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Archimedes' Cattle Problem

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ARCHIMEDES' CATTLE PROBLEM

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If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

- Archimedes

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Archimedes, born nearly 2,300 years ago, is a man of legacy within the mathematical and scientific world. Known for his work in the field of hydrostatics, mathematical calculus, and geometry, Archimedes' mathematical advancements went unrivaled for 2,000 years until the time of Isaac Newton [5]. Archimedes' Cattle Problem is equally a test of mathematical prowess in modern history as it was during its inception. Born in Syracuse, many scholars believe that Archimedes studied in Alexandria as a young man. It was there that he was taught by the followers of Euclid and grew to know many of his life-long contemporaries such as Eratosthenes of Cyrene and Conon of Samos [4]. It is Eratosthenes, alongside other Alexandrian mathematicians, to whom he posed his famous "Cattle Problem":

"If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions…"

In his letter, Archimedes begins by describing the four herds of cattle on the Thrinacian island of Sicily. It is believed by many of that age that the herds of cattle belonged to the sun god Helios, an idea that is further backed by the 8th century B.C. piece Homer's Odyssey. Nearly 600 years before the time of Archimedes, Homer writes within the tale of Odysseus' journey an accounting of the cattle: "And you [Odysseus] will come to the island of Thrinacia. There in great numbers feed the cattle of Helios and his sturdy flocks, seven herds of cattle and as many fine flocks of sheep, and fifty in each…" [2, XII, p. 200].

The problem was presented to Eratosthenes in the form of a poetic letter and is broken up into two smaller problems. For the first problem, Archimedes provides a system of seven linear equations, each of which relates the number of a certain color and gender of cattle to another two. The system of linear equations can be derived from the letter as follows:

$$W = (\tfrac{1}{2} + \tfrac{1}{3})\, B + Y$$

$$B = (\tfrac{1}{4} + \tfrac{1}{5})\, D + Y$$

$$D = (\tfrac{1}{6} + \tfrac{1}{7})\, W + Y$$

$$\omega = (\tfrac{1}{3} + \tfrac{1}{4})\, B + \beta$$

$$\beta = (\tfrac{1}{4} + \tfrac{1}{5})\, D + \delta$$

$$\delta = (\tfrac{1}{5} + \tfrac{1}{6})\, Y + \psi$$

$$\psi = (\tfrac{1}{6} + \tfrac{1}{7})\, W + \omega$$

In this system of linear equations, W represents the number of white bulls, B the number of black bulls, Y the number of yellow bulls, D the number of dappled bulls, ω the number of white cows, β the number of black cows, ψ the number of yellow cows, and δ the number of dappled cows.

The system of linear equations was provided in such a form because Archimedes' method of writing fractions utilized only simple reciprocals (unit fractions). By modern terms, this is not a difficult computation. The simplest method of solving the system of linear equations in order to attain positive integer solutions, however, is via software. The software that I chose in order to solve the system is *Mathematica 8*, which is available in Western Washington University's Math Lab.

$$\text{Allow } S = (W, B, D, Y, \omega, \beta, \delta, \psi)\ W$$
$$\text{to be the one-dimensional solution vector, parametrized by } W$$

Utilizing *Mathematica 8*'s "Solve" function with the system of linear equations presented above computes the solution:

$$S = \left(1, \tfrac{267}{371}, \tfrac{297}{742}, \tfrac{790}{1113}, \tfrac{171580}{246821}, \tfrac{815541}{1727747}, \tfrac{83710}{246821}\right) W$$

Fractions of cattle is nonsensical, therefore I had to then compute the least-common multiple of the denominators of each value in the solution in order to solve for the lowest integer solution. *Mathematica 8* proved useful for this task as well using its "LCM" function.

$$LCM\ (1, 371, 742, 1113, 246821, 1727747, 355494, 246821) = 10, 366, 582$$

Each value in the fraction-solution is then multiplied by the least common multiple of its denominators, yielding the solution:

$$S = (10366482, 7460514, 4149387, 7358060, 7206360, 4893246, 539)$$

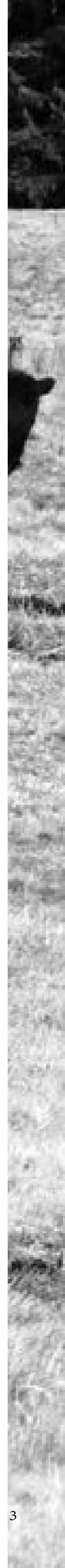
As such, this is the least positive integer solution of the first half of Archimedes' Cattle Problem, and the problem has an infinite number of solutions (each complete solution could be multiplied by any positive integer k and still be a solution).

The solution to the first part of Archimedes' problem, therefore, is any positive integer multiple of 50,389,082 cattle proportioned as described above [3]. In the letter, Archimedes acknowledges the intellect of anyone capable of solving the first part of this problem and goes on to test the wisdom of the mathematicians to a further extent with a second part:

"If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise. But come, understand also all these conditions regarding the cattle of the Sun."

The second problem continues to state that when mingled, the black and white bulls form a perfect square (that is, the sum of the number of black and white bulls is equal to a square number) and that the yellow and dappled bulls form an isosceles right triangle (that is, the sum of yellow and dappled bulls is a triangular number). These additional constraints add a further constriction to the valid values from the previous solution. First, we will address the constraint of the square number:

$$W + B = 10{,}366{,}482k + 7{,}460{,}514k = 17{,}826{,}996k = \text{a square number}$$



This number can be broken down and simplified using the Fundamental Theorem of Arithmetic (also known as the Unique Prime-Factorization Theorem), which states that every integer greater than 1 is either prime itself or is the product of primes. 17,826,996 is not prime; therefore it can be broken down as such using many methods. My method of choice is the simple Factor function on the TI-89 calculator. We then know that:

$$W + B = 17{,}826{,}996k = 2^2(3)(11)(29)(4657)k = \text{a square number, so}$$

$$k = 3(11)(29)(4657)r^2 = 4{,}456{,}749r^2$$

Next, we will address the constraint of the triangular numbers:

$$Y + D = (18492776362863, 32793026546940)\ r^2 = 51285802909803r^2 = m^2 + m/2 = \text{a triangular number}$$

This can be manipulated out to:

$$m^2 + m - 2(51285802909803r^2) = 0,$$

$$\text{which only has a solution if}$$

$$1 + 4(2)(51285802909803r^2) = \text{a square number}$$

And then simplified to a form of Pell's Equation:

$$1 + 410286423278424r^2 = v^2,$$
where v^2 is a square

Once again, using the Fundamental Theorem of Arithmetic, we can simplify it to:

$$1 + 2^3(3)(7)(11)(29)(353)(4657^2) = v^2,$$
and further to
$$1 = v^2 - 2^3(3)(7)(11)(29)(353)(4657^2)$$
given that v^2 is a square, it may absorb the square terms on the right side

The final equation is in a form known historically as Pell's equation, a Diophantine equation in the form of $x^2 - ny^2 = 1$ [1]. In the case of Archimedes' problem, the subject is faced with the simplified Pell equation $v^2 - 4{,}729{,}494r^2 = 1$ after all squares are absorbed [4]. When solved, the final answer in terms of the minimum number of total cattle possible is 202,545 digits [7].

It is interesting that Archimedes constructed this conundrum and posed it to his peers for several reasons. Solving the first problem was not out of the realm of possibilities for the intellect of the scholars at the Library of Alexandria. However, they would have struggled in the areas of representing large numbers and complicated Pell equations due to their limited resources. Their work would have been further restricted by limitation of the fraction notation used at the time [1]. These restrictions, however, may have been one of Archimedes' motivating points when constructing the problem because he had been fascinated with the possibility of representing large numbers. In his work *The Sand Reckoner*, Archimedes utilized a numbering system of base myriad (that is, base 100,000,000) in order to attempt to calculate the number of grains of sand that it would require to fill both the Earth and the universe [6]. It is unclear, however, that the mathematicians at Alexandria had any true hope of being able to solve the second half of the problem and being deemed truly wise, as the letter finishes:

"If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom."

Some historians believe that it "is hardly likely that the famous geometer of Syracuse or the Alexandrian mathematicians came anywhere near its solution… they probably displayed the equations involved and left the matter at that" [1, p. 237]. This is further backed by the notion that the earliest speculated claim of a solution was attributed to the famous mathematician and physicist Carl Gauss in 1830 [4], and that no complete solution to the problem was verified until 1965 when a research team from the University of Waterloo utilized a scientific computer in order to calculate their result [8].

Another issue encountered in the solving of his problem lies in the real-world logistics of the solution. The sheer magnitude of the number of cattle could not have fit within the confines of the Earth, let alone the island of Sicily [1]. In the computed solution, the number of bull outnumber the number of cows to such an extent that they could not possibly prosper. Lastly, the solution does not fit with the history of Sicily. Homer records seven herds of cattle on the island, contradictory to the four noted by Archimedes. The Cattle Problem has several real-world inconsistencies, and posed immense mathematical challenges to those at Alexandria, and Archimedes himself. Nevertheless, Archimedes' name has made it into households as a man who revolutionized modern mathematics, puzzled the greatest minds of his time and challenged even our own.

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