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Topics in Extremal Graph Theory: Ramsey Numbers and the Turan Function

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TOPICS IN EXTREMAL GRAPH THEORY: RAMSEY NUMBERS AND THE TURAN FUNCTION

DAMON J. GULCZYNSKI

1. PRELIMINARIES

Below are some relevant definitions and notational explanations. To understand the topics discussed in the following pages best a certain degree of mathematical maturity is needed. It is my belief that upper-division undergraduate students will find these notes dense but accessible. In all I think that these notes can be aptly described as the perfect gift for the person who already has everything.

Depending on background the reader may, at this point, want to skip this preliminary section, perhaps referencing it as needed.

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote, as usual, the set of natural numbers, integer numbers and real numbers, respectively.

A plus or minus used as a superscript on a set denotes, respectively, the positive or negative members of that set, i.e. $\mathbb{Z}^+ = \{x \in \mathbb{Z}\colon x \geq 1\}$.

The cardinality of a set A, is denoted as $|A|$. $|\mathbb{N}|$ and $|\mathbb{R}|$ are denoted as $\aleph_0$ and c, respectively.

The set $\{x \in \mathbb{Z}^+\colon 1 \leq x \leq n\}$ is denoted by $[n]$, for any $n \in \mathbb{Z}^+$.

The set $\{A \subseteq S\colon |A| = \alpha\}$ is denoted as $S^{(\alpha)}$ for any set S and any cardinality $\alpha \leq |S|$.

A *hypergraph* is an ordered pair, (V, E), containing a vertex set and an edge set. The vertex set, V, can be any finite set (usually it is the set $[n]$ for some n) while the edge set, E, is a set of subsets of V. V must be nonempty. We will allow E to be empty, however we will not allow the empty set to be an element of E. That is to say a *hypergraph* can have no edges, but it can't have an empty edge. If the *hypergraph* is given a name like G we'll write $V(G)$ and $E(G)$ to mean the vertex set of G and the edge set of G, respectively. $|V(G)|$ is called the order of G, while $|E(G)|$ is called the size of G.

Given hypergraphs G and H, we say that G and H are *equal via isomorphism*, or often simply *equal* if there exists a 1-1, onto, function $f : V(G) \to V(H)$, such that $\{v_1, v_2, ..., v_n\} \in E(G)$ if and only if $\{f(v_1), f(v_2), ..., f(v_n)\} \in E(H)$.

Given a hypergraph G, we call H a sub-hypergraph of G if H is equal via isomorphism to some graph F where $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$.

Given a hypergraph G we say that $v, u \in V(G)$ are *adjacent* via $e \in E(G)$ if $v, u \in e$.

A *k-uniform* hypergraph is one in which $|e| = k \ \forall \ e \in E$.

A 2-uniform hypergraph is called a *graph*. Notice that in a *graph* the edges can be considered unordered pairs. Edges of *graphs* are generally written as (u, v) or even vu as opposed to the more formal $\{u, v\}$. Since the edges are unordered pairs uv and vu are the same edge.

Given a vertex x of some hypergraph, G, the *degree of* x, denoted $deg(x)$ is $|\{e \in E(G) : x \in e\}|$. Notice that for a graph $deg(x)$ is the number of vertices adjacent to x. We denote by $\Delta(G)$, the greatest degree of the vertices of G.

A *complete r-uniform hypergraph on n vertices* is denoted as K_n^r. It is a hypergraph with $V = [n]$ and $E = [n]^{(r)}$ where $r \leq n$. When $r = 2$ we simply denote it as K_n. This is called the *complete graph on n vertices*, or often just "kay - n ".

A *k-partite graph*, G, is a graph where $V(G)$ can be partitioned into *partite classes*, $S_1, S_2, ..., S_k$ in such a way that $uv \notin E(G)$ if both u and v are in the same *partite class*. If it is the case that $uv \notin E(G)$ if **and only if** both u and v are in the same *partite class* then we say that G is a *complete k-partite graph* and we denote G by $K_{s_1, s_2, ..., s_k}$ where $|S_i| = s_i \ \forall \ i \in [k]$.

If H is a subgraph of G and H is complete with respect to its vertex set ($E(H) = V(H)^{(r)}$ for some r), then H is called a *clique* of G. A *clique* of order k is called a *k-clique*.

Given a hypergraph, a *cycle* is an alternating sequence of edges and vertices, $u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k$, such that $u_{i-1}, u_i \in e_i$ for $i \in [k]$, $u_0 = u_k$, and $e_i \neq e_j$ and $u_i \neq u_j$ $\forall$ $i, j \in [k]$. k is said to be the *length* of the cycle. A cycle of length k is called a *k-cycle*. Notice that when dealing with graphs explicitly listing the edges of a cycle is not necessary since we have only one choice.

A *k-coloring* of a set S is a function $\chi: S \to [k]$. Given $s \in S$ we call $\chi(s)$ the color of s. We call $A \subseteq S$ *monochromatic* if $\chi(a_1) = \chi(a_2)$ $\forall$ $a_1, a_2 \in A$. The *color classes* of S induced by χ, are the sets $\chi_i = \{s \in S: \chi(s) = i\}$, $i \in [k]$. Sometimes just for fun if k is small we use actual colors instead of numbers for our coloring. The author is partial to goldenrod, periwinkle, mauve and red.

Sometimes, when the meaning is clear, we shall not be so formal in our notation. For instance given a hypergraph G and a vertex x, we shall write $G - x$ to represent the hypergraph with vertex set $V(G) - \{x\}$ and edge set $\{e - x: e \in E(G)\} - \emptyset$. Likewise we shall write $G - e$, where e is an edge to mean the hypergraph with vertex set $V(G)$ and edge set $E(G) - \{e\}$. If G is a graph then $G - x$ is the graph with vertex set $V(G) - x$ and edge set $E(G) - \{e \in E(G): x \in e\}$.

WLOG (often pronounced Way-Log) stands for Without Loss of Generality.

With the bulk of the formalities taken care of we can now proceed to the good stuff. The stuff dreams are made of.

2. Ramsey Numbers

Suppose that you have organized a wrestling tournament with six or more wrestlers.

Claim. There are in the tournament either three wrestlers who pairwise have all wrestled each other once before, or there are in the tournament three wrestlers who pairwise have never wrestled each before. □

Proof. Consider a single wrestler a. Since there are at least five wrestlers other than a, it must be that either there are at least three wrestlers all of whom a has wrestled before, or there are at least three wrestlers all of whom a has not wrestled before.

Assume the former – there are three people b, c and d all of whom a has wrestled before. If any two of these people have previously wrestled, then this pair and a make three people that pairwise have all wrestled each other before. If no two of these people have previously wrestled then b, c and d make three people that pairwise have never wrestled each other before.

The situation is entirely symmetric if we assume in the first place the latter condition. □

Without explicitly stating it, we have just proven that $R(3) \leq 6$ where R is called the *Ramsey function.*

Definition 1. Given an integer $n \geq 2$ let $R(n)$ (*Ramsey n*) be the smallest positive integer with the following property: if the elements of $[R(n)]^{(2)}$ are 2-colored arbitrarily it is necessarily the case that there exists some set $A \subseteq [R(n)]$ such that $|A| = n$ and $[A]^{(2)}$ is monochromatic.

After some careful thought one can now see why the claim above proves that $R(3) \leq 6$.

Graph theory offers an alternative and often times simpler definition of the Ramsey function.

Definition 2. Given an integer $n \geq 2$ let $R(n)$ be the smallest positive integer such that if the edges of $K_{R(n)}$ are 2-colored arbitrarily there necessarily exists a clique of order n whose edge set is monochromatic.

The reader should convince herself (or himself depending on how stringently politically correct we wish to be) that the two definitions above are equivalent. They should notice also that from the graph theory perspective $R(3) \leq 6$ translates as: an arbitrary 2-coloring of the edges of K_6 will always result in a monochromatic triangle (clique of order 3.) One can, however with little difficulty, 2-color the edges of K_5 in such a way that yields no monochromatic triangle. Thusly we can make the stronger statement $R(3) = 6$.

As it currently stands, there is a glaring shortcoming in the given definitions of the Ramsey function. Given an n how do we know that $R(n)$ exists? This question is the motivation for most fundamental result in all of Ramsey theory.

Theorem 1. *The Ramsey function is well-defined.*

Proof. It suffices to show that given n, $R(n) \leq 2^{2n-1}$. With that said, let χ be an arbitrary 2-coloring on the edges of $G = K_{2^{2n-1}}$. Let x_1 be an arbitrary vertex of G. Consider the *majority-coloring set* of x_1, that is to say the greater of the two sets $\{v \in V(G) : \chi(v, x_1) = 1\}$ and $\{v \in V(G) : \chi(v, x_1) = 2\}$. Call this majority-coloring set S_1. Pick $x_2 \in S_1$ and consider the *majority-coloring set of x_2 restricted to S_1* that is to say the greater of the two sets, $\{v \in S_1 : \chi(v, x_2) = 1\}$ and $\{v \in S_1 : \chi(v, x_2) = 2\}$. Call this set S_2. Next pick $x_3 \in S_2$ and call its majority-coloring set when restricted to S_2, S_3.

Repeating this process, since we have 2^{2n-1} total vertices and each majority-coloring set "eliminates "at most half the remaining vertices we are always ensured at least $2n - 1$ vertices: $x_1, x_2, ..., x_{2n-1}$ and at least $2n - 2$ sets $S_1 \supset S_2 \supset S_3 \supset ... \supset S_{2n-2}$. Furthermore for each $i \in [2n - 2]$ x_i has associated with it a color γ_i, where $\chi(x_i, v) = \gamma_i$ for all $v \in S_i$. Since we have $2n - 1$ vertices one of the following two sets: $\{x_i : \gamma_i = 1\}$ and $\{x_i : \gamma_i = 2\}$ must have at least n elements. Call this set A. By how we selected the x's and S's above, it must be the case that for all $x_i, x_j \in A, \chi(x_i, x_j) = \gamma_i$. It follows that any n elements of A form a monochromatic clique of order n. $\square$

As we can see the bound 2^{2n-1} is not sharp, for we know that $R(3) = 6 \neq 32$. In fact it is a lousy bound, much better ones have been developed. A lot of effort has gone into finding bounds for Ramsey numbers because establishing an exact value is practically impossible. We know that $R(3) = 6$ and $R(4) = 18$, but that's all. Paul Erdös once said something to the effect of the following regarding $R(5)$ and $R(6)$:

Suppose that some evil presence was going to destroy the human race unless we could tell him (and provide proof for) an exact value for $R(5)$. If every mathematician in the world devoted every waking hour of their day to such a task, with the help of superpower computers, we could have a solution within a year. If on the other hand the evil presence had asked for an exact value for $R(6)$ instead of $R(5)$, the best course of action would be for every mathematician in the world to devote every waking hour of their day, using superpower computers, in an attempt to build a machine that would kill the evil presence!

There are some very natural extensions to Ramsey numbers. In particular there is no reason why we should limit ourselves to only two colors, or to only one n, or only to coloring pairs (edges of cardinality two.) Below is a more "beefed-up"definition of the Ramsey function. It is the Ramsey function on steroids (or at least creatine.)

Definition 3. Let integers $n_1, n_2, ..., n_k$ all be greater than or equal to another given integer $r \geq 1$. Let $R^{(r)}(n_1, n_2, ..., n_k) = R$ be the smallest positive integer such that if the set $[R]^{(r)}$ is k-colored then it is necessarily the case that there exists a set $A \subseteq [R]$ such that $|A| = n_i$ and $A^{(r)}$ is monochromatic with color i.

The beefed-up Ramsey function also has a hypergraph-dependent definition.

Definition 4. Let integers $n_1, n_2, ..., n_k$ all be greater than or equal to another given integer $r \geq 1$. Let $R^{(r)}(n_1, n_2, ..., n_k) = R$ be the smallest positive integer such that if the edges of K_R^r are k-colored then it is necessarily the case that there exists a clique of order n_i whose edge set is monochromatic with color i.

Notice that according to these definitions our standard $R(n)$ most formally should be written as $R^{(2)}(n,n)$. However it is common notational practice to drop the superscript when $r = 2$, and to drop one of the n's when $r = 2$, $k = 2$, and $n_1 = n_2$. Thus $R^{(2)}(3,4)$ is written as $R(3,4)$ and $R^{(2)}(4,4)$ is written as $R(4)$. Admittedly this is a bit unsettling since $R(4)$ really should be used in the case when $k = 1$. However this is such a trivial case that we can almost ignore it. To avoid confusion if we are assuming only one argument (i.e. $k = 1$) then it will always be stated explicitly.

With our stronger Ramsey numbers now defined we need to prove their existence with a stronger theorem. To this extent we need a lemma that states Ramsey numbers exist whenever $k = 2$.

Lemma 1. *$R^{(r)}(n_1, n_2)$ exists for all n_1, n_2, r.*

Proof. The proof is done by induction on r and on n_1 and n_2. The base case $r = 1$ is handled by the equality $R^{(1)}(n_1, n_2) = n_1 + n_2 - 1$, while the other base cases are provided by the equalities $R^{(r)}(n_1, r) = n_1$ and $R^{(r)}(r, n_2) = n_2$. The reader should verify these equalities before moving on. We are now at liberty to assume that $R^{(r-1)}(x,y)$ exists for all x, y and that $R^{(r)}(n_1 - 1, n_2)$ and $R^{(r)}(n_1, n_2 - 1)$ exist for all r, n_1, n_2.

The strategy is now to show that $R^{(r)}(n_1, n_2) \leq R^{(r-1)}(R^{(r)}(n_1 - 1, n_2), R^{(r)}(n_1, n_2 - 1)) + 1$, thus completing the proof. To this end denote by $R + 1$ the right side of the inequality above. Let χ be a 2-coloring of the edges of $G = K_{R+1}^r$. Let x be an arbitrary vertex of G. Consider the hypergraph $G - x = K_R^{r-1}$. Let χ^* be the *shadow-coloring* of the edges of this hypergraph with respect to x, that is to say given $e \in E(G - x)$ let $\chi^*(e) = \chi(e \cup \{x\})$. By how R is defined χ^* must induce on $G - x$ either a clique of order $R_1 = R^{(r)}(n_1 - 1, n_2)$ whose edges are all of color 1, or a clique of order $R_2 = R^{(r)}(n_1 - 1, n_2)$ whose edges are all of color 2. WLOG assume the former.

Let $A = K^{r-1}_{R_1}$ be the monochromatic clique. Consider now the coloring that χ induces on the $K^r_{R_1}$ that has the same vertex set as A. By how R_1 is defined there exists either a clique of order $n_1 - 1$ of color 1, or a clique of order n_2 of color 2. In the latter case we have instantly reached our goal, so assume the former case. Let $S = K^r_{n_1-1}$ be the monochromatic clique. It then follows by the shadow-coloring, that the complete graph on $S \cup \{x\}$ is a clique of order n_1 whose edges are all colored 1. The proof is finished. □

Now we have the tools to prove the sought after stronger theorem.

Theorem 2. *The beefed-up Ramsey function is well-defined.*

Proof. The proof is done by induction on k. The base case $k = 1$ is trivial. By the inductive hypothesis it suffices to show that $R^{(r)}(n_1, n_2, ..., n_k) \leq R^{(r)}(n_1, ..., n_{k-2}, R^{(r)}(n_{k-1}, n_k))$. Notice that the above Lemma is needed to ensure the existence of $R^{(r)}(n_{k-1}, n_k)$. Denote by R the right side of the above inequality. Let χ be a k-coloring of the edges of $G = K^r_R$. Let $\chi_1, \chi_2, ..., \chi_k$ be the respective color classes. Consider the (k-1)-coloring of G induced by the color classes $\chi_1, \chi_2, ..., \chi_{k-1} \cup \chi_k$.

By how R is defined either one of the color classes χ_i contains the edges of a clique of order n_i, or $\chi_{k-1} \cup \chi_k$ contains the edges of a clique of order $R_1 = R^{(r)}(n_{k-1}, n_k)$. The former case completes the proof immediately, so let us assume the latter case. This leaves us with a $K^r_{R_1}$ whose edges are 2-colored by the color classes χ_{k-1} and χ_k. By how R_1 has been defined this 2-coloring produces either a clique of order n_{k-1} whose edges are all of color $k-1$, or a clique of order n_k whose edges are all of color k. In either case we reach our goal. □

With this existence theorem at hand we can now list some other Ramsey Number theorems. I have omitted the proofs, save the occasional remark, for the sake of brevity. The reader can do them as exercises; most of them are pretty fun. Not as much fun as a game of bocci, or making love to a beautiful woman, but fun nonetheless.

Theorem 3. $R(p,q) \leq R(p-1,q) + R(p,q-1)$ *(The inequality can be made strict if both of $R(p-1,q)$ and $R(p,q-1)$ are even.)*

Theorem 4. $R(p,q) \leq \binom{p+q-2}{p-1}$

Proof. Use induction and theorem 3. □

Theorem 5. $R^{(r)}(n_1, n_2, ...n_k) \leq R^{(r-1)}(R^{(r)}(n_1-1, n_2, ..., n_k), R^{(r)}(n_1, n_2 - 1, ..., n_k), ..., R^{(r)}(n_1, n_2, ..., n_k - 1)) + 1$

Proof. Same basic strategy as Lemma 1. □

Theorem 6. $2^{(r)} \leq R(3, 3, \ldots, 3) \leq 3r!$ *(assuming r arguments)*

Theorem 7. $R(n) \leq R^{(3)}(6, n)$

At this point we can round out the list of all known non-trivial Ramsey Numbers. $R(3) = 6$. $R(4) = 18$. $R(3,4) = 9$. $R(3,5) = 14$. $R(3,6) = 18$. $R(3,7) = 23$. $R(3,8) = 28$. $R(3,9) = 36$. The only known non-trivial value for three colors is $R(3,3,3) = 17$.

Up to this point we have been dealing with finite quantities. Ramsey numbers are finite numbers defined using finite sets and finite colorings. But what if we want to go beyond the finite. What if we long for the infinite? Must we then abandon Ramsey theory altogether? The answer is no. Cantor fans can rest easy. We can have Ramsey theory and also satisfy our yen for the infinite.

Definition 5. Given cardinalities α, β we say α *arrows* β denoted as $\alpha \to \beta$, if whenever $|A| \geq \alpha$ and $A^{(2)}$ is 2-colored, there exists $B \subseteq A$, $B \geq \beta$ with $B^{(2)}$ monochromatic.

If we use our imaginations for a minute and stretch our notion of graphs into the infinite, then the above definition says that if the edges of K_α are 2-colored then there always exist a monochromatic clique of order β.

Theorem 8. *For any finite cardinality n, $\aleph_0 \to n$.*

Proof. This is a corollary of Theorem 1! ☐

For the next three theorems the Axiom of Choice is assumed. The proof of Theorem 11 has been omitted because it requires too many tools not in the scope of these notes. It can be found in Graham, Rothschild and Spencer [2].

Theorem 9. $\aleph_0 \to \aleph_0$

Proof. Let $|A| \geq \aleph_0$. Let χ be a 2-coloring of $A^{(2)}$ with colors mauve and red. Consider an arbitrary element $x_1 \in A$. Let $A_1 = \{a \in A : \chi(x_1, a) = \text{mauve}\}$, $B_1 = \{a \in A : \chi(x_1, a) = \text{red}\}$. One of these two sets must be infinite, let's say it is A_1. Let $A_1 = S_1$. Pick $x_2 \in S_1$. Let $A_2 = \{a \in S_1 : \chi(x_2, a) = \text{mauve}\}$, $B_2 = \{a \in S_1 : \chi(x_2, a) = \text{red}\}$. One of these sets must be infinite, let's say B_2. Let $B_2 = S_2$. Pick $x_3 \in S_2$.

Repeating this process we get an infinite sequence of subsets $S_1 \supseteq S_2 \supseteq S_3 \ldots$. In this sequence one of either A_i, or B_i must be repeated an infinite number of times. WLOG assume A_i is repeated for $i = k_1, k_2, k_3, \ldots$. It then follows that the set $X = \{x_{k_1}, x_{k_2}, x_{k_3}, \ldots\}$ has cardinality $\aleph_0$ and $X^{(2)}$ is monochromatic. □

Theorem 10. $c \nrightarrow c$

Proof. Let $<$ be the usual ordering of $\mathbb{R}$ and $\prec$ a well-ordering. 2-color $\mathbb{R}^{(2)}$ by $\chi(x,y) = \begin{cases} \text{goldenrod}, & \text{if } x \prec y \\ \text{periwinkle}, & \text{if } y \prec x \end{cases}$ where $x < y$. Assume that the pairs of a given set $S \subseteq \mathbb{R}$ are monochromatic under χ. Further assume that they all have color goldenrod. It must be that S is well-ordered by $<$. It then follows that for all $x \in S$ (except possibly a single maximum element) there exists a minimal $x^* \in S$ such that $x < x^*$. Thus S does not contain any elements from the open interval (x, x^*). Given $n \in \mathbb{Z}^+$ let $A_n = \{x : x^* - x > \frac{1}{n}\}$. Since the elements in A_n must be at least the fixed distance $\frac{1}{n}$ apart it must be that A_n is countable. It follows that $S = \bigcup_{n=1}^{\infty} A_n$ is also countable. If we assume in the first place that the elements of $S^{(2)}$ are all periwinkle then S is well-ordered by $>$, and a symmetric argument forces S to be countable. □

Theorem 11. *For all cardinalities β there exists an α such that $\alpha \rightarrow \beta$.*

Concerning infinite extensions of Ramsey theory, I have been fiddling with coloring the pairs of various infinite sets with infinitely many colors with the goal being monochromatic triangles.

Definition 6. Given cardinalities α, β we say that α *triangles* β denoted $\alpha \rhd \beta$ if whenever $|A| = \alpha$ and $A^{(2)}$ is colored with β many colors, there exist $S \subseteq A$ where $|S| = 3$ and $S^{(2)}$ is monochromatic.

Theorem 12. $2^\alpha \not\rhd \alpha$, *for all* α.

Proof. Let $|A| = \alpha$. Consider the power set of A, P. It is well known that $|P| = 2^\alpha$. Color $P^{(2)}$ by $\chi(S,T) = x$ where x is an arbitrary element in $S \cup T - S \cap T$ (which is always non-empty since $S \neq T$). Let W, U, V be an arbitrary triangle from P. Suppose $\chi(U,V) = a$. WLOG we can assume that $a \in U$ and $a \notin V$. If $\chi(V,W) = a$ as well, then since $a \notin V$ it must be that $a \in W$. But now we get that $a \in U \cap W$ and thus $\chi(U,W) \neq a$. □

Notice that the left inequality in Theorem 6 is an immediate corollary of the theorem above. I was happily surprised when I discovered that.

Theorem 13. $c \not\rhd \aleph_0$

Proof. Since $2^{\aleph_0} = c$ this result is another corollary of the above theorem. As an alternate proof consider the following integer coloring of $\mathbb{R}^{(2)}$: given $a, b \in \mathbb{R}$ let $\chi(a,b) = n$ where $n \in \mathbb{Z}$ and $2^{n-1} < |a-b| \leq 2^n$. Clearly a unique n always exists. Furthermore given reals $x > y > z$ if $2^{n-1} < x - y \leq 2^n$, and $2^{n-1} < y - z \leq 2^n$, then it follows that $2^n < x - z \leq 2^{n+1}$, and so under this coloring no monochromatic triangles exist. $\square$

I am currently working on finding a cardinality a such that $a \triangleright \aleph_0$. Up to this point I have been unsuccessful. Mainly I have been trying to figure out if $2^c \triangleright \aleph_0$, but have not yet. If you figure it out, or develop any other results along this line, I would love to hear about them. I can be reached at gulczyd@hotmail.com. What I would really like to see is a triangle version of theorem 11.

3. Turan Function

Definition 7. Let H be a graph of order n, and let $p \geq n$ be a positive integer. Define $T(H,p)$ (Turan H-p) as the smallest positive integer such that any graph G, with vertex set $[p]$ and size $T(G,p)$ necessarily contains H as a subgraph.

Notice that we do not need an in-depth proof, as was the case with the Ramsey function, to realize that the Turan function is well-defined. One has only to note that $T(H,p) \leq \binom{p}{2}$, for the only the graph on $[p]$ with size $\binom{p}{2}$ is K_p and every graph with an order less than or equal to p is a subgraph of K_p. With that said, we are almost ready to present perhaps the most famous theorem dealing with the Turan function. First we need a definition and an easy Lemma.

Definition 8. A *near regular complete k-partite graph of order p* denoted $F(p,k)$ is a complete k-partite graph of order p, $K_{s_1,s_2,\ldots,s_k}$, where $s_1 \leq s_2 \leq \ldots \leq s_k$ and $s_k \leq s_1 + 1$. Notice that p, k uniquely determine $F(p,k)$.

Lemma 2. *Given positive integers $p \geq k$, the size of $F(p,k)$ is $m(p,k)$, where $m(p,k) = \binom{p}{2} - r\binom{q+1}{2} - (k-r)\binom{q}{2}$ where $p = qk + r$ and $0 \leq r < k$.*

Proof. Let $p \geq k$ be integers where $p = qk + r$ and $0 \leq r < k$. Consider $F(p,k)$. It has all the edges of a K_p with the exception of those edges connecting two vertices of the same partition class. There must be r partition classes with $q+1$ vertices and $k - r$ partition classes with q vertices. Thus the number of edges of $F(p,k)$ is exactly $m(p,k)$. $\square$

Theorem 14. *For all positive integers $p \geq n$, $T(K_n, p) = m(p, n-1) + 1$.*

Proof. The strategy for this proof is to show, by induction on n, that every graph of order p and size at least $m(p, n-1)+1$ contains K_n as a subgraph. Furthermore, the only graph of order p and size $m(p, n-1)$ that does not contain K_n as a subgraph is $F(p, n-1)$.

The base case $n = 2$ is true, for then $F(p, n-1) = F(p, 1)$ which is simply a graph on $[p]$ with no edges. Certainly this is the only graph of order p with $m(p, 1) = 0$ edges.

Assume for $n \geq 3$, that every graph of order $s \geq n-1$ and size at least $m(s, n-2)+1$ contains as a subgraph K_{n-1}. Further assume that the only K_{n-1}-free graph of order s and size $m(s, n-2)$ is $F(s, n-2)$.

With that said, for $p \geq n$ let G be of order p and have the maximum size possible without containing a K_n. Consider a vertex v of G such that $deg(v) = \Delta(G) = \Delta$. Since G does not contain K_n, the set $N = \{x \in V(G) \colon xv \in E(G)\}$ does not contain a K_{n-1}. It follows that the size of N is at most $m(\Delta, n-2)$, the size of $F(\Delta, n-2)$.

It must be the case that $\Delta \geq n-1$. To see this assume that $\Delta \leq n-2$. It then follows that, since $p \geq n$, there exists a vertex u, such that u and v are not adjacent. Since G is the K_n free graph of maximum size, the graph $G + uv$ must contain a K_n. However all vertices of G, with the possible exception of u and v, have degree at most $\Delta = n-2$. Since K_n is (n-1)-regular and $p \geq 3$, we get a contradiction. $\Delta \geq n-1$ as claimed.

Let $U = \{u_1, u_2, ..., u_t\} = V(G) - V(N)$. Since each vertex u_i $(i \in [t])$ has degree at most Δ, the largest $|E(G)|$ can be is $(t+1)\Delta + m(\Delta, n-2)$. Furthermore, this will happen only if:

a) $N = F(\Delta, n-2)$

b) $u_i u_j \notin E(G) \; \forall \, i \neq j \in [t]$

Condition a) is easy to see. To see condition b) notice that if, in fact, $u_i u_j \in E(G)$ for some $i, j \in [t]$ then in counting the degrees of the vertices of U we would have double counted the edge $u_i u_j$. Thus the number of distinct edges present in such a counting would be strictly less than $(t+1)\Delta$.

Now, keeping all this in mind consider the graph $G' = N' + U'$ where $N' = F(\Delta, n-2)$, $V(U') = [t+1]$, and for all $x \in N'$, $y \in U'$, $xy \in E(G')$, but no two vertices of U' are adjacent. Such a graph is of order p and size $(t+1)\Delta + m(\Delta, n-2)$. Furthermore it is an (n-1)-partite graph and thus does not contain a K_n. By how G is defined it follows that $|E(G')| \leq |E(G)|$. From which we establish $|E(G)| = (t+1)\Delta + m(\Delta, n-2)$. This forces conditions a) and b) above to be true. But notice that if conditions a) and b) are true, then it follows that $N = N'$, $U \cup \{v\} = U'$ and thus G is exactly the graph G'!

G is an (n-1)-partite graph whose partite classes are $U \cup \{v\}$ augmented with the partite classes of $F(\Delta, n-2)$. Thus we can say that $G = K_{t+1,k_1,k_2,\dots,k_{n-2}}$. At this point all that remains to be shown is that G is nearly regular. We know that $F(\Delta, n-2)$ is nearly regular, so WLOG we can assume that $k_1 \leq k_2 \leq \dots \leq k_{n-2}$ and $k_{n-2} \leq k_1 + 1$. Since v is a vertex of maximum degree in G and v is in the $U \cup \{v\}$ partite class, it must be that $t + 1 \leq k_1$.

Now we only have to show that $k_{n-2} \leq t+2$. Suppose for a contradiction that $k_{n-2} > t+2$. Consider the graph $H = K_{t+2,k_1,k_2,\dots k_{n-3},k_{n-2}-1}$. H is a K_n-free graph, but $|E(H)| - |E(G)| = (k_{n-2} - 1) - (t+1) > 0$ an impossibility considering the defining property of G. It must be that $k_{n-2} \leq t+2$ and so $G = F(p, n-1)$. □

The proof of the next theorem is one that I came up with on my own. This proof depends on the following lemma.

Lemma 3. *Given a graph G of order n and size greater than or equal to $\binom{n-1}{2} + 2$ either there exists $x \in V(G)$ such that $\frac{n-1}{2} < deg(x) \leq n-2$, or $G = K_n$.*

Proof. Let G be a graph of order n and size greater than or equal to $\binom{n-1}{2} + 2$. Suppose that there exists no vertex with degree between $\frac{n-1}{2}$ and $n-2$. We can then partition $V(G)$ into two sets, those that have degree $n-1$ and those that have degree less than or equal to $\frac{n-1}{2}$. Call the latter set T and the former set S. Let $|T| = k$, and thus $|S| = n-k$. It then follows that

$$2(\binom{n-1}{2} + 2) \leq \sum_{v \in V(G)} deg(v) \leq (n-1)(n-k) + k(\frac{n-1}{2})$$

from which we can deduce $(n-1)(n-2) + 4 \leq (n-1)(n - \frac{k}{2})$, which means that $0 \leq k \leq 3$.

With nothing more than a little thought we can see that the case $k = 1$ is impossible. If $k = 2$ then any vertex in T since it is adjacent to every vertex in S has degree greater than or equal to $n-2$, a contradiction. If $k = 3$ then it must be that $n \leq 6$, in which case $(n-1)(n-2) + 4 > (n-1)(n - \frac{k}{2})$, another contradiction. The only non-contradictory case is $k = 0$, but this exactly the case $G = K_n$. □

Theorem 15. $T(H, n) = \binom{n-1}{2} + 2$ *where H is an n-cycle.*

Proof. First we will show by induction on n that $T(H, n) \leq \binom{n-1}{2} + 2$. The base case $n = 3$ can be verified by the reader.

Assume it to be true for $n = m-1$. Consider an arbitrary graph on $[m]$ call it G with size $\binom{m-1}{2} + 2$. If $G = K_m$ then obviously H is a subgraph of G, so assume $G \neq K_m$. Let x be the vertex guaranteed by Lemma 2, $\frac{m-1}{2} < deg(x) \leq m-2$. Consider the graph $G-x$. It must have size at least $(\binom{m-1}{2}+2)-(m-2)$, which very conveniently is equal to exactly $\binom{m-2}{2}+2$. By the inductive hypothesis $G-x$ has an $(m-1)$-cycle.

Let $v_0, v_1, ..., v_{m-1}$ be this cycle. Since $deg(x) > \frac{m-1}{2}$, by the pigeonhole principle x must be adjacent in G to both v_i and v_{i+1} for some i. It follows immediately that $v_0, v_1, ..., v_i, x, v_{i+1}, ..., v_{m-1}$ is an m-cycle in G, thus completing the induction.

To complete the proof we still have to show that $T(H,n) \geq \binom{n-1}{2}+2$, but this is not too difficult, for we simply have to consider K_{n-1} augmented by a single new vertex and a single new edge. Such a graph is of order n and size $\binom{n-1}{2}+1$ and it does not contain H as a subgraph. □

References

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