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Hyperplanes Equipartition with Cascading Makeev

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Abstract

Given finite number of masses in the Euclidean space, one could ask is if possible to equipartition these masses into equal parts. Fixing the collection of masses, and the amount of hyperplanes, the equipartition-ability depends on the dimension, and there exists a dimension of such equipartition is possible. In this paper, topology and combinatorics method are used for estimating the lower bound and upper bound of the dimension. In particular, we are looking equipartition problem together with Cascading Makeev Constrain: Given two vector in $\mathbb{Z}^k$, $\vec{m} = (m_1, \ldots, m_k)$ and $\vec{\ell} = (\ell_1, \ldots, \ell_k)$ such that $1 \leq \ell_i \leq k + 1 - i$, so that for any ℓ_i of $\{H_i, \ldots, H_k\}$ hyperplanes equipartition each of the m_i measures.

Contents

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Introduction

In this chapter, we are going build up the tools we are using to solve the Hyperplane equipartition problem. First, there are many different way to represent collection of hyperplanes in the Euclidean space. Section 1.1 is about the some background of the Equipartition problem. We are going introduces some the simple case of the hyperplane parturition problem, i.e. what is call “ham sandwich problem”. Also, we are going to mention the hyperplane equipartition In the section 1.2, we are going to topologize the space of hyperplane. Such topology over the hyperplanes will introduces senses of symmetry, i.e. we are going to discuses group actions on the set of hyperplane. In section 1.3, we are going to discuss the general frame work for Fourier analysis for finite Abelian group. We are going to study about function from finite Abelian group G to $\mathbb{C}$. Then, introduce the characters, which offer an orthonormal basis for the function space. Section 1.4 is some result and tools follow from the generalized theory of Fourier analysis for finite Abelian group. In particular, in this section, we are going to study the finite Abelian group $\mathbb{Z}_2^k$, for $k \in \mathbb{N}$. In section 1.5 we shall introduce measures, i.e. collection of masses. Then, we shall offer conditions when the given collection of hyperplanes equipartition the collection of masses. In section 1.6, we are going to introduces some machinery form equivariant topology. This allowed us the solve some of the mass equipartition problems.

1.1 Background

Hyperplane equipartition question in topology and measure theory. It is has many applications to combinatorial geometry and discrete geometry. It is a generalization of "Ham Sandwich Problem" [10], which as follows:

Question 1: *Let $\mu_1, \ldots, \mu_n$ be measures in $\mathbb{R}^n$, it is possible to divide all of μ_i in half with a single $(n-1)-$dimensional hyperplane?*

In question 1, when we choose $n = 3$, we are considering three masses μ_1, μ_2, μ_3 in $\mathbb{R}^3$, together with plane. *(The three measures could be viewed as two chunks of bread with a chunk of ham — a sandwich, with a single cut that bisect the three piece simultaneously.)* The case for $n = 3$, is proposed by *Hugo Steinhaus* and proved by *Stefan Banach*. The proof relies on the Borsuk-Ulam Theorem [6].

Theorem 1.1.1 (Borsuk-Ulam Theorem)**.** *Let $f : S^n \to \mathbb{R}^n$ be a continuous function. Then, there exists an $x \in S^n$ such that $f(-x) = f(x)$.*

The proof of **Question 1,** uses the fact that $(n-1)-$dimensional hyperplane could be represented by points on the sphere S^{n-1}. This allowed us to define map from S^{n-1} to $\mathbb{R}^{n-1}$, and apply Borsuk-Ulam Theorem. *(We are going to proof* ***Question 1*** *in section 1.2.)* The Ham Sandwich Problem, could be generalized into the following problem, which is Hyperplane Equipartition Problem [1]:

Question 2. (Grünbaum-Ramos.) *Let $\mathcal{M} = \{\mu_1, \ldots, \mu_m\}$ be collection of measures on $\mathbb{R}^n$. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a collection of hyperplane in $\mathbb{R}^n$. What is the minimal dimension n such that $H_i \in \mathcal{H}$ divide $\mu_j \in \mathcal{M}$ simultaneously, for all $1 \leq i \leq k$ and $1 \leq j \leq m$.*

Observe that Question 1 is a special case for Question 2, where when the collection of hyperplane $|\mathcal{H}| = 1$, i.e. we are only considering one hyperplane in $\mathbb{R}^n$. The solution for Question 2 could be deduced by using Question 1. The main different between Question 1 and Question 2 is that the number of regions are different. In general, if the dimension is large enough, given the collection $\mathcal{H} = \{H_1, \dots, H_k\}$, there will be 2^k distinct regions. However, similar method could be applied.

1.2 Configuration Space: the space of Hyperplane

In this section, we are going to study configuration space [1, Section 3.1]. We shall fix $n \in \mathbb{N}$, the dimension of the Euclidean space. We are going to consider hyperplanes in $\mathbb{R}^n$. Let H be hyperplane in $\mathbb{R}^n$. Notice that H is a $n-1$ dimensional linear subspace of $\mathbb{R}^n$. Then, we have $H^\perp$ is one dimensional, which span by some unit vector $\vec{N}$. The unit vector $\vec{N}$ satisfies the properties that $\langle \vec{v}, \vec{N} \rangle = 0$, for all $\vec{v} \in H$. Notice that H is not necessary a linear space, in particular H might not go through $\vec{0}$. Then, the distance of H from the origin is $|\mathrm{Proj}_{\vec{N}}(\vec{p})|$, for some point $p \in H$. In particular, we could write $H = \vec{N}^\perp + c\vec{N}$, where $c = |\mathrm{Proj}_{\vec{N}}(\vec{p})|$.

Now, we could partition the points of $\mathbb{R}^n$ into three category, based on the value of $\langle \vec{v}, \vec{N} \rangle$, for the given $\vec{N}$. *(Observe that $\langle \vec{v}, \vec{N} \rangle$ could less than 0, greater than 0, or equal to 0.)* Let H be a hyperplane in $\mathbb{R}^d$. Let $\vec{N}$ be the unit normal vector of H. We denote

$$H^0 = \{\vec{u} \mid \langle \vec{N}, \vec{u} \rangle \geq c\}, \qquad \text{and} \qquad H^1 = \{\vec{u} \mid \langle \vec{N}, \vec{u} \rangle \leq c\}.$$

We call $\{H^0, H^1\}$ the half spaces indexed by $\mathbb{Z}_2$. *(defined in* [1, Section 3]*)* Notice that there are two distinct unit normal vector for given hyperplane H, call them $\vec{N}, \vec{N}'$. We know that $\vec{N} = -\vec{N}'$. We know that $\langle \vec{v}, \vec{N}' \rangle = \langle \vec{v}, -\vec{N} \rangle = -\langle \vec{v}, \vec{N} \rangle$. This means, the two half spaces $\{H^0, H^1\}$ of H are dependent on the choice of the normal unit vector of H. *(In general, when we consider $\mathcal{H} = \{H_1, \dots, H_k\}$ a collection of hyperplanes, there is also a permutation action on the set of hyperplanes. Together with negation action, the full action denoted as $\mathfrak{S}_k^\pm$ in* [1]*.)*

Proposition 1.2.1. *Let H be a hyperplane in $\mathbb{R}^d$. Let $\{H^0, H^1\}$ bet the half spaces of H. Then $\mathbb{Z}_2$ acts on $\{H^0, H^1\}$ by negation.*

Proof. Let $\vec{N}, \vec{N}'$ be the two unit normal vector of H, such that $\vec{N} = -\vec{N}'$. Then, there exists $c \in \mathbb{R}$ such that $H = \vec{N}^\perp + c\vec{N}$ or $H = \vec{N}'^\perp - c\vec{N}'$. Let $\{H^0, H^1\}$ and $\{H'^0, H'^1\}$ be the half spaces of represented by $\vec{N}$ and $\vec{N}'$, respectively. Without loss of generality, let $\vec{u} \in H^0$. Then, we have

$\langle \vec{N}, \vec{u} \rangle \geq c$. Then, when we negate, we have $\langle \vec{N}', \vec{u} \rangle = -\langle \vec{N}, \vec{u} \rangle \leq -c$. Hence, we have $\vec{u} \in H'^1$. Similarly, we have $H^1 = H'^0$. Thus, we have negation $\mathbb{Z}_2$ acts on $\{H^0, H^1\}$ by negation. $\square$

Since we know that $H = \vec{N}^{\perp} + c\vec{N}$, we have $\vec{N} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, and $c \in \mathbb{R}^n$. We shall consider $\vec{h} = (a_1, a_2, \ldots, a_n, c) \in \mathbb{R}^{n+1}$. Then, we have H could be represented by vector in $\vec{h}/\|\vec{h}\| \in S^n$. By proposition 1.2.1, such vector representation is not unique, if $\vec{h}$ is a representation of H, then, we have $-\vec{h}$ is also representation of H. Thus, pairs of antipodal points on S^n in $\mathbb{R}^{n+1}$, where we could view the space of the hyperplane in $\mathbb{R}^n$ as the $S^n/\mathbb{Z}^2$. Furthermore, we could view the two half space could be viewed as a parametrization by the space S^n. This is also introduced in [1], for $\vec{h} = (\vec{N}, c) \in S^n$.

$$H^0(\vec{h}) = \{\vec{u} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{N} \rangle \geq c\}; \qquad H^1(\vec{h}) = \{\vec{u} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{N} \rangle \geq c\}.$$

In some sense, we are abusing the fact that space of hyperplane $H \subseteq \mathbb{R}^n$ is in fact represented by $\vec{N} \times \mathbb{R} \in S^{n-1} \times \mathbb{R}$, up to negation actions on $S^{n-1} \times \mathbb{R}$. Observe that $S^{n-1} \times \mathbb{R}$ infinite cylinder, which is not isomorphic to S^n. *(We have to add to points to the infinite cylinder, the two points at infinity.)* Thus, when we look at the south pole and the north pole of S^n, i.e. the points $(\vec{0}, 1)$ and $(\vec{0}, -1)$. Notice that

$$H^0(\vec{0}, 1) := \{\vec{u} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{0} \rangle \geq 1\} = \emptyset; \quad H^1(\vec{0}, 1) := \{\vec{u} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{0} \rangle \leq 1\} = \mathbb{R}^n.$$

This is could be view as the "hyperplane" at infinity, *(which is not quite a hyperplane.)* Similarly, we have $(\vec{0}, -1)$ is also represents the hyperplane at infinity, with a different choices the normal vector.

In general, we are going to consider finite collect hyperplanes. We shall use $\mathcal{H} = \{H_1, \ldots, H_k\}$ to denote the collection of hyperplanes. Since H_i could be view as point in S^n, we could view $\mathcal{H}$ as points in $(S^n)^k$, i.e. k-fold product of n-sphere. The hyperplane H_i gives us half spaces $\{H_i^0, H_i^1\}$ indexed by $\mathbb{Z}_2$, for $i \in \{1, \ldots, k\}$. We have $H_1, H_2, \ldots, H_k$ partition the $\mathbb{R}^d$ into 2^k disjoint regions. Each regions could be indexed by $\epsilon \in \bigoplus_{i=1}^k \mathbb{Z}_2$, where for $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$, for some $\epsilon_i \in \mathbb{Z}_2$. Let $\{\mathcal{R}_g\}_{g \in \mathbb{Z}_2^k}$ be disjoint regions determined by $\mathcal{H}$. Since $\{\mathcal{R}_g\}_{g \in \mathbb{Z}_2^k}$ determined by $\mathcal{H}$, hence $\{\mathcal{R}_g\}_{g\in\mathbb{Z}_2^k}$ is parametrized by space $(S^n)^k$, where for $\vec{h} = (\vec{h_1}, \ldots, \vec{h_k}) \in (S^n)^k$.

$$\mathcal{R}_g(\vec{h}) := \bigcap_{i=1}^{k} H^{g_i}(\vec{h_i}). \tag{1.2.1}$$

Similarly, we could talk about symmetry of the space $(S^n)^k$. By Proposition 1.2.1, we have the group $\bigoplus_{i=1}^k \mathbb{Z}_2$ acts on $(S^n)^k$. Geometrically, the action of $\bigoplus_{i=1}^k \mathbb{Z}_2$ interchanges the i^{th} half spaces *(reflection along the hyperplane i^{th})* when $\epsilon_i = 1$, and fix the i^{th} half spaces when $\epsilon_i = 0$. Thus we have $\bigoplus_{i=1}^k \mathbb{Z}_2$ acts on the space of $(S^n)^k$, where

$$(\vec{h_1}, \vec{h_2}, \ldots, \vec{h_k})^\epsilon = \big((-1)^{\epsilon_1}\overrightarrow{h_1}, (-1)^{\epsilon_2}\overrightarrow{h_2}, \ldots, (-1)^{\epsilon_k}\overrightarrow{h_k}\big), \tag{1.2.2}$$

for $\epsilon \in \oplus_{i=1}^k \mathbb{Z}_2$ and $(\vec{h_1}, \vec{h_2}, \ldots, \vec{h_k}) \in (S^n)^k$. *(Again, since we are not considering permutation action over the collection of hyperplane, this is a simplification of the action $\mathfrak{S}_k^{\pm}$ over $\{\mathbb{R}_g\}_{g\in\mathbb{Z}_2^k}$, provided in* [1].*)* Observe that the action $\oplus_{i=1}^k \mathbb{Z}_2$ on $(S^n)^k$ induces the action $\oplus_{i=1}^k \mathbb{Z}_2$ on the regions $\{\mathcal{R}_g\}_{g\in\mathbb{Z}_2^k}$, where $(\mathcal{R}_g)^\epsilon = \mathcal{R}_{g+\epsilon}$. *(There is $g+\epsilon$ is the addition over $\bigoplus_{i=1}^k \mathbb{Z}_2$.)* Intuitively, action $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \bigoplus_{i=1}^k \mathbb{Z}_2$ is reflection of regions along the n^{th} hyperplane, where $\epsilon_n = 1$.

1.3 Fourier Analysis for Finite Abelian Groups

In this section, we are going to develop the basic Fourier analysis for finite Abelian groups. We are going to following the We shall fix G be a finite abelian group. The following proposition ([7, Theorem 0.1]) is classification of all finite Abelian groups:

Proposition 1.3.1. *Every finite Abelian group G is isomorphic to the group $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$ for some positive integers $N_1, \ldots, N_k$.*

We shall define $L^2(G) := \{f \mid f : G \to \mathbb{C}\}$ be the collection of complex-valued function. *(In fact $L_2(G)$ is a Hilbert space, we are going use the fact that $L^2(G)$ is inner product space, but we are not going to use the completeness.)*

Proposition 1.3.2. *The set $L^2(G)$ is a vector space over $\mathbb{C}$, under function addition and scalar multiplication over $\mathbb{C}$. Furthermore, we have $L^2(G)$ is $|G|$ dimensional vector space.*

Proof. Fix $f, h \in L^2(G)$ and $\alpha \in \mathbb{C}$. Then, we shall define $f + h : G \to \mathbb{C}$ by $(f + h)(g) = f(g) + h(g)$. We shall define $\alpha \cdot f : G \to \mathbb{C}$ as $(\alpha \cdot f)(g) = \alpha(f(g))$. Notice that $\mathbb{C}$ is Field, hence, we have $L^2(G)$ is Associative and Commutative. The constant function $z : G \to \mathbb{C}$ where $z(g) = 0$ is the additive identity. The negation of f could be defined as $-f : G \to \mathbb{C}$ where $(-f)(g) = -(f(g))$.

To show that $L^2(G)$ is a $|G|$ dimensional vector space. We shall prove by constructing a explicit basis of $L^2(G)$. Let $g \in G$. Let $\varphi_g : G \to \mathbb{C}$ be a function such that

$$\varphi_g(h) = \begin{cases} 0, & \text{if } h \neq g; \\ 1, & \text{if } h = g. \end{cases}$$

Claim that φ_g and φ_h are linear independent for any $g \neq h$. Because if $\varphi_g = C\varphi_h$, for some $C \in \mathbb{C}$. We have $1\varphi_g(g) = C\varphi_h(g) = 0$. This is a contradiction. Thus, we have φ_g and φ_h are linear independent over $\mathbb{C}$. To show that $\{\varphi_g\}_{g\in G}$ span the vector space $L^2(G)$. Let $f \in L^2(G)$. We shall observe that

$$f(x) = \sum_{g\in G} f(g) \cdot \varphi_g(x).$$

Thus, we have $\{\varphi_g\}_{g\in G}$ span $L^2(G)$. Therefore, we have $L^2(G)$ is $|G|-$dimensional vector space.

□

Thus, we have $L^2(G)$ is a vector space over $\mathbb{C}$. We could defined the usual inner product over the space $L^2(G)$. *(The space $L_2(G)$ is in fact a Hilbert space* [7]*).* Let $\langle \cdot, \cdot \rangle : L^2(G) \times L^2(G) \to \mathbb{C}$, such that for $f, h \in L^2(G)$

$$\langle f, h \rangle := \frac{1}{|G|} \sum_{g\in G} f(g)\overline{h(g)}. \tag{1.3.1}$$

Here, we shall verify 1.3.1 is a valid inner product.

Proposition 1.3.3. *Let G be a finite abelian group. The inner product (1.3.1) is well define over $L^2(G)$.*

Proof. We shall check all properties of inner product. Let $f, h, t \in L^2(G)$. Let $\alpha \in \mathbb{C}$ be a constant. *Conjugate symmetry,* notice that conjugate over a sum is equal to sum of conjugations, then

$$\langle f, h\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{h(g)} = \frac{1}{|G|}\overline{\sum_{g\in G} h(g)\overline{f(g)}} = \overline{\langle h, f\rangle}.$$

Linearity in the first argument,

$$\langle \alpha f, h\rangle = \frac{1}{|G|}\sum_{g\in G} \alpha f(g)\overline{h(g)} = \frac{\alpha}{|G|}\sum_{g\in G} f(g)\overline{h(g)} = \alpha\langle f, h\rangle.$$

$$\langle f+h, t\rangle = \frac{1}{|G|}\sum_{g\in G}(f(g)+h(g))\overline{t(g)} = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{t(g)} + \frac{1}{|G|}\sum_{g\in G} h(g)\overline{t(g)} = \langle f, t\rangle + \langle h, t\rangle.$$

Positive-definiteness, We shall observe that $(a+ib)(a-ib) = a^2+b^2 \geq 0$. Thus, we have $f(g)\overline{f(g)} \geq 0$ for all $g \in G$. Then, we have

$$\langle f, f\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{f(g)} \geq 0;$$

In particular, if $f(g) = 0$ for all $g \in G$, we have sum of 0's, where $\langle f, f\rangle$ if and only if $f = 0$. $\square$

Since the inner product is well defined, in particular using the positive-definiteness of the inner product, we could defined a norm on $L^2(G)$, where $\|f\| = \sqrt{\langle f, f\rangle}$. *(This ia called the L^2-norm.)* In the Fourier analysis on groups, characters will play a important role. *(This is similar to the standard Fourier analysis over $L^2(\mathbb{R})$, by using trigonometric functions.)*

Definition 1.3.4. A **character** of G is a group homomorphism $\chi : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the unit circle in $\mathbb{C}$, i.e. we have $\mathbb{C}^\times := \{e^{i\theta} \mid \theta \in \mathbb{R}\}$.

We shall denote $\widehat{G}$ be the set of characters of G. Let $\chi \in \widehat{G}$. Notice that χ is group homomorphism, i.e. we have for all $g_1, g_2 \in G$, we have $\chi(g_1g_2) = \chi(g_1)\chi(g_2)$. Notice that $\chi : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times \subseteq \mathbb{C}$. Hence, we have $\chi \in L^2(G)$, and $\widehat{G} \subseteq L^2(G)$. The following Proposition 1.3.5 is equivalent to [7, Theorem 1.1].

Proposition 1.3.5. *The set $\widehat{G}$ is a group with the binary operation $* : \widehat{G} \times \widehat{G} \to \widehat{G}$, where $(\chi_1 * \chi_2)(a) = \chi_1(a)\chi_2(a)$ for all $\chi_1, \chi_2 \in G$ and $a \in G$.*

Proof. Fix $\chi_1, \chi_2 \in \widehat{G}$. We shall check *Closure, Associativity, Identity element, and Inverse element.* For *closure*, shall check if $\chi_1\chi_2$ is group homomorphism. Let $g_1, g_2 \in G$. Then, we have

$$\begin{aligned}(\chi_1\chi_2)(g_1g_2) = \chi_1(g_1g_2)\chi_2(g_1g_2) &= \chi_1(g_1)\chi_1(g_2)\chi_2(g_1)\chi_2(g_1)\\ &= \chi_1(g_1)\chi_2(g_1)\chi_1(g_2)\chi_2(g_1)\\ &= (\chi_1\chi_2)(g_1)(\chi_1\chi_2)(g_2).\end{aligned}$$

Hence, we have $\chi_1\chi_2 \in \widehat{G}$. For *Associativity,* this is inherent by the fact that $\mathbb{C}^\times$ is Associative. For *identity element*, we shall first observe that $\chi_e : G \to \mathbb{C}^\times$, where $\chi_e(g) = 1$ for all $g \in G$, is a homomorphism. (Because, we have $\chi_e(g_1g_2) = 1 = \chi_e(g_1)\chi_e(g_2)$.) Claim that χ_e is identity element. Because we know that 1 is identity element of the group $\mathbb{C}^\times$, we have

$$(\chi_e\chi_1)(g) = \chi_e(g)\chi_1(g) = \chi_1(g) = \chi_1(g)\chi_e(g) = (\chi_1\chi_e)(g).$$

For *Inverse*, we shall consider $\chi^{-1}(g) = (\chi(g))^{-1}$, ($(\chi(g))^{-1}$ is well defined, because $\chi(g) \in \mathbb{C}^\times$, where is has the form $e^{i\theta}$). Notice that

$$\chi^{-1}(g_1g_2) = (\chi(g_1g_2))^{-1} = (\chi(g_1)\chi(g_2))^{-1} = \chi(g_1)^{-1}\chi(g_2)^{-1} = \chi^{-1}(g_1)\chi^{-1}(g_2).$$

Hence, we have $\chi^{-1} \in \widehat{G}$. Furthermore, we have $\chi\chi^{-1}(g) = \chi_e(g) = \chi^{-1}\chi(g)$, for all $g \in G$. □

Our next goal will be to prove that the characters form an orthonormal basis for the space $L^2(G)$. First, we shall prove a simple Lemma *(*[7, Lemma 1.2]*)*.

Lemma 1.3.6. *Let G be a finite Abelian group, and χ be a non-principal character of G (i.e. χ is not χ_e the identity of the group $\widehat{G}$.) Then, we have $\sum_{g\in G}\chi(g) = 0$.*

Proof. We shall proof by contradiction, suppose that $\sum_{g\in G}\chi(g) \neq 0$. Let $g' \in G$ be arbitrary. Then, we have

$$\chi(g')\sum_{g\in G}\chi(g) = \sum_{g\in G}\chi(g'+g) = \sum_{g\in G}\chi(g).$$

(Because $g' + G = G$.) Hence, we have $\chi(g') = 1$. Since $g' \in G$ Is arbitrary, we have $\chi = \chi_e$, i.e. it is the principal character. This is a contradiction. □

Using Lemma 1.3.6, we are able to show that orthogonality properties for the characters. *(Also see* [7, Lemma 1.3]*.)*

Lemma 1.3.7. *The characters $\widehat{G}$ of a finite Abelian group G are orthonormal functions in $L^2(G)$.*

Proof. Let $\chi \in \widehat{G}$. First, we shall observe that $\chi(g) \in \mathbb{C}^\times$, where $|\chi(g)| = 1$. We have

$$\|\chi\|^2 = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi(g)} = \frac{1}{|G|}\sum_{g\in G}|\chi(g)| = \frac{|G|}{|G|} = 1.$$

Thus, we have χ are unit vectors in $L^2(G)$. To show that the vectors of $\widehat{G}$ are orthogonal. Let $\chi_1, \chi_2 \in \widehat{G}$. First, we shall observe that $\chi_1^{-1} = \overline{\chi_1}$. Since $\chi_2 \neq \chi_1$. Then, we have $\chi_1\overline{\chi_2} \neq \chi_e$. We have

$$\langle \chi_1, \chi_2\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} = \frac{1}{|G|}\sum_{g\in G}\chi(g) = 0.$$

(The last equality is Lemma 1.3.6). Thus, we have χ is orthonormal functions in $L^2(G)$. □

In order to show that $\widehat{G}$ forms a basis of $L^2(G)$. First, we shall offer a classify of the group $\widehat{G}$. By Proposition 1.3.1, we know that $G = \bigoplus_{i=1}^k \mathbb{Z}_{N_1}$. *(Proposition 1.3.8 and Lemma 1.3.9 stated as a fact in* [7]*. However, we are going to verify them in this paper.)* Here, we shall start with simple example, where G is the trivial product, i.e. when $G = \mathbb{Z}_N$ for some $N \in \mathbb{Z}^+$. Let $z \in \mathbb{Z}_N$. Let $\chi_z \in L^2(\mathbb{Z}_N)$ such that

$$\chi_z(g) = e^{zg\cdot 2\pi i/N}. \tag{1.3.2}$$

(We shall always use $+$ denote the binary operation of $\mathbb{Z}_N$. Here zg refers to the multiplicative operation over $\mathbb{Z}_N$.)

Proposition 1.3.8. *Given $G = \mathbb{Z}_N$. For all $z \in \mathbb{Z}_N$, the $L^2(\mathbb{Z}_N)$ function χ_z define in* (1.3.2) *is character, i.e. we have $\chi_z \in \widehat{\mathbb{Z}_N}$. (i.e. we have χ_z is a character of $\mathbb{Z}_N$.) Furthermore, we have $\widehat{\mathbb{Z}_N} = \{\chi_z \mid z \in \mathbb{Z}_N\}$, and $\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N$.*

Proof. Let $z_1, z_2 \in \mathbb{Z}_N$. We shall observe that

$$\chi_z(z_1 + z_2) = e^{z(z_1+z_2)2\pi i/N} = e^{zz_1\cdot 2\pi i/N}e^{zz_2\cdot 2\pi i/N} = \chi_z(z_1)\chi_z(z_2).$$

Furthermore, we have $\chi_z(0) = e^0 = 1$. Thus, we have χ_z is a homomorphism. By Proposition 1.3.2, we know that $L^2(\mathbb{Z}_N)$ is a $|\mathbb{Z}_N|-$dimensional vector space. Notice that $\{\chi_z \mid z \in \mathbb{Z}_N\}$ has exactly $|\mathbb{Z}_N|$ orthonormal function (vectors), hence these are all the group homomorphisms. Thus, we have $\widehat{\mathbb{Z}_N} = \{\chi_z \mid z \in G\}$. To show that $\mathbb{Z}_N \simeq \widehat{\mathbb{Z}_N}$. Let $z_1, z_2 \in \mathbb{Z}_N$. Then, we have

$$\chi_{z_1}(g)\chi_{z_2}(g) = e^{z_1 g \cdot 2\pi i/N} e^{z_2 g \cdot 2\pi i/N} = e^{(z_1+z_1)g\cdot 2\pi i/N} = \chi_{z_1+z_2}(g).$$

Observe that the binary operation over $\widehat{\mathbb{Z}_N}$ is equivalent to addition of the subscript over $\mathbb{Z}_N$. Hence, we have $\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N$. □

Now, we shall generalize the proof above to all finite abelian group $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$. Let $z = (z_1, \ldots, z_k) \in G$, we shall define $\chi_z \in L^2(G)$, where for $g = (g_1, g_2, \ldots, g_k) \in G$.

$$\chi_z(g) = \prod_{i=1}^{k} e^{z_i g_i \cdot 2\pi i/N_i}. \tag{1.3.3}$$

First, we shall observe that the given function in (1.3.3) is a generalization of the function described in (1.3.2), i.e. when $G = \mathbb{Z}_N$ the two definition agrees with each other.

Lemma 1.3.9. *Let $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$ be a finite Abelian group. For all $z \in g$, the $L^2(G)$ function χ_g define in* (1.3.3) *is character, i.e. we have $\chi_g \in \widehat{G}$. (i.e. we have χ_g is a character of G.) Furthermore, we have $\widehat{G} = \{\chi_g \mid g \in G\}$, and $G \simeq \widehat{G}$.*

Proof. Fix $g = (g_1, \ldots, g_k) \in G$. First, we shall verify χ_g is a group homomorphism. Let $a, b \in G$. We shall write $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$. We have

$$\chi_g(a+b) = \prod_{i=1}^{k} e^{g_i(a_i+b_i)\cdot 2\pi i/N_i} = \prod_{i=1}^{k} e^{g_i a_i \cdot 2\pi i/N_i} \prod_{i=1}^{k} e^{g_i b_i \cdot 2\pi i/N_i} = \chi_g(a)\chi_g(b).$$

Furthermore, we have $\chi_g(0) = \prod_{i=1}^{k} e^0 = 1$, which is the identity in $\mathbb{C}^\times$. Thus, we have χ_g is a homomorphism. By Proposition 1.3.2, we have $L^2(G)$ is a $|G|$-dimensional vector space. Notice that $\{\chi_g \mid g \in G\}$ has exactly $|G|$ orthonormal functions, hence these are all the group homomorphisms. Therefore, we have $\widehat{G} = \{\chi_g \mid g \in G\}$. To show that $G \simeq \widehat{G}$. Let $a, b, g \in G$. We have

$$\chi_a(g)\chi_b(g) = \prod_{i=1}^{k} e^{a_i g_i \cdot 2\pi i/N_i} \prod_{i=1}^{k} e^{b_i g_i \cdot 2\pi i/N_i} = \prod_{i=1}^{k} e^{(a_i+b_i)g_i \cdot 2\pi i/N_i} = \chi_{a+b}(g)$$

Observe that the binary operation over $\widehat{G}$ is equivalent to addition of the subscript over G. Hence, we have $\widehat{G} \simeq G$. $\square$

The Lemma 1.3.9 offers a classification of $\widehat{G}$. Furthermore, we could use Lemma 1.3.9 and Lem 1.3.7 to show that $\widehat{G}$ forms a orthonormal basis of $L^2(G)$. *(Corollary 1.3.10 is equivalent* [7, Theorem 1.4]*.)*

Corollary 1.3.10. *Let G be a finite abelian group. The collection of character $\widehat{G}$ is orthonormal basis of $L^2(G)$.*

Proof. By Lemma 1.3.7, we have $\widehat{G}$ is a set of orthonormal vectors, hence linear independent. By Lemma 1.3.9 show that $|\widehat{G}| = |G|$. Notice that $L^2(G)$ $|G|$-dimensional vector space. Then, we have $\widehat{G}$ span $L^2(G)$. Therefore, we have $\widehat{G}$ is a orthonormal basis. $\square$

Since $\widehat{G}$ is a orthonormal basis of $L^2(G)$, in particular, because $\widehat{G}$ is a basis for the vector space $L^2(G)$, then the set of function $\widehat{G}$ span $L^2(G)$. Notice that $|\widehat{G}| < \infty$, i.e. we have $L^2(G)$ is finite dimensional vector space. Moreover, given set of basis $\widehat{G}$ of $L^2(G)$, the function $f \in L^2(G)$ could be written as a unique linear combination over $\widehat{G}$. We shall call

$$f = \sum_{g \in G} c_g \chi_g \tag{1.3.4}$$

the **Fourier transform** of the function f. We call c_g the **Fourier coefficient**. In the following Proposition, we are going to compute the explicit formula for the Fourier coefficient *(given some $f \in L^2(G)$).*

Proposition 1.3.11. *Let $f \in L^2(G)$. The Fourier Transform of f is unique. Furthermore, the Fourier coefficient $c_g = \langle f, \chi_g \rangle$.*

Proof. We shall observe that

$$\langle f, \chi_g \rangle = \left\langle \sum_{h \in G} c_h \chi_h, \chi_g \right\rangle = \sum_{h \in G} c_h \langle \chi_h, \chi_g \rangle = c_g.$$

(The first equality just substitution; the second equality, we used Proposition 1.3.3 Linearity in the first argument; The third equality, we used the fact that characters are orthonormal, which is Lemma 1.3.7.) Thus we have Fourier transformation for f is unique, and $c_g = \langle f, \chi_g \rangle$. $\square$

Remark. Since we know that c_g are unique, for the given function $f \in L^2(G)$. We could view the Fourier transform as a function $\xi : L^2(G) \to L^2(\widehat{G})$, such that $\xi(f) = \widehat{f}$ where $\widehat{f}(g) = \langle f, \chi_g \rangle$.

1.4 Special Case for Finite Fourier Analysis

In the previous section, we developed the theorem of Fourier Analysis of $L^2(G)$ over $\mathbb{C}$. In this section, we are going to study of the special case:

1. We shall fix our finite Abelian group $G = \mathbb{Z}_2^k$.

2. Furthermore, we are going to consider $L^2_{\mathbb{R}}(\mathbb{Z}_2^k) := \{f \mid \mathbb{Z}_2^k \to \mathbb{R}\}$, i.e. the real vector space. Instead of the Complex vectors space $L^2(\mathbb{Z}_2^k)$.

We are going to show that the set of characters $\widehat{\mathbb{Z}_2^k}$ still forms a basis over $L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$. However, since we move from $\mathbb{C}$ to $\mathbb{R}$, we shall first verify if the $\widehat{\mathbb{Z}_2^k}$ is well-defined.

(Well-defined?) To show $\widehat{\mathbb{Z}_2^k}$ is well-defined over $\mathbb{R}$. First, we shall recall the definition of $\widehat{\mathbb{Z}_2^k}$ over $\mathbb{C}$: the set of group homomorphism $\chi : \mathbb{Z}_2^k \to \mathbb{C}$. Let $g = (g_1, \ldots, g_k) \in \mathbb{Z}_2^k$, by Lemma 1.3.9, we have

$$\chi_g(a) = \prod_{i=1}^{k} e^{a_i g_i \pi i} = \prod_{i=1}^{k} (-1)^{a_i g_i} \in \mathbb{R} \subseteq \mathbb{C}. \tag{1.4.1}$$

Hence, we have the character of $\mathbb{Z}_2^k$ could be embedded within $\mathbb{R}$. *(This is the special property for $\mathbb{Z}_2$, because we are using the roots of unity in $\mathbb{C}^\times$, the second roots of unity are $\{-1, 1\}$, i.e. they are in fact real value.)* Since $\mathbb{Z}_2^k$ has representation over $\mathbb{R}$, we could restrict the $L^2(\mathbb{Z}_2)$ over $\mathbb{R}$.

By Corollary 1.3.10, we shown that $\widehat{\mathbb{Z}_2^k}$ is orthonormal basis of $L^2(G)$. We shall observe that $L^2_{\mathbb{R}}(\mathbb{Z}_2^k) \subseteq L^2(\mathbb{Z}_2^k)$. Thus, for all function $f \in L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$ has a Fourier transform, with respect to $\widehat{\mathbb{Z}_2^k}$. Furthermore, since we are considering the $f \in L^2\mathbb{R}(\mathbb{Z}_2^k)$, we have

$$c_g = \langle f, \chi_g \rangle = \frac{1}{|\mathbb{Z}_2^k|} \sum_{h \in \mathbb{Z}_2^k} f(h)\overline{\chi_g(h)} = \frac{1}{|\mathbb{Z}_2^k|} \sum_{h \in \mathbb{Z}_2^k} f(h)\chi_g(h) \in \mathbb{R},$$

(Here we are using the fact that $\chi_g : g \to \mathbb{R}$, where $\chi_g(h) \in \mathbb{R}$ implies that $\overline{\chi_g(h)} = \chi_g(h)$.) Thus, for all function $f \in L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$, the Fourier transform of f is real, i.e. all the Fourier coefficient of f are real valued. We shall state the following remark, to summarize all statement above:

Remark 1.4.1. Let $L^2_{\mathbb{R}}(\mathbb{Z}_2^k) := \{f \mid f : \mathbb{Z}_2^k \to \mathbb{R}\}$. Let $f, g \in L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$. Then, we have $L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$ satisfies the following properties:

1. $L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$ is a vector space over $\mathbb{R}$.
2. The inner produce $\langle f, g\rangle$ in (1.3.1) is well-defined. Furthermore $\langle f, g\rangle \in \mathbb{R}$.
3. We have $\widehat{\mathbb{Z}_2^k} = \{\chi_g \mid \chi_g : \mathbb{Z}_2^k \to \{\pm 1\}\}$, i.e. we have $\widehat{\mathbb{Z}_2^k}$ is a set of real valued function (1.4.1).
4. $\widehat{\mathbb{Z}_2^k}$ is orthonormal basis for the vector space $L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$.
5. The Fourier transform $f = \sum_{g\in G} c_g \chi_g$ over $\widehat{\mathbb{Z}_2^k}$ is well-defined.
6. The Fourier coefficient c_g for $f \in L^2_{\mathbb{R}}(\mathbb{Z}_2^k)$ is real, i.e. we have $c_g \in \mathbb{R}$ for all $g \in \mathbb{Z}_2^k$.

In particular, Remark 1.4.1.5 and Remark 1.4.1.6, are two important properties in this project. This section offer the foundation for us the explore the mass equipartition problem.

1.5 Equivariant map, Target Space and Test Map

In this section, we are going to fix $\mathbb{R}^n$, for some $n \in \mathbb{N}$. In section 1.2, we discuss about the topological set up of the space of hyperplane. Given a collection of hyperplanes $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$, we shall recall the definition of disjoint region $\{\mathcal{R}_g\}_{g\in\mathbb{Z}_2^k}$ defined in Equation (1.2.1), we label the disjoint region by element $g \in \mathbb{Z}_2^k$. Now, we shall introduce masses: a positive, finite Borel measure, that is absolutely continuous with respect to the Lebesgue measure. *(This is general set up, mentioned in* [1, 1.1 Historcial Summary]*.)* We shall fix μ on $\mathbb{R}^n$ be a mass. We are interested in $\mu(\mathcal{R}_g)$, for all $g \in \mathbb{Z}_2^k$. Notice that $\mu(\mathcal{R}_g) \in \mathbb{R}$, and each region uniquely defined by its subscript $g \in \mathbb{Z}_2^k$. We shall define the following function, with respect to μ on $\mathbb{R}$:

Definition 1.5.1. Let μ be an absolute continuous measure. Let $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ be hyperplanes in $\mathbb{R}^n$. Suppose that $\mathcal{R}$ is indexed by $\mathbb{Z}_2^k$. The test map is a function $f_\mu : \mathbb{Z}_2^k \to \mathbb{R}$ with respect to μ and $\mathcal{H}$, such that $f_\mu(g) = \mu(\mathcal{R}_h)$. We call f_μ The test map for μ.

(Remark. Definition 1.5.1 is a simplification of [3, Equation (1.1)]*.)* Notice that, if $\mathcal{H}$ equipartition μ, the measure restricted to each open region should be exactly $\mu(\mathbb{R}^n)/2^k$. Since we know that the function f_μ depends on $\mathcal{H}$, for each fix μ, we could view this problem as mapping $\mathcal{H}$ to the corresponding μ-test map. Recalling that $\mathcal{H} = \{H_1, \dots, H_k\}$ could be represented by points on $(S^n)^k$, and, μ-test map is a function in $L^2(\mathbb{Z}_2^k)$. Thus, we could also view function $F_\mu^* : (S^n)^k \to L^2(\mathbb{Z}_2^k)$. In this case, an equipartition will correspond to a constant function in $f_\mu^* \in F_\mu^*((S^n)^k) \subseteq L^2(\mathbb{Z}_2^k)$, where $f_\mu^*(g) = \mu(\mathbb{R}^n)/2^k$. *(This is a different interpretation, however, this is a different approach of the hyperplane equipartition problem. So, we shall stick with the test map.)* In general, there exists region $\mathcal{R}_g$ such that $\mu(\mathcal{R}_g) \neq \mu(\mathbb{R}^n)/2^k$, for some $g \in \mathbb{Z}_2^k$. By Remark 1.4.1, the test map has $f_\mu : \mathbb{Z}_2^k \to \mathbb{R}$ has a Fourier decomposition over the set of characters of $\mathbb{Z}_2^k$

$$f_\mu(x) = \sum_{g \in \mathbb{Z}_2^k} c_g \chi_g(x), \tag{1.5.1}$$

where the $\chi_g \in \widehat{\mathbb{Z}_2^k}$, and $c_g \in \mathbb{R}$ are the Fourier coefficients. There are some properties and advantage, when we study the Fourier decomposition of the function f_μ.

Definition 1.5.2. Let $f : X \to Y$. Let G be a group, such that G act on X and Y. We say the function f is $G-$equivariant if

$$f(g \cdot x) = g \cdot f(x),$$

for all $g \in G$ and $x \in X$.

Test map returns the measure within sectors, which also allow us to capture how does the group $\epsilon \in \bigoplus_{i=1}^k \mathbb{Z}_2$ acts the 2^k-regions. Such map has a Fourier decomposition with respected to the action $\epsilon \in \bigoplus_{i=1}^k \mathbb{Z}_2$. Recalling that $\mathbb{Z}_2^k$ acts on the $\{\mathcal{R}_g\}_{g \in \mathbb{Z}_2^k}$, such that $(\mathcal{R}_g)^\epsilon = \mathcal{R}_{g+\epsilon}$, for $\epsilon \in \mathbb{Z}_2^k$. Since we defined $f_\mu(g) = \mu(\mathcal{R}_g)$, we expect $\epsilon \in \mathbb{Z}_2^k$ actions on $f_\mu(g)$ by $(f_\mu(g))^\epsilon = \mu(\mathcal{R}_{g+\epsilon})$, for all $g \in \mathbb{Z}_2^k$.

If f_μ is a constant map, i.e in this case, we have the corresponding hyperplanes equipartition the given measure μ, we shall observe that $\mathbb{Z}_2$ acts trivially on the Then we have This implies that f_μ is fixed under $\mathbb{Z}_2^k$ action. *(When f_μ is a constant map, the corresponding collection of hyperplanes $\mathcal{H}$ equipartition the given measure.)*

Lemma 1.5.3. *Let f_μ be the test map of μ, with respect to $\mathcal{H}$. Let $f_\mu = \sum_{g\in\mathbb{Z}_2^k}$ The Fourier coefficient $c_{(0,0,\ldots,0)}$ for f is $\mu(\mathbb{R}^d)/2^k$.*

Proof. Let $\mathbf{0} = (0,0,\ldots,0)$. We have

$$c_{\mathbf{0}} = \langle f, \chi_{\mathbf{0}} \rangle = \frac{1}{|\mathbb{Z}_2^k|} \sum_{g\in\mathbb{Z}_2^k} f(g)\chi_{\mathbf{0}}(g) = \frac{1}{|\mathbb{Z}_2^k|} \sum_{h\in\mathbb{Z}_2^k} \mu(\mathcal{R}_h) = \frac{1}{|\mathbb{Z}_2^k|}\mu(\mathbb{R}^d) = \frac{1}{2^k}.$$

Therefore, we have $c_{\mathbf{0}} = 1/2^k$. □

This is the special property for $\chi_{\mathbf{0}}$, because it is the identity map. *(We have $\chi_{\mathbf{0}}(g) = 1$ for all $g \in \mathbb{Z}_2^k$.)* In general, for $g \in \mathbb{Z}_2^k$ and $g \neq \mathbf{0}$, there exists some $h \in \mathbb{Z}_2^k$, such that $\chi_g(h) = -1$. In this case, if $c_g \neq 0$ for some $g \neq \mathbf{0}$, there exists $h \in \mathbb{Z}_2^k$ action on the disjoint regions, such that f_μ is not fixed under the h action. Hence, it is not equipartition. *(i.e. we could view equipartition as function the is fix under $\mathbb{Z}_2^k$ action.)* We shall compute some simple example.

Example 1.5.4. Let square S centered at $(0,0)$ with side length 2. Let μ be area measure on S and 0 otherwise. Let $\mathcal{H} = \{H_1, H_2\}$ be collection of hyperplane, such that H_1 and H_2 is represented by points in S^3

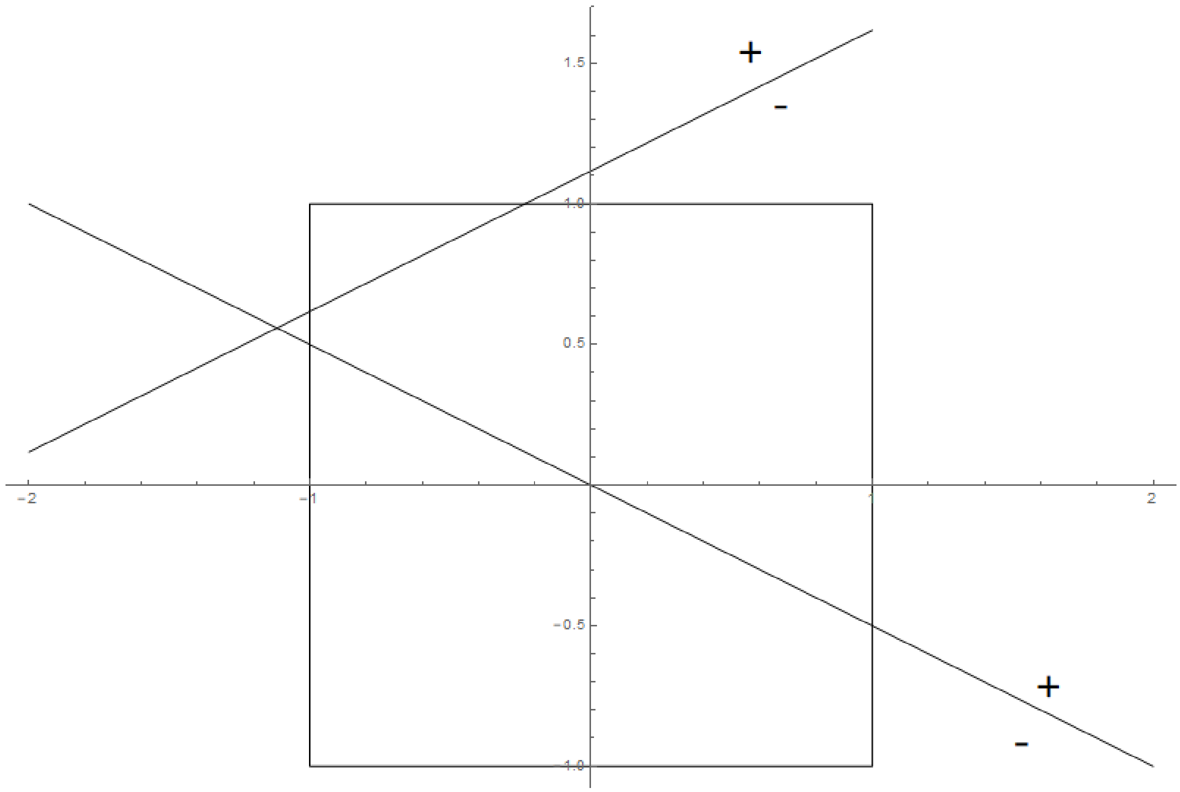
$$s_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right); \qquad s_2 = \left(-\frac{1}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{1}{\sqrt{2}}\right),$$

respectively. In this example, we shall compute:

1. Compute f_μ, and compute the corresponding Fourier expansion.

2. Let $h = (0,1) \in \mathbb{Z}_2^2$. We shall consider the how h is acting on all regions.

Proof. 1. First of all, since we are in $\mathbb{R}^2$. There hyperplanes are in fact, lines. For H_1, its corresponding unit normal vector is $N_1 = (1,2)/\sqrt{5}$, and the distance from the origin is 0. We could write H_1 with $y = -x/2$. Similarly, the unit normal vector for H_2 is $N_2 = (-1, 2)/\sqrt{5}$. *(We know that* $\sqrt{2}s_2 = (-1/\sqrt{5}, 2/\sqrt{5}, 1)$, *where* $H_2 = \vec{N_2}^{\perp} + \vec{N_2}$*)* We have H_2 could be represented by the equation $(x, y) \cdot \vec{N_2} = 1$, where $y = (x + \sqrt{5})/2$. The picture below show a picture of the measure, and the given hyperplane H_1 and H_2.



The $+$ and $-$ in the picture represents H^0 and H^1 respectively. There are four regions. Hence, the four regions are indexed by $\mathbb{Z}_2^2$, which are $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. *(In the picture the* $+$ *corresponds to a* 0 *and* $-$ *corresponds to* 1.*)* Now we shall compute measure of the each disjoint section, and this corresponds to the values for $f_\mu : \mathbb{Z}_2 \to \mathbb{R}$, where

$$f_\mu(0,0) = \frac{7 - 3\sqrt{5}}{2}; \qquad f_\mu(1,0) = \frac{-3 + 3\sqrt{5}}{2}; \qquad f_\mu(0,1) = 0; \qquad f_\mu(1,1) = 2.$$

We shall now compute the Fourier transform of the function f_μ. We compute the Fourier series be using the definition of character over $\mathbb{Z}_2^2$, the formula is introduced in (1.4.1), where

$$
\begin{aligned}
c_{(0,0)} &= \frac{\mu(\mathbb{R}^2)}{4} = 1; \\
c_{(1,0)} &= \frac{1}{4}\left(\frac{7-3\sqrt{5}}{2} - \frac{-3+3\sqrt{5}}{2} + 0 - 1\right) = \frac{3-3\sqrt{5}}{4}; \\
c_{(0,1)} &= \frac{1}{4}\left(\frac{7-3\sqrt{5}}{2} + \frac{-3+3\sqrt{5}}{2} - 0 - 1\right) = 0; \\
c_{(1,1)} &= \frac{1}{4}\left(\frac{7-3\sqrt{5}}{2} - \frac{-3+3\sqrt{5}}{2} - 0 + 1\right) = \frac{7-3\sqrt{5}}{4}.
\end{aligned}
$$

Thus, we have the Fourier transform of f_μ is

$$f_\mu = 1 + \frac{3-3\sqrt{5}}{4}\chi_{(1,0)} + \frac{7-3\sqrt{5}}{4}\chi_{(1,1)}.$$

(We could just plug in different values, to check the Fourier transform is actually agree with the original function.)

2. We shall now consider the action $h = (0,1) \in \mathbb{Z}_2^2$ on $\{\mathcal{R}_g\}_{g\in\mathbb{Z}_2^2}$. Observe that the action h is reflection along H_2. Notice that this action is not fixed under $\chi_{(0,1)}$ and $\chi_{(1,1)}$, i.e. the action h will negate the Fourier Coefficient of $c_{(0,1)}$ and $c_{(1,1)}$. *(In this case, we have* $c_{(0,1)} = 0$. *Hence, only we shall replace* $c_{(1,1)}$ *by* $-c_{(1,1)}$.*)* Hence, we have

$$(f_\mu)^h = 1 + \frac{3-3\sqrt{5}}{4}\chi_{(1,0)} - \frac{7-3\sqrt{5}}{4}\chi_{(1,1)}.$$

□

From this example above we shall observe that f_μ is a linear combination of equivariant map, *(in general)* with respect to $\mathbb{Z}_2^k$ action.

Proposition 1.5.5. *Let f_μ be test map. Let $\mathcal{F} : \mathbb{Z}_2^k \to \mathbb{R}$ map of Fourier coefficient, such that $\mathcal{F}(g) = c_g$, for all $g \in \mathbb{Z}_2^k$. Let $\epsilon, h \in \mathbb{Z}_2^k$. Then, we have*

$$\mathcal{F}(h \cdot \epsilon) = \chi_\epsilon(h) \cdot \mathcal{F}(\epsilon).$$

Hence, we have $\mathcal{F}$ is $\mathbb{Z}_2^k$-equivariant.

Proof.

$$\mathcal{F}(h \cdot \epsilon) = \frac{1}{2^k} \sum_{g \in \mathbb{Z}_2^k} f_\mu(hg)\chi_\epsilon(g) = \frac{\chi_\epsilon(h)}{2^k} \sum_{g \in \mathbb{Z}_2^k} f_\mu(hg)\chi_\epsilon(gh) = \chi_\epsilon(h) \cdot \mathcal{F}(\epsilon).$$

□

Proposition 1.5.5 shows that if f_μ is fix under the action $h \in \mathbb{Z}_2$. Then, for all $\epsilon \in \mathbb{Z}_2^k$, we have $\chi_\epsilon(h) = -1$ implies that $c_\epsilon = 0$. Now, we shall view the equipartition problem as Fourier coefficient annihilation problem.

1.6 Polynomial Condition for Coefficient Annihilation

As we see in Example 1.5.4, we have shall observe that H_2 is if fact equipartition the given mass μ, and we shall observe that $c_{(0,1)} = 0$ vanishes. The following Lemma is a generalization of case in Example 1.5.4, when there are a sub-collection of hyperplane equipartition the given mass. *(This section is a simplification of* [1, Secition 6.1]*, together with Finite Fourier transform in* [3]*.)*

Lemma 1.6.1. *Let f_μ be a test map with respect to μ on $\mathbb{R}^n$ and $\mathcal{H}$. Let $\mathcal{E} \subseteq \mathcal{H}$ be a sub-collection of hyperplanes. Suppose that $\mathcal{E} = (H_{i_1}, \ldots, H_{i_s})$ equipartition the given measure μ. Let $g \in \mathbb{Z}_2^k$ be a action, such that the g-action fix $\mathcal{H} \backslash \mathcal{E}$. Then, we have the Fourier coefficient $c_g = 0$.*

Proof. Without loss of generality, we shall assume that $\mathcal{E} = \{H_1, \ldots, H_s\}$ is the first s hyperplanes among the collection $\mathcal{H}$. We shall fix $g_1^*, \ldots, g_s^* \in \mathbb{Z}_2$. First, we shall observe that

$$\sum_{h_{s+1},\ldots,h_k \in \mathbb{Z}_2} f_\mu(h_1^*, \ldots, h_s^*, h_{s+1}, \ldots, h_k) = \frac{\mu(\mathbb{R}^n)}{2^s},$$

for some $h_1^*, \ldots, h_s^* \in \mathbb{Z}_2$. *(This sum on the left hand side of the equation could be view as measure of the region $(h_1^*, \ldots, h_s^*)$, by ignoring all $\mathcal{H} \setminus \mathcal{E}$. Since we ignore $\mathcal{H} \setminus \mathcal{E}$, by assumption we have $\mathcal{E}$ equipartition the measure μ, this implies that each region has exactly $\mu(\mathbb{R}^n)/2^s$ measure, which is the right hand side of the equality.)* Now, let $g = (g_1^*, g_2^*, \ldots, g_s^*, 0, 0, \ldots, 0) \in \mathbb{Z}_2^k$. We shall denote $g^* = (g_1^*, g_2^*, \ldots, g_s^*) \in \mathbb{Z}_2^s$ and $h^* = (h_1^*, \ldots, h_s^*)$ and $h' = (h_{s+1}, \ldots, h_k)$. Let we shall consider the the Fourier coefficient for c_g, we have

$$\begin{aligned} c_g = \langle f, \chi_g \rangle = \frac{1}{2^k} \sum_{h \in \mathbb{Z}_2^k} f(h)\chi_g(h) = \frac{1}{2^k} \sum_{h^* \in \mathbb{Z}_2^s} \sum_{h' \in \mathbb{Z}_2^{k-s}} & f_\mu\big((h^*, h')\big)\chi_g\big((h^*, h')\big) \\ =&\frac{1}{2^k} \sum_{\vec{h}^* \in \mathbb{Z}_2^*} \frac{\mu(\mathbb{R}^n)}{2^s} \chi_{g^*}(h^*) \\ =&0. \end{aligned}$$

(sum over the characters is 0*, when $g^* \neq \boldsymbol{0}$. But, we know that g^* is an non-trivial action over $\mathcal{E}$, hence it is not $\boldsymbol{0}$.)* □

In Lemma 1.6.1, we show that what are the Fourier coefficient we have to annihilate in order to have have $\mathcal{E} \subseteq \mathcal{H}$ equipartition the given measure μ. In general, we don't we could consider the multiple masses, i.e. a collection of $\mathcal{M} = \{\mu_1, \ldots, \mu_m\}$. Then, we could define a collection of test map $f_{\mu_i} : \mathbb{Z}_2^k \to \mathbb{R}$ that measures disjoint regions $\{\mathcal{R}_g\}_{h \in \mathbb{Z}_2^k}$ corresponds to μ_i. Then, we could talk about if the given set of hyperplane equipartition the collection of measures, or sub-collection of hyperplanes of $\mathcal{H}$ equipartition the some sub-collection of $\mathcal{M}$.

Now, we shall introduce techniques of equivariant topology. For $\mathcal{H} = \{H_1, \ldots, H_k\}$ in $\mathbb{R}^n$, we shall consider polynomial ring $\mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n+1}, x_2^{n+1}, \ldots, x_k^{n+1})$. For $g \in \mathbb{Z}_2^k$, we shall assign g to the polynomial $g \cdot (x_1, x_2, \ldots, x_k)$. *(The $\cdot$ here the usual dot product.)* for simplicity, we shall define a function

$$\mathfrak{P} : \mathbb{Z}_2^k \to \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n+1}, x_2^{n+1}, \ldots, x_k^{n+1}), \text{ such that } \mathfrak{P}(g) = g \cdot (x_1, \ldots, x_k)$$

The following the Main theorem, for the polynomial condition *(We are not going to proof the following Theorem. This main Theorem is the $\mathbb{Z}_2^k$ case for real-valued measures of* [3, Theorem 3.1]*.)*

Theorem 1.6.2. *Let $\mathcal{M} = \{\mu_1, \ldots, \mu_m\}$ be a collection of measures. For each $i \leq m$ and μ_i, let $g_{i,1}, \ldots, g_{i,t_i} \in \mathbb{Z}_2^k$, and let*

$$h^i(x_1, \ldots, x_k) = \prod_{s=1}^{t_i} \mathfrak{P}(g_{i,s}).$$

If $\prod_{i=1}^{m} h^i(x_1, \ldots, x_k) \neq 0$, there exists a collection of hyperplane $\mathcal{H} = \{H_1, \ldots, H_k\}$ such that the Fourier coefficient for f_{μ_i} indexed by $g_{i,1}, \ldots, g_{i,t_i}$ vanishes.

Put together Theorem 1.6.2 and Lemma 1.6.1 this allowed us the compute the exact polynomial condition for hyperplane equipartition problem.

Corollary 1.6.3. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a collection of hyperplane. Suppose that $\mathcal{H}$ equipartition measure μ, the polynomial condition is $\prod_{g \in \mathbb{Z}_2^k \setminus \{0\}} \mathfrak{P}(g)$.*

Checking if the explicit computation for polynomial condition vanish could be hard, especial when we in higher dimension. However, in lower dimension, for instance $\mathbb{R}^3$, we could do some explicit computation of the polynomial condition for hyperplane equipartition problem. We shall look at an example, using the Corollary 1.6.3. We are going to compute the explicit polynomial condition in the following example, and test whether the given polynomial condition vanishes.

Example 1.6.4. Let $\mathcal{H} = \{H_1, H_2\}$ be a collection of hyperplanes. Let μ be a measure on $\mathbb{R}^2$. Compute the polynomial condition if $\mathcal{H}$ equipartition the given measure μ.

Proof. Let μ be the test map. By Lemma 1.6.1, we have the Fourier coefficient $c_{(0,1)}$, $c_{(1,0)}$ and $c_{(1,1)}$ vanishes. This is equivalent to the polynomial condition

$$P = x_1 x_2 (x_1 + x_2) = x_1^2 x_2 + x_1 x_2^2.$$

Observe that P does not vanishes in $\mathbb{Z}_2[x_1, x_2]/(x_1^3, x_2^3)$. Hence, it is possible to equipartition the given measure in $\mathbb{R}^2$ by two hyperplane. □

Example 1.6.4 is the one of the simplest example among all equipartition problem. The following example we are going to look at equipartition in $\mathbb{R}^3$, and the result shows the case that the polynomial condition vanishes.

Example 1.6.5. Let $\mathcal{H} = \{H_1, H_2, H_3\}$ be a collection of hyperplane. Compute the polynomial condition if $\mathcal{H}$ equipartition a given measure μ, in dimension 3.

Proof. Let f_μ be the test map. By Lemma 1.6.1, we have the Fourier coefficient $c_{(1,0,0)}$, $c_{(0,1,0)}$, $c_{(1,1,0)}$, $c_{(0,0,1)}$, $c_{(1,0,1)}$, $c_{(0,1,1)}$, and $c_{(1,1,1)}$ of f_μ should vanish. In this case, each vanishing coefficient will give us a polynomial in $\mathbb{Z}_2[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^4)$. We have

$$P = \prod_{g \in \mathbb{Z}_2^3 \setminus \{\mathbf{0}\}} \mathfrak{P}(g) = x_1x_2x_3(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3)$$

We have shall check if P vanish under $\mathbb{Z}_2[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^4)$. We shall expand the product into sums. Then, we have

$$\begin{aligned}
P =& x_1x_2x_3(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3) \\
=& (x_1 + x_2 + x_3)(x_1x_2^2x_3^3 + x_1x_2^3x_3^2 + x_1^3x_2x_3^2 + x_1^3x_2^2x_3 + x_1^2x_2x_3^3 + x_1^2x_2^3x_3) \\
=& x_1^2x_2^2x_3^3 + x_1^2x_2^3x_3^2 + \underline{x_1^4x_2x_3^2} + \underline{x_1^4x_2^2x_3} + x_1^3x_2x_3^3 + x_1^3x_2^3x_3 \\
& + x_1x_2^3x_3^3 + \underline{x_1x_2^4x_3^2} + x_1^3x_2^2x_3^2 + x_1^3x_2^3x_3 + x_1^2x_2^2x_3^3 + \underline{x_1^2x_2^4x_3} \\
& + \underline{x_1x_2^2x_3^4} + x_1x_2^3x_3^3 + x_1^3x_2x_3^3 + x_1^3x_2^2x_3^2 + \underline{x_1^2x_2x_3^4} + x_1^2x_2^3x_3^2 \\
=& 2x_1^3x_2^3x_3 + 4x_1^3x_2^2x_3^2 + 4x_1^2x_2^3x_3^2 + 2x_1^3x_2x_3^3 + 4x_1^2x_2^2x_3^3 + 2x_1x_2^3x_3^3 \\
=& 0.
\end{aligned}$$

Thus, we in this case, the polynomial vanishes. Hence, we cannot tell if there exists $\mathcal{H}$ equipartition the given measure μ. □

1.7 Some Explicit Formula for Polynomial Condition

Given $\mathcal{H} = \{H_1, \ldots, H_k\}$. Notice that Theorem 1.6.2 always offer us product of sum. It is a hard to tell the given product of sum vanishes in the given polynomial ring. Hence, we would love the convert the product of summations into summation of product $x_1^{p_1}x_2^{p_2}\cdots x_k^{p_k}$.

Most of the polynomial condition are hard to compute. *(First of all, the conditions could be random, which is not interesting.)* In general, we shall consider a set of symmetric condition. For example, the full equipartition prosperity are symmetric. *(Because that is the maximum among of condition we can put on a measure.)* The full equipartition is equivalent of annihilation of all non-zero Fourier coefficient. Thus, its corresponding polynomial condition is

$$\prod_{g\in\mathbb{Z}_2^k\setminus\{\mathbf{0}\}}\mathfrak{P}(g)=\begin{vmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2^{k-1}} & x_2^{2^{k-1}} & \cdots & x_k^{2^{k-1}} \end{vmatrix}=\sum_{\sigma\in S_k} x_{\sigma(1)}^{2^{k-1}}x_{\sigma(2)}^{2^{k-2}}\cdots x_{\sigma(k-1)}^{2}x_{\sigma(k)}. \tag{1.7.1}$$

In equation (1.7.1), we rewrite the product into summation over S_k. Here S_k denote the symmetric group of k-elements. Similarly, there is an other symmetric condition: any two of the hyperplanes in $\mathcal{H}$ equipartition the measure. This is represented by polynomial:

$$\prod_{1\leq g_1+\cdots+g_k\leq 2}\mathfrak{P}(g_1,g_2,\ldots,g_k)=\begin{vmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k} & x_2^{k} & \cdots & x_k^{k} \end{vmatrix}=\sum_{\sigma\in S_k} x_{\sigma(1)}^{k}x_{\sigma(2)}^{k-1}\cdots x_{\sigma(k-1)}^{2}x_{\sigma(k)}. \tag{1.7.2}$$

(Both Equation 1.7.1 and Equation 1.7.2 used Vandermonde determinate, more detail could be find in [11]*.)* In general, the conditions are unpredicted so that, we cannot easily compute the sum. *(The two cases above are special cases, so that the expansion of the produce is in fact a sum over symmetric group.)*

2

Cascading Makeev with $\vec{\ell} \in \{0,2\}^k$

Given a symmetric condition, mentioned in section 1.7, it is likely that S_n acts on the subscript of the terms of the polynomial condition, i.e. we could write the polynomial condition as a sum over symmetric group. However, if the polynomial condition is too symmetric, since for all $n \in \mathbb{Z}^+$, we have $|S_n| = 2k$ for some $k \in \mathbb{Z}$, it is possible that all terms of the polynomial condition have even coefficient, hence vanishes. Thus, the condition we consider should be skewed symmetric. In this Chapter, we are going to study hyperplane equipartition with Cascading Makeev condition. In section 2.1, we are going to talk about what is Cascading Makeev condition. In section 2.2, we are going to talk about parity of multinomial coefficient. Even though, this is also a well-know result, we are going to provide classification of all even multinomial coefficient. Section 2.3 is the main result of this project. We are going to look at some special case of the Cascading Makeev condition, namely when $\vec{\ell} \in \{0,2\}^k$. This is first non-trivial cases, of the Cascading Makeev condition. We are going to study the minimal dimension for such hyperplane equipartition is possible. Furthermore, we are going to make condition as tight as possible, so that there is no more room for more Cascading Makeev condition.

2.1 Introduction

In the paper By V. V. Makeev [2], there following hyperplane equipartition problem is being studied:

Question 3: *(Makeev.)* Let μ be finite absolute continuous measure *(with respect to Lebesgue measure)*. Is there a collection of hyperplane $\mathcal{H} = \{H_1, \dots, H_k\}$, such that any two of them equipartition μ.

The question above is being studied extensively by Makeev. Also, a generalization of the Makeev problem is proposed and studied by Pavle V.M. Blagojević & Roman Karasev, *(the condition proposed by Makeev, together with odd continuous map form from S^{n-1} to S^{n-1}. See* [4]*)*. In this project, we are going to look at an other generalization of Makeev condition: the Cascading Makeev Condition.

Cascade: Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a collection of hyperplane. A Cascade over $\mathcal{H}$ is a sequence of nest sub-collection of hyperplanes,

$$\mathcal{H} = \mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \cdots \supseteq \mathcal{H}_k,$$

where $\mathcal{H}_i = \{H_i, H_{i+1} \dots, H_k\}$.

(Remake. The size of $\mathcal{H}_i$ decreases when i increases. In some sense, this is looks like a "Cascade".) The Cascading Makeev Condition is applying some kind of Makeev condition to each layer $\mathcal{H}_i$. Furthermore, we are going to relax the constrain of one mass. *(We are going to look as collection of masses.)* Also, instead to any two of $\mathcal{H}$, we are going generalized to any ℓ of $\mathcal{H}$. *(This is one of the generalization of the Makeev condition.)*

Definition 2.1.1. Let $k \in \mathbb{Z}^+$. Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a collection of hyperplane. Let $\{\mathcal{H}_1, \dots, \mathcal{H}_k\}$ be a cascade of $\mathcal{H}$. Let $\vec{m} = (m_1, m_2, \dots, m_k)$ be a vector of collections of measures. Let $\vec{\ell} = (\ell_1, \ell_2, \dots, \ell_k)$ such that $1 \le \ell_i \le k - i + 1$ for all $i \in \{1, 2, \dots, k\}$. Then, the cascading Makeev condition is a triple $(\mathcal{H}, \vec{\ell}, \vec{m})$, such that any ℓ_i of $\mathcal{H}_i$ equipartition each of m_i measures.

2.2 Parity of Multinomial Coefficient

Since we are considering polynomial ring over $\mathbb{Z}_2$. Consider the polynomial $(x_1+x_2+\cdots+x_k)^n$, when we expand such polynomial, we would love to answer what are the terms that are non-vanishing over $\mathbb{Z}_2[x_1,\ldots,x_k]$, i.e. the coefficient is odd. Notice that we could use multinomial theorem to expanding the polynomial $(x_1+x_2+\cdots+x_k)^n$, where

$$(x_1+x_2+\cdots+x_k)^n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_m = k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}.$$

In particular, when the multinomial coefficient is divisible by 2, the given term vanishes in $\mathbb{Z}_2[x_1,\ldots,x_k]$.

In this section, we are going to develop the theory of parity of multinomial coefficient. This could be view as a generational of Lucas's Theorem, (i.e. the divisibility of binomial coefficient by prime number.) On the other hand, divisibility condition of multinomial coefficient is being studied by Shigeki Akiyama [5]. However, since we are only considering module 2, we shall proof the parity condition ourself.

Definition 2.2.1. Let $r \in \mathbb{N}$. Define function $\nu : \mathbb{N} \to \mathbb{N}$ where $\nu(r) = \sum_{n=1}^{\infty} \lfloor r/2^n \rfloor$.

Here is more intuitively definition, or equivalent definition of the function ν.

Proposition 2.2.2. *Let $r \in \mathbb{N}$. Then we have $\nu(r) = \max_{k\in\mathbb{N}}\{2^k \mid r!\}$.*

Proof. Observe that $\lfloor r/2^n \rfloor$ is the value $\{1, 2, \ldots, r\}$ that is divisible by 2^n, for every $n \in \mathbb{N}$. Hence, we counted every number that is divisible by $2, 2^2, 2^3, \ldots$ exactly once, which is exactly the number of 2 we could divide out from $r!$, i.e. it is $\max_{k\in\mathbb{N}}\{2^k \mid r!\}$. □

When we consider choose or multi-choose, we will have $r!/k$, for some $k \in \mathbb{N}$ and $k \mid r$. The reason we care about if $r!/k$ is non-vanishing in $\mathbb{Z}_2$, i.e. we have $r!/k$ is odd. In this case, we must have $\nu(r) \mid k$. In this case, we shall study what are the multinomial coefficients $\binom{r}{r_1, r_2, \ldots, r_n}$, where $\nu(r) \mid \prod_{i=1}^{n} r_i!$.

Lemma 2.2.3. *Let $r \in \mathbb{N}$, and $r_1, r_2, \ldots, r_n \in \mathbb{N}$ such that $\sum_{i=1}^{n} r_i = r$. Then $2 \nmid \binom{r}{r_1, r_2, \ldots, r_n}$ if and only if $\sum_{i=1}^{n} \nu(r_i) = \nu(r)$.*

Proof. ($\Rightarrow$) Suppose that $2 \nmid \binom{r}{r_1,\ldots,r_n}$, i.e. we have $r!/\prod_{i=1}^n r_i!$ is odd. Proof by contradiction, Suppose that $\sum_{i=1}^n \nu(r_i) < \nu(r)$. Notice that $2^{\sum_{i=1}^n \nu(r_i)} \mid 2^{\nu(r)}$, such that $2^\nu(r)/(2^{\sum_{i=1}^n \nu(r_i)}) \geq 2$. Then, we have $\binom{r}{r_1,\ldots,r_n}$ is even, which is a contradiction. On the other hand, suppose that $\sum_{i=1}^n \nu(r_i) > \nu(r)$. Then, we have $2^{\sum_{i=1}^n \nu(r_i)} \nmid 2^{\nu(r)}$. But, we know that $2^{\sum_{i=1}^n \nu(r_i)} \mid \prod_{i=1}^n r_i!$. This is a contradiction. Thus, we have $\sum_{i=1}^n \nu(r_i) = \nu(r)$.

($\Leftarrow$) Suppose that $\sum_{i=1}^n \nu(r_i) = \nu(r)$. By proposition 2.2.2, we have $\max_{k\in\mathbb{N}}\{2^k \mid r!\} = \sum_{i=1}^n \max_{k\in\mathbb{N}}\{2^k \mid r_i!\}$. By the fundamental theorem of arithmetic, we have $r! = 2^k C$ and $\prod_{i=1}^n r_i! = 2^k C'$, for some $k \in \mathbb{N}$ and $C, C' \in \mathbb{Z}^+_{\text{odd}}$. Then, we have

$$\frac{C}{C'} = \frac{2^k C}{2^k C'} = \binom{r}{r_1,\ldots,r_n} \in \mathbb{N}.$$

Notice that C, C' are odd, hence, we have C/C' is also odd. Therefore, we have $2 \nmid \binom{r}{r_1,\ldots,r_n}$. $\square$

Lemma 2.2.3 give the necessary condition when a multinomial coefficient being odd. This could be use to classify all the multinomial coefficient that are odd. Thus, we shall study when $\sum_{i=1}^n \nu(r_i) = \nu(r)$.

Lemma 2.2.4. *Let $r \in \mathbb{N}$. Suppose $r_1, \ldots, r_n \in \mathbb{N}$ such that $\sum_{i=1}^n r_i = r$, then $\sum_{i=1}^n \nu(r_i) \leq \nu(r)$, with equality if and only if $\sum_{i=1}^n \lfloor r_i/2^j \rfloor = \lfloor r/2^j \rfloor$ for all $j \in \mathbb{N}$.*

Proof. By definition of ν, we have

$$\sum_{i=1}^n \nu(r_i) = \sum_{i=1}^n \sum_{j=1}^\infty \left\lfloor \frac{r_i}{2^j} \right\rfloor = \sum_{j=1}^\infty \sum_{i=1}^n \left\lfloor \frac{r_i}{2^j} \right\rfloor \leq \sum_{j=1}^\infty \left\lfloor \frac{\sum_{i=1}^n r_i}{2^j} \right\rfloor = \sum_{j=1}^\infty \left\lfloor \frac{r}{2^j} \right\rfloor = \nu(r).$$

The inequality become equality, precisely when $\sum_{i=1}^n \lfloor r_i/2^j \rfloor = \lfloor r/2^j \rfloor$ for all $j \in \mathbb{N}$. $\square$

Since we are considering polynomial over $\mathbb{Z}_2$. Binary expansion are important for determinate odd multinomial coefficient. We shall view binary expansion of $r \in \mathbb{N}$ as a sequence in $\vec{b} = (b_0, b_1, b_2, \ldots) \in \{0,1\}^\omega \subseteq \mathbb{N}^\omega$, such that $r = \sum_{i=0}^\infty 2^i b_i$. For all $r \in \mathbb{N}$, the binary expansion of r is unique. Even though binary expansion is a sequence $\vec{b} \in \{0,1\}^\omega$, we shall still view binary expansion as vector over $\mathbb{N}^\omega$, and define the addition where for $\vec{a} = (a_i)_{i\in\mathbb{N}}$ and $\vec{b} = (b_i)_{i\in\mathbb{N}}$, we have $\vec{a} + \vec{b} = (a_i + b_i)_{i\in\mathbb{N}}$. However, the binary map from $\mathbb{N}$ to $\mathbb{N}^\omega$ is not surjective anymore.

Definition 2.2.5. Let $r \in \mathbb{N}$. We shall define $\varphi : \mathbb{N} \to \mathbb{N}^\omega$ such that $\varphi(r)$ is the binary expansion of r, for all $r \in \mathbb{N}$. We shall define $\psi : \mathbb{N}^\omega \to \mathbb{N}$ such that for all $\vec{b} = (b_i)_{i\in\mathbb{N}}$, we have $\psi(\vec{b}) = \sum_{i=1}^{\infty} 2^i b_i$.

Notice that φ is injective and $\mathrm{Im}(\varphi) = \bigcup\limits_{n\in\mathbb{N}} \{0,1\}^n$. Also, we have $\psi\mid_{\mathrm{Im}(\varphi)}$ is bijective.

Definition 2.2.6. Let $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega$. We say that $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ if and only if $a_i < b_i$ for all $i \in \mathbb{N}$.

Theorem 2.2.7. *Let $r \in \mathbb{Z}^+_{even}$. Let $r_1, \ldots, r_n \in \mathbb{N}$, such that $\sum_{i=1}^n r_i = r$. Then $2 \nmid \binom{r}{r_1,\ldots,r_n}$ if and only if $\sum_{i=1}^n \varphi(r_i) = \varphi(r)$.*

Proof. ($\Rightarrow$) Suppose that $2 \nmid \binom{r}{r_1,\ldots,r_n}$. By Lemma 2.2.3, we have $\sum_{i=1}^n \nu(r_i) = \nu(r)$. Let $\sum_{i=1}^n \varphi(r_i) = (a_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega$ and $\varphi(r) = (b_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega$. Suppose that $\sum_{i=1}^n \varphi(r_i) \neq \varphi(r)$. Notice that $\psi(\sum_{i=1}^n \nu(r_i)) = r$, but $(a_i)_{i\in\mathbb{N}}$ is not the binary expansion sequence. Then, there we have $\sum_{i=1}^\infty a_i < \sum_{i=1}^\infty b_i$. Then, we have

$$\sum_{i=1}^n \nu(r_i) = \sum_{i=1}^n a_i(2^i - 1) = \sum_{i=1}^\infty a_i 2^i - \sum_{i=1}^\infty a_i < \sum_{i=1}^\infty b_i 2^i - \sum_{i=1}^\infty b_i = \nu(r).$$

This contradiction, because $\sum_{i=1}^n \nu(r_i) = \nu(r)$.

($\Leftarrow$) Suppose that $\sum_{i=1}^n \varphi(r_i) = \varphi(r)$. Let $\sum_{i=1}^n \varphi(r_i) = (a_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega$ and $\varphi(r) = (b_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega$. Then we have the equality $\sum_{i=1}^\infty a_i < \sum_{i=1}^\infty b_i$. Hence, we have

$$\sum_{i=1}^n \nu(r_i) = \sum_{i=1}^n a_i(2^i - 1) = \sum_{i=1}^\infty a_i 2^i - \sum_{i=1}^\infty a_i = \sum_{i=1}^\infty b_i 2^i - \sum_{i=1}^\infty b_i = v(r).$$

By Lemma 2.2.3, we have $2 \nmid \binom{r}{r_1,\ldots,r_n}$. $\square$

Here are some examples. Consider the $r = 263$. Notice that we could write $242 = 128 + 64 + 32 + 16 + 2$, which corresponds to the binary sequence $\vec{b} = (0, 1, 0, 0, 1, 1, 1, 1, 0, \ldots)$. Now we could choose $r_1, \ldots, r_n$ whose binary sequence sum up to, for example the following set of sequence

$$\vec{r_1} = (0,1,0,0,0,0,0,0,0,\ldots) \quad \rightarrow \quad r_1 = 2.$$

$$\vec{r_2} = (0,0,0,0,1,0,1,0,0,\ldots) \quad \rightarrow \quad r_2 = 16 + 64 = 80.$$

$$\vec{r_3} = (0,0,0,0,0,1,0,0,0,\ldots) \quad \rightarrow \quad r_3 = 32.$$

$$\vec{r_4} = (0,0,0,0,0,0,0,1,0,\ldots) \quad \rightarrow \quad r_4 = 128.$$

Then, we have $\binom{263}{2,80,32,128} = 16334714770361286933187424916658815345594549734348689954292025952790918104236217325568597502619767727715$, which is odd.

Corollary 2.2.8. *Let $k \in \mathbb{N}$. Then $\binom{2^k}{r_1,\ldots,r_n}$ is odd if and only if $\binom{2^k}{r_1,\ldots,r_n} = \binom{2^k}{2^k} = 1$.*

Proof. Notice that 2^k corresponds to the binary sequence $\vec{b} = (0,0,,\ldots,0,1,0,\ldots)$, i.e. there is a 1 on the k^{th} position and 0 otherwise. Then, the only sequence that is less than equal to $\vec{b}$ is either $\vec{b}$ or $\vec{0}$, in $\mathbb{N}^{\{\omega\}}$. Hence, we have $\binom{2^k}{r_1,\ldots,r_n}$ if and only if $\binom{2^k}{r_1,\ldots,r_n} = \binom{2^k}{2^k} = 1$. □

Intuitively, when we consider the polynomial $(x_1 + \ldots + x_n)^{2^k}$ in $\mathbb{Z}_2[x_1, x_2, \ldots, x_n]$, it is equivalent to consider $x_1^{2^k} + \cdots + x_n^{2^k}$.

2.3 Cascading Makeev with Greedy Algorithm

In this section, we shall study hyperplane equipartition with cascading Makeev, given that $\vec{\ell} = (2,2,\ldots,2,0,\ldots,0)$. We will study and compute the minimal dimension such that the polynomial condition does not vanishes. Given $\mathcal{H} = \{H_1, \ldots, H_2\}$ a set of hyperplane, any two of among $\mathcal{H}$ equipartition a mass could be represented by the polynomial $(\prod_{i=1}^n x_i)(\prod_{1 \le i < j \le n}(x_i + x_j))$. Recalling the Vandermonde determinate *(Equation 1.7.2)*.

$$\prod_{1 \le g_1 + \cdots + g_k \le 2} \mathfrak{P}(g_1, g_2, \ldots, g_k) = \begin{vmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \cdots & x_k^k \end{vmatrix} = \sum_{\sigma \in S_k} x_{\sigma(1)}^k x_{\sigma(2)}^{k-1} \cdots x_{\sigma(k-1)}^2 x_{\sigma(k)}.$$

It is early to handle the polynomial when we consider the sum of as a sum. We shall use Proposition Notice that S_n acts on the index. We know that $|S_n| = n!$, such that $2 \mid |S_n|$. We shall first study the simplest case, when there are k hyperplanes $\mathcal{H} = \{H_1, \ldots, H_k\}$ with vector $\vec{m} = (r, 0, 0, \ldots, 0)$.

Theorem 2.3.1. *Let $\vec{m} = (r, 0, 0, \ldots, 0) \in \mathbb{N}^k$ be a vector of measures, for some $r \in \mathbb{N}$. Let $r = 2^{r_1} + 2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}$ be the binary expansion, such that $r_i > r_{i+1}$. Then the polynomial representation for $(\mathcal{H}, \vec{\ell}, \vec{m})$ does not vanishes in $k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$ dimension.*

Proof. Consider the polynomial representation for $\vec{m} = (r, 0, \ldots, 0)$

$$\left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x^k_{\sigma(k)} \right)^r = \prod_{i=1}^{n} \left(\sum_{\sigma \in S_k} x^{2^{t_i}}_{\sigma(1)} \cdots x^{k2^{t_i}}_{\sigma(k)} \right) \tag{2.3.1}$$

Notice that $r_1 > r_2 > \cdots > r_n$. Based on the binary expansion, we have $2^{r_1} > 2^{r_2} + \cdots + 2^{r_n}$. We have

$$A = (x_1^{k2^{r_1}} x_2^{(k-1)2^{r_1}} \cdots x_k^{2^{r_1}}) \prod_{i=2}^{n} (x_1^{2^{r_i}} x_2^{2 \cdot 2^{r_i}} \cdots x_k^{k2^{r_i}}) = \prod_{i=1}^{k} x_i^{(k-i+1)2^{r_1} + i(2^{r_2} + \cdots + 2^{r_n})},$$

which is non-vanishing in dimension $k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$. Notice that such polynomial could be obtained in $\binom{r}{2^{r_1}, 2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}} + C$ different ways to obtain this polynomial, where $C \geq 0$ is some sum of multinomial coefficient.

We shall show that $\binom{r}{2^{r_1}, 2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}} + C$ is odd. Claim that C is sum of even multinomial coefficient, i.e. the multinomial coefficient $\binom{r}{2^{r_1}, 2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}}$ is the unique odd coefficient. Let $\binom{r}{s_1, \ldots, s_m}$ be such multinomial coefficient. By the Theorem 2.2.7, we know that multinomial coefficient is odd if and only if $\sum_{i=1}^{m} \varphi(s_i) = \varphi(r)$. It is equivalent to consider what are $\sigma_1, \ldots, \sigma_m \in S_k$ such that

$$\prod_{i=1}^{m} \left(x_{\sigma_i(1)} x^2_{\sigma_i(2)} \cdots x^k_{\sigma_i(k)} \right)^{2^{r_i}} = A.$$

Claim that $\sigma_1(t) = k - t + 1$ and $\sigma_j(t) = t$ for all $j \in \{2, \ldots, m\}$. Proof by finite induction on t. Base case $t = k$, consider $x^{k2^{r_1}}_{\sigma_1(k)}$. We have $x_1^{k2^{r_1} + (2^{r_2} + \cdots + 2^{r_n})}$ is only term that has power higher then $k2^{r_1}$. Hence we have $\sigma_1(k) = 1$, and $\sigma_2(1) = \sigma_3(1) = \cdots = \sigma_m(1) = 1$. Hence, the base case holds. Induction step, suppose that $\sigma_1(t) = k - t + 1$ and $\sigma_j(t) = t$, for all $1 \neq N \leq t \leq k$. We shall show that $N - 1$ holds. If $N - 1 = 1$, we are done. Suppose that $N - 1 \neq 1$. Then, consider $x^{(N-1)2^{r_1}}_{\sigma_1(N-1)}$. Notice that $x^{(N-1)2^{r_1} + (k-N+2)(2^{r_2} + \cdots + 2^{r_n})}_{k-N+2}$ is the only remain terms that has power higher then $(N-1)2^{r_1}$. Hence, we have $\sigma_1(N-1) = k - N + 2$ and $\sigma_2(N-1) = \cdots = \sigma_m(N-1) = N-1$. But this is exactly corresponds to the multinomial coefficient $\binom{r}{2^{r_1}, 2^{r_2} + \cdots + 2^{r_n}}$. Hence, we have C is even, and $\binom{r}{2^{r_1}, 2^{r_2} + \cdots + 2^{r_n}}$ is the unique odd multinomial coefficient. Therefore, we have the polynomial representation does not vanishes. $\square$

Notice that the polynomial condition of $\vec{m} = (r, 0, \ldots, 0)$ is $\prod_{i=1}^{k} x_i^{(k-i+1)2^{r_1} + i(2^{r_2} + \cdots + 2^{r_n})}$, which is not tight in dimension $k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$. Notice that the power on the term is $\prod_{i=1}^{k} x_i^{(k-i+1)2^{r_1} + i(2^{r_2} + \cdots + 2^{r_n})}$ arithmetic sequence, decreases by $2^{r_1} - (2^{r_2} + 2^{r_3} + \cdots + 2^{r_n})$. Let $S(n)$ denote as the n^{th} ***triangular number***. Then, we have room for $S(k-1) \cdot (k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n})$ conditions. Naturally, we shall consider adding more masses to the vector $\vec{m} = (r, 0, 0, \ldots, 0)$.

Corollary 2.3.2. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$, for some $k \in \mathbb{N}$. Let $r \in \mathbb{N}$, with the binary expansion $r = \sum_{i=1}^{n} 2^{r_i}$. Then $2^{r_1} + \sum_{i=2}^{n} 2^{r_i}$ is the minimal dimension, such that polynomial condition of $\vec{m} = (r, 0, \ldots, 0)$ does not vanishes.*

Proof. By Theorem 2.2.7, for each $i \in \{1, \ldots, n\}$ we shall choose 2^{r_i} copies of a term from the sum $\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)}^k$. We shall write

$$\left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)}^k \right)^r = \prod_{i=1}^{n} \left(\sum_{\sigma \in S_k} x_{\sigma(1)}^{2^{r_i}} \cdots x_{\sigma(k)}^{k2^{r_i}} \right).$$

Without loss of generality, we shall fix $x_1^{k2^{r_1}}$, then, the minimal power can only be $x_1^{k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}}$, when we multiply out. $\square$

Now, we shall consider adding more mass to the vector $\vec{m} = (r, 0, \ldots, 0)$. By Corollary 2.3.2, we cannot increases r. However, we could increases the second coordinate, and consider the $\vec{m} = (r, s, 0, \ldots, 0)$, with $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ hyperplanes, in dimension $k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$, where $r = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$.

Theorem 2.3.3. *Let $k \geq 2$. Let $r \in \mathbb{N}$, with binary expansion $r = \sum_{i=1}^{n} 2^{r_i}$. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a collection of hyperplane. Let $s = 2^{r_1} - (2^{r_2} + \cdots + 2^{r_n} + 1)$. Then $\vec{m} = (r, s, 0, \ldots, 0)$ does not vanishes in dimension $k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$.*

Proof. Let $d = k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$. Notice that the corresponding polynomial condition for $\vec{m} = (r, s, 0, \ldots, 0)$ is represented by

$$\underbrace{\left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x^k_{\sigma(k)}\right)^r}_{A} \underbrace{\left(\sum_{\gamma \in S_{k-1}} x_{\gamma(1)} \cdots x^{k-1}_{\gamma(k-1)}\right)^s}_{B}$$

Consider expanding A, every term consist highest power d after the expansion. Thus, if a term done not consists x_k^d in A, then there exists some x_i^d, for some $i \in \{1, \ldots, k\}$. When we multiply out any terms in B, it must consists x_i^a, for some $a \geq 1$. Then, we have x_i^{d+a} has degree greater than d, hence vanishes. Therefore, every non-vanishing term must consists x_k^d, hence we could factor out x_k^d, where

$$\begin{aligned}
&\underbrace{\left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x^k_{\sigma(k)}\right)^r}_{A} \underbrace{\left(\sum_{\gamma \in S_{k-1}} x_{\gamma(1)} \cdots x^{k-1}_{\gamma(k-1)}\right)^s}_{B} \\
=&\prod_{i=1}^{n} \left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x^k_{\sigma(k)}\right)^{2^{r_i}} \underbrace{\left(\sum_{\gamma \in S_{k-1}} x_{\gamma(1)} \cdots x^{k-1}_{\gamma(k-1)}\right)^s}_{B} \\
=&x_k^d \left(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x^{k-1}_{\sigma(k-1)}\right)^{2^{r_1}} \underbrace{\prod_{i=2}^{n} \left(\sum_{\sigma \in S_k} x^2_{\sigma(1)} \cdots x^k_{\sigma(k-1)}\right)^{2^{r_i}}}_{C} \underbrace{\left(\sum_{\gamma \in S_{k-1}} x_{\gamma(1)} \cdots x^{k-1}_{\gamma(k-1)}\right)^s}_{B}. \\
=&x_k^d \underbrace{\prod_{i=2}^{n} (x_1 \cdots x_{k-1})^{2^{r_i}}}_{\text{factor of } C} \underbrace{\left(\sum_{\gamma \in S_{k-1}} x_{\gamma(1)} \cdots x^{k-1}_{\gamma(k-1)}\right)^{s+r}}_{D}.
\end{aligned}$$

Notice that $\varphi(s + r) = (1, 1, 1, \ldots, 1, 0, 0, \ldots)$, where there are exactly $r_1 + 1$ many leading 1. Then, we shall rename binary decomposition of $s+r$ as $2^{r_1} + 2^{r_1 - 1} + \cdots + 2^0$. By Theorem 2.2.7, we shall consider the term when expanding D:

$$(x_1 x_2 \cdots x^{k-1}_{k-1})^{2^{r_1}} \prod_{i=2}^{r_1} (x_1^{k-1} x_2^{k-2} \cdots x^1_{k-1})^{2^{r_1 - i}}.$$

Similar to Theorem 2.3.1, this term can be shown that it does not vanishes. (By finite induction.) Altogether, we have the term

$$\left(x_k^d \prod_{i=2}^{n} (x_1 \cdots x_{k-1})^{2^{r_i}}\right) (x_1 x_2 \cdots x^{k-1}_{k-1})^{2^{r_1}} \prod_{i=2}^{r_1} (x_1^{k-1} x_2^{k-2} \cdots x^1_{k-1})^{2^{r_1 - i}}.$$

Notice that $2^{r_1} - (2^{r_1-1} + 2^{r_1-2} + \cdots + 2^1 + 1) = 1$. Observe that the power is arithmetic sequence:

$$\begin{aligned}
\underline{(2^{r_2} + \cdots + 2^{r_n})} + k2^{r_1} &= d;\\
\underline{(2^{r_2} + \cdots + 2^{r_n})} + (k-1)2^{r_1} + \sum_{i=1}^{r_1} 2^{r_1-i} &= d-1;\\
\underline{(2^{r_2} + \cdots + 2^{r_n})} + (k-2)2^{r_1} + 2\sum_{i=1}^{r_1} 2^{r_1-i} &= d-2;\\
&\vdots\\
\underline{(2^{t_2} + \cdots + 2^{t_n})} + 2^{r_1} + k\sum_{i=1}^{r_1} 2^{r_1-i} &= d-k+1.
\end{aligned}$$

(The underlined section is a constant.) Therefore, this term is has odd coefficient and has highest degree d. Hence, it does not vanishes. □

We can conclude few special cases, by using the Theorem 2.3.3. These special cases are simpler, could be proved by using different method. However, these Corollary are also implied by the previous theorem, through few observations.

Corollary 2.3.4. *Let $\vec{m} = (2^t, 2^t, 0, \ldots, 0) \in \mathbb{N}^k$ be a vector of measures, for some $t \in \mathbb{N}$. The polynomial representation for $(\mathcal{H}, \vec{\ell}, \vec{m})$ vanishes in $k2^t$-dimension, unless $\vec{m} = (2^t, 2^t) \in \mathbb{N}^2$. Further more, the result it tight when $k = 2$.*

Proof. Let $\vec{m} = (r, s, 0, 0, \ldots, 0)$. Let $r = 2^{r_1} + \cdots + 2^{r_n}$. Suppose that we are in $d = k2^{r_1} + (2^{r_2} + \cdots + 2^{r_n})$ dimension. By Theorem 2.3.3, we have $s \leq 2^{r_1} - (2^{r_2} + 2^{r_3} + \cdots + 2^{r_n} + 1)$. Notice that

$$r = 2^{r_1} + \underbrace{2^{r_2} + \cdots + 2^{r_n}}_{A} > 2^{r_1} - (\underbrace{2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}}_{A} + 1) = s.$$

Then, we have $\vec{m} = (2^t, 2^t, 0, 0, \ldots, 0)$ vanishes. However, when we consider the case when $k = 2$, we could simply compute the corresponding polynomial condition

$$(x_1x_2^2 + x_2x_1^2)^{2^t}(x_2)^{2^t} = (x_1^{2^t}x_2^{2\cdot 2^t} + x_1^{2\cdot 2^t}x_2^{2^t})x_2^{2^t} \equiv x_1^{2\cdot 2^t}x_2^{2\cdot 2^t} \qquad \text{in } \mathbb{Z}_2[x_1, x_2]/(x_1^{2\cdots 2^t}, x_2^{2\cdots 2^t}).$$

Therefore, we $k = 2$ is a special case. □

Corollary 2.3.5. *Let $k > 2$. Let $t, r \in \mathbb{N}$. Let $\vec{m} = (2^t, s, 0, \ldots, 0) \in \mathbb{N}^k$ be a vector of measures. The polynomial representation for $(\mathcal{H}, \vec{\ell}, \vec{m})$ non-vanishes in $k2^t$-dimension, if $s \leq 2^t - 1$.*

Proof. Let $\vec{m} = (r, s, 0, 0, \dots, 0)$. Let $r = 2^{r_1} + \cdots + 2^{r_n}$. Suppose that we are in $d = k2^{r_1} + (2^{r_2} + \cdots + 2^{r_n})$ dimension. By Theorem 2.3.3, we have $s \leq 2^{r_1} - (2^{r_2} + 2^{r_3} + \cdots + 2^{r_n} + 1)$. Notice that

$$r = 2^{r_1} + \underbrace{2^{r_2} + \cdots + 2^{r_n}}_{A} > 2^{r_1} - (\underbrace{2^{r_2} + 2^{r_3} + \cdots + 2^{r_n}}_{A} + 1) = s.$$

Now, we shall observe that the term $A \geq 0$. Hence, we have

$$2^{r_1} - 1 \geq 2^{r_1} - 1 - A = s,$$

with equality, when $A = 0$. □

Example 2.3.6. Any vector $\vec{m} = (2, 1, 0, 0, \dots, 0)$, will not vanish in $2k-$dimension.

Proof. Consider polynomial condition for $\vec{m} = (2, 1, 0, 0, \dots, 0)$ is given by

$$\left(\sum_{\sigma \in S_k} x^2_{\sigma(1)} x^4_{\sigma(2)} x^6_{\sigma(3)} \cdots x^{2k}_{\sigma(k)} \right) \left(\sum_{\tau \in S_k} x_{\tau(2)} x^2_{\tau(3)} x^3_{\tau(4)} \cdots x^{k-1}_{\tau(k)} \right).$$

After expending the polynomial, there is a unique term obtained by $(x_1^{2k} x_2^{2k-2} \cdots x_k^2)(x_2 x_3^2 \cdots x_k^{k-1})$ can be unique obtained and does not vanish in $\mathbb{Z}_2[x_1, \dots, x_k]/(x_1^{2k+1}, \dots, x_k^{2k+1})$. □

We shall continuous the process, i.e. add more mass to the vector $\vec{m} = (r, s, 0, \dots, 0)$ greedily under the same dimension $d = k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$, mentioned in Theorem 2.3.1. Unfortunately, the polynomial gets complicated. Even through the statement are not very complicated, proving it might require some different observation. The following theorem uses a relatively simple argument by combinatoric methods.

Lemma 2.3.7. *Let $\vec{m} = (r, s, 0, 0, \dots, 0)$ be a vector of mass in $\mathbb{N}^k$. Consider expanding the corresponding polynomial condition $P \in \mathbb{Z}_2[x_1, \dots, x_k]/(x_1^d, \dots, x_k^d)$. If A be a non-vanishing term after the expansion, then $A = x_1^d x_2^{a_2} \cdots x_k^{a_k}$, where $a_i \neq a_j$ for all $i, j \in \{2, \dots, k\}$.*

Proof. Proof by contradiction, Suppose that $a_i = a_j$ for some $i, j \in \{2, \dots, k\}$. Since we know that A is a term of the product

$$\left(\prod_{\sigma \in S_k} x_{\sigma(1)} x^2_{\sigma(2)} \cdots x^k_{\sigma(k)} \right)^r \left(\prod_{\sigma \in S_{k-1}} x_{\sigma(2)} x^2_{\sigma(3)} \cdots x^{k-1}_{\sigma(k)} \right)^s.$$

Since we $a_i = a_j$, we could interchange the index x_i and x_j while choose the terms from the product. Hence, in additional there is $|S_2|$ acting on the index. Hence, we have $|S_2| = 2$ divides the coefficient, i.e. there are even ways to chooses this particular term. Hence, we have A vanishes in $\mathbb{Z}_2$. $\square$

The Lemma 2.3.7 shows that any non-vanishing term cannot have repeated power, *(for example the term $x_1^5 x_2^4 x_3^4 x_4^2$ will definitely has even coefficient.)*

Theorem 2.3.8. *Let $k \geq 5$. Let $r = 2^{r_1} + \cdots + 2^{r_n}$, $s = 2^{r_1} - (2^{r_2} + \cdots + 2^{r_n} + 1)$, and $t \in \mathbb{N}$. Let $\vec{m} = (r, s, t, 0, \ldots, 0)$ be a vector of mass in $\mathbb{N}^k$. Consider dimension $d = k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$. Suppose that the polynomial condition for $\vec{m} = (r, s, t, 0, \ldots, 0)$ does not vanishes, then $t = 0$.*

Proof. First, we shall recall the result from Theorem 2.3.3, such that any non-vanishing term of given polynomial representation satisfies the following condition:

1) Non-vanishing term the power for x_1 must be d, otherwise vanishes.

2) The total degree is $d + (d-1) + (d-2) + \cdots + (d-k+1)$. There are totally $1 + 2 + \cdots + k - 1 = S(k-1)$ degree remain.

Suppose that $t \geq 1$. We shall consider the minimal case when $t = 1$, where we are adding $S(k-2)$ many conditions. Hence, there are $S(k-1) - S(k-2) = k - 1$ degree left. We shall proof by contradiction. Suppose that there exists a non-vanishing term of A the polynomial condition $\vec{m} = (r, s, 1, 0, \ldots, 0)$. Since $t = 1$ does not relates to the the first hyperplane H_1. Then x_1 of the term A has degree d. Without loss of generality, suppose that x_2 of A has the highest degree d, as well. This implies that $x_3, x_4, \ldots, x_k$ has total degree $(k-2)d - (k-2) = (k-2)(d-1)$. We shall recall that $k \geq 5$, and $d = k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n} \geq k$. By Lemma 2.3.7, we have $x_3, x_4, \ldots, x_k$ must have distinct power, i.e. the maximum degree we could have is $d, d-1, d-2, \ldots, d-(k-2)$. There are $S(k-2)$ degree left, but $S(k-2) > k - 1$, for all $k \geq 5$. Therefore, the term A must vanishes, by Lemma 2.3.7. This is a contradiction. $\square$

Theorem 2.3.8 tells us that if the vector $\vec{m} = (r, s, 0, 0, \dots, 0)$ is long enough, i.e. the number of hyperplanes is large. Then, the given value $r = 2^{r_1} + \cdots + 2^{r_n}$ and $s = 2^{r_1} - (2^{r_2} + \cdots + 2^{r_n} + 1)$, is "temporarily" tight in the given dimension $d = k2^{r_1} + 2^{r_2} + \cdots + 2^{r_n}$, *(i.e. If any* 2 *of* $\mathcal{H}$ *equipartition* r *masses, and any* 2 *of* $\mathcal{H}_2$ *equipartition* s *masses, we cannot impose any Makeev condition on* $\mathcal{H}_3$*).* However, the final polynomial condition is

$$\sum_{\sigma \in S_k} x^d_{\sigma(1)} x^{d-1}_{\sigma(2)} \cdots x^{d-k+1}_{\sigma(k)} = x_1^{d-k} x_2^{d-k} \cdots x_3^{d-k} \underline{\sum_{\sigma \in S_k} x^k_{\sigma(1)} x^{k-1}_{\sigma(2)} \cdots x_{\sigma(k)}} \tag{2.3.2}$$

Observe that the underline section is exactly polynomial condition for the traditional Makeev problem, i.e. any two of $\mathcal{H}$ equipartition the given mass. Moreover, we have total $S(k-1)$ degrees of freedom. This implies that we could add more conditions. It is natural to ask, given $\vec{m} = \{1, 0, 0, \dots, 0\}$, where could impose an other Makeev condition? How many masses could any 2 of $\mathcal{H}_t$ equipartition?

Proposition 2.3.9. *Let* $\vec{m} = (1, 0, \dots, 0, 1, 0, \dots, 0)$ *with length* k*. The corresponding polynomial is non-vanishing when the second* 1 *has position greater than* $\lceil k/2 \rceil$*.*

Proof. Consider polynomial condition for $\vec{m} = (1, 0, \dots, 0, 1, 0, \dots, 0)$ is given by

$$\left(\sum_{\sigma \in S_k} x_{\sigma(1)} x^2_{\sigma(2)} x^3_{\sigma(3)} \cdots x^k_{\sigma(k)} \right) \left(\sum_{\tau \in S_{k-m}} x_{\tau(m)} x^2_{\tau(m+1)} x^3_{\tau(m+2)} \cdots x^{k-m}_{\tau(k)} \right).$$

Consider the polynomial obtained by $(x_1^k x_2^{k-1} \cdots x_k)(x_m^{k-m} x_{m+1}^{k-m-1}, \dots, x_k)$ is unique and does not vanish. $\square$

Conjecture 2.3.10. *Let* $\vec{m} = (1, 0, \dots, 0, 1, 0, \dots, 0)$ *with length* k*. The corresponding polynomial condition vanishes, when the second* 1 *has position less than* $\lceil k/2 \rceil$*.*

3

Cascading Makeev with $\vec{\ell} \in \{0, 1, 2, 3\}^k$

In this chapter, we are going to study when $\vec{\ell} \in \{0, 1, 2, 3\}^k$, for $k = |\mathcal{H}|$ is the number of hyperplanes. Having 3 in the vector ℓ means "any three subsets of collection of hyperplane equipartition the given set of masses in $\vec{m}$."

3.1 Computation Example for $\vec{\ell} = (3, 0, 0, 0)$

We shall consider the collection of hyperplane $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. We shall consider the case when $\vec{\ell} = (3, 0, 0, 0)$ and $\vec{m} = (1, 0, 0, 0)$, i.e. any three subset of $\mathcal{H}$ equipartition the mass. We shall compute polynomial condition explicitly.

First, we shall relax the condition of dimension, by assuming that $d \in \mathbb{N}$ is arbitrarily large. Since we are considering $\mathbb{Z}_2[x_1, x_2, x_3, x_4]/\{x_1^d, x_2^d, x_3^d, x_4^d\}$, because d is large enough, then any terms of the polynomial wouldn't vanish due to the degree. Hence it is equivalent to consider the polynomial under the ring $\mathbb{Z}_2[x_1, x_2, x_3, x_4]$. Notice that the polynomial representation is

$$\underbrace{x_1x_2x_3x_4 \prod_{1 \le i < j \le 4} (x_i + x_j)}_{A} \underbrace{\prod_{1 \le i < j < k \le 4} (x_i + x_j + x_k)}_{B}.$$

We shall observe that the product A is the Vandermonde determinate, where we could write as

$$A = \sum_{\sigma \in S_4} x_{\sigma(1)} x_{\sigma(2)}^2 x_{\sigma(3)}^3 x_{\sigma(4)}^4. \tag{3.1.1}$$

Now we shall expand B, where we have

$$\begin{aligned}
B =&(x_1^2 + x_2^2 + \underline{2x_1x_2} + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4) \\
=&(x_1^3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_4 + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 \\
&+ x_1^2x_3 + x_2^2x_3 + x_1x_3^2 + x_1x_3x_4 + x_2x_3^2 + x_2x_3x_4 + x_3^2x_4 \\
&+ x_1^2x_4 + x_2^2x_4 + x_1x_3x_4 + x_1x_4^2 + x_2x_3x_4 + x_2x_4^2 + x_3x_4^2)(x_2 + x_3 + x_4) \\
=&(x_1^3 + x_1x_2^2 + \underline{2x_1^2x_3} + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + \underline{2x_1^2x_4} + x_2^2x_4 + x_3^2x_4 + x_1x_4^2 + x_2x_4^2 + x_3x_4^2 \\
&+ x_1x_2x_3 + x_1x_2x_4 + 3x_1x_3x_4 + \underline{2x_2x_3x_4})(x_2 + x_3 + x_4) \\
=&(\underline{\underline{x_1^3x_2 + x_1x_2^3 + x_2^3x_3 + x_2^3x_4 + x_1^3x_2 + x_1x_3^3 + x_2x_3^3 + x_3^3x_4 + x_1^3x_4 + x_1x_4^3 + x_2x_4^3 + x_3x_4^3}} \\
&+ 2x_2^2x_3^2 + 2x_2^2x_4^2 + 2x_3^2x_4^2 \\
&+ 2x_1x_2x_3^2 + 2x_2x_3^2x_4 + 2x_1x_2x_4^2 + 2x_2x_3x_4^2 + 2x_1x_2^2x_3 + 2x_1x_2^2x_4 + 2x_2^2x_3x_4 + 2x_1x_3x_4^2 + 2x_1x_3^2x_4 \\
&+ 3x_1x_2x_3x_4)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
B = &x_1^3x_2 + x_1x_2^3 + x_2^3x_3 + x_2^3x_4 + x_1^3x_2 + x_1x_3^3 \\
&+ x_2x_3^3 + x_3^3x_4 + x_1^3x_4 + x_1x_4^3 + x_2x_4^3 + x_3x_4^3
\end{aligned} \tag{3.1.2}$$

We shall observe that the double-underlined sum has exactly 12 terms, and they are distinct. There are 12 different ways to choose two distinct elements from $\{1, 2, 3, 4\}$. Unfortunately, this is not action over S_4, i.e. it is not a sum over S_4. *(Because we know that $|S_4| = 24$, but the underlined summation does not have 24 terms.)* Here are we shall introduces the terminology of group stabilizer and group orbit [9, Chapter 4].

Definition 3.1.1. Let G be a group, such that G acts on a set X. The orbit of the element $x \in X$ is the set of elements in X, denote as $Orb_G(x)$, such that

$$Orb_G(x) := \{x^g \mid g \in G\}. \tag{3.1.3}$$

Furthermore, if $x^g = y$ for some $y \in X$, then, we say y is in the orbit of x under G.

Definition 3.1.2. Let G be a group, such that G acts on a set X. The stabilizer subgroup of, with respect to $x \in X$, denote as $Stab_G(x)$, such that

$$Stab_G(x) = \{g \in G \mid x^g = x\}. \tag{3.1.4}$$

Furthermore, if $x^g = x$, we say x is fixed point of g, or, the element g fixes x.

Proposition 3.1.3. *(Orbit Stabilizer Lemma.) Let G be a group which acts on a finite set X. Let $x \in X$. Then, we have*

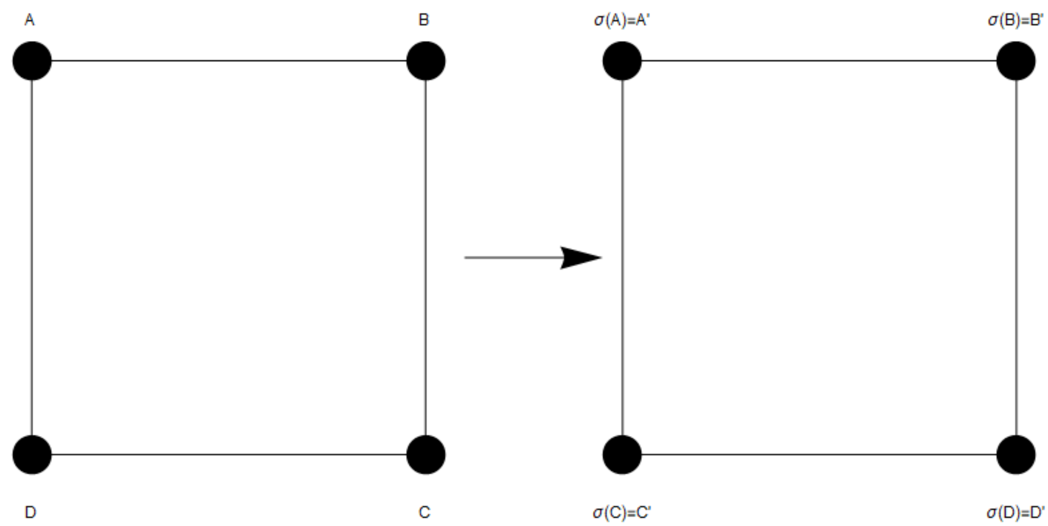
$$|Orb(x)| = \frac{|G|}{|Stab(x)|}.$$

Proof. Notice that $Stab(x) \leq G$. Now we shall consider the distinct right coset of $Stab(x)$, denote as C. Let $f : C \to Orb(x)$, such that $f(c) = x^c$. Claim that f is bijection. Notice that for all $g \in G$, we could write $g = sc$ for some $s \in Stab(x)$ and $c \in C$. Then, we have $x^g = x^{sc} = x^c$. Hence, each distinct right coset represents an element in $Orb(x)$. Thus, we have f is bijective. Notice that $|Orb(x)| = |C| = |G|/|Stab(x)|$. $\square$

In this section, we are going to talk about S_4. For simplicity, we shall omit the subscript, and use Orb(x) and Stab(x) instead. Now, we shall use the idea of orbit and stabilizer to generalize represent the sum above. Recalling that we use summation in (3.1.1). Notice that this is a summation over $\sigma \in S_4$. This mean that we are considering the group S_4. Notice that we are summing over product, such that S_4 is acting on the subscript. Observe that subscript could be viewed as permutation of $\{1, 2, 3, 4\}$, we shall denote $\mathcal{P}(\{1, 2, 3, 4\})$. We shall will element in $\mathcal{P}(\{1, 2, 3, 4\})$ as a order $4-$tuple. *(For example, the element* $(1, 2, 3, 4) \in \mathcal{P}(\{1, 2, 3, 4\})$*.)* So we have S_4 action over $\mathcal{P}(\{1, 2, 3, 4\})$, such that for $\sigma \in S_4$ we have $(a, b, c, d)^\sigma = (\sigma(a), \sigma(b), \sigma(c), \sigma(d))$.

Claim 3.1.4. *The action S_4 on the set $\{(a, b, c, d) \mid a \neq b \neq c \neq d;$ and $a, b, c, d \in \{1, 2, 3, 4\}\}$ is transitive.*

Proof. For (A, B, C, D) and $(A', B', C', D') \in \{(a, b, c, d) \mid a \neq b \neq c \neq d;$ and $a, b, c, d \in \{1, 2, 3, 4\}\}$ we shall consider following map, given by the picture below.



$\square$

In particular, for this sum above in (3.1.1), we the sum could be written as $(1, 2, 3, 4)^{S_4}$. Hence, we have

$$\sum_{\sigma \in S_4} x_{\sigma(1)} x^2_{\sigma(2)} x^3_{\sigma(3)} x^4_{\sigma(4)} = \sum_{(a,b,c,d) \in (1,2,3,4)^{S_4}} x_a x_b^2 x_c^3 x_d^4 \tag{3.1.5}$$

We shall observe that $Orb(1, 2, 3, 4)$ is has order 24. Hence, there are exactly 24 term in the right hand side of the equation.

Now, we shall take a look at the polynomial:

$$x_1^3 x_2 + x_1 x_2^3 + x_2^3 x_3 + x_2^3 x_4 + x_1^3 x_2 + x_1 x_3^3 + x_2 x_3^3 + x_3^3 x_4 + x_1^3 x_4 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3.$$

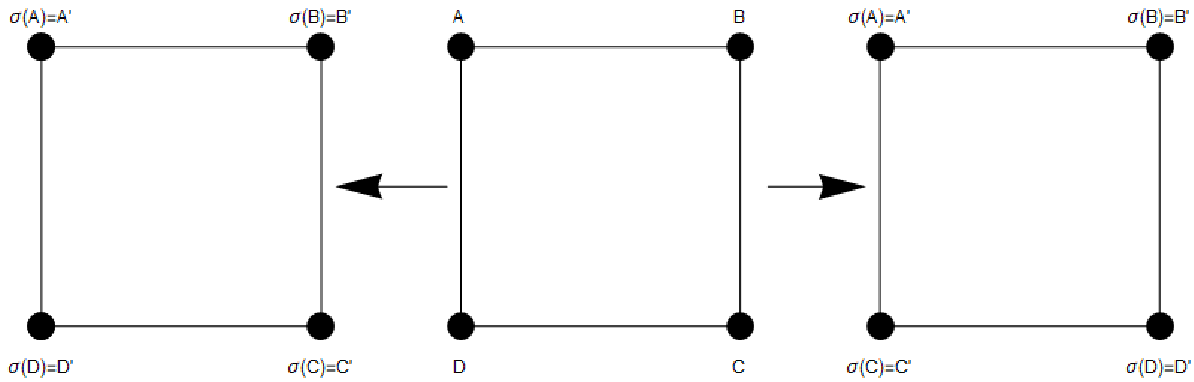
The set of subscript is $\{(a, b) \mid a \neq b \text{ and } a, b \in \{1, 2, 3, 4\}\}$. We know that there are exactly 12 distinct subscripts. Notice that S_4 act on the set of subscript, such that for subscript (a, b) and $\sigma \in S_4$, we have $(a, b)^\sigma = (\sigma(a), \sigma(b))$.

Claim 3.1.5. *The group S_4 acts on $\{(a, b) \mid a \neq b$ and $a, b \in \{1, 2, 3, 4\}\}$ transitively.*

Proof. We shall denote $\mathcal{S} = \{(a, b) \mid a \neq b \text{ and } a, b \in \{1, 2, 3, 4\}\}$. Since we know that $|S_4| = 24$, and $|\mathcal{S}| = 12$. We shall view $\mathcal{S}$ as restrictions of

$$\mathcal{S}' = \{(a, b, c, d) \mid a \neq b \neq c \neq d, \text{ and } a, b, c, d \in \{1, 2, 3, 4\}\}$$

(The $\mathcal{S}'$ is the labeling of square) into the first two coordinate. Then, in term for $\mathcal{S}'$ we shall consider $(A,B,C,D) \to (A',B',C',D')$ and $(A,B,C,D) \to (A',B',D',C')$.



Hence, we have the action is transitive. Also, there are exactly two element in S_4 that map $(A,B) \mapsto (A',B')$. *(An other interpretation is by using the orbit stabilizer Lemma 3.1.3, there are exactly two element in S_4 fix (A,B). Then, we have $|Orb(A,B)| = |S_4|/|Stab(A,B)| = 12$, which is exactly the size of $\mathcal{S}$.)* $\square$

Hence, we have could write the sum B as

$$B = \sum_{(a,b)\in(1,2)^{S_4}} x_a^3 x_b$$

Now, we could compute the product AB. Where we could write

$$\begin{aligned} AB &= \Big(\sum_{\sigma\in S_4} x_a x_b^2 x_c^3 x_d^4\Big)\Big(x_1x_2x_3x_4 + \sum_{(a,b)\in Orb(1,2)} x_a^3 x_b\Big) \\ &= 3\sum_{\sigma\in S_4} x_{\sigma(1)}^2 x_{\sigma(2)}^3 x_{\sigma(3)}^4 x_{\sigma(4)}^5 + \sum_{\sigma\in S_4} x_{\sigma(1)} x_{\sigma(2)}^2 x_{\sigma(3)}^5 x_{\sigma(4)}^6 + \sum_{\sigma\in S_4} x_{\sigma(1)} x_{\sigma(2)}^3 x_{\sigma(3)}^4 x_{\sigma(4)}^6 + \\ &+ \sum_{\sigma\in S_4} x_{\sigma(1)} x_{\sigma(2)}^2 x_{\sigma(3)}^4 x_{\sigma(4)}^7. \end{aligned} \tag{3.1.6}$$

(Remark, we since we are summing over $(a,b) \in Orb(1,2)$, we shall split into different cases, when we expanding the product. Again, if some of the power for $x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}$ are equal, when we sum over S_4 it vanishes. Hence, we are only considering the product that give distinct power, which are the for terms above.) So far, we only consider the case when the coefficient is odd. Now, we shall consider the dimension constrain. In particular, we shall observe that AB

does not vanish when $d = 5$. We could still ask the question if we can throw in anther layer of cascade condition. We have

$$AB = \sum_{\sigma \in S_4} x^2_{\sigma(1)} x^3_{\sigma(2)} x^4_{\sigma(3)} x^5_{\sigma(4)} \in \frac{\mathbb{Z}_2[x_1, x_2, x_3, x_4]}{(x_1^6, x_2^6, x_3^6, x_4^6)}.$$

Appendix A
Computer Programming

Sometime it is hard to see the pattern without examples. Also, it is good to check solutions by some examples. However, it is computationally intensive to expand all the terms by hand. On the other hand, it is way faster to let a computer to do the job.

A.1 Coding in Mathematica for Polynomial Condition

The following piece of code is a function *HyperMain* where given $\{x_i, x_{i+1}, \ldots, x_j\}$ and $m \in \mathbb{Z}^+$ find the product $\prod_{i \le t_1 \le \cdots \le t_m \le j}(x_{t_1} + \cdots + x_{t_m})$.

```
HyperMain[min_, max_, m_] :=
Module[
     {vanish = 1, index = {}, indexM = {}, a = {}, b = {}, polys = {}},
     For[i = min, i <= max, i++, AppendTo[polys, Subscript[x, i]]];
     For[i = 1, i <= max - min + 1, i++, AppendTo[index, i]];
     For[i = min, i <= max, i++, AppendTo[a, 0]];
     index = Subsets[index, {m}];
     For[i = 1, i <= Length[index], i++,
          b = a;
          For[j = 1, j <= m, j++, Part[b, Part[Part[index, i], j]] = 1];
          AppendTo[indexM, b];
     ];
     For[i = 1, i <= Length[indexM], i++,
     vanish = vanish*indexM[[i]].polys];
     vanish
];
```

Example A.1.1.

$$HyperMain[1,4,3] = (x_1 + x_2 + x_3)(x_1 + x_2 + x_4) + (x_1 + x_3 + x + 4)(x_2 + x_3 + x_4);$$

$$HyperMain[1,4,2] = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 + x_4)(x_2 + x_4)(x_3 + x_4);$$

The following piece of code is a function *Equipartition* where given $\mathcal{H} = \{H_i, H_2, \ldots, H_k\}$ with m outputs the polynomial condition for for any m subset of $\mathcal{H}$ equipartition a measure. (*This function uses HyperMain.*)

```
Equipartition[min_, max_, m_] :=
Module[
     {vanish = 1, count = m},
     While[count != 0, vanish = vanish*HyperMain[min, max, count]; count--];
     vanish
]
```

Example A.1.2.

$$Equipartition[1,4,2] = x_1x_2x_3x_4(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 + x_4)(x_2 + x_4)(x_3 + x_4)$$

The following piece of code is a function *FullOrthogonality* where given $\mathcal{H} = \{H_1, \ldots, H_k\}$ a collection of hyperplanes, returns the Full orthogonality condition in the polynomial ring.

```
FullOrthogonality[min_, max_] :=
Module[{Mods = {}},
     For[i = min, i <= max, i++, AppendTo[Mods, Subscript[x, i]]];
     Equipartition[min, max, 2]/Apply[Times, Mods]
]
```

Example A.1.3.

$$FullOrthogonality[1,4] = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 + x_4)(x_2 + x_4)(x_3 + x_4)$$

The following piece of code is a function *Modding*, given the number of hyperplanes k and the dimension d return polynomial we should mod out, i.e. the set of polynomial $\{2, x_1^d, x_2^d, \ldots, x_k^d\}$.

```
Modding[k_, d_] :=
Module[
     {Mods = {2}},
     For[i = 1, i <= k, i++, AppendTo[Mods, Subscript[x, i]^(d + 1)]];
     Mods
]
```

Example A.1.4.

$$Modding[8,7] = \left\{2, x_1^8, x_2^8, x_3^8, x_4^8, x_5^8, x_6^8, x_7^8, x_8^8\right\}.$$

Here are few built in function we are going to use for computation. The function *PolynomialMod* is a built in function, that takes a polynomial and a set of modules, outputs the remained.

A.2 Coding with Cascading Makvee

Combining all the code above, this allowed us to compute the polynomial condition with Cascading Makvee condition. There are few sample computations.

Example 3.2.1. Let $\vec{m} = (2,1,0,1)$ and $\vec{\ell} = \{2,2,0,2\}$. Then, we have the corresponding polynomial condition is

$$\begin{aligned} &Equipartition[1,4,2]^2 Equipartition[2,4,2] Equipartition[4,4,2] \\ =& x_1^2 x_2^3 x_3^3 x_4^4 (x_1+x_2)^2 (x_1+x_3)^2 (x_2+x_3)^3 (x_1+x_4)^2 (x_2+x_4)^3 (x_3+x_4)^3 \end{aligned}$$

Suppose that we are in $\mathbb{R}^8$, then we are in the polynomial ring $\mathbb{Z}_2[x_1,x_2,x_3,x_4]/\{x_1^9,x_2^9,x_3^9,x_4^9\}$. Then,

$$\begin{aligned} &PolynomialMod[Equipartition[1,4,2]^2 * Equipartition[2,4,2] Equipartition[4,4,2], Modding[4,8]] \\ =& x_2^3x_3^8x_4^8x_1^8 + x_2^5x_3^6x_4^8x_1^8 + x_2^6x_3^5x_4^8x_1^8 + x_2^8x_3^3x_4^8x_1^8 + x_2^5x_3^7x_4^7x_1^8 + \\ & x_2^7x_3^5x_4^7x_1^8 + x_2^6x_3^7x_4^6x_1^8 + x_2^7x_3^6x_4^6x_1^8 + x_2^7x_3^8x_4^4x_1^8 + x_2^8x_3^7x_4^4x_1^8 \end{aligned}$$

Hence, when $d = 8$ the polynomial representation does not vanish. On the other hand, Suppose that we are in $\mathbb{R}^7$. Then, we are in the polynomial ring $\mathbb{Z}_2[x_1,x_2,x_3,x_4]/\{x_1^8,x_2^8,x_3^8,x_4^8\}$, where

$$PolynomialMod[Equipartition[1,4,2]^2 * Equipartition[2,4,2] Equipartition[4,4,2], Modding[4,8]] = 0,$$

Therefore, we can conclude that the minimal dimension with this particular set up is $d = 8$.

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