56	4.56	2.34
r1=4	1	5.00	4.25	4.11	2.78	2.03	1.89	2.00	1.25	1.11
	2	8.00	5.00	4.44	5.78	2.78	2.22	5.00	2.00	1.44
	3	13.00	6.25	5.00	10.78	4.03	2.78	10.00	3.25	2.00
	4	20.00	8.00	5.78	17.78	5.78	3.56	17.00	5.00	2.78

Table 2. Variance of the CBWD with different parameters(r=γ, b=β)

$$E[Z-E(Z)]^3$$

$$= E(Z^3) - 3E(Z)E(Z^2) + 2E(Z)^3$$

$$=6\gamma_1^{2}\gamma_2\beta_1^{-2}\beta_2^{-1} + 6\gamma_1\gamma_2^{2}\beta_1^{-1}\beta_2^{-2} + 6\gamma_2^{3}\beta_2^{-3} + 6\gamma_1^{3}\beta_1^{-3} - 3\left(\gamma_1\beta_1^{-1} + \gamma_2\beta_2^{-1}\right)\left(2\gamma_1\gamma_2\beta_1^{-1}\beta_2^{-1} + 2\gamma_2^{2}\beta_2^{-2} + 2\gamma_1^{2}\beta_1^{-2}\right) + \left(\gamma_1\beta_1^{-1} + \gamma_2\beta_2^{-1}\right)^3$$

$$= 6\gamma_1^{2}\gamma_2\beta_1^{-2}\beta_2^{-1} + 6\gamma_1\gamma_2^{2}\beta_1^{-1}\beta_2^{-2} + 6\gamma_2^{3}\beta_2^{-3} + 6\gamma_1^{3}\beta_1^{-3} - 6\gamma_1^{2}\gamma_2\beta_1^{-2}\beta_2^{-1} - 6\gamma_1\gamma_2^{2}\beta_1^{-1}\beta_2^{-2} - 6\gamma_2^{3}\beta_2^{-3} - 6\gamma_1^{2}\gamma_2\beta_1^{-2}\beta_2^{-1} + 2(3\gamma_1^{2}\gamma_2^{1}\beta_1^{-2}\beta_2^{-1} + 3\gamma_1\gamma_2^{3}\beta_1^{-1}\beta_2^{-3} + \gamma_2^{3}\beta_2^{-3} + \gamma_1^{3}\beta_1^{-3})$$

$$= 2\left[\left(\frac{\gamma_1}{\beta_1}\right)^3 + \left(\frac{\gamma_2}{\beta_2}\right)^3\right]. \qquad (5^{**})$$

$E[Z-E(Z)]^4$

$= E(Z^4) - 4E(Z)E(Z^3) + 6E(Z)^2E(Z^2) - 3E(Z)^4$

$=24(\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} + \gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} + \gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + \gamma_1{}^4\beta_2{}^{-4} + \gamma_2{}^4\beta_2{}^{-4}) -$

$24(\gamma_1\beta_1{}^{-1} + \gamma_2\beta_2{}^{-1})(\gamma_1{}^2\gamma_2\beta_1{}^{-2}\beta_2{}^{-1} + \gamma_1\gamma_2{}^2\beta_1{}^{-1}\beta_2{}^{-2} + \gamma_2{}^3\beta_2{}^{-3} + \gamma_1{}^3\beta_1{}^{-3}) +$

$6(\gamma_1{}^2\beta_1{}^{-2} + \gamma_2{}^2\beta_2{}^{-2} + 2\gamma_1\gamma_2\beta_1{}^{-1}\beta_2{}^{-1})(2\gamma_1\gamma_2\beta_1{}^{-1}\beta_2{}^{-1} + 2\gamma_2{}^2\beta_2{}^{-2} + 2\gamma_1{}^2\beta_1{}^{-2}) -$

$3(\gamma_1{}^4\beta_1{}^{-4} + 4\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} + 6\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + 4\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} + \gamma_2{}^4\beta_2{}^{-4})$

$=24\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} + 24\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} + 24\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + 24\gamma_1{}^4\beta_2{}^{-4} + 24\gamma_2{}^4\beta_2{}^{-4} -$

$24\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} - 24\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} - 24\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} - 24\gamma_1{}^4\beta_2{}^{-4} - 24\gamma_2{}^4\beta_2{}^{-4}$

$-24\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} - 24\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} - 24\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} + 12\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} +$

$12\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + 12\gamma_2{}^4\beta_2{}^{-4} - 3\gamma_1{}^4\beta_2{}^{-4} - 12\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1} - 18\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} -$

$12\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} - 3\gamma_2{}^4\beta_2{}^{-4} + 12\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} + 12\gamma_2{}^4\beta_2{}^{-4} + 12\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} +$

$24\gamma_1\gamma_2{}^3\beta_1{}^{-1}\beta_2{}^{-3} + 24\gamma_1{}^3\gamma_2\beta_1{}^{-3}\beta_2{}^{-1}$

$= 9\gamma_1{}^4\beta_1{}^{-4} + 6\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + 9\gamma_2{}^4\beta_2{}^{-4}$.

Using corresponding results above, expressions of measures of skewness and kurtosis are given respectively as,

$$\text{Skewness}(Z) = \frac{E[Z-E(Z)]^3}{Var(Z)^{3/2}} = 2\,\frac{\gamma_2{}^3\beta_2{}^{-3} + \gamma_1{}^3\beta_1{}^{-3}}{(\gamma_1{}^2\beta_1{}^{-2} + \gamma_2{}^2\beta_2{}^{-2})^{\frac{3}{2}}}$$

$$\text{Kurtosis}(Z) = \frac{E[Z-E(Z)]^4}{Var(Z)^2}$$

$$= \frac{9\gamma_1{}^4\beta_1{}^{-4} + 6\gamma_1{}^2\gamma_2{}^2\beta_1{}^{-2}\beta_2{}^{-2} + 9\gamma_2{}^4\beta_2{}^{-4}}{(\gamma_1{}^2\beta_1{}^{-2} + \gamma_2{}^2\beta_2{}^{-2})^2}.$$

Table 3 and Table 4 show values for the skewness and kurtosis for various values of the parameters.

Skewness(Z)										
	r2	b1=2			b1=3			b1=4		
		b2			b2			b2		
		1	2	3	1	2	3	1	2	3
r1=1	1	1.61	1.41	1.49	1.77	1.49	1.41	1.85	1.61	1.46
	2	1.85	1.61	1.46	1.93	1.77	1.61	1.96	1.85	1.73
	3	1.93	1.77	1.61	1.97	1.88	1.77	1.98	1.93	1.85
	4	1.96	1.85	1.73	1.98	1.92	1.85	1.99	1.96	1.91
r1=2	1	1.41	1.61	1.77	1.49	1.46	1.61	1.61	1.41	1.49
	2	1.61	1.41	1.49	1.77	1.49	1.41	1.85	1.61	1.46
	3	1.77	1.49	1.41	1.88	1.66	1.49	1.93	1.77	1.61
	4	1.85	1.61	1.46	1.93	1.77	1.61	1.96	1.85	1.73
r1=3	1	1.49	1.77	1.88	1.41	1.61	1.77	1.46	1.49	1.66
	2	1.46	1.49	1.66	1.61	1.41	1.49	1.73	1.46	1.42
	3	1.61	1.41	1.49	1.77	1.49	1.41	1.85	1.61	1.46
	4	1.73	1.46	1.42	1.85	1.61	1.46	1.91	1.73	1.56
r1=4	1	1.61	1.85	1.93	1.46	1.73	1.85	1.41	1.61	1.77
	2	1.41	1.61	1.77	1.49	1.46	1.61	1.61	1.41	1.49
	3	1.49	1.46	1.61	1.66	1.42	1.46	1.77	1.49	1.41
	4	1.61	1.41	1.49	1.77	1.49	1.41	1.85	1.61	1.46

Table 3. Skewness of the CBWD with different parameters(r=γ, b=β)

Kurtosis(Z)										
	r2	b1=2			b1=3			b1=4		
		b2			b2			b2		
		1	2	3	1	2	3	1	2	3
r1=1	1	7.44	8.25	10.76	8.10	8.15	10.50	8.43	8.16	9.73
	2	4.38	4.56	5.56	4.43	4.46	4.92	4.46	4.44	4.65
	3	3.01	3.15	3.60	3.00	3.05	3.24	3.00	3.02	3.11
	4	2.28	2.39	2.65	2.26	2.31	2.42	2.26	2.28	2.34
r1=2	1	8.25	12.84	15.21	8.15	13.61	18.60	8.16	12.75	19.38
	2	4.56	7.13	10.33	4.46	5.99	9.38	4.44	5.28	7.89
	3	3.15	4.42	6.75	3.05	3.65	5.27	3.02	3.33	4.32
	4	2.39	3.12	4.64	2.31	2.63	3.48	2.28	2.45	2.92
r1=3	1	10.76	15.21	16.63	10.50	18.60	22.50	9.73	19.38	26.17
	2	5.56	10.33	13.50	4.92	9.38	14.64	4.65	7.89	13.59
	3	3.60	6.75	10.19	3.24	5.27	9.00	3.11	4.32	7.28
	4	2.65	4.64	7.53	2.42	3.48	5.80	2.34	2.93	4.47
r1=4	1	12.84	16.30	17.20	13.61	21.54	24.28	12.75	24.36	29.79
	2	7.13	12.66	15.17	5.99	13.02	18.42	5.28	11.63	18.96
	3	4.42	9.14	12.60	3.65	7.68	12.83	3.33	6.12	11.25
	4	3.12	6.56	12.12	2.63	4.91	8.81	2.45	3.84	6.97

Table 4. Kurtosis of the CBWD with different parameters(r=γ, b=β)

5.3 Cumulant generating function

In probability theory and statistics, the cumulants κ_n of a probability distribution are a set of quantities that provide an alternative approach for calculating the moments of the distribution. The moments determine the cumulants, and vice-versa, in the sense that any two probability distributions whose moments are identical will have identical cumulants as well. In some cases, theoretical treatments of problems in terms of cumulants are simpler than those using moments.

The cumulants κ_n of a random variable X are defined via the cumulant-generating function

$$C(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}.$$

If $M_z(t)$ is the moment generating function of the random variable Z, the cumulant generating function is basically the natural logarithm of $M_z(t)$.

The cumulants of a distribution are closely related to the distribution's moments. For example, if a random variable X admits an expected value $\mu = E(X)$ and a variance $\sigma^2 = E[(X-\mu)^2]$, then the first two cumulants: $\kappa_1 = \mu$ and $\kappa_2 = \sigma^2$.

Generally, the cumulants can be extracted from the cumulant generating function by differentiating $C(t)$ and set $t = 0$. In other words, the cumulants appear are the coefficients in the Maclaurin series of $C(t)$.

The cumulant generating function of the CBWD can be obtained as

$$C_z(t) = \log[M_z(t)]$$

$$= \log\left\{\left[\left(1-\frac{\gamma_1}{\beta_1}t\right)\left(1-\frac{\gamma_2}{\beta_2}t\right)\right]^{-1}\right\}$$

$$= -\log\left(1 - \frac{\gamma_1}{\beta_1} t\right) - \log\left(1 - \frac{\gamma_2}{\beta_2} t\right)$$

$$= \frac{\gamma_1}{\beta_1} t + (\frac{\gamma_1}{\beta_1})^2 \frac{t^2}{2} + (\frac{\gamma_1}{\beta_1})^3 \frac{t^3}{3} + (\frac{\gamma_1}{\beta_1})^4 \frac{t^4}{4} + \cdots$$

$$+ \frac{\gamma_2}{\beta_2} t + \left(\frac{\gamma_2}{\beta_2}\right)^2 \frac{t^2}{2} + \left(\frac{\gamma_2}{\beta_2}\right)^3 \frac{t^3}{3} + (\frac{\gamma_2}{\beta_2})^4 \frac{t^4}{4} + \cdots$$

$$= \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right) t + \left[\left(\frac{\gamma_1}{\beta_1}\right)^2 + \left(\frac{\gamma_2}{\beta_2}\right)^2\right] \frac{t^2}{2} + \left[\left(\frac{\gamma_1}{\beta_1}\right)^3 + \left(\frac{\gamma_2}{\beta_2}\right)^3\right] \frac{t^3}{3}$$

$$+ \left[\left(\frac{\gamma_1}{\beta_1}\right)^4 + \left(\frac{\gamma_2}{\beta_2}\right)^4\right] \frac{t^4}{4} + \cdots$$

$$= \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right) t + \left[\left(\frac{\gamma_1}{\beta_1}\right)^2 + \left(\frac{\gamma_2}{\beta_2}\right)^2\right] \frac{t^2}{2!} + 2! \left[\left(\frac{\gamma_1}{\beta_1}\right)^3 + \left(\frac{\gamma_2}{\beta_2}\right)^3\right] \frac{t^3}{3!}$$

$$+ 3! \left[\left(\frac{\gamma_1}{\beta_1}\right)^4 + \left(\frac{\gamma_2}{\beta_2}\right)^4\right] \frac{t^4}{4!} + \cdots$$

$$= \sum_{n=1}^{\infty} (n-1)! \left[\left(\frac{\gamma_1}{\beta_1}\right)^n + \left(\frac{\gamma_2}{\beta_2}\right)^n\right] \frac{t^n}{n!}$$

By the definition of cumulants, the cumulants κ_n of are the coefficients of $\frac{t^n}{n!}$ in $C_z(t)$.

That is,

$$\kappa_n = (n-1)! \left[\left(\frac{\gamma_1}{\beta_1}\right)^n + \left(\frac{\gamma_2}{\beta_2}\right)^n\right],$$

The first four cumulants are given by,

$$\kappa_1 = \mu = \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}, \qquad \text{(ref. Equation(5))}$$

$$\kappa_2 = \sigma^2 = \left(\frac{\gamma_1}{\beta_1}\right)^2 + \left(\frac{\gamma_2}{\beta_2}\right)^2, \quad \text{(ref. Equation(5}^*\text{))}$$

$$\kappa_3 = 2\left[\left(\frac{\gamma_1}{\beta_1}\right)^3 + \left(\frac{\gamma_2}{\beta_2}\right)^3\right], \quad (\text{ref. Equation}(5^{**}))$$

and

$$\kappa_4 = 6\left[\left(\frac{\gamma_1}{\beta_1}\right)^4 + \left(\frac{\gamma_2}{\beta_2}\right)^4\right].$$

5.4 Mean Deviations

The amount of scatteredness in a set of data is measured to some extent by the deviations from the mean and median. The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(\mu) = E|Z-\mu| = \int_0^\infty |z-\mu| f(z)dz$$

and

$$\delta_2(M) = E|Z-M| = \int_0^\infty |z-M| f(z)dz$$

respectively, where $\mu = E(Z)$ and M denotes the median. These measures can be calculated using the relationships that,

$$\delta_1(\mu) = \int_0^\mu (\mu-z)f(z)dz + \int_\mu^\infty (z-\mu)f(z)dz$$

$$= \int_0^\mu (\mu-z)f(z)dz + [\int_0^\infty (z-\mu)f(z)dz - \int_0^\mu (z-\mu)f(z)dz]$$

$$= 2\left[\mu F(\mu) - \int_0^\mu z f(z)dz\right], \quad (6)$$

and

$$\delta_2(M) = \int_0^M (M-z)f(z)dz + \int_M^\infty (z-M)f(z)dz$$

$$= 2MF(M) - M - \int_0^M zf(z)dz + \int_M^\infty zf(z)dz$$

$$= E(Z) + 2MF(M) - M - 2\int_0^M zf(z)dz.$$

The integral term in Equation (6) is obtained as

$$\int_0^\mu zf(z)dz$$

$$= \int_0^\mu z \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)dz$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \int_0^\mu z \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)dz$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\int_0^\mu z\, e^{-\frac{\beta_2}{\gamma_2}z}\, dz - \int_0^\mu z\, e^{-\frac{\beta_1}{\gamma_1}z}\, dz\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\int_0^\mu z\, de^{-\frac{\beta_1}{\gamma_1}z} - \frac{\gamma_2}{\beta_2}\int_0^\mu z\, de^{-\frac{\beta_2}{\gamma_2}z}\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\left(z\, e^{-\frac{\beta_1}{\gamma_1}z}\,\Big|_0^\mu - \int_0^\mu e^{-\frac{\beta_1}{\gamma_1}z}\, dz\right) - \frac{\gamma_2}{\beta_2}\left(z\, e^{-\frac{\beta_2}{\gamma_2}z}\,\Big|_0^\mu - \int_0^\mu e^{-\frac{\beta_2}{\gamma_2}z}\, dz\right)\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\left(z\, e^{-\frac{\beta_1}{\gamma_1}z}\,\Big|_0^\mu + \frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}z}\,\Big|_0^\mu\right) - \frac{\gamma_2}{\beta_2}\left(z\, e^{-\frac{\beta_2}{\gamma_2}z}\,\Big|_0^\mu + \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}z}\,\Big|_0^\mu\right)\right]. \quad (7)$$

Substituting Equation (5) ($\mu = \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}$) into Equation (7), we have,

$$\frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\left(z\, e^{-\frac{\beta_1}{\gamma_1}z}\,\Big|_0^\mu + \frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}z}\,\Big|_0^\mu\right) - \frac{\gamma_2}{\beta_2}\left(z\, e^{-\frac{\beta_2}{\gamma_2}z}\,\Big|_0^\mu + \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}z}\,\Big|_0^\mu\right)\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left\{\frac{\gamma_1}{\beta_1}\left[\left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right) e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} + \frac{\gamma_1}{\beta_1}\left(e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - 1\right)\right]\right.$$

$$\left. - \frac{\gamma_2}{\beta_2}\left[\left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right) e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} + \frac{\gamma_2}{\beta_2}\left(e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} - 1\right)\right]\right\}$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\left(2\frac{\gamma_1}{\beta_1} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} + \frac{\gamma_2}{\beta_2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \frac{\gamma_1}{\beta_1}\right)\right.$$

$$\left. - \frac{\gamma_2}{\beta_2}\left(2\frac{\gamma_2}{\beta_2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} + \frac{\gamma_1}{\beta_1} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} - \frac{\gamma_2}{\beta_2}\right)\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(2\frac{{\gamma_1}^2}{{\beta_1}^2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} + \frac{\gamma_1\gamma_2}{\beta_1\beta_2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \frac{{\gamma_1}^2}{{\beta_1}^2} - 2\frac{{\gamma_2}^2}{{\beta_2}^2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right.$$

$$\left. - \frac{\gamma_1\gamma_2}{\beta_1\beta_2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} + \frac{{\gamma_2}^2}{{\beta_2}^2}\right)$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[2\frac{{\gamma_1}^2}{{\beta_1}^2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - 2\frac{{\gamma_2}^2}{{\beta_2}^2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} + \frac{\gamma_1\gamma_2}{\beta_1\beta_2}\left(e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right)\right.$$

$$\left. + \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right)\left(\frac{\gamma_2}{\beta_2} - \frac{\gamma_1}{\beta_1}\right)\right]$$

$$= \frac{2\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\left(\frac{{\gamma_1}^2}{{\beta_1}^2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \frac{{\gamma_2}^2}{{\beta_2}^2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right) + \frac{\gamma_1\gamma_2}{\beta_1\beta_2}\left(e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right) + \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right).$$

Similarly, substituting M in place of μ into Equation (7), we can obtain

$$\int_0^M z f(z) dz$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1}\left(z\, e^{-\frac{\beta_1}{\gamma_1}z} |_0^M + \frac{\gamma_1}{\beta_1} e^{-\frac{\beta_1}{\gamma_1}z} |_0^M\right) - \frac{\gamma_2}{\beta_2}\left(z\, e^{-\frac{\beta_2}{\gamma_2}z} |_0^M + \frac{\gamma_2}{\beta_2} e^{-\frac{\beta_2}{\gamma_2}z} |_0^M\right)\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left[\frac{\gamma_1}{\beta_1} M e^{-\frac{\beta_1}{\gamma_1}M} + \frac{{\gamma_1}^2}{{\beta_1}^2} e^{-\frac{\beta_1}{\gamma_1}M} - \frac{{\gamma_1}^2}{{\beta_1}^2} - \frac{\gamma_2}{\beta_2} M e^{-\frac{\beta_2}{\gamma_2}M} - \frac{{\gamma_2}^2}{{\beta_2}^2} e^{-\frac{\beta_2}{\gamma_2}M} + \frac{{\gamma_2}^2}{{\beta_2}^2}\right]$$

$$= \frac{\beta_2 \gamma_1 M + \beta_2{\gamma_1}^2/\beta_1}{\beta_1\gamma_2 - \beta_2\gamma_1}\, e^{-\frac{\beta_1}{\gamma_1}M} - \frac{\beta_1\gamma_2 M + \beta_1{\gamma_2}^2/\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\, e^{-\frac{\beta_2}{\gamma_2}M} + \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}.$$

It follows that the expressions for the mean deviation about the mean and the mean deviation about the median may be written respectively as

$$\delta_1(\mu) = 2\left[\mu F(\mu) - \int_0^{\mu} z f(z) dz\right]$$

$$= 2\left[\left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right)\left(\frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}} + 1\right) - \frac{2\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \cdot\right.$$

$$\left.\left(\frac{{\gamma_1}^2}{{\beta_1}^2} e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \frac{{\gamma_2}^2}{{\beta_2}^2} e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right) - \frac{\gamma_1\gamma_2}{\beta_1\beta_2}\left(e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right) - \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right)\right]$$

$$= 2\left\{\left[\frac{\beta_1\gamma_1\gamma_2 - \beta_2{\gamma_1}^2}{\beta_1(\beta_1\gamma_2 - \beta_2\gamma_1)} - \frac{\gamma_1\gamma_2}{\beta_1\beta_2}\right] e^{-1-\frac{\beta_1\gamma_2}{\beta_2\gamma_1}} - \left[\frac{\beta_2\gamma_1\gamma_2 - \beta_1{\gamma_2}^2}{\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)} - \frac{\gamma_1\gamma_2}{\beta_1\beta_2}\right] e^{-1-\frac{\beta_2\gamma_1}{\beta_1\gamma_2}}\right\},$$

and

$$\delta_2(M)$$

$$= E(Z) + 2MF(M) - M - 2\int_0^M z f(z) dz$$

$$= \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2} + 2M\left(\frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_1}{\gamma_1}M} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_2}{\gamma_2}M} + 1\right) - M$$

$$-2\left(\frac{\beta_2\gamma_1 M + \frac{\beta_2{\gamma_1}^2}{\beta_1}}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_1}{\gamma_1}M} - \frac{\beta_1\gamma_2 M + \frac{\beta_1{\gamma_2}^2}{\beta_2}}{\beta_1\gamma_2 - \beta_2\gamma_1} e^{-\frac{\beta_2}{\gamma_2}M} + \frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right)$$

$$= M - \left(\frac{\gamma_1}{\beta_1} + \frac{\gamma_2}{\beta_2}\right) - 2\left[\frac{\beta_2{\gamma_1}^2}{\beta_1(\beta_1\gamma_2 - \beta_2\gamma_1)} e^{-\frac{\beta_1}{\gamma_1}M} - \frac{\beta_1{\gamma_2}^2}{\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)} e^{-\frac{\beta_2}{\gamma_2}M}\right].$$

6. Entropy and Asymptotic Behaviors

6.1 Rényi entropy

The Rényi entropy of a random variable Z is one of a family of functions for quantifying the uncertainty or randomness in a system. The Rényi entropy has been used in various situations in science and engineering. Rényi entropy is defined by,

$$\Re(\mathrm{s}) = \frac{1}{1-s}\log[f^s(z)dz],$$

where $s > 0$ and $s \neq 1$.

For the pdf of the CBWD given by Equation (1), we have,

$$\Re(\mathrm{s}) = \frac{1}{1-s}\log\left\{\int_0^\infty \left[\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\left(e^{-\frac{\beta_2}{\gamma_2}z}-e^{-\frac{\beta_1}{\gamma_1}z}\right)\right]^s dz\right\}$$

$$= \frac{1}{1-s}\log\left[\int_0^\infty \frac{{\beta_1}^s{\beta_2}^s}{(\beta_1\gamma_2-\beta_2\gamma_1)^s}\left(e^{-\frac{\beta_2}{\gamma_2}z}-e^{-\frac{\beta_1}{\gamma_1}z}\right)^s dz\right]$$

$$= \frac{1}{1-s}\log\left[\frac{{\beta_1}^s{\beta_2}^s}{(\beta_1\gamma_2-\beta_2\gamma_1)^s}\int_0^\infty \left(e^{-\frac{\beta_2}{\gamma_2}z}-e^{-\frac{\beta_1}{\gamma_1}z}\right)^s dz\right]$$

$$= \frac{1}{1-s}\left\{\log\left[\frac{{\beta_1}^s{\beta_2}^s}{(\beta_1\gamma_2-\beta_2\gamma_1)^s} + \log\int_0^\infty \left(e^{-\frac{\beta_2}{\gamma_2}z}-e^{-\frac{\beta_1}{\gamma_1}z}\right)^s dz\right]\right\}$$

$$= \frac{1}{1-s}\left\{s\log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\right) + \log\left[\int_0^\infty \sum_{i=0}^{s}\binom{s}{i}(-1)^i\left(e^{-\frac{\beta_2}{\gamma_2}z}\right)^{s-i}\left(e^{-\frac{\beta_1}{\gamma_1}z}\right)^i dz\right]\right\}$$

$$= \frac{1}{1-s}\left\{s\log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\right) + \log\left[\sum_{i=0}^{s}\binom{s}{i}\int_0^\infty (-1)^i e^{-\frac{\beta_2(s-i)}{\gamma_2}z}\, e^{-\frac{\beta_1 i}{\gamma_1}z}\, dz\right]\right\}$$

$$= \frac{1}{1-s}\left\{s\log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\right) + \log\left[\sum_{i=0}^{s}\binom{s}{i}\int_0^\infty (-1)^i e^{-\left(\frac{\beta_2(s-i)}{\gamma_2}+\frac{\beta_1 i}{\gamma_1}\right)z}\, dz\right]\right\}$$

$$= \frac{1}{1-s}\left\{\log \sum_{i=0}^{s} \binom{s}{i} \left[(-1)^{i+1} \frac{\gamma_1\gamma_2}{\beta_2\gamma_1 s - \beta_2\gamma_1 i + \beta_1\gamma_2 i} e^{-\left(\frac{\beta_2(s-i)}{\gamma_2}+\frac{\beta_1 i}{\gamma_1}\right)z}\right]_0^\infty \right.$$

$$\left. + s \log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\right)\right\}$$

$$= \frac{1}{1-s}\left\{s \log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\right) + \log\left[\sum_{i=0}^{s} \binom{s}{i} (-1)^{i+1} \frac{\gamma_1\gamma_2}{\beta_2\gamma_1 s - \beta_2\gamma_1 i + \beta_1\gamma_2 i}(0-1)\right]\right\}$$

$$= \frac{1}{1-s}\left\{s \log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\right) + \log\left[\sum_{i=0}^{s} \binom{s}{i} (-1)^{i} \frac{\gamma_1\gamma_2}{\beta_2\gamma_1 s + (\beta_1\gamma_2 - \beta_2\gamma_1) i}\right]\right\}, s \neq 1.$$

6.2 Asymptotic Behaviors

The asymptotic properties of the convoluted beta-Weibull distribution are investigated by considering the behavior of $\lim_{z\to 0} f(z)$ and $\lim_{z\to\infty} f(z)$ as follows:

Considering the situation when $z \to 0$ and $z \to \infty$ respectively in Equation (1), we have,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \lim_{z\to 0}\left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} (1-1)$$

$$= 0,$$

and

$$\lim_{z\to \infty} f(z) = \lim_{z\to \infty} \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)$$

$$= \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} \lim_{z\to \infty}\left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right)$$

$$= \frac{\beta_1 \beta_2}{\beta_1 \gamma_2 - \beta_2 \gamma_1} \ (0 - 0)$$

$$= 0.$$

These results are in agreement with the unimodality of the distribution as shown in Figure (2).

7. Parameter Estimation

We consider the process of estimation of the parameters of CBWD in this section by the method of maximum likelihood estimation. Let $z_1, z_2, z_3 \ldots z_n$ be a random sample from n independent and identically distributed random variables each with density function given in Equation (1). Then the likelihood function for the random variables is given as

$$\mathrm{L}\left(\tilde{Z}|\beta_1,\beta_2,\gamma_1,\gamma_2\right) = \left(\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\right)^n \cdot \prod_{j=1}^{n}\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]. \quad (8)$$

The values of the parameters that maximize the likelihood function also maximize the log likelihood. Taking the logarithm of Equation (8), we have

$$\ell = \log\mathrm{L}\left(\tilde{Z}|\beta_1,\beta_2,\gamma_1,\gamma_2\right)$$

$$= \mathrm{n}\log\left(\frac{\beta_1\beta_2}{\beta_1\gamma_2-\beta_2\gamma_1}\right) + \sum_{j=1}^{n}\log\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]$$

$$= \mathrm{n}\log\left(\beta_1\beta_2\right) - \mathrm{n}\log\left(\beta_1\gamma_2-\beta_2\gamma_1\right) + \sum_{j=1}^{n}\log\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]. \quad (9)$$

Now taking the partial derivatives of this Equation (9) with respect to $\beta_1, \beta_2, \gamma_1$and γ_2 respectively to have,

$$\frac{\partial\ell}{\partial\beta_1} = n\frac{\beta_1\gamma_2-\beta_2\gamma_1}{\beta_1\beta_2}\,\frac{\beta_2\left(\beta_1\gamma_2-\beta_2\gamma_1\right)-\beta_1\beta_2\,\gamma_2}{\left(\beta_1\gamma_2-\beta_2\gamma_1\right)^2} + \sum_{j=1}^{n}\frac{z_j\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]\gamma_1}$$

$$= \mathrm{n}\frac{1}{\beta_1} - n\frac{\gamma_2}{\beta_1\gamma_2-\beta_2\gamma_1} + \sum_{j=1}^{n}\frac{z_j\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]\gamma_1}. \quad (10)$$

$$\frac{\partial \ell}{\partial \beta_2} = n\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1\beta_2}\,\frac{\beta_1(\beta_1\gamma_2 - \beta_2\gamma_1) - \beta_1\beta_2\,\gamma_1}{(\beta_1\gamma_2 - \beta_2\gamma_1)^2} - \sum_{j=1}^{n}\frac{z_j\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]\gamma_2}$$

$$= \mathrm{n}\frac{1}{\beta_2} - n\frac{\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1} - \sum_{j=1}^{n}\frac{z_j\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]\gamma_2}. \qquad (11)$$

$$\frac{\partial \ell}{\partial \gamma_1} = \frac{n\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1} - \sum_{j=1}^{n}\frac{\beta_1 z_j\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]{\gamma_1}^2}. \qquad (12)$$

and

$$\frac{\partial \ell}{\partial \gamma_2} = -\frac{n\beta_1}{\beta_1\gamma_2 - \beta_2\gamma_1} + \sum_{j=1}^{n}\frac{\beta_2 z_j\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]{\gamma_2}^2}. \qquad (13)$$

The maximum likelihood estimates $\hat{\beta}_1, \hat{\beta}_2, \hat{\gamma}_1$ and $\hat{\gamma}_2$ are solution of Equations (10) - (13) when equated to zero.

For interval estimations of the set of $(\beta_1, \beta_2, \gamma_1, \gamma_2)$, and their tests of hypotheses, the Fisher information $\mathrm{I_n}(\cdot)$ symmetric matrix is required. The elements of this matrix consist of the excepted values of the second partial derivatives of the negative log likelihood. That is,

$$\mathrm{I_n}(\beta_1, \beta_2, \gamma_1, \gamma_2) = \begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial^2\beta_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\beta_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\gamma_1}\right) & -E\left(\frac{\partial^2\ell}{\partial\gamma_1\partial\gamma_2}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\beta_1\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta_2\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial\gamma_1\partial\gamma_2}\right) & -E\left(\frac{\partial^2\ell}{\partial^2\gamma_2}\right) \end{bmatrix}$$

Continue from the first derivatives in Equation (10)-(13), the corresponding second partial derivatives are obtained as follows:

$$\frac{\partial^2 \ell}{\partial^2 \beta_1} = \sum_{j=1}^{n} \frac{z_j}{\gamma_1} \frac{-\frac{z_j}{\gamma_1}\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)] - \frac{z_j}{\gamma_1}[\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)]^2}{[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)]^2}$$

$$-n{\beta_1}^{-2} + n{\gamma_2}^2\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^{-2}$$

$$= -n{\beta_1}^{-2} + n{\gamma_2}^2\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^{-2} - \sum_{j=1}^{n} \frac{{z_j}^2}{{\gamma_1}^2} \frac{\exp\left[-\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)z_j\right]}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]^2} \quad .$$

$$\frac{\partial^2 \ell}{\partial^2 \beta_2}$$

$$= -\sum_{j=1}^{n} \frac{z_j}{\gamma_2} \frac{-\frac{z_j}{\gamma_2}\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right] + \frac{z_j}{\gamma_2}[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)]^2}{[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)]^2}$$

$$-n{\beta_2}^{-2} + n{\gamma_1}^2\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^{-2}$$

$$= -n{\beta_2}^{-2} + n{\gamma_1}^2\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^{-2} - \sum_{j=1}^{n} \frac{{z_j}^2}{{\gamma_2}^2} \frac{\exp\left[-\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)z_j\right]}{\left[\exp\left(-\frac{\beta_2}{\gamma_2}z_j\right)-\exp\left(-\frac{\beta_1}{\gamma_1}z_j\right)\right]^2} \quad .$$

$$\frac{\partial^2 \ell}{\partial^2 \gamma_1}$$

$$= \sum_{j=1}^{n} \beta_1 z_j \frac{\{2\gamma_1[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)] - {\gamma_1}^2 \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) \beta_1 z_j {\gamma_1}^{-2}\} \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)}{[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2 {\gamma_1}^4}$$

$$- \sum_{j=1}^{n} \beta_1 z_j \frac{\exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) \beta_1 z_j {\gamma_1}^{-2} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)] {\gamma_1}^2}{[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2 {\gamma_1}^4} + \frac{n{\beta_2}^2}{\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^2}$$

$$= \frac{n{\beta_2}^2}{\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^2} - \sum_{j=1}^{n} \beta_1 z_j \frac{(\beta_1 z_j - 2\gamma_1) \exp\left[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j\right] + 2\gamma_1 \exp\left(-\frac{2\beta_1}{\gamma_1} z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^2 {\gamma_1}^4}.$$

$$\frac{\partial^2 \ell}{\partial^2 \gamma_2}$$

$$= -\sum_{j=1}^{n} \beta_2 z_j \frac{\left\{2\gamma_2 \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right] + {\gamma_2}^2 \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j {\gamma_2}^{-2}\right\} \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right)}{\left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^2 {\gamma_2}^4}$$

$$+ \sum_{j=1}^{n} \beta_2 z_j \frac{\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j {\gamma_2}^{-2} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)] {\gamma_2}^2}{[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2 {\gamma_2}^4} + \frac{n{\beta_1}^2}{\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^2}$$

$$= \frac{n{\beta_1}^2}{\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^2} + \sum_{j=1}^{n} \beta_2 z_j \frac{(\beta_2 z_j - 2\gamma_2) \exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j] + 2\gamma_2 \exp\left(-\frac{2\beta_2}{\gamma_2} z_j\right)}{[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2 {\gamma_2}^4}.$$

$$\frac{\partial^2 \ell}{\partial\beta_1\partial\beta_2}$$

$$= \sum_{j=1}^{n} \frac{z_j \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)}{\gamma_1} \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^{-2} \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \frac{z_j}{\gamma_2} - n\gamma_1\gamma_2(\beta_1\gamma_2 - \beta_2\gamma_1)^{-2}$$

$$= \sum_{j=1}^{n} \frac{\exp -[\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j]}{\gamma_1\gamma_2} {z_j}^2 \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^{-2} - n\gamma_1\gamma_2(\beta_1\gamma_2 - \beta_2\gamma_1)^{-2}.$$

$$\frac{\partial^2 \ell}{\partial\beta_1\partial\gamma_1}$$

$$= \sum_{j=1}^{n} z_j \frac{\exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) \beta_1 z_j {\gamma_1}^{-2} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]\gamma_1}{{\gamma_1}^2 [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2}$$

$$- \sum_{j=1}^{n} z_j \frac{\exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) - \gamma_1 \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) \beta_1 z_j {\gamma_1}^{-2}]}{{\gamma_1}^2 [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2}$$

$$- n\gamma_2\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)^{-2}$$

$$= \sum_{j=1}^{n} z_j \frac{\left(\beta_1 z_j {\gamma_1}^{-1} - 1\right) \left\{\exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j]\right\} + \exp\left(-2\frac{\beta_1}{\gamma_1} z_j\right)}{{\gamma_1}^2 [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2} - n\gamma_2\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)^{-2}.$$

$$\frac{\partial^2 \ell}{\partial\beta_1\partial\gamma_2}$$

$$= -\sum_{j=1}^{n} \frac{z_j \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)}{\gamma_1} \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^{-2} \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j {\gamma_2}^{-2}$$

$$- n \frac{\beta_1\gamma_2 - \beta_2\gamma_1 - \beta_1\gamma_2}{\left(\beta_1\gamma_2 - \beta_2\gamma_1\right)^2}$$

$$= -\sum_{j=1}^{n} \frac{\beta_2 {z_j}^2}{\gamma_1 {\gamma_2}^2} \exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j][\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^{-2} + n \frac{\beta_2 \gamma_1}{(\beta_1 \gamma_2 - \beta_2 \gamma_1)^2}.$$

$$\frac{\partial^2 \ell}{\partial \beta_2 \partial \gamma_1}$$

$$= -\sum_{j=1}^{n} \frac{z_j \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right)}{\gamma_1} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^{-2} \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) \beta_1 z_j {\gamma_1}^{-2}$$

$$+ n \frac{\beta_1 \gamma_2 - \beta_2 \gamma_1 + \beta_2 \gamma_1}{(\beta_1 \gamma_2 - \beta_2 \gamma_1)^2}$$

$$= -\sum_{j=1}^{n} \frac{\beta_1 {z_j}^2}{{\gamma_1}^2 \gamma_2} \exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j] \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^{-2} + n \frac{\beta_1 \gamma_2}{(\beta_1 \gamma_2 - \beta_2 \gamma_1)^2}.$$

$$\frac{\partial^2 \ell}{\partial \beta_2 \partial \gamma_2}$$

$$= -\sum_{j=1}^{n} z_j \frac{\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j {\gamma_2}^{-2} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)] \gamma_2}{{\gamma_2}^2 [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2}$$

$$+ \sum_{j=1}^{n} z_j \frac{\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right) - \gamma_2 \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j {\gamma_2}^{-2}]}{{\gamma_2}^2 [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^2}$$

$$- n \gamma_1 \beta_1 (\beta_1 \gamma_2 - \beta_2 \gamma_1)^{-2}$$

$$= -\sum_{j=1}^{n} z_j \frac{(1 - \beta_2 z_j {\gamma_2}^{-1}) \left\{\exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j]\right\} - \exp\left(-2 \frac{\beta_2}{\gamma_2} z_j\right)}{{\gamma_2}^2 \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^2} - n \gamma_1 \beta_1 (\beta_1 \gamma_2 - \beta_2 \gamma_1)^{-2}.$$

and

$$\frac{\partial^2 \ell}{\partial \gamma_1 \partial \gamma_2}$$

$$= \sum_{j=1}^{n} z_j \frac{\beta_1 z_j \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)}{{\gamma_1}^2} \left[\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)\right]^{-2} \exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) \beta_2 z_j \, {\gamma_2}^{-2}$$

$$- \frac{n\beta_1\beta_2}{(\beta_1\gamma_2 - \beta_2\gamma_1)^2}$$

$$= \sum_{j=1}^{n} \frac{\beta_1\beta_2 {z_j}^2}{{\gamma_1}^2 \, {\gamma_2}^2} [\exp\left(-\frac{\beta_2}{\gamma_2} z_j\right) - \exp\left(-\frac{\beta_1}{\gamma_1} z_j\right)]^{-2} \exp[-\left(\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right) z_j] - \frac{n\beta_1\beta_2}{(\beta_1\gamma_2 - \beta_2\gamma_1)^2}.$$

The expressions for the Fisher Information matrix are not simple analytically, and we do not intend to pursue this further. In the next chapter, we discuss the method of simulation by the Markov Chain Monte Carlo. And an attempt is made to generate random variates from the convoluted beta-Weibull distribution.

8. Simulation

Markov Chain Monte Carlo (MCMC) methods encompass a general framework of methods introduced by Metropolis et al. (1953) and Hastings (1970) for Monte Carlo integration. The Monte Carlo integration estimates the integral

$$\int_A g(t)\,dt$$

with a sample mean by restating the integration problem as an expectation with respect to some density function $f(\cdot)$. The integration problem then is reduced to find a way to generate samples form the target density $f(\cdot)$. According to Maria (2008), the MCMC approach to sampling from $f(\cdot)$ is to construct a Markov chain with stationary distribution $f(\cdot)$, and run the chain for a sufficiently long time until the chain coverges to its stationary distribution. Simply, the Monte Carlo estimate of

$$\mathrm{E}[g(\theta)] = \int g(\theta)\, f_{\theta|x}(\theta)d\theta$$

is the sample mean

$$\bar{g} = \frac{1}{m}\sum_{i=1}^{m} g(x_i)$$

where $x_1, x_2, \cdots, x_m$ is a sample from the distribution with density $f_{\theta|x}$.

The Metropolis-Hastings (M-H) algorithms are a class of MCMC methods, one of which is the Metropolis sampler. The main idea is to generate a Markov Chain $\{X_t \mid t = 0, 1, 2 \cdots\}$ such that its stationary distribution is the target distribution. The algorithm must specify, for a givenX_t, how to generate the next state X_{t+1}. In all of the Metropolis-Hastings sampling algorithms, there is a candidate point Y generated from a proposal distribution $g(\cdot\,|X_t)$. If this candidate point is accepted, the chain moves to state Y at time t+1 and $X_{t+1} = Y$; otherwise the chain stays in state X_t and $X_{t+1} = X_t$. The choice of proposal distribution is very flexible, but the chain generated by this choice must satisfy certain regularity conditions. The proposal distribution must be chosen so that the generated chain will converge to a stationary distribution- the target distribution.

The algorithms or steps required in generating a Markov chain $\{X_0, X_1, X_3 \cdots\}$ by the Metropolis-Hastings sampler are as follows, (See Maria (2008)):

1) Choose a proposal distribution $g(\cdot\,|X_t)$ (subject to regularity conditions stated above).
2) Generate X_0 from a distribution g.
3) Repeat (until the chain has converged to a stationary distribution according to some criterion):
   (a) Generate Y from $g(\cdot\,|X_t)$.
   (b) Generate U from Uniform (0,1).
   (c) If
   $$U \leq \frac{f(Y)g(X_t|\mathrm{Y})}{f(X_t)g(Y|X_t)}$$
   accept Y and set $X_{t+1} = Y$; otherwise set $X_{t+1} = X_t$.
4) Increment t.

Following the procedure described above, we generated a simulation of the convoluted beta-Weibull distribution with parameters $\beta_1 = 1, \beta_2 = 4, \gamma_1 = 3, \gamma_2 = 2$. The histogram of the simulated data and the curve of empirical density function of the CBWD with same parameters are shown in Figure (4). Also, the R script for MCMC samples of CBWD is provided in the Appendix.

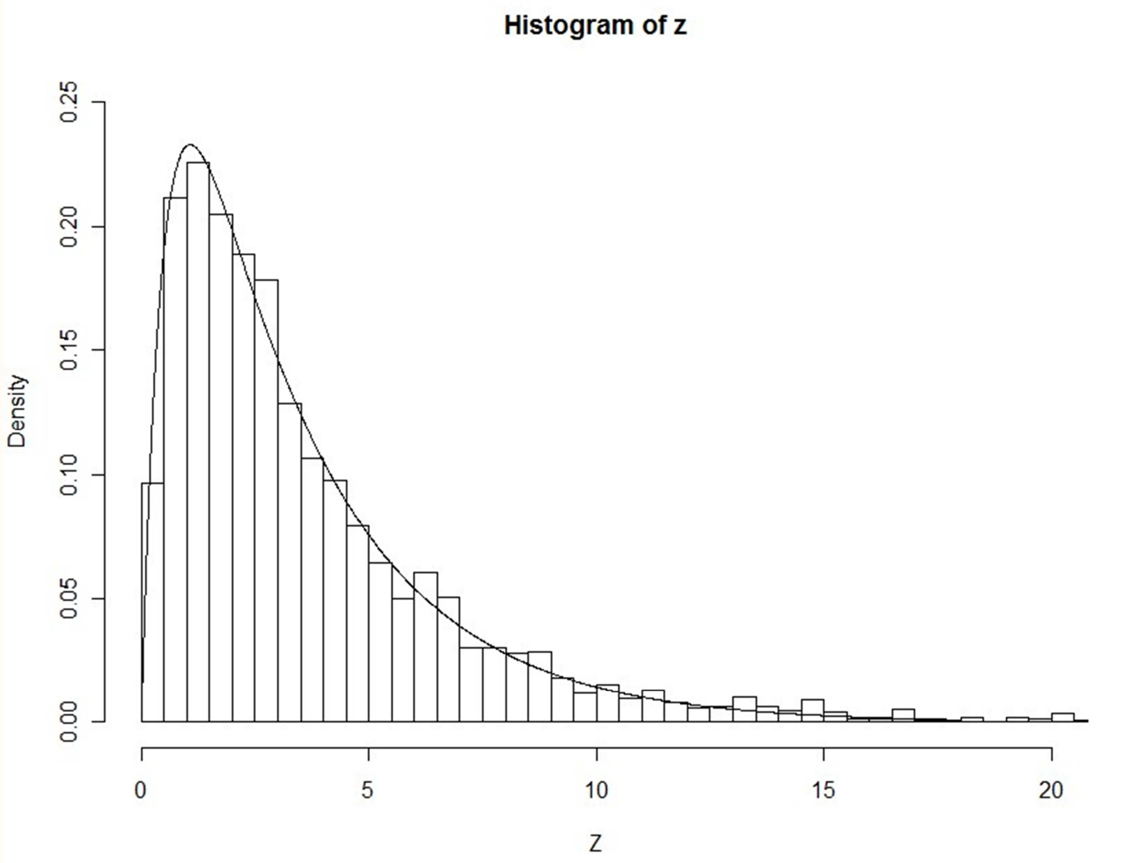

Fig 4. The histogram of the simulated CBWD and PDF of the CBWD

9. Conclusion

The convoluted beta-Weibull distribution was defined and studied in this work. Various properties of the distribution were discussed. These include the moment generating function, characteristic function, mean, variance, shewness, kurtosis, the mean deviation about the mean, and the mean deviation about the median. Also discussed are the Rényi entropy, asymptotic behaviors, estimation of parameters by the method of maximum likelihood. A simulated random variates of the distribution were generated by the method of Markov Chain Monte Carlo (MCMC). R statistical software program was used in the implementation of our results.

Appendix

i. The code in R program to generate the graph of CDF of the convoluted beta-Weibull distribution with paramenters $\beta_1 = 2, \beta_2 = 3,\ \gamma_1 = 2, \gamma_2 = 4$.

```
b1=2
b2=3
r1=2
r2=4
z=seq(0,15,.01)
K=(b1*b2/(b1*r2-b2*r1))
c1=(b2*r1)/(b1*r2-b2*r1)
c2=(b1*r2)/(b1*r2-b2*r1)
cdf=function(x){
c1*exp(-(b1/r1)*z)-c2*exp(-(b2/r2)*z)+1}
plot(z,cdf(z),type="l",xlab="Z ")
```

ii. The code in R program to generate the graph of PDF of the convoluted beta-Weibull distribution for different parameters values of $\beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2$ ($\beta_1 = 3, \beta_2 = 1,\ \gamma_1 = 4, \gamma_2 = 3$; $\beta_1 = 4, \beta_2 = 1,\ \gamma_1 = 3, \gamma_2 = 2$; $\beta_1 = 3, \beta_2 = 4,\ \gamma_1 = 2,\ \gamma_2 = 3$)

```
b1=3
b2=1
r2=3
x=seq(0,10,.01)
bw1.pdf=function(x,b1,b2,r1,r2){
k=b1*b2/(b1*r2-b2*r1)
f=k*(exp(-b2/r2*x)-exp(-b1/r1*x))
}
```

```
plot(x,bw1.pdf(x,3,1,4,3),type='l',ylim=c(0,.6),xlab="Z",ylab=
"pdf")
lines(x,bw1.pdf(x,4,1,3,2),lty=2)
lines(x,bw1.pdf(x,3,4,2,3),lty=4)
legend("topright",inset=0.02,legend=c("b1=3,b2=1,r1=4,r2=3
","b1=4,b2=1,r1=3,r2=2","b1=3,b2=4,r1=2,r2=3"),lty=1:2:4)
```

iii. The code in R program to generate the graph of hazard rate function of the convoluted beta-Weibull distribution with parameters $\beta_1 = 2, \beta_2 = 3,\ \gamma_1 = 2, \gamma_2 = 4$

```
b1=2
b2=3
r1=2
r2=4
x=seq(0,15,.01)
c1=(b2*r1)/(b1*r2-b2*r1)
c2=(b1*r2)/(b1*r2-b2*r1)
hrf=function(x){
(b1*b2)*(exp((-b2/r2)*x)-exp((-b1/r1)*x))/(b1*r2*exp((-
b2/r2)*x)-b2*r1*exp((-b1/r1)*x))}
plot(x,hrf(x),type="l",xlab="Z",ylab="The Hazard Rate
Function (h(z))")
```

iv. The code in R program to generate a histogram of the simulated CBWD and PDF of the CBWD with parameters $\beta_1 = 1, \beta_2 = 4,\ \gamma_1 = 3, \gamma_2 = 2$

```
K=(b1*b2/(b1*r2-b2*r1))
c1=(b2*r1)/(b1*r2-b2*r1)
c2=(b1*r2)/(b1*r2-b2*r1)
f=function(x,b1,b2,r1,r2){
```

```
if(any(x<0)) return(0)
stopifnot(b1>0,b2>0,r1>0,r2>0)
return(K*(exp(-(b2/r2)*x)-exp(-(b1/r1)*x)))
}

m=10000
b1=1
b2=4
r1=3
r2=2
x=numeric(m)
x[1]=rrayleigh(1,1)
k=0
u=runif(m)

for(i in 2:m){
xt=x[i-1]
z=rrayleigh(1,xt)
num=f(z,b1,b2,r1,r2)*drayleigh(xt,z)
den=f(xt,b1,b2,r1,r2)*drayleigh(z,xt)
if(u[i]<=num/den) x[i]=z else{
x[i]=xt
k=k+1
}
}
print(k)

b=1001
z=x[b:m]
hist(z,breaks="scott",freq=F,xlim=c(0,20),ylim=c(0,.25),xlab=
"Z - value")
t=seq(0,18,0.01)
lines(t,K*(exp(-(b2/r2)*t)-exp(-(b1/r1)*t)),type="l")
```

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