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A Strength Test for the Borda Count

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Recommended Citation

Waring, Jade Monroe, "A Strength Test for the Borda Count" (2019). *Senior Projects Spring 2019*. 122.
https://digitalcommons.bard.edu/senproj_s2019/122

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Bard

A Strength Test for the Borda Count

A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

by
Jade Waring

Annandale-on-Hudson, New York
May, 2019

Abstract

When running an election with more than two candidates, there are many ways to choose the winner. A famous theorem of Arrow states that the only mathematically fair way to choose is to do so at random. Because this is not a desirable way to choose a winner of an election, many mathematicians have devised alternate ways of aggregating ballots. In my project I consider one of these ways – the Borda Count, considered to be one of the most desirable from both the point of view of mathematics and economics – and came up with a method to test the mathematical fairness of an arbitrary voting system against the known fairness of the Borda Count.

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Dedication

I dedicate this senior project to my beloved siblings; Chanty, Violet, and Tai.

Acknowledgments

I would like to acknowledge the help I received from my loved ones and the Bard Mathematics faculty. I want to sincerely thank my senior project advisors; Lauren Rose and John Cullinan. Lauren; thank you for constantly supporting me, believing in me, and helping me believe in myself and my potential. Your kind words and constant encouragement are sincerely appreciated. John; thank you for guiding me, supporting me, and for sharing endless knowledge with me. You have been so incredibly sweet and helpful this year, I could not have done it without you. To my family, friends, and especially parents; thank you so much for always being there for me, believing in me, and inspiring me. Lastly, I would like to sincerely thank Ethan Bloch; thank you so much for all of your coding help! You are so incredibly smart, and your LaTeX assistance has been very much so appreciated.

1

Introduction

Voting, from a mathematical perspective, is the process of ranking candidates in a way that attempts to describe the preference of the individual casting the vote. There are many different methods of voting that can produce different winners within the same election. These methods range in their fairness and difficulty as well. In this project we will be looking at ranked methods of voting. This means that voters will be asked to list the candidates in the election in order of their preference. This method of ranking candidates gives more information about the voter's entire preference, which is very important and should always be taken into consideration.

Definition 1.0.1. A **Social Choice Procedure** is a function for which a typical input is a sequence of lists, of some set A (the set of alternatives) and the corresponding output is either an element of A , a subset of A , or there is no winner. [1] p. 28. △

There are numerous Social Choice Procedures that exist today. Some decide the winner of an election based on eliminating candidates who fall unfavorable in their rankings. Others have a point system where each candidate is awarded a certain number of points depending on where they have been ranked. The others run on numerous other mathematical criteria that help determine the winner of the given election.

An interesting question is one that asks if there is a "best" Social Choice Procedure that declares the winner of an election. If there are only two candidates running in an election, there seems to be only one (other than a Dictatorship) way to decide the winner. If there are two candidates running in an election, candidate A and B , there are only two possible outcomes, A wins and B comes in second, or B wins and A comes in second. There exists a Social Choice Procedure where the candidate with the greatest number of 1st place votes wins. This Social Choice Procedure is called Majority Rule.

Definition 1.0.2. Majority Rule is the Social Choice Procedure for two alternatives in which an alternative is a winner if it appears at the top of at least half of the individual preference lists (equivalently, if at least half of the voters vote for that alternative).[1] p. 4. \triangle

Questions may arise regarding if there are alternative methods for choosing a winner for a 2-candidate election, and if there are, is there a method that satisfies the same properties as Majority Rule? There is a theorem by Kenneth May that looks into this question.

Theorem 1.0.3. *(May, 1952) If the number of people is odd and each election produces a unique winner, then Majority Rule is the only Social Choice Procedure for two alternatives that satisfies the following three conditions:*

1. *It treats all the voters the same: If any two voters exchange ballots, the outcome of the election is unaffected.*
2. *It treats both alternatives the same: If every voter reverses his or her vote (changing a vote for A to a vote for B and vice-versa), then the election outcome is reversed as well.*
3. *It is monotone: If some voter were to change his or her ballot from a vote for the loser to a vote for the winner, then the election outcome would be unchanged.*

This theorem tells us that for two alternatives, the search for the best Social Choice Procedure is quite easy. Majority Rule is clearly the only rational option. Things change drastically when

we move to the case of three or more alternatives. The search for the most efficient Social Choice Procedure becomes exponentially harder.

Some of the Social Choice Procedures we will be exploring in later chapters are Plurality, Anti-Plurality, Instant Runoff, Coombs, Borda Count, and a Dictatorship. There are numerous examples of elections that have been run with 3 or more candidates where different Social Choice Procedures will produce different winners. Lets take a look at an example with 5 candidates, (a, b, c, d, e) and 5 voters. [1] p. 36. All five voters $(p_1, p_2, p_3, p_4, p_5)$ have ranked the candidates in the order of their preference and the results are below.

p_1	p_2	p_3	p_4	p_5
a	b	c	d	e
b	c	b	c	d
e	a	e	a	c
d	d	d	e	a
c	e	a	b	b

The rules of each Social Choice Procedure will be explained in Chapter 2. For now we will just state which candidate each Social Choice Procedure would choose as the winner of this election. The Plurality Social Choice Procedure would declare a five way tie for this election, denoted as $a - b - c - d - e$. Meanwhile Anti-Plurality and Coombs would choose candidate d as the winner. Then the Borda Count would declare candidate c as the winner, while Instant Runoff would choose candidate b as the winner. Lastly, the Dictatorship could declare candidate a as the winner. Therefore, all of the Social Choice Procedures (except Coombs and Anti-Plurality) will produce different winners for this election. This results in some confusion as to which of the Social Choice Procedures produces the strongest and most mathematically fair winner?

This project will be looking into the previously mentioned Social Choice Procedures and others to determine the strength of one of the strongest Social Choice Procedures; The Borda Count. Is there a Social Choice Procedure that is even stronger? If so, what would it be? These are some of the questions that will be analyzed and explored in the upcoming chapters within this project.

2

Background

2.1 Preliminaries

In this section we will be defining some of the basic terminology for discussing Social Choice Procedures and their properties.

Definition 2.1.1. **Alternatives** are the candidates/choices in an election. They are a set, A , where $A = \{\text{candidate 1, candidate 2, candidate 3...}\}$. \triangle

Example 2.1.2. Suppose an election was held that asked voters which color they believe is the most appealing; red, orange, blue, or green. In this election, red, orange, blue, and green would be the alternatives. \diamond

Definition 2.1.3. A **Ballot**, B , is an ordering (linear or partial) of the set of alternatives. \triangle

Example 2.1.4. Here we will show an example of a linearly ordered ballot. The Bard Womens Lacrosse team voted on the ideal practice time for next season. The set of alternatives were $A = \{8:00\text{am, } 1:00\text{pm, } 5:00\text{pm, } 7:00\text{pm}\}$. The team members were asked to rank the practice times in order of most preferred to least. One person's ballot, p_1 , is below.

$$\begin{array}{c}
 p_1 \\
 5 : 00pm \\
 | \\
 1 : 00pm \\
 | \\
 7 : 00pm \\
 | \\
 8 : 00am
 \end{array}$$

◇

Example 2.1.5. This is an example of a partially ordered ballot. Lets take the same set A as example 2.1.4. Team members were again asked to rank the practice times in order of most preferred to least, but they may allow for ties. Another person's ballot, p_2 is below.

$$\begin{array}{c}
 p_2 \\
 1 : 00pm \quad 5 : 00pm \\
 \quad \searrow \quad \swarrow \\
 7 : 00pm \\
 | \\
 8 : 00am
 \end{array}$$

Note that 1:00pm and 5:00pm are tied for 1st place, while 7:00pm is in 2nd place and 8:00am is in 3rd place.

◇

Definition 2.1.6. A **Profile**, P , is a set of ballots. [3] p. 9.

△

Example 2.1.7. Suppose $A = \{a, b, c\}$ and there are four voters. This will result in four ballots. Then profile P is the set of all four ballots. P is below.

$$P = \begin{array}{cccc}
 a & a & b & c \\
 | & | & | & | \\
 b & c & a & a \\
 | & | & | & | \\
 c & b & c & b
 \end{array}$$

◇

Definition 2.1.8. A **Bucket Order**, also known as the Societal Preference Order, on a set of alternatives is a partial order in which the alternatives are partitioned into subsets A_1, A_2, \dots, A_n where every candidate in A_i is preferred to everyone in A_j when $i < j$. [4] p. 11. \triangle

Example 2.1.9. Suppose there is an election for a new governor, and $A = \{a, b, c\}$. After the votes are counted, it is determined that b is the winner of the election, c comes in second 2^{nd} place, and a comes in 3^{rd} place. Then the corresponding Bucket Order would be

$$\begin{array}{c} b \\ | \\ c \\ | \\ a \end{array}$$

Suppose the election was run again, and a and c tied for 2^{nd} , but b was still the winner. Then the corresponding Bucket order would be

$$\begin{array}{c} b \\ \wedge \\ c \ a \end{array}$$

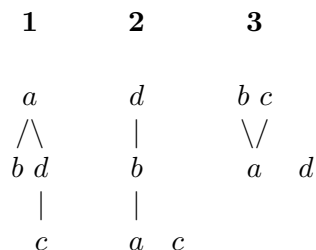
\diamond

Definition 2.1.10. Let A be a set and let R be a relation on A . Then the pair (A, R) is called a **Poset** if the following properties hold:

- 1) Reflexivity: aRa for all $a \in A$.
- 2) Antisymmetry: If aRb and bRa , then $a = b$ for all $a, b \in A$.
- 3) Transitivity: If aRb and bRc , then aRc for all $a, b, c \in A$. [4] p. 3. \triangle

In other words, a Poset is a partial order on A , where candidates can be ranked above one another, tied with one another, and in no relation to one another. Posets allow for ties and can represent when voters have no preference on a candidate.

Example 2.1.11. Here are three examples of potential posets that could be seen in this project where $A = \{a, b, c, d\}$.



Note that Posets are similar to Bucket Orders. The main difference is that Posets can depict when a voter has no preference on a candidate, like candidate c in the second poset or candidate d in the third poset. Posets also can depict cases like the first example, where c is positioned on the ballot. Candidate c is below d but not below b . ◇

2.2 Social Choice Procedures

This section will define and give examples of many Social Choice Procedures. These Social Choice Procedures differ in their approaches to selecting the winner of an election.

To reiterate, a Social Choice Procedure is a function that inputs a profile, and outputs either a single alternative, a set of alternatives if ties are allowed, or the Bucket Order. Social Choice Procedures are also known as Voting Systems. When discussing these Voting Systems we refer to the output as the winner.

Definition 2.2.1. Plurality is the Social Choice Procedure that declares the winner as the alternative with the greatest number of 1st place rankings in the individual ballots. [1] p. 6. △

Example 2.2.2. Suppose $A = \{a, b, c\}$. Voters must rank the candidates from their most favorable to their least favorable. There are six voters, and the profile, P , is below.

$$P = \begin{array}{cccccc} & a & \mathbf{b} & \mathbf{b} & c & \mathbf{b} & a \\ & | & | & | & | & | & | \\ & c & a & c & a & a & c \\ & | & | & | & | & | & | \\ & b & c & a & b & c & b \end{array}$$

When analyzing this profile using Plurality we only will be looking at the voter's first choices. We can see that candidate \mathbf{b} got three 1st place votes, candidate a got two 1st place votes, and candidate c got one 1st place vote. Therefore, candidate \mathbf{b} is the winner of this election. \diamond

Definition 2.2.3. **Anti-Plurality** is the Social Choice Procedure that declares the winner as the alternative with the fewest number of last place votes on the individual ballots. \triangle

Similarly to Plurality, Anti-Plurality is only concerned with one position on the ballot. In this case the position is the last place slot instead of the 1st place slot.

Example 2.2.4. Suppose there is an election being held to determine the new capital of New York. Let the set of alternatives be $A = \{NYC, Albany, Utica\}$, and suppose 100 people are voting in this election. The profile, P , is below.

$$P = \begin{array}{cccc} & \mathbf{20} & \mathbf{40} & \mathbf{10} & \mathbf{30} \\ & NYC & Albany & Utica & NYC \\ & | & | & | & | \\ Albany & | & NYC & Albany & Utica \\ & | & | & | & | \\ Utica & Utica & NYC & Albany & \end{array}$$

When using the Anti-Plurality Social Choice Procedure we will only be looking at the last place position on the ballots. We see that Utica received 60 last place votes, Albany received

30 last place votes, and NYC received only 10 last place votes. Therefore, NYC would be the winner in this election. \diamond

In the next part of this chapter we will be analyzing three Social Choice Procedures that utilize and consider every position on the ballots, (i.e 1^{st} , 2^{nd} , ... last place). The positioning of the candidates will result in either a certain amount of points, or potentially dropping out of the election. These Social Choice Procedures include, **Coombs**, **Instant Runoff**, and the **Borda Count**.

Definition 2.2.5. **Coombs** first looks to see if an alternative has the majority of 1^{st} place votes, if an alternative does then they are the winner. If no alternative has the majority of 1^{st} place votes, then the alternative with the greatest number of last place votes is eliminated from the ballots, and the process is repeated until an alternative has the majority of 1^{st} place votes. [3] p. 16. \triangle

Example 2.2.6. The Bard Women's Lacrosse Team voted on the ideal practice time. Let $A = \{8:00am, 1:00pm, 5:00pm, 7:00pm\}$. Seven team members voted, resulting in the following profile, P .

	Emily	Caleigh	Nicole	Kady	Melissa	Rachael Y.	Izabelle
$P =$	7 : 00pm	1 : 00pm	1 : 00pm	1 : 00pm	5 : 00pm	5 : 00pm	5 : 00pm
	8 : 00am	7 : 00pm	5 : 00pm	5 : 00pm	8 : 00am	7 : 00pm	7 : 00pm
	5 : 00pm	5 : 00pm	8 : 00am	8 : 00am	7 : 00pm	1 : 00pm	8 : 00am
	1 : 00pm	8 : 00am	7 : 00pm	7 : 00pm	1 : 00pm	8 : 00am	1 : 00pm

Since there is no majority winner, we will eliminate the candidate (practice time) with the greatest number of last place votes. 1:00pm received 3 last place votes, 8:00pm and 7:00pm

both received 2 last place votes, and 5:00pm received zero last place votes. Therefore, 1:00pm received the greatest number of last place votes. Now we eliminate 1:00pm and re-write the ballots, creating the new profile below, P' .

	Emily	Caleigh	Nicole	Kady	Melissa	Rachael Y.	Izabelle
$P' =$	7 : 00pm	7 : 00pm	5 : 00pm	5 : 00pm	5 : 00pm	5 : 00pm	5 : 00pm
	8 : 00am	5 : 00pm	8 : 00am	8 : 00am	8 : 00am	7 : 00pm	7 : 00pm
	5 : 00pm	8 : 00am	7 : 00pm	7 : 00pm	7 : 00pm	8 : 00am	8 : 00am

We now see that 5:00pm has received the majority of 1st place votes (5/7). This means that the process of eliminating the candidate with the most last place votes is stopped, and 5:00pm is declared to be the winner of this election. \diamond

Next we will look at a Social Choice Procedure that is very similar to Coombs. Instead of eliminating the candidate with the greatest number of last place votes, this Social Choice Procedure eliminates the candidate with the least number of 1st place votes.

Definition 2.2.7. Instant Runoff first looks to see if an alternative has the majority of 1st place votes, if an alternative does then they are the winner. If no alternative has the majority of 1st place votes, then the alternative with the least number of 1st place votes is eliminated from the ballots, and the ballots are rewritten until an alternative has the majority of 1st place votes. [3] p. 18. \triangle

Example 2.2.8. Lets look at the same profile from example 2.2.6, but lets use Instant Runoff this time. Because there is no majority winner with the first round of ballots, we will now eliminate the candidate with the least number of 1st place votes. Since 8:00am got zero 1st place votes we will eliminate 8:00am and rerun the process. The profile, P' , is below.

	Emily	Caleigh	Nicole	Kady	Melissa	Rachael Y.	Izabelle
$P' =$	$7:00pm$	$1:00pm$	$1:00pm$	$1:00pm$	$5:00pm$	$5:00pm$	$5:00pm$
	$5:00pm$	$7:00pm$	$5:00pm$	$5:00pm$	$7:00pm$	$7:00pm$	$7:00pm$
	$1:00pm$	$5:00pm$	$7:00pm$	$7:00pm$	$1:00pm$	$1:00pm$	$1:00pm$

Now we see that there is still no majority winner. So we do the process again, creating profile P'' . We now eliminate 7:00pm because it only has one 1st place vote, while both of the other candidates have three 1st place votes. The profile P'' is below.

	Emily	Caleigh	Nicole	Kady	Melissa	Rachael Y.	Izabelle
$P'' =$	$5:00pm$	$1:00pm$	$1:00pm$	$1:00pm$	$5:00pm$	$5:00pm$	$5:00pm$
	$1:00pm$	$5:00pm$	$5:00pm$	$5:00pm$	$1:00pm$	$1:00pm$	$1:00pm$

Now we have a winner! 5:00pm has received four 1st place votes while 1:00pm received only three 1st place votes, making 5:00pm our winner. ◇

Another method to select the winner of an election can be done by giving each candidate a certain number of points based on their position on the given ballot. We will be discussing this method next.

Definition 2.2.9. The **Borda Count** gives each candidate $2n - 2i$ points for each ballot that ranks the candidates in the i^{th} row, where $n =$ the number of alternatives. These points are totaled for each candidate, and the candidate with the most points wins. △

Note that the winner of a Borda Count election is called the Borda winner.

Example 2.2.10. Lets use the same original profile we used in example 2.2.6. Lets see if by using the Borda Count we still receive 5:00pm as our winner. Note that in this example $n = 4$ because there are four practice times to choose from. For the Borda Count each position in the ballot receives a certain number of points. The table below will determine how many points each candidate will receive using the Borda Count when $n = 4$.

Position on the Ballot	Borda Score ($2n - 2i$)	Number of Points
1 st	$2(4) - 2(1)$	6
2 nd	$2(4) - 2(2)$	4
3 rd	$2(4) - 2(3)$	2
4 th	$2(4) - 2(4)$	0

Therefore, 1st place is awarded 6 points, 2nd place is awarded 4 points, 3rd place is awarded 2 point, and 4th place is awarded 0 points. The finalized points for P are below.

$$8:00\text{am} = 6(0) + 4(1) + 2(3) + 0(2) = 10$$

$$1:00\text{pm} = 6(3) + 4(0) + 2(1) + 0(3) = 20$$

$$5:00\text{pm} = 6(3) + 4(2) + 2(2) + 0(0) = \mathbf{30}$$

$$7:00\text{pm} = 6(1) + 4(3) + 2(1) + 0(2) = 20$$

Therefore, 5:00pm would remain the winner because it received the most points from the Borda Count. ◇

Lastly we will look at a Social Choice Procedure that is not as commonly used, but is still a valid Social Choice Procedure. This Social Choice Procedure has been and is still currently utilized in some elections today.

Definition 2.2.11. A **Dictatorship** takes a profile P , and selects one voter to be the dictator, d . Then whoever is the winner on d 's ballot is the winner of the election. [1] p. 8. △

Example 2.2.12. Suppose there is an election being held and the alternatives are $A = \{a, b, c\}$ and there are three voters (p_1, p_2, p_3) . Let profile P be below.

$$P = \begin{array}{ccc} & p_1 & p_2 & p_3 \\ & a & b & b \\ & | & | & | \\ & c & a & c \\ & | & | & | \\ & b & c & a \end{array}$$

Suppose the dictator is p_2 . Then the winner of the election is candidate b , because b is on the top of p_2 's ballot. ◇

2.3 Fairness Criteria

In this section we will be looking at a group of mathematical criteria; the various Fairness Criteria. We will analyze the fairness of elections through different Social Choice Procedures by using the following criteria; Majority, Condorcet, Majority Loser, Condorcet Loser, Monotonicity, Independence of Irrelevant Alternatives, and Pareto.

Definition 2.3.1. A Social Choice Procedure satisfies the **Majority Criteria** if it chooses the candidate who receives greater than 50% of 1st place votes as the winner. We call this winner the Majority winner. [3] p. 23. △

An election *violates* the Majority Criterion if a candidate loses the election, despite having the majority of 1st place votes.

Example 2.3.2. In 2018, the United States Senate election in Maine used Plurality to elect its winner. Voters ranked the candidates in order of their preference. The results of the election are below.

General election for U.S Senate, Maine			
Candidate	Percent	Votes	r/d/i
Angus King	54.3 %	344,575	Independent
Eric Brakey	35.2 %	223,502	Republican
Zak Ringelstein	10.4 %	66,268	Democrat

Angus King was the winner of this election. He received 54.3% of 1st place votes, while Eric Brakey received 35.2%, and Zak Ringelstein received 10.4%. Therefore, this election satisfies the Majority Criteria because the candidate that was selected as the winner has the majority, more than 50%, of 1st place votes. This makes Angus King a Majority winner. \diamond

Plurality, Coombs, and Instant Runoff all satisfy the Majority Criteria. Meanwhile the Borda Count and Anti-Plurality violate the Majority Criteria. [3] p. 23.

Definition 2.3.3. Let $x \in X$. A Social Choice Procedure satisfies the **Condorcet Criteria** if $x \in f(P)$, such that $x > y$ for all $y \in A - x$ by more voters than $y > x$. We call x the Condorcet winner. [3] p. 24. \triangle

In other words, a Social Choice Procedure that satisfies the Condorcet Criteria would select the candidate who would win a two candidate election against every other candidate.

Example 2.3.4. Suppose there was an election held to determine what high school student's favorite subjects are. Suppose 100 students voted and $A = \{Science, Math, English\}$. The profile, P , is below.

	8	35	40	17
$P =$	<i>Math</i>	<i>English</i>	<i>Math</i>	<i>Science</i>
	<i>English</i>	<i>Science</i>	<i>English</i>	<i>English</i>
	<i>Science</i>	<i>Math</i>	<i>Science</i>	<i>Math</i>

Now we need to compare the candidates to each other in a head to head election. In the table below note that the number next to the subject represents the number of people who prefer that subject to the subject either above or below it. The results are below.

Comparing Subjects Head-to-Head		
{English, Math}	{English, Science}	{Science, Math}
52 English	83 English	52 Science
48 Math	17 Science	48 Math

From these results we can see that English is preferred by more voters over every other subject. In other words, English would win a 2-candidate election verse any other subject. Therefore, any Social Choice Procedure that does select English as the winner will satisfy the Condorcet Criteria. \diamond

None of the Social Choice Procedures we are looking at satisfy the Condorcet Criteria. [3] p. 25. But, there does exist a Loser Condorcet Criteria (as well as a Majority Loser Criteria) that is satisfied by some of the Social Choice Procedures we are looking at. Therefore, it is still important to understand the Condorcet Criteria even though it is not satisfied by any of the Social Choice Procedures we are analyzing.

Definition 2.3.5. A Social Choice Procedure satisfies the **Majority Loser Criteria** if it never selects an alternative who the majority prefers every other alternative over it as the winner. [3] p. 25. \triangle

Example 2.3.6. Lets use the profile from example 2.3.4, and use the Borda Count to determine the winner. The profile and Borda Count calculations are bellow.

8	35	40	17
<i>Math</i>	<i>English</i>	<i>Math</i>	<i>Science</i>
<i>English</i>	<i>Science</i>	<i>English</i>	<i>English</i>
<i>Science</i>	<i>Math</i>	<i>Science</i>	<i>Math</i>

$$\mathbf{English} = 4(35) + 2(65) + 0(0) = 270.$$

$$\mathbf{Math} = 4(48) + 2(0) + 0(52) = 192.$$

$$\mathbf{Science} = 4(17) + 2(35) + 0(48) = 138.$$

Therefore, by using the Borda Count, we see that English would be the winner. We can also note that Math is the Majority Loser. In this example the Borda Count satisfies the Majority Loser Criteria because it does not choose the Majority loser, Math, as the winner. \diamond

The Social Choice Procedures that generally satisfy the Majority Loser Criteria are Borda Count, Instant Runoff, Coombs, and Anti-Plurality. Meanwhile, Plurality does not always satisfy the Majority Loser Criteria. [3] p. 25.

Example 2.3.7. This is an example where Plurality does not satisfy the Majority Loser Criteria. Suppose 7 people (p1-p7) vote on which of these four countries they would like to go to the most and $A = \{Japan, Italy, Germany, France\}$. The profile is below.

p1	p2	p3	p4	p5	p6	p7
<i>France</i>	<i>France</i>	<i>France</i>	<i>Germany</i>	<i>Italy</i>	<i>Japan</i>	<i>Italy</i>
<i>Germany</i>	<i>Japan</i>	<i>Japan</i>	<i>Italy</i>	<i>Japan</i>	<i>Germany</i>	<i>Japan</i>
<i>Japan</i>	<i>Italy</i>	<i>Germany</i>	<i>Japan</i>	<i>Germany</i>	<i>Italy</i>	<i>Germany</i>
<i>Italy</i>	<i>Germany</i>	<i>Italy</i>	<i>France</i>	<i>France</i>	<i>France</i>	<i>France</i>

Using Plurality we see that France has three 1st place votes, Italy has two 1st place votes, and both Germany and Japan only have one 1st place vote. Therefore, France would be the winner of this election. Note that France is also the Majority Loser with four last place votes. Therefore, this is an example where Plurality does **not** satisfy the Majority Loser Criteria, it violates it. \diamond

Definition 2.3.8. Let $x \in X$. A Social Choice Procedure satisfies the **Condorcet Loser Criteria** if $x \notin f(P)$, such that $x > y$ for all $y \in X - x$ by more voters than $y > x$. We call x the Condorcet loser. [3] p. 26. \triangle

Example 2.3.9. We will use the data from example 2.3.4 to create a table showing the head to head winners of a 2-candidate election.

Comparing Subjects Head-to-Head		
{English, Math}	{English, Science}	{Science, Math}
48 Math	17 Science	48 Math
52 English	83 English	52 Science

From this table we can see that the least preferred candidate is Math. This results in Math being the Condorcet loser. Therefore, any Social Choice Procedure that chooses Math as the winner would be violating the Condorcet Loser Criteria. \diamond

Definition 2.3.10. A Social Choice Procedure satisfies the **Monotonicity Criteria** if the following holds for every alternative x : If x is the winner of the election and a voter changes his or her preference list by moving x up one spot (that is exchanging x 's position with that of the alternative immediately above x on his or her ballot), then x should still be the winner of the election. [1] p. 12. \triangle

In other words, ranking a winning candidate higher should never cause that candidate to lose nor should ranking a losing candidate lower cause that candidate to win.

Example 2.3.11. Suppose there is an election held with four voters and where $A = \{a, b, c\}$. The profile P is below.

$$P = \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ & \mathbf{a} & \mathbf{a} & b & c \\ & | & | & | & | \\ & b & c & c & \mathbf{a} \\ & | & | & | & | \\ & c & b & \mathbf{a} & b \end{array}$$

If we determine the winner of this election using the Borda Count, we would see that candidate a is the winner with 10 points, while candidate b has 6 points and candidate c has 8 points. Now suppose the original profile, P , is changed in favor of candidate a (p_3 puts candidate a over candidate c). The new profile, P' , is below.

$$P' = \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ & \mathbf{a} & \mathbf{a} & b & c \\ & | & | & | & | \\ & b & c & \mathbf{a} & \mathbf{a} \\ & | & | & | & | \\ & c & b & c & b \end{array}$$

Doing the Borda Count on P' would result in a having 12 points, b having 6 points, and c having 6 points. Therefore, candidate a would still be the winner. Therefore, in this example the Borda Count satisfies the Monotonicity Criteria. \diamond

The Social Choice Procedures which satisfy the Monotonicity Criteria are Plurality, Ant-Plurality and the Borda Count. Meanwhile, Coombs and Instant Runoff violate the Monotonicity Criteria. [3] p. 28.

Definition 2.3.12. A Social Choice Procedure satisfies the **Independence of Irrelevant Alternatives Criteria (IIA)** if the following holds for every pair of alternatives x and y : If the winner (or set of winners, if there is a tie) of the election includes x but not y , and one or more voter changes their preference, but no one changes his or her mind about whether x is preferred to y or y to x , then the winner (or set of winners) should not change so as to include y . [1] p. 12. \triangle

In other words, when a winner is decided in an election, this winner should remain the winner if one of more of the losing candidates drops out.

Example 2.3.13. The IIA Criteria is violated by the Borda Count, Instant Runoff, Coombs, Plurality, and Anti-Plurality. [3] p. 29. Here is an example where Plurality violates IIA. Suppose there are five voters and $A = \{a, b, c, d\}$. The Profile, P , is below.

$P =$	$p1$	$p2$	$p3$	$p4$	$p5$
	a	a	b	c	d
	b	c	c	b	b
	c	b	a	a	a
	d	d	d	d	c

Note that the winner in P is candidate a because it received the most 1st place votes. Suppose candidates c and d drop out of the election. The new Profile, P' , is below.

$P' =$	$p1$	$p2$	$p3$	$p4$	$p5$
	a	a	b	b	b
	b	b	a	a	a

We can now conclude that Plurality violates **IIA** because the original winner, candidate a , is no longer the winner with profile P' . Candidate b is the winner in P' with three 1st place votes. \diamond

Definition 2.3.14. The **Pareto Condition** states that if everyone prefers candidate a over candidate b , then candidate b should not be among the winners of the election. [1] p. 11. \triangle

Example 2.3.15. Suppose there is an election where there are three voters and $A = \{x, y, z\}$. The profile, P , is below.

$$P = \begin{array}{ccc} x & x & y \\ | & | & | \\ z & y & x \\ | & | & | \\ y & z & z \end{array}$$

In order for a Social Choice Procedure to satisfy Pareto with this profile, candidate z could never be the winner. This is because candidate x is preferred to candidate z by everyone. \diamond

Table 2.3.1 includes all of the Social Choice Procedures and Fairness Criteria discussed in this project so far. It shows if a given Social Choice Procedure satisfies the specific Fairness Criteria or not. The following two proofs are that Anti-Plurality and Coombs satisfy Pareto. For all other proofs on this table, see [1] or [3].

Proof. By definition of Pareto, if everyone prefers candidate x to candidate y , then y can never be the winner. Anti-Plurality chooses the winner of the election to be the candidate with the fewest number of last place votes. Therefore, the winner cannot be y because x will always have fewer last place votes than y . Therefore, Pareto is satisfied by Anti-Plurality. \square

	Majority	Condorcet	Majority Loser	Condorcet Loser	Monotonicity	IIA	Pareto
Plurality	Yes	No	No	No	Yes	No	Yes
Anti-Plurality	No	No	Yes	Yes	Yes	No	Yes
Coombs	Yes	No	Yes	No	No	No	Yes
Instant Runoff	Yes	No	Yes	Yes	No	No	Yes
Borda Count	No	No	Yes	Yes	Yes	No	Yes
Dictatorship	No	No	No	No	Yes	Yes	Yes

Table 2.3.1.

Proof. By definition of Pareto, if everyone prefers candidate x to candidate y , then y can never be the winner. Coombs removes the candidate with the greatest number of last place votes. Therefore if everyone prefers x to y , then y will be removed before x will be removed, resulting in x being the winner. Therefore, Pareto is satisfied by Coombs. \square

In the next chapter we will be examining a Social Choice Procedure that I believe is among the strongest of all Social Choice Procedures. We will be giving definitions and doing examples as well. This Social Choice Procedure will be used to help examine the strength of the Borda Count.

3

Partial Borda Count

3.1 Definition and Examples

The Partial Borda Count is similar to the Borda Count in the sense that the winner of the election is declared to be the candidate with the greatest number of points. The difference between the Borda Count and the Partial Borda Count is that the Partial Borda Count also accounts for ties and for when voters have no preference on a candidate. More specifically, it allows for any partial ranking of alternatives. This results in the Partial Borda Count being a very advanced and well represented Social Choice Procedure. Voters can express their opinions on the candidates extremely accurately because the Partial Borda Count allows for partial ranking.

Definition 3.1.1. The **Partial Borda Count** takes a ballot, B , which is a partial order on the set of alternatives, A , and gives each alternative the following score: 2 times the number of alternatives below it, plus the number of alternatives that are incomparable to it. \triangle

There are numerous ways that the Partial Borda Count allows voters to rank the alternatives in an election. Here are four examples of how voters can rank candidates when the Partial Borda Count is being used on three alternatives.

			a	
	b	b		
	∧		b	$a b c$
	$a c$	$c a$		
			c	
	1	2	3	4

Example 1 is illustrating the ballot where the voter prefers candidate b as their 1st choice, and then candidate a and candidate c tie for their 2nd choice.

Example 2 is illustrating the ballot where the voter prefers candidate b over candidate c but has no preference on candidate a .

Example 3 is illustrating the ballot where the voter prefers candidate a to candidate b and candidate b to candidate c . This is how we usually see ranked ballots.

Example 4 is illustrating the ballot where the voter likes each candidate equally, resulting in a three way tie.

Example 3.1.2. Suppose there is an election being held where $A = \{a, b, c, d\}$ and there are three voters. In this election all three voters are allowed to rank the candidates with ties if they wish. The profile is below.

p_1	p_2	p_3
a	d	$b c$
∧		∨
$b d$	b	a
	∧	
c	$c a$	d

We will evaluate, explain, and score p_1 's ballot below. Note that each candidate, x , receives 2 times the number of alternatives below x plus the number of alternatives that are incomparable to x .

p_1 's ballot evaluation:

Candidate a receives 2 times the number of alternatives below a (which are b, d , and c) + the number of alternatives that are incomparable to a (which is no one). Hence a receives $2(3)+0 = 6$ points. Therefore, a receives 6 points from p_1 .

Candidate b receives 2 times the number of alternatives below b (which is no one) + the number of alternatives that are incomparable to b (which is c). Hence b receives $2(0) + 1 = 1$ point. Therefore, b receives 1 point from p_1 .

Candidate c receives 2 times the number of alternatives below c (which is no one) + the number of alternatives that are incomparable to c (which is no one). Hence c receives $2(0)+0 = 0$ points. Therefore, c receives 0 points from p_1 .

Candidate d receives 2 times the number of alternatives below d (which is c) + the number of alternatives that are incomparable to d (which is no one). Hence d receives $2(1) + 0 = 2$ points. Therefore, d receives 2 points from p_1 .

Using the same process for scoring all three voters (p_1, p_2, p_3), the points are tallied up below for the results of the Partial Borda Count. The points are added in sections of parentheses, the first being the points from p_1 , the second from p_2 and the third from p_3 .

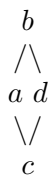
$$a = (2(3) + 0) + (2(0) + 0) + (2(1) + 0) = 8$$

$$b = (2(0) + 1) + (2(2) + 0) + (2(2) + 0) = \mathbf{9}$$

$$c = (2(0) + 0) + (2(0) + 0) + (2(2) + 0) = 4$$

$$d = (2(1) + 0) + (2(3) + 0) + (2(0) + 0) = 8$$

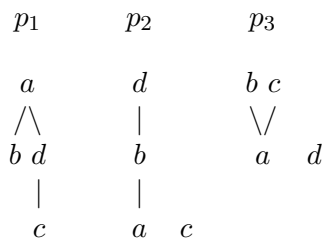
Now we can determine that the winner of this election is candidate b . Note that because candidate b received 9 points, candidate a and candidate d received 8 points, and candidate c received 4 points, the resulting Bucket Order would be



◇

Next we will be giving an example where voters utilize their voting right within the Partial Borda Count to rank candidates with ties and with no preference at all. The scoring will be analyzed and explained as well.

Example 3.1.3. In this example let there be three voters and $A = \{a, b, c, d\}$. The profile is below.



We will evaluate, explain, and score p_3 's ballot this time. Note that each candidate, x , receives 2 times the number of alternatives below x plus the number of alternatives that are incomparable to x . Also note that in p_3 's ballot they have no preference on candidate d and in p_2 's ballot they have no preference on candidate c .

p_3 's ballot evaluation:

Candidate a receives 2 times the number of alternatives below a (which is no one) + the number of alternatives that are incomparable to a (which is d). Hence a receives $2(0) + 1 = 1$ point. Therefore, a receives 1 point from p_3 .

Candidate b receives 2 times the number of alternatives below b (which is a) + the number of alternatives that are incomparable to b (which is d). Hence b receives $2(1) + 1 = 3$ points. Therefore, b receives 3 points from p_3 .

Candidate c receives 2 times the number of alternatives below c (which is a) + the number of alternatives that are incomparable to c (which is d). Hence c receives $2(1) + 1 = 3$ points. Therefore, c receives 3 points from p_3 .

Candidate d receives 2 times the number of alternatives below d (which is no one) + the number of alternatives that are incomparable to d (which is a, b , and c). Hence d receives $2(0) + 3 = 3$ points. Therefore, d receives 3 points from p_3 .

Similarly to the previous example, using the same process for scoring all three voters, the points are tallied up below for the results of the Partial Borda Count. The points are added in sections of parenthesis, the first being the points from p_1 , the second from p_2 and the third from p_3 .

$$a = (2(3) + 0) + (2(0) + 1) + (2(0) + 1) = 8$$

$$b = (2(0) + 1) + (2(1) + 1) + (2(1) + 1) = 7$$

$$c = (2(0) + 0) + (2(0) + 3) + (2(1) + 1) = 6$$

$$d = (2(1) + 0) + (2(2) + 1) + (2(0) + 3) = \mathbf{10}$$

Now we can determine that the winner of this election is candidate d , and the resulting Bucket Order would be

$$\begin{array}{c} d \\ | \\ a \\ | \\ b \\ | \\ c \end{array}$$

◇

The Partial Borda Count is a versatile Social Choice Procedure that allows the voters to fully express their opinions of candidates. In my opinion the Partial Borda Count is among the best

of the Social Choice Procedures analyzed in this paper. This is why I have chosen it to be a step in the Social Choice Procedure that I created for testing the strength of the Borda Count. This topic will be explained and analyzed in the following chapter.

4

The Borda Average

4.1 Strength of the Borda Count

After analyzing various Social Choice Procedures, one can conclude that the Borda Count and Partial Borda Count are among the strongest of the Social Choice Procedures. Now my question is, how strong is the Borda Count? If a Social Choice Procedure could be constructed to test the Borda Count's strength, would the Borda Count prove to be as strong as we think it is? This chapter is dedicated to answering this question and analyzing the results.

Lets say we were to run an election using any Social Choice Procedure and declare a winner. Then we run the same election using the Borda Count and declare a winner. Then after running these two Social Choice Procedures, perform the Partial Borda Count on their Bucket Orders. It would be interesting to see if the original Borda winner is the same as this new winner. What would a Social Choice Procedure look like in order to do this? First lets start by defining some important terminology.

Definition 4.1.1. We say a Social Choice Procedure F **Breaks the Borda** on a profile P if, when we perform the Partial Borda Count on the output of F on P and the Borda Count on P , the winner of the Partial Borda Count differs from the winner of the Borda Count. \triangle

Lets now recall definition 2.1.8. that defines a **Bucket Order** on a set of alternatives as a partial order in which the alternatives are partitioned into subsets A_1, A_2, \dots, A_n where every candidate in A_i is preferred to everyone in A_j when $i < j$.

Also note that the output of a Social Choice Procedure on a profile P is a Bucket Order. Below we will provide another example of a Bucket Order.

Example 4.1.2. Let $A = \{a, b, c\}$, and let there be four voters. Let the profile be below, as well as the tallied points for each alternative using the Borda Count.

a	b	b	c
b	a	c	a
c	c	a	b

$$a = 4(1) + 2(2) + 0(1) = 8$$

$$b = 4(2) + 2(1) + 0(1) = 10$$

$$c = 4(1) + 2(1) + 0(2) = 6$$

When looking at the results we see that candidate b is the winner, candidate a is in 2^{nd} place, and candidate c came in 3^{rd} place. Therefore, the Bucket Order would look like this,

$$\begin{array}{c}
 b \\
 | \\
 a \\
 | \\
 c
 \end{array}$$

◇

Next we will be defining the Social Choice Procedure that I created which tests the strength of the Borda Count.

Definition 4.1.3. The **Borda Average** is a Social Choice Procedure that takes the Bucket Order of two Social Choice Procedures, B_1 and B_2 , where B_1 must always be the Borda Count, and B_2 can be any Social Choice Procedure. It takes the two Bucket Orders of B_1 and B_2 and does the Partial Borda Count on them. The Partial Borda Count winner is therefore the winner of the election. \triangle

Example 4.1.4. Suppose the students of Bard College were to vote on which building on their campus is their favorite to do homework in. There are 4 buildings to choose from, $A = \{Library, Preston, Kline, Olin\}$ and 100 voters. The profile, P , is below.

		16	27	24	33
		<i>Preston</i>	<i>Library</i>	<i>Library</i>	<i>Kline</i>
P	=	<i>Kline</i>	<i>Kline</i>	<i>Preston</i>	<i>Preston</i>
		<i>Olin</i>	<i>Olin</i>	<i>Kline</i>	<i>Olin</i>
		<i>Library</i>	<i>Preston</i>	<i>Olin</i>	<i>Library</i>

Now we must find the two Bucket Orders, B_1 and B_2 , for this profile. Lets start with B_1 , because we know we must use the Borda Count. The tallied points and Bucket Order, B_1 is below.

$$\text{Kline} = 6(33) + 4(43) + 2(24) + 0(0) = 418$$

$$\text{Preston} = 6(16) + 4(57) + 2(0) + 0(27) = 324$$

$$\text{Library} = 6(51) + 4(0) + 2(0) + 0(49) = 306$$

$$\text{Olin} = 6(0) + 4(0) + 2(76) + 0(24) = 152$$

$$B_1 = \begin{array}{c} \textit{Kline} \\ | \\ \textit{Preston} \\ | \\ \textit{Library} \\ | \\ \textit{Olin} \end{array}$$

Now for our B_2 we can choose any of the Social Choice Procedures that have been analyzed in this paper (Plurality, Anti-Plurality, Instant Runoff, Coombs, Dictatorship, Borda Count, Partial Borda Count). It would not really make sense to choose the Borda Count or the Partial Borda Count, but one could theoretically choose either of those if they wished, but the results would not be as interesting. For this example lets choose B_2 to be the Bucket Order of Instant Runoff on P .

First Olin will be removed from the ballots because it has the least number of 1st place votes, creating P' , which is below.

$$P' = \begin{array}{cccc} 16 & 27 & 24 & 33 \\ \textit{Preston} & \textit{Library} & \textit{Library} & \textit{Kline} \\ | & | & | & | \\ \textit{Kline} & \textit{Kline} & \textit{Preston} & \textit{Preston} \\ | & | & | & | \\ \textit{Library} & \textit{Preston} & \textit{Kline} & \textit{Library} \end{array}$$

Now we can conclude that Preston has the least number of 1st place votes, which when removed from P' will create P'' , which is below.

$$P'' = \begin{array}{cccc} 16 & 27 & 24 & 33 \\ \textit{Kline} & \textit{Library} & \textit{Library} & \textit{Kline} \\ | & | & | & | \\ \textit{Library} & \textit{Kline} & \textit{Kline} & \textit{Library} \end{array}$$

Lastly, we can remove Kline because it has fewer 1st place votes than the Library, resulting in the Library being our winner. Now to create the Bucket Order for P , we will create B_2 by listing the candidates in the order they were removed, from the bottom to the top, where the top candidate is the winner. B_2 is below.

$$B_2 = \begin{array}{c} \textit{Library} \\ | \\ \textit{Kline} \\ | \\ \textit{Preston} \\ | \\ \textit{Olin} \end{array}$$

Now we will take B_1 and B_2 and do the Partial Borda Count on them. The results for the tallying is below.

$$\textit{Olin} = 0 + 0 = 0$$

$$\textit{Library} = 6 + 2 = 8$$

$$\textit{Preston} = 2 + 4 = 6$$

$$\textit{Kline} = 4 + 6 = \mathbf{10}$$

Therefore, Kline is the winner of this election because it received the most points. \diamond

The Borda Average tests the strength of the Borda Count by averaging it with the other Bucket Order and seeing if the Borda Average winner remains the same as the original Borda winner. If it is, like in our last example, then the Borda winner is therefore a strong winner. But, it is not always the case where the Borda winner is the same as the Borda Average winner. Below we will show an example of this.

Example 4.1.5. Let $A = \{a, b, c\}$ and there are five voters. Profile P , is below.

$$P = \begin{array}{ccc} 2 & 2 & 1 \\ a & b & c \\ | & | & | \\ c & c & b \\ | & | & | \\ b & a & a \end{array}$$

Let B_1 be the Borda Count and B_2 be Instant Runoff. The tallied points and Bucket Order for B_1 is below.

$$a = 4(2) + 2(0) + 0(3) = 8$$

$$b = 4(2) + 2(1) + 0(2) = 10$$

$$c = 4(1) + 2(4) + 0(0) = 12$$

$$B_1 = \begin{array}{c} c \\ | \\ b \\ | \\ a \end{array}$$

For B_2 we need to remove the candidate with the fewest number of 1st place votes. In P that would be candidate c . If candidate c is removed from P , then candidate b is declared the winner with the majority (3/5) of 1st place votes, resulting in the following Bucket Order.

$$B_2 = \begin{array}{c} b \\ | \\ a \\ | \\ c \end{array}$$

The next step is to do the Partial Borda Count on B_1 and B_2 to determine the winner of the election. The results are below.

$$a = 2 + 0 = 2$$

$$b = 4 + 2 = \mathbf{6}$$

$$c = 0 + 4 = 4$$

Therefore, the winner of this election would be candidate b ! The original Borda winner was candidate c , therefore showing that the original Borda winner is not always the same as the Borda Average winner, enforcing that the Borda Count is not necessarily as strong as we thought. \diamond

Now we have shown that the Borda Average winner is not always the same as the original Borda winner. We can conclude from this that the strength of the Borda Count is not necessarily as strong as we thought it was. Now a question that came up was, is there a single profile P , where it will always produce a different Borda Average winner than the Borda winner for any Social Choice Procedure B_2 , that satisfies certain Fairness Criteria? The answer to this question is yes.

We will look for the smallest profile to break the Borda. Here is our profile, P .

$$P = \begin{array}{ccc} c & c & a \\ | & | & | \\ a & a & b \\ | & | & | \\ b & b & c \end{array}$$

There are three voters and three candidates. Using this profile, and satisfying certain Fairness Criteria that will be stated later, you can always break the Borda using any Social Choice Procedure for B_2 by using the Borda Average.

Lets take a look at who the Borda winner is for P .

$$a = 4(1) + 2(2) + 0(0) = 8$$

$$b = 4(0) + 2(1) + 0(2) = 2$$

$$c = 4(2) + 2(0) + 0(1) = 8$$

The resulting Bucket Order is below.

$$\begin{array}{c} a \ c \\ \setminus / \\ b \end{array}$$

Other Ballot (B_2)	PBC Score	Result
a b c	$a = 7$ $b = 2$ $c = 3$	a c b
a c b	$a = 7$ $b = 0$ $c = 5$	a c b
b a c	$a = 5$ $b = 4$ $c = 3$	a b c
b c a	$a = 3$ $b = 4$ $c = 5$	c b a
c a b	$a = 5$ $b = 0$ $c = 7$	c a b
c b a	$a = 3$ $b = 2$ $c = 7$	c a b

Table 4.1.1.

Now because we claim that B_2 can be any Social Choice Procedure, we must look at all of the possible Bucket Orders that can result from P , a chart showing these different Bucket Orders is in Table 4.1.1 and Table 4.1.2.

The only instances where we could not break the Borda would be if B_2 outputted $a - b - c$ or



Other Ballot (B_2)	PBC Score	Result
$a \ b$ $\backslash /$ c	$a = 6$ $b = 3$ $c = 3$	a $/ \backslash$ $b \ c$
$b \ c$ $\backslash /$ a	$a = 3$ $b = 3$ $c = 6$	c $/ \backslash$ $a \ b$
a $/ \backslash$ $b \ c$	$a = 6$ $b = 1$ $c = 4$	a $ $ c $ $ b
b $/ \backslash$ $a \ c$	$a = 4$ $b = 3$ $c = 4$	$a \ c$ $\backslash /$ b
c $/ \backslash$ $a \ b$	$a = 4$ $b = 1$ $c = 6$	c $ $ a $ $ b
$a - b - c$	$a = 5$ $b = 2$ $c = 5$	$a \ c$ $\backslash /$ b

Table 4.1.2.

because in Table 4.1.1 and Table 4.1.2 these possible Bucket Orders for B_2 result in the Borda Average winner being the same as the Borda winner, therefore **not** breaking the Borda.

Theorem 4.1.6. *Let F be a Social Choice Procedure on three alternatives. If F satisfies any of the following Fairness Criteria (Pareto, Majority, Condorcet, Majority Loser, or Condorcet Loser), then F will not output $(a-b-c)$ or*

$$\begin{array}{c} b \\ \wedge \\ a \quad c \end{array}$$

from the profile P .

Proof. Case 1. Suppose F satisfies the Pareto Condition. By definition if everyone prefers x to y then y cannot be among the winners. But in profile P , everyone prefers a to b . So b can never be among the winners of the election.

Case 2. Suppose F satisfies the Majority Criteria. By definition if x receives more than 50% of the 1st place votes, then x is the winner. In profile P , c receives about 66% of 1st place votes (2/3), so c will be the winner.

Case 3. Suppose F satisfies the Condorcet Criteria. By definition if x would win a two candidate election against every other candidate, then x is the winner. In profile P , candidate c would win a two candidate election against every other candidate, resulting in c being the winner.

Case 4. Suppose F satisfies the Majority Loser Criteria. By definition if x receives the majority of last place votes, then x cannot be the winner. In profile P , candidate b receives greater than 50% of last place votes, resulting in b never being able to be the winner.

Case 5. Suppose F satisfies the Condorcet Loser Criteria. By definition if x would lose a two candidate election against every other candidate, then x could never win the election. In profile P , candidate b would lose a two candidate election against every other candidate, resulting in b never being able to be the winner of the election. \square

Note that Theorem 4.1.6 does **not** include IIA in its Fairness Criteria. It is **not** true that if a Social Choice Procedure satisfies IIA, then it cannot output

$$\begin{array}{c} b \\ \wedge \\ a \ c \end{array}$$

or $(a - b - c)$. In other words, there exists examples of Social Choice Procedures that satisfy IIA that can output b as the winner. For example, suppose the winner of the election is declared to be the oldest candidate, and candidate b is the oldest, resulting in b being the winner. Note that this Social Choice Procedure would violate many of the Fairness Criteria, is unfair, and does not consider the preferences of the voters.

Corollary: Let F be the Borda Average. Then F will break the Borda Count when using any Social Choice Procedure satisfying (Pareto, Majority, Condorcet, Majority Loser, or Condorcet Loser).

Proof. The results of Table 4.1.1 and Table 4.1.2 list every possible outcome for Bucket Orders from profile P . There are only two instances where we will get the same Bucket Order as the Borda Count. These two instances are when there is a three way ties between all of the candidates, or when

$$\begin{array}{c} b \\ \wedge \\ a \ c \end{array}$$

□

Therefore, the only instance when the Borda Count could not be broken would be when b is the winner, or when there is a three way tie. What kind of Social Choice Procedure would produce one of these results?

Example 4.1.7. Let F be the Social Choice Procedure that chooses the tallest candidate to be the winner of the election and b is the tallest. Therefore the Bucket Order would be

$$\begin{array}{c} b \\ \wedge \\ a \quad c \end{array}$$

◇

Example 4.1.8. Let F be a Dictatorship. Once the dictator is chosen, suppose b was on the top of their ballot. Then b would be the winner of the election and the Bucket Order would be

$$\begin{array}{c} b \\ \wedge \\ a \quad c \end{array}$$

◇

Example 4.1.9. Let F be a Social Choice Procedure that chooses the winner of the election by whoever went to a college that starts with the letter B. Suppose a went to Bard, b went to Brandeis, and c went to Barnard. Then there would be a three way tie, $a - b - c$. ◇

Note that these examples are unrealistic and unfair. The kinds of examples of Social Choice Procedures that would output b as the winner or a three way tie are strange and skewed procedures. In my opinion, this results in them being so called "weak" Social Choice Procedures. Therefore, in order to not break the Borda, you would have to use an unrealistic, rare, or absurd Social Choice Procedure that would most likely violate many of the Fairness Criteria.

We can now conclude that we have completed a strength test for the Borda Count. The Borda Count is a strong Social Choice Procedure, but when you average it with another Social Choice Procedure's Bucket Order (use the Borda Average), the Borda winner is not always the same as the Borda Average winner. Breaking the Borda can easily be done when satisfying the above Fairness Criteria. Although there do exist Social Choice Procedures that would not break the Borda (i.e Dictatorship, tallest candidate wins, etc.), these Social Choice Procedures would never be used in rational election.

5

Future Work

There are two things that I would analyze and like to complete if I had more time with this project. The first thing would be to figure out if I could include Monotonicity in Theorem 4.1.6. I have already spent an ample amount of time thinking about this question, and going back and forth between trying to prove if Monotonicity could be included in the Fairness Criteria in Theorem 4.1.6. or not. To reiterate, the Theorem says

Let F be a Social Choice Procedure on three alternatives. If F satisfies any of the following criteria (Pareto, Majority, Condorcet, Majority Loser, or Condorcet Loser), then from profile P , F will not output $(a - b - c)$ or

$$\begin{array}{c} b \\ \wedge \\ a \quad c \end{array}$$

It is hard to determine if F satisfies Monotonicity. Would it be able to output a three way tie or b as the winner? Monotonicity says if candidate x wins the election, and one of the ballots were changed to raise x 's position on this ballot, then x should remain the winner. Moving a candidate higher in a ballot should not change the outcome of the election. It is hard to determine if we could prove that Monotonicity should be included in Theorem 4.1.6 or not because we would

not know the exact Social Choice Procedure that would be determining the winner. In other words, we would not know how the winner is decided, resulting in an uncertainty of who the winner should be. I believe that Monotonicity could not be added to Theorem 4.1.6, but I am still unsure on how exactly to prove it. Therefore, this would be something that I would certainly try to prove if I had more time with this project.

If I was able to prove that Monotonicity could or could not be added to Theorem 4.1.6, and I still had more time then I would try to do the following. I would analyze various other Social Choice Procedures and determine if I could add other Fairness Criteria in Theorem 4.1.6. Some examples of additional Social Choice Procedures would be the Condorcet Method and Sequence Pairwise Voting with a Fixed Agenda. Additionally, some of the other Fairness Criteria that I could add would be; Always-a-Winner, Faithfulness, Neutrality, Consistency, and the Cancellation Property.

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