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The Coordinated Effects of Mergers in Differentiated Products Markets[1]

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[1] An early version of this paper was circulated as Kühn and Motta (1999): "The Economics of Joint Dominance". I am deeply indebted to Massimo Motta, whose original idea of studying the effects of product varieties as transferable assets in a collusion setting, is developed in this paper. Without him this paper would not have been written. Unfortunately, Massimo felt that his contribution to the final product did not warrant his appearing as a co-author. I would like to thank Bruno Jullien, Patrick Rey, and Ennio Stachetti for extremely helpful discussions on this paper. I have also benefited from comments by seminar audiences at the Federal Trade Commission, Free University of Brussels, Michigan State University, University of Toronto, and the University of Toulouse. The capable research assistance of Michael Rimler on the numerical part of the paper is gratefully acknowledged.

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Abstract

The paper addresses the issue of coordinated effects of mergers in the framework of a differentiated products model. Firms' assets are product varieties that can be sold individually or entirely transfered to another firm in a merger. We show that under symmetric optimal punishment schemes the highest feasible collusive price declines from any asset transfer to the largest firm as long as the size of the smallest firm is unchanged. In contrast, for fully optimal punishment schemes the prices of firms that get larger increase and those of firms that get smaller decrease. However, in all cases mergers are unprofitable unless the length of product lines is very asymmetric. We discuss the implications of the analysis for merger policy

JEL.: D43, K21, L13, L41.

Keywords: collusion, product lines, mergers, coordinated effects, joint dominance.

1 Introduction

In merger policy concerns about the impact of market consolidation on the likelihood of collusion have been more prominent in recent years. In the US this discussion is expressed as a concern about "coordinated effects" of mergers. In Canada and in Europe the same issue is raised under the label "joint dominance". Especially in European merger policy, regulators have attempted to use the joint dominance concept aggressively for blocking mergers in such prominent - and often controversial - cases as *Nestle-Perrier*, *Kali und Salz*, *Gencor-Lonrho*, and *Airtours*. The judgement by the European Court of First Instance annulling the European Commission decision to block a merger in *Airtours* has clarified that "joint dominance" analysis in European mergers has to be treated as coordinated effects analysis and thus has to be consistent with collusion theory.[1]

It has, however, been difficult to find empirically sound approaches to assessing the coordinated effects of mergers. In contrast to unilateral effects analysis in which there exists a sophisticated set of empirical techniques to approach the assessment of unilateral effects, no such consistent approach exists for coordinated effects. This has remained true even after a comprehensive recent review of such analysis by the Department of Justice in the US. These difficulties arise because of a fundamental difference between unilateral effects and coordinated effects. For unilateral effects analysis a firm is assumed to be short run profit maximizing before and after the merger. This allows a simple analysis of the change of the incentives to raise prices unilaterally after the merger. However, in coordinated effects analysis we are interested in the degree to which firms can move away from short run profit maximization towards monopoly pricing. This makes both the theoretical and empirical analysis more difficult. Indeed, we are largely lacking a proper theoretical framework that would allow us to make empirical inferences from market data.

In this paper we attempt a systematic analysis of the issues concerning the coordinated effects of asset transactions between firms, including mergers, in the framework of a differentiated goods model. For the assessment of coordinated effects we are interested in the mapping between asset distribution and the incentives to collude. The assets of interest in our model are brands or varieties that are owned by a specific firm. We model collusion as an infinitely repeated game. Since there is a very large set of equilibria for such game there is no sense in which we could do compara-

[1] For discussions concerning the issues raised in this debate see Kühn (2002 a, b).

tive statics for a particular equilibrium. For this reason we have to perform comparative statics for the complete set of equilibrium outcomes (see Abreu, Pearce, Stacchetti (1990) or Kandori (1992)). Unfortunately, such comparative statics can only be done analytically when equilibrium value sets that are generated from changing the parameters are nested. This is not the case for our model.

We therefore proceed in two steps.We first generate some intuition for the forces at work in the model by analyzing symmetric optimal punishment equilibria. Then we determine the comparative statics of the full equilibrium value set numerically for one particular example.

For symmetric optimal punishment strategies we show that the scope for collusion is determined only by the incentives of the largest firm to deviate from the most severe punishment price and of the smallest firm to deviate from the most collusive price. We go on to show that asset acquisitions by the smallest firm in the market (including mergers) will facilitate collusion and raise the most profitable collusive price, while asset acquisitions by the largest firm will tend to undermine collusion and lower the most profitable collusive price. The reason is that a small firm reduces its incentive to deviate from a high price by becoming bigger because the amount of demand it wins over from a given price cut decreases when it becomes bigger relative to the market. Conversely, a large firm will have greater difficulty to credibly punish when it becomes larger relative to the market. Hence, the effectiveness of punishments are undermined. For these reasons mergers by the largest firm can lead to the counterintuitive result that the merger reduces the highest achievable collusive price. Furthermore, asymmetry increasing mergers will not be profitable in such settings because the joint profits of the merging companies are decreased.

We then discuss the robustness of these comparative statics results relative to fully optimal punishments. For this purpose we numerically calculate the effects of asset transfers in a duopoly model. To obtain a clear set of comparative statics results, we calculate the profit, price and welfare changes, when firms select the equilibrium played using a Nash Bargaining solution from the equilibrium value set. We show that, as predicted by the symmetric optimal punishment equilibrium, profits decline as fairly symmetric firms become more asymmetric. However, for sufficient asymmetry the increase in asymmetry leads to profit increases as firms are exhibiting close to best response pricing.

The comparative statics of prices are more complicated. When the smaller firm gets smaller its prices always decline. However, the prices of the larger firm increase starting from fairly symmetric distributions of varieties.

In markets in which collusion is relatively easy, this may lead to average prices rising above the monopoly price. Because of this effect, all prices may fall when markets with very asymmetric asset holdings are further consolidated. Under collusion the consolidation of very asymmetric markets to a monopoly may therefore even be welfare improving.

We then explore the policy implications of the theory for merger policy. The most important result is that firms have no interest to induce slight asymmetries through asset transactions since this is profit reducing. This means that asymmetry increasing mergers are unlikely to occur for market power reasons. But there are also further lessons for policy. First, we show that if a firm is large enough relative to the rest of the market it cannot credibly participate in a collusive scheme. This gives new meaning to the legal definition of a dominant firm as a firm "that can act independently of the market". Indeed a dominant firm in our context will not be able to collude precisely because it always has an incentive to unilaterally keep prices up, making it impossible to punish smaller firms for deviating from collusive outcomes. This implies that for practical competition policy purposes single firm dominance and multi-firm dominance should be mutually exclusive. Similarly one can show that it always pays all firms not to include very small firms in a collusive arrangement. This corresponds to the insight that optimal collusion would allow a small firm to price arbitrarily close to its short run best response, even if larger firms increase prices signifciantly above the best response price. These two results can be taken as building blocks for a policy rule by which joint dominance is assessed for a subset of firms that are much larger than others and relatively similar within the group. This justifies the focus of policy on two-firm, three-firm, or four firm dominance in markets that may have many more firms.

We can also show that coordinated effects should not be analyzed market by market. By arguments analogous to the analysis of the impact of multi-market contact on collusion (see Bernheim and Whinston 1990), asymmetries in one market may be compensated by offsetting asymmetries in another market. We give a simple example in which a merger would produce no overlap between the firms, but where it has strong adverse coordinated effects. Our model also allows us to analyze the persistence of dominant or jointly dominant constellations. In static models there is a tendency for the largest firm to be the most likely to add new products to the product line, reinforcing single firm dominance. If firms play optimally collusive equilibria, then smaller firms within the optimal collusive group will have larger incentives to add a product to the product line because this reduces the incentive problems for collusion and increases the price. Hence, joint dominance is also self reinforcing.

Previous research has done very little to provide theoretical or empirical underpinnings for dealing with the issue of coordinated effects of mergers. While the issue of asset transfers in mergers has been discussed for static models in the work of Farrell and Shapiro (1990 a,b) and McAfee and Williams (1992), asymmetries between firms in models of collusion have traditionally only been based on parametric differences in costs (Harrington 1991, Rothschild 1999) or discount factors Harrington (1989).

The first paper that discusses coordinated effects of mergers based on asset transfers is Compte, Jenny, and Rey (2002). Their paper is inspired by the Nestle-Perrier transaction. They look at the effects of asset transfers and mergers in the context of collusion in a Bertrand-Edgeworth homogeneous goods model with capacity constraints and calibrate their model to the data from the Nestle-Perrier case. However, their model assumes uniform reservation prices for all customers. For this reason optimal collusion only occurs at the reservation price or no collusion can be sustained at all. This excludes the possibility of meaningful price effects. Vasconcelos (2004) re-examines the analysis of capacity transfers of Compte et al. (1992) when cost functions are of the type in McAfee and Williams (1992) and firms set quantities instead of prices. This paper also has no analysis of the price effects of mergers.

Our paper in contrast deals with *brands* as assets. This means that we necessarily have to work in a differentiated products model on the demand side. While the paper is similar in spirit to Compte et al. (2002), our setting allows us to study the price effects of mergers. Instead of focusing the analysis on whether collusion is "more or less likely" this analysis is in the spirit that one always has to consider the whole set of dynamic equilibria. Therfore the more important question to ask is how collusive price change when market structure changes. Our paper is the first paper that derives results for the whole equilibrium set, i.e. it considers full optimal punishment equilibria and shows that the comparative statics in price for firms that grow larger and those that get smaller typically go in opposite directions. This cautions against the use of variants of symmetric optimal punishment schemes in asymmetric models. With this analysis the paper is also the first to use computational methods to obtain comparative statics results for changes in the collusive price for asymmetric models. We demonstrate the practical applicability of the Abreu-Pearce-Stacchetti algorithm for applied modelling in industrial organization.

The model is introduced in section 2. Section 3 contains the main analysis of the model. Section 4 presents the numerical analysis of the full equilibrium value set. Section 5 discusses implications for merger policy. Section 6 concludes.

2 The Model

Consumer demand for n varieties of the product is given by the set of demand functions $D_i(\mathbf{p})$, $i = 1, ..., n$, where $\mathbf{p}$ is the n-dimensional vector of prices. The demand system is symmetric in prices. Individual demand is decreasing in own price and increasing in the prices of other varieties. There are m firms in the market, $2 \leq m \leq n$. A firm k can produce multiple varieties, each under the same constant marginal cost $c > 0$. It is characterized by its "product line", namely the set K of varieties it owns. We denote the length of the product line, i.e. the number of varieties in K, by n_K. The set K^c is the set of varieties firms other than k have in their product lines. A variety is a proprietary asset of a firm, so that no two firms can produce the same variety, i.e. $K \cap K^c = \emptyset$ and $n_K + n_{K^c} = n$ for all K. By the assumption of demand symmetry, firms differ only in the length of their product lines. We denote the largest (smallest) firm in the market by $\bar{k}$ ($\underline{k}$), which produces $\bar{n}$ ($\underline{n}$) varieties.

Let $\mathbf{p}_K$ be the n_K-dimensional vector of prices of all varieties in K so that $\mathbf{p} = (\mathbf{p}_K, \mathbf{p}_{K^c})$. We assume that for any set of products K, the total revenue $\mathbf{p}_K \cdot \mathbf{D}_K(\mathbf{p}_K, \mathbf{p}_{K^c})$ is strictly concave in $\mathbf{p}_K$ for every $\mathbf{p}_{K^c}$.[2] This assumption implies that there is a unique monopoly price $p^m(n)$ that a monopolist controlling all n varieties would set for all varieties. We also assume strategic complementarity and that best responses are contraction mappings. To state this formally, let $D^K(p_K, \mathbf{p}_{K^c})$ be the demand for variety $i \in K$ if all products $j \in K$ are priced at p_K. Then assumption 1 guarantees these properties[3]:

Assumption 1: *For all K, demand satisfies the following properties: (i) For every n_{K^c}-vector $\varepsilon > 0$, $\frac{\partial^2[\ln D^K(p_K,\mathbf{p}_{K^c}+a\boldsymbol{\varepsilon})]}{\partial p_k \partial a}\Big|_{a=0} > 0$ (strategic complementarity) and (ii) $\left|\frac{\partial \ln D^K(p_K,\mathbf{p}_{K^c})}{\partial p_K}\right| > \sum_{j\in K^c} \frac{\partial \ln D^K(p_K,\mathbf{p}_{K^c})}{\partial p_j}$ and $\left|\frac{\partial^2 \ln D^K(p_K,\mathbf{p}_{K^c})}{\partial p_K^2}\right| > \sum_{j\in K^c} \frac{\partial^2 \ln D^K(p_K,\mathbf{p}_{K^c})}{\partial p_k \partial p_j}$ (own price effects dominate cross price effects).*

In each period all firms first observe a public signal $\sigma \in [0, 1]$, which

[2] Note that by the symmetry of demand we can always relable demand for each firm K in such a way that the first K arguments in the demand vector $\mathbf{D}_K(\mathbf{p}_K, \mathbf{p}_{K^c})$ refer to the prices of firm K.

[3] See Vives (1999) for an overview of such results.

is not payoff relevant for that period.[4] Then each firm k simultaneously selects a price vector $\mathbf{p}_K \in [0, \bar{P}]^{n_K}$, where $\bar{P} >> p^m(1)$, the price a monopolist in a market with only one variety would set.[5] For a given period the payoff per variety of firm k can be written as $\Pi^k(\mathbf{p}_K, \mathbf{p}_{K^c}) = \frac{1}{n_K}\sum_{j\in K}(p_j - c)D_j(\mathbf{p}_K, \mathbf{p}_{K^c})$. By the symmetry and concavity assumptions on demand, the short run best response function $\mathbf{p}^*_K(\mathbf{p}_{K^c})$ in the stage game has the property that the firm sets the same price for every one of its varieties, i.e. $p^*_i(\mathbf{p}_{K^c}) = p^*_j(\mathbf{p}_{K^c})$ for all $i, j \in K$. Assumption 1 implies that this stage game has a unique Bertrand equilibrium for any distribution of varieties across the m firms.

We analyze the set of equilibria for the infinitely repeated version of this game. A history of the game up to time t, $H(t) \in \left[0, \bar{P}\right]^{n\cdot(t-1)} \times [0, 1]^{t-1}$, includes all past prices set for all products and all past realizations of the public signal σ. A strategy for firm K is a sequence of functions $\widetilde{p}_K = \{\mathbf{p}_K(H(t), \sigma_t)\}_{t=1}^{\infty}$, $\mathbf{p}_K(H(t), \sigma_t) : \left[0, \bar{P}\right]^{n\cdot(t-1)} \times [0, 1]^t \to \left[0, \bar{P}\right]^{n_K}$, for all $t \geq 1$. Firms discount future profits with discount factor $\delta \in (0, 1)$. The per variety average discounted value to firm k from a given strategy profile $\widetilde{p}$ is given by

$$v_k(\widetilde{p}) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\frac{1}{n_K}E\left\{\sum_{j\in K}(p_j(t) - c)D_j(\mathbf{p}_K(t), \mathbf{p}_{K^c}(t))\right\}, \qquad (1)$$

where E is the expectations operator taken over future realizations of the signal σ. We are interested in analyzing the set of per variety average value vectors for firms, V, that can be sustained in subgame perfect equilibria in this game. In particular, we want to analyze how the "most collusive" outcomes of this game change as we transfer varieties from one firm to another. Formally, this means that we want to perform a comparative statics analysis of the Pareto frontier of the equilibrium value set and of the price vectors that support those values. While the relevant theory for the derivation of equilibrium value sets is well-established (see Abreu 1988; Abreu, Pearce, and Stacchetti 1990), it is often impossible to derive analytical results about the equilibrium value set. In the literature such comparative statics have

[4] We introduce this public randomization device for purely technical reasons. It guarantees a convex equilibrium value set, which allows us to encompass cases in which one period punishments cannot be made arbitrarily large as in the linear differentiated goods model or versions of the logit model. The assumption also permits us to use the Abreu, Pearce, Stachetti (1991) algorithm for computing equilibrium value sets.

[5] This is equivalent to the best response price if rivals charge infinite prices. It is an upper bound to the price any firm would charge in the market.

only been derived in cases in which the value sets are nested (see Abreu, Pearce, and Stacchetti 1990, Kandori 1992, Kühn and Rimler 2004). This is not the case for our analysis. We have therefore chosen a two step approach to analyzing the problem. We first analyze the restricted set of "symmetric optimal punishment equilibria". This is the set of equilibria in which all firms set the same price in their equilibrium strategies in every given period in every continuation equilibrium of the game. We fully analyze this constrained set of equilibria. We then discuss the limitations of the analysis and show numerically to what extent the basic insights carry over to unrestricted optimal punishment schemes.

3 Asset Distributions and the Incentives to Collude

3.1 Symmetric Optimal Punishment Schemes

In symmetric optimal punishment equilibria the analysis of incentives to deviate from the equilibrium strategy is particularly simple. We only need to consider two prices in any period: the price p_K that firm k has to decide on and the price p that all other firms set for all other varieties according to the proposed equilibrium strategy. We can therefore work with the simplified profit function $\pi^k(p_K, p) = (p_K - c)D(p_K, p, n^K)$, where $D(p_K, p, n^K)$ represents the demand where all rival varieties are priced at p and firm k sets price p_K for all its n^K varieties. Since, by symmetry, $\pi^K(p,p) = \pi^{K'}(p,p)$ for all K' we drop the superscript for symmetric price vectors. In symmetric optimal punishment equilibria the profits per variety are therefore the same for all firms and all firms have the same profit ranking over the equilibrium value set.

We now show that the set of symmetric optimal punishment equilibria has a particularly simple characterization. Let $\underline{v}$ be the average per variety value in the lowest continuation value equilibrium and let P^c be the price charged in every period of the highest value continuation equilibrium.[6] Furthermore, let $p^*(p, n^K)$ be the short run best response price of a firm controlling the varieties in K to a price p charged by all others. Then the incentive condition for the highest value equilibrium can be written for each firm k as:

$$\pi(P^c, P^c) \geq (1-\delta)\pi^K(p^*(P^c, n^K), P^c, n^K) + \delta\underline{v}. \tag{2}$$

[6] The lowest average continuation value is then given by $\frac{\underline{v}}{1-\delta}$. This is a standard way in the literature on repeated game to simplify notation.

The average profit from complying with setting the most profitable collusive price P^c must exceed the average profits from obtaining best response profits to P^c for one period and then reverting to the lowest value continuation equilibrium. The incentive condition for the worst equilibrium can be written for each firm k as

$$\underline{v} \geq (1-\delta)\pi^K(p^*(p_L, n^K), p_L, n^K) + \delta \underline{v}, \tag{3}$$

where sticking to p_L in the first period in which punishment is triggered and obtaining $\underline{v}$ must be more profitable than optimally deviating from p_L and restarting punishment at continuation value $\underline{v}$. Without loss of generality we can decompose $\underline{v}$ into the profits from charging p_L in the first period and then obtaining some continuation value $\hat{v}$ in the future: $\underline{v} \equiv (1-\delta)\pi(p_L, p_L) + \delta\hat{v}$. We then have a very simple characterization of optimal punishment strategies:

Lemma 1 *The incentive constraint for the lowest value equilibrium is always binding for at least one firm k. Generically, we have either $\hat{v} = \pi(P^c, P^c)$ and $p_L > 0$ or $\underline{v} < \hat{v} < \pi(P^c, P^c)$ and $p_L = 0$ for that firm.*

Proof. See the Appendix. ■

To see the basic argument suppose that $p_L > 0$ and $\hat{v} < \pi(P^c, P^c)$. For any given $\underline{v}$ we can then generate the same value by using a lower p_L and a higher $\hat{v}$. The incentive constraint for P^c is not affected, but incentive constraint (3) would be strictly relaxed because, by the envelope theorem, $\pi(p^*(p_L), p_L)$ strictly increases in p_L. However, this would contradict the assertion that $\underline{v}$ is the lowest equilibrium value. This implies that all optimal punishments are always maximally front loaded: optimal collusion requires to generate $\underline{v}$ by punishing with the lowest feasible p_L in the period after a deviation.[7]

To sustain any price p in a given period in a symmetric optimal punishment equilibrium, the incentive to deviate,

$$\phi^K(p^*(p, n^K), p, n^K) \equiv \pi^K(p^*(p, n^K), p, n^K) - \pi(p, p), \tag{4}$$

has to be smaller than the profit loss from switching from the best collusive equilibrium with per period profits $\pi(P^c, P^c)$ for ever to an equlbirium with the worst average continuation value of $\underline{v} = (1-\delta)\pi(p_L, p_L) + \delta\hat{v}$. The

[7] The property $p_L > 0$ is always guaranteed under the assumption $\lim_{p\to 0} D(p, p) = \infty$, which holds when the marginal utility of consumers at zero consumption is strictly positive.

incentive compatibility conditions for each firm K for sustaining the highest and the lowest possible collusive price can then be rewritten as:

$$\phi^K(p^*(P^c, n^K), P^c, n^K) \leq \frac{\delta}{1-\delta}[\pi(P^c, P^c) - \underline{v}] \tag{5}$$

and:

$$\phi^K(p^*(p_L, n^K), p_L, n^K) \leq \delta[\hat{v} - \pi(p_L, p_L)], \tag{6}$$

where $\hat{v} = \pi(P^c, P^c)$ if $p_L > 0$ and $\hat{v} < \pi(P^c, P^c)$ otherwise. The incentive constraint (6) must be an equality for the firm k with the greatest incentive to deviate $\phi^K(p^*(p_L, n^K), p_L, n^K)$ by Lemma 1. Similarly (5) must be binding for the firm with the largest incentive to deviate from that price whenever $P^c < p^m(n)$. Note that only the incentives to deviate ϕ^K are affected by the relative length of the product line in our model, namely through the properties of the short run best response function. In the next subsection we characterize the incentives to deviate before we present our main results in the following subsection.

3.2 Firm Size and the Incentives to Deviate from Collusion

The difference in the length of the product line only affects the incentives to deviate $\phi^K(p^*(p, n^K), p, n^K)$ for any given price p firms may want to sustain. The incentives for deviation from the prices P^c and p_L will therefore depend primarily on the properties of the *short run* best responses $p^*(p, n^K)$ for any firm K. These properties follow directly from the assumption of strategic complementarity and the fact that the best response function of firm K is a contraction:

Lemma 2 *The best response price $p^*(p, n^K)$ is strictly increasing in p (whenever $D(p^*(p, n^K), p, n^K) > 0$), $p^m > p^*(p^m, n^K)$, and $p^*(c, n^K) > c$. For each n^K there exists a unique price $\hat{p}(n^K) > c$ such that $p^*(\hat{p}(n^K), n^K) = \hat{p}(n^K)$. It is strictly increasing in n^K.*

Proof. See the Appendix. ■

By strategic complementarity, the higher the prices of competitors are, the higher a price the firm will charge. The boundary properties of the best response function are also obvious from our assumptions but important to note here: First, if the prices of competitors are too high the firm has an incentive to undercut rivals. If the prices are too low, then it will have an incentive to charge a higher price. The uniqueness of $\hat{p}(\hat{n})$ then follows trivially from the assumption that own effects dominate cross effects in demand so that best response functions are contraction mappings. The result that the price $\hat{p}(n)$ is increasing in n is also based on simple intuition: The larger the firm the more pricing externalities between products the firm takes into account and the less aggressive its pricing policy is.

The previous Lemma allows a simple characterization of the short run gains from undercutting at the best response prices that will be central to the characterization of optimal symmetric punishment schemes:

Lemma 3 *Suppose* $n^K > n^{K'}$, *then*
(a) $\phi^K(p^*(p, n^K), p, n^K) > \phi^{K'}(p^*(p, n^{K'}), p, n^K)$, *if* $p < \hat{p}(n^{K'})$ *and*
(b) $\phi^K(p^*(p, n^K), p) < \phi^{K'}(p^*(p, n^K), p)$, *if* $p > \hat{p}(n^K)$.

Proof. See the Appendix. ■

Lemma 3 describes the short run incentives of firms to deviate from a common symmetric price across all firms and varieties. This incentive depends on the level of the price p. At high levels of the price p all firms have an incentive to deviate to a lower price. However, the incentives of a small firm to cut the price are larger than the incentives of a large firm. The small firm gains customers from more varieties than the large firm does and therefore generates a larger quantity effect from a price decrease from a given price cut than a larger firm does. For exactly the same reason this situation is reversed when the price is sufficiently low so that all firms have an incentive to unilaterally raise the price above p. The larger firm loses proportionately less demand to rivals from a price increase than a smaller rival so that the profits per variety are higher when a large firm raises the price should it be pricing below its best response price.

However, for a complete description of the incentives to deviate we also have to consider the possibility that P^c or p_L are in the range $(\underline{n}, \bar{n})$. In this case some firms have incentives to deviate upwards and some downwards. We show in our first proposition that this slight complication does not materially affect the basic insight that only two incentive constraints matter for our later analysis: that of the smallest firm to stick to the most profitable sustainable collusive price P^c and that of the largest firm to stick to the optimal punishment price p_L:

Proposition 1 *In any symmetric optimal punishment equilibrium: (a) the smallest firm* $\underline{k}$ *always has the largest incentive to deviate from the most profitable collusive price* $P^c(\delta)$*, and (b) the largest firm* $\bar{k}$ *has the greatest incentive to deviate from the optimal punishment price* p_L.

Proof. *See the Appendix.* ■

Proposition 1 captures the fact that a collusive price maintained by all firms must always relax competition relative to the firm that has the largest incentives to undercut in the short run. In our model it is the small firm that always has a greater potential to capture customers from other firms when deviating from the collusive price. The smallest firm is, in our model, the firm with the smallest degree of market power, since it controls the smallest number of varieties in the market. As a result, it also has the smallest incentive to contribute to the public good of raising prices.

For exactly symmetric reasons, the incentives to deviate are exactly reversed when the equilibrium calls for punishments. Now the largest firm has the greatest incentive to deviate from the punishment strategy. The reason is that in the short run it can make the highest profits by deviating to a *higher* price exploiting its market power. The large firm can extract more rent from consumers because it controls a larger share of varieties internalizing more externalities in price setting. It therefore has to give up more rent per variety than smaller competitors when agreeing to comply with a punishment strategy.

In any optimal punishment equilibrium some firm must have an incentive to raise the price in the short run, otherwise the punishment could be made more severe without violating incentive constraints. But then the firm that has the largest incentives to raise prices will be the one that will put a limit on the severity of equilibrium punishments. This is in our case the largest firm. In our model a larger firm has more market power in the sense of controlling more varieties, and as a consequence has a greater incentive to raise prices. The difficulty for collusion that arises from the presence of a large firm is that such a firm cannot as credibly threaten smaller firms to punish them for deviations from collusion. Punishment is much more costly for a large firm than for a small firm. It is precisely the market power of the large firm that undermines its ability to collude with a small firm.

Note that these results appear to be in stark contrast to those of Compte et al. (2002) on homogeneous goods markets with capacity constraints. In their model a large firm has the largest incentive to deviate from collusion, while small firms have smaller ability to credibly punish the larger firm. The reason is that large firms have large incentives to fill capacity, giving them greater incentives to undercut the collusive price. On the other hand, small firms are capacity constrained and cannot credibly drop the price as much as the large firm in a punishment period. Our results are driven by the opposite incentive structure. They indicate that it depends very much on the assets under consideration which incentive problem is more important for the large or the small firms. However, as we show below the policy implications for the analysis of assets transfers as in mergers and remedies based on asset sales will still be very similar.

3.3 The Scope for Collusion

The results of the previous subsection have shown that the only incentive constraints that matter for symmetric optimal punishment schemes are those of the smallest firm not to deviate from the optimal collusive price and of the largest firm not to deviate from the optimal punishment price. In this subsection we characterize first characterize the optimal collusive price for every δ, $P^c(\delta)$. On the basis of this characterization result we can discuss in the following subsection how asset transfers and asset acquisitions change the maximally collusive price and the overall scope for collusion.

In an optimal symmetric punishment equilibrium the highest profit equilibrium is obtained by charging $P^c(\delta)$ until some firm deviates from that price. After any deviation in some period firms charge p_L in the next period and then obtain an average continuation value $\hat{v} \leq \pi(P^c(\delta), P^c(\delta))$ in the following period. A direct approach to this problem would therefore be to fix δ and find the most profitable $P^c(\delta)$ achievable under the incentive constraints by adjusting p_L (and $\hat{v}$). It turns out to be easier to look at a dual problem: Fix some pricep^c, $\hat{p}(n_{\underline{K}}) \leq p^c \leq p^m$, to be charged when no deviation has occured and search for the lowest δ, denoted as $\bar{\delta}(p^c)$, for which this price is sustainable under the relevant incentive constraints by adjusting p_L (and $\hat{v} \leq \pi(p^c, p^c)$, where $\hat{v} = \pi(p^c, p^c) \iff p^c > 0$). The relevant incentive constraints are given by (5) and (6) where p^c replaces P^c. The highest $p^c \leq p^m$ that is just sustainable at $\bar{\delta}$ is $P^c(\bar{\delta})$.

As we have shown in proposition 1 we only need to focus on the incentive condition (5) for the smallest firm and the incentive condition (6) for the largest firm. Rewrite (5) by substituting $\underline{v} = \pi^{\bar{K}}(p^*(p_L, n_{\bar{K}}), p_L, n_{\bar{K}})$ as $\phi^{\underline{K}} = \frac{\delta}{1-\delta}[\pi(p^c, p^c) - \pi^{\bar{K}}(p^*(p_L, n_{\bar{K}}), p_L, n_{\bar{K}})]$. The locus of this constraint in (p_L, δ) space for a given price p^c, denoted by $p_L^{\underline{K}}(\delta)$, is plotted as the monotonically increasing schedule in Figure 1. Clearly, as $\delta \to 1$, incentive constraint (5) is just binding in the limit only if $\pi(p^c, p^c) - \pi^{\bar{K}}(p^*(p_L, n_{\bar{K}}), p_L, n_{\bar{K}}) \to 0$, so that that $\lim_{\delta \to 1} p_L^{\underline{K}}(\delta) \in [\hat{p}(\underline{n}), p^c)$ in the limit. As δ is decreased the punishment price has to decrease to leave firm $\underline{K}$ indifferent between charging the collusive price or deviating to the short run best response. Clearly, there will be some $\delta > 0$ at which the $p_L = 0$ in order to just make firm $\underline{K}$ indifferent between charging p^c and deviating to a best response. For lower δ no collusion is feasible. For all combinations of δ and p_L to the right and below $p_L^{\underline{K}}(\delta)$ no firm has an incentive to deviate

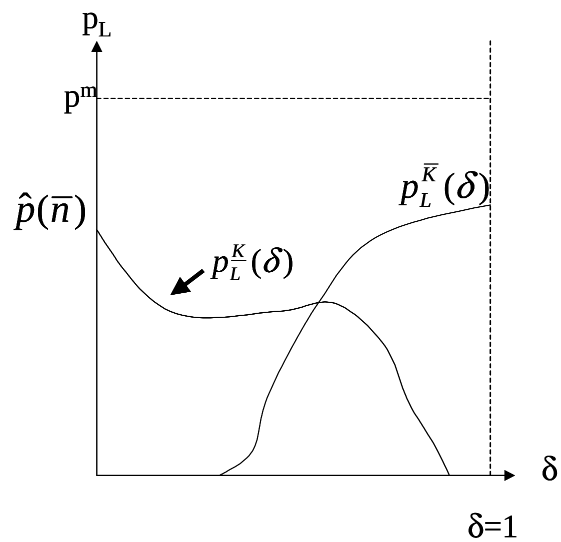

Figure 1:

from p^c, for combinations of δ and p_L to the left and above this schedule p^c cannot be sustained.

Now consider the severest punishment that can be undertaken so that the largest firm $\bar{K}$ is still willing to charge the punishment price instead of deviating to a higher price. Consider the schedule generated by $\phi^{\bar{K}} = \delta[\pi(p^c, p^c) - \pi(p_L, p_L)$. This is represented by the non-monotonic schedule $p_L^{\bar{K}}(\delta)$ in figure 2. At $\delta = 0$ the only sustainable price is the fixed point in the largest firm's best response function $\hat{p}(\bar{n})$. By Lemma 3 we know that the punishment price has to fall strictly below $\hat{p}(\bar{n})$, so that $p_L^{\bar{K}}(\delta)$ must be initially falling close to $\delta = 0$.[8] In contrast, it can be shown that for δ close enough to 1, $p_L^{\bar{K}}(\delta)$ must be arbitrarily close to 0. In Figure 2 we have drawn a case in which $p_L^{\bar{K}}(\delta) = 0$ for some $\delta < 0$. For preferences with strictly positive marginal utility at zero $p_L^{\bar{K}}(\delta)$ approaches zero as δ goes to 1. Note that for all combinations of δ and p_L below and to the left of $p_L^{\bar{K}}(\delta)$ no collusion can be sustained.

We now show that if the schedules $p^{\bar{K}}(\delta)$ and $p_L^K(\delta)$ intersect there is only one intersection, which determines the lowest δ at which p^c can be

[8] Indeed, It can be shown that $p_L^{\bar{K}}(\delta)$ decreases at an infinite rate at $\delta = 0$.

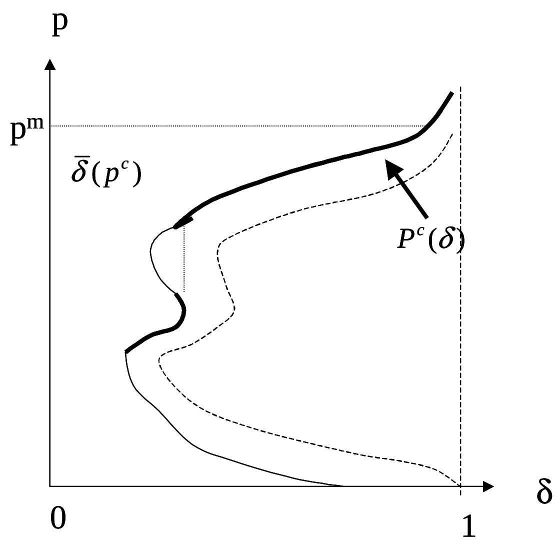

Figure 2:

sustained, denoted by $\bar{\delta}(p^c)$. If they do not $\bar{\delta}(p^c)$ is given by the δ at which $p_{\frac{K}{L}}(\delta) = 0$. Note that any p^c that can be sustained forever at $\bar{\delta}(p^c)$, can be sustained for any higher δ as well.

Lemma 4 *For every $p^c > c$ there exists a unique $\bar{\delta}(p^c)$ such that this price can be sustained forever at a maximal punishment equilibrium if and only if $\delta \geq \bar{\delta}(p^c)$.*

Proof. See the Appendix. ■

Figure 2 shows as a dotted line the function $\bar{\delta}(p^c)$. Note that this function does not necessarily have a unique minimum. It is perfectly possible that there are multiple local minima of $\bar{\delta}(p^c)$ because locally reducing the price p^c may tighten the incentive constraint of the large firm more than it relaxes the constraint of the small firm. However, we nevertheless can obtain an unambiguous result about the change in the optimally collusive price $P^c(\delta)$ as a function of δ. We have drawn this schedule as the solid line in Figure 2.

Note that by Lemma 4 we can sustain a price p^c for ever if and only if δ is weakly above the locus $\bar{\delta}(p^c)$. But then Figure 2 allows us to read off the most profitable collusive price P^c by fixing δ and moving up in the figure.

Either we get to a point at which p^m is sustainable as a collusive price or we get to a point on the locus $\bar{\delta}(p^c)$ such that no higher price can be sustained at that discount factor. Note that for $\delta \geq \bar{\delta}(p^m)$, the optimally collusive price will be p^m, the monopoly price. For lower δ, $P^c(\delta)$ will increase in δ, with the possibility of upward jumps to local minima of the function $\bar{\delta}(p^c)$. The next proposition states this result formally:

Proposition 2 *There exists* $\underline{\delta}$ *and* $\bar{\delta}(p^m)$ *with* $0 < \underline{\delta} < \bar{\delta}(p^m) < 1$ *such that there exists an optimal symmetric punishment equilibrium if and only if* $\delta \in [\underline{\delta}, 1]$*. The optimal collusive price* $P^c(\delta)$ *is equal to* p^m *for all* $\delta \in [\bar{\delta}(p^m), 1]$ *and it is strictly increasing on* $\delta \in [\underline{\delta}, \bar{\delta}(p^m)]$*, piecewise continuous with at most a finite number of upward jumps. The optimal punishment price* $P_L(\delta)$ *is strictly decreasing in* δ *on* $\delta \in [\underline{\delta}, 1]$*.*

Proof. See the Appendix. ■

Proposition 2 gives the intuitive result that the optimally collusive price will fall as δ is decreased. It gets harder to give the small firm incentives not to deviate from a high collusive price and at the same time it gets more difficult to make the large firm enforce harsh punishments on a small deviating firm. This intuitive monotonic relationship between the price level and δ, the parameter that measures the ease of collusion, is a nice property of our model that allows us to study the price effects of asset transactions, something that is not possible in the model of Compte et al.(2002).

3.4 The Impact of Asset Transfers and Mergers on the Optimal Collusive Price

We can now state the central results in the paper that characterize the effect of asset transfers and mergers on the optimal collusive outcome. In this model we will not only obtain results on the change in the range of discount factors for which some collusion is sustainable at a symmetric optimal punishment equilibrium, but also obtain insights into the direction of price changes in the highest profit equilibrium that result from the asset transfers. Nevertheless, we will start with the more traditional question whether the range of discount factors for which some degree of collusion can be sustained changes with the distribution of varieties among the firms.

This question comes down to a comparative statics exercise on $\underline{\delta}$. Intuitively, asset transactions should only matter if they change the size of the largest or smallest firm in the industry. If the largest firm increases the number of varieties leaving the number of varieties owned by the smallest firm unchanged, then the largest firm has less of an incentive for punishments and $\underline{\delta}$ increases. Conversely , if the size of the smallest firm increases, it will have less of an incentive to cut the price and hence, collusion will be sustainable even at a lower δ. To make this argument formal let κ_1 and κ_2 be the capacity distributions before and after the transaction respectively and define $\underline{\delta}(\kappa_l)$ as the lowest discount factor at which some collusion is sustainable for asset distribution κ_l. Then we have:

Proposition 3 *(a) Suppose that $\underline{K}$ is the same under κ_1 and κ_2. Then $\underline{\delta}(\kappa_1) > \underline{\delta}(\kappa_2)$ if and only if $\bar{K}_1 > \bar{K}_2$.*

(b) Suppose that $\bar{K}$ is the same under κ_1 and κ_2. Then $\underline{\delta}(\kappa_1) > \underline{\delta}(\kappa_2)$ if and only if $\underline{K}_1 < \underline{K}_2$.

(c) If both $\underline{K}$ and $\bar{K}$ are the same under κ_1 and κ_2, then $\underline{\delta}(\kappa_1) = \underline{\delta}(\kappa_2)$

Proof. See the Appendix. ■

This proposition shows that increases in the size of the largest firm will make collusion more difficult to sustain in the market, while increases in the size of the smallest firm will make it easier to sustain collusive outcomes. This should be expected from our basic intuition.

However, we can now go one step further and show how asset transactions impact on the most profitable collusive prices that are sustainable for any given discount factor. This is the more important comparative static since there is no reason to believe that discount factors should change as a result of a merger. Obviously, a comparison of the most collusive prices under two asset distributions only makes sense when before and after the transaction some collusion was feasible in the industry. Similarly, there will be no impact on the price if the monopoly price can be sustained before and after the transaction. To limit our discussion to the relevant range of discount factors, define $\underline{\delta}_{\max} \equiv \max\{\underline{\delta}(\kappa_1), \underline{\delta}(\kappa_2)\}$ as the lowest discount factor for which collusion can be sustained both before and after the asset transaction. Similarly, let $\bar{\delta}(p^m, \kappa)$ be the be the lowest discount factor for which the monopoly price can be sustained under asset distribution κ and define $\bar{\delta}_{\min} \equiv \min\{\bar{\delta}(p^m, \kappa_1), \bar{\delta}(p^m, \kappa_2)\}$. We are interested in the range of discount factors $\delta \in (\underline{\delta}_{\max}, \bar{\delta}_{\min})$.

To gain intuition let us start with asset transactions that do not change the size ranking of the largest and smallest firms. Consider first asset acquisitions by the largest firm $\bar{K}$ from other firms than the smallest firm, leaving the identity of the smallest firm unchanged. This means the incentive constraint for the smallest firm is unchanged. However, the increase in the number of varieties offered by $\bar{K}$ will tighten its incentive constraint on complying with punishments for deviations. If, for a given $\delta \leq \bar{\delta}(p^m)$, the price under the original distribution is still to be sustained as a collusive price, the lowest punishment price sustainable must be increased. This implies that $P^c(\kappa_1)$ cannot be sustained anymore (and also any higher price that would be closer to the monopoly price). Hence the highest sustainable price in a symmetric optimal punishment equilibrium must go down as the size of the largest firm is increased and the size of the smallest firm is unchanged.[9]

The argument is exactly reversed if the smallest firm increases its product line by buying varieties from other firms, leaving the size of the largest firm unchanged and leaving the identity of the smallest firm unchanged. As the product line is increased, the incentive constraint at the previous optimal collusive price will be relaxed for the smallest firm. Hence, the optimal collusive price can be increased leading to higher profits. The next proposition states these results more formally:

Proposition 4 *Consider a transaction that changes the distribution of varieties in the market from κ_1 to κ_2 and fix $\delta \in [\underline{\delta}_{\max}, \bar{\delta}_{\min})$. Then:*

(a) If $\bar{K}_1 < \bar{K}_2$ and $\underline{K}_1 = \underline{K}_2$ then $P^c(\delta, \kappa_1) > P^c(\delta, \kappa_2)$.

(b) If $\underline{K}_1 < \underline{K}_2$ and $\bar{K}_1 = \bar{K}_2$ then $P^c(\delta, \kappa_1) < P^c(\delta, \kappa_2)$

(c) Among all redistributions of assets between two firms, those from the largest to the smallest firm will increase P^c most, as long as these do not make the previously smallest firm into the largest firm.

Proof. See the Appendix. ■

Note that the comparative statics that proposition 4 derives for the most profitable collusive price between two different asset distributions holds equally for the change in profits per variety. Hence, asset transactions that would tighten the collusive constraint would be unprofitable under the assumption that the most collusive outcome could be obtained. This is the case because then there would be no possible deal between two firms that could increase their joint profits through the transaction if the most profitable collusive price were to fall as a result.

[9] The basic idea of the argument can also be seen by looking at Figures 1 and 2 above. If the largest firm gets larger the locus of the binding constraint for the large firm moves up in Figure 1. This means that for any p^c the lowest δ at which it is sustainable, $\bar{\delta}(p^c)$ is increased. Since this holds for any p^c this means that the schedule $\bar{\delta}(p^c)$ in Figure 2 simply moves to the right. But then the highest sustainable price before the asset transaction cannot be sustainable anymore (and no prices that were previously not sustainable can now be). Hence, the highest sustainable price has to go down.

Clearly, in our setting only the incentives of the largest and the smallest firm matter for determining the comparative statics effects. This does not mean that the result in proposition 4 requires that the identity of the largest or smallest firm is unchanged. For example, if the largest firm sells varieties it may become smaller than the previously second largest firm. Proposition 4 still holds, but with the identity of the largest firm changed between the two distributions of assets considered. Note, that proposition 4 implies that under unconstrained trading of varieties, firms should trade up to the point where all own the same number of varieties.

What proposition 4 does not consider is the case of a firm that sells all of its varieties. The reason is that a firm that has no varieties is not a small firm, but one that has exits the market. For example, if the smallest firm in the market sells all its varieties to the largest firm, the second smallest remaining firm has to be considered the smallest firm under the new asset distribution. This is obviously the relevant case when considering mergers. Given this slight modification the result of proposition 4 can be directly translated into a result about mergers;

Proposition 5 *If the smallest firm merges with any other firm than the largest firm, the highest sustainable price strictly increases. If the largest firm merges with any other than the smallest firm, the highest sustainable price falls. Mergers of the largest firm with the smallest firm are more likely to decrease prices the smaller the second smallest firm.*

Proof. Follows directly from proposition 4. ■

The fact that mergers of the largest firm tend to depress prices in the market may at first appear counterintuitive given standard assessments of mergers. Why would market performance improve when the largest firm gets larger? And why does this result not imply that full monopolization leads to the best outcome? The error in this type of reasoning stems from the fact that we usually think about mergers in terms of single firm dominance or the unilateral effects of the merger. In unilateral effects analysis a firm that increases its product line will always gain in market power and has an incentive to raise the price unilaterally. However, our analysis is about the impact of mergers on the incentives to jointly raise prices. As long as collusion is feasible an increase in the heterogeneity between firms will make collusion more difficult and consequently lower the highest sustainable prices in models with product differentiation. This does not mean that any merger of the largest firm will lead to lower prices. If the firm gets so large that collusion is not sustainable, any further increase in its share of the varieties in the market will lead to increased market power and higher prices because now unilateral effects determine the market price.

This analysis therefore suggests that there is a sharp distinction in merger analysis between markets in which single firm dominance matters and markets in which joint dominance is the primary concern. What we have shown is that the comparative statics of asset transfers go in opposite directions, so that decision rules on mergers and on potential remedies will have to condition on a finding of joint or single firm dominance. In other words, it is essential for the merger policy to determine whether unilateral or coordinated effects determine the price setting in the market. We will develop this theme more formally in section 5. Before we will analyze more carefully which of the conclusions of this section can be expected to carry over when analyzing the complete equilibrium set, dropping the restriction to *symmetric* strategies.

4 Fully Optimal Punishment Schemes

The symmetric optimal punishment equilibria analyzed in the previous section are, of course, only a selection from the overall set of subgame perfect Nash equilibria of the repeated price game between firms in the market. They will only give insights into the comparative statics of collusion if they qualitatively capture the changes in the equilibrium value set and the associated set of sustainable prices. Kühn and Rimler (2004) have shown this will typically be the case for symmetric games.[10] However, they also show that for asymmetric games the analysis of the comparative statics of symmetric optimal punishment equilibria can be misleading. To see the problem in the context of our model, notice that the only way that incentive constraints for collusion can be relaxed is by reducing the highest symmetric collusive price and increasing the lowest available symmetric punishment price. Since reallocations of products between firms that increase asymmetries tighten the relevant incentive constraints, the highest sustainable price in a symmetric optimal punishment equilibrium must fall when asymmetries increase. However, it is questionable whether this would be the case for subgame perfect equilibria that are Pareto optimal among the set of firms. Consider, for example, the smallest firm getting smaller through a transfer of a product to the ownership of a larger firm. The small firm now has a larger incentive

[10] For a more systematic discussion of the relationship between the comparative statics of the equilibrium value set and rerstricted equilibrium value sets see the discussion in Kühn and Rimler (2004).

to deviate from any given collusive price. Instead of relaxing the incentive constraint by reducing all prices (and leaving market shares constant), the incentive constraint of the smaller firm could be relaxed by allowing it to charge a lower price than the other colluding firms. Indeed, one may expect that the highest industry profits from collusion could be obtained when the smallest firm reduces the price and the largest firms *increases* its price. This could generate the same effect on the incentive compatibility constraints, but at higher average prices. Our results on the price reducing effects of asymmetry increasing mergers may, therefore, be misleading.

For this reason we extend, in this section, the analysis of our model to the full equilibrium value set. However, this analysis cannot be done analytically since the equilibrium value sets for different distributions of assets across firms are not nested in our model. We, therefore, limit ourselves to a numerical analysis using the Abreu-Pearce-Stacchetti algorithm for determining the equilibrium value set (see Abreu, Pearce, and Stacchetti 1991). For computational reasons we also limit the analysis to asset reallocations in a duopoly. This procedure allows us to compute the full comparative statics of the equilibrium value set as we change the distribution of varieties between the two firms. We can show that for sufficient asymmetry the value sets are nested and the Pareto frontier of the value set unambiguously shifts outwards. However, around symmetry the value sets are not nested. It is then more difficult to interpret the comparative statics of collusion both with respect to profits and prices. However, as discussed by Kühn and Rimler (2004) one can construct consistent comparative statics results by assuming a method of selecting the equilibrium played through a method that is consistent across asset distributions. In particular, we consider the equilibrium that would be selected by a Nash bargaining solution over the equilibrium value set. This selection is consistent across different parametrizations of the problem and well established in the literature for modelling contexts in which it is important to generate a unique selection from the equilibrium value set.[11] We will now discuss the numerical setup and then show an example for the results.

[11] See Jehiel (1992) for an example in which this assumption is made to evaluate the impact of investment decisions on later collusion. In the context of the analysis of renegotiation proof equilibria Abreu, Pearce, and Stacchetti (2003) use the same selection device to generate unique renegotiation proof equilibrium value sets.

4.1 The numerical setup

For this numerical computation we use the quadratic utility function:

$$U(q) = a\sum_i q_i - \frac{1}{2}\left(\sum_i q_i\right)^2 - \frac{n}{2(1+\phi)}\left[\sum_i q_i^2 - \frac{1}{n}\left(\sum_i q_i\right)^2\right].$$

Whenever there is positive demand for all varieties at price vector $\mathbf{p}$, the associated demand function for variety i is:

$$D_i(\mathbf{p}) = \frac{1}{n}[a - p_i - \phi(p_i - p)]$$

where $p = \frac{1}{n}\sum_i p_i$ is the average of all prices of varieties in the market. At such a price vector $\mathbf{p}$, the average price p is a sufficient statistic for the overall level of the market price and there is a well defined aggregate demand given by:

$$\sum_i D_i(\mathbf{p}) = a - p$$

Note that for general punishment equilibria we not only have to allow for different firms setting different prices, but also for each firm setting different prices for each one of its products. This could be quite inconvenient for numerical analysis. Increasing the number of varieties increases the dimensionality of the price set to be considered and therefore could considerably lengthen the time to convergence of the algorithm. Fortunately, it turns out that, for the linear model, the equilibrium value set generated when restricting firms to price all varieties at the same price is the same as the one generated when there is no such restriction:

Lemma 5 *Suppose that $p_L^K \neq 0$ for all K. Then the equilibrium value set of the full game under quadratic preferences is the same as the equilibrium value set of a game in which the firms are restricted to charging the same price on each variety they control.*

Proof. See the Appendix. ■

This Lemma reduces the duopoly problem we want to calculate to a more standard asymmetric duopoly with a one dimensional strategy for each firm. We can change the degree of asymmetry parametrically to calculate the impact of asymmetries on the Pareto frontier of the equilibrium value set. The impact of the length of the product line enters the model only in terms of the share of the product line, so that we do not have to worry about the effects of total varieties available.

To implement the algorithm of Abreu, Pearce, and Stacchetti (1991), we have to resort to a discrete price grid instead of the continuous strategy space we have assumed in the theoretical part of the paper. We select a feasible strategy set for each firm of 400 equally distanced prices between 0 and a. The set of potential price vectors $\mathcal{P}_0$ is a set of 400×400 price pairs. We also select a convex set of candidate average continuation values that is known to contain the equilibrium average value set. Denote the convex hull of this set by $\mathcal{H}_0$. The algorithm maps pairs of candidate price sets and candidate average continuation value sets $(\mathcal{P}_\tau, \mathcal{H}_\tau)$ to another pair of such sets $(\mathcal{P}_{\tau+1}, \mathcal{H}_{\tau+1})$. It proceeds as follows: For every price vector $\mathbf{p} \in \mathcal{P}_\tau$ we check whether the vector satisfies the incentive constraints for some selection of possible continuation values from $\mathcal{H}_\tau$.[12] Note that for each price vector we can choose different continuation value vectors depending on whether the firms have charged $\mathbf{p}$ or, otherwise, which firm deviated from the price vector. If the price vector cannot be sustained by any combination of feasible values in $\mathcal{H}_\tau$, it is eliminated from $\mathcal{P}_{\tau+1}$. If it can be sustained it is included in $\mathcal{P}_{\tau+1}$. Furthermore, for every pair of continuation values $\mathbf{v} \in \mathcal{H}_\tau$ for which the price vector can be sustained we calculate $(1-\delta)\pi(\mathbf{p},\mathbf{p}) + \boldsymbol{\delta} v_{i\tau}$ for each firm i. All vectors resulting from this for all price pairs are recorded in a set $\mathbf{V}_{\tau+1}$. The set $\mathcal{H}_{\tau+1}$ is the convex hull of $\mathbf{V}_{\tau+1}$. By theorem 5 in Abreu, Pearce, and Stacchetti (1990) this algorithm converges to the equilibrium value set. Since the average per variety profit of any firm can never exceed the per variety profits of a monopolist and since firms can always guarantee themselves at least zero profits, we choose $\mathcal{H}_0$ as the set of all average per variety profits that are non-negative and in which the average profits across all varieties are no larger than the monopoly profits. Note that for any value pair on the Pareto frontier of the (average) equilibrium value set there is a unique price that generates this value pair.

4.2 Numerical Results

We have run this algorithm for a number of different parameterizations. For all parametrizations the qualitative outcome is the same. We present here results for a single parametrization with $a = 100$, $c = 40$, $\phi = 1$, and $\delta = \frac{1}{3}$.

[12] We limit the search to the convex hull of the set of average continuation values since any price vector that can be sustained by appropriate continuation values strictly in the set can also be sustained by values on the convex hull of the equilibrium value set. See Abreu, Pearce, and Stacchetti (1991).

We construct the maximal average continuation value for all products from the per variety monopoly profit given by $\max_p (p-c)(a-p) = \left(\frac{a-c}{2}\right)^2 = 900$. For this example, we have fully calculated the equilibrium average value sets and the associated price sets. We have performed comparative statics in changing the distribution of varieties between the firms. It shows that the Pareto frontier of the value set is not ordered in the distribution of varieties when distributions are relatively symmetric. However, for very asymmetric distributions it monotonically moves towards more profitable outcomes. To get more concrete insights into the comparative statics we then selected, for each distribution of varieties, the equilibrium that is the outcome of Nash bargaining over the equilibrium value set. The Nash bargaining solution takes the one shot Nash equilibrium played forever as the threat point. We will call this the Nash bargaining solution to the collusion problem.

Figures 3 and 4 illustrate the comparative statics of the Nash bargaining solution to the collusion problem both for prices and equilibrium value sets as we increase the proportion of varieties controlled by firm 1. The discreteness of prices matters for the computational results. Generically, the incentive constraints will not hold with equality for any firm given the discreteness of the strategy space. For a small change in relative size it may therefore be the case that the same prices will still satisfy the incentive compatibility constraints of the firms. This induces the flat portions in the graph for prices. We have verified that graphs do indeed get smoother as we make the grid size finer, but required computer time rises very fast in the number of prices considered.

Note also, that the incentive constraint for the larger firm is typically not binding when setting the collusive price for the equilibrium chosen by the Nash bargaining solution. However, for the smaller firm this constraint is binding. Indeed, for any fixed incentives from future profits the collusive price of the smaller firm must be adjusted downward towards its best response price. This will guarantee that the collusive price for the smaller firm must fall. Since the price that the larger firm sets in the collusive equilibrium is essentially not constrained, it will tend to rise around the symmetric outcome in order to reduce the degree to which the price of the smaller firm has to fall. While this gives the smaller firm greater market share this is compensated by a higher price level overall from the point of view of the firms.

Figure 3 confirms this intuition. Optimal collusion always involves the smaller firm setting a lower price than the larger firm. The tightening of incentive constraints due to more asymmetric distributions of varieties across firms lead to the smaller firm reducing the price. But the larger firm does

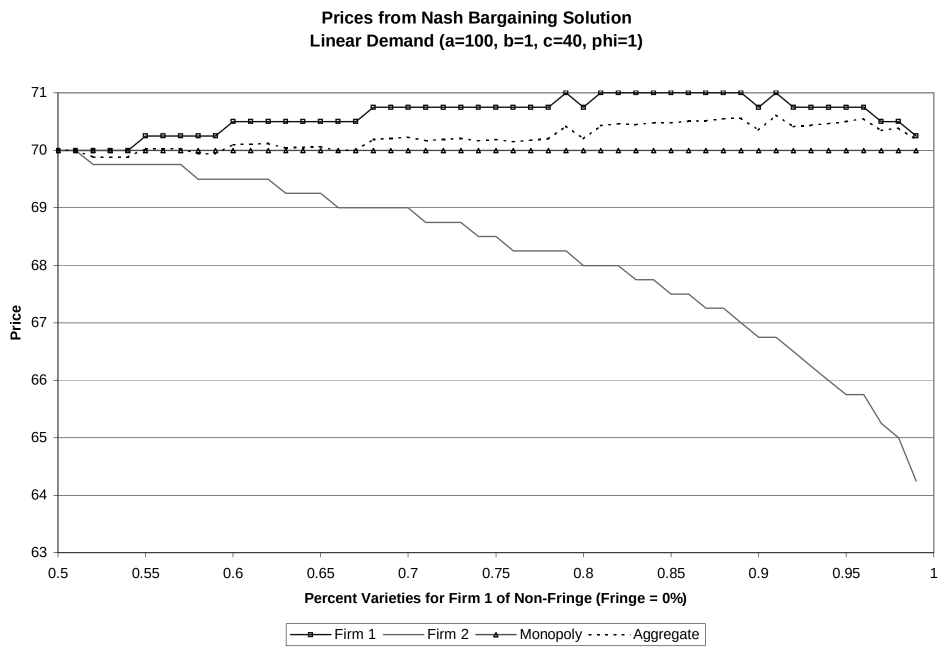
Prices from Nash Bargaining Solution
Linear Demand (a=100, b=1, c=40, phi=1)
Price
71
70
69
68
67
66
65
64
63
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
Percent Varieties for Firm 1 of Non-Fringe (Fringe = 0%)
Firm 1
Firm 2
Monopoly
Aggregate

Figure 3:

not just reduce the price by less but even increases the price. We have chosen the discount factor in this example such that full collusion is just possible at a symmetric allocation of varieties. As a result, the optimal collusive scheme selected by Nash bargaining will make the larger firm charge a price exceeding the monopoly price as soon as the asymmetry is large enough. Note that the average price charged in the market initially slightly falls because the price of the smaller firm falls faster than the price of the larger firm increases. But since the price effect of the larger firm has more weight over all purchases, the average price soon starts to increase above the monopoly price as well. It is therefore not the case that all prices in the market should be expected to decrease from an asymmetry increasing merger close to symmetry.

However, note also that both prices do decrease when very asymmetric distributions of assets get even more asymmetric. Indeed, this would indicate that if firm 1 already owns 90% of the varieties in the market, a move to complete monopolization welfare dominates the very asymmetric duopoly market structure. Figure 4 shows that this is indeed the case when we look at welfare:Welfare initially slightly increases but then rapidly falls. However, once the market is sufficiently concentrated full monopolization increases welfare because the monopolist does not keep prices above the monopoly price anymore in order to give the small firm greater incentives to collude. This is a result that could not be obtained with symmetric optimal punishment schemes and shows that the restriction to symmetry significantly affects the comparative statics of prices - even qualitatively. Not all parametrizations of the problem will have the property that all prices will fall from a merger to monopoly when firms are sufficiently asymmetric. If the highest collusive price under symmetry is sufficiently below the monopoly price, we expect to see the large firm's price always to increase. Overall, one should note, however, that the price effects of the asset transactions are small as long as full collusion is achievable for a symmetric distribution of varieties.

While the comparative statics of equilibrium prices yield a picture that suggests that policy rules are hard to obtain from such analysis, a focus on the comparative statics of equilibrium values suggests otherwise.

Figure 5 shows that the conclusions of our analysis of symmetric optimal punishment equilibria do hold qualitatively when we look at profits. The profits per variety of the larger firm decrease over a wide range of distributions of varieties between the firms. Only when the asymmetry is large enough does the further increase in varieties held by the largest firm lead to profits increasing. This simply reflects the fact that collusive prices are

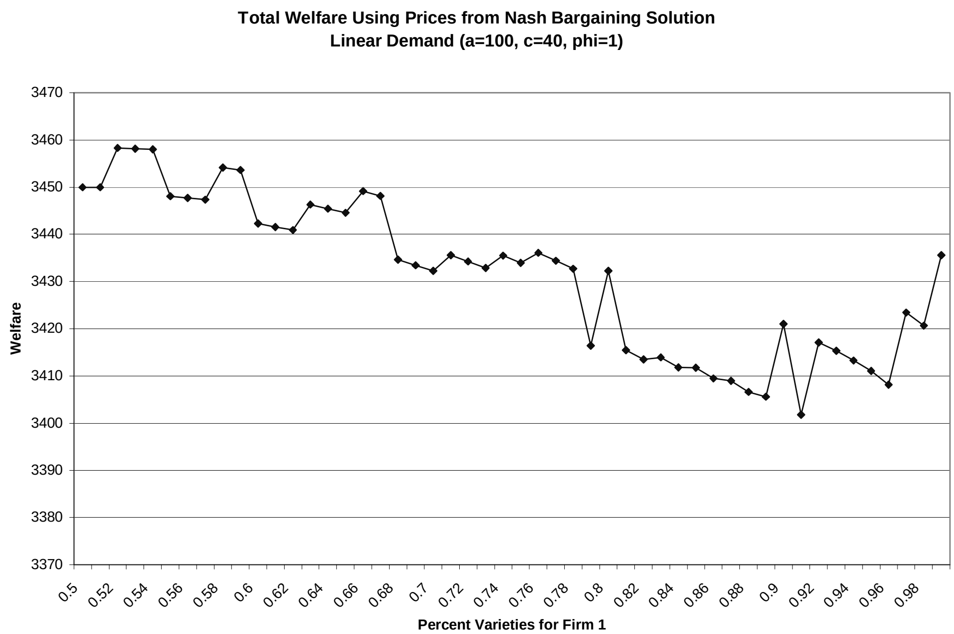
Total Welfare Using Prices from Nash Bargaining Solution
Linear Demand (a=100, c=40, phi=1)
Welfare
3470
3460
3450
3440
3430
3420
3410
3400
3390
3380
3370
0.5
0.52
0.54
0.56
0.58
0.6
0.62
0.64
0.66
0.68
0.7
0.72
0.74
0.76
0.78
0.8
0.82
0.84
0.86
0.88
0.9
0.92
0.94
0.96
0.98
Percent Varieties for Firm 1

Figure 4:

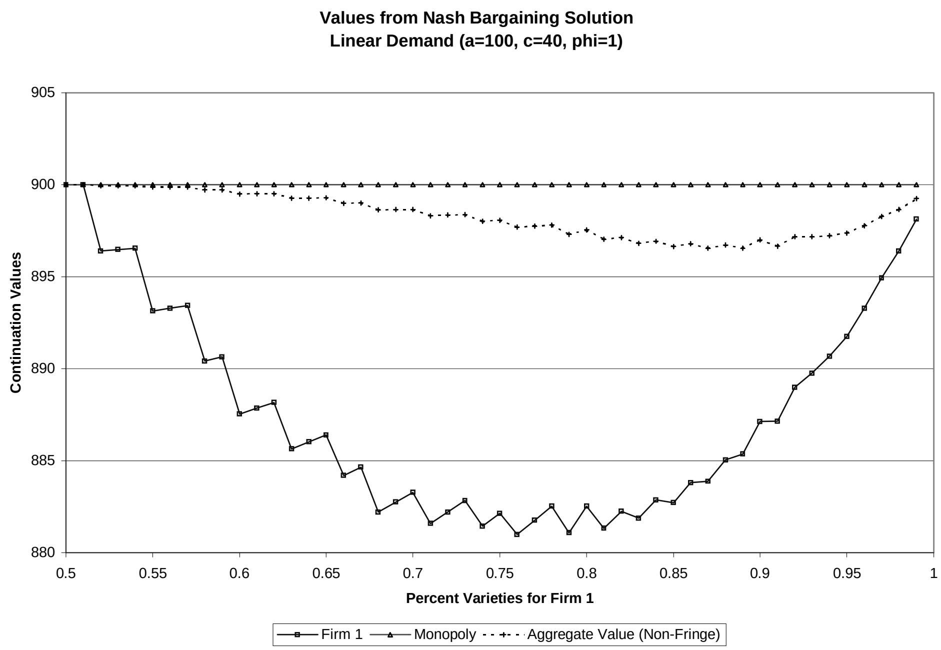
Values from Nash Bargaining Solution
Linear Demand (a=100, c=40, phi=1)
905
900
895
890
885
880
Continuation Values
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
Percent Varieties for Firm 1
Firm 1
Monopoly
Aggregate Value (Non-Fringe)

Figure 5:

then close to best response prices so that the profits move essentially in the same direction as Nash profits when asymmetry is increased.

Note that over a very wide range average profits across all varieties go down when the smaller firm sells a variety to a larger firm. This implies that asymmetry increasing asset transactions are not profitable unless the asymmetry is already large. This would create a presumption that asymmetry increasing asset transaction in markets that were previously fairly symmetric would likely have some efficiency effects that justify them when it is believed that collusive outcomes are achieved. These transactions cannot generate a presumption that the goal is to facilitate collusion in the post-merger market situation. Interestingly, joint profits per variety only increase in this example when welfare is increasing in further consolidation as well. Hence, conditional on the firms always achieving a Nash Bargaining solution over the equilibrium value set, the only mergers observed aer welfare improving.

5 Applying Coordinated Effects Theory to Merger Analysis

In this section we discuss how the framework developed in this paper can be used to derive rules for the treatment of coordinated effects in merger analysis. There are three issues we will deal with. First there is the question whether coordinated effects and unilateral effects should ever be found at the same time. In anti-trust regimes like those of Canada or Europe this should be considered the same as asking the question whether single firm dominance or joint dominance could be present at the same time. Secondly, we will show that new issues arise in the assessment of coordinated effects when there are mergers involving firms operating in two separate markets. Finally, we show that there is a tendency for joint dominance to persist so that intervention in merger proceedings is a relevant concern.

5.1 The Conflict between Dominance and Joint Dominance

For practical purposes there are two important questions for merger analysis that we have not addressed. First, we ask: is it sensible to find both joint dominance and single firm dominance at the same time in a merger proceeding? We will show below that there is a precise sense in our model in which joint and single firm dominance are mutually exclusive. A closely related question is how to interpret the above theory in practice. In real markets the smallest firms are virtually never involved in collusive agreements and the concern is mostly about coordination of behavior between the largest firms in the industry. However, the analysis of symmetric optimal punishment strategies seems to suggest that the incentives of the very smallest firm in an industry are crucial for merger analysis. We resolve this issue below by showing that there are good reasons to believe that joint dominance analysis should be applied to a subset of firms.

First note that collusion essentially disappears in our model when firms are sufficiently asymmetric[13]: under fully optimal punishment equilibria both the large firm and the small firm converge to pricing arbitrarily close to their short run best response. Note that this does *not* arise because one of the firms becomes very small relative to the rest of the market. To see this, consider a limit of our model with an arbitrary number of varieties each owned by a different firm. In that context every firm becomes arbitrarily small relative to the market and a monopolistically competitive equilibrium arises as the Nash equilibrium of the one shot game. As Green (1980) has shown, collusion is still feasible at some discount factor strictly below 1. Although each firm in a one shot equilibrium perceives no impact on the profits of others it will consider its behavior pivotal for triggering a switch from a collusive outcome to a monopolistically competitive outcome. Collusion can therefore still be sustained for some discount factors in our model even if all firms are arbitrarily small relative to the market.

What limits collusion under very asymmetric distributions of varieties between firms is the market power of the largest firm. When a firm owns most of the varieties it can never be made to punish very harshly. Since a very small proportion of varieties have very small impact on the large firm's profits the gain from giving a small firm incentives to stick to a collusive price are very small. However, in order to induce strong punishments there has to be a large price reduction. This becomes very costly to a large firm. As a result, a large firm has to price close to its best response price whatever the remaining firms do. Indeed, when the firm is large this will be close to the monopoly price. In turn this means that the small firm will face very small differences between collusive and punishment prices of its rival and charge a price very close to its best response price as well.

Note that this comes very close to the way lawyers define "single firm dominance", namely pricing behavior that is essentially independent of the

[13] For symmetric optimal punishment equilibria this can be shown to hold exactly for a wide class of product differentiation models. We have omitted this result from this paper because it is an artefact of restricting equilibrium strategies to be symmetric. In general a small amount of collusion in the sense of raising the lowest price above the best response price is always achievable in general optimal punishment schemes.

pricing behavior of smaller firms in the market. But note that it is precisely the market dominance that makes it impossible to keep smaller firms from raising prices significantly above the (monopolistically) competitive outcome. That means dominance precludes significant collusion or joint dominance. This is the precise sense in which in our model single firm dominance and joint dominance are mutually exclusive.

Unfortunately, our example in the previous section shows that the comparative statics of behavior may still depend on whether firms are colluding or not. When firms do not collude, the price charged by the largest firm would unambiguously rise under the linear demand model discussed in the previous section. However, it decrease under the same circumstances in our example. The comparative statics of prices due to asset transactions under joint dominance therefore do not necessarily coincide with the comparative statics in the absence of collusion - even in cases in which the actual market behavior is indistinguishable from non-collusive behavior.

A closely related problem is the question of how to treat coordinated effects in practice for markets in which there are a few very large firms and a significant number of very small firms. Should joint dominance analysis focus on the large firms only or should it include all firms in the market?

We now show for symmetric optimal punishment equilibria that there is a precise sense in which a large enough subset of firms would have an incentive to act independently of other, smaller firms, in the market. We show that larger firms have an incentive to drop small firms from the collusive agreement when the small firm is small enough relative to the market. This shows that only the largest firms would have an incentive to form a collusive agreement suggesting that joint dominance or coordinated effects of mergers should only be considered for a subset of firms in the market. In the Appendix we formally prove the following result:

Proposition 6 *Suppose the smallest firm becomes arbitrarily small relative to the size of the second smallest firm. Then it is optimal for the larger firms to exclude it from an agreement on an optimal symmetric punishment equilibrium.*

Proof. See the Appendix. ■

The intuition is very simple. Take any initial distribution of varieties in a market. Now consider a sequence of markets in which the number of varieties is replicated and demand normalized to leave aggregate demand unaffected. In this sequence the number of varieties for the smallest firm is kept fixed, while the rest of the varieties is distributed to keep the relative size of the remaining firms fixed. Clearly, along the sequence of markets, $\underline{n}/n$ goes to zero. But then dropping the smallest firm will make virtually no difference to other firms at any finite prices, because a price reduction will have arbitrarily small effect on demand for each of the larger firms. However, keeping the small firm in the collusive agreement limits the level of the collusive price. By dropping the smallest firm from the collusive agreement the collusive price can be strictly increased, making all firms better off. Small enough firms will therefore not be part of a symmetric optimal punishment equilibrium.

Note that this intuition must survive for fully optimal punishment schemes as well. The smaller the firm is relative to the others, the smaller the impact of the price of that firm on the profits of other firms. Hence, other firms under optimal collusion will be less willing to raise the price to incentivize a higher price by the smallest firm. As a result, very small firms must be pricing close to their best response price in any equilibrium that is on the Pareto frontier of the equilibrium value set. While there may be equilibria in which the smallest firm raises prices significantly above the best response price to the price vector charged by other firms (even when it is very small), these equilibria will be Pareto dominated. Note that this is different from the case of one very large firm. In that case all firms must be close to best responding in all equilibria.

The observations we have made above have some important implications for the application of joint dominance in general and coordinated effects analysis in mergers in particular. It shows that it makes sense to consider joint dominance only of a limited collusive groups as long as some firms are much smaller than the rest. Hence, a policy that looks for a group that is relatively homogeneous within and heterogeneous relative to smaller firms outside the group appears to be sensible.

5.2 The Impact of Multimarket Contact on the Evaluation of Joint Dominance

In competition policy practice the European Commission has argued that joint dominance should be assessed market by market if the geographic markets involved in the merger are found to be separate. The analysis in this paper suggests that this practice may not be sensible and could lead to counterproductive remedies when coordinated effects are a concern.

Generally in competition policy cases the average market structure across geographic markets is quite distinct from the market structure market by market. However, from the literature on multi-market contact (see Bernheim and Whinston, 1990) we know that optimal collusive schemes lead to pooling of incentive constraints across markets. For example, if the market structure across markets is much more heterogeneous than on average then the market by market analysis may seriously underestimate the problem of joint dominance overall.

Consider an example in which there are two markets, A and B, of the type described in our model that are identical except for the ownership structure. There is one firm that owns 70% of the varieties in market A and 30% of the varieties in market B. It faces one firm in each of the two markets that own the rest of the varieties but are independent from each other. We will discuss the impact on collusion when the two independent firms in the two separate markets merge.

Let us assume that each of the independent firms can perfectly observe the pricing behavior in the market in which they are not present. Then, in an optimal punishment equilibrium, punishment in both markets will be induced if there is a deviation from collusion in any one of the markets. To simplify the discussion we will consider optimal punishment schemes that are symmetric in each of the markets. The incentive constraint for the large firm is then given by:

$$\begin{aligned} & \frac{\bar{n}}{\bar{n}+\underline{n}}\phi^{\bar{K}}(p^*(p^A,\bar{n}),p^A,\bar{n})+\frac{\underline{n}}{\bar{n}+\underline{n}}\phi^{\underline{K}}(p^*(p^B,\underline{n}),p^B,\underline{n}) \qquad (7)\\ \leq\ & \delta[\frac{\bar{n}}{\bar{n}+\underline{n}}\left(\pi(P^{cA},P^{cA})-\pi(p^A_L,p^A_L)\right)+\frac{\underline{n}}{\bar{n}+\underline{n}}\left(\pi(P^{cB},P^{cB})-\pi(p^B_L,p^B_L)\right)] \end{aligned}$$

where p^i refers to the price (either collusive or punishment) that is to be sustained and P^{ci} refers to the most profitable sustainable price and p^i_L to the most severe punishment in market i. Note, that this incentive constraint does not change for the firm active in both markets after the merger.

After the merger the two independent firms have the same incentive constraint (7) with only the indices for the largest and smallest firm inverted. This means that after the merger there will be the same price charged in both markets both under optimal collusion and under optimal punishments. Before the merger, the independent firm in market A is the smaller firm. Its incentive constraint is given by:

$$\phi^{\underline{K}}(p^*(p^A,\underline{n}),p^A,\underline{n})\leq\delta[\pi(P^{cA},P^{cA})-\pi(p^A_L,p^A_L)] \qquad (8)$$

In market B the independent firm is the larger firm and its incentive constraint is the appropriate analogue to (8). To understand the impact of the merger it is convenient to first consider the equilibrium after the merger.

Let P^c and p_L be the respective prices relevant for equilibrium behavior after the merger. We will discuss whether these prices are sustainable before the merger. Note that the left hand side of (7) will be the same for all firms at these prices due to the symmetry in the post merger equilibrium. Now note that $\phi^{\underline{K}}(p^*(P^c,\underline{n}),P^c,\underline{n}) > \frac{\bar{n}}{\bar{n}+\underline{n}}\phi^{\underline{K}}(p^*(P^c,\underline{n}),P^c,\underline{n}) + \frac{\underline{n}}{\bar{n}+\underline{n}}\phi^{\bar{K}}(p^*(P^c,\bar{n}),P^c,\bar{n})$. This implies that the independent firm in market A has a greater incentive to deviate from the collusive price than the firm present in both markets. By a symmetric argument it has less of an incentive when considering deviations from the punishment price. This implies that incentive constraints in market A are strictly tightened relative to the post merger market. By a symmetric argument we can establish that the same is true in market B relative to the incentive constraint for optimal punishments. This implies that the most profitable collusive price must (at least weakly) be lower in both markets relative to optimal collusion after the merger.

This analysis suggests that joint dominance or coordinated effects of mergers should not be assessed market by market. Given the strong asymmetry between the firms in each of the markets one would conclude that coordinated effects were not relevant in a market by market analysis. However, this would be incorrect. The merger is, in this example, symmetry increasing and would lead to higher prices as long as collusion is sustainable.

5.3 Product Innovation and the Persistence of Joint Dominance

One of the features of single dominant firm positions is that there is a powerful force that leads to the persistence of such dominance over time as has been shown in the work of Gilbert and Newbery (1984). The argument is that the payoff gain by maintaining monopoly profits over one duopoly profit is greater than the payoff gain in obtaining one duopoly profit over nothing. Hence, if a new variety appears firms with the greater number of varieties will tend to have the greater incentive to bid for the new product. Generally, such bidding for innovations will lead to the innovation being obtained by the firm in who's hands it maximizes industry profits. Since greater asymmetry in static Bertrand equilibrium will tend to increase industry profits, the analysis in the absence of collusion therefore suggests a tendency towards increasing single firm dominance in the type of market we are discussing in this paper.

However, in the context of our analysis the question arises whether there is any sense in which jointly dominant positions can also be considered to be persistent. An answer can be obtained directly from our numerical analysis of fully optimal collusion in the duopoly case. As we have shown in the previous section an increase in asymmetry in relatively symmetric firms leads to reduced industry profits, while it leads to increased industry profits when the asymmetry is large enough. Hence, for relatively symmetric market structures the smaller firm will not only obtain the benefit of the extra revenue stream from the new product, but also increase the per-variety profitability of the market. In contrast, the larger firm's acquisition of the new variety would lower the per-variety profitability. Hence, industry profits will be maximized by the smaller firm acquiring the product. There is therefore a tendency towards persistence of joint dominance. Conversely, if the firms are already in the region of single firm dominance, industry profitability is lower when the smallest firm acquires the new product and there is a tendency for the largest firm to become more dominant. Again, single firm dominance persists.

Formally, these results can be seen in our numerical model directly. The addition of another product simply changes the share of the firms in the total number of products, which is the only relevant parameter of the model. Hence, the results derived about the effects of increasing heterogeneity directly applies to the acquisition of new products.

6 Conclusions

In this paper we have analyzed the impact of acquisitions of brands by firms in a differentiated products oligopoly and their impact on the likelihood of collusion. In particular, we have been able to derive results on the impact of mergers on the likelihood of collusion and the level of collusive prices that can be attained in the market. The analysis shows that mergers do not simply eliminate a firm from the market but change the size distribution of remaining firms. As a result mergers involving the largest firms will typically reduce the likelihood of collusion and the highest achievable collusive prices due to the increased heterogeneity of firms in the market.

This analysis has important consequences for merger analysis. First, it gives an instrument to assess the coordinated effects of mergers. Second, it shows that either unilateral effects of mergers should be considered as determinant for the competitive impact of mergers or coordinated effects, but not both. Indeed, the remedies for coordinated effects in terms of asset sell offs will typically be the opposite to those that would be appropriate under unilateral effects analysis.

However, there are also significant limitations to this analysis. Our results, in a strict sense, only compare the scope for collusion given one market structure with the scope of collusion in another. The model does not model the impact of a merger in an ongoing collusive regime. Such an analysis would generate a large number of new questions that we have been able to avoid here. For example, how should we deal with the problem that firms will anticipate the possibility of mergers in the future? What change in the market generates a merger in the first place? We have abstracted from these questions in the paper because they have not even been satisfactorily addressed in the theoretical analysis of unilateral effects. Instead we have focused only on the incentive effect of particular asymmetries generated through asset distributions between firms. However, even this modest approach suggests that coordinated effects analysis in mergers is a very delicate exercise that is difficult to capture in simple policy rules.

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7 Appendix A

In this Appendix we derive all of the results under the assumptions that $p_L > 0$, $D(p^*(p_L, n^K), p_L, n^K) > 0$ and $D(p^*(P^c, n^K), P^c, n^K) > 0$ for all K unless specifically stated otherwise. These assumptions facilitate the exposition of the argument. We discuss at the end of the appendix which proofs would have to be adapted and how to cover the corner cases. None of the results depend on these assumptions.

Lemma 1: *The incentive constraint for the lowest value equilibrium is always binding. Generically, we have either $\hat{v} = \pi(P^c, P^c)$ and $p_L > 0$ or $\underline{v} < \hat{v} < \pi(P^c, P^c)$ and $p_L = 0$.*

Proof: We first show that the incentive constraint for the lowest value equilibrium must be binding. Suppose it is not. Then change the first period punishment slightly from p_L to achieve a first period punishment payoff of $\pi(p_L, p_L) - \varepsilon$. Maintain $\hat{v}$ as the outcome of the continuation equilibrium. Then the continuation value after a deviation would be $\pi(p_L, p_L) - \varepsilon + \delta\hat{v} < \underline{v}$ and the incentive constraint on P^c would be relaxed. However, since the incentive constraint in the punishment equilibrium was supposed to be slack we can always choose ε small enough such that this constraint remains slack. But then $\underline{v}$ is not the lowest equilibrium value achievable, a contradiction.

Now suppose that $p_L > 0$ and $\hat{v} < \pi(P^c, P^c)$. we will show that this contradicts the definition of p_L and $\hat{v}$. Now choose a $p_L^- < p_L$ and a $\hat{v}^+ > \hat{v}$ such that $\pi(p_L^-, p_L^-) + \delta\hat{v}^+ = \underline{v}$. Given the public signal σ the equilibrium value set is convex and, therefore, such a choice of $\hat{v}^+$ is always available. But then $\pi(p^*(p_L), p_L)$ is, by the envelope theorem, strictly reduced, making the incentive constraint on punishments slack. But then a more severe punishment is available,. contradicting the definitions of $\underline{v}$, p_L, and $\hat{v}$. It follows that either $\hat{v} = \pi(P^c, P^c)$ or $p_L = 0$. QED.

Lemma 2: *The best response price $p^*(p, n^K)$ is strictly increasing in p (whenever $D(p^*(p, n^K), p, n^K) > 0$), $p^m > p^*(p^m, n^K)$, and $p^*(c, n^K) > c$. For each n^K there exists a unique price $\hat{p}(n^K) > c$ such that $p^*(\hat{p}(n^K), n^K) = \hat{p}(n^K)$. It is strictly increasing in n^K.*

Proof: For the proof suppose first that $D(p^*(p(p, n^K), p, n^K) > 0$. Then, by concavity of the profit function, the best response is unique. Now note that $p^*(p, n^K)$ maximizes $\ln(p_K - c) + \ln D(p_K, p, n^K)$. Since by assumption 1 $\frac{\partial^2 \ln D(p_K, p, n^K)}{\partial p_K \partial p} > 0$ it follows by standard monotone comparative statics arguments that $p^*(p, n^K)$ is increasing in p. Now note that by symmetry of the demand functions $D(c, c) > 0$. Then by continuity of demand there exists $p' > c$, such that $D(p', c) > 0$. Hence, a firm would prefer setting p' to setting c, which shows that $p^*(c, n^K) > c$. We now show that $p^m > p^*(p^m, n^K)$ for all $n^K < n$. Suppose otherwise. Note that for $(p_K, p) = (p^m, p^m)$ we have that

$$\begin{aligned} \frac{\partial \pi(p^m, p^m, n^K)}{\partial p^m} &= D(p^m, p^m) + (p^m - c)\frac{\partial D(p^m, p^m, n^K)}{\partial p_K} \\ &= -(p^m - c)\frac{\partial D(p^m, p^m, n^K)}{\partial p_{K^c}} < 0 \end{aligned}$$

where the equality follows from the definition of the monopoly price and the inequality follows from demand increasing in rival prices. By concavity of the profit function it follows that $p^*(p^m, n^K) < p^m$. Furthermore, since the objective function is continuously differentiable it follows by the implicit function theorem that $p^*(p, n^K)$ is continuous. By Assumption 1 it is also a contraction mapping. Hence, there exists a unique fixed point of the best response function as claimed.

We now show that $\hat{p}(n^K)$ is strictly increasing in n^K. It is convenient to define for this purposes $D(\mathbf{p}_K, p_{n^K+1}, \mathbf{p})$ as the demand function where the first n^K goods are priced at price $\mathbf{p}_K$, the n^K+1st good priced at p_{n^K+1} and all other goods at p. Assume $p \leq \hat{p}(n^K+1)$. Suppose, for contradiction that $p^*(p, n^K) \geq p^*(p, n^K+1)$ and consider:

$$
\begin{aligned}
& \frac{\sum_{j=1}^{n_K+1} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), \mathbf{p})} - \frac{\sum_{j=1}^{n^K} \frac{\partial D(\mathbf{p}_K^*(p,n^K), p, \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K), p, \mathbf{p})} \\
\geq \quad & \frac{\sum_{j=1}^{n_K+1} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), \mathbf{p})} - \frac{\sum_{j=1}^{n^K} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p, \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p, \mathbf{p})} \\
= \quad & \frac{\sum_{j=1}^{n_K} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), p)} \\
& + \frac{\frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), p)}{\partial p_{n^K+1}}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), p)} - \frac{\sum_{j=1}^{n^K} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p, \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p, \mathbf{p})} \\
\geq \quad & \frac{\sum_{j=1}^{n_K} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p, p)}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), p)} \\
& + \frac{\frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), p)}{\partial p_{n^K+1}}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), p)} - \frac{\sum_{j=1}^{n^K} \frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p, \mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p, n^K), p, \mathbf{p})} \\
= \quad & \frac{\frac{\partial D(\mathbf{p}_K^*(p,n^K+1), p_K^*(p,n^K+1), p)}{\partial p_{n^K+1}}}{D(\mathbf{p}_K^*(p, n^K+1), p_K^*(p, n^K+1), p)} > 0
\end{aligned}
$$

where the weak inequality in the second line follows from the assumption that $p^*(p, n^K) \geq p^*(p, n^K+1)$ and the log-concavity of demand. The weak inequality in the penultimate line follows from the strategic complementarity assumption (Assumption 1(i)). It implies that the cross-derivative of log-demand in p_K and p_{n^K+1} must be strictly positive. But now note that by the first order condition of profit maximization it follows that

$$\frac{\sum_{j=1}^{n_K+1}\frac{\partial D(\mathbf{p}_K^*(p,n^K+1),p_K^*(p,n^K+1),\mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p,n^K+1),p_K^*(p,n^K+1),\mathbf{p})} - \frac{\sum_{j=1}^{n^K}\frac{\partial D(\mathbf{p}_K^*(p,n^K),p,\mathbf{p})}{\partial p_j}}{D(\mathbf{p}_K^*(p,n^K),p,\mathbf{p})}$$
$$= \quad -\frac{1}{p_K^*(p,n^K+1)-c}+\frac{1}{p_K^*(p,n^K)-c}>0$$

which implies $p_K^*(p,n^K+1) > p_K^*(p,n^K)$, a contradiction to our assumption that $p_K^*(p,n^K+1) \leq p_K^*(p,n^K)$. Hence, $p_K^*(p,n^K+1) > p_K^*(p,n^K)$ when $p \leq \hat{p}(n^K+1)$. But then it follows directly that $\hat{p}(n^K+1) > \hat{p}(n^K)$, because $p^*(p,n^K)$ must have crossed the 45^0-line to be strictly below $p^*(p,n^K+1)$ at $\hat{p}(n^K+1)$.

Lemma 3: *Suppose $n^K > n^{K'}$, then $\phi^K(p^*(p,n^K),p) > \phi^{K'}(p^*(p,n^{K'}),p)$, if $p < \hat{p}(n^{K'})$ and $\phi^K(p^*(p,n^K),p) < \phi^{K'}(p^*(p,n^K),p)$, if $p > \hat{p}(n^K)$.*

Proof: By Lemma 2, $\hat{p}(n^K)$ increases in n^K. Hence, $\hat{p}(n^K) > \hat{p}(n^{K'})$. First, consider the case $p < \hat{p}(n^{K'})$. Then $p^*(p,n^J) > p$ for $J = K, K'$.Therefore, $\phi^K(p^*(p,n^K),p) \geq \phi^K(p^*(p,n^{K'}),p) > \phi^{K'}(p^*(p,n^{K'}),p)$, where the first inequality follows because $p^*(p,n^K)$ is optimal for firm K and the second inequality is implied by the fact that profit functions for each variety controlled by K and K' respectively differ only on the prices of $n^K - n^{K'}$ goods. These goods are priced at p in the case of firm K' but at price $p^*(p,n^{K'}) > p$ for firm K. Since the profits of each variety at a short run optimal price are increasing in the prices of other goods the inequality must hold. The argument for the case $p > \hat{p}(n^K)$ is analogous: In this case both firms' best response is a price strictly below p. We have $\phi^{K'}(p^*(p,n^{K'}),p) \geq \phi^{K'}(p^*(p,n^K),p) > \phi^K(p^*(p,n^K),p)$, where the last inequality comes from the fact that each variety controlled by firm K faces $n^K - n^{K'}$ varieties less charging $p > p^*(p,n^K)$.

Lemma 3A: *For all K the incentive for firm K to deviate from price p, $\phi^K(p^*(p,n^K),p)$, is strictly quasi-convex in p.*

Proof: By definition of $p^*(p,n^K)$, the function$\phi^K(p^*(p,n^K),p)$ achieves a minimum at $\hat{p}(n^K)$. To prove quasi-convexity we only have to show that $\phi^K(p^*(p,n^K),p)$ is increasing at $p > \hat{p}(n^K)$ and decreasing for $p < \hat{p}(n^K)$.

Suppose first that $p > \hat{p}(n^K)$. Then:

$$\begin{aligned}
\frac{d\phi^K(p^*(p,n^K),p)}{dp} &= \pi^K_{p_{K'}}(p^*(p,n^K),p) - \pi^K_{p_{K'}}(p,p) - \pi^K_{p_K}(p,p) \\
&= -\int_{p^*(p,n^K)}^{p} \pi^K_{p_{K'}p_K}(z,p)dz - \pi^K_{p_K}(p,p) \\
&= -\int_{p^*(p,n^K)}^{p} \left[\pi^K_{p_{K'}p_K}(z,p) + \pi^K_{p_K p_K}(z,p)\right] dz - \pi^K_{p_K}(p^*(p,n^K),p) \\
&= -\int_{p^*(p,n^K)}^{p} \left[\pi^K_{p_{K'}p_K}(z,p) + \pi^K_{p_K p_K}(z,p)\right] dz > 0
\end{aligned}$$

where the last equality follows from the definition of $p^*(p,n^K)$ and the last inequality follows from assumption 1 since own-effects are assumed to dominate cross-effects effects. To complete the proof note that $p < \hat{p}(n^K)$ implies that in the second line above $\pi^K_{p_K}(p,p) > 0$, so that by strategic complementarity it follows that $\frac{d\phi^K(p^*(p,n^K),p)}{dp} < 0$.

Proposition 1: *In any symmetric optimal punishment equilibrium: (a) the smallest firm always has the largest incentive to deviate from the most profitable collusive price P^c, and (b) the largest firm has the greatest incentive to deviate from the optimal punishment price p_L.*

Proof: We first prove part (a) of the proposition. Let Υ be the set of firms that have a strict incentive to deviate upwards at P^c and let Δ be the set of firms that have a weak incentive to deviate downwards. If $P^c \geq \hat{p}(\bar{n})$, then $\Upsilon = \emptyset$ and $\underline{K} \in \Delta$, so that by Lemma 3 the proposition holds trivially. Suppose $P^c < \hat{p}(\bar{n})$. Then $\bar{K} \in \Upsilon$ and $\underline{K} \in \Delta$ or $\Delta = \emptyset$. Hence, by Lemma 3, either $\bar{K}$ or $\underline{K}$ has the highest incentive to deviate from P^c. Suppose it were $\bar{K}$. By Lemma 3A, $\phi^{\bar{K}}(p^*(p,\bar{n}),p)$ is strictly quasi-convex in p, and the incentive constraint of $\bar{K}$ can be relaxed by setting $P^c + \varepsilon$. Since all other incentive constraints (5) were slack on P^c we can always choose ε small enough, so they remain slack. But then all incentive constraints (6) on p_L can still be satisfied because $\hat{v}$ must still be feasible if $\pi(P^c,P^c)$ increases. Hence, all firms make higher profits (since $\hat{p}(\bar{n}) < p^m$) by strict concavity of revenue. This contradicts the definition of P^c as the most profitable sustainable collusive price. Hence, $\underline{K}$ has the highest incentive to deviate from P^c.

To prove part (b) of the proposition, suppose $0 \leq p_L < p(\underline{n})$. Then $\Delta = \emptyset$ and by Lemma 3 firm $\bar{K}$ has the largest incentive to deviate upwards. Suppose $p_L \geq p(\underline{n})$. Then $\underline{K} \in \Delta$ and either $\bar{K} \in \Upsilon$ or $\Upsilon = \emptyset$. Hence, by Lemma 2, either $\underline{K}$ or $\bar{K}$ has the highest incentive to deviate.

Suppose it is $\underline{K}$ and therefore $p_L > p(\underline{n}) > 0$. Then, by quasi-convexity of $\phi^{\underline{K}}(p^*(p, n^{\underline{K}}), p)$ in p (Lemma 3A), the incentive constraint of $\underline{K}$ with respect to p_L can be strictly relaxed by lowering p_L slightly and maintaining $\hat{v}$ without making other constraints binding. Furthermore, all incentive constraints on P^c would be strictly relaxed, contradicting the definition of p_L. Hence, only the incentive of the largest firm to deviate determines the punishment price p_L.

Lemma 4: *For every $p^c > c$ there exists a unique $\bar{\delta}(p^c)$ such that this price can be sustained forever at a maximal punishment equilibrium if and only if $\delta \geq \bar{\delta}(p^c)$.*

Proof: Since all expressions in (**??**) and (6) are continuously differentiable in δ and p, it follows by the implicit function theorem that the two schedules $p_L^{\bar{K}}(\delta)$ and $p_L^{\underline{K}}(\delta)$ are continuous. First, assume that $p_L^{\bar{K}}(1) > p_L^{\underline{K}}(1)$. Then, given that $p_L^{\bar{K}}(0) > p_L^{\underline{K}}(0)$, there exists at least one $\bar{\delta}$, such that $p_L^{\bar{K}}(\bar{\delta}) = p_L^{\underline{K}}(\bar{\delta})$. Clearly at this point the incentives of the largest firm to deviate from p_L and the incentives of the smallest firm to deviate from p^c must be equal, so that:

$$\phi^{\underline{K}}(p^*(p^c, \underline{n}), p^c) = \phi^{\bar{K}}(p^*(p_L^{\bar{K}}(\bar{\delta}), \bar{n}), p_L^{\bar{K}}(\bar{\delta})). \tag{9}$$

Suppose that there exists another intersection. Then there exists either a $\delta > \bar{\delta}$ such that $p_L^{\bar{K}}(\delta) \geq p_L^{\underline{K}}(\delta)$ or a $\delta < \bar{\delta}$ such that $p_L^{\bar{K}}(\delta) \leq p_L^{\underline{K}}(\delta)$. Consider the first case. Then $\delta\left[\pi^{\bar{K}}(p_L^{\bar{K}}(\delta), p_L^{\bar{K}}(\delta)) - \pi^{\bar{K}}(p^c, p^c)\right] > \phi^{\underline{K}}(p^*(p^c, n^{\underline{K}}), p^c)$. But since by Lemma 2, $p_L^{\bar{K}}(\delta) < \hat{p}(\bar{n})$ for all δ and by Lemma 3 $\phi^{\bar{K}}(p^*(p, \bar{n}), p)$ is decreasing at $p < \hat{p}(\bar{n})$, the incentive constraint of $\bar{K}$ must be slack at $p_L^{\bar{K}}(\delta)$, a contradiction. By a symmetric argument it is shown that the incentive constraint is violated if $p_L^{\bar{K}}(\delta) \leq p_L^{\underline{K}}(\delta)$ at some $\delta < \bar{\delta}$, again contradicting the definition of p_L. Hence, there is a unique intersection between the schedules $p_L^{\bar{K}}(\delta)$ and $p_L^{\underline{K}}(\delta)$ as shown in Figure 2. Now suppose that $p_L^{\bar{K}}(1) \leq p_L^{\underline{K}}(1)$. Then, by monotonicity of $p_L^{\underline{K}}(1)$ there is no intersection between the two schedules. In fact, whenever (5) is satisfied, (6) must be strictly slack. Hence, $\bar{\delta}$ solves $p_L^{\underline{K}}(\bar{\delta}) = 0$ in this case. Now note that all incentive constraints are relaxed when δ is increased. Hence, since by definition p^c can just be sustained forever at $\bar{\delta}(p^c)$, it can be sustained if and only if $\delta \geq \bar{\delta}(p^c)$.

Proposition 2: *There exists $\underline{\delta}$ and $\bar{\delta}(p^m)$ with $\underline{\delta} < \bar{\delta}(p^m) < 1$ such that for all $\delta \in [\underline{\delta}, 1]$ there exists an optimal symmetric punishment equilibrium. The optimal collusive price $P^c(\delta)$ is equal to p^m for all $\delta \in [\bar{\delta}(p^m), 1]$ and it is strictly increasing on $\delta \in [\underline{\delta}, \bar{\delta}(p^m)]$, piecewise continuous with at most* *a finite number of upward jumps. The optimal punishment price $p_L(\delta)$ is decreasing in δ on $\delta \in [\underline{\delta}, 1]$ and strictly decreasing when $p_L(\delta) > 0$.*

Proof: By our argument above there exists a function $\bar{\delta}(p^c)$ that relates the price p^c to the lowest discount factor at which it can be sustained and hence has a global minimum on $[\underline{p}, p^m]$. Since both incentive conditions are continuously differentiable it follows from the implicit function theorem that $\bar{\delta}(p^c)$ is continuous. Let P^δ be the set of prices that are sustainable at δ. Since $\bar{\delta}(p^c)$ is continuous and for every pair (δ, p^c) that is weakly above the schedule p^c can be sustained at δ, it follows that this set is bounded and closed so that $P^c(\delta) \equiv \min\{p^m, \max \mathsf{P}^\delta\}$, whenever P^δ is non-empty. Since for δ close to 1 a punishment price p_L arbitrarily close to 0 is sustainable for any $p^c > c$ and gains from deviation become arbitrarily small relative to the losses from continuing with p_L, the incentive constraint for $\underline{k}$ must become slack for $\delta \to 1$. Hence, $\bar{\delta}(p^m) < 1$.

Now note that $\mathsf{P}^{\bar{\delta}(p^m)}$ contains a set of prices below p^m. To see this note that a small change in p^c at p^m has only a second order effect on $\pi(p^c, p^c)$. Hence, any accommodating change in p_L to leave the incentive constraint of $\bar{k}$ unchanged has a second order effect on $\pi(p_L, p_L)$. But this means that any small change in p^c that leaves $\bar{k}$ on his incentive constraint will have only a second order effect on $\underline{k}$'s losses from inducing punishments. At the same time, $\phi(p^*(p^c, n^{\underline{K}}), p^c, n^{\underline{K}})$ is strictly increasing at $p^c = p^m$, a first order effect. Hence, the incentive constraint of $\underline{k}$ can be made strictly slack at $\bar{\delta}(p^m)$ for some small interval of prices $(\hat{p}^c, p^m)$. It follows that some price $p^c < p^m$ from that interval can be sustained at some $\hat{\delta} < \bar{\delta}(p^m)$ and for all higher δ. Hence, there exists $\underline{\delta} \geq 0$ such that $\underline{\delta} < \bar{\delta}(p^m)$ such that collusion at some price p^c forever is sustainable for all $\delta \geq \underline{\delta}$. Furthermore, Since for some $\hat{\delta} > 0$ we have $p_L^{\frac{K}{L}}(\hat{\delta}) = 0$ and $\hat{\delta} \leq \underline{\delta}$ from the proof of Lemma 4 it follows that $\underline{\delta} > 0$.

Now consider any $\delta \geq \underline{\delta}$ and $\delta^+ > \delta$. Clearly, at δ^+, both incentive constraints are slack for $(P^c(\delta), p_L(\delta))$, so that some price strictly above $P^c(\delta)$ is sustainable. Hence, $P^c(\delta)$ is strictly increasing in δ on $(\underline{\delta}, \bar{\delta}(p^m))$ and is constant above that. To complete the proof of the properties of $P^c(\delta)$ note that, by continuity of $\bar{\delta}(p^c)$, $P^c(\delta)$ must change continuously unless there is a strict upward jump to a local minimum of $\bar{\delta}(p^c)$. Since there must be generically only a finite number of minima of this function $P^c(\delta)$ will therefore be continuous except possibly for a finite number of upward jumps.

To prove the properties of $p_L(\delta)$ first note that at $\bar{\delta}(p^c) \leq \bar{\delta}(p^m)$ we must have

$$\phi^{\underline{K}}(p^*(p^c, \underline{n}), p^c) = \phi^{\bar{K}}(p^*(p_L, \bar{n}), p_L) = 0,$$

unless $p_L = 0$. Since by Lemma 3 and proposition 1 the function $\phi^{\underline{K}}(p^*(p^c, \underline{n}), p^c)$ is increasing in p^c and $\phi^{\bar{K}}(p^*(p_L, \bar{n}), p_L)$ is decreasing in p_L, $p_L(\delta)$ has to be lower when $P^c(\delta)$ is higher as long as $p_L(\delta) > 0$. Hence, $p_L(\delta)$ inherits the monotonicity properties of $P^c(\delta)$ on $(\underline{\delta}, \bar{\delta}(p^m))$, as long as $p_L(\delta) > 0$. If $p_L(\delta) = 0$ for some δ it will be zero for all higher δ. For $\delta > \bar{\delta}(p^m)$, $P_L(\delta) = P_L^{\bar{K}}(\delta)$ and therefore decreasing in δ.

Proposition 3: *(a) Suppose that $\underline{K}$ is the same under κ_1 and κ_2. Then $\underline{\delta}(\kappa_1) > \underline{\delta}(\kappa_2)$ if and only if $\bar{K}_1 > \bar{K}_2$. (b) Suppose that $\bar{K}$ is the same under κ_1 and κ_2. Then $\underline{\delta}(\kappa_1) > \underline{\delta}(\kappa_2)$ if and only if $\underline{K}_1 < \underline{K}_2$.*

Proof: By definition $\underline{\delta}(\kappa_1)$ is the lowest discount factor for which collusion can be sustained under κ_1. Let the corresponding collusive price be $P^c(\kappa_1)$ and optimal punishment price $P_L(\kappa_1)$. We now show that an equilibrium with $p^c = P^c(\kappa_1)$ and $p_L = P_L(\kappa_1)$ can still be sustained under κ_2, but leaves the incentive constraint for firm $\underline{k}$ unchanged. Since $\underline{K}_1 = \underline{K}_2$, it is immediate that firm $\underline{k}$'s incentive constraint still holds with equality under κ_1. Since $\bar{K}_1 > \bar{K}_2$, the incentives have changed for firm $\bar{k}$. Since the number of varieties of the largest firm goes down when switching to distribution κ_2, by Lemma 3A, the incentive to deviate for the largest firm goes down at given collusive and punishment prices. Hence, the incentive constraint of $\bar{k}$ becomes strictly slack. Then p_L could be slightly decreased to $\hat{p}_L = p_L - \varepsilon$ and the incentive condition for $\bar{k}$ would still be slack. But the incentive condition for $\underline{k}$ would now also be strictly slack. Hence, collusion at price $p^c = P^c(\kappa_1)$ and $p_L = \hat{p}_L$ can also be sustained for some $\delta < \underline{\delta}(\kappa_1)$. Then $\underline{\delta}(\kappa_1) > \underline{\delta}(\kappa_2)$, proving part (a). The argument for part (b) is symmetric and omitted.

Proposition 4: *Consider a transaction that changes the distribution of varieties in the market from κ_1 to κ_2 and fix $\delta \in [\underline{\delta}_{\max}, \bar{\delta}_{\min})$. Then: (a) If $\bar{K}_1 < \bar{K}_2$ and $\underline{K}_1 = \underline{K}_2$ then $P^c(\delta, \kappa_1) > P^c(\delta, \kappa_2)$, (b) If $\underline{K}_1 < \underline{K}_2$ and $\bar{K}_1 = \bar{K}_2$ then $P^c(\delta, \kappa_1) < P^c(\delta, \kappa_2)$, and (c) Among all redistributions of assets between two firms, those from the largest to the smallest firm will increase P^c most, as long as these do not make the previously smallest firm into the largest firm.*

Proof: To prove part (a), we first show that an equilibrium with $p^c = P^c(\delta, \kappa_2)$ and $p_L = P_L(\delta, \kappa_2)$ can still be sustained under κ_2. Since $\underline{K}_1 = \underline{K}_2$, it is immediate that firm $\underline{k}$'s incentive constraint still holds with equality under κ_1. Since $\bar{K}_1 < \bar{K}_2$, the incentives have changed for firm $\bar{k}$: the number of varieties of the largest firm goes down when switching to distribution κ_1. By Lemma 3A, the incentive to deviate for the largest firm goes down at given collusive and punishment prices when switching from κ_2

to κ_1. Hence, the incentive constraint of $\bar{k}$ becomes strictly slack. Then p_L could be slightly decreased to $\hat{p}_L = p_L - \varepsilon$ and the incentive condition for $\bar{k}$ would still be slack. But then the incentive condition for $\underline{k}$ would now also be strictly slack. Hence, collusion at price $p^c = P^c(\kappa_1) + \gamma$ and $p_L = \hat{p}_L$ can also be sustained for the same δ. It follows that $P^c(\delta, \kappa_1) > P^c(\delta, \kappa_2)$, proving part (a). The argument for part (b) is symmetric and omitted. To prove (c) note that the most a single transaction can change the incentives of the largest firm is by making it as large as the second largest firm. Similarly, the most that the size of the smallest firm can be increased is to the size of the second smallest firm. any transaction between the largest and the smallest firm will therefore relax both the upward and the downward incentive constraints, as long as the transaction does not turn the smallest firm into the largest. The claim then follows from parts (a) and (b).

Lemma 5: *Suppose that $p_L^K \neq 0$ for all K. Then the equilibrium value set of the full game under quadratic preferences is the same as the equilibrium value set of a game in which the firms are restricted to charging the same price on each variety they control.*

If $p_L^K \neq \mathbf{0}$ for all K any value in the value set can be generated from the profits generated by a price vector $\mathbf{p}$ that satisfies the incentive constraints in one period and the corresponding continuation value. In this case the continuation value will always be from the Pareto frontier of the equilibrium value set or, after a deviation, the values that correspond to the lowest equilibrium value for the deviating firm. This means that we can generate the whole equilibrium value set from a problem in which firms are restricted to set the same price for all products in their product line if prices that support equilibria on the Pareto frontier of the equilibrium set or prices that support optimal punishments have the equal price property. We now show that this is the case.

We first show that an equilibrium in which some firm charges different prices for products in its product line cannot generate a value on the Pareto frontier of the equilibrium value set. Note that any price vector that supports an equilibrium that generates a point on the Pareto frontier of the equilibrium value set must involve this price vector being set forever. Assume that the price vector involves one firm charging different prices for products in its product line. Then the price vector $\mathbf{p}_K$ has to satisfy:

$$\sum_{i \in K} \pi_i(\mathbf{p}_K, \mathbf{p}_{k^c}) = (1 - \delta) \sum_{i \in K} \pi_i(\mathbf{p}_K^*(\mathbf{p}_{K^c}), \mathbf{p}_{K^c}) + \delta \underline{v}_K$$

where $\underline{v}_K$ is the lowest average continuation profit achievable in any equilibrium for firm K. By symmetry of the profit function the best response prices in the vector $\mathbf{p}_K^*(\mathbf{p}_{K^c})$ are the same for all products in the product line of K. Now note that under the linear demand system demand for good i only depends on its own price and the averages of the prices in each product line. Hence, if we leave the average price for firm K unchanged no incentive constraints for other firms are affected. On the other hand the incentive constraints for firm K can be strictly relaxed if $\sum_{i\in K}\pi_i(\mathbf{p}_K,\mathbf{p}_{K^c})$ can be increased while leaving the average price in the product line K unchanged. Since $\sum_{i\in K}\pi_i(\mathbf{p}_K,\mathbf{p}_{K^c})$ is quasi-concave in $\mathbf{p}_K$ it has a unique maximum relative to a linear constraint. Furthermore, by symmetry of the objective and the constraint this maximum is achieved at equal prices. Hence, a price vector $\mathbf{p}_K$ can only support an equilbirium that generates a point on the Pareto frontier of the equilibrium value set if the prices for all products in product line K are the same.

Now consider the worst equilibrium for firm K. First consider whether a firm other than firm K could ever charge different prices in an equilibrium that generates the most severe punishment for K. Let us call that firm J. Then the incentive constraint for J is given by:

$$(1-\delta)\sum_{i\in J}\pi_i(\mathbf{p}_{LK}^J,\mathbf{p}_{LK}^{J^c})+\delta\bar{v}_{JK}\geq(1-\delta)\sum_{i\in K}\pi_i(\mathbf{p}_J^*(\mathbf{p}_{LK}^{J^c}),\mathbf{p}_{LK}^{J^c})+\delta\underline{v}_J$$

where p_{LK}^j is the the punishment price vector charged by firm j when firm K has deviated in the previous period. $\bar{v}_{JK}$ is the continuation value for J when $\mathbf{p}_{LK}$ is charged. It must be on the Pareto frontier of the value set given the assumptionn of the Lemma. Finally, $\underline{v}_J$ is the continuation value should firm J deviate from $\mathbf{p}_{LK}^J$. As before fix the average price in the price vector $\mathbf{p}_{LK}^J$. By the same argument as before it is shown that the incentive constraint is strictly relaxed if firm J charges the average price for all products in the product line. Since the average price is unchanged, no other incentive constraint is affected. But then firm J can lower its average price with its incentive constraint still slack. But lowering $\mathbf{p}_{LK}^J$ will relax the incentive constraints for all other firms. Hence, the punishment for firm K can be made more severe, violating the assumption that $\mathbf{p}_{LK}$ was the vector supporting optimal punishments for firm K.

Finally, we need to consider whether firm K would ever charge different prices for its products when it is punished. Suppose it did. As in the previous arguments we know that a firm can always get higher profits from having the same prices for a given average price. Hence, it could obtain the same profits as with different prices at some lower average price which is set for all firms. But given that all other firms charge symmetric punishment prices it is straightforward to show that the incentive constraints for all firms $J \neq K$ are strictly relaxed. Hence, the punishment prices charged by firms $J \neq K$ can be reduced, which implies that an even lower punishment continuation value for K is achievable. It follows that the optimal punishment equilibrium for K does not have any firm charge different prices on its product line.

Proposition 6: *Suppose the smallest firm becomes arbitrarily small relative to the size of the second smallest firm. Then it is optimal for the larger firms to exclude it from an agreement on an optimal symmetric punishment equilibrium.*

Proof: Consider a situation with $m+1$ firms. We consider a sequence of industries along which we keep the number of varieties for the smallest firm fixed but replicate the number of varieties for all other firms along this sequence. In other words for all firms $K \neq \underline{K}$ we have $n_K^t = 2n_K^{t-1}$, while $\underline{n}^t = \underline{n}$ for all t where t goes to infinity. Note that along this sequence the number of varieties evolves as $n^t = 2(n^{t-1} - \underline{n}) + \underline{n}$ so that the total number of varieties goes to infinity as well. By the assumption that

$$\left| \frac{\partial \ln D^K(p_K, \mathbf{p}_{K^c})}{\partial p_K} \right| > \sum_{j \in K^c} \frac{\partial \ln D^K(p_K, \mathbf{p}_{K^c})}{\partial p_j} \tag{10}$$

it then must be the case that $\sum_{j \in \underline{K}} \frac{\partial D^K(p_K, \mathbf{p}_{K^c})}{\partial p_j}$ converges to zero at least at rate $\frac{1}{t}$ as $t \to \infty$. This is the main property we use for this proof.

Now consider two possible equilibria in the market. First, the equilibrium in which all firms collude according to the equilibrium in section 3 of the paper. This is characterized by the punishment price p_L^{all} and the highest sustainable collusive price P_{all}^c. Second, consider the equilibrium that is obtained when all firms except for the smallest optimally collude conditional on the smallest firm simply using a best response. We will show that the latter equilibrium will have a strictly higher collusive price $p^c = P_{all}^c + \varepsilon$ can be sustained and as a result strictly higher profits whenenver the smallest firm is small enough relative to the next larger firm.

Let $\widetilde{K}$ be the second smallest firm. We now construct an equilibrium with p^c for the collusive price and p_L for the punishment price, such that this can be sustained with firm $\underline{K}$ best responding. We simplify the proof by assuming that demand approaches infinity at zero price. Let $\pi(p, p_{\underline{K}}^*(p))$ be the profit per variety of firms other than $\underline{K}$ when $\underline{K}$ best responds to price p, and let $\pi^K(p^*(p), p, p_{\underline{K}}^*(p))$ be the best response profit of firm K when others play p except for firm $\underline{K}$ who best responds to everyone esle charging p. Note that $\lim_{t\to\infty}[\pi(p,p) - \pi(p, p_{\underline{K}}^*(p))] = 0$ and

$\lim_{t \to 0} \left[\pi^K(p^*(p), p, p) - \pi^K(p^*(p), p, p^*_{\underline{K}}(p)) \right] = 0$. If the incentive compatibility constraint for the second largest firm is satisfied, we would have:

$$\phi^{\widetilde{K}}(p^*(p^c), p^c, p^*_{\underline{K}}(p^c)) \geq \delta \left[\pi(p^c, p^*_{\underline{K}}(p^c)) - \pi(p_L, p_L, p^*_{\underline{K}}(p_L)) \right] \tag{11}$$

Taking limits for $t \to \infty$ on both sides we have:

$$\phi^{\widetilde{K}}(p^*(p^c), p^c, p^c) \geq \delta \left[\pi(p^c, p^c) - \pi(p_L, p_L, p_L) \right] \tag{12}$$

Now note that at $p^c = P^c$, this incentive constraint is strictly slack because firm $\widetilde{K}$ is strictly larger than firm $\underline{K}$. Hence, for some $t < \infty$ there will still exist $\varepsilon > 0$ such that the constraint is slack. Now consider the incentive comnstraint for the largest firm, which takes into account that p^c is the new collusive price that can be sustained:

$$\phi^{\widetilde{K}}(p^*(p_L), p_L, p^*_{\underline{K}}(p_L)) \geq \delta \left[\pi(p^c, p^*_{\underline{K}}(p^c)) - \pi(p_L, p_L, p^*_{\underline{K}}(p_L)) \right]$$

Taking limits on both sides yields:

$$\phi^{\widetilde{K}}(p^*(p_L), p_L, p_L) \geq \delta \left[\pi(p^c, p^c)) - \pi(p_L, p_L, p_L) \right]$$

This is the same constraint as in the case where firm $\underline{K}$ is included in the collusive scheme, only that $p^c > P^c$. Hence, relative to that benchmark the incentive constraint at the limit is strictly relaxed. We can therefore find a t large enough and an $\varepsilon > 0$ such that both incentive constraints are saistified. Hence, for t large enough some $p^c > P^c$ is sustainable. Furthermore since

$$\lim_{t \to \infty} [\pi(p^c, p^*_{\underline{K}}(p^c)) - \pi(P^c, P^c)] = \pi(p^c, p^c) - \pi(P^c, P^c) > 0$$

the colluding firms strictly benefit from excluding the smallest firm for large enough t. Hence, if the smallest firm becomes arbitrarily small relative to the second smallest, the other firms benefit by letting that firm best respond and limiting the collusive scheme to themselves.