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An algorithmic approach to detect non-injectivity of the Partial Borda Count

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An algorithmic approach to detect non-injectivity of the Partial Borda Count

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ABSTRACT

Voting is how we elect today's voices, faces, and leaders in our country. It is argued to be a very essential right we have as a people. A voter votes, by listing their preferences. Their preferences are relating the candidates to one each other (i.e. whether they prefer candidate A to candidate B or if they are indifferent between the two). There are many different social choice functions that can be used to calculate the results of an election. This project glances over the theory of Condorcet, Borda, Arrow, and Young, all of whom had a great impact on voting theory and social choice theory. I experiment with partially-ordered preferences using the Partial Borda Count.

The Partial Borda Count switches from being injective (one-to-one) to non-injective (multiple posets going to the same score vector) for all elections with 5-elements or higher. I created an algorithm that determines certain posets that go to the same score vectors for n-candidate elections (if $n \geq 5$). My algorithm was able to detect all of the failures of injectivity for a 5-candidate election. I then use this algorithm to see if I can predict which posets go to the same score vector, for a 6-candidate election, without having to construct a 6-element database. It turns out my algorithm proved successful in locating some of the injectivity failures of 6-element elections.

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To my POSSE, waaaaahhhhhhhhhh.

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CHAPTER 1: <u>Introduction</u>

It is known that for a simple two-alternative election, the most optimal way of voting is majority rules**.** Life is not this simple, most elections have more than two candidates, policies, or alternatives to vote for. When an election exceeds two alternatives, majority rules is no longer optimal**.** A more formal definition is given in the next chapter, but majority rules is essentially won by a candidate receiving more than half of *first*-preference votes [10]. There is no way to guarantee that any one candidate will get more than half of the voters to prefer it over the others when there are more than two options for the voters to choose from. Not to say it is impossible to reach a majority, but it is definitely not too likely or common.

So, what happens for elections with more than two candidates? This paper explores the ideologies of Condorcet, Borda, Arrow, and Young. All of whom had an impact on voting and social choice theory, specifically for when there are more than two alternatives to vote on. It turns out that there is no voting system that can satisfy certain fairness axioms constructed by Arrow. Out of the voting systems explored in the next chapter, this paper hones in on the Partial Borda Count in conjunction with indifferent voting preferences.

For elections with up to 4 candidates, the Partial Borda Count is injective. Injectivity, meaning one-to-one, each poset has a unique score vector that it produces. When there are 5 candidates in an election there are 4 pairs of different posets that produce the same score, this fails injectivity. I generated an algorithm that can predict certain posets that produce the same

score vector. Although my algorithm may not find *all* injectivity failures for elections exceeding 4 voting options, it does at least inform voters that even though their preferences could differ, the score they give each candidate may not. There is a chance that two voters, with completely different preferences and ballots can give the candidates the same scores. Involving indifferent preferences expands the pool of posets a voter can produce. There is a much greater range of opportunities for voters' preferences when giving them the freedom to feel indifferent about candidates on a ballot. This also allows for a more accurate representation of voter preferences, since voters can list their true preferences. As opposed to Condorcet where a preference *must* be given.

CHAPTER 2: <u>Context</u>

I.   SOCIAL CHOICE THEORY OVERVIEW

Social choice theory, "is the study of collective decision processes and procedures" [15]. It aims to answer questions about analyzing individual preferences and social choice procedures, then deciding which procedure is best for collecting these preferences.

Definition 2.1.1 Let there be two sets, X and Y, where X is the set of individual voting preferences and Y is the set of candidates or alternatives. Let there be a function from X to Y. A **social choice procedure** obtains all of the inputs from X and translates it into a single output (or string of outputs in the case of a tie) in Y. Also called a **social welfare function**. [16]

There are five properties of a social choice procedure that are deemed desirable:

1. *Always-A-Winner (AAW)*: each voter's preferences must produce at least one winner.

2. *Condorcet Winner Criterion*: if there is a Condorcet winner (will be explained later), that winner is also the *sole* social choice.

3. *Pareto Condition*: if everyone prefers candidate A over candidate B, then candidate B is not a social choice, for all candidates A, B.

4. *Monotonicity*: if a voter changes their ballot from the loser to the winner, the results will stay the same.

5. *Independence of Irrelevant Alternatives (IIA)*: let candidate A be in the social choice set, but candidate B is not. If one or more voters change their preferences, but do not change the relation preference of A to B, then the social choice set should still not include B.

Note the following theories in this chapter fall under the umbrella of social choice theory.

II.   MAJORITY RULES

Majority Rules is a long-established method of voting when there are two candidates.

Definition 2.2.1 Majority rules is a social choice procedure for two candidates. A candidate wins when they rank higher than the other for at least half of the votes. [16]

When majority rules was established it was the only voting method that was considered desirable, or completely fair at the time. Questions of what makes majority rules a desirable voting system, and if there was another voting system that could be equally desirable started to arise. This is what lead to May's Theorem [16]:

Theorem 2.2.1 MAY'S THEOREM. If there is an odd number of voters, and the election produces a unique winner, then majority rules is the only social choice procedure that satisfies:

- *Anonymity*: all voters are treated the same, meaning if any two voters exchange ballots, the results will stay the same.

- *Monocity*: if a voter changes their ballot from the loser to the winner, the results will stay the same.

- *Neutrality*: both candidates are treated the same, meaning if all ballots were revered, the election results will also be reversed.

- *Universal Domain:* the domain must consist all logically possible profiles of votes.

[16]

There are some inefficiencies and concerns about the theoretical aspect of majority rules. Maskin [11] lists some if these faults: intransitivity (a majority can prefer A to B, and B to C, but still prefer C to A), indecisiveness (there is no guarantee there will be a winner, there are multiple reasons for a majority not being reached), and susceptibility to manipulation, which I will get into later in this chapter.

III.   CONDORCET

The Condorcet voting method is a method that uses a series of pairwise comparisons in order to determine the winner of an election. The Condorcet method is named after Marquis de Condorcet, a man of many accomplishments, which include, permanent secretary of the Academy of Sciences and member of the French Academy. Condorcet is said to occupy, "a special place in the history of French thought. He is the last of the philosophes…He did not conceive a completely original system, but he did create a synthesis of all the theories of his predecessors" [8].Condorcet's philosophy is based on a society where truth matters and voters have the ability to determine what the truth is. In Condorcet's eyes, there people are utility maximizing, as it is in their best interest to do so. Condorcet's beliefs view voters as accurate decision makers who can make correct judgements when it comes to right vs wrong. These judgements made by the voters are seen as objective. Since Condorcet comes from this philosophical background, the purpose of his method is to find the collective truth through series of majority rules games. The method is designed this way to maximize the voters' probability of making the 'correct' (socially optimizing) decision.

Social Choice Theory originated in the French Academy of the Sciences, and it was the rediscovery of Condorcet's work that started its development. However, a downside of Condorcet's research is it allowed for other economists to selectively pick and choose from his theory. Economists took partial information and formed arguments that are different from the theory's original purpose--to make the best collective decision.

Definition 2.3.1 If a candidate beats all other candidates in a pairwise election they are known as the **Condorcet winner**.

Example 2.3.1 For instance, if there are three candidates (A, B, C), A is only the Condorcet winner if A > B, AND A > C by majority rules.

Definition 2.3.2 When using a social choice procedure and the election has a Condorcet winner, if the social choice function chooses the Condorcet winner, that election is said to have **Condorcet Consistency (CC)**.

Condorcet Consistency allows for ties between candidates. However, if a voter feels indifferent about a candidate and chooses not to rank them, that candidate(s) is assumed to be ranked under the candidates the voter did rank [5].

Even though Condorcet's theory was a very important stepping stone for social choice theory, his voting system was not just accepted and then put into practice. There are certain criteria set in place that can determine if Condorcet's voting system is an adequate voting system.

Definition 2.3.3 Electoral criteria are features required for a voting system to be deemed as 'successful'.

Definition 2.3.4 If the purpose of a criterion is to advocate for the candidate that best fits majority voters, it is said to be **optimizing**.

Definition 2.3.5 If the purpose of a criterion is to reject voting systems that allow electoral manipulation, it is called an **anti-manipulation** criterion. Anti-manipulation criteria is also considered to be optimizing. If a system fails any anti-manipulation criterion, it is not likely to be the best voting system. Therefore, these criteria can be called optimizing criteria. Note, there are criteria that are only optimizing and do not share the anti-manipulation characteristic, but *all* anti-manipulation criteria share the optimizing characteristic. [5]

The following are all examples of anti-manipulation (optimizing) criteria: [5]

No show- **this is violated if a voter can benefit from abstaining their preferences**

> Let's say there is an election between three candidates (A, B, C). Voter i's preferences are as follows: B > C > A. If it is better for voter i to not vote, meaning if voter i abstaining from voting gives candidate B an advantage, then this violates no-show.

Twin- **this is violated if two people that have the same preferences would benefit if one of them abstained from voting**

> Let there be a voter z whose preferences are also B > C > A. It follows voter z and voter i have the same preferences. If B gets an advantage when voter i does not vote, but voter z does (or vice versa) then this would violate the twin criterion.

Truncation- **this is violated if a voter can benefit from not listing all of their preferences**

> Let there be a voter j whose preferences are: A > C > B. If A gets an advantage by voter j only listing A > B on their ballot, then this violates truncation as j left C out of their preferences.

The more candidates involved in an election, the more complex things get, and the less likely that one candidate will beat every other alternative. Therefore, not every election produces a Condorcet winner.

Definition 2.3.6 If there is an election where each candidate beats the other ⅔ of the time, there is no Condorcet winner, and this is called a **Condorcet Paradox.**

Example 2.3.2 Suppose there are 3 candidates and 6 voters. If the ballots are as follows:

1. A > B > C
2. A > C > B
3. B > A > C

4. B > C > A
5. C > A > B
6. C > B > A

A is preferred to either B and/or C 4 out of the 6 ballots, which shows that it beats its other opponents ⅔ of the time. Similarly for candidates B and C. This is a Condorcet paradox.

Due to the conditions of being a Condorcet winner, it can be pretty frequent that elections do not have a Condorcet winner. Especially if there is a large number of candidates to choose from. For example, it is easier for A to be a Condorcet winner if there are only two other

candidates, but when there are 10 other candidates, A *must* win all 10. This brings to question the efficiency of the Condorcet method.

Condorcet's method runs on *completeness*.

Definition 2.3.7 A ballot is considered **complete** if it allows voters to express their true preferences between each pair of candidates.

This assumption inaccurately represents voters. There is a field in political science that looks into voter rationality. Mathematical social choice theorists believe voters are rational, meaning voters know and understand how to maximize their utility. Therefore, voters can state their true best preferences in an election. This belief corresponds with Condorcet's above assumption of completeness. However, political scientists and economists question voter rationality. Does anyone truly know what is best for them? If they do not, then their choices cannot be rational, hence their preferences not complete. Under this way of thought, voter's preferences can never be complete.

Definition 2.3.8 Condorcet developed a **majority principle**, stating that if there exists a candidate that has a simple majority over every other alternative, then that candidate should be ranked first.

Theorem 2.3.1 CONDORCET'S JURY THEOREM. If each member of a jury has an equal and independent chance at deciding whether or not the defendant is guilty, then the probability of the

majority of jurors being correct is greater than the probability each individual juror is correct. Therefore, the probability of majority jurors being correct approaches 1 as the jury size increases.

Let n (an odd number) be the number of jurors, let p be the probability a juror makes the 'correct' decision. Let $J_n(p)$ be the probability a majority of the jurors make the right decision. It follows if $p > \frac{1}{2}$ and $n > 3$:

1. $J_n(p) > p$

2. $J_n(p) \rightarrow 1$ as $n \rightarrow \infty$.

This theorem makes three main assumptions so there is always the question of whether or not it is impartial or bias to certain conditions. Condorcet's Jury Theorem is based on the jurors having a shared goal, voting being statistically independent (probability of joint occurrence is equivalent to probability of individual occurrence), and that the jurors vote correctly more than ½ of the time.

It first assumes there is a common goal. When assuming there is a shared goal, what is better for one is better for all, influencing $p > \frac{1}{2}$. This assumption creates an incentive for jurors to choose correctly, which is restrictive and may not always be the case. In fact, jurors are purposefully chosen so that there is no bias, so there should be no common goal. It can be argued that the common goal of the jurors is to find justice. This could be an incentive for the jurors to chose correctly. However, whether or not justice is found in the end, the jurors remained unaffected. A goal cannot incentivize if there are no benefits.

It also assumes statistical independence contradicts social sciences. This would mean the jurors did not communicate with each other, did not fall into peer pressure, and did not

share common information. It is self explanatory how this might be the exact opposite of what really goes on in a jury situation. [12]

There are also multiple versions of this theorem. For example, the above theorem proves Condorcet's Jury Theorem for groups with homogeneous behaviors/reliability. A statistician, Hoeffding, came up with the following version of Condorcet's Jury Theorem for heterogeneous groups:

Theorem 2.3.2 Let S be the number of successes, n is the independent trials, and p_i is the probability of success on the ith trial. If c is a positive integer such that:

$$p = (p_1 + ... + p_n)/n \geq (c/n)$$

Then

$$P(S>c) > \sum_{i=c}^{n} (n \ i)(p)^i (1-p)^{n-i}$$

Then from here we can see if c=m+1 when n+2m+1, then the following theorem is made:

Theorem 2.3.3 If n>3 and p = average voter competency> ½ + 1/2n, then

(a) $h_n(p) = h_n(p_1, ..., p_n) > p$ and

(b) $h_n(p) \rightarrow l$ as $n \rightarrow \infty$

IV.   BORDA

The Borda Count is a voting method that gives each candidate a score based on each voter's preferences. The candidate with the most points wins. The Borda Count is also seen as the arch competitor of Condorcet's method. It allows for pluralism and was introduced by Jean-Charles de Borda in 1170 [1]. Borda noticed that the alternative that is in first-place most frequent, is not necessarily the optimal choice overall.

Definition 2.4.1 Suppose there are n candidates. Using the **Borda count**, the candidate in first place gts 2n-2 points, while the candidate in second place get 2n-4 points, and so on until the candidate in last place gets 0 points. The total number of points given on each voter's ballot is n(n-1). The candidate with the most points overall wins. An example with candidates is given to the left. The poset to the left produces the following score vector {6, 4, 2, 0}.

Example 2.4.1 Suppose there are 5 candidates: A, B, C, D, E.

Voter 3's preferences: D > C > B > A > E.

Borda Count point vector: However, the point vector that best portrays

{8, 6, 4, 2, 0}   Voter 3's preferences is {10, 5, 4, 1, 0}.

A criticism of this voting method is that it weighs voter preferences equally spaced, when a voter may feel differently as shown in the example above. Voter 3 strongly prefers D to C, but feels less strongly about C to B. Another criticism of Borda Count is, like Condorcet, it does not

allow for indifferent voting preferences. Also like Condorcet's method, the Borda Count does not have anything in place to deal with ties between any number of candidates.

Condorcet even came up with the following counterexample to Borda's rule:

Example 2.3.2

# of voters	30	1	29	10	10	1
Preference	ABC	ACB	BAC	BCA	CAB	CBA

[20]

It follows that with Borda's Count, the winning preference is BAC, with B as the winner. However, A has a simple majority over B and C, therefore with Condorcet's method A would have been the Condorcet winner. It seems like the main difference between Borda and Condorcet, is Condorcet believes that candidates should be compared based on the relationships with every other candidate individually, not taking into.

Overall, note that the Borda Count is near perfect. It is able to avoid Condorcet's paradox, however, it violates one of Arrow's axioms, which will be mentioned a little later on.

Definition 2.4.2 Again, suppose there are n candidates. Using the **Partial Borda Count**, each candidate automatically receives n-1 points. Then, each candidates loses a point for each alternative ranked higher, and gains a point for each alternative ranked lower. All score vectors should sum to the same total amount n(n-1). See the example below.

Example 2.4.3 *Simple Partial Borda Count Example:*

There are five candidates, so each candidate starts off with 4 points, meaning the score vector should equal 20.
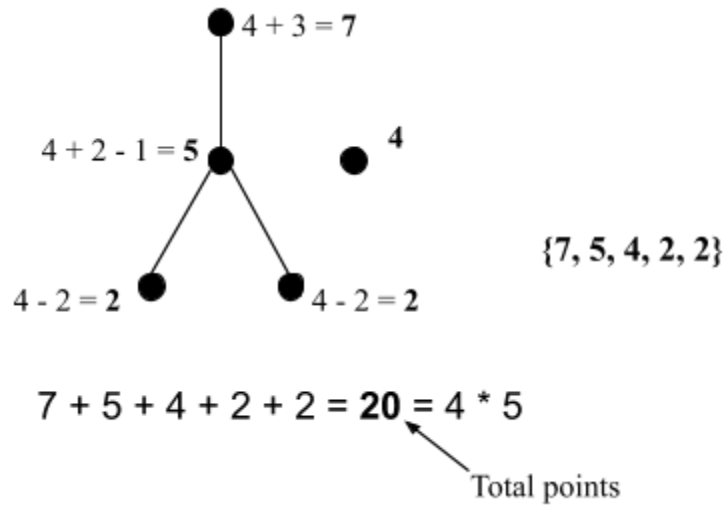
V.   ARROW

Arrow pointed out certain social choice conditions that are deemed desirable for a voting system. He discovered that no such system can satisfy all of the conditions at once. Arrow created new territory. Unlike Condorcet, and Borda, he started off by determining what a voting system should do, what it should abide by. He then came up with certain properties a 'fair' voting system should have, and tried to concoct a voting system that could satisfy these properties.

Arrow paid close attention to voters' preferences. He realized that preferences strictly needed to be ordinal, meaning the weight of preferences did not matter. Mainly because weighted preferences are subjective and relative to a voter's own personal scale. For example, let's say voter i gives candidate k a score of 10. Let another voter j give candidate k a score of 10. These two 10s can carry a completely different weight to each voter. Voter i's 10 can be a 10 out of 60, while voter j's 10 is a 10 out of 11. Even though both voters have the same weighted score, the actual weight put on each score is different and unique to each voter. There is no generic rubric to follow.

In his theory, Arrow portrays preferences as binary relations notated as, $R_{\{i\}}$ [15]. Therefore, if voter i has the following preference: A > C, voter i has the following, equivalent, binary relation $aR_{\{i\}}c$.

Definition 2.5.1 A **binary relation** is transitive (if $aR_{\{i\}}b$ and $bR_{\{i\}}c$, then $aR_{\{i\}}c$) and connected $(aR_{\{i\}}b, bR_{\{i\}}a$, or both).

Definition 2.5.2 A voter has **weak preference** when they either prefer a to b with some indifference or reservations. This is represented by $aR_{\{i\}}b$.

Definition 2.5.3 If voter i has a **strict preference**, they prefer a to b with *no indifference*. This is represented by $aP_{\{i\}}b$.

Definition 2.5.4 A **social welfare function** is a function that assigns each profile to its binary relation.

Definition 2.5.5 For some relation R, and any set X, the **restriction** is denoted R|X (restrictions of R to X), and isolates the relationship(s) between the set of candidates chosen.

Example 2.5.1 RESTRICTION EXAMPLE. Let there be an election with 4 candidates (a, b, c, and d), and 3 voters. Let's look at the following profile:

R *Preferences*	R|{a, b, c} *The restriction to the set {a,b,c}:*	R|{a, b} *The restriction to the set {a, b}:*
Voter i - acdb	acb	ab
Voter j - bdac	bac	ba
Voter k - cdba	cba	ba

As mentioned earlier, Arrow started his process by listing some characteristics of what qualifies as a just voting system. These characteristics are listed below along with explanations. These characteristics are more appropriately known as 'axioms' because they were automatically believed to be true.

ARROW'S SIX AXIOMS:

I. *Voting system rationality*: the voting system only outputs total orderings

II. *Determinism*: the results are solely based off of the preferences of the voters (i.e. there are no randomizations)

III. *Consensus*: if all of the voters prefer a certain ranking, then the voting system must also prefer that same ranking (e.g. if all the voters prefer A to B, then the voting system must also prefer A to B)

IV. *Impartiality*: all of the voters are treated equally

V. *Independence of a third alternative*: the relative ranking of two alternatives is completely independent of a third alternative

VI. *No dictators*: the preferences of one voter does not outweigh the preferences of every other voter

[16]

Theorem 2.5.1 ARROW'S THEOREM. If there are more than two candidates, there is no voting system that simultaneously satisfies the six axioms above [16]

The above theorem is also known as the Arrow Impossibility Theorem, as it seems finding a 'perfect' voting system is impossible. Tao [17] mentions in his paper that the first axiom that usually has to be sacrificed is the *Independence of a third alternative*. Ironically, a dictatorship would actually satisfy 5 of 6 axioms, as it of course violates the *No dictators* policy.

Arrow's theorem is argued to be impossible because he is, "trying to do too much with too little information" [15]. Therefore, his succeeders need to do at least one of three things, try to do a little less (i.e. maybe not try to find a 'perfect' voting system), get more information on voters' preferences, or redefine what a 'perfect' voting system is, or a combination.

VI.   YOUNG

Overall, Young did not agree with Condorcet's voting theory, and believed that Borda gives a more accurate appraisal of the best candidate in an election. Young challenges Condorcet's idea of there being a 'best choice'. He believes that the value of each candidate is relative and subjective. What is 'best' for one voter may not be 'best' for the next. While, as mentioned earlier, Condorcet sees the majority's choice as the optimal choice. Condorcet strongly believed in finding the collective 'truth'. Again, this deals with the dilemma of majority rules. Why should 51% of voters be able to dictate the choice of the remaining 49%? Or even 50.1% able to carry the decision over the 49.9% of the population?

Young does not completely disagree with Condorcet, as he acknowledges that Condorcet's method could prove true for certain circumstances. However, "if the objective is simply to reach the correct decision with highest probability, then clearly this is not the best one can do" [19]. Young also does not completely agree with Condorcet's method for three or more alternative elections, he does agree with Condorcet's theory behind his theory. For example, they both align on their view on a correct alternative may not exist for every election consisting of three or more candidates.

Young views the maximum likelihood method as the best method for determining the collective truth.

Definition 2.6.1 A **vote graph** (see Example 2.6.1 below) is a visualization of the three alternate dilemma. The dots represent an alternative, and the arrows and lines show the direction of voting preferences.

Example 2.6.1
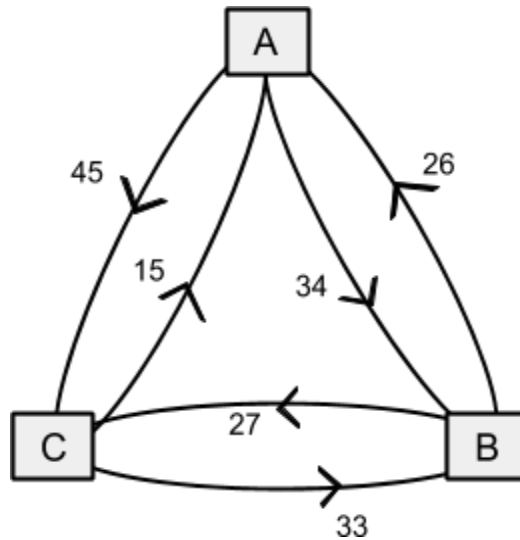
In the diagram above, 34 voters prefer A to B, 26 prefer B to A and so on. It follows that the probability of a ballot will preference $A > B > C$ or ABC is $A > B + B > C + A > C = 26 + 33 + 15 = 74$. The complete results of the example is listed below:

ABC	$\Rightarrow$ 74	BCA	$\Rightarrow$ 112$
ACB	$\Rightarrow$ 68	CAB	$\Rightarrow$ 98
BAC	$\Rightarrow$ 82	CBA	$\Rightarrow$ 106

Definition 2.6.2 It follows that the preference with the 'maximum pairwise support' is BCA, which is also known as **Condorcet's rule of three** [19]. In the example above, the Condorcet rule of three would be BCA.

CHAPTER 3: <u>Voter Preferences and Posets</u>

Posets are a visual representation of voters' preferences. Therefore, before getting into the details of posets, one should first get a better grasp on voter choice theory. When I say voter, I am referring to the average person that has the right to vote. Think of voting as listing preferences, the order in which a voter ranks or prefers a candidate over another. It is possible for a voter to not have a preference about a candidate in relation to the others, the voter is said to be indifferent.

Definition 3.1.1 An **indifferent preference** means the voter feels neutral about the alternative and cannot rank it in relation to the other alternatives. This is shown in a poset by an isolated dot.

This project involves indifferent voting preferences. Meaning a voter has the freedom to feel indifferent about any candidate(s). Voting systems like the Condorcet method, and the original Borda Count, do not allow for indifferent preferences; voters are *required* to rank all candidates. This not only restricts the number of potential ballots that can be produced by voters, but it is also causes some inaccurate representations of preferences. If a voter feels indifferent about A and B, but *has* to rank one over the other, then the voter is not able to list their true preferences.

When I mention true preferences, I am using *true* interchangeably with *best* preferences. So true preferences are the preferences that serve the voter's views and beliefs the best way

possible; the most optimal ballot for the voter. Giving a voter the opportunity to list their preferences as they believe are best is important. However, most economic theory involving people's decisions (e.g. game theory) mention that people are not rational. The average person simply does know how or does not prioritize maximizing utility, which is the most rational thing to do.

"'Precious few Americans make sophisticated use of political abstraction. Most are mystified by or at least indifferent to standard ideological concepts, and not many express consistently liberal, conservative, or centrist positions on government policy… the depth of ignorance demonstrated by modern mass publics can be quite breathtaking' and 'the number of Americans who can garble the most elementary points is … impressive'... most voters 'know jaw-droppingly little about politics'" [2].

The above quote while a little harsh, gives a brief and accurate idea of how people and voters are viewed in economic theory. There is an argument to be made that if people are irrational, then voters' do not know their *true* preferences. While I will not go into the details about what this means, it is definitely something interesting to note.

Now that there is some context, it is time to get to the purpose of this project.

Definition 3.1.2 A **poset** is a visual representation of a voter's preferences using dots and lines. Each dot represents a candidate, and the lines represent the relations between the candidates. If a dot is disjoint (isolated), then the voter feels indifferent about that candidate. A poset is equivalent to a voter's ballot.

NOTE: I will be using the words *poset* and *ballot* interchangeably.*

Definition 3.1.3 A poset that connects all of the dots in one single relation (no disjoints or gaps in the relation) is called a **complete poset**.

Definition 3.1.4 A poset that has a disjoint dot, or gap in its relation is called a **disjoint poset**.
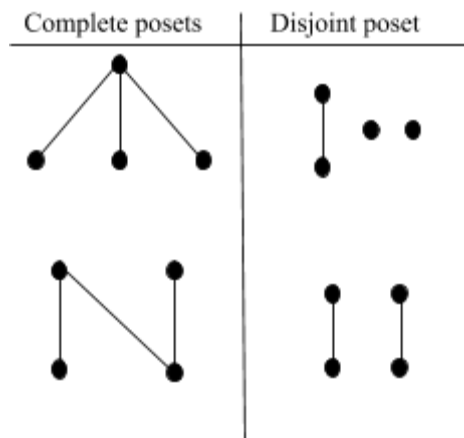
I will be working with the Partial Borda Count (*Definition 2.4.2*). Here is a quick recap: all candidates are given a score of n-1, points are gained by the amount of alternatives a candidate is above, points are taken by the amount of alternatives a candidate is below, the total amount of points given is always n(n-1).

Example 2.4.3 *Simple Partial Borda Count Example:*

There are five candidates, so each candidate starts off with 4 points, meaning the score vector should equal 20.

$4 + 3 = 7$

$4 + 2 - 1 = 5$

4

$\{7, 5, 4, 2, 2\}$

$4 - 2 = 2$ $4 - 2 = 2$

$7 + 5 + 4 + 2 + 2 = 20 = 4 * 5$

Total points
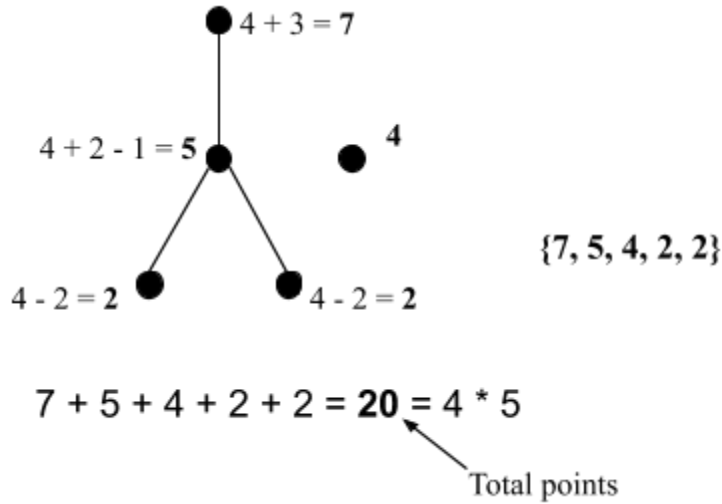
Theorem 3.1.1 A score vector $\{a_1, a_2, \ldots, a_n\}$, where n is the number of alternatives, must satisfy the following conditions:

1) $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$

2) $a_1 + a_2 + \ldots + a_n = n(n-1)$

3) $a_1 \leq 2(n-1)$

Lemma 3.1.1 If there is a n-1 score in the first or last place of a score vector, then all of the scores in the score vector must be n-1 in order to satisfy the second condition of *Theorem 3.1.1*.

Proof. Let there be n candidates. It follows, the highest score in the vector (the first score) is n-1. This means the vector takes the form {n-1, ...}. Let one point be n-2, producing a score vector of {n-1, n-1,..., n-2}. It follows n-1 + n-1 + … + n-2 = n-1(n-1) + n-2 < n(n-1). This violates condition 1 of *Theorem 3.1.1*. Therefore, the only way n-1 can be the highest score is if all of the scores in the vector are also n-1 to produce the sum n(n-1).

Now let n-1 be the lowest score in the vector, taking a form of {..., n-1}. Let one of the scores be n, to produce a vector {n, n-1, …, n-1}. It follows, n + n-1 + … + n-1 = n + n-1(n-1) > n(n-1). Again, this violates the first condition of *Theorem 3.1.1*. The only way to produce a proper score vector with n-1 being the lowest score, all of the scores must also equal n-1.

Therefore, if there is a n-1 score in the first or last place of a score vector, then all of the scores in the score vector *must* be n-1.

Below is a visualizations of all the possible posets (ballots) for 0- to 4-candidate elections:

All of the posets above have unique score vectors, meaning none of the posets produce the same score. However, when illustrating the 5-candidate posets (listed in the appendix), there were four pairs of posets that each pair goes to the same score vector (listed below). If thinking

about the Partial Borda Count as a social choice function, it is injective until the domain exceeds

4.

5-candidate posets with same score vector:

$\{7, 5, 4, 3, 1\}$

$\{7, 5, 4, 2, 2\}$

$\{6, 5, 4, 3, 2\}$

$\{6, 6, 4, 3, 1\}$
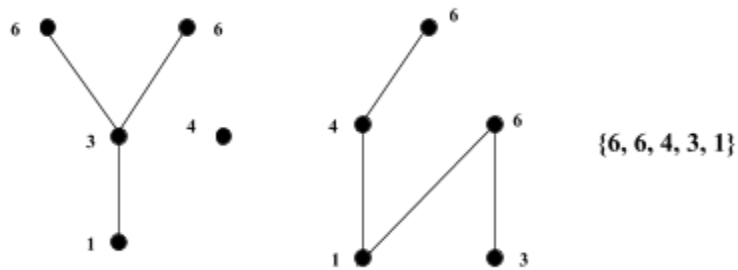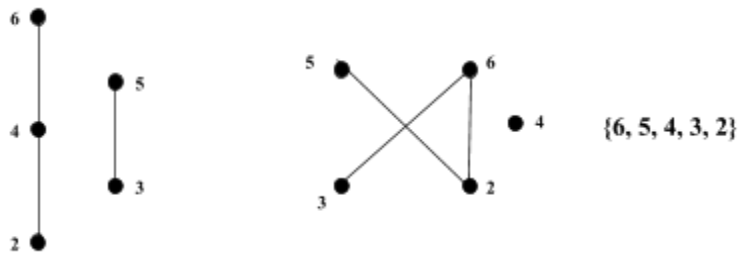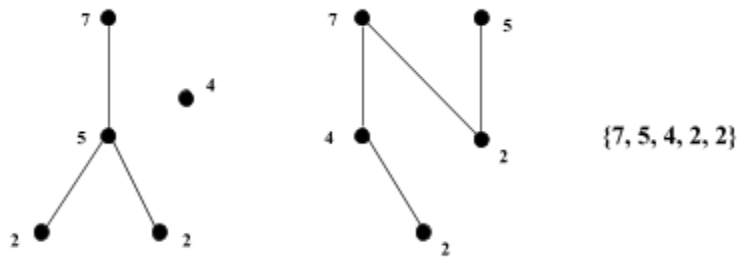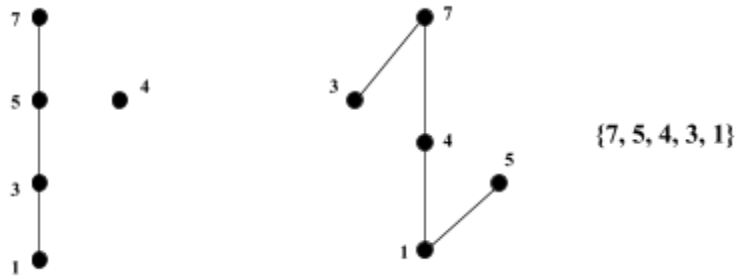
I noticed that for each of these groups of matching posets, there can be a generalization made. Let's look at the first group listed.
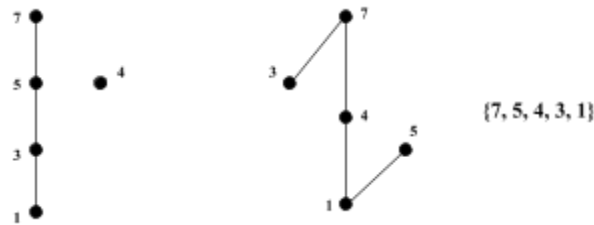
We can easily make these 5-candidate ballots, 6-candidate ballots by adding one disjoint point to each, as seen below.
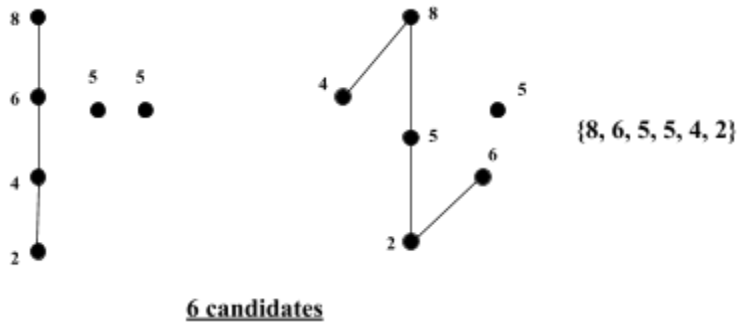
6 candidates

Notice how these posets still go to the same score vector. This is because adding one indifferent candidate does not affect the relationship of the ordered candidates.

Example 3.1.1 The ballot below represent voter i's preferences for 5 candidates, A, B, C, D, and E (black). There is news of a new candidate, F, entering the election. Voter i is completely indifferent about this candidate for whatever reason. Voter i's new ballot is also represented in the diagram below (red).

{7, 5, 4, 3, 1}

↘

{8, 6, 5, 5, 4, 2}
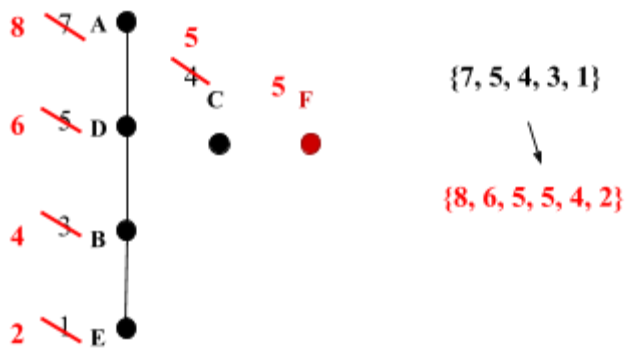
Notice the 5-candidate poset before F entered the election. It is a disjoint 5-candidate poset. This disjoint poset is the combination of a complete 4-candidate poset of relationship between A, B, D, and E, and a disjoint candidate, C. Therefore, when F joins as an indifferent, disjoint, candidate. It has no effect on the relationships between A, B, D, and E. Hence, this complete 4-candidate poset remains unchanged in terms how many points each receive and give away. The only thing the additional candidate does affect is increasing initial score of all the candidates, which results in the score of each candidate increasing by one.

{9, 7, 6, 6, 6, 5, 3}
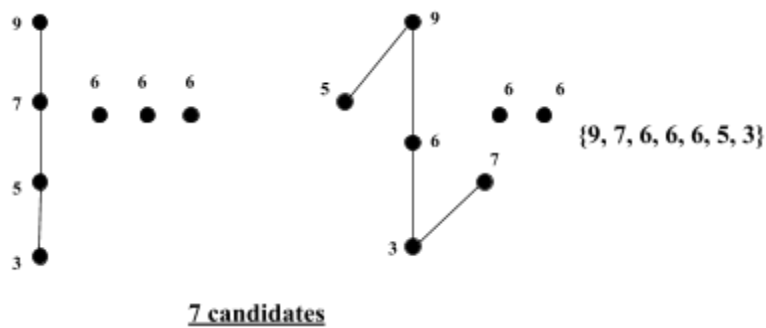
7 candidates

To test this theory, I tried increasing the candidate number to 7, by adding another disjoint dot to each. As seen above, the two posets still go to the same score vector.

Theorem 3.1.1 Let there be a m-candidate election. There is a group of posets that go to the same score vector. Let $n > m$, and let $n-m = k$. It follows this group of posets will go to the same score vector for all n-candidate elections, as long as k disjoint dots are added to each poset.

Lets apply this to the two posets mentioned earlier:
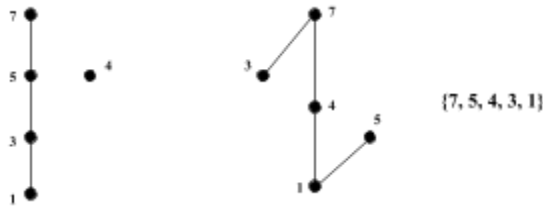
We know these two posets *match* (i.e. go to the same score vector) in a 5-candidate election. We've already seen that they still match for 6, and 7-candidate elections, and based on Theorem 3.1.1 they match for all elections with greater than 5 candidates when adding disjoint points. Therefore, we can make the following generalization below for these two posets.
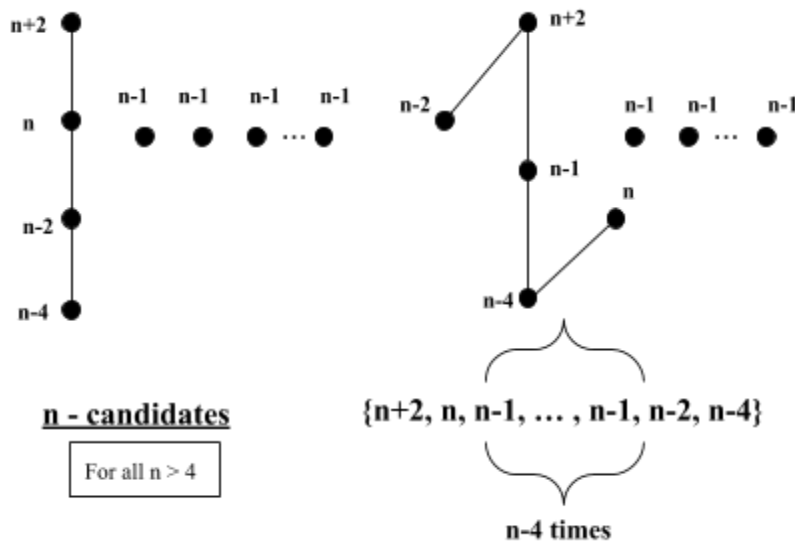
I then went ahead and made this same generalization for the other three groups of posets that

match in 5-candidate elections.

n candidates

For all n > 4

$\{n+2, n, n-1, \ldots n-1, n-3, n-3\}$

n-4 times
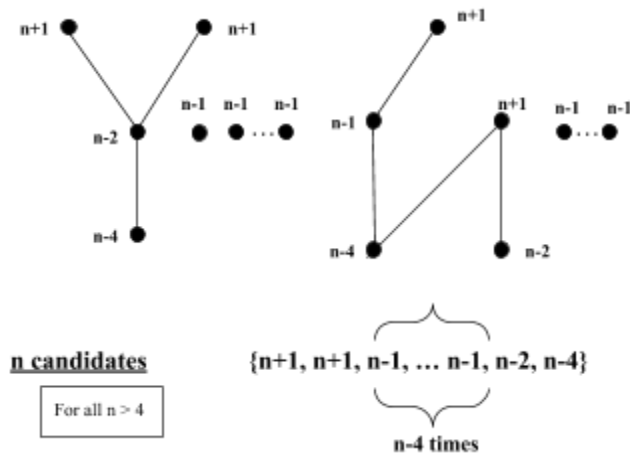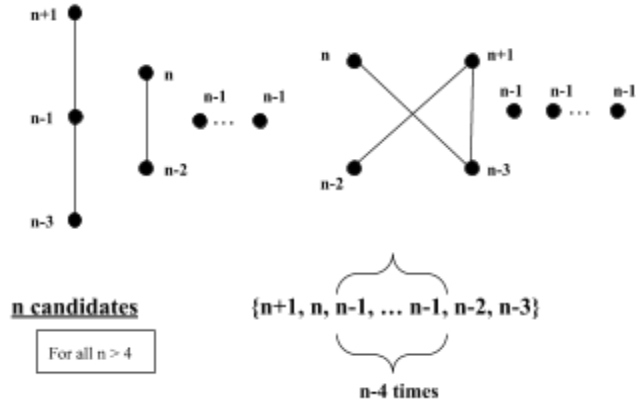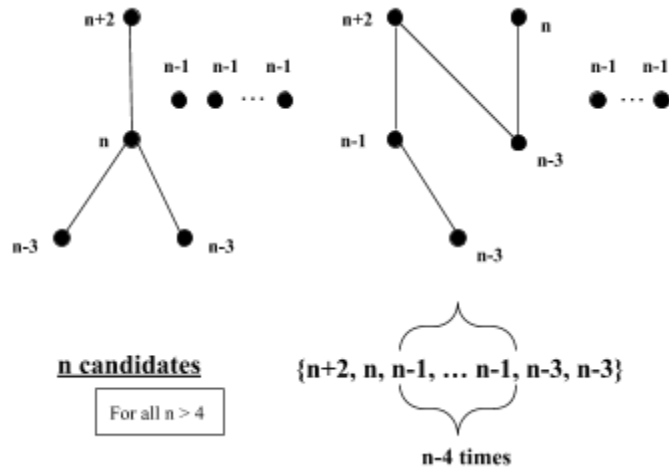
CHAPTER 4: <u>Results</u>

The next questions to answer are what does this mean and how can we predict which posets have the same score. I was able to construct an algorithm that can match certain complete 6-candidate posets to certain disjoint 6-candidate posets, then generalized it to work for any complete n-candidate posets.

First I composed a database with all of the 5-candidate posets and their score vectors. From there I looked at the groups of posets that had the same score vectors:
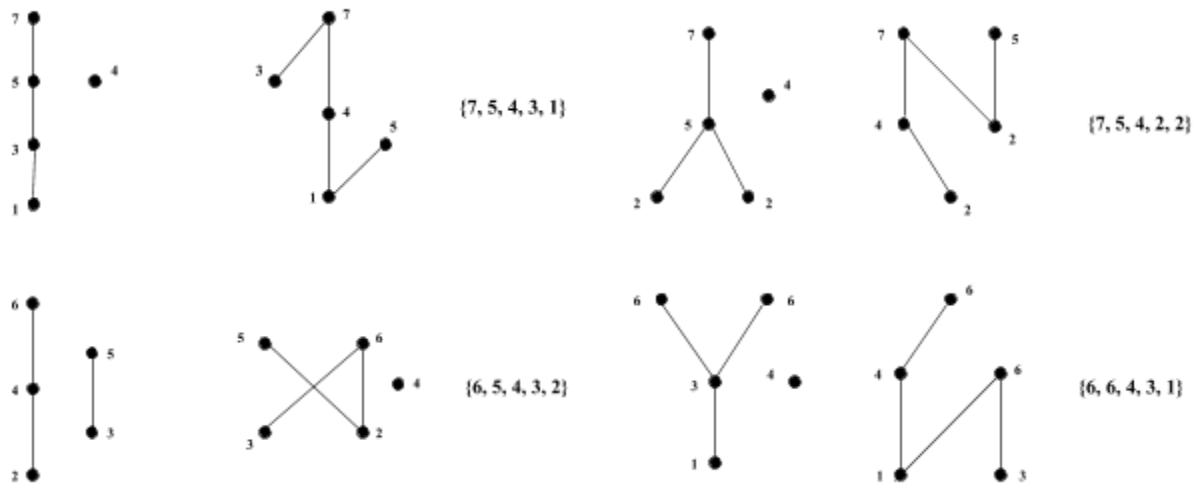
These four score vectors are the only ones that have multiple posets for 5-candidate elections. Looking at this, I noticed that three of the four groups of posets that produce the same score has one complete poset and one disjoint poset. From there, I wanted to see if there was some kind of correlation or connection that I can use for 6-candidate posets, as composing all of the near 400 posets would be extremely tedious.

I then used one of the groups above to create the following algorithm. I know the disjoint 5-candidate poset can also be looked at as a complete 4-candidate poset with an extra (indifferent) point that has a score of n-1 (the score all alternatives start off with), which in this case is 4. Therefore, what happens if we remove this 4-point from the complete 5-candidate poset?

Complete 4-candidate poset

Score is n-1 (in this case 4) because it is disjoint

This is the n-1 point (as it gains as many points as it gives away). What happens when we remove this?
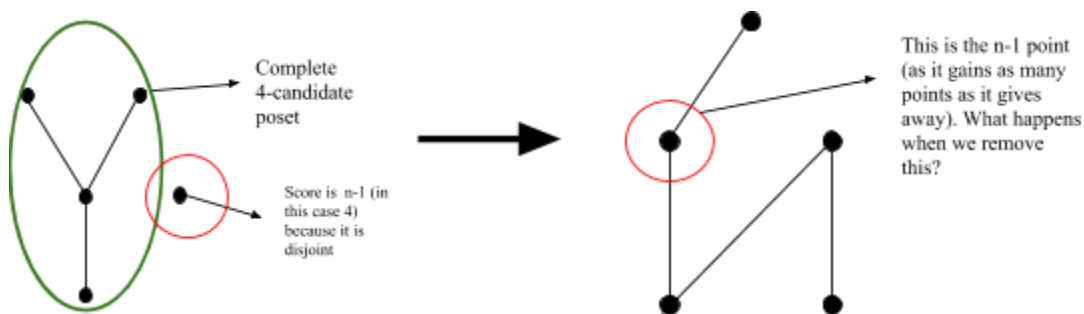
The main question is, can we reverse the arrow and find the disjoint poset using the complete poset?

This lead me to the following method:

1) Identify a complete 5-candidate poset with at least one score of 4

2) Remove that point from the score vector

3) Subtract 1 point from all other scores

4) Match this score vector to a complete 4-candidate poset

5) Add an indifferent, disjoint point (4-points)

6) Add 1 point back to every other score

7) This new score vector should match the score vector from Step 1
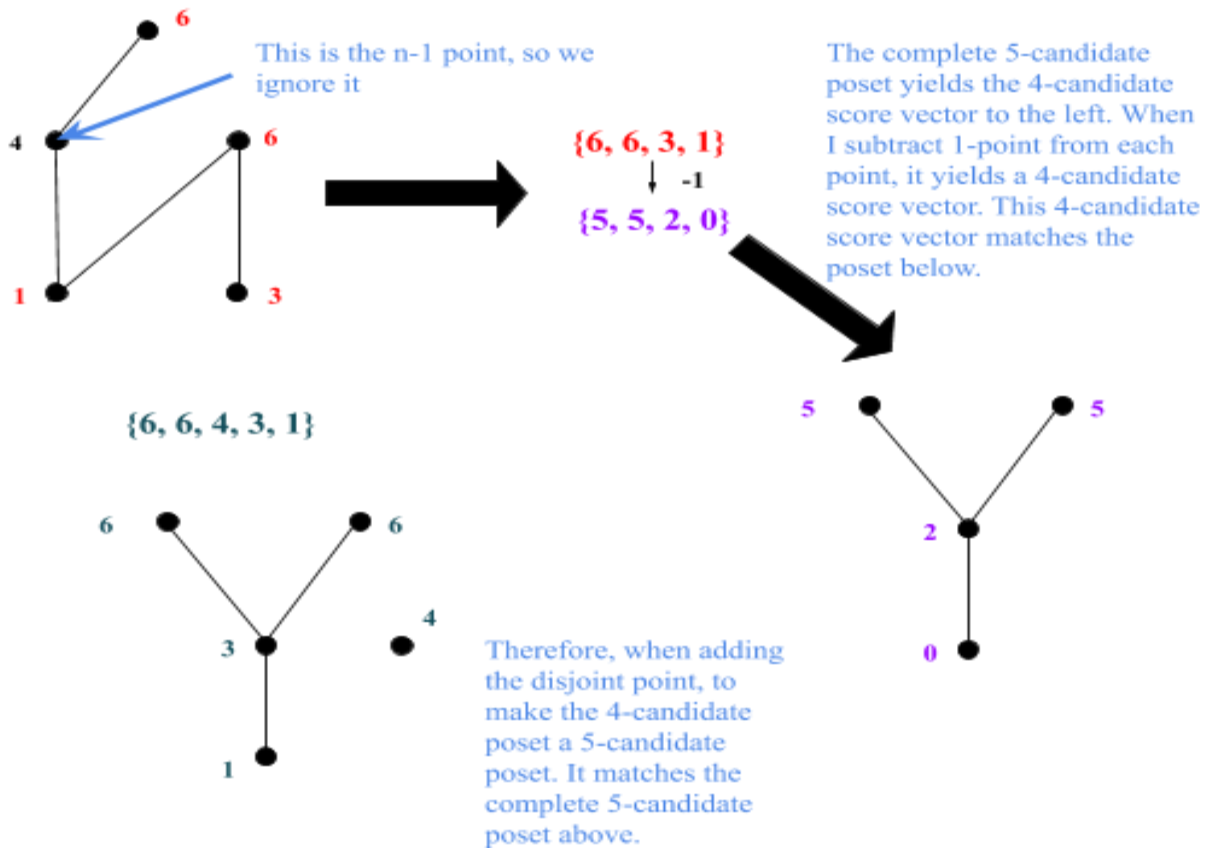
The method mentioned above can be generalized for a complete *n*-candidate poset with a n-1 point. The steps are as follows:

- *Step 1*- Identify a complete n-candidate poset with at least one n-1 point

- *Step 2*- Remove one n-1 point for the poset

- *Step 3*- Subtract one point from the remaining scores

- *Step 4*- Identify a n-1 complete poset that matches this new poset

- *Step 5-* If a match is found, add a disjoint vertex (n-1 point) to the poset, add one point back to the other four scores, and it becomes a disjoint n-candidate poset that has the same score vector as the poset from Step 1

I found the following posets with the same vector using the algorithm above:

{9, 6, 5, 5, 3, 2}

{8, 5, 4, 2, 1}

{9, 6, 5, 5, 3, 2}

n-candidates

For all n > 5.

{n+3, n, n-1, n-1, ... n-1, n-3, n-4}

n-5 times

{8, 7, 5, 5, 4, 1}

{7, 6, 4, 3, 0}

{8, 7, 5, 5, 4, 1}

n-candidates

For all n > 5.

{n+2, n+1, n-1, n-1, ... n-1, n-2, n-5}

n-5 times
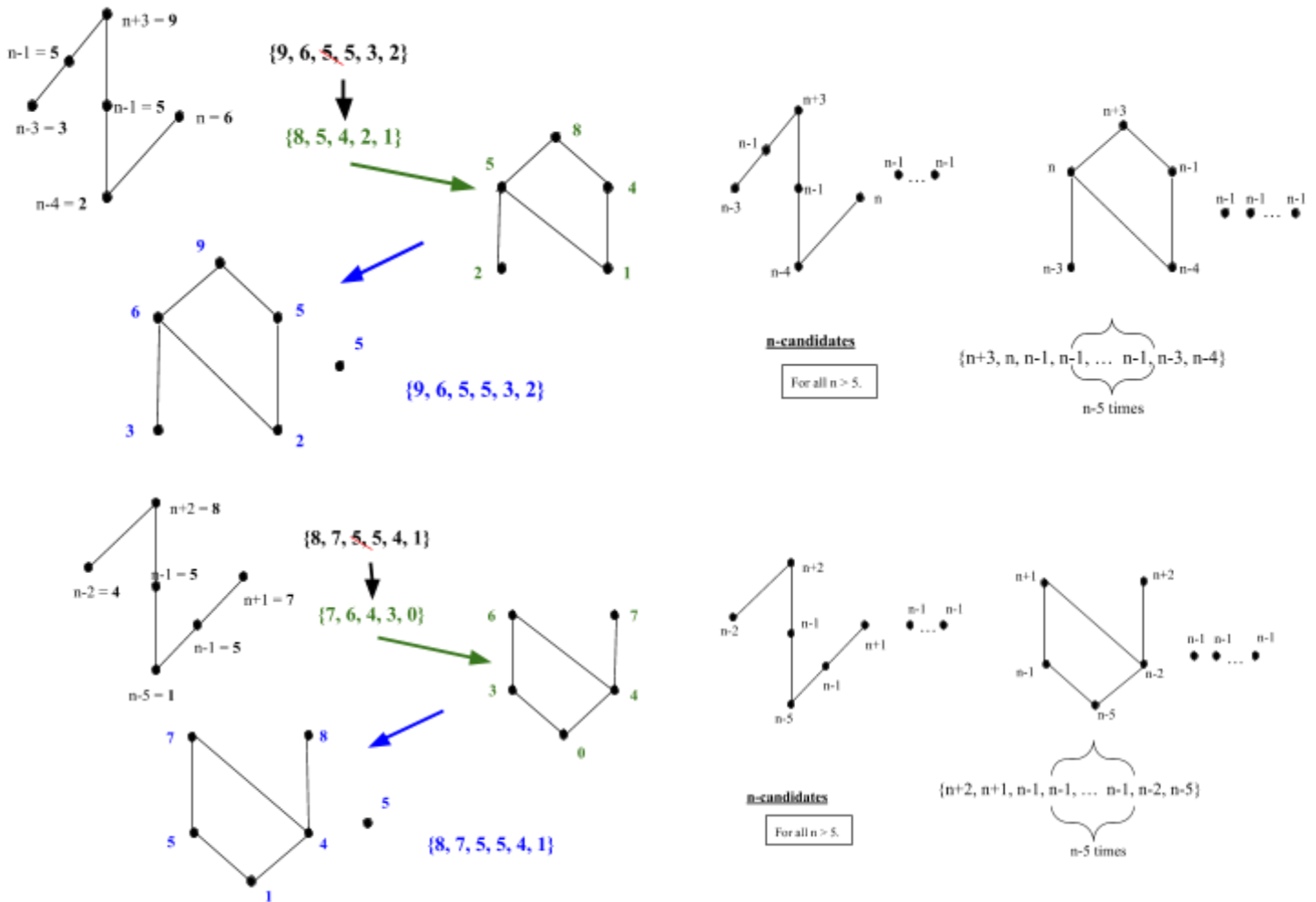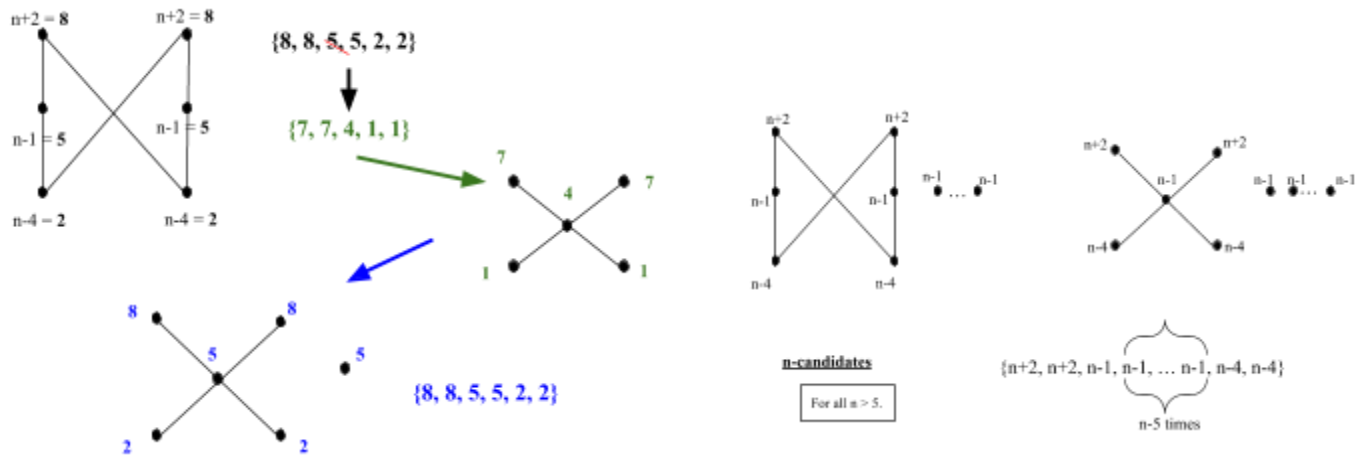
The method above helps find disjoint n-candidate posets using complete n-candidate posets. However, when using this method, I had to semi-randomly put together a complete 6-candidate poset with a n-1 point and then see if it matched a 5-candidate poset that I already had access to. What happens if we reverse this method? Can use the database I already have to predict matches with complete 6-candidate vectors? The answer is yes, and the generalization of the reverse of this method is as follows:

Reverse of Method 1:

- *Step 1*- Identify a n-candidate score vector $\{a_1, a_2, a_3, \ldots, a_n\} = n(n-1)$

- *Step 2*- Add one point to each score $\{a_1+1, a_2+1, a_3+1, \ldots, a_n+1\} = n(n)$ ✗

- *Step 3*- Identify a spot to place a score of n-1 $\{a_1+1, a_2+1, \ldots n-1, \ldots\} = n+1(n)$ ✔

- *Step 4*- See if there is a way to construct a n+1 poset that matches the score vector

Using this reverse method, I compiled a collection of all of the 5-candidate (n) score vectors, and applied the above method to it. I used a more arbitrary method to collect more potential 6-candidate (n+1) score vectors. From there it was trial and error to see which score vectors actually yield a poset. I was able to make the following observations.

Theorem 4.1.1 In order for an alternative to have n-1 points it *must* satisfy **one** of the following conditions:

1) It has the same amount of alternatives above as below it
2) It is a disjoint point

Since I am trying to find complete posets, I can ignore the second condition and only work with the first. Knowing this, I noticed that a complete poset cannot be constructed when there was a 5 in the fourth or fifth place of the 6-alternative score vector. Why? There is not an even number of alternatives above as below the n-1 point. When the score vector is set up like the following {_,_,_,5,_,_} or {_,_,_,_,5,_} and the poset is a complete vector, the n-1 alternative must give a point to all of the alternatives above it, however, there is not enough alternatives below it to keep it n-1. What about when a score vector has a form of {_,5,_,_,_,_} or {_,_,5,_,_,_}? There is still an uneven number of candidates above and below the n-1 point in the score vector, however, all of the candidates below the 5-point, does not have to give a point to it. Meaning, a score vector {8,7,**5**,4,3,3} can produce the following complete 6-candidate poset:
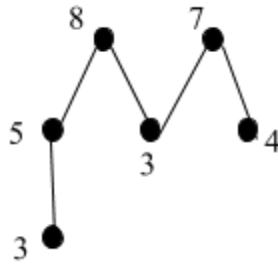
A trend that I noticed, is all score vectors that started with 10 points were complete posets. I then made the following conclusions:

Lemma 4.1.1 The highest score a n-candidate poset can have is 2(n-1).

Proof. Let there be an election with n candidates. It follows for a voter, one candidate is above the rest. The candidate on top starts with n-1 points, and gains 1-point from every other candidate. Note, this is the best any candidate can do. Therefore, it receives a score of (n-1)+ 1(n-1) = 2(n-1). Since this is the best a candidate can do, 2(n-1) is the highest score a candidate can receive in a n-candidate election.

Lemma 4.1.2 If a score vector starts off with a score of 2(n-1), then it *must* be a complete poset, **and** there is a unique alternative on top.

Proof. Let there be an election with n policy alternatives to vote on. It follows from the lemma above, if an alternative receives 2(n-1) points, it must get a point from every other alternative. This means in a poset, every other alternative must be connected to it, and below it. Since it is connected to every other alternative, the poset is complete. Since it is gaining a point from every other alternative, it is the only one on top and shares indifference or is below no other alternative.

Below is a brief display of the results discussed earlier:

There are 6 places a point can be placed in a 6-candidate score vector. In order to find all 6-candidate posets with a 5-point, I tried putting a 5 in all 6 places of the score vector. I then realized only two of the cases were even relevant to what I was looking for.

Case 1 - {5,_,_,_,_,_} ⇒ From the characteristics of a score vector, I know the only score vector that can come from this case is {5,5,5,5,5,5}.

Case 2 - {_,5,_,_,_,_}

Case 3 - {_,_,5,_,_,_}

Case 4 - {_,_,_,5,_,_} ⇒ Could not construct usable score vectors.

Case 5 - {_,_,_,_,5,_} ⇒ Could not construct usable score vectors.

Case 6 - {_,_,_,_,_,5} ⇒ Equivalent to case 1.

Case 2: {_,5,_,_,_,_}

1st # of score vector	**Potential Complete 6-candidate score vectors**					
6	{_,5,5,5,5,4}					
7	{_,5,5,5,5,3}	{_,5,5,5,4,4}				
8	{_,5,5,5,5,2}	{_,5,5,5,4,3}	{_,5,5,4,4,4}			
9	{_,5,5,5,5,1}	{_,5,5,5,4,2}	{_,5,5,5,3,3}	{_,5,5,4,4,3}	{_,5,4,4,4,4}	
10	{_,5,5,5,5,0}	{_,5,5,5,4,1}	{_,5,5,5,3,2}	{_,5,5,4,4,2}	{_,5,5,4,3,3}	(_,5,4,4,4,3,)

*** Trend: I was only able to construct complete posets with the score vectors that started with 10***

Posets I was able to construct from the above vectors:
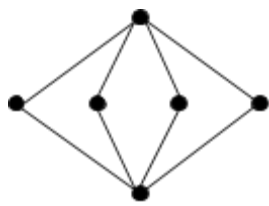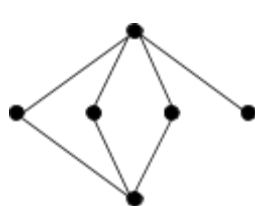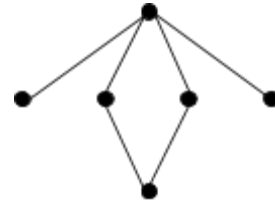
{10, 5, 5, 5, 5, 0}   {10, 5, 5, 5, 4, 1}   {10, 5, 5, 4, 4, 2}
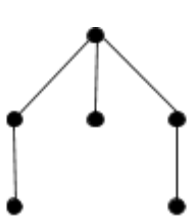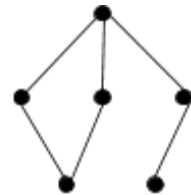
{10, 5, 5, 4, 3, 3}   {10, 5, 5, 5, 3, 2}   {10, 5, 4, 4, 4, 3}

Case 3: {_,_,5,_,_,_}

<u>**1st 2 #s of vector**</u>   <u>**Potential Complete 6-candidate score vectors**</u>

66	{_,_,5,5,5,3}	{_,_,5,5,4,4}			
76	{_,_,5,5,5,2}	{_,_,5,5,4,3}	{_,_,5,4,4,4}		
77, 86	{_,_,5,5,5,1}	{_,_,5,5,4,2}	{_,_,5,5,3,3}	{_,_,5,4,4,3}	
87, 96	{_,_,5,5,5,0}	{_,_,5,5,4,1}	{_,_,5,5,3,2}	{_,_,5,4,3,3}	{8.7,5,4,4,2}
88, 97, 10 6	{_,_,5,5,4,0}	{_,_,5,5,3,1}	{_,_,5,5,2,2}	{_,_,5,4,3,2}	{_,_,5,4,4,1} {_,_,5,3,3,3}
98, 10 7	{_,_,5,5,3,0}	{_,_,5,5,2,1}	{_,_,5,4,4,0}	{_,_,5,4,3,1}	{_,_,5,4,2,2} {_,_,5,3,3,2}
99, 10 8	{_,_,5,5,2,0}	{_,_,5,5,1,1}	{_,_,5,4,3,0}	{_,_,5,4,2,1}	{_,_,5,4,3,1} {_,_,5,3,2,2}

Trend: 10 9 and 10 10 score vectors cannot be constructed

Posets I was able to construct from the above vectors:
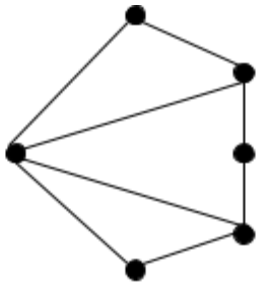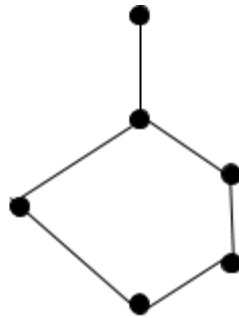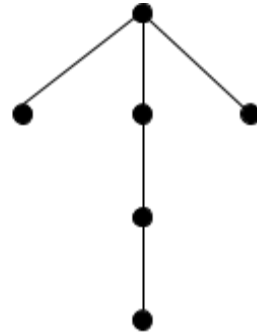
{10, 8, 5, 5, 2, 0} {10, 8, 5, 4, 3, 0} {10, 8, 4, 3, 3, 2}
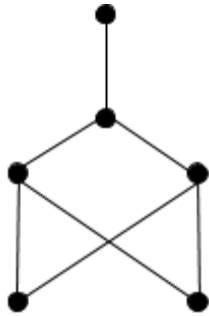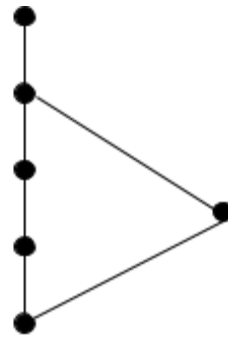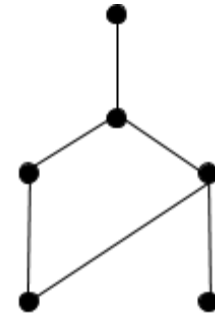
{10, 8, 5, 5, 1, 1} {10, 8, 5, 4, 3, 0} {10, 8, 5, 4, 2, 1}
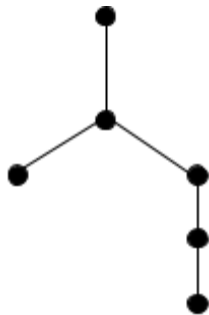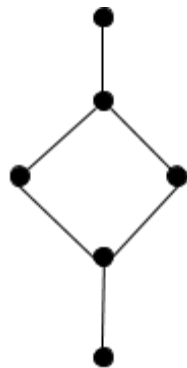
{10, 8, 5, 3, 3, 1} {10, 8, 5, 5, 2, 0} {10, 7, 5, 4, 3, 1}
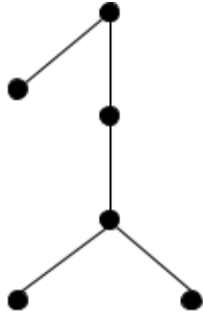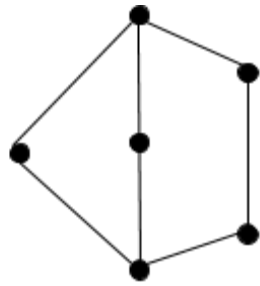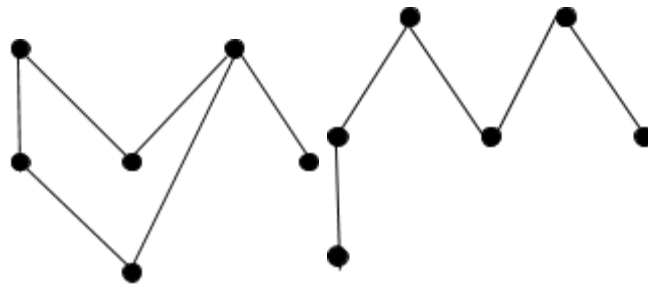
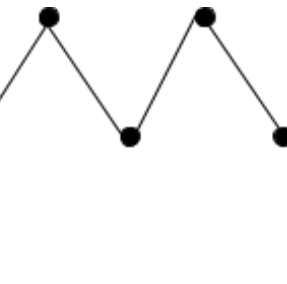{10, 7, 5, 4, 2, 2}   {10, 6, 5, 5, 4, 0}
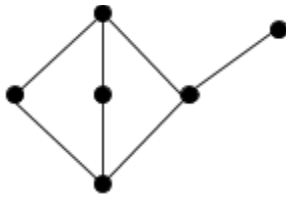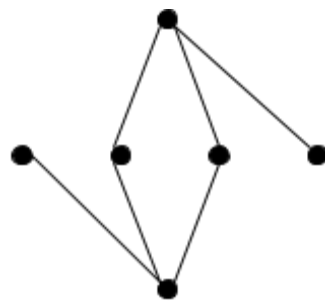
{9, 7, 5, 5, 4, 0}   {9, 6, 5, 5, 4, 1}
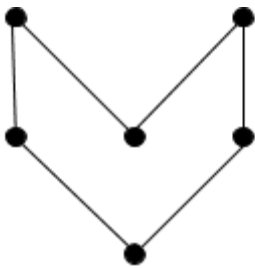
{8, 8, 5, 5, 4, 0}   {8, 8, 5, 4, 3, 2}   {8, 7, 5, 4, 3, 3}

Once I constructed the posets above, the next move was to see if any of these 6-candidate posets have a match. The 6-candidate score vector undergoes the algorithm, a 5-candidate score vector is produced, I used my 5-candidate poset database to see if the score vector it produced actually exists, if it did then you found a poset match, if not the poset's score vector is unique. When looking at the score vectors for each poset, I was able to get rid of all score vectors with a score higher than 9 and posets with a score of 0.

We know from Chapter 3, that the highest score a score vector can have is 2(n-1). Therefore, the highest score a 5-candidate poset can have is 8. So the 6-candidate posets with a score of 10 points will never have a 5-candidate 'match' as when going through my algorithm, 10-points would become 9 (> 8) in a 5-candidate vector, which is not possible. If a 6-candidate score vector has a 0 point, when it goes through my algorithm and 1 point is subtracted, it would become a 5-candidate score vector with a negative point, is also not possible.

I was able to find the following matches from the posets above:
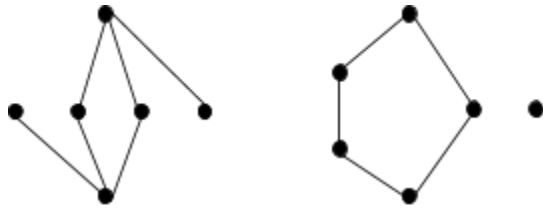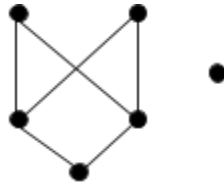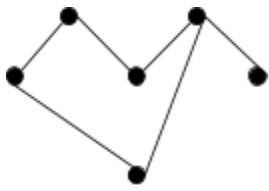
6-candidates

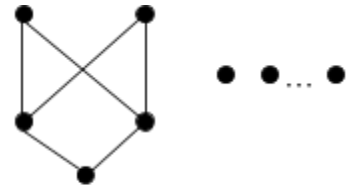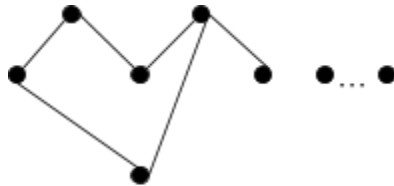{9, 6, 5, 5, 4, 1}
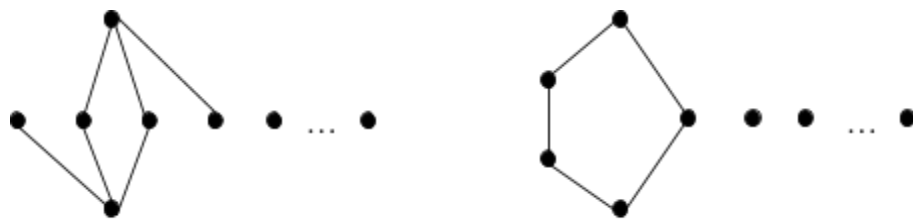
N-candidates

For all n ≥ 6.

{n+3, n, n-1, … n-1, n-2, n-4}
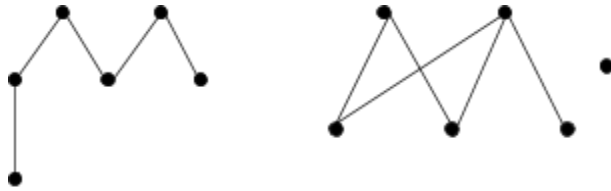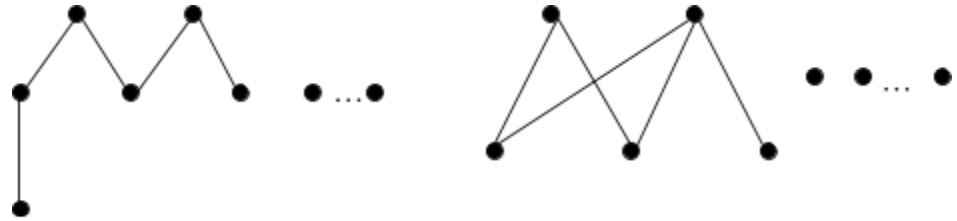
6-candidates
{8, 8, 5, 4, 3, 2}

N-candidates

For n ≥ 6

{n+2, n+2, n-1, … n-1, n-2, n-3, n-4}

6-candidates

{8, 7, 5, 4, 3, 3}

N-candidates

For $n \geq 6$

{n+2, n+1, n-1, … , n-1, n-2, n-3, n-3}

Most of my research and experimentation was very trial and error, and situational. The most solid and consistent finding that came from my experimentation was the method I created for matching posets. Something to note is that the injectivity failure, that originates in 5-candidate elections, is only built upon. Meaning once posets match, they will *always* match for all elections with a greater number of candidate (see Theorem 3.1.1). The injectivity failure only increases and builds as the number of candidates increase; the quantity of elements/inputs that fail injectivity.
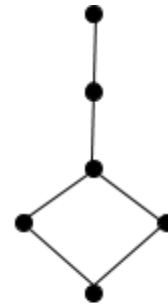
CHAPTER 5: <u>Conclusion</u>

My algorithm did prove useful in finding failures of injectivity. However, the matches that it can produce is limited, as it only applies to posets with a n-1-point, while I'm sure there are many 6-candidate posets without a 5-point. Here are a few examples that I was able to construct:
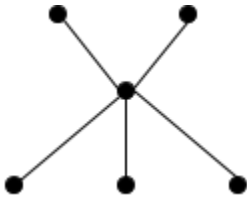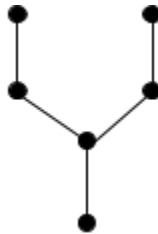
{9, 9, 6, 2, 2, 2}   {8, 8, 6, 6, 2, 0}   {10, 8, 6, 3, 3, 0}

My algorithm would not work for the above score vectors. A great way to further the research in this project would be to find a broader way of predicting poset matches, or finding another condition (i.e. having at least one n-1 point, as required for my method). In creating this method, it further proves that the function, Partial Borda Count is only injective when voting with four options or less.

Once, a fifth option enters the equation, the function fails injectivity with four score vectors that have multiple posets attached to it. Through *Theorem 3.1.1*, we know these four anomalies are also present in every election with an option-size greater than 5. Meaning, when

there are 6 options in an election, there are anomalies that are unique to 6-candidate elections, like the poset matches listed in the previous chapter, plus the total anomalies from the 5-candidate elections that carry over. Therefore the total anomalies in 7-candidate elections is the unique 7-candidate anomalies + 6-candidate anomalies (which includes the 5-candidate anomalies). And so forth to theoretically infinity-candidate elections. This means that the failure of injectivity just keeps accumulating and collecting more and more as the amount of candidates increase.

For all n > 4, there are unique anomalies that fail injectivity. However, the total amount of anomalies for all elections with candidates greater than 5 is dependent on the total anomalies of all election-sizes before it. For example, you cannot know how many injectivity failures there are in a 7-candidate election without knowing how many are in 6-candidate elections, which is dependent on 5-candidate elections. 5-candidate elections are in fact the only elections in which the total number of anomalies is independent, as it is when the existence of the failures start.

If voters can take anything from this project, it should be that when voting with the Partial Borda Count, any two or group of random voters with completely different preferences, can wind up giving the candidates the same score. Posets or ballots are not unique, they can share the same result.

Appendix

5-candidate posets (63 in total)
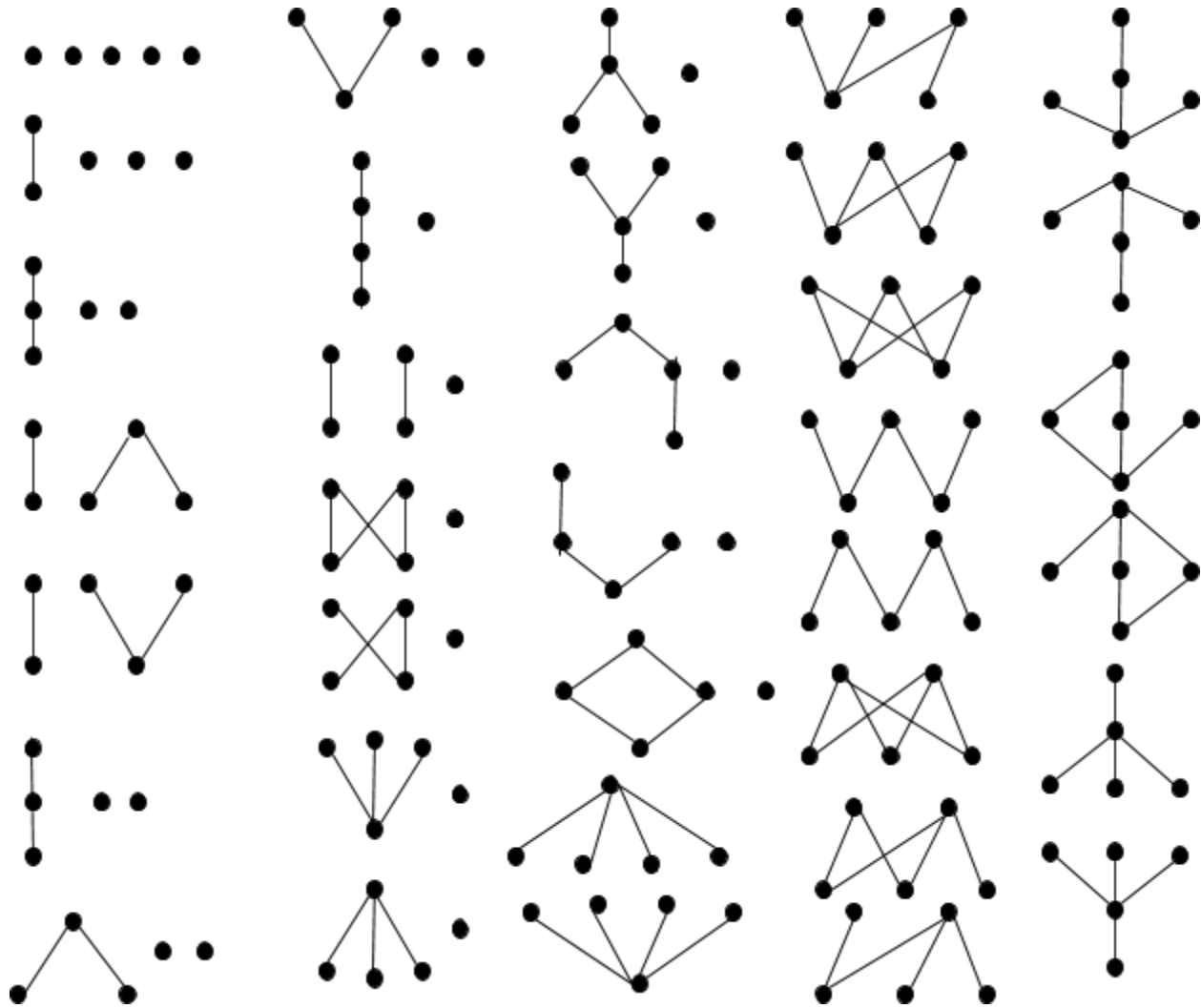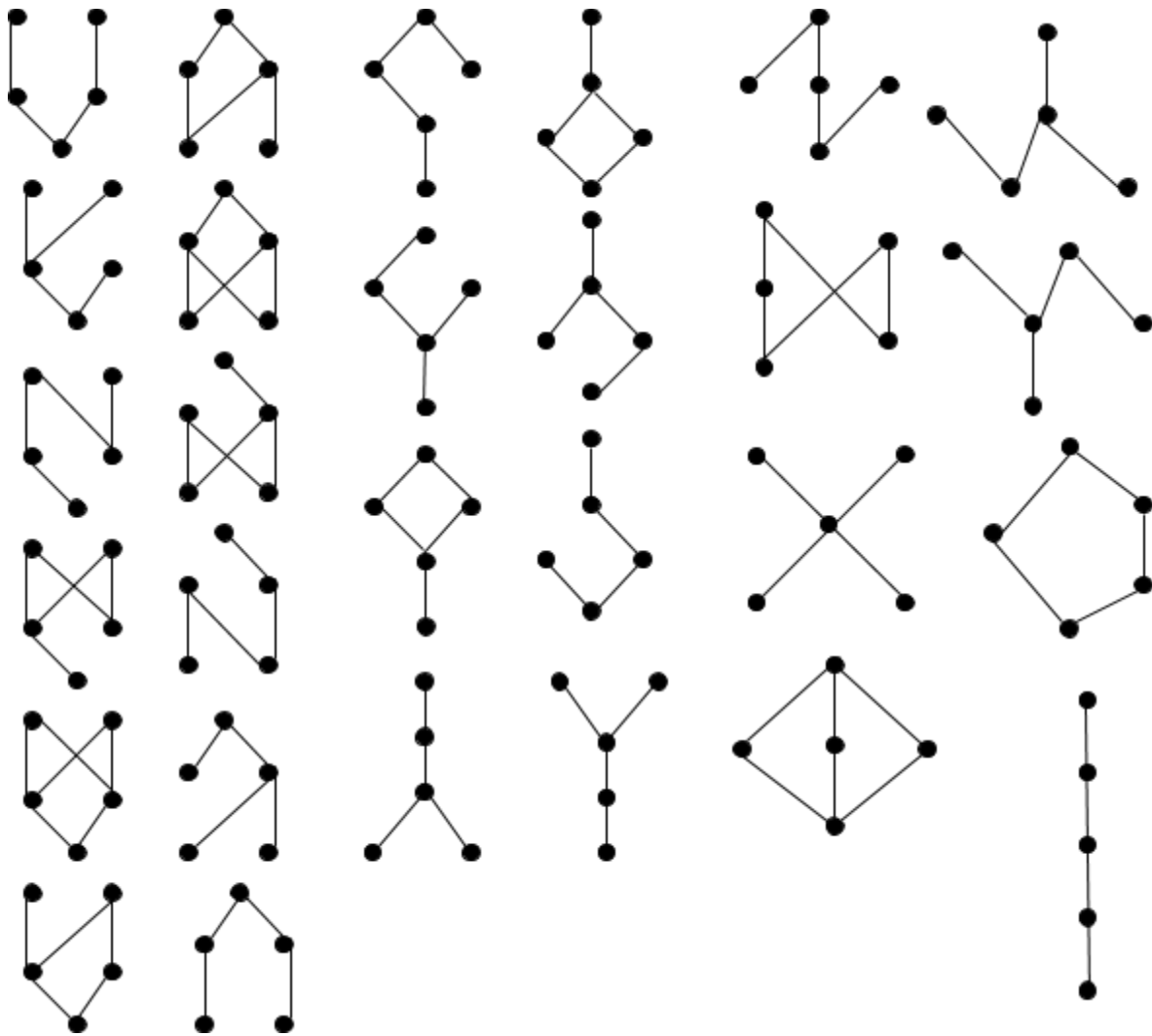
<u>**References**</u>

[1] Black, D. (1976). Partial Justification of the Borda Count. *Public Choice, Vol. 28, pages 1-15*. Retrieved from:
https://www.jstor.org/stable/pdf/30022913.pdf?refreqid=excelsior%3A9a3a591463fb0e6eef5e8d1d750f2593

[2] Bendor, J., Diermeier, D., Diegel D. A. & Ting M. M. (2011). *A Behavioral Theory of Elections.* Retrieved from
https://www.jstor.org/stable/pdf/j.ctt7shf3.8.pdf?ab_segments=0%252Fdefault-2%252Fcontrol&refreqid=excelsior%3Af1b9b00dea88a910ad65b08664d2877d\

[3] Boland, P. J. (1989). Majority Systems and the COndorcet Jury Theorem. *Journal of the Royal Statistical Society. Series D (The Statistician), Vol. 38, No.*, pp. 181-189. Retrieved from:
https://www.jstor.org/stable/pdf/2348873.pdf?ab_segments=0%252Ftbsub-1%252Frelevance_config_with_defaults&refreqid=excelsior%3A08f0d4dffa2bc63f701592be9a11e9d0

[4] Crepel, P. & Rieucau, J. (2005) Condorcet's Social Mathematics, A Few Tables. *Social Choice and Welfare, Vol. 25, (Issue 2/3), pages 243-285*. Retrieved from:
https://www.jstor.org/stable/pdf/41106700.pdf?refreqid=excelsior%3A2a36aa928ef463471c9d6e84d7c8e77e

[5] Darlington, R. B. (2018). *Are Condorcet and Minimax Voting Systems the Best?*. Retrieved from: https://arxiv.org/pdf/1807.01366.pdf

[6] Darlington, R. B. (2018). *Why Condorcet Consistency is Essential.* Retrieved from:
https://arxiv.org/pdf/1706.01841.pdf

[7] Emerson, P. (2013). The original Borda count and partial voting. *Social Choice and Welfare, Vol. 40 (No. 2), pages 353-358*. Retrieved from:
https://www.jstor.org/stable/pdf/23360957.pdf?refreqid=excelsior%3A6769f375a1313839e73ed3531301bca0

[8] Gardner, R. (1977). The Borda Game. *Public Choice, Vol. 30, pages 43-50*. Retrieved from:
https://www.jstor.org/stable/pdf/30022965.pdf?refreqid=excelsior%3Af6339ac7211e80f24e6dc1120d2e9c54

[9] Hamlin, A. (2017, March 30). *The Illusion of Majority Rule in US Elections.* Retreieved from
https://ivn.us/2017/03/30/illusion-majority-rule-elections/

[10]Koyre, A. (1948). Condorcet. *Journal of the History of Ideas, Vol. 9, No. 2,* pp. 131-152. Retrieved from:
https://www.jstor.org/stable/pdf/2707225.pdf?refreqid=excelsior%3Addc280452eb7720c1cde57 75a2c6edb2

[11] Maskin, E. (1979, February). *Voting for Public Alternatives: Some Notes on Majority Rule.* Retrieved from:
https://scholar.harvard.edu/files/maskin/files/voting_for_public_alternatives_some_notes_on_ma jority_rule.pdf

[12] McGann, A. (2016, August). *Voting CHoice and Rational Choice.* Retrieved from
http://oxfordre.com/politics/view/10.1093/acrefore/9780190228637.001.0001/acrefore-97801902 28637-e-79

[13] Ladha, K. K. (1993). Condorcet's jury theorem in light of de Finett's theorem: Majority-rule voting with correlated votes. *Social Choice and Welfare, Vol. 10, No. 1*, pp. 69-85. Retrieved from:
https://www.jstor.org/stable/pdf/41106045.pdf?ab_segments=0%252Ftbsub-1%252Frelevance_c onfig_with_defaults&refreqid=excelsior%3A5fc65aabf4c7fd5b6e118c3c8cf49a3c

[14] Ratliff, T. C. (2002). A Comparison of Dodgson's Method and the Borda Count. *Economic Theory, Vol. 20, pages 357-372.* Retrieved from:
https://www.jstor.org/stable/pdf/25055533.pdf?refreqid=excelsior%3Affe7a748d336db8258174e 3e559368f2

[15] Risse, M. (2005). Why the count de Borda cannot beat the Marquis de Condorcet. *Social Choice and Welfare, Vol. 25 (No 1), pages 95-113.* Retrieved from:
https://www.jstor.org/stable/pdf/41106687.pdf?refreqid=excelsior%3A5e5f0acce3b52593a4f692 01663a2521

[16] Stanford Encyclopedia of Philosophy. (2014, 10 13). *Arrow's Theorem.* Retrieved from:
https://plato.stanford.edu/entries/arrows-theorem/

[17] Stanford Encyclopedia of Philosophy. (2013, 12 18). *Social Choice Theory.* Retrieved from:
https://plato.stanford.edu/entries/social-choice/

[18] Tao, T. Arrow's Theorem. Los Angeles, CA: Department of Mathematics, UCLA. Retrieved from: https://www.math.ucla.edu/~tao/arrow.pdf

[19] Taylor, A. D. & Pacelli A. M.(2008). *Mathematics and Politics.*New York, NY: Springer Science+Business Media, LLC.

[20] Young, H. P. (1988). Condorcet's Theory of Young. *The American Political Science Review, Vol. 82. (No 4), pages 1231-1244.* Retrieved from:
https://www.jstor.org/stable/pdf/1961757.pdf?refreqid=excelsior%3Adc7027a4d0bfd7b203024554aadb3ade

[21] Young, H. P. (1995). Optimal Voting Rules. *The Journal of Economic Perspectives, Vol. 9 (No 1) pages 51-64.* Retrieved from:
https://www.jstor.org/stable/pdf/2138354.pdf?refreqid=excelsior%3A2898da2e44a22695189d7387f6edb0a4