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Existence of Positive Solutions to a Family of Fractional Two-Point Boundary Value Problems

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DEDICATION

I dedicate this thesis to my best friend Joseph and to my other friends and family members who have supported me.

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ABSTRACT

In this paper we will consider an nth order fractional boundary value problem with boundary conditions that include a fractional derivative at 1. We will develop properties of the Green's Function for this boundary value problem and use these properties along with the Contraction Mapping Principle, and the Schuader's, Krasnozel'skii's, and Legget-Williams fixed point theorems to prove the existence of positive solutions under different conditions.

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Chapter 1

Introduction

Taking an nth order derivative of a function, where n is a positive integer, is an easy to grasp concept. To achieve this result, n derivatives of the function must be taken. However, when α is a positive value that is not an integer, taking α derivatives of a function is not as simple to visualize. The same thing can be said about taking α integrals of a function. For this reason and other contributing factors, the development of the theory behind fractional calculus was slow. Since the concept of the fractional derivative was first brought into question in 1695, many distinguished mathematicians have had a part in the development of this field.

Until recently, many of the definitions that were developed for fractional operations were very narrow in scope, encompassing only certain types of functions. There were also definitions that seemed to contradict one another. Today, there is a unifying system for taking fractional derivatives and integrals, and many of the properties of this system mirror the properties of integer order calculus.

1.1

History of Fractional Calculus

The study of *fractional* calculus began as an exploration into whether the meaning of a derivative $\dfrac{d^n y}{dx^n}$ of integer order could be extended to have meaning when n was a fractional value. Since then, the question of whether n can be irrational or complex has also been posed. Because it was discovered that the meaning could be extended to fractional n, the term *fractional calculus* was adopted and has since become a misnomer, given that the meaning can be extended to not only fractional values, but to irrational and complex values as well.

In 1695, L'Hôpital posed a question about what would happen if n were $\frac{1}{2}$ to Liebniz, who had invented the notation $\dfrac{d^n y}{dx^n}$. In response, Liebniz said that the quantity $d^{1/2}\overline{xy}$ or $d^{1:2}\overline{xy}$ could be represented by an infinite series, although infinite series permitted only the use of exponents that were positive and negative integers. He then stated (see [15]) that $d^{\frac{1}{2}}x$ would be equal to $x\sqrt{dx : x}$ and that useful consequences would one day be drawn from this. This prediction was correct.

Euler (see [15]) contributed to fractional calculus in 1730 when he commented that, in the case that n is a positive integer, repeated differentiation would achieve the result of $\dfrac{d^n p}{dx^n}$, but that a way such as this to achieve the result when n was a fractional value was not evident. He suggested the use of interpolation as was described in the same paper. In 1731, Euler, while considering fractional differentiation, extended the well-known relation

$$\frac{d^n z^p}{dz^n} = \frac{p!}{(p-n)!} z^{p-n} \tag{1.1}$$

to $n = \alpha$ for arbitrary α as

$$\frac{d^\alpha z^p}{dz^\alpha} = D_z^\alpha z^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} z^{p-\alpha}. \tag{1.2}$$

This formula was in fact (see [12]) what led Euler to invent the Gamma function for factorials of fractional values.

In 1812, L. S. Laplace defined the fractional derivative via an integral, and, finally, the first appearance of a derivative of arbitrary order appeared in a text by S. F. Lacroix, although only 2 pages of the 700-page volume was devoted to the topic (see [15]). In his text, Lacroix develops the nth derivative of $y = x^m$, writing

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, \qquad \text{where } m \geq n. \tag{1.3}$$

Notice that this definition is stated exactly as that of Euler, with the added stipulation that $m \geq n$.

Fractional calculus was also contributed to indirectly by J. L. Lagrange when he developed the law of indices for differential operators of integer order and Joseph B. J. Fourier when he used his integral representation of $f(x)$ to obtain a formula for $\dfrac{d^u}{dx^u} f(x)$ where u is of arbitrary order.

The first to make use of fractional operations was Niels Henrik Abel in 1823. He applied fractional calculus (see [15]) to solve the integral equation

$$\kappa = \int_0^x (x-t)^{-1/2} f(t)\, \mathrm{d}t \tag{1.4}$$

when he noticed that the right-hand side of this equation was a case of fractional integration with order $1/2$ without the multiplicative factor $\dfrac{1}{\Gamma\left(\frac{1}{2}\right)}$. Following this use of fractional calculus, there were no developments to the subject for almost a decade, and then the works of Joseph Liouville began to appear.

Liouville was the first to make a major study of fractional calculus. He developed two definitions given by

$$D^v \sum_{n=0}^{\infty} c_n e^{a_n x} = \sum_{n=0}^{\infty} c_n a_n^v e_n^x, \qquad \text{where } \operatorname{Re} a_n > 0, \tag{1.5}$$

and

$$D^v x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a-v}, \qquad \text{where } a > 0. \tag{1.6}$$

Liouville was also the first to attempt (see [15]) to solve differential equations that involved fractional operators.

Because the definitions given by Lacroix and Liouville were essentially different, there was much controversy over which system to accept, and many mathematicians became distrustful of fractional operations. While Lacroix's definition dealt with functions of the form x^a, where $a > 0$, Liouville's definition dealt with functions of the form x^{-a}, where $a > 0$. It was observed by William Center that, using Lacroix's definition,

$$\frac{d^{1/2}}{dx^{1/2}} x^0 = \frac{1}{\sqrt{\pi x}},$$

but, using Liouville's definition,

$$\frac{d^{1/2}}{dx^{1/2}} x^0 = 0,$$

because $\Gamma(0) = \infty$, even though in both definitions it was assumed that $a > 0$, and hence $\dfrac{d^u x^0}{dx^u}$ was not defined (see [15]). The debate then became, "What is $\dfrac{d^u x^0}{dx^u}$?" It was suggested (see [12]) by Augustus De Morgan that these definitions could very well be a part of a system that would incorporate both, and he proved to be correct.

As a student, G. F. Bernhard Riemann developed his version of the fractional integral, which he published in 1892. He set out to generalize the Taylor series to the fractional integral and obtained

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t-s)^{\nu-1} f(s)\, \mathrm{d}s + \Psi(t). \tag{1.7}$$

Here, Riemann added a complementary function $\Psi(x)$ because of the ambiguity in the lower limit of integration c. He was concerned about the measure of deviation caused by the law of exponents in the case ${}_cD_t^{-\mu}\,{}_{c'}D_t^{-\nu} f(t)$ when $c \neq c'$. The law of exponents mentioned here is given by

$${}_cD_t^{-\mu}\,{}_cD_t^{-\nu} f(t) = {}_cD_t^{-\mu-\nu} f(t), \tag{1.8}$$

where t and c denote the limits of integration. The question of whether or not this complementary function existed caused confusion among many, including A. Cayley who commented that the meaning of this complementary function, which would contain an "infinity of arbitrary constants," was the greatest difficulty of Riemann's theory. Cayley later noted (see [15]) that Riemann was greatly entangled with the existence of this complementary function.

Liouville also commented on the existence of a complementary function when he gave an explicit evaluation of his interpretation of this function. However, he did not consider a special case, which led to a contradiction, thus proving his evaluation wrong. Peacock, who agreed that the Lacroix definition of the fractional derivative was the correct form, also made two errors on the topic of fractional calculus. This added to the uneasiness surrounding the topic of fractional calculus.

1.2 Riemann-Liouville Fractional Calculus

The work that ultimately led to the Riemann-Liouville definition, which is most commonly used today, appeared in a paper by N. Ya. Sonin in 1869 that was entitled "On differentiation with arbitrary index," whose starting point was Cauchy's integral formula. A paper written by A. V. Letnikov in 1872 later followed that was an extension of this paper. In their work, it was noted that the nth derivative of Cauchy's integral formula is given by

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \, \mathrm{d}\xi. \tag{1.9}$$

Generalizing $n!$ to arbitrary n can easily be done using the Gamma function. However, when n is no longer an integer, a problem occurs. The integrand of (1.9) no longer contains a pole. A branch cut would be required for a contour, which was discussed but not included (see [15]) in the work of either Sonin or Letnikov.

In 1884, H. Laurent published a paper which also used Cauchy's integral formula as its starting point. The contour that he used was an open circuit on a Riemann surface. Using this definition of contour integration, Laurent was led to

$$_cD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t-s)^{\nu-1} f(s) \, \mathrm{d}s, \qquad \operatorname{Re}\nu > 0, \tag{1.10}$$

for an integral of arbitrary order. Notice that, when $t > c$, this formula is the same as that of Riemann, without the addition of a complementary function.

Now, a sufficient condition that (1.10) converges is that

$$f\left(\frac{1}{t}\right) = O(t^{1-\epsilon}), \qquad \text{for some } \epsilon > 0, \tag{1.11}$$

meaning that there exist constants C and $r > 0$ such that

$$\left| f\left(\frac{1}{t}\right) \right| \leq C|t^{1-\epsilon}|, \qquad \text{for some } \epsilon > 0. \tag{1.12}$$

Functions that are integrable and satisfy condition (1.12) are referred to as functions of the Riemann class.

If $c = -\infty$, then (1.10) becomes

$${}_{-\infty}D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{t} (t-s)^{\nu-1} f(s)\,\mathrm{d}s, \qquad \operatorname{Re}\nu > 0. \tag{1.13}$$

A sufficient condition that (1.13) converges is that

$$f(-t) = O(t^{-\nu-\epsilon}), \qquad \text{for some } \epsilon > 0, \tag{1.14}$$

meaning that there exist constants C and $r > 0$ such that

$$|f(-t)| \leq C|t^{-\nu-\epsilon}|, \qquad \text{for some } \epsilon > 0. \tag{1.15}$$

Functions that are integrable and satisfy condition (1.14) are known as functions of the Liouville class. Notice that both the definitions given by Lacroix and by Liouville, which had previously sparked a debate in the mathematical community, hold true under (1.10) and (1.13).

The most frequently used version of (1.10) is when $c = 0$,

$${}_{0}D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-s)^{\nu-1} f(s)\,\mathrm{d}s, \qquad \operatorname{Re}\nu > 0. \tag{1.16}$$

This is referred to as the standard Riemann-Liouville fractional integral.

In the case that the upper limit of integration is ∞, the Weyl fractional integral

$${}_{t}W_{\infty}^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t}^{\infty} (s-t)^{\nu-1} f(s)\,\mathrm{d}s, \qquad \operatorname{Re}\nu > 0, \tag{1.17}$$

is often used (see [15]) in place of (1.10).

Notice now that we have defined $D^{-\nu}$ for $\operatorname{Re}\nu > 0$, but we have not yet defined D^{ν} for $\operatorname{Re}\nu > 0$, or, in other words, the fractional derivative. In this case, let n be the smallest integer greater than $\operatorname{Re}\nu$, and let $v = n - \nu$. Then

$0 < \operatorname{Re} v \leq 1$, and the fractional derivative of $f(t)$ of arbitrary order ν is

$$ {}_cD_t^{\nu} f(t) = {}_cD_t^{n}[{}_cD_t^{-v} f(t)], \tag{1.18} $$

where $t > 0$, and ${}_0D_t^{v} f(t)$ will denote the standard Riemann-Liouville fractional derivative of $f(t)$.

For the purposes of this paper, ${}_0D_{\infty}^{-\nu} f(t)$ will be denoted by $I_{0+}^{\nu} f(t)$ and ${}_0D_{\infty}^{\nu} f(t)$ will be denoted by $D_{0+}^{\nu} f(t)$.

1.3

Modern Uses of Fractional Differential Equations

Alhough at first it seemed that there was no practical use for fractional operations, today fractional differential equations are used in almost every branch of science. The study of fractional Boundary Value Problems (BVP's) is used (see [14, 15]) in the fields of physics, biology, medicine, control theory, fluid flow, rhealogy, diffuse transport akin to diffusion, electrical networks, electromagnetic theory, and probability. In fact, fractional differential equations model certain situations, such as the study of heredity and memory problems, better (see [14]) than differential equations of integer order.

Today, there are an increasing number of papers relating to differential equations of arbitrary order being published. The use of fixed point theory and cone-theoretic techniques to show the existence of solutions to difference equations, ordinary differential equations, and singular boundary value problems is abundant, (see [1, 6, 8]) but still far less work has been done to develop the existence of solutions to fractional, or arbitrary order differential equations, as in [2, 3, 9, 14].

Chapter 2

The Existence of Positive Solutions to a Family of Fractional Two-Point Boundary Value Problems

Let $n \geq 2$ denote an integer, and let α and β be positive reals such that $n-1 < \alpha \leq n$ and $0 \leq j \leq \beta \leq n-1$, for some $j \in \{0,\ 1, \ldots,\ n-2\}$. We will consider the boundary value problem for the fractional differential equation given by

$$D_{0+}^{\alpha} u + f(t, u, u', \ldots, u^{(j)}) = 0, \qquad \text{where } 0 < t < 1, \tag{2.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \text{ for } i = 0, 1, \ldots, n-2, \qquad \text{and } D_{0+}^{\beta} u(1) = 0, \tag{2.2}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives.

2.1

Preliminary Definitions

In this section, we will collect some essential definitions.

Definition 2.1. For $0 < t < \infty$, the Gamma Function, $\Gamma(t)$, is defined by

$$\Gamma(t) = \int_0^{\infty} s^{t-1} e^{-s} \, \mathrm{d}s. \tag{2.3}$$

The Gamma Function has the following two properties:

(i) For each $t \in (0, \infty)$, $\Gamma(t+1) = t\Gamma(t)$.

(ii) For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

Property (i) can be used to extend Definition 2.1 to negative non-integer numbers. Notice that if n is a nonpositive integer, then $\Gamma(n)$ is not defined. In this case, the convention that $\dfrac{1}{\Gamma(n)} = 0$ will be adopted.

Definition 2.2. Let $v > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0+}^{\nu} u$, is defined as

$$I_{0+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\,\mathrm{d}s, \tag{2.4}$$

provided that the right-hand side exists.

Definition 2.3. Let n denote a positive integer, and assume that the positive real α satisfies $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\,\mathrm{d}s \\ &= D^n I_{0+}^{n-\alpha} u(t), \end{aligned} \tag{2.5}$$

provided the right-hand side exists.

2.2

The Green's Function

The Green's Function for the boundary value problem (2.1), (2.2) is given by (see [5])

$$G(\beta; t, s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \leq s \leq t < 1, \\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & \text{if } 0 \leq t \leq s < 1. \end{cases} \tag{2.6}$$

Thus, u is a solution of (2.1), (2.2) if and only if

$$u(t) = \int_0^1 G(\beta; t, s) f(s, u(s), u'(s), \dots, u^{(j)}(s))\,\mathrm{d}s, \qquad 0 \leq t \leq 1.$$

We will develop properties of (2.6) to prove the existence of positive solutions to (2.1), (2.2).

Lemma 2.1. *Let β be a positive real and $j \in \{0, 1, \dots, n-2\}$ be an integer, satisfying $0 \leq j \leq \beta \leq n-1$. The kernel, $G(\beta; t, s)$, satisfies the following properties:*

$$\frac{\partial^i}{\partial t^i} G(\beta; t, s) \geq 0, \;\; (t,s) \in [0,1] \times [0,1), \qquad \textit{for } i = 0, 1, \dots, j, \tag{2.7}$$

$$\max_{0\le t\le 1}\int_0^1 \frac{\partial^i}{\partial t^i}G(\beta;t,s)\,\mathrm{d}s = \frac{(\alpha-i)t_i^{\alpha-i-1}-(\alpha-\beta)t_i^{\alpha-1}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)} := \overline{G}_i, \tag{2.8}$$

$$\textit{where } t_i = \min\left\{\frac{(\alpha-1-i)}{(\alpha-\beta)},\ 1\right\}.$$

Proof. Define, for $0 \le s \le t < 1$, the function g_1 by

$$g_1(\beta;t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{2.9}$$

and define, for $0 \le t \le s < 1$, the function g_2 by

$$g_2(\beta;t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)},\ 0 \le t \le s < 1. \tag{2.10}$$

In order to prove (2.7), let s and t be positive reals such that $0 \le s \le t < 1$, and let $i \in \{0,\ 1, \ldots,\ j\}$. Then

$$\begin{aligned}
\frac{\partial^i}{\partial t^i}G(\beta;t,s) &= \frac{\partial^i}{\partial t^i}g_1(\beta;t,s)\\
&= \frac{\partial^i}{\partial t^i}\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&= \frac{1}{\Gamma(\alpha)}\frac{\partial^i}{\partial t^i}[t^{\alpha-1}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1}]\\
&= \frac{1}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-1-\beta}\Gamma(\alpha)t^{\alpha-1-i}}{\Gamma(\alpha-i)} - \frac{\Gamma(\alpha)(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)}\right]\\
&= \frac{1}{\Gamma(\alpha-i)}[(1-s)^{\alpha-1-\beta}t^{\alpha-1-i}-(t-s)^{\alpha-1-i}]\\
&\ge \frac{1}{\Gamma(\alpha-i)}[(1-s)^{\alpha-1-i}t^{\alpha-1-i}-(t-s)^{\alpha-1-i}]\\
&= \frac{1}{\Gamma(\alpha-i)}[(t-ts)^{\alpha-1-i}-(t-s)^{\alpha-1-i}].
\end{aligned}$$

But $ts < s$, and hence $\frac{1}{\Gamma(\alpha-i)}[(t-ts)^{\alpha-1-i}-(t-s)^{\alpha-1-i}] > 0$, implying that $\frac{\partial^i}{\partial t^i}g_1(\beta;t,s) \ge 0$.

Next, let $0 \le t \le s < 1$ and $i \in \{0,\ 1, \ldots,\ j\}$. Then

$$\begin{aligned}
\frac{\partial^i}{\partial t^i}G(\beta;t,s) &= \frac{\partial^i}{\partial t^i}g_2(\beta;t,s)\\
&= \frac{\partial^i}{\partial t^i}\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{(1-s)^{\alpha-1-\beta}\Gamma(\alpha)t^{\alpha-1-i}}{\Gamma(\alpha-i)} \\
&= \frac{1}{\Gamma(\alpha-i)}(1-s)^{\alpha-1-\beta}t^{\alpha-1-i} \\
&\geq 0,
\end{aligned}
$$

and hence $\frac{\partial^i}{\partial t^i}G(\beta;t,s) \geq 0$, when $i \in \{0,\ 1,\dots,\ j\}$. This proves (2.7).

Now,

$$
\begin{aligned}
\int_0^t \frac{\partial^i}{\partial t^i} g_1(\beta;t,s)\,\mathrm{d}s &= \int_0^t \frac{\partial^i}{\partial t^i}\left(\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\mathrm{d}s \\
&= \int_0^t \frac{t^{\alpha-1-i}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)}\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha-i)}\left[\frac{-t^{\alpha-1-i}(1-s)^{\alpha-\beta}}{\alpha-\beta}+\frac{(t-s)^{\alpha-i}}{\alpha-i}\right]_0^t \\
&= \frac{1}{\Gamma(\alpha-i)}\left[\frac{-t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\alpha-\beta}+\frac{t^{\alpha-1-i}}{\alpha-\beta}-\frac{t^{\alpha-i}}{\alpha-i}\right] \\
&= \frac{1}{\Gamma(\alpha-i)}\left[\frac{t^{\alpha-1-i}-t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\alpha-\beta}-\frac{t^{\alpha-1}}{\alpha-i}\right] \\
&= \frac{t^{\alpha-1-i}(\alpha-i)-t^{\alpha-1-i}(1-t)^{\alpha-\beta}(\alpha-i)-t^{\alpha-i}(\alpha-\beta)}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_t^1 \frac{\partial^i}{\partial t^i} g_2(\beta;t,s)\,\mathrm{d}s &= \int_t^1 \frac{\partial^i}{\partial t^i}\left(\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}\right)\mathrm{d}s \\
&= \int_t^1 \frac{t^{\alpha-1-i}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha-i)}\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha-i)}\left[\frac{-t^{\alpha-1-i}(1-s)^{\alpha-\beta}}{\alpha-\beta}\right]_t^1 \\
&= \frac{1}{\Gamma(\alpha-i)}\left[\frac{t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\alpha-\beta}\right] \\
&= \frac{t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\Gamma(\alpha-i)(\alpha-\beta)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\,\mathrm{d}s \\
&\quad= \frac{t^{\alpha-1-i}(\alpha-i)-t^{\alpha-1-i}(1-t)^{\alpha-\beta}(\alpha-i)-t^{\alpha-i}(\alpha-\beta)}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}+\frac{t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\Gamma(\alpha-i)(\alpha-\beta)} \\
&\quad= \frac{t^{\alpha-i-1}(\alpha-i)-t^{\alpha-i}(\alpha-\beta)}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}.
\end{aligned}
$$

Now, by properties of the first derivative of any function, $\max_{t\in[0,1]} \int_0^1 \frac{\partial^i}{\partial t^i} G(\beta; t, s)\,\mathrm{d}s$ occurs when

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\int_0^1 \frac{\partial^i}{\partial t^i} G(\beta; t, s)\,\mathrm{d}s\right] &= \frac{\partial}{\partial t}\frac{(\alpha-i)t^{\alpha-i-1}-(\alpha-\beta)t^{\alpha-i}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}\\
&= \frac{(\alpha-1-i)(\alpha-i)t^{\alpha-2-i}-(\alpha-i)(\alpha-\beta)t^{\alpha-1-i}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}\\
&= (\alpha-1-i)(\alpha-i)t^{\alpha-2-i}-(\alpha-i)(\alpha-\beta)t^{\alpha-1-i}\\
&= 0,
\end{aligned}$$

which occurs when $t = \frac{\alpha-1-i}{\alpha-\beta}$. Note that if, $\frac{\alpha-1-i}{\alpha-\beta} > 1$, then by (2.7), the maximum occurs when $t = 1$. It follows that

$$\max_{0\le t\le 1} \int_0^1 \frac{\partial^i}{\partial t^i} G(\beta; t, s)\,\mathrm{d}s = \frac{(\alpha-i)t_i^{\alpha-i-1}-(\alpha-\beta)t_i^{\alpha-i}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}, \tag{2.11}$$

where $t_i = \min\left\{\frac{\alpha-i-1}{\alpha-\beta},\ 1\right\}$, which proves (2.8). □

2.3

Contraction Mapping Principle

The theory behind the use of the Contraction Mapping Principle in proving the existence of fixed points for differential equations has been studied in papers such as [16]. These authors make use of the Contraction Mapping Principle, stated below, to show the existence of solutions to differential equations of integer order in partially ordered and ordered metric spaces. The existence and uniqueness of solutions to a nonlinear fractional Cauchy problem in a special Banach space is developed in [4]. We will develop a theorem and proof for the existence and uniqueness of solutions of problem (2.1), (2.2) in the Banach space $C^{(j)}[0,1]$.

Definition 2.4. Let $\langle X, d\rangle$ be a metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $\alpha \in [0,1)$ such that $d(T(u), T(v)) \le \alpha d(u,v)$ for all $u,\ v \in X$.

Theorem 2.1 (Contraction Mapping Principle)**.** *[16] Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$ and let $T : \mathcal{B} \to \mathcal{B}$ be such that there exists $k \in [0,1)$ such that $\|Tu - Tv\| \le k\|u - v\|$ for all $u, v \in \mathcal{B}$. Then T has a unique fixed point in B. Moreover, if $u \in \mathcal{B}$, the sequence $\{Tu^n\}_{n=0}^{\infty}$ converges to the unique fixed point.*

Theorem 2.2. *Let $f(t, y_0, y_1, \ldots, y_j) : [0,1] \times \mathbb{R}^{j+1} \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition $|f(t, y_0, y_1, \ldots, y_j) - f(t, z_0, z_1, \ldots, z_j)| \leq k \sum_{i=0}^{j} |y_i - z_i|$ on $[0,1] \times \mathbb{R}^{j+1}$. Then, if $k \sum_{i=0}^{j} \overline{G}_i < 1$, (1.1), (1.2) has a unique solution.*

Proof. Consider the Banach space $C^{(j)}[0,1]$ with norm $\|u\| = \sum_{i=0}^{j} |u^{(i)}|_0$, where $|u|_0 = \max_{t \in [0,1]} |u(t)|$. Define the mapping $T : C^{(j)}[0,1] \to C^{(j)}[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s), \ldots, u^{(j)}(s)) \, \mathrm{d}s \tag{2.12}$$

for all $t \in [0,1]$ and $u \in C^{(j)}[0,1]$. Thus, if $\hat{u}$ is a fixed point of T, $\hat{u}$ solves (2.1), (2.2).

Now, the metric $d(u,v) = \|u - v\| = \sum_{i=0}^{j} |u^{(i)} - v^{(i)}|_0$ on $C^{(j)}[0,1]$ is induced by the norm on $C^{(j)}[0,1]$. Let $u,\ v \in C^{(j)}[0,1]$. Then, by (2.8)

$$\begin{aligned}
&|Tu - Tv|(t) \\
&\quad = \left| \int_0^1 G(\beta; t,s) f(s, u(s), \ldots, u^{(j)}(s)) \, \mathrm{d}s - \int_0^1 G(\beta; t,s) f(s, v(s), \ldots, v^{(j)}(s)) \, \mathrm{d}s \right| \\
&\quad = \left| \int_0^1 G(\beta; t,s) \left[f(s, u(s), \ldots, u^{(j)}(s)) - f(s, v(s), \ldots, v^{(j)}(s)) \right] \, \mathrm{d}s \right| \\
&\quad \leq \int_0^1 |G(\beta; t,s)| \, k \sum_{i=0}^{j} |u^{(i)} - v^{(i)}| \, \mathrm{d}s \\
&\quad \leq \int_0^1 |G(\beta; t,s)| k \sum_{i=0}^{j} |u^{(i)} - v^{(i)}|_0 \, \mathrm{d}s \\
&\quad = \int_0^1 |G(\beta; t,s)| k \|u - v\| \, \mathrm{d}s \\
&\quad \leq k \overline{G}_0 \|u - v\|.
\end{aligned}$$

Similarly, $|(Tu)^{(i)}(t) - (Tv)^{(i)}(t)| \leq k \|u - v\| \displaystyle\int_0^1 \frac{\partial^i G(\beta; t,s)}{\partial t^i} \, \mathrm{d}s \leq k \overline{G}_i \|u - v\|$, and, consequently,

$$\begin{aligned}
\|Tu - Tv\| &= \sum_{i=0}^{j} |Tu^{(i)} - Tv^{(i)}|_0 \\
&\leq k \sum_{i=0}^{j} \overline{G}_i \|u - v\|.
\end{aligned}$$

Hence, since $k \sum_{i=0}^{j} \overline{G}_i < 1$, T is a contraction mapping on $C^{(j)}[0,1]$, and thus T has a unique fixed point $\hat{u}(t)$ satisfying (2.12), which is the unique solution of (2.1), (2.2). □

2.4

Schauder Fixed Point Theorem

Let $n \geq 2$ denote an integer, and let α and β be positive reals such that $n-1 < \alpha \leq n$ and $0 \leq j \leq \beta \leq n-1$, for some $j \in \{0,\ 1, \dots,\ n-2\}$. We will consider the boundary value problem for the fractional differential equation given by

$$D^{\alpha}_{0^+}u + a(t)f(u, u', \dots, u^{(j)}) = 0, \qquad \text{where } 0 < t < 1, \tag{2.13}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0,\ \text{ for } i = 0, 1, \dots, n-2, \qquad \text{and } D^{\beta}_{0^+}u(1) = 0, \tag{2.14}$$

where $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ are the standard Riemann-Liouville fractional derivatives. Notice that the Green's Function for (2.13), (2.14) is given by (2.6) and Lemma 2.1 holds.

The Schauder Fixed Point Theorem has been utilized in the study and proof of existence of solutions to fractional order differential equations and systems of fractional order differential equations as well, see [10, 17]. We will use the Schauder fixed point theorem to show the existence of positive solutions of (2.13), (2.14). To this end, define $|a|_\infty = \operatorname*{ess\,sup}_{t\in[0,1]} |a(t)|$ to be the essential supremum of $|a|$.

We make the following assumptions on the functions f and a:

(A1) $f : [0, \infty) \to [0, \infty)$ is continuous, and

(A2) $a : [0, 1] \to [0, \infty)$ with $a \in L^\infty[0, 1]$.

Theorem 2.3 (Schauder Fixed Point Theorem [10])**.** *If $\mathcal{M}$ is a closed, bounded, convex subset of a Banach space $\mathcal{B}$ and $T : \mathcal{M} \to \mathcal{M}$ is completely continuous, then T has a fixed point in $\mathcal{M}$.*

Define the Banach space $\mathcal{B} = \{u \in C^{(j)}[0, 1] : u(0) = u'(0) = \cdots = u^{(j-1)}(0) = 0\}$ to be the space of all j times differentiable functions whose j^{th} derivative is continuous on the interval $[0, 1]$ which satisfy $u(0) = u'(0) = \cdots = u^{(j-1)}(0) = 0$, endowed with the norm $\|u\| = \max_{0\leq t\leq 1} |u^{(j)}(t)| = |u^{(j)}|_0$. Notice that, for $i = 1,\ 2, \dots, j$,

$$|u^{(j-i)}(t)| = |u^{(j-i)}(t) - u^{(j-i)}(0)|$$

$$
\begin{aligned}
&= \left| \int_0^t u^{(j+1-i)}(s)\, \mathrm{d}s \right| \\
&\le \int_0^t \left| u^{(j+1-i)}(s) \right| \, \mathrm{d}s \\
&\le |u^{(j+1-i)}(t)| \\
&\le |u^{(j+1-i)}|_0 .
\end{aligned}
$$

Therefore, $|u|_0 \le |u'|_0 \le \cdots \le |u^{(j-1)}|_0 \le |u^{(j)}|_0 = \|u\|$.

Define an operator $T : \mathcal{B} \to \mathcal{B}$ by

$$
(Tu)(t) = \int_0^1 G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(j)}(s))\, \mathrm{d}s.
$$

Again, if $\hat{u}$ is a fixed point of T, $\hat{u}$ solves (2.13), (2.14).

Lemma 2.2. *The operator T is completely continuous on $\mathcal{M}$, where for fixed $N > 0$, the set $\mathcal{M}$ is defined to be $\mathcal{M} = \{u \in \mathcal{B} : \|u\| < N\}$.*

Proof. Fix $N > 0$, and let $\mathcal{M} = \{u \in \mathcal{B} : \|u\| < N\}$. Let

$$
L = \max_{(x_0, x_1, \ldots, x_j) \in [0,N]^{j+1}} |f(x_0, x_1, \ldots, x_j)|.
$$

Then, for $u \in \mathcal{M}$,

$$
\begin{aligned}
\|Tu(t)\| &= \left| \int_0^1 \frac{\partial^j}{\partial t^j} G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(j)}(s))\, \mathrm{d}s \right| \\
&\le \int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t,s) \right| |a(s)| \left| f(u(s), u'(s), \ldots, u^{(j)}(s)) \right| \, \mathrm{d}s \\
&\le |a|_\infty L \int_0^1 \frac{\partial^j}{\partial t^j} G(t,s)\, \mathrm{d}s \\
&\le |a|_\infty L \overline{G}_j .
\end{aligned}
$$

Hence, $\|Tu\| \le L|a|_\infty \overline{G}_j$ for all $u \in \mathcal{M}$. So, $T(M)$ is uniformly bounded.

Now, let $\epsilon > 0$, and define $\delta = \left(\dfrac{\epsilon \Gamma(\alpha - j)(\alpha - \beta)}{|a|_\infty L(\alpha - \beta + 1)} \right)^{\dfrac{1}{\alpha - j - 1}}$. Let $t_1,\ t_2 \in [0,1]$, with $t_1 < t_2$ and $t_2 - t_1 < \delta$. Then, for $u \in \mathcal{M}$,

$$
\begin{aligned}
&|Tu^{(j)}(t_2) - Tu^{(j)}(t_1)| \\
&\qquad = \left| \int_0^1 \frac{\partial^j}{\partial t^j} G(t_2,s)a(s)f(u(s), u'(s), \ldots, u^{(j)}(s))\, \mathrm{d}s \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \int_0^1 \frac{\partial^j}{\partial t^j} G(t_1, s) a(s) f(u(s), u'(s), \ldots, u^{(j)}(s))\,\mathrm{d}s \Big| \\
&= \left| \int_0^1 \left[\frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right] a(s) f(u(s), u'(s), \ldots, u^{(j)}(s))\,\mathrm{d}s \right| \\
&\le \int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right| |a(s)| |f(u(s), u'(s), \ldots, u^{(j)}(s))|\,\mathrm{d}s \\
&\le |a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right| \mathrm{d}s.
\end{aligned}
$$

There are three cases we must consider. First, let $0 \le s \le t_1 < t_2 < 1$. Then

$$
\begin{aligned}
&|a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right| \mathrm{d}s \\
&\quad = |a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} g_1(t_2, s) - \frac{\partial^j}{\partial t^j} g_1(t_1, s) \right| \mathrm{d}s \\
&\quad = |a|_\infty L \int_0^1 \frac{1}{\Gamma(\alpha - j)} \big| (1-s)^{\alpha-\beta-1} t_2^{\alpha-j-1} \\
&\qquad\qquad - (t_2 - s)^{\alpha-j-1} - (1-s)^{\alpha-\beta-1} t_1^{\alpha-j-1} + (t_1 - s)^{\alpha-j-1} \big| \,\mathrm{d}s \\
&\quad = \frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 \big| (1-s)^{\alpha-1-\beta} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \\
&\qquad\qquad - ((t_2 - s)^{\alpha-j-1} - (t_1 - s)^{\alpha-j-1}) \big| \,\mathrm{d}s \\
&\quad \le \frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 \big| (1-s)^{\alpha-1-\beta} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \big| \\
&\qquad\qquad + \big| (t_2 - s)^{\alpha-j-1} - (t_1 - s)^{\alpha-j-1} \big| \,\mathrm{d}s \\
&\quad \le \frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 \big| (1-s)^{\alpha-1-\beta} (t_2 - t_1)^{\alpha-j-1} \big| \\
&\qquad\qquad + \big| (t_2 - s - t_1 + s)^{\alpha-j-1} \big| \,\mathrm{d}s \\
&\quad = \frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 (1-s)^{\alpha-1-\beta} (t_2 - t_1)^{\alpha-j-1} \\
&\qquad\qquad + (t_2 - t_1)^{\alpha-j-1} \,\mathrm{d}s
\end{aligned}
$$

since $(1-s)^{\alpha-1-\beta}$ and $(t_2 - t_1)^{\alpha-j-1} > 0$. Now,

$$
\begin{aligned}
&\frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 (1-s)^{\alpha-1-\beta} (t_2 - t_1)^{\alpha-j-1} + (t_2 - t_1)^{\alpha-j-1} \,\mathrm{d}s \\
&\qquad = \frac{|a|_\infty L}{\Gamma(\alpha - j)} \int_0^1 (t_2 - t_1)^{\alpha-j-1} ((1-s)^{\alpha-1-\beta} + 1) \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{|a|_\infty (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-1-\beta} + 1 \, \mathrm{d}s \\
&= \frac{|a|_\infty (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \left[\frac{-(1-s)^{\alpha-\beta}}{\alpha-\beta} + s \right]_0^1 \\
&= \frac{|a|_\infty (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \left(1 + \frac{1}{\alpha-\beta} \right) \\
&= \frac{|a|_\infty (\alpha-\beta+1)(t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&\leq \frac{|a|_\infty (\alpha-\beta+1)\delta^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&= \epsilon.
\end{aligned}
$$

Second, we must consider $0 \leq t_1 \leq s \leq t_2 < 1$ where $t_1 \neq t_2$. Here,

$$
\begin{aligned}
&|a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right| \mathrm{d}s \\
&= |a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} g_1(t_2, s) - \frac{\partial^j}{\partial t^j} g_2(t_1, s) \right| \mathrm{d}s \\
&= |a|_\infty L \int_0^1 \frac{1}{\Gamma(\alpha-j)} \big| (1-s)^{\alpha-\beta-1} t_2^{\alpha-j-1} \\
&\quad - (t_2 - s)^{\alpha-j-1} - (1-s)^{\alpha-\beta-1} t_1^{\alpha-j-1} \big| \, \mathrm{d}s \\
&= \frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 \left| (1-s)^{\alpha-\beta-1} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) - (t_2 - s)^{\alpha-j-1} \right| \mathrm{d}s \\
&< \frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \, \mathrm{d}s
\end{aligned}
$$

since $(1-s)^{\alpha-\beta-1}$, $(t_2^{\alpha-j-1} - t_1^{\alpha-j-1})$, and $(t_2 - s)^{\alpha-j-1} > 0$. Now,

$$
\begin{aligned}
&\frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \, \mathrm{d}s \\
&\qquad \leq \frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} (t_2 - t_1)^{\alpha-j-1} \, \mathrm{d}s \\
&\qquad = \frac{|a|_\infty L (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-\beta-1} \, \mathrm{d}s \\
&\qquad = \frac{|a|_\infty L (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \left[\frac{-(1-s)^{\alpha-\beta}}{\alpha-\beta} \right]_0^1 \\
&\qquad = \frac{|a|_\infty L (t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&\qquad \leq \frac{|a|_\infty L (\alpha-\beta+1)(t_2 - t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{|a|_\infty L(\alpha-\beta+1)\delta^{\alpha-j-1}}{\Gamma(\alpha-j-1)(\alpha-\beta)} \\
&= \epsilon.
\end{aligned}
$$

Last, we consider $0 \leq t_1 < t_2 \leq s < 1$. Here,

$$
\begin{aligned}
|a|_\infty L &\int_0^1 \left| \frac{\partial^j}{\partial t^j} G(t_2, s) - \frac{\partial^j}{\partial t^j} G(t_1, s) \right| \mathrm{d}s \\
&= |a|_\infty L \int_0^1 \left| \frac{\partial^j}{\partial t^j} g_2(t_2, s) - \frac{\partial^j}{\partial t^j} g_2(t_1, s) \right| \mathrm{d}s \\
&= |a|_\infty L \int_0^1 \frac{1}{\Gamma(\alpha-j)} \left| (1-s)^{\alpha-1-\beta} t_2^{\alpha-j-1} - (1-s)^{\alpha-1-\beta} t_1^{\alpha-j-1} \right| \mathrm{d}s \\
&= \frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-1-\beta} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \,\mathrm{d}s
\end{aligned}
$$

since $(1-s)^{\alpha-1-\beta}(t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) > 0$. Now,

$$
\begin{aligned}
\frac{|a|_\infty L}{\Gamma(\alpha-j)} &\int_0^1 (1-s)^{\alpha-1-\beta} (t_2^{\alpha-j-1} - t_1^{\alpha-j-1}) \,\mathrm{d}s \\
&\leq \frac{|a|_\infty L}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-1-\beta} (t_2 - t_1)^{\alpha-j-1} \,\mathrm{d}s \\
&= \frac{|a|_\infty L (t_2-t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \int_0^1 (1-s)^{\alpha-1-\beta} \,\mathrm{d}s \\
&= \frac{|a|_\infty L (t_2-t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)} \left[\frac{-(1-s)^{\alpha-\beta}}{\alpha-\beta} \right|_0^1 \\
&= \frac{|a|_\infty L (t_2-t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&\leq \frac{|a|_\infty L(\alpha-\beta+1) (t_2-t_1)^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&\leq \frac{|a|_\infty L(\alpha-\beta+1)\delta^{\alpha-j-1}}{\Gamma(\alpha-j)(\alpha-\beta)} \\
&= \epsilon.
\end{aligned}
$$

Hence $|Tu^{(j)}(t_2) - Tu^{(j)}(t_1)| \leq \epsilon$ for all $u \in \mathcal{M}$, and thus T is equicontinuous on $\mathcal{M}$. Therefore, by the Arzelà-Ascoli Theorem, T is completely continuous. □

Theorem 2.4. *Let N be fixed, and let*

$$
\mathcal{B} = \{u \in C^{(j)}[0,1] : u(0) = u'(0) = \cdots = u^{(j-1)}(0) = 0\}
$$

and $T : \mathcal{M} \to \mathcal{M}$ *be defined by*

$$(Tu)(t) = \int_0^1 G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(j-1)}(s))\,\mathrm{d}s,$$

where $\mathcal{M} = \{u \in \mathcal{B} : \|u\| < N\}$ *and* $\|u\| = |u^{(j)}|_0$. *Then* T *has a fixed point in* $\mathcal{M}$.

Proof. By definition, $\mathcal{M}$ is bounded.

To see that $\mathcal{M}$ is closed, let $\{h_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, and let $h_0 \in \mathcal{B}$ be such that $\|h_i - h_0\| \to 0$ as $i \to \infty$. Then $h_i^{(j)} \to h_0^{(j)}$ on $[0, 1]$. Thus, since $h_i \in \mathcal{M}$ for all i, $|h_i^{(j)}| \leq N$ for all i, and $|h_0^{(j)}(x)| \leq N$ on $[0, 1]$. So, $\|h_0\| \leq N$, and $h_0 \in \mathcal{M}$. Hence, $\mathcal{M}$ is closed.

Let $h,\ g \in \mathcal{M}$, and, for real λ with $0 \leq \lambda \leq 1$, consider $\lambda h + (1+\lambda)g$. Well, since $h,\ g \in \mathcal{M}$, we have

$$\begin{aligned}
|\lambda h^{(j)}(x) + (1-\lambda)g^{(j)}(x)| &\leq |\lambda h^{(j)}(x)| + |(1-\lambda)g^{(j)}(x)| \\
&= \lambda|h^{(j)}(x)| + (1-\lambda)|g^{(j)}(x)| \\
&\leq \lambda N + (1-\lambda)N \\
&= N.
\end{aligned}$$

Hence $\lambda h + (1-\lambda)g \in \mathcal{M}$ for all $h,\ g \in \mathcal{M}$, and $\mathcal{M}$ is convex.

From Lemma 3.2, T is completely continuous on $\mathcal{M}$. Hence, the assumptions of the Schauder Fixed Point Theorem are met, and thus T has a fixed point in $\mathcal{M}$ which is a solution of (2.13), (2.14). $\square$

Chapter 3

Existence of Multiple Positive Solutions to a Family of Fractional Two-Point Boundary Value Problems

Let $n \geq 2$ denote an integer, and let α and β be positive reals, satisfying $n-1 < \alpha \leq n$ and $n-2 < \beta \leq n-1$, for some $j = n-2$. We will consider the boundary value problem for the fractional differential equation given by

$$D_{0+}^{\alpha} u + a(t) f(u, u', \ldots, u^{(n-2)}) = 0, \qquad \text{where } 0 < t < 1, \tag{3.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \text{ for } i = 0, 1, \ldots, n-2, \qquad \text{and } D_{0+}^{\beta} u(1) = 0, \tag{3.2}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives.

Notice that the Green's function for (3.1), (3.2) is given by (2.6) and that Lemma 2.1 holds.

Lemma 3.1. *Let γ and s be fixed nonnegative reals, with $0 \leq \gamma \leq s < 1$, and let β be a positive real such that $n-2 < \beta \leq n-1$. The kernel, $G(\beta; t, s)$, satisfies the following properties:*

$$\begin{aligned} \overline{G}_{n-2} &= \max_{t \in [0,1]} \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t, s)\, \mathrm{d}s \\ &= \frac{(\alpha-n+1)^{\alpha-n+1}(\alpha-n+2) - (\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)}, \end{aligned} \tag{3.3}$$

where $\overline{G}_{n-2}$ is the specific case of $\overline{G}_i$ as defined in Lemma 2.1 where $i = n-2$, and

$$\min_{\gamma \leq t \leq 1} \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) \geq [1 - (1-\gamma)^{\beta-1}] \gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s, s). \tag{3.4}$$

Proof. Let $i = n-2$. Notice that $t_{n-2} = \dfrac{\alpha-n+1}{\alpha-\beta}$ since $\alpha - \beta \geq \alpha - n + 1$, implying that $\dfrac{\alpha-n+1}{\alpha-\beta} \leq 1$. Hence,

$$\max_{0 \leq t \leq 1} \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t, s)\, \mathrm{d}s = \frac{\dfrac{(\alpha-n+1)^{\alpha+1-n}}{(\alpha-\beta)^{\alpha+1-n}}(\alpha-n+2) - \dfrac{(\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)}$$

$$= \frac{(\alpha-n+1)^{\alpha-n+1}(\alpha-n+2)-(\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)},$$

which proves (3.3).

To prove (3.4), note that

$$\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s) = \frac{(1-s)^{\alpha-1-\beta}t^{\alpha-n}-(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}.$$

Now,

$$\begin{aligned}(1-s)^{\alpha-1-\beta}t^{\alpha-n}-(t-s)^{\alpha-n} &= (1-s)^{\alpha-\beta-1}t^{\alpha-n}-\left(t\left(1-\frac{s}{t}\right)\right)^{\alpha-n}\\ &= t^{\alpha-n}\left((1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-n}\right).\end{aligned}$$

Note that, if $t=0$, then $s=0$, and thus, $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)=0$. If $0<s\le t<1$, then $\frac{1}{t}>1$, and since, s is positive, $\frac{s}{t}>s$. This implies that $1-\frac{s}{t}<1-s$, and, since $-1<\alpha-n\le 0$ and $n-1<\beta+1\le n$, $\left(1-\frac{s}{t}\right)^{\alpha-n}>(1-s)^{\alpha-n}\ge(1-s)^{\alpha-\beta-1}$. Therefore $(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-n}<0$, and, consequently, $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)<0$. Also note that

$$\frac{\partial^{n-1}}{\partial t^{n-1}}g_2(t,s) = \frac{(1-s)^{\alpha-1-\beta}t^{\alpha-n}}{\Gamma(\alpha-n+1)} > 0,$$

since $(1-s)^{\alpha-1-\beta}t^{\alpha-n}>0$.

Since $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)<0$, $\frac{\partial^{n-2}}{\partial t^{n-2}}g_1(t,s)$ is a decreasing function of t. Hence, for $0\le\gamma\le s<1$,

$$\begin{aligned}\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}g_1(t,s) &= \frac{\partial^{n-2}}{\partial t^{n-2}}g_1(1,s)\\ &= \frac{(1-s)^{\alpha-1-\beta}-(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\\ &= \frac{(1-s)^{\alpha-1-\beta}[1-(1-s)^{\beta-n+2}]}{\Gamma(\alpha-n+2)}\\ &\ge \frac{(1-s)^{\alpha-1-\beta}[1-(1-\gamma)^{\beta-n+2}]}{\Gamma(\alpha-n+2)}\\ &\ge \frac{(1-s)^{\alpha-1-\beta}[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s^{\alpha-n+2}}{\Gamma(\alpha-n+2)}\end{aligned}$$

$$
\begin{aligned}
&= [1-(1-\gamma)^{\alpha-n+2}]\gamma^{\alpha-n+1}s\frac{(1-s)^{\alpha-1-\beta}s^{\alpha-n+1}}{\Gamma(\alpha-n+2)} \\
&= \ [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s).
\end{aligned}
$$

Note that $\dfrac{\partial^{n-2}}{\partial t^{n-2}}g_2(t,s)$ is an increasing function of t.
Hence, for $0 \le \gamma \le s < 1$,

$$
\begin{aligned}
\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}g_2(t,s) &= \frac{\partial^{n-2}}{\partial t^{n-2}}g_2(\gamma,s) \\
&= \frac{(1-s)^{\alpha-1-\beta}\gamma^{\alpha-n+1}}{\Gamma(\alpha-n+2)} \\
&\ge \frac{(1-s)^{\alpha-1-\beta}\gamma^{\alpha-n+1}[1-(1-\gamma)^{\beta-n+2}]s^{\alpha-n+2}}{\Gamma(\alpha-n+2)} \\
&= [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s).
\end{aligned}
$$

Thus, $\displaystyle\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;t,s) \ge [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s)$ for all $\gamma \le s < 1$, which proves (3.4). $\square$

3.1

Kraznosel'skii's Fixed Point Theorem

In this section, we will use a well-known fixed point theorem for operators acting on a cone in a Banach space. Some authors have used Kraznosel'skii's fixed point theorem to show the existence of solutions of ordinary differential equations, difference equations, and dynamic equations on time scales; however, few papers have been published that were devoted to the study of boundary value problems of fractional order as in [2, 3, 9], where the authors develop proofs for the existence of positive solutions to the nonlinear fractional boundary value problems

$$D^\alpha u + a(t)f(u) = 0, \qquad 0 < t < 1,\ 1 < \alpha \le 2,$$

and

$$D^\alpha u + a(t)f(u) = 0, \qquad 0 < t < 1,\ 3 < \alpha \le 4,$$

satisfying boundary conditions

$$u(0) = 0, \qquad u'(1) = 0,$$

and

$$u(0) = 0 = u'(0) = u''(0) = u'(1) = 0,$$

respectively, which are two specific cases of problem (3.1), (3.2). We seek to show the existence of positive solutions of (3.1), (3.2) where for arbitrary positive integer n and positive real α, $n-1 < \alpha \leq n$.

Theorem 3.1 (Krasnosel'skii's Fixed Point Theorem [11])**.** *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{K} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open sets with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{K} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{K} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{K} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{K} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Definition 3.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{K}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{K}$, for all $u,\ v \in \mathcal{K}$ and all $\alpha,\ \beta \geq 0$, and

(ii) if $u \in \mathcal{K}$ and $-u \in \mathcal{K}$, then $u = 0$.

Define the Banach Space $\mathcal{B} = \{u \in C^{(n-2)}[0,1] : u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0\}$ to be the space of all $n-2$ times differentiable functions whose $(n-2)$nd derivative is continuous on the interval $[0,1]$ which satisfy $u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0$ endowed with the norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u^{(n-2)}(t)| = |u^{(n-2)}|_0$. Notice that for $i = 1, 2, \ldots n-2$,

$$\begin{aligned}
\left|u^{(n-2-i)}(t)\right| &= \left|u^{(n-2-i)}(t) - u^{(n-2-i)}(0)\right| \\
&= \left|\int_0^t u^{(n-1-i)}(s)\,\mathrm{d}s\right| \\
&\leq \int_0^t \left|u^{(n-1-i)}(s)\right|\,\mathrm{d}s \\
&\leq \left|u^{(n-1-i)}(t)\right| \\
&\leq \left|u^{(n-1-i)}\right|_0.
\end{aligned}$$

Therefore, $|u|_0 \leq |u'|_0 \leq \cdots \leq |u^{(n-3)}|_0 \leq |u^{(n-2)}|_0 = \|u\|$.

Define an operator $T : \mathcal{B} \to \mathcal{B}$ by

$$(Tu)(t) = \int_0^1 G(t,s)a(s)f(u(s), u'(s), \dots, u^{(n-2)}(s))\,\mathrm{d}s.$$

Now, if u is a fixed point of T, then u solves (3.1), (3.2).

Lemma 3.2. *The operator T is completely continuous on $\mathcal{M}$ where for fixed $N > 0$, the set $\mathcal{M}$ is defined to be $\mathcal{M} = \{u \in \mathcal{B} : \|u\| < N\}$.*

Proof. Similar to the proof of Lemma 2.2. □

We make assumptions (A1), (A2) and the following on the functions f and a:

(A3) There exists a $\gamma \in (0,1)$ and an $m > 0$ such that $a(t) > m$ a. e. on $[\gamma, 1]$.

Lemma 3.3. *The set $\mathcal{K} = \{u \in \mathcal{B} : u^{(n-2)}(t) \geq 0$ for all $t \in [0,1]\}$ is a cone.*

Proof. By definition, $\mathcal{K} \in \mathcal{B}$. Also note that any polynomial funtion with positive coefficients is in $\mathcal{K}$, and hence $\mathcal{K}$ is nonempty.

Let $u,\ v \in \mathcal{K}$, and $\alpha,\ \beta \in \mathbb{R}$, with $\alpha,\ \beta \geq 0$. Then $u^{(n-2)}(t),\ v^{(n-2)}(t) \geq 0$ for all $t \in [0,1]$. Hence, $\alpha u^{(n-2)}(t) + \beta v^{(n-2)}(t) \geq 0$, and thus $\alpha u + \beta v \in \mathcal{K}$. Now, let $u,\ -u \in \mathcal{K}$. Then $u^{(n-2)}(t),\ -u^{(n-2)}(t) \geq 0$ for all $t \in [0,1]$, implying $u^{(n-2)}(t) \equiv 0$. But $u^{(n-3)}(0) = 0$ since $u \in \mathcal{B}$ and, thus $u^{(n-3)}(t) = 0$ for all $t \in [0,1]$.

Similarly, since $u^{(i)}(0) = 0,\ i = 0, 1, \dots, n-4$, the function u satisfies $u^{(i)}(t) \equiv 0$. Hence $u \equiv 0$. By Definition 3.1, $\mathcal{K}$ is a cone. □

Theorem 3.2. *Suppose that (A1) and (A2) are satisfied and that there exists a $\gamma \in (0,1)$ such that (A3) is satisfied. Let $M = |a|_\infty$, and let $A, B \in \mathbb{R}$ with $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2}M}$ and $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}\displaystyle\int_\gamma^1 s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s)\,\mathrm{d}s\right]^{-1}$. If there exist positive constants r and R with $r < R$ and $Br < AR$, such that f satisfies*

(H1) $f(x_0, x_1, \dots, x_{n-2}) \leq AR$ for all $(x_0, x_1, \dots, x_{n-2}) \in [0,R]^{n-1}$, and

(H2) $f(x_0, x_1, \dots, x_{n-2}) \geq Br$ for all $(x_0, x_1, \dots, x_{n-2}) \in [0,r]^{n-1}$,

then (3.1), (3.2) has at least one positive solution u with $r < \|u\| < R$.

Proof. It has been shown that T is completely continuous and that the set

$$\mathcal{K} = \{u \in \mathcal{B} : u^{(n-2)}(t) \geq 0 \text{ for all } t \in [0,1]\} \tag{3.5}$$

is a cone.

Define the open set $\Omega_2 = \{u \in \mathcal{B} : \|u\| < R\}$. Let $u \in K \cap \partial\Omega_2$. Then assumption (H1) and (2.8) give

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\leq \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) \right| |a(s)||f(s,u(s),\ldots,u^{(n-2)}(s))|\,\mathrm{d}s \\
&\leq MAR \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) \right| \mathrm{d}s \\
&\leq MAR\overline{G}_{n-2} \\
&\leq R \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$ for all $u \in \mathcal{K} \cap \partial\Omega_2$.

Next, define the open set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}$. Let $u \in K \cap \partial\Omega_1$. Then, using (A1)-(A3), assumption (H2) and (3.4), we have that

$$\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq mBr \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\geq mBr \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&= mBr[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&\geq r \\
&= \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_1$. Since $0 \in \Omega_1 \subset \Omega_2$, the contractive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. So, there exists at least one solution of u of (3.1), (3.2) with $r < \|u\| \leq R$. □

Theorem 3.3. *Suppose that (A1) and (A2) are satisfied and that there exists* $\gamma \in (0,1)$ *such that (A3) is satisfied. Let* $M = |a|_\infty$*, and let* $A, B \in \mathbb{R}$ *with* $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2}M}$ *and* $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \displaystyle\int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \right]^{-1}$. *If there exist positive constants* r *and* R *such that* $r < R$ *and* $Ar < BR$*, such that* f *satisfies*

(H3) $f(x_0, x_1, \ldots, x_{n-2}) \geq BR$ *for all* $(x_0, x_1, \ldots, x_{n-2}) \in [0,R]^{n-1}$ *and*

(H4) $f(x_0, x_1, \ldots, x_{n-2}) \leq Ar$ *for all* $(x_0, x_1, \ldots, x_{n-2}) \in [0,r]^{n-1}$.

then (3.1), (3.2) *has at least one positive solution* u *with* $r < \|u\| < R$.

Proof. We will employ the completely continuous operator T and the cone $\mathcal{K}$ as in the previous proof.

Define the open set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}$. Let $u \in \mathcal{K} \cap \partial\Omega_1$. Then assumption (H4) and (2.8) we have that

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\leq MAr \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \right| \\
&\leq MAr\overline{G}_{n-2} \\
&\leq r \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$ for all $u \in \mathcal{K} \cap \partial\Omega_1$.

Next, define the open set $\Omega_2 = \{u \in \mathcal{B} : \|u\| < R\}$. Let $u \in \mathcal{K} \cap \partial\Omega_2$. Then, by (A1)-(A3), assumption (H3) and (3.4),

$$\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq mBR \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\geq mBR \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&= mBR[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&\geq R \\
&= \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in \mathcal{K} \cap \partial\Omega_2$. Since $0 \in \Omega_1 \subset \Omega_2$, the expansive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. So, there exists at least one solution of u of (3.1), (3.2) with $r < \|u\| \leq R$. $\square$

Theorem 3.4. *Suppose that (A1) and (A2) are satisfied and that there exists $\gamma \in (0,1)$ such that (A3) is satisfied. Let $M = |a|_\infty$, and let $A, B \in \mathbb{R}$ with $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \right]^{-1}.$$

If there exist positive constants r_i and R_i for $i = 1, 2, \cdots, k$ such that $r_1 < R_1 < r_2 < R_2 < \cdots < r_k < R_k$ for some k and $Br_i < AR_i$ for all i such that f satisfies

(H5) $f(x_0, x_1, \ldots, x_{n-2}) \leq AR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, R_i]^{n-1}$, and

(H6) $f(x_0, x_1, \ldots, x_{n-2}) \geq Br_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, r_i]^{n-1}$,

then (3.1), (3.2) has at least k positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Proof. We will again employ the use of the completely continuous operator T and the cone $\mathcal{K}$.

Define open sets $\Omega_{2_i} = \{u \in \mathcal{B} : \|u\| < R_i\}$ for $i = 1, \ldots, k$. Fix i and let $u \in K \cap \partial\Omega_{2_i}$. Then for any i, (H5) and (2.8) give

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\leq MAR_i \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\leq MAR_i \overline{G}_{n-2} \\
&\leq R_i \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$ for all $u \in K \cap \partial\Omega_{2_i}$.

Next, define the open sets $\Omega_{1_i} = \{u \in \mathcal{B} : \|u\| < r_i\}$ for $i = 1, \ldots, k$. Fix i and let $u \in K \cap \partial\Omega_{1_i}$.

Then, using (A1)-(A3), assumption (H6) and (3.4), we have that

$$\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geq mBr_i \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\geq mBr_i \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&= mBr_i[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&\geq r_i \\
&= \|u\|.
\end{aligned}
$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_{1_i}$. Since $0 \in \Omega_{1_i} \subset \Omega_{2_i}$, the contractive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_{2_i} \backslash \Omega_{1_i})$ for each i. So, there exists at least one solution of u_i of (3.1), (3.2) with $r_i < \|u\| \leq R_i$ for each $i = 1, \cdots, k$. □

Theorem 3.5. *Suppose that (A1) and (A2) are satisfied and that there exists $\gamma \in (0,1)$ such that (A3) is satisfied. Let $M = |a|_\infty$, and let $A, B \in \mathbb{R}$ with $0 \leq A \leq \frac{1}{\overline{G}_{n-2}M}$ and $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s\right]^{-1}$. If there exist positive constants r_i and R_i for $i = 1, 2, \cdots, k$ such that $r_1 < R_1 < r_2 < R_2 < \cdots < r_k < R_k$ for some k and $BR_i < Ar_i$ for all i such that f satisfies*

(H7) $f(x_0, x_1, \ldots, x_{n-2}) \geq BR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, R_i]^{(n-1)}$, and

(H8) $f(x_0, x_1, \ldots, x_{n-2}) \leq Ar_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, r_i]^{(n-1)}$,

then (3.1), (3.2) has at least k positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Proof. We will again employ the use of the completely continuous operator T and the cone $\mathcal{K}$.

Define open sets $\Omega_{1_i} = \{u \in \mathcal{B} : \|u\| < r_i\}$ for $i = 1, \ldots, k$. Let $u \in K \cap \partial\Omega_{1_i}$. Then for any i, (H8) and (2.8) give

$$
\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\leq MAr_i \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\leq MAr_i \overline{G}_{n-2} \\
&\leq r_i \\
&= \|u\|.
\end{aligned}
$$

So, $\|Tu\| \leq \|u\|$ for all $u \in K \cap \partial\Omega_{1_i}$.

Next, define the open sets $\Omega_{2_i} = \{u \in \mathcal{B} : \|u\| < R_i\}$ for $i = 1, \ldots, k$. Let $u \in K \cap \partial\Omega_{2_i}$.
Then, using (A1)-(A3), assumption (H7) and (3.4), we have that

$$
\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq mBR_i \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\geq mBR_i \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&= mBR_i [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&\geq R_i \\
&= \|u\|.
\end{aligned}
$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_{2_i}$. Since $0 \in \Omega_{1_i} \subset \Omega_{2_i}$, the expansive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_{2_i} \backslash \Omega_{1_i})$ for each i. So, there exists at least one solution of u_i of (3.1), (3.2) with $r_i < \|u\| \leq R_i$ for each $i = 1, \cdots, k$. □

Theorem 3.6. *Suppose that (A1) and (A2) are satisfied and that there exists $\gamma \in (0,1)$ such that (A3) is satisfied. Let $M = |a|_\infty$, and let $A, B \in \mathbb{R}$ with $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2}M}$ and $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \displaystyle\int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s\right]^{-1}$. If there exist positive constants r_i and R_i for $i = 1, 2, \cdots$, such that $r_1 < R_1 < r_2 < R_2 < \cdots$ and $Br_i < AR_i$ for all i such that f satisfies*

(H9) $f(x_0, x_1, \ldots, x_{n-2}) \leq AR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, R_i]^{n-1}$, and

(H10) $f(x_0, x_1, \ldots, x_{n-2}) \geq Br_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, r_i]^{n-1}$,

then (3.1), (3.2) has infinitely many positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Proof. We will again employ the use of the completely continuous operator T and the cone $\mathcal{K}$.

Define open sets $\Omega_{2_i} = \{u \in \mathcal{B} : \|u\| < R_i\}$ for all i. Fix i and let $u \in K \cap \partial\Omega_{2_i}$. Then for any i, (H5) and (2.8) give

$$
|Tu^{(n-2)}|(t) = \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \right|
$$

$$
\begin{aligned}
&\leq MAR_i \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\leq MAR_i \overline{G}_{n-2} \\
&\leq R_i \\
&= \|u\|.
\end{aligned}
$$

So, $\|Tu\| \leq \|u\|$ for all $u \in K \cap \partial\Omega_{2_i}$.

Next, define the open sets $\Omega_{1_i} = \{u \in \mathcal{B} : \|u\| < r_i\}$ for $i = 1, 2, \ldots$. Fix i and let $u \in K \cap \partial\Omega_{1_i}$.

Then, using (A1)-(A3), assumption (H6) and (3.4), we have that

$$
\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\,\mathrm{d}s \\
&\geq mBr_i \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,\mathrm{d}s \\
&\geq mBr_i \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&= mBr_i [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \\
&\geq r_i \\
&= \|u\|.
\end{aligned}
$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_{1_i}$. Since $0 \in \Omega_{1_i} \subset \Omega_{2_i}$, the contractive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_{2_i} \backslash \Omega_{1_i})$ for each i. So, there exists at least one solution of u_i of (3.1), (3.2) with $r_i < \|u\| \leq R_i$ for each i. □

Theorem 3.7. *Suppose that (A1) and (A2) are satisfied and that there exists $\gamma \in (0,1)$ such that (A3) is satisfied. Let $M = |a|_\infty$, and let $A, B \in \mathbb{R}$ with $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2} M}$ and $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \right]^{-1}$. If there exist positive constants r_i and R_i for $i = 1, 2, \ldots,$ such that $r_1 < R_1 < r_2 < R_2 < \cdots$ for some k and $BR_i < Ar_i$ for all i such that f satisfies*

(H11) $f(x_0, x_1, \ldots, x_{n-2}) \geq BR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, R_i]^{(n-1)}$, and

(H12) $f(x_0, x_1, \ldots, x_{n-2}) \leq Ar_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0, r_i]^{(n-1)}$,

then (3.1), (3.2) has at least k positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Proof. We will again employ the use of the completely continuous operator T and the cone $\mathcal{K}$.

Define open sets $\Omega_{1_i} = \{u \in \mathcal{B} : \|u\| < r_i\}$ for $i = 1, 2, \ldots$. Let $u \in K \cap \partial\Omega_{1_i}$. Then for any i, (H8) and (2.8) give

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \right| \\
&\leq MAr_i \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\, \mathrm{d}s \\
&\leq MAr_i \overline{G}_{n-2} \\
&\leq r_i \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$ for all $u \in K \cap \partial\Omega_{1_i}$.

Next, define the open sets $\Omega_{2_i} = \{u \in \mathcal{B} : \|u\| < R_i\}$ for $i = 1, 2, \ldots$. Let $u \in K \cap \partial\Omega_{2_i}$.
Then, using (A1)-(A3), assumption (H7) and (3.4), we have that

$$\begin{aligned}
Tu^{(n-2)}(t) &\geq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&\geq \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&\geq mBR_i \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\, \mathrm{d}s \\
&\geq mBR_i \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \\
&= mBR_i [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \\
&\geq R_i \\
&= \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_{2_i}$. Since $0 \in \Omega_{1_i} \subset \Omega_{2_i}$, the expansive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_{2_i} \backslash \Omega_{1_i})$ for each i. So, there exists at least one solution of u_i of (3.1), (3.2) with $r_i < \|u\| \leq R_i$ for each $i = 1, 2, \ldots$. □

3.2

The Leggett-Williams Fixed Point Theorem

In this section, we will consider (3.1) and (3.2) along with the Banach space $\mathcal{B}$, the cone $\mathcal{K}$, and the operator T defined in the previous section. To again show the existence of multiple solutions, we will use the Leggett-Williams fixed point theorem, as in [9]. In order to do this, for α a positive concave functional, we define the following subsets of $\mathcal{K}$:

$$\mathcal{K}_c = \{u \in \mathcal{K} : \|u\| < c\},$$

$$\mathcal{K}_a = \{u \in \mathcal{K} : \|u\| < a\},$$

$$\mathcal{K}(\alpha, b, d) = \{u \in \mathcal{K} : b \leq \alpha(u),\ \|u\| \leq d\},\ \textit{and}$$

$$\mathcal{K}(\alpha, b, c) = \{u \in \mathcal{K} : b \leq \alpha(u),\ \|u\| \leq c\}.$$

Theorem 3.8 (Leggett-Williams [13])**.** *Suppose that $T : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ is completely continuous, and suppose there exists a concave positive funtioncal α on $\mathcal{K}$ such that $\alpha(u) \leq \|u\|$ for $u \in \overline{\mathcal{K}}_c$. Suppose there exist constants $0 < a < b < d \leq c$ such that*

(B1) $\{u \in \mathcal{K}(\alpha,\ b,\ d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ if $u \in \mathcal{K}(\alpha,\ b,\ d)$;

(B2) $\|Tu\| < u$ if $u \in \mathcal{K}_a$; and

(B3) $\alpha(Tu) > b$ for $u \in \mathcal{K}(\alpha,\ b,\ c)$ with $\|Tu\| > d$.

Then T has at least three fixed points $u_1,\ u_2,$ and u_3 such that $\|u_1\| < a$, $b < \alpha(u_2)$, and $\|u_3\| > a$ with $\alpha(u_3) < b$.

Theorem 3.9. *Define the continuous positive concave functional $\alpha : \mathcal{B} \to \mathcal{B}$ by $\alpha(u) = \min_{\gamma \leq t \leq 1} |u^{(n-2)}(t)|$, and let $\gamma \in (0,1)$, $M = \|a\|_\infty$, $0 < A \leq \dfrac{1}{M\overline{G}_{n-2}}$ and*

$$B \geq \left[m[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1} \int_0^\gamma s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\,\mathrm{d}s \right]^{-1}.$$

Let $a,\ b,$ and c be such that $0 < a < b < c$. Assume that the following hold:

(L1) $f(u(t), u'(t), \ldots, u^{(n-2)}(t)) < Aa$ for all $(t, u^{(n-2)}(t)) \in [0,1] \times [0,a]$,

(L2) $f(u(t), u'(t), \ldots, u^{(n-2)}(t)) > Bb$ for all $(t, u^{(n-2)}(t)) \in [\gamma,1] \times [b,c]$,

(L3) $f(u(t), u'(t), \ldots, u^{(n-2)}(t)) \leq Ac$ *for all* $(t, u^{(n-2)}(t)) \in [0, 1] \times [0, c]$.

Then (3.1), (3.2) *has at least three positive solutions* $u_1,\ u_2,\ u_3 \in K$ *satisfying*

$$\|u_1\| < a,$$

$$b < \alpha(u_2),\ \textit{and}$$

$$a < \|u_3\| \textit{ with } \alpha(u_3) < b.$$

Proof. Let $u \in \mathcal{K}_c$. Then $\|u\| < c$ and by $(L3)$ and (2.8),

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \right| \\
&\leq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) |a(s)| |f(u(s), u'(s), \ldots, u^{(n-2)}(s))|\, \mathrm{d}s \\
&\leq M \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) |f(u(s), u'(s), \ldots, u^{(n-2)}(s))|\, \mathrm{d}s \\
&< AcM \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s)\, \mathrm{d}s \\
&\leq AcM\overline{G}_{n-2} \\
&= c.
\end{aligned}$$

Hence, $\|Tu\| < c$ and $T : \mathcal{K}_c \to \mathcal{K}_c$.

Similarly, let $u \in \mathcal{K}_a$. Then $\|u\| < a$, and by $(L1)$ and (2.8),

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \right| \\
&\leq \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) \right| |a(s)| |f(u(s), u'(s), \ldots, u^{(n-2)}(s))|\, \mathrm{d}s \\
&\leq M \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) |f(u(s), u'(s), \ldots, u^{(n-2)}(s))|\, \mathrm{d}s \\
&< MAa \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s)\, \mathrm{d}s \\
&= M\overline{G}_{n-2}Aa \\
&= a.
\end{aligned}$$

So, $T : \mathcal{K}_a \to \mathcal{K}_a$.

Let d be a constant such that $b < d \leq c$. Then, for $u(t) = \dfrac{d}{(n-2)!} t^{n-2}$, $\alpha(u) = d > b$ and $u \in \mathcal{K}(\alpha,\ b,\ d)$. Thus $\mathcal{K}(\alpha,\ b,\ d) \neq \emptyset$. Hence, $\|Tu\| < u$ if $u \in \mathcal{K}_a$, and condition $(B2)$ of (3.8) holds.

Let $u \in \mathcal{K}(\alpha,\ b,\ d)$. Then $\|u\| \leq d \leq c$ and $\alpha(u) = \min\limits_{\gamma \leq t \leq 1} |u^{(n-2)}(t)| = \min\limits_{\gamma \leq t \leq 1} u^{(n-2)}(t) \geq b$. Now, by $(L2)$ and (3.4),

$$\begin{aligned}
\alpha(Tu) &= \min_{\gamma \leq t \leq 1} \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&> \min_{\gamma \leq t \leq 1} \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&> Bbm[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1} \int_0^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \\
&= b.
\end{aligned}$$

Thus, for all $u \in \mathcal{K}(\alpha,\ b,\ d)$, we have that $\alpha(Tu) > b$. So, condition $(B1)$ of (3.8) holds.

Finally, let $u \in \mathcal{K}(\alpha,\ b,\ c)$ with $\|Tu\| > d$. Then $\|u\| \leq c$ and $\alpha(u) = \min\limits_{0 \leq t \leq 1} |u^{(n-2)}(t)| = \min\limits_{\gamma \leq t \leq 1} u^{(n-2)}(t) \geq b$. From assumption $(L2)$ and (3.4),

$$\begin{aligned}
\alpha(Tu) &= \min_{\gamma \leq t \leq 1} \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&> \min_{\gamma \leq t \leq 1} \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \ldots, u^{(n-2)}(s))\, \mathrm{d}s \\
&> Bbm[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1} \int_0^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \\
&= b.
\end{aligned}$$

This shows that condition $(B3)$ of (3.8) holds.

Thus, from (3.8), T has at least three fixed points $u_1,\ u_2,\ u_3$ such that $\|u_1\| < a,\ b < \alpha(u_2)$, and $a < \|u_3\|$ with $\alpha(u_3) < b$. These fixed points are solutions of (3.1), (3.2). $\square$

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