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Applications of Equivariant Topology In Cascading Makeev Problems

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Abstract

Many solutions to problems arising in discrete geometry have come from insights in equivariant topology. Configuration-Space/Test Map (CS/TM) type setups, pioneered by Živaljević, offer reductions of combinatorial or geometric facts to showing the nonexistence of certain G-equivariant maps $f : X \to V \setminus Z$. In particular, partitions of objects by arcs, planes, and convex sets, and Tverberg theorems have been particularly amenable to topological methods [1], since their solutions affect the global structure of the relevant topological objects. However, there have been limits to the method as demonstrated by a failure to solve of the celeberated and now settled Topological Tverberg conjecture [2] and, more generally, difficulty in finding sharp bounds for various conjectures. Nonetheless, we seek to employ characteristic classes, a cohomological invariant common to Borsuk-Ulam type problems, since these allow us to use explicit polynomial calculations to sharpen results to related problems. While determining sharp topological results for equipartition problems is a hard problem, there has been recent success in finding precise solutions by adding geometric constraints to the problem of plane equipartitions.[4] This suggests that the polynomial method still has its use in related problems, and employ these methods said results to"cascading Makeev" type problems.

Contents

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1
Introduction

This project is concerned at a fundamental level with how we can apply topological considerations to problems from discrete geometry. Heuristically the basic questions are as follows: given n dimensional masses in space, can we find some $n-1$ dimensional hyperplane that cuts each of their volumes in half simultaneously? If we instead consider k planes, can we find an "equipartition" so that each of the planes simultanously cut each mass into $\frac{1}{2^k}$ pieces? How about k planes equiparting m_1 masses and $k-1$ equiparting another m_2 masses?

These problems are generalizations of the famous Ham-Sandwich theorem, which gives a positive result for a bisection of k masses by a single hyperplane in $\mathbb{R}^k$. (see 3.1.2) However, these results are less geometrically surprising if one were to ask for bisections in dimension $k+1$. For example, it is unsurprising that one can bisect a mass in $\mathbb{R}^2$ with a single line (mean value theorem) or that 2 masses in $\mathbb{R}^3$ can be bisected simultaneously. Indeed, the widely open question is the following due to Grünman in 1960, but generalized by Ramos in 1996:

Question 1: *What is the minimum dimension d such that any m mass distributions $\mu_1, \ldots \mu_m$ on $\mathbb{R}^d$ can be simultaneously equiparted by k hyperplanes?*[1]

[1] all relevant definitions are made in 3

The primary methods to tackle such questions will arise from equivariant topology, which is a field that seeks to utilize notions of continuous maps, but also pays attention to symmetries that arise naturally from an essentially geometric problem. More formally, we consider some finite group G and build up the category whose objects are topological spaces X equipped with a G-action, and whose morphisms are G-equivariant continuous maps.

Specifically: given a geometric problem P, we define the *configuration space*, X, which parametrizes all associated solutions to the problem (such as points, lines, or arcs.) Additionally, we consider a *test space* $Z \subset V$ and a continuous map $f : X \to V$ where $p \in X$ is a solution to a problem if and only if $f(p) \in Z$. With this setup in mind, we further require that f be G-equivariant, where G acts on both X and Y. From this, the typical method of proof is to show the nonexistence of maps $f : X \to_G Y \setminus Z$, ensuring the existence of a geometric solution.

However, topological methods only allow us to demonstrate situations where we can *guarantee* a solution to a geometric problem, but it does not provide necessary conditions. In the traditional Ham-Sandwich theorem the topological upper bound for a solution to the problem is geometrically tight. This will not always be the case, so following S. Simon [4] we will also consider a variety of different geometric conditions, such as orthogonality, to drive the geometric lower bound up to the topological bounds we can find. It will turn out that the topological methods for these problems will be exactly analogous to the straight ahead equipartition problems by utilizing a representation-theoretic point of view.

The class of questions we consider generalize question 1, and were first introduced by Makeev [11] but generalized in [10]. One can ask for a stronger condition so that given any k hyperplanes $H_1, \dots H_k$, can we have any ℓ of k hyperplanes equiparting m_1 measures? We consider cases most notably when $\ell = 2, 3$ and prove bounds for this question with more than one measure, and also considering the "cascades

Finally, section 2 covers the algebraic topology and representation theory requisite to understand the full strength of methods we employ. However, if it is either familiar, or the reader is anxious to begin thinking about these problems, the reader is welcome to go directly to section 3. We also note that computations appearing in the `verbatim` environment were done in SageMath.

2
Preliminaries

2.1 Representation Theory For Finite Abelian Groups

2.1.1 Introduction

This appendix is a short introduction to the representation of finite abelian groups, and is intended to provide precisely the requisite language to understand the "fourier" decomposition of our test functions. All of material presented is completely standard and can be found in any introductory text in representation theory. Let G be a finite abelian group throughout this chapter. We will assume basic knowledge of linear algebra.

Definition 2.1.1. A representation of G is a group homomorphism $\rho : G \to GL(V)$ where V is an n-dimensional vector space, and $GL(V)$ is the collection of linear automorphisms of V. n is said to be the dimension of the representation.

A different way to say this is that we equip a vector space V with a G-module structure. With this in mind, a G-module homomorphism is precisely what we require to have a morphism of representations (a linear map $f : V \to W$ that commutes with the linear action of G.) In other words, if $\rho : G \to GL(V)$ and $\phi : G \to GL(W)$ are two representations, then $f \circ \rho = \phi \circ f$, then f is a G-linear map.

Example 2.1.2. Viewing $\mathbb{R}^2$ as a vector space, $\mathbb{Z}_2 := \{0, 1\}$ may act on it my multiplication by ± 1. In particular, we have the homomorphism

$$g \mapsto \phi_g := \begin{pmatrix} (-1)^g & 0 \\ 0 & (-1)^g \end{pmatrix}$$

which is a perfectly good representation of $\mathbb{Z}_2$.

More generally, we can construct a representation for $(\mathbb{Z}_2)^{\oplus k}$ on $\mathbb{R}^{2^k}$ by letting $g := (g_1, \dots, g_k) \in Z_2^{\oplus k}$ where g_i are either 0 or 1 and considering

$$g \mapsto \phi_g := \begin{pmatrix} (-1)^{g_1 + \cdots g_k} & 0 \\ 0 & (-1)^{g_1 + \cdots + g_k} \end{pmatrix}$$

is another possible representation.

In general, we will want the simplest possible description of a representation. This will occur when a subspace is left invariant by all elements of a group G, or accordingly, when the matrix form can be put into block diagonal form:

Definition 2.1.3. Given a representation $g \mapsto \phi_g \in GL(V)$, we say that a subspace W is G-invariant if $\phi_g(W) \subset W$ for all $g \in G$. In this case, we define the restriction of each ϕ_g to W to be a subrepresentation, where $(\phi \restriction_W)_g(w) = \phi_g(w)$ for all $w \in W$. A representation is said to be irreducible if it has no nontrivial subrepresentations.

From now on, we will abuse notation by fixing G and denoting its representation by the underlying vector space it acts on, V, and we write gv for $\phi_g(v)$.

Theorem 2.1.4. *If W is a subrepresentation of V, then the complement of W is also a subrepresentation as well.*

Proof. Consider the linear projection $\pi_* : V \to W$. From this, we form $\pi : V \to W$ given by

$$\pi(v) := \frac{1}{|G|} \sum_{g \in G} g(\pi_*(g^{-1}v)).$$

Note that $\pi(w) = w$ for all $w \in W$, and for all $v \in V$, we also have that

$$\pi(hv) = \frac{1}{|G|} \sum_{g \in G} g(\pi_*(g^{-1}h(v))) = h \cdot \sum_{h^{-1}g \in G} h^{-1}g(\pi_*(h^{-1}g)^{-1}v) = h\pi(v)$$

so π is in fact a G-linear map, and in particular, $\ker(\pi)$ is invariant under G. □

This tells us that in fact, every representation V of a finite group (we didn't use abelian anywhere) G can be written as $V \cong V_1 \oplus \cdots \oplus V_n$, where each V_i is irreducible, by repeating the process in the previous theorem. In particular, the matrix form for every representation can be written block diagonally across all group elemeents. We can in fact do better for Abelian groups:

Theorem 2.1.5. *(Schur's Lemma) If V, W are both irreducible representations of G, and $\phi : V \to W$ is a G-linear map, then ϕ is either an isomorphism of trivial.*

Proof. Note that if $v \in \ker \phi$, then $\phi(gv) = g\phi(v) = 0$, so $gv \in \ker \phi$ so that $\ker \phi$ is G-invariant. Similarly, the image is invariant. Hence, if V, W are both irreducible, then the claim follows immediately. □

Corollary 2.1.6. *If $V = W$ is an irreducible representataion, then any G-linear homomorphism $\phi = \lambda \cdot I$ for $\lambda \in C$.*

Proof. Since ϕ has an eigenvalue, so $\phi - \lambda I(v) = 0$ has nontrivial kernel and $\phi - \lambda I$ is also a G-linear map, so by Schur's Lemma, $\phi = \lambda I$. □

With the preceeding corollary in mind, we can now prove the main theorem:

Theorem 2.1.7. *Every irreducible representation of our abelian group G is in fact one dimensional*

Proof. multiplication by any group element $g \in G$ provides a G-linear map, since $gh(v) = hg(v)$, and hence $g = \lambda \cdot I$. In other words, every subspace of V is invariant, so if V is irreducible, then it must be one dimensional □

Hence, every representation $V \cong \oplus_{\alpha \in A} V_\alpha$ where all the V_α are one dimensional. In particular, the matrix form for every representation is similar (isomorphic) to a diagonal matrix.

2.1.2 General Characters and Fourier Decomposition

In this section we will describe "Fourier Analysis" on finite groups. This essentially amounts to looking for a decomposition of a function $f : G \to \mathbb{C}$ as conveniently as possible. We will do this via representation theory.

Definition 2.1.8. If V is a representation of G, its character χ_V is a function on the group given by $\chi_V(g) = \mathrm{Tr}(g)$, where $\mathrm{Tr}g$ is the trace of the function.

A first consequence of this fact is that $\chi_V(hgh^{-1}) = \chi_V(g)$, so the character depends only on conjugacy classes. Furthermore, since the trace of linear maps depend only on eigenvalues,we have further that $\chi_{V\oplus W} = \chi_V + \chi_W$ and $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

Moreover, we can consider the space of functions $V := \{f : G \to \mathbb{C} \mid f(ghg^{-1}) = f(h)\}$ as a vector space. In the case where G is abelian, this is just the space of all functions. Hence, we can consider a representation of G on V.

The key construction will be the introduction of an inner product on V defined by

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g\in G} \overline{\alpha(g)}\beta(g).$$

If we pause for the sake of intuition we should note that more generally $\frac{1}{|G|}\sum_{g\in G} g$ is in fact a projection onto the space of vectors fixed by G. Combining all of this, we get the following (amazing) argument:

Let $\mathrm{Hom}(V,W)^G$ denote the space of G linear maps. Recall that $\mathrm{Hom}(V,W) \cong V^* \otimes W$ via the isomorphism $v^* \otimes w \mapsto (x \mapsto v^*(x)w)$ which is an isomorphism of representations as well. Using the fact that characters are multiplicative over tensor product, we can check that $\chi_{V^*\otimes W} = \overline{\chi_V} \cdot \chi_W$.

From this, we can see that $\dim(\mathrm{Hom}(V,W)^G) = \dim(V^* \otimes W)^G = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V} \cdot \chi_W$

where the last equality comes from the fact that

$$\frac{1}{|G|} \sum_{g\in G} \overline{\chi_V} \cdot \chi_W$$

is a projection onto $(V^* \otimes W)^G$, so it acts by identity on the subspace, and hence the trace agrees with the dimension.

In other words, by Schur's lemma, we have that if V, W are irreducible, then $\mathrm{DimHom}(V, W) = 1$ if they are isomorphic and zero otherwise. Using the preceeding argument, we see that characters of irreducible representations are orthonormal!

When G is finite abelian, there are exactly $|G|$ characters, since the characters depend only on conjugacy classes of the group (each element), and hence they in fact form an orthonormal basis for the space of functions $f : G \to \mathbb{C}$.[1] In particular, we can define the "fourier transform" of a function $f : G \to \mathbb{C}$ by taking $\langle f, \chi_i \rangle$. Hence, we obtain a decomposition (fourier inversion formula):

$$f = \sum_{g \in G} \langle f, \chi_g \rangle \chi_g$$

which is exactly what we wanted.

However, we can actually rewrite this in a slightly more convenient way. First, we have to note that since G is finite, there is some $n \in \mathbb{N}$ so that $g^n = 1 \in G$. Since $\chi : G \to \mathbb{C}^\times$ is a group homomorphism, we also have that $\chi(g^n) = \chi(g)^n = 1$ for some n for each $g \in G$, we know that χ is actually unitary, in the sense that $\overline{\chi}(g) = \chi^{-1}(g)$, and $\chi : G \to S^1 \subset \mathbb{C}$, which is an easier way to compute the "fourier coefficients."

2.1.3 Characters of Finite Abelian Groups

Here, we provide the most essential computations needed to understand the characters for $\mathbb{Z}_2^{\oplus k}$, and more generally a finite abelian group G.

To begin with, let $H^1(G) := \{\chi : G \to \mathbb{C}^\times\}$ be the space of characters on G. First, suppose that $G = \mathbb{Z}_m$ is finite cyclic. Then we can construct exactly m characters, by letting ω be a primitive m^{th} root of unity, and letting $g \mapsto \omega^g$ be an assignment of roots of unity to each group element. We then define $\chi_g(h) := \omega^g h$. Since there are exactly n characters here, it will suffice to check that each has norm 1 under the inner product, which is an easy verification. Similarly,

[1] This result can be strengthened if one considers class functions, in which case characters always form an orthonormal basis

$\langle \chi_g, \chi_j \rangle = 0$ whenever $g \neq j$, since the inner product is just

$$\frac{1}{m} \sum_{h \in \mathbb{Z}_m} \omega^{gh} \omega^{-jh},$$

but this is a sum over roots of unity, but it is also well known that this is zero, so these are orthogonal one dimensional representations. Note further that $\mathbb{Z}_m \cong H^1(\mathbb{Z}_m)$. Furthermore, we can check that a homomorphism $f : G \oplus H \to S^1$ splits over direct sum by the universal property of (co)products. Hence, by the structure theorem for finite abelian groups, we have that $G \cong \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}$, so we can define $\chi_\epsilon(g_1, \ldots g_n) = \prod_{j=1}^n \chi_{\epsilon_j}(g_i)$, with ϵ_j the standard basis. In particular, this defines an isomorphism $G \cong H^1(G)$.

2.2 Vector Bundles

In this section, we will provide the definition of a G-bundle or vector bundle. For the sake of unifying the G-bundle and vector bundle definition, we will introduce both through the slightly more general notion of a fiber bundle.

A common way to construct a new topological space from ones we know already is the cartesian product, $M \times N$. Fiber bundles are a generalization of this concept that still mantain enough structure so that allow us to pass from local information to a global picture in a way that is still useful. Here is the definition:

Definition 2.2.1. A fiber bundle is a a triple of spaces (E, B, F) equipped with a continuous surjection $p : E \to B$ so that for each $x \in B$, there exists some neighborhood U of x so that $\rho^{-1}(U)$ is homeomorphic to the cartesian product $E \times F$.

In this case we often say that U_x are trivializing neighborhoods, i.e: there exists a homeomorphism $U_x \times F \to \rho^{-1}(U_x)$ so that the following diagram commutes:

$$\begin{array}{ccc} U_x \times F & \xrightarrow{\alpha_u} & p(U_x) \\ & \searrow^{\pi} & \downarrow \rho \\ & & U \end{array}$$

where π is just projection onto the first factor. We usually say that α_u are “local trivializations” of the bundle.

This notion allows us to make the following definitions: a real vector bundle is just a fiber bundle where the fiber F is also an n-dimensional real vector space. However, since we now consider some additional structure, it should be remarked that the local trivializations α_u : $p^{-1}(U) \to U \times \mathbb{R}^n$ should know also be linear isomorphisms at each fiber, which we will sometimes denote V_x . In this case n is the dimension of the vector space. if $n = 1$, we usually say that the vector bundle is a line bundle.

The general definition of a *G-bundle* is a fiber bundle $p : E \to B$ equipped with a continuous action of G so that G perserves fibers: $g(p^{-1}(x)) \subset p^{-1}(x)$. The most important examples will be when each fiber is a topological group acting on itself in a way compatible with the bundle structure.

Example 2.2.2. Let $\mathbb{R}P^n$ denote the space of all linear subspaces

Example 2.2.3. We consider line Bundles over the circle. There is of course the trivial bundle $S^1 \times \mathbb{R}$, where projection is given by $(s, x) \mapsto s$, so that each fiber is a copy of the real line. There is a slightly different bundle known as the mobius bundle, where the quotient map that induces $(s + 1, t) \sim (s, -t)$ on $\mathbb{R} \times \mathbb{R}$ gives a bundle structure over S^1 in a natural way, since $S^1 \cong \mathbb{R}/\sim_1$ where $x \sim_1 x + 1$. If we denote the map $p : E \to S^1$, given by $(s, x) \mapsto s$, then this is a vector bundle (and it is a mobius band with boundary deleted.) Indeed, if $a \in S$, then let $U = \{s \in S \mid a - \frac{1}{2} < x < a + \frac{1}{2}\}$, then $\alpha_U : p^{-1}(U) \to U \times \mathbb{R}$ is given by regarding $([s, t]) \in E$ as $([s], t)$.

Now, there are some natural questions to ask: is the mobius bundle equivalent to the trivial bundle over S^1? What would it even mean to be equivalent? The next part of this section seeks to answer these questions by introducing some new vocabulary:

Definition 2.2.4. Given two bundles $p : E \to B$ and $p' : E' \to B'$, we say that $\tilde{f} : E \to E'$ is a morphism of bundles if it descends to a map $f : E \to B$, where $f \circ p = p' \circ \tilde{f}$, or in other words, the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \downarrow{p} & & \downarrow{p'} \\ B & \xrightarrow{f} & B' \end{array}$$

and the restriction to each fiber $\tilde{f} : V_x \to V_{f(x)}$ is linear for all $x \in E$. In the case that $f : B \to B'$ is a homeomorphism , we can simplify the diagram, by replacing $f \circ p$ by a single arrow to E', and we say that two bundles are isomorphic if $\tilde{f}$ is a diffeomorphism, or its restriction to each fiber is an isomorphism of vector spaces (which are equivalent conditions.

We can see via this definition that the mobius bundle M is not isomorphic to $S^1 \times \mathbb{R} \to S^1$, since the total spaces are not even homeomorphic. This can be observed by removing $S^1 \times \{0\}$, which results in two disconnected components, while the removal of any image of this in the mobius band, $M \setminus S^1$ is connected.

However, there is an easier way to tell when a bundle is trivial, that also gives us the opportunity to make an essential definition.

Definition 2.2.5. Given a vector bundle $p : E \to B$, we define a section of our bundle to be a right inverse $s : B \to E$ so that $p \circ s = id$

The intuition behind this definition is that a section $s : B \to E$ assigns to every $b \in B$ some vector $s(b) \in p^{-1}(b)$. In particular, since $0 \in p^{-1}(x)$ for each $x \in B$, we always have a zero section for every vector bundle.

Example 2.2.6. Given a smooth manifold M, one can always form the bundle $p : TM \to M$, where TM is the collection of all tangent spaces to a manifold, and the bundle map p assigns to each tangent space the point to which it is tangent. Sections of this bundle are precisely vector fields.

We will especially care about nonvanishing sections throughout this thesis. For example, the bundle $p : TS^n \to S^n$ admits a nonvanishing section if and only if n is odd. This is the content of the hairy ball theorem, if the reader is familiar with it.

It turns out that the structure of sections in a bundle gives a very nice criteria for triviality:

Theorem 2.2.7. *an n-dimensional vector bundle $p : E \to B$ is isomorphic to the trivial bundle if and only if there exist n sections $\{s_1, \dots s_n\}$ so that $s_1(x), \dots s_n(x)$ constitute a basis for each fiber V_x.*

we omit the proof here, but the interest reader can see [7] for details.

In particular, we will know that if a vector bundle $p : E \to B$ does not admit a nowhere vanishing section, then this is an obstruction to triviality! The theory of characteristic classes provides necessary conditions for the existence of nowhere vanishing sections, and we will see that $w_1(M)$ will be an algebraic invariant that detects such obstructions. The Mobius bundle is a first example of this, as it can be seen (and is shown in Appendix C) that $w_1(M) \neq 0$, so the mobius bundle cannot be trivial.

We conclude this section with the important generalization of the mobius bundle that is used throughout this document.

Example 2.2.8. Let $B := \mathbb{R}P^n$. We would like to think of $\mathbb{R}P^n$ as the collection of 1 dimensional subspaces in $\mathbb{R}^{n+1}$, or for those more algebraically inclined, $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$. In this way, we can consider $\gamma_n := \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in \ell\}$, and consider projection $\gamma_n \to \mathbb{R}P^{n+1}$. This is a smooth line bundle, with fiber $\ell \in \mathbb{R}P^n$. This is usually called the tautological line bundle over $\mathbb{R}P^n$. Note that $\mathbb{R}P^1 \cong S^1$, and that $\gamma_1 \to S^1$ is nothing but the mobius bundle.

2.2.1 Direct Sums

In the following sections, we give examples of "constructing new from the old," in the sense that we will talk about important constructions and operations on vector bundles. All of these will have easy induced maps on the level of cohomology and characteristic classes. The first part of this section seeks to carry out the anologues of typical constructions in linear algebra for vector bundles. We will then move into important constructions for the classification of vector bundles.

The most basic way to study the structure of a vector space is to consider its subspaces. In particular, we wish to construct a vector sub-bundle of $p : E \to B$. It turns out that we need only make the usual notion of subspace compatible with the bundle structure to obtain a sub-bundle:

Definition 2.2.9. Given a bundle $p : E \to B$, we say that $E_0 \subset E$ is a sub-bundle if it intersects each fiber in a subspace so that the restriction $p : E_0 \to B$ is a vector bundle

Given two vector spaces V_1, V_2, there is a natural operation, called the direct sum (categorical coproduct) that allows us to construct a third vector space, $V_1 \oplus V_2$. Given two vector bundles, (E_1, V_1, B) and (E_2, V_2, B) we want to create a new bundle with fiber $V_1 \oplus V_2$. We can do this by taking the direct sum of vector bundles.

Definition 2.2.10. The direct sum of (E_1, V_1, B) and (E_2, V_2, B) is a bundle $E_1 \oplus E_2 \to B$ with total space

$$E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1)\}$$

and $p(e_1, e_2) = b \in B$ precisely if $p_1(e_1) = p_2(e_2)$.

In principle, one should check that this is still a vector bundle, but we will omit this verification.

It can be shown further by constructing suitable inner products, compatible with the bundl structure that for each sub bundle $E_0 \subset E$, there is a vector subbundle $E_0^{\perp} \subset E$ so that $E \cong E_0 \oplus E_0^{\perp}$.

2.2.2 Tensor Product

Given two vector spaces V_1, V_2, one often wishes to consider $V_1 \otimes V_2$, the smallest vector space so that every pair of bilinear maps $f : V_1 \times V_2 \to W$ factor through the linear map $\tilde{f} : V_1 \otimes V_2 \to W$ for all choices of vector spaces W. Similarly, we will often care about bundle maps that are linear by each fiber, so we wish to carry this construction over to vector bundles. Heuristically, given two bundles (E_1, V_1, B) and (E_2, V_2, B), we will want to construct a new bundle $E_1 \otimes E_2 \to B$ so that the fiber lying above each point is precisely $V_1 \otimes V_2$. The difficulty here is topologizing this set, so we instead turn to a more natural definition, but one can find an alternative treatment in [7].

Recall that for a vector bundle $E \to B$ with fiber $\mathbb{R}^k$ we have the existence of "trivializing neighborhoods" $\{U_i\}$ that cover B, so that on each neighborhood, there exist local diffeomorphisms (that are linear on each fiber) $\alpha_i : p^{-1}(U) \to U_i \times \mathbb{R}^k$. However, it is the case that when

$x \in U_{ij} := U_i \cap U_j$, there are two different transition functions defined on the intersection. Hence, we can obtain a "gluing" function that tells us how to connect this local data, and in particular, we should have that $\alpha_i \circ \alpha_j^{-1} : U_{ij} \times \mathbb{R}^k \to U_{ij} \times \mathbb{R}^k$ given by $(x, v) \mapsto (x, \alpha_i \circ \alpha_j^{-1}(v))$ where $\alpha_i \circ \alpha_j^{-1}$ is an element of $GL(\mathbb{R}^k)$. The standard way of recording this data is by taking recording $g_{ij} : U_i \cap U_j \to GL_n(\mathbb{R})$. These satisfy what is called the "cocyle condition," $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$. Then, it necessarily follows that a vector bundle E is just $\left(\coprod_{i \in I} U_i \times \mathbb{R}^k\right) / \sim$ where $(x, v) \sim g_{ij}(x, v)$ for some $i, j \in I$. Note that this construction shows how to recover vector bundles as a G bundle. We take the associated data $(E, B, P, \mathbb{R}^k)$, and have $GL_k(\mathbb{R})$ act on fibers via the transition maps.

Now, we are prepared to make the following definition:

Definition 2.2.11. Given two bundles $E_1 \to B$ with fiber V_1 and $E_2 \to B$ with fiber V_2, we refine some cover of B until there exist local trivializations for both bundles, and call this $\{U_i\}$. Then, let $\{(g_1)_{ij}\}$ and $\{(g_2)_{ij}\}$ be the two different transition maps for each element of the open cover. We take $h_{ij} := (g_1)_{ij} \otimes (g_2)_{ij} : U_i \cap U_j \to GL_{nm}(\mathbb{R})$ and define

$$E_1 \otimes E_2 := \left(\coprod_{i \in I} U_i \times \mathbb{R}^{n \times m} \right) \Big/ \sim$$

where $(x, v) \sim h_{ij}(x, v)$ for some $i, j \in I$.

One should check that this is still a bundle, and that fibers are indeed the tensor product (although this is more or less by construction.) Also, we remark that tensoring some n-dimensional bundle by a line bundle still gives an n-dimensional bundle.

2.2.3 Pullbacks

Given a space B, one can ask that up to bundle isomorphism, what kind of n-dimensional bundles appear over B? In other words, can we classify all bundles over a space B? Indeed, there is a category $Vect^n(B)$ of n-dimensional bundles over B, with bundle maps as morphisms. This observation will lead us to the classification of vector bundles, since maps $f : A \to B$ induce functors $f^* : Vect^n(B) \to Vect^n(A)$, known as pullbacks, and this assignment is essentially unique. In fact, the entire category $Vect^n(B)$ can be recovered from knowing the homotopy class of a map out of B into a suitable space, and really, all of its objects will be pullbacks

The following is a definition-theorem, whose proof can be found in [7].

Definition 2.2.12. Given a vector bundle $p : E \to B$ and a map $f : A \to B$, we say that pullback bundle along f is the bundle $p^* : f^*E \to A$, where $f^*E := \{(a, e) \in A \times E \mid f(a) = p(e)\}$, and p^* is nothing but projection. Additionally, there is a bundle map f' induced by this construction, given by $f(a, e) = e$, and this carries fibers over $a \in A$ isomorphically to fibers over $f(a)$. In other words, the following diagram commutes:

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

We can motivate this definition in several ways. The first is that this is precisely the same as the categorical pullback, but really this is precisely the construction needed to replace transition functions $f^*h_{ij} := h_{ij} \circ f$, so the "structure" of this bundle is literally given by the pullback of f.

First of all, the assignment respects composition, in the sense that $(f \circ g)^*E = g^*f^*(E)$, and the identity $B \to B$ returns the same bundle back (this assignment is functorial.) Secondly, pullback commutes with both direct sum and tensor product, so that $f^*(E_1 \oplus E_2) = f^*(E_1) \oplus f^*(E_2)$ and $f^*(E_1 \otimes E_2) = f^*(E_1) \otimes f^*(E_2)$.

2.2.4 Classification Of Vector Bundles

We will now turn to the final section on vector bundles. We will omit almost every proof here, but try to motivate the classifying maps for bundles. We begin by providing how to pass from pullbacks of maps to pullbacks of homotopy classes of maps. Then, we will define the universal bundle and state the main result of this section. Throughout, we assume that all base spaces are paracompact (for technical reasons.)

Finally, we claim that pullbacks are homotopy invariant in the following sense:

Theorem 2.2.13. *Given a vector bundle and two homotopic maps $f_0, f_1 : A \to B$, then the pullback bundles $f_0^* \cong f_1^*(E)$ as vector bundles.*

The proof of which can be found in [7].

Now, we will construct the so-called Universal Bundle , which will give a bijection between $[B, X]$, the homotopy classes of maps into some base space X and $Vect_n(B)$, and the bijection will be given via the pullback operation.

Definition 2.2.14. The Grassman manifold $G_n(\mathbb{R}^k)$ with $n \leq k$ is the collection of n-dimensional vector subspaces of $\mathbb{R}^k$. The Stiefel Manifold $V_n(\mathbb{R}^k)$ is the collection of orthonormal n frames in $\mathbb{R}^k$.

Note that $G_1(\mathbb{R}^k) = \mathbb{R}P^{k-1}$. We topologize $V_n(\mathbb{R}^k)$ by considering it a subspace of $(S^{k-1})^{\oplus n}$ and giving it the subspace topology. Note that the Stiefel manifold is compact, since it is a closed subspace (as orthogonality is an algebraic condition.) There is a natural surjection $p : V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending an n-frame to its span. G_n gets the quotient topology. Furthermore, we can take $G_n(\mathbb{R}^\infty) := \cup_k G_n(\mathbb{R}^k)$, with inclusions as attaching maps.

Now, consider $E_n(\mathbb{R}^k) := \{(\ell, v) \in G_n(\mathbb{R}^k \times \mathbb{R}^k \mid v \in \ell\}$. Similarly to before, we take $E_n(\mathbb{R}^\infty)$ to be the CW complex coming from inclusion. $p : E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ given by $p(\ell, v) = \ell$ in fact defines a vector bundle, and indeed we obtain the universal bundle this way:

Theorem 2.2.15. *The map $[X, G_n(\mathbb{R}^\infty)] \to Vect^n(X)$ given by $[f] \mapsto f^*(E_n(\mathbb{R}^\infty))$ is a bijection.*

From now on, we will suppress notation and refer to the universal bundle as $\pi : E_n \to G_n$.

Example 2.2.16. Consider the map $S^1 \to S^1$ given by $t \mapsto (\cos(\pi t), \sin(\pi t)$, which passes to $\mathbb{R}P^1$ via the usual identification, and since it is an odd function, it is nontrivial on the level of homology, and hence not homotopic to the constant map, and hence nontrivial. In particular, note that the pullback is precisely the mobius bundle.

2.3 Characteristic Classes

In this section, we will assume working knowledge of cohomology. We will care especially about Stiefel Whitney Classes, so it will not hurt to take cohomology with coefficients in $\mathbb{Z}_2$. We spend the majority of this section discussing majority, but merely state the relevant results and definitions required to read this document.

Suppose we start by asking whether some n dimensional bundle $p : E \to B$ is trivial. In other words, $E \cong B \times \mathbb{R}^n$. Clearly, this means that the classifying map $f : B \to G_n$ must be nullhomotopic, since we need that $E \cong \{(b, v) \mid f(b) = \pi(v)\}$, where $\pi : E_n \to G_n$, but this only occurs up to homotopy if $E \cong B \times \pi^{-1}(g)$ for a single $g \in G$. This is in practice prohibitively difficult to verify, so we can instead ask, how about the induced map $f^* : H^*(G_n) \to H^*(B)$? If this is nontrivial, then f cannot be nullhomotopic. Charcteristic classes will be a way of detecting when this map is nontrivial, since they will be elements of the cohomology ring for B that respect pullbacks: $w_i(f^*(E)) = f^*(w_i(E))$, where the second $f^*(w_i(E))$ is the map induced on cohomology.[2] Thus, characteristic classes will be obstructions to triviality.

Example 2.3.1. A bundle is said to be orientable if transition functions preserve the orientation of vector spaces at each fiber. Clearly, every trivial bundle is orientable, since transition functions are identity for $B \times \mathbb{R}^n$. One obstruction to triviality is orientability. Let $p : M \to S^1$ be the mobius bundle. We can detect orientability by constructing a homomorphism $\pi_1(S^1) \to \mathbb{Z}_2$, assigning it 0 if it preserves orientations of fibers around each loop in S^1, and 1 otherwise. Clearly, this homomorphism factors through the abelianization of $\pi_1(S^1)$, aka $H_1(S^1)$. However, a map $H_1 \to \mathbb{Z}_2 \in \text{Hom}(H_1, \mathbb{Z}_2) = H^1(S^1)$. This is $w_1(M)$, the first Stiefel Whitney class. For the mobius bundle, this is nontrivial, which is a different way of proving that M is nontrivial.

We can in fact replace $p : M \to S^1$ in the example above by $p : E \to B$ to obtain the general definition for the first stiefel whitney class. Recasting the above, we can assume that B is a CW complex. Then A vector bundle over the 1-skeleton of B is trivial if and only if it is orientable

[2] One can rephrase this condition to look more natural by demanding that a characteristic class (with $\mathbb{Z}_2$ coefficients and of degree k) is a natural transformation from the functor $B \mapsto Vect^n(B)$ to $B \mapsto H^q(B, \mathbb{Z}_2)$.

(the restriction of the bundle to $B^(1)$ is trivial.) For higher dimensions, recall that a bundle is trivial if and only if we can find linearly independent orthonormal sections over B. In other words, if $w_1(E)$ vanishes, we can find n orthonormal sections over B. Can these sections be extended over each 2 cell? This is what the second Stiefel whitney class measures, and it turns out to be an element of $H^2(B, \mathbb{Z})$, but we reduce coefficients mod 2 via the universal coefficient theorem for technical reasons.

The following definition is an axiomatic treatment of the Stiefel-Whitney Classes. We will not prove the existence of such functions, or that these axioms uniquely determine them. One can consult [8] or [7] for serious treatments of these things.

Definition 2.3.2. There is a unique sequence of functions $w_1, w_2, \ldots$ where the assignements $w_i : Vect^i(B) \to H^i(B, \mathbb{Z}_2)$ have the following properties:

(i) (naturality) $w_i(f^*E)) = f^*(w_i(E))$ for a pullback $f^*(E)$.

(ii) (Whitney sum) $w_i(E_1 \oplus E_2) = \sum_{k+j=n} w_k(E_1)w_j(E_2)$, where multiplication is the usual cup product in $H^*(B)$.

(iii) (Well-definedness) $w_i(E)$ vanishes for all $i > \dim(E)$.

(iv) (nontriviality) If $E \to \mathbb{R}P^1$ is the canonical line bendle, then $w_1(E)$ generates $H^1(\mathbb{R}P^1)$.

Example 2.3.3. The trivial bundle $B \times \mathbb{R}^n$ has trivial stiefel whitney classes in all dimensions, since it is the pullback of a bundle over a point, so the naturality axiom implies that every stiefel whitney class must be zero for all $i > 0$.

Theorem 2.3.4. *If $p : E \to B$ is a vector bundle with a nowhere zero cross section, then $w_n(E) = 0$.*

Proof. Let $s : B \to E$ be a nonvanishing section. At each fiber, take $s^\perp$ as the collection of vectors orthogonal to $s(b)$. We let E' be the union of all such fibers. Then $E \cong E' \oplus \mathbb{R}$ (the actual bundle verification is 3.3 in [8]), and we can see the rest of the claim by Theorem B.1.7. By the Whitney Sum axiom, its also true that $w_n(E) = \sum_{k+j=n} w_k(E')w_j(\mathbb{R}) = 0$. □

Recall that the tensor product of line bundles is again a line bundle. In fact, we can form a natural "inverse" for $Vect^1(B)$, by providing a line bundle with an inner product, so that all gluing functions take value in ± 1, so that $E \otimes E$ has gluing functions that are squares of each gluing function, which is trivial. Hence, a line bundle is its own inverse. Since the tensor construction is associative (and abelian for line bundles), this in fact gives us a natural group structure on $Vect^1(B)$. This brings us to the following proposition, which we will not prove, but is utterly beautiful to this author:

Theorem 2.3.5. *The function $w_1 : Vect^1(X) \to H^1(X, \mathbb{Z}_2)$ is a homomorphism (and an isomorphism when X is a CW-complex.*

Proof. See [7] Proposition 3.1.0 □

In particular, we have that for line bundles $w_1(\gamma_1 \otimes \gamma_2) = w_1(\gamma_1) + w_1(\gamma_2)$. For the case of $B = S^1$, we can actually verify this in a reasonable way. The collection $[S^1, \mathbb{R}P^\infty] = \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}_2$, so there are in fact only two bundles over S^1, and we can see readily that the bundle is fully characterized by orientability, or equivalently, by w_1 as discussed in example 2.3.1. Hence, the homomorphism is truly an isomorphism in this case.

We complete this section by recalling the Kunneth formula for Cohomology. There is a map $\Phi : H^*(X) \times H^*(Y) \to H^*(X \times Y)$, induced by projections $p_1, p_2 : X \times Y \to X, Y$ and then taking the cup product in cohomology: $\Phi(x, y) = p_1^*(x) \smile p_2^*(y)$, which is bilinear since the cup product is distributive. By the universel property of tensors products, this map factors through the tensor product, and we have the kunneth formula:

Theorem 2.3.6. *$\tilde{\Phi} : H^*(x) \otimes H^*(Y) \to H^*(X \times Y)$, given by $\Phi(x \otimes y) = p_1^*(x) \smile p_2^*(y)$ is an isomorphism whenever X, Y are CW complexes.*

Proof. See [6] Appendix 3B. □

3

Background

3.1 Demonstrating The Ham Sandwich Theorem

In this section, we demonstrate the typical reduction that occurs under the CS/TM paradigm, by applying it to the classical Ham Sandwich Theorem in excruciating detail. The construction used here will be promptly generalized in the following section.

Definition 3.1.1. Let *mass* in $\mathbb{R}^n$ is a positive, finite Borel measure on $\mathbb{R}^n$ such that it is absolutely continuous with respect to Lebsegue Measure

Instead of absolutely continuous, we could equivalently insist that the a codimension 1 flat has measure zero. A good example to have in mind is n dimensional Lebesgue measure that is characteristic on a compact subset $A \subset \mathbb{R}^n$, i.e:

$$\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \chi_A d\mu,$$

for any measurable $B \subset \mathbb{R}^n$.

Theorem 3.1.2. *(Ham- Sandwich): Given n masses $\mu_1, \ldots \mu_n \in \mathbb{R}^n$, there exists a hyperplane that equiparts each mass simultaneously.*

The “half-space” construction is the key observation to determining the appropriate configuration space. Note that a general hyperplane is the collection of $(x_1, \ldots, x_n)$ that satisfy

$a_1x_1 + \cdots + a_nx_n - a_{n+1} = 0$, meaning that we can parametrize a hyperplane by the coefficients $(a_1, \cdots a_{n+1}) \in \mathbb{R}^{n+1}$. In particular, after scaling appropriately, we obtain an identification with S^n. Furthermore, to each hyperplane, we assign

$$H^+(u) := \{u \in \mathbb{R}^n \mid \langle u, a\rangle \geq a_{n+1}\} \qquad H^-(u) := \{u \in \mathbb{R}^n \mid \langle u, a\rangle \leq a_{n+1}\}$$

where $a = (a_1, \ldots, a_{n+1}) \in S^n$. When $x = \pm(0, 1)$, we let $H^0(0, 1) = \mathbb{R}^d$ and $H^1(0, 1) = \emptyset$, the "hyperplanes at infinity." This makes the assignment surjective.

We now construct the test map and test space. Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be given by

$$f(a) := \left(\mu_1(H^+(a) - \frac{1}{2}\mu_1(\mathbb{R}^n), \mu_n(H^+(a)) - \frac{1}{2}\mu_2(\mathbb{R}^n)\right).$$

This construction says that $\mathbb{R}^n$ is the test space and $\{0\} \in \mathbb{R}^n$ is the solution space. We first note that f is clearly $\mathbb{Z}_2$ equivariant, this is because in each component, we have that $-f_i(a) = \frac{1}{2}\mu_1(\mathbb{R}^n) - \mu_i(H^+(a) = \mu_i(H^-(a))$, and $H^+(-a) = H^-(a)$ by definition. Putting these facts together, we see that $f(-a) = -f(a)$, exactly as claimed.

We can also check that f is continuous by applying Lebesgue's dominated convergence theorem:

It will suffice to show that for any $u_n \to u \in S^n$, we also have that $\mu_i(u_n) \to \mu_i(u)$. Note that for some $x \in \mathbb{R}^n$ that is not on the boundary $\partial H^+(a)$ (which has measure zero since it is a hyperplane), we will have for sufficiently large n, $x \in H^+(u_n)$ if and only if $x \in H^+(u)$. This in turn means that for the characteristic function χ_u on $H^+(U)$, we have that $\chi_{u_n} \to \chi(u)$ almost everywhere.

Hence by the finiteness assumption on our measures, we can apply the dominated convergence theorem:

$$\mu_i(H^+u_n) = \lim_{n\to\infty} \int \chi_{u_n} d\mu_i \to \int \chi_u d\mu_i = \mu_i(H^+(u)),$$

as desired.

Hence, we can apply the final step in the CS/TM method. We must show the nonexistence of a continuous and equivariant $f : S^n \to_{\mathbb{Z}_2} \mathbb{R}^n \setminus \{0\}$. This is precisely the classical Borsuk- Ulam theorem, which we will prove here for completeness.

Theorem 3.1.3. *(Borsuk Ulam): there does not exist a continuous, nonvanishing, $\mathbb{Z}_2$ equivariant map $g : S^n \to \mathbb{R}^n$.*

Proof. Suppose there were such a map. Then, this would define a map $h : S^n \to S^{n-1}$ given by $h(x) := \frac{g(x)}{\|g(x)\|}$. Since h is also $\mathbb{Z}_2$ equivariant, we can pass to $\tilde{h} : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$). First note that $\tilde{h}_* : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^{n-1})$ cannot be trivial, since if it were, it would lift back to S^{n-1}, an immediate contradiction. By the hurewicz theorem, we deduce that $\tilde{h}$ induces an isomorphism on the first integral homology, and by the universal coefficient theorem, an isomorphism $\tilde{h}^* : H^1(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \to H^1(\mathbb{R}P^n, \mathbb{Z}_2)$. However, recall that $H^*(\mathbb{R}P^k) = \mathbb{Z}_2[x]/(x^{n+1})$. If we let α, β be generators for the cohomology ring of $H^*(\mathbb{R}P^{n-1})$ and $H^*(\mathbb{R}P^n)$ respectively, then

$$0 = \tilde{h}^*(\alpha^n) = \beta^n \neq 0,$$

a contradiction. □

The previous cohomological argument will be generalized in the coming sections by using Steifel Whitney classes.

3.2 The Full Configuration Space/Test Map Set Up

More generally than before, we can ask given k hyperplanes and m masses, and some target dimension d, for which triples (k, m, d) can we guarantee an equipartition? One can also impose geometric contstraints, such as orthogonality, on the hyperplanes. In this section, we consider the more general set up for such problems.

3.2.1 Equipartitions

If we are given k hyperplanes, we can still make the half space constructions

$$H^+(u) := \{u \in \mathbb{R}^d \mid \langle u, a\rangle \geq a_{d+1}\} \qquad H^-(u) := \{u \in \mathbb{R}^d \mid \langle u, a\rangle \leq a_{d+1}\}$$

for each hyperplane, which are each parametrized by S^d, making the full parametrization the k fold product $(S^d)^{\oplus k}$. With these in mind, we can form regions

$$\mathcal{R}_g := \bigcap_{i=1}^{k} H^{g_i}(u_i) = \bigcap_{i=1}^{k} H^{+}(-g_i u_i)$$

where $g_1, \ldots, g_k = g \in \mathbb{Z}_2^{\oplus k} = \{+1, -1\}^k$ and $u = (u_1, \ldots u_k) \in (S^d)^{\oplus k}$.

It is important to know that each one of these “orthants” or “regions” are indexed my members of the group $\mathbb{Z}_2^{\oplus k}$ for convenience.

Definition 3.2.1. We say that m masses $\mu_1, \ldots, \mu_m$ are equipartitioned by k hyperplanes if $\mu_i(\mathcal{R}_g) - \frac{1}{2^k}\mu_i(\mathbb{R}) = 0$ for each $i \in \{1, \ldots, m\}$.

One can then check that the measure of each $\mathcal{R}_g$, takes a value in $\mathbb{R}[\mathbb{Z}_2]^{\oplus k}$, which indexes each copy of $\mathbb{R}$ by group elements, and also comes equipped with a natural action of $\mathbb{Z}_2^{\oplus k}$. Then, for each of the m masses, suggesting then that we should define the function $\phi_M = (\phi_1, \ldots, \phi_m) : (S^d)^{\oplus k} \to \mathbb{R}[\mathbb{Z}_2^{\oplus k}]$ where

$$\phi_i(x) = \sum_{h \in \mathbb{Z}_2^{\oplus k}} \left(\mu_i(\mathcal{R}_h(x)) - \frac{1}{2^k}\mu_i(\mathbb{R}^d) \right) h$$

which one can check is indeed $\mathbb{Z}_2^{\oplus k}$ equivariant, and also whose zeroes correspond precisely to an equipartition.

We can actually improve the previous situation slightly, by removing the diagonal (trivial representation), Δ, since if all values are equal, then they are all zero, so we are justified in removing it, to obtain $U_k := \mathbb{R}^{2^k} \setminus \Delta$.

Indeed one can check equivariance just as we did in example 1.0.1 that the map ϕ_M is indeed $\mathbb{Z}_2^{\oplus k}$-equivariant, and so we have the reduction of our problem to showing the nonexistence of an equivariant map $f : (S^d))^{\oplus k} \to U_k \setminus \{0\}$.

We will see the purpose of the representation-theoretic language in Section 2.3

3.2.2 Orthogonality

Orthogonality has in some sense an equivalently natural set up. All the relevant notions here are due to Steven Simon [4]

Example 3.2.2. As a toy problem, suppose we want to prove that there exist orthogonal lines in $\mathbb{R}^2$. As before, we can parametrize all lines by S^2, so we obtain $(a_1, a_2, a_3) \times (b_1, b_2, b_3) \in S^2 \times S^2$ representing two different planes. These planes are orthogonal precisely when $\langle (a_1, a_2), (b_1, b_2) \rangle = 0$ Consider the projections $\pi : S^2 \times S^2$ that sends each vector in S^2 to its first two co-ordinates followed by the bilinear map $(a_1, a_2, a_3) \times (b_1, b_2, b_3) \mapsto a_1 b_1 + a_2 b_2$. There composition provides us a map

$$f : S^2 \times S^2 \to \mathbb{R}$$

whose zeroes determine a solution to our problem. Indeed, f is equivariant, since the usual antipodal action on $S^2 \times S^2$ corresponds to multiplication by -1 in $\mathbb{R}$. For example, the action of $\mathbb{Z}_2^{\oplus 2}$ acts on $\mathbb{R}$ by $g_1 \cdot g_2$, and this clearly agrees with the action on $S^2 \times S^2$. Hence, we have a reduction to showing the nonexistence of an equivariant map $f : (S^2)^2 \to_{\mathbb{Z}_2^2} \mathbb{R}$.

We can now consider some collection of k hyperplanes where we want some of them to be orthogonal. Our Configuration space is the same as before, namely $(S^d)^{\oplus k}$. We will again index our test space by $\mathbb{Z}_2^{\oplus k}$, with standard basis $e_1, \ldots, e_k$. However, the target space will just be copies of $\mathbb{R}^2$, so the important thing will be to construct the action $\mathbb{Z}_2^{\oplus k}$ in order to ensure equivariance. Borrowing notation from [4], we can allow select pairs $(r, s) \in \mathcal{O} := \{(r, s) \mid 1 \leq r < s \leq k\}$ we want to be orthogonal, say some collection of k-tuples in $\mathbb{Z}_2^{\oplus k}$

$$A(\mathcal{O}) := \{(\alpha_1, \ldots, \alpha_k) \mid \alpha_r = \alpha_s = 1 \,\text{and}\, \alpha_i = 0 \,\text{otherwise}\}$$

[1]. With this action in mind, we obtain an equivariant map

$$g : (S^d)^{\oplus k} \to \bigoplus_{(r,s)} V_{r,s}$$

whose nonvanishing is equivalent to finding a solution, by taking the first $d-1$ co-ordinates of corresponding spheres S_r^d, S_s^d and taking inner products.

[1] we should think of $\alpha_r = \alpha_s$ as acting nontrivially, and the rest of the α_i as acting trivially, so that we get basically the desired action of $\mathbb{Z}_2^{\oplus k}$ on our vector space. This will become clearer in Section 2.3 when we discuss the representation theoretic interpretation of this set up

We will also see in Section 2.2 that both orthogonality and equiparition problems can be unified and generalized into showing the nonexistence of certain equivariant maps to more carefully calculated target spaces.

3.2.3 Cascades

Cascades are an additional geometric problem used in [4] to generalize equipartition problems. In essence, one asks that in addition to an equipartition of m_1 masses by $H_1, \dots H_k$ hyperplanes, we can demand further that $H_2, \dots H_k$ equipart another collection of masses m_2, or further that $H_3, \dots H_k$ equipart another mass m_3, and so on.

We now provide the equivariant set up:

Let $\mathcal{M}_i$ be collections of masses $\mu_{i,1}, \dots \mu_{i,m_i}$ with $1 \leq i \leq k$ in $\mathbb{R}^d$. Likewise, we form the projections

$$\pi_i : (S^d)^{\oplus k} \to (S^d)^{\oplus (k-i+1)},$$

which provide some collection of $(k-1+1)$ hyperplanes, and of course provide regions

$$\mathcal{R}_{g_{k-i+1},\dots,g_k}(\pi_i(x)) = \bigcap_{\ell=k-i+1}^{k} H^{g_\ell}(x_\ell)$$

in the plane once again indexed by $(g_{k-i+1}, \dots g_k) \in \mathbb{Z}_2^{\oplus(k-i+1)}$. The equivariant maps $\phi_{\mathcal{M}_i}$: $(S^d)^{\oplus k}$ $toU_{k,i}^{\oplus m_i}$ are as before, but induced by first taking projection, and then applying the previous maps, so equivariance is automatic.

3.3 Finite Fourier Analysis and Geometric Conditions as Representations

3.3.1 equipartition

We now provide the representation theoretic approach to the problem, which will allow us to employ finite fourier analysis in order to have greater control over the CS-TM setup for a wider range of geometric problems. This is ultimately equivalent theoretically, but in practice, the quickest way to find the necessary computation is by considering certain decompositions of our functions. We will assume familiarity with the representation theory of finite abelian groups, and recall the fourier inversion formula, the background for which can be found in Appendix A. All the work discussed here is due to Steven Simon and can be found in [4].

We first consider the vector space $L^2(\mathbb{Z}_2)^{\oplus k}) := \{f : (\mathbb{Z}_2)^{\oplus k} \to \mathbb{R}\}$, the space of all real valued functions. Again, we equip this space with an inner product

$$\langle f_1, f_2 \rangle = \frac{1}{2^k} \sum_{g \in \mathbb{Z}_2^{\oplus k}} f_1(g) f_2^{-1}(g)$$

and under this inner product, the space of characters $\chi_g : \mathbb{Z}_2^{\oplus k} \to \mathbb{R}$ provide an orthonormal basis for $L^2(G)$. Moreover, $g \mapsto \chi_g$ provides an isomorphism between the group of characters and G. We can provide an explicit description of each character by taking for each $(g_1, \ldots g_k) = g$, the character

$$\chi_g(h) = \prod_{j=1}^{k} (-1)^{g_1 h_1 + \cdots + g_k g_k}.$$

Furthermore, $L^2(G)$ can be replaced by $\mathbb{R}[\mathbb{Z}_2^{\oplus k}] := \{\sum_{g \in \mathbb{Z}_2^{\oplus k}} \lambda_g \cdot g \mid \lambda_g \in \mathbb{R}\}$, since the values of each function is completely determined by the values it takes on group elements. With this in mind, we can decompose each function in $L^2(G)$ with its decomposition

$$f = \sum_{g \in \mathbb{Z}_2^{\oplus k}} c_g \chi_g,$$

where each fourier coefficient c_g is nothing but $\langle f, \chi_g \rangle$, the value of f on each character.

Example 3.3.1. We will see here the most trivial set up for our CS-TM set up. Let μ_1 be a single mass in $\mathbb{R}^2$. Let ℓ be a line in the plane, and recall that we can take H^0 and H^1, which correspond to $\{0, 1\} = \mathbb{Z}_2$. Let $f : \mathbb{Z}_2 \to \mathbb{R}$ be the function given by $g \mapsto \mu_1(H^g)$. We have two characters χ_0 which is trivial, and χ_1 which is simply $\chi_1(0) = 1$ and $\chi_1(1) = -1$. Hence, we compute that

$$c_0 = \langle f, \chi_0 \rangle = \frac{1}{2} \left(\mu_1(H^+) + \mu_1(H^-) = \right) \frac{1}{2} \mu_1(\mathbb{R}).$$

And similarly,

$$c_1 = \langle f, \chi_1 \rangle = \frac{1}{2} \left(\mu_1(H^+) - \mu_1(H^-) \right).$$

One can verify that $f = c_0\chi_0 + c_1\chi_1$, and so $0 = f(1) - f(0)$, if and only if c_1 vanishes, or in other words, $f = c_0\chi_0 = \frac{1}{2}$, which is equivalent to the vanishing of our function ϕ_M from 3.2.1

Example 3.3.2. We now compute a slightly more complicated example. Let μ be a single mass in the plane, and specialize our symmetry group to $G = \mathbb{Z}_2^{\oplus 2}$. Given two independent lines ℓ_1, ℓ_2, let the regions they determine be given by quadrants R_g, for each $g \in \mathbb{Z}_2^{\oplus 2}$. Consider the function $f : \mathbb{Z}_2^2 \to \mathbb{R}$, and for each $g \in G$, evaluate the measure of each corresponding quadrant R_g., so in other words $f : g \mapsto \mu(R_g)$. We compute the fourier coefficients c_g for each $g \in \mathbb{Z}_2^{\oplus k}$. We can calculate the characters for the group by taking $\chi_{(g_1,g_2)}(h_1,h_2) = -1^{g_1h_1+g_2h_2}$. Here is the table of characters, which indicate values taken on each group element:

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$\chi_{(0,0)}$	1	1	1	1
$\chi_{(1,0)}$	1	-1	1	-1
$\chi_{(0,1)}$	1	1	-1	-1
$\chi_{(1,1)}$	1	-1	-1	1

From these, we compute the fourier coefficients:

$$
\begin{aligned}
c_{0,0} &= \frac{1}{4}(\mu(\mathbb{R}^2)) \\
c_{(1,0)} &= \frac{1}{4}\left(\mu(R_{0,0})) - \mu(R_{(1,0)} + \mu(R_{0,1}) - \mu(R_{1,1})\right) \\
c_{(0,1)} &= \frac{1}{4}\left(\mu(R_{0,0})) + \mu(R_{(1,0)} - \mu(R_{0,1}) - \mu(R_{1,1})\right) \\
c_{(1,1)} &= \frac{1}{4}\left(\mu(R_{0,0})) - \mu(R_{(1,0)} - \mu(R_{0,1}) + \mu(R_{1,1})\right)
\end{aligned}
$$

and so, we can write

$$
f = \frac{1}{4}(\mu(\mathbb{R}^2)) + \sum_{g \in \mathbb{Z}_2 \setminus (0,0)} c_i\chi_i
$$

which tells us that $f : \mathbb{Z}_2^{\oplus 2} \to \mathbb{R}^4$ is an equipartition if and only if c_g vanish for all nontrivial $g \in \mathbb{Z}_2^{\oplus 2}$, or in other words, $f = \frac{1}{4}\mu(\mathbb{R}^2)$ is constant.

We can in fact tell more here. For example, if we want only ℓ_1 to bisect the mass. This is simply asking that $\frac{\mu(\mathbb{R}^2)}{2} = f(0,0) + f(1,0) = \frac{\mu(\mathbb{R}^2)}{2} + c_{(0,1)}[\chi_{(0,1)}(0,0) + \chi_{(0,1)}(1,0)]$, which is equivalent to $c_{(0,1)}$ vanishing. Similarly, if $c(1,1)$ vanishes, then $\mu(0,0)+\mu(1,1)=\mu(0,1)+\mu(1,0)$, so we have another "partial equipartition."

The purpose of the previous was only to show how using the representation theoretic construction for our set up gives us greater control over the geometric problem, in a way that is quite natural. For a problem p, the equivariant maps we care about generalize the construction from 3.2.1 in a natural way.

We now describe the procedure for the equipartition problem with representation-theoretic language:

We can indeed check that $\{R_g\}_{0\neq g\in G}$ is an equipartition of a mass if and only if $c_g=0$ for all nontrivial group elements in the fourier decomposition of $f: g\mapsto \mu(R_g)$. Indeed, we can still remove the diagonal (trivial representation) since $f\in \mathbb{R}[G]$, in a natural way, and indeed

$$\{\sum_{g\in G} r_g\cdot g \mid g=0\} = \{\sum_{0\neq\epsilon\in G} a_\epsilon \chi_\epsilon\}$$

so we are in fact dealing with the regular representation $U_k := \mathbb{R}[\mathbb{Z}_2^{\oplus k}]$ from before, so for a problem with m masses in $\mathbb{R}^d$ with k hyperplanes, we have actually defined a map $(S^d)^{\oplus k}\to U_k^{\oplus m}$, and we can remove the diagonal by taking

$$\phi_i = \sum_{h\in\mathbb{Z}_2^{\oplus k}} \left(\mu_i(R_h(x) - \frac{1}{2^k}\mu_i(\mathbb{R}^d)\right) h.$$

and constructing $\phi_M = (\phi_1,\dots,\phi_m)$ as before, whose vanishing is equivalent to an equipartition.[2] Checking equivariance amounts to checking that the assignment $\phi_M(hx)=\chi_\epsilon(h)f(x)$, which is done basically by checking that the averaging map is G-linear as in 2.1.4.

3.3.2 orthogonality as an enlarged representation

We arrive now at the problem of orthogonality, and in the firm tradition of motivating technical definitions in this document, we begin with an example:

[2]in the example, we asked that ϕ_M was constant, but here we include a correction term, so that we throw out the trivial representation.

Example 3.3.3. Recall the map $f : \mathbb{Z}_2^{\oplus 2} \to \mathbb{R}$ from 3.2.2. We have seen from the previous section that we can rewrite f in terms of characters, or one dimensional representations, so we have a map

$$\phi : S^2 \times S^2 \to \bigoplus_{\epsilon \neq 0} V_\epsilon.$$

Recall from 3.2.2 that there is a $\mathbb{Z}_2^{\oplus 2}$-equivariant map $g : S^2 \times S^2 \to \mathbb{R}$ given by $g((a_1, b_1), (a_2, b_2)) = a_1 \cdot a_2$ whose vanishing implies orthogonality. The following question is important: under what action is it invariant? It is a one dimensional representation, so it must be a character, and indeed, we require that for each $x \in \mathbb{R}$, we also have that $(1,0) \cdot x = -x$ and $(0,1) \cdot x = -x$, so it is invariant under the action of $\chi_{(1,1)}$. Hence, we can actually consider the new target space $V_{(1,1)}$, and take the obvious map $\tilde{\phi}$ that evaluates the inner product of two hyperplanes and sends it to $V_{(1,1)}$, which is again equivariant.

The general procedure is exactly analogous to this example and we preceed as before, but keeping in mind that we obtain copies of this 1 dimensional representation for orthogonality with various inner products.

3.4 From Equivariant Maps to Characteristic Classes

3.4.1 The General Situation

The main point of this section is to establish a bijective correspondence between equivariant maps $f : X \to_G Y$ and sections of a certain vector bundle. This will allow us to employ the theory of characteristic classes to prove the nonexistence of nonvanishing maps. We assume working knowledge of Vector Bundles as well as characteristic classes, which can be found in appendices B and C at the end of this document.

We will follow the exposition of Matschke [5] quite closely here.

Given a G-equivariant map $f : X \to_G Y$, this induces an obvious map

$$s_f : X/G \to (X \times Y)/G \qquad [x] \mapsto [x, f(x)]$$

where $X \times Y$ gets the diagonal action, $g(x,y) = (g(x), g(y)$, and the relevant bundle is $p : (X \times Y)/G \to X/G$, with fiber Y and projection given by $p([x,y]) = [x]$.[3]

We can then specialize to our situation $f : (S^d)^{\oplus k} \to U^k$, which is equivariant under the action of $\mathbb{Z}_2^{\oplus k}$. The key point here, is that S^d is a free $\mathbb{Z}_2^{\oplus k}$ space, and $\mathbb{R}^n$ is the fiber (a real vector space) where $\mathbb{Z}_2^{\oplus k}$ acts linearly. Hence, we in fact have a *vector* bundle $p : (X \times Y)/G \to X/G$, and we want to show is that the fiber of each bundle is just a vector space, in which case the following theorem applies:

Theorem 3.4.1. *Suppose $p : E \to B$ is a vector bundle of rank n that admits a nowhere vanishing cross-section. Then the n^{th} Stiefel-Whitney class $\omega_n(p) \in H^n(B, \mathbb{Z}_2)$ is trivial.*

Proof. See Theorem 2.3.4. □

With this in mind, we note that a nonvanishing map $f : X \to_G Y$ induces a nonvanishing section if and only if it is itself nonvanishing. Hence, if we can show the top stiefel whitney class to be nontrivial, this will suffice to show that f cannot exist.

3.4.2 Cohomology of $\mathbb{R}P^d$

The base space from the previous section X/G for our problem is exactly $(S^d)^{\oplus k}/\mathbb{Z}_2^{\oplus k}$, equipped with the diagonal action, or in other words $(\mathbb{R}P^d)^{\oplus k}$, the k fold product of projective spaces. Hence, all stiefel whitney classes will belong to the cohomology ring $H^*((\mathbb{R}P^d)^{\oplus k})$, and so by the Kunneth formula [4],

$$H^n((\mathbb{R}P^d)^{\oplus k}) \cong \bigoplus_{j_1+\cdots+j_k=n} H^{j_1}(\mathbb{R}P^d) \otimes \cdots \otimes H^{j_k}(\mathbb{R}P^d),$$

as graded $\mathbb{Z}_2$ vector spaces, so it will suffice to understand the polynomial ring of $\mathbb{R}P^d$.

As groups, this is easy, it's not difficult to see that $H^i(\mathbb{R}P^n) = \mathbb{Z}_2$. There is an obvious CW structure for $\mathbb{R}P^n$ given by a single cell in each dimension with attaching map $S^{k-1} \to \mathbb{R}P^{k-1}$,

---

[3] there is a technicality we are supressing here. The assignment s_f actually does not map to $(X \times Y)/G$ generally. Since f is equivariant, if $g \in G_x$ (the isotropy subgroup for $x \in X$), then by equivariance, we have that $g(f(x)) = f(g(x)) = f(x)$, so in fact f will map to the collection of $[x,y] \in (X \times Y)/G$ such that $G_x \subset G_y$, but our actions will be free and so the two spaces will coincide.

[4] see the 2.3.6

which is nothing but the quotient map. One can then compute that $H_i(\mathbb{R}P^n) = \mathbb{Z}_2$ for $i \leq n$, and zero otherwise, and by poincare duality, $H^i = \mathbb{Z}_2$ as well. The multiplicative structure also follows from Poincare duality and induction. On one hand, $H^*(\mathbb{R}P^1)$ is clearly $\mathbb{Z}_2[x]/x^2$, but we also know that $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$ induces isomorphisms on homology for $i < n-1$, so we need only show that in $\mathbb{R}P^n$ we have that $x \smile x^{n-1} \neq 0$, since x generates $H^1(\mathbb{R}P^n)$, which turns out to be true. See [6] 3.40 for details.

Hence we have:

Theorem 3.4.2. $H^*(\mathbb{R}P^n) \cong \mathbb{Z}_2[x]/(x^{n+1})$. *Moreover, if* $\gamma : E \to \mathbb{R}P^n$ *is the canonical line bundle, then* $w_1(E) = x$.

Proof. The latter claim follows from the axioms provided in definition 2.3.2, where axiom 4 provides that $E \to \mathbb{R}P^1$, and the naturality axiom ,since the inclusion $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ induces an isomorphism on cohomology, so $w_1(E) \neq 0$, and hence it is x. □

Finally, note that by algebraic considerations, we have that

$$\mathbb{Z}_2[x_1]/(x_1^{d+1}) \otimes \mathbb{Z}_2[x_2]/(x_2^{d+1}) = \mathbb{Z}_2[x_1, x_2]/(x_1^{d+1}, x_2^{d+1})$$

3.4.3 Obstructions to equivariant maps $f : (S^d)^{\oplus k} \to \bigoplus_{\epsilon \neq 0} V_\epsilon$.

Here we provide the actual set up for passing from some bundle arising from an equivariant map $f : (S^d)^{\oplus k} \to \bigoplus_{\epsilon \neq 0} V_\epsilon$, and the respective top stiefel whitney class (which is the obstruction to nonvanishing.)

Recall that $f : X \to_G Y \setminus \{0\}$ correspond bijectively to some nonvanishing section in the bundle $p : X \times_G Y \to X/G : [x, y] \mapsto [x]$, where the section is simply $[x] \mapsto [x, f(x)]$. In particular, we have that $Y = \oplus_{\epsilon \neq 0} V_\alpha$. [5] We can let $p_\alpha : X \times_G V_\alpha \to X/G$ be the restriction of this bundle to a single subspace.

Note how excellent this situation is, since the whitney sum formula tells that

$$w_n(X \times_G Y) = w_n\left(\bigoplus_\alpha (X \times_G V_\alpha)\right) = \prod_\alpha w_1(p_\alpha)$$

[5]this is our lazy way of writing the decomposition into irreducible representations

so it will suffice to understand the line bundle $p_\alpha : (S^d)^{\oplus k} \times_G V_\alpha \to (RP^d)^{\oplus k}$, which is the k-fold tensor product of line bundles over $\mathbb{R}P^1$, so in particular, $w_1(P_\alpha) = w_1(p^1_\alpha)x_1 + \cdots + w_1(p^k_\alpha)x_k$ by theorem 2.3.5. These characteristic classes are either trivial, or 1, since these line bundles are either $S^1 \times \mathbb{R}$ or the mobius bundle, depending on the representation on V_α.

If one wishes deep down to formalize the last part of the preceeding paragraph, we examine instead the inclusion

$$i_h : S^1 \cong \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^d \hookrightarrow (\mathbb{R}P^d)^{\oplus k}$$

which induces on cohomology the map $H^1(\mathbb{R}P^d)^{\oplus k}) \to H^1(\mathbb{R}P^1)$ given by $\lambda_1 x_1 + \ldots \lambda_k x_k \mapsto \lambda_h x_h$. By naturality, the pullback of said inclusion is just $\lambda_h = 0, 1$ depending on the one dimensional $\mathbb{Z}_2$ representation in the h^{th} factor. Pulling back, we see that this bundle is indeed the tensor product along each of these inclusions, since this is exactly what is required for "bilinearity" [add details here, I didn't want to just yet. Still trying to think about it].

In other words, if the h^{th} factor in the fourier decomposition vanishes, it is not included in the polynomial. On the other hand, the rest of these guys end up in the n^{th} stiefel whitney class, and indeed we find that

$$w_n\left((S^d)^{\oplus k} \times_G (\bigoplus_{\epsilon \neq 0} V_\alpha)\right) = \prod_{0 \neq a \in \mathbb{Z}_2^k} (a_1 x_1 + \ldots a_k x_k) \in \mathbb{Z}_2[x_1, \ldots x_k]/(x_1^{d+1}, \ldots, x_k^{d+1})$$

3.5 The Polynomial Method For Plane Equipartitions

We have the following theorem due to S. Simon [4]:

Theorem 3.5.1. *Let $m = kd$. Let $h(u_1, \ldots u_k) = \prod_{i=1}^{kd}(a_{i,1}u_1 + \ldots a_{i,k}u_k) \in \mathbb{Z}_2[u_1, \ldots u_k]/(u_1^{d+1}, \ldots u_k^{d+1})$. If $h(u_1, \ldots, u_k) = u_1^d \cdots u_k^d$, then any $\mathbb{Z}_2^{\oplus k}$ equivariant map $f : (S^d)^{\oplus k} \to \oplus_{i=1}^{kd} V_{(\alpha_{i,1}, \ldots \alpha_{i,k})}$ has a zero. Equivalently, given m masses on $\mathbb{R}^d$, there exists k hyperplanes so that $c_{i,a_i} = 0$ for all $1 \leq i \leq m$ for the corresponding fourier expansion.*

Proof. See Proposition 6.2 in [4]. ∎

The point here, is that we take the calculation from the previous section, we get that for a particular kd dimensional representation of $\mathbb{Z}_2^{\oplus k}$, with basis $\epsilon_i := (\epsilon_{i,1}, \dots, \epsilon_{i,k})$ for $1 \leq i \leq kd$, then the corresponding polynomial will be

$$\prod_{i=1}^{kd}(\epsilon_{i,1}x_1 + \cdots + \epsilon_{i,k}x_k) \in \mathbb{Z}_2[x_1, \dots, x_k]/(x_1^{d+1} \cdots x_k^{d+1})$$

and we want to ask when this vanishes. On one hand, we know that once we multiply out, we should get some sum of homogeneous polynomials $ax_1^{\alpha_1} \cdots x_k^{\alpha_k}$ with $\sum_{i=1}^k \alpha_i = kd$, but as soon as $\alpha_i \neq d$ for some choice of i, another one must be greater than d, so the monomial will vanish. In this way, the generator $x_1^d \cdots x_k^d$ is the only term that survives.

The following explicit can be found in [4] as well.

Theorem 3.5.2. *(Simon, 2017) Given k hyperplanes and some subcollection $\mathcal{O}_k := \{(r,s)j \leq r < s \leq k\}$ where s, r are orthogonal, the corresponding polynomial is*

$$\sum_{\sigma \in S_{k-j+1}} u_{\sigma(j)}^{k-j} \cdot u_{\sigma(j+1)}^{k-j-1} \cdots u_{\sigma(k)}^{0}$$

.

Proof. Then we know that this representation is nothing but $V_{\mathcal{O}} := \bigoplus_{(r,s)\in\mathcal{O}} V_{e_r+e_s}$, we can see that these arise from subrepresentations of the form $(0, \dots, 1, \dots, 1, \dots 0)$, that are nontrivial in only two places, so each direct summand comes to the polynomial $u_r + u_s$, and the whitney product formula (or the previvous theorem, tells that the corresponding polynomial is precisely

$$\prod_{(r,s)\in\mathcal{O}} (u_r + u_s) = \sum_{\sigma \in S_{k-j+1}} u_{\sigma(j)}^{k-j} \cdot u_{\sigma(j+1)}^{k-j-1} \cdots u_{\sigma(k)}^{0}$$

since this is exactly the Vandermonde determinant. □

We also obtain similar methods for cascades from [4], using the setup discussed in 3.2.3:

Theorem 3.5.3. *(Simon, 2017): The polynomial corresponding to a $\mathcal{M}$, denoted $p_k()$ where is some k-tuple consisting of collections of masses, is precisely*

$$P_k() = P_{k,1}^{m_1} \cdots P_k^{m_k}.$$

In other words, the polynomial corresponding to cascades is nothing but the multiplication of polynomials for each subcollection we seek to consider.

We now prove the Borsuk Ulam Theorem in our new language.

Example 3.5.4. For each choice of k masses, there exists a hyperplane in dimension $d = k$ that equiparts all of this collection. Take our usual map $\phi_M : S^d \to U_k^m$, we can see that the corresponding polynomial $\prod_{i=1}^{d} e_1 x = x^d \neq 0 \in \mathbb{Z}[x]/(x^{d+1})$.

3.6 Geometric Lower Bounds and Optimizing Results

One of the nicest things about the Ham-Sandwich theorem is that it is an *optimal* result in the following sense:

Definition 3.6.1. given k hyperplanes and m masses, we say that the minimal dimension d in which we can ensure a solution to a geometric problem P is an optimal result. When it is unambigious what the prolbem is, we write $d = \Delta(k, m)$.

Lemma 3.6.2. *given* 1 *hyperplane and* k *masses, we have that* $\delta(k, 1) = k$.

Proof. We have already proven in 3.5.4 that k is an upper bound on $\Delta(k, 1)$. However, we can see that it is optimal, since if $d < m$, we can form a d-simplex in $\mathbb{R}^d$, and place small masses on each vertex, in which case, no hyperplane will simultaneously intersect all of them.

□

In fact, using the Ham-sandwich theorem alone to find an upper bound on the equipartition problem. Indeed, $\Delta(k, m) \leq 2^{k-1}m$. This is true, since we can bisect all of the m masses with one hyperplane, and inductively, we can bisect $2m$ remaining masses in dimension $2m$, and continuing this way, we will finish after k steps.

Similarly, there is a "trivial" or automatic lower bound on the equipartition problem, shown by Ramos:

Theorem 3.6.3. $\Delta(k, m) \geq m \cdot \frac{2^k - 1}{k}$

Proof. see [9] for a full proof or [5] Lemma 2.5 for a proof sketch. □

One should note, that the trivial upper bound grows an order of magnitude above Ramos' lower bound for the problem above, although it is in fact conjectured that $\Delta(m, k) = \lceil m \cdot \frac{2^k-1}{k} \rceil$, and indeed this has been confirmed for all known values of $\Delta(m, k)$ [4].

The best known upper bound on general the problem is

$$\Delta(m, k) \leq 2^{q+k-1} + r$$

where $m = 2^q + r$ and $0 \leq r < 2^q$. [4] However, as r tends to zero, the conjectured lower bound and this upper bound .

Geometrically, one can deduce the conjectured lower bound heuristically. In particular, one can guess that kd should be greater than the number of equations for a geometric problem. Indeed, for each mass m, there are $2^k - 1$ coefficients that need to vanish in the fourier expansion, and so we can "guess" a lower bound for the equipartition problem. In the following chapters, we will use this heuristic as a conjectured lower bound for each geometric problem P, and aim to find topological upper bounds that are as close to possible for the lower bound.

4
Main Results

4.1 Generalized Makeev-Type Problems

We begin by considering the generalized Makeev problem as outlined in [10]:

Definition 4.1.1. We say that the tuple of natural numbers (d, m, k, l) with $l \leq k$ is *admissible* if for every collection of m masses in $\mathbb{R}^d$, there exist k mutually orthogonal hyperplanes $H_1, \ldots H_k$ such that any l of them equipart all of the measures.

The problem is to find admissible tuples. [1] We will relax the orthogonality assumption so that we require only that two subsets subset $A, B \subset \{H_1, \ldots H_k\}$ are orthogonal to each other. (Indeed, we will see that full orthogonality is an equivalent to asking that any 2 of $k-1$ hyperplanes equipart a new mass with the polynomial method.)

When $m = 1$ and $l = 2$, there is the following known result, shown by Makeev [11], whose proof we will outline using the polynomial method:

Theorem 4.1.2. *(Makeev, 2007) Given k hyperplanes and* 1 *mass, we can guarantee that any* 2 *among them equipart the mass in dimension $d = k$. In other words, the tuple $(k, 1, k, 2)$ is admissible.*

[1] note that when $l = k$, we recover the classical equipartition problem.

Proof. One sees that in the fourier decomposition for $f : \mathbb{Z}_2^{\oplus k} \to \mathbb{R}$, $g \mapsto \mu(\mathcal{R}_g)$, we require that all coefficients associated to nontrivial group elements of the form $(*, \dots, i, \cdots *)$ must vanish as i ranges through $\{1, \dots, k\}$. In particular, this implies that the corresponding polynomial is

$$\prod_{j=1}^{k} x_i \prod_{i<j}(x_i + x_j) = \prod_{j=1}^{k} x_i \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} x_i^{\sigma(i)-1} \right) = \left(\sum_{\sigma \in S_n} \prod_{i=1}^{k} x_i^{\sigma(i)} \right)$$

which clearly does not vanish in the ring $\mathbb{Z}_2[x_1, \dots, x_k]/(x_1^{k+1}, \dots, x_k^{k+1})$, which is the cohomology ring for the k-fold product of $\mathbb{R}P^k$. □

One goal here will be to generalize this result for when $m > 1$.

Additionally consider "Cascading" Makeev-type problems, as described in [4] whenever possible. In particular, given a Makeev-type solution for (d_1, m_1, k, l_1), can we for the same collection, find a solution so that any l_2 of $k-1$ hyperplanes equipart some mass m_2 as well? In other words, we are looking for a dimension d so that (d, m_1, k, l_1) and $(d, m_2, k-1, l_2)$ can be solved simultaneously for the same collection of k hyperplanes.

For the most part, we will restrict our attention to the special cases of $l = 2, 3$ where computations remain manageable.

4.2 Equipartitions by any 2 of k hyperplanes

First, we illustrate the general method here with the following small dimension:

Example 4.2.1. let μ_1 be a mass on $\mathbb{R}^d$. Given three hyperplanes H_1, H_2, H_3, and that we want any two of them to equipart the mass into quarters H_1, H_2. If we restrict our attention to H_1 and H_2, we first note that

$$H_1^{\pm} \cap H_2^{\pm} = (H_1^{\pm} \cap H_2^{\pm} \cap H_3^{+}) \cup (H_1^{\pm} \cap H_2^{\pm} \cap H_3^{-})$$

This allows us to view $\mathbb{Z}_2^{\oplus 2} \hookrightarrow \mathbb{Z}_2^{\oplus 3}$ via the decomposition $\mathbb{Z}_2^{\oplus 3} = \mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_2$. In other words, we "forget" the third hyperplane (the representation is trivial) we want all nontrivial (associated) coefficients $C(*, *, 0)$ to die. In particular, let $f : \mathbb{Z}_2^{\oplus 3} \to \mathbb{R}$ be as in the theorem. Then,

$$f := C_{(0,0,0)} + \sum_{\epsilon \neq 0} C_\epsilon \chi_\epsilon,$$

and since the assignment i additive, we can deduce that $\mathcal{R} = \mathcal{R}_{(g_1,g_2,0)} \cup \mathcal{R}_{g_1,g_2,1}$ implies that $f(g_1, g_2, 0) + f(g_1, g_2, 1) = 1/4$, but we also see that this forces every other coefficeint to be zero in the decomposition. A different way of viewing this, is to consider all nontrivial elements of the form $(g_1, g_2, 0)$ to be a full equipartition, which we have already seen required that they all die. In other words, we obtain polynomials $x_1, x_2, (x_1 + x_2)$, and likewise for the other two choices H_2, H_3 and H_1, H_3. Eliminating "double counting", we see that the following coefficients need to die:

$$\{(1, 0, 0), (1, 1, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1), (0, 1, 1)\}$$

corresponding to the polynomial assignment

$$x_1x_2x_3(x_1 + x_2)(x_1 + x_3)(x_2 + x_3).$$

Next, we consider the problems quadruple $(d, 2, 3, 2)$ with hyperplanes H_1, H_2, H_3 along with a simultaneous solution for $(d, 1, 2, 2)$ for the remaining two hyperplaes H_2, H_3. In particular, we show that this can be done in $d = 6$ with some degrees of freedom. To make the problem geometrically tight, we also impose orthogonality conditions on H_1, H_2 as well as one additional bisection for H_3. Either of the latter two conditions may be omitted to obtain a solution for the weaker problem.

To see the conjectured lower bound, note that we need $kd \geq$ number of conditions. In particular, the number of conditions is 12 for the first problem (which can be seen heuristically, or from the previous example), 2 for the second problem, along with an additional 3 equations, which gives a total of 17, so in particular, we expect $d = 6$ to be optimal geometrically, and indeed this gives a topological upper bound:

Theorem 4.2.2. *Given 3 hyperplanes, if any 2 of them equipartition 2 masses μ_1, μ_2, while H_2, H_3 also equipart a mass and both H_1, H_2 are orthogonal to H_1, we are guaranteed a solution in $d = 2 \cdot 3$. In the case that H_3 bisects a single mass, the result is sharp with respect to our conjectured lower bound and can be done in $d = 6$.*

Proof. Recall that the orthogonality conditions correspond to the polynomial $(x_1+x_2)(x_1+x_3)$ from as in 3.5.2. We can also check that equiparting both masses corresponds to squaring a polynomial, which is just the frobenius endomorphism over $\mathbb{Z}_2$, so we simply square each element in the polunymial. From these facts, and the first result, we see that the relevant polynomial is

$$\begin{aligned}
&\left(\sum_{\sigma\in S_3} x^6_{\sigma(1)}x^4_{\sigma(2)}x^2_{\sigma(3)}\right)\cdot\left(\sum_{\tau\in S_2} x^2_{\tau(2)}x_{\tau(3)}\right)(x_1+x_2)(x_1+x_3)\\
&= x_1^6\left(\sum_{\sigma\in S_3} x^4_{\sigma(2)}x^2_{\sigma(3)}\right)\cdot\left(\sum_{\tau\in S_2} x^3_{\tau(2)}x^2_{\tau(3)}\right)\\
&= x_1^6\left(\sum_{\sigma\in S_3} x^6_{\sigma(2)}x^5_{\sigma(3)}\right)
\end{aligned}$$

where S_2 is acting on $\{2,3\}$.

□

Continuing in this way, we can examine the same set up, but now with $(d,2,4,2)$ and $(d,1,3,2)$ for $H_1,\dots H_4$, we can check that the number of equations is now $20+6$, so we get that $4d\geq 26$, or $d\geq 7$. Unfortunately, the corresponding polynomial is already degree 8, and so we may attempt full orthogonality, which would impose an additional 6 constraints, implying a conjectured lower bound of $d\geq 8$, but unfortunately the polynomial vanishes here. Instead we impose weaker conditions analogous to the previous theorem 4.2.2, so we require that all hyperplanes are orthogonal to H_1, with H_3,H_4 orthogonal to H_2(so all but H_3 and H_4 are orthogonal.) This gives an additional 5 conditions, and if we require that either H_3 or H_4 bisect a mass, then we get precisely 32 geometric conditions, and indeed we have the following definition/theorem:

Definition 4.2.3. Let $(i_1,\dots,i_k)$ denote the problem where i_1 in the j^{th} position indicates that any 2 of $H_j,\dots H_k$ equipart i_j masses.

Theorem 4.2.4. *In* $(2,1,0,0)$*, if we require that* H_2,H_3,H_4 *are perpendicular to* H_1 *and* H_3,H_4 *are perpendicular to* H_2*, then this can be done in* $d=8$*, and if we require that either* H_3 *or* H_4 *bisect one of the masses, the result is sharp in the conjectured lower bound.*

Proof. Similarly to the previous case, in $d = 8$, we have the polynomial

$$x_1^8 \left(\sum_{\sigma \in S_3} x_{\sigma(2)}^6 x_{\sigma(3)}^4 x_{\sigma(4)}^2 \right) \cdot \left(\sum_{\tau \in S_3} x_{\tau(2)}^3 x_{\tau(3)}^2 x_{\tau(4)} \right) \left(\prod_{i=2}^4 (x_1 + x_i) \right) (x_2 + x_3)(x_2 + x_4)$$
$$= x_1^8 \left(\sum_{\sigma \in S_4} x_{\sigma(2)}^6 x_{\sigma(3)}^4 x_{\sigma(4)}^2 \right) \cdot \left(\sum_{\tau \in S_3} x_{\tau(2)}^4 x_{\tau(3)}^3 x_{\tau(4)}^2 \right) (x_2 + x_3)(x_2 + x_4).$$

The key reduction here is that bounding the degree by 8 forces many summands to vanish, and hence there is the decomposition

$$\left(\sum_{\sigma \in S_4} x_{\sigma(2)}^6 x_{\sigma(3)}^4 x_{\sigma(4)}^2 \right) \cdot \left(\sum_{\tau \in S_3} \sigma_{\tau(2)}^3 x_{\tau(3)}^2 x_{\tau(4)} \right)$$
$$= \sum_{\sigma \in S_3} x_{\sigma(2)}^8 x_{\sigma(3)}^5 x_{\sigma(4)}^6 + \sum_{\sigma \in S_3} x_{\sigma(2)}^8 x_{\sigma(3)}^7 x_{\sigma(4)}^6$$

and the left summand vanishes since there are repeated degrees and we are summing over the symmetric group. Hence, we can see by this reduction and keeping track of degrees that the full polynomial simplifies to

$$x_1^8 (x_2^8 x_4^7 x_3^8 + x_3^7 x_4^8 x_2^8)$$

which does not vanish in $d = 8$. Of course, one can now ask for a bisection by either H_3 or H_4 will yield the polynomial $x_1^8 x_2^8 x_3^8$, which is topologically tight for $d = 8$, and accomplishes the geometric lower bound. □

Unfortunately, this pattern does not continue for $k > 4$, but this is in some sense optimal for small k, by [Eric's result].

One might hope that we could get a full cascade in the $k = 4$ case, so that we have any 2 of H_1, H_2, H_3, H_4 equiparrt, along with any 2 of H_2, H_3, H_4 and finally any 2 of H_3, H_4. Indeed, this can be donee, but it is possible in minimal dimension $d = 6$ with reduced polynomial

```
x1^4*x2^6*x3^6*x4^4 + x1^3*x2^6*x3^6*x4^5 + x1^4*x2^6*x3^4*x4^6 + x1^3*x2^6*x3^5*x4^6
+ x1^4*x2^6*x3^5*x4^4 + x1^3*x2^6*x3^6*x4^4 + x1^4*x2^6*x3^4*x4^5 + x1^4*x2^4*x3^6*x4^5
+ x1^3*x2^5*x3^6*x4^5 + x1^2*x2^6*x3^6*x4^5 + x1^3*x2^6*x3^4*x4^6 + x1^4*x2^4*x3^5*x4^6
+ x1^3*x2^5*x3^5*x4^6 + x1^2*x2^6*x3^5*x4^6 + x1^4*x2^6*x3^5*x4^3 + x1^3*x2^6*x3^6*x4^3
```

```
+ x1^3*x2^6*x3^5*x4^4 + x1^2*x2^6*x3^6*x4^4 + x1^4*x2^6*x3^3*x4^5 + x1^3*x2^6*x3^4*x4^5
+ x1^3*x2^6*x3^3*x4^6 + x1^2*x2^6*x3^4*x4^6
```

which is indeed quite complicated. One could instead impose some geometric conditions that are still allow for a successful computation in $d = 6$ such as orthogonality, but this author was not able to raise the geometric lower bound above $d = 5$ for a tight result.

Instead, we seek to change the number of masses, first to an arbitrary 2^k and then any positive integer in the first and second co-ordinate to find closer to optimal results.

However, sufficient conditions can be given for cases where we require $(2^k, r, 0 \ldots, 0)$, with $r < 2^k$, and full orthogonality of n hyperplanes in dimension $d = 2^k \cdot n$.

Theorem 4.2.5. *We can guarantee an equipartition for 2^k masses by any 2 of n hyperplanes, an equipartition by of another r masses with $r < 2^k$, and full orthogonality of all masses in $d = 2^k \cdot n$ granted that $n < \frac{2^k+r+1}{r+1}$.*

Proof. We find that the polynomial corresponding to the problem is precisely

$$\left(\sum_{\sigma\in S_n} x_{\sigma(1)}^{2^k n} \cdots x_{\sigma(n)}^{2^k}\right)\left(\sum_{\tau\in S_{n-1}} x_{\tau(2)}^{n-1} \cdots x_{\tau(n)}^{1}\right)^r \left(\sum_{\sigma\in S_n} x_{\sigma(1)}^{0} \cdots x_{\sigma(n)}^{n-1}\right)$$

after applying applying "freshman's dream" for 2^k. Furthermore, the assumption that $d = 2^k \cdot n$ bounds the possibilities for $\sigma(1)$, and indeed we see that $\sigma(1) = 1$, since all other terms vanish after multiplying with the second polynomial. Hence, we obatain the polynomial

$$x_1^{2^k n} \cdot \left(\sum_{\sigma\in S_{n-1}} x_{\sigma(2)}^{2^k(n-1)} \cdots x_{\sigma(n)}^{2^k}\right)\left(\sum_{\tau\in S_{n-1}} x_{\tau(2)}^{n-1} \cdots x_{\tau(n)}^{1}\right)^r \left(\sum_{\sigma\in S_n} x_{\sigma(1)}^{0} \cdots x_{\sigma(n)}^{n-1}\right).$$

However, we also note that all the terms in the rightmost factor with any $\tau(i) = 1$ vanish by our assumption on degree, so we obtain the equality

$$\begin{aligned} & x_1^{2^k n} \cdot \left(\sum_{\sigma \in S_n} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1 \right)^{2^k} \left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1 \right)^r \left(\sum_{\sigma \in S_n} x_{\sigma(1)}^0 \cdots x_{\sigma(n)}^{n-1} \right) \\ &= x_1^{2^k n} \cdot \left(\sum_{\sigma \in S_n} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1 \right)^{2^k} \left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1 \right)^r \left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)} \cdots x_{\sigma(n)}^{n-1} \right) \\ &= x_1^{2^k n} \left(\sum_{\sigma \in S_n} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1 \right)^{2^k + r} \left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)} \cdots x_{\sigma(n)}^{n-1} \right) \end{aligned}$$

which does not vanish whenever $(n-1)(2^k + r + 1) < 2^k n$, or $n < \frac{2^k + r + 1}{r+1}$.

□

We obtain the following corollary, since $\left(\sum_{\sigma \in S_n} x_{\sigma(1)}^0 \cdots x_{\sigma(n)}^{n-1} \right)$ reduces to $\left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)} \cdots x_{\sigma(n)}^{n-1} \right)$ in the previous theorem:

Corollary 4.2.6. *We can guarantee an equipartition for 2^k masses by any 2 of n hyperplanes, an equipartition by of another $r + 1$ masses with $r < 2^k$, and full orthogonality of all masses in $d = 2^k \cdot n$ granted that $n < \frac{2^k + r + 1}{r+1}$.*

Proof. The previous theorem yields the same polynomial by the assumption on degree, so we obtain precisely the same bounds.

In particular, we have the orthogonality polynomial

$$\sum_{\sigma \in S_n} x_{\sigma(1)}^0 \cdots x_{\sigma(k)}^{k-1}$$

but since we require that $\sigma(1) = 1$, we recover the same polynomial as the previous theorem.

□

We remark that this result is "asymptotically" pretty good with respect to the conjectured lower bound. Indeed, we can calculate that the the number of equations is

$$2^k \left(\binom{n}{2} + n \right) + r \left(\binom{n-1}{2} \right) + n = n \left(2^k \cdot \frac{n+1}{2} + r \frac{n-2}{2} + 1 \right)$$

so the requirement that $n \cdot d > n\left(2^k \cdot \frac{n+1}{2} + r\frac{n-2}{2} + 1\right)$ implies that

$$d > 2^k \cdot \frac{n+1}{2} + r\frac{n-2}{2} + 1 = 2^{k-1}(n+1) + r \cdot \frac{n-2}{2} + 1.$$

Using the same technique now to generalize the case $m_1 = 2^k + r$ for some arbitrary integer m_1. We ask that either there is one additional mass so that any 2 of $n-1$ remaining hyperplanes equipart a single mass, or additionally that we have full orthogonality. We will do the case of one additional mass for clarity, and also to show that we really need at least oe additional constraint for the proof to go through, but the proof generalizes readily for the arbitrary case

Theorem 4.2.7. *Given n hyperplanes $H_1, \ldots H_n$ hyperplanes and $m_1 = 2^k + r$ masses in the plane, we can guarantee an equipartion by any 2 of n hyperplanes with full orthogonality in dimension $m_1 \cdot n$ granted that $n < 2^k + r + 1$.*

Proof. we have the polynomial

$$x_1^{2^k n} \cdot \left(\sum_{\sigma \in S_n} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1\right)^{2^k} \left(\sum_{\sigma \in S_{n-1}} x_{\sigma(2)}^{n-1} \cdots x_{\sigma(n)}^1\right)^r \left(\sum_{\sigma \in S_n} x_{\sigma(1)}^0 \cdots x_{\sigma(n)}^{n-1}\right)$$

by our assumption on degree. In particular, this yields the sufficient condition that $(n-1)(2^k + r + 1) < (2k + r)n$ or $n < 2^k + r + 1$.

□

Note that this problem is analogous to the last, where $r = 0$ here, and in the previous problem, which give the same sufficient condition. We of course, also have a corollary replacing the orthogonality condition with any 2 of $n-1$ hyperplanes equiparting a single additional mass m_2. We generalize this in the following theorem:

Theorem 4.2.8. *Given $m_1 = 2^q + r$ masses, and n hyperplanes, we can guarantee an equipartition of each m_1 by any 2 of n, along with an equipartition of an additional collection of k masses by any 2 of $n-1$ masses in dimension $m_1 \cdot n$ granted that $nk < 2^1 + r + k$.*

Proof. replace 1 in the previous proof by k.

□

We can optimize this bound (topologically) when m_1 is actually a power of 2, and in fact we can guarantee an equipartition by any 2 of $n-1$ remaining hyperplanes of $m-1$ in this case.

Corollary 4.2.9. *Let $H_1, \dots H_n$ be a collection of n hyperplanes and $m_1 = 2^q$ and $m_2 = 2^q - 1$. We can guarantee an equipartition of m_1 masses by any 2 of n hyperplanes, and an equipartition of m_2 by any 2 of $n-1$ hyperplanes in diension $d = 2^q \cdot n$, granted that $n \leq \frac{2^{q+1}-1}{2^q-1}$.*

This is the most we can require, since as soon as there are 2^q second masses, we would need $(n-1)\cdot(2^{r+1}) < 2^r \cdot n$, or $(n-1) < 1/2$ which occurs only for $n = 1$, which is a trivial problem. On the other hand, this result recovers 4.2.2 as a corollary, barring some of the additional geometric considerations, since we obtain that we need $n \leq 3$, which would show that the result can be obtained in dimension 6.

4.3 Equipartitions by any 3 of k hyperplanes

We begin by examining equipartitions by any 3 of 4 hyperplanes. First of all, by discussions in the previous section, note that the full polynomial corresponding to this problem:

$$P_{(3,4)} := \prod_{i=1}^{4} x_i \left(\sum_{\sigma \in S_4} x_1^0 x_2^1 x_3^3 x_4^4 \right) \left(\prod_{1 \leq i < j < k \leq 4} (x_i + x_j + x_k) \right)$$

Theorem 4.3.1. *$P_{(3,4)}$ does not vanish in dimension 5.*

Proof. However, we start by formally rewriting the third factor with $t = \sum x_i$:

$$\prod_{1 \leq i < j < k \leq 4} (x_i + x_j + x_k) = \prod_{i=1}^{4} (t - x_i)$$

Note that x_i are roots of this polynomial.

and we apply Viete's formula: given a monic polynomial of degree n, and roots $x_{i_1}, \dots x_{i_k}$, we have that

$$\sum_{i \leq i_i < \dots \leq n} x_{i_1} \cdots x_{i_k} = (-1)^k a_{n-k}$$

where of course, the $(-1)^k$ is not going to matter over $\mathbb{Z}_2$.

This allows us to rewrite our polynomial in the following way:

$$t^4 + \left(\sum_{i=1}^{4} x_i\right) t^3 + \left(\sum_{i \le i < j \le 4} x_i x_j\right) t^2 + \left(\sum_{i=1}^{4} \prod_{j \ne i} x_j\right) t + \prod_{i=1}^{4} x_i.$$

The first thing to note is that $\sum_{i=1}^{4} x_i = t$, which implies that so the the polynomial reduces to

$$\begin{aligned} P(t) &= t^4 + t \cdot t^3 + \left(\sum_{i \le i < j \le 4} x_i x_j\right) t^2 + \left(\sum_{i=1}^{4} \prod_{j \ne i} x_j\right) t + \prod_{i=1}^{4} x_i \\ &= \left(\sum_{i \le i < j \le 4} x_i x_j\right) t^2 + \left(\sum_{i=1}^{4} \prod_{j \ne i} x_j\right) t + \prod_{i=1}^{4} x_i. \end{aligned}$$

which follows since $2t^4$ vanishes over $\mathbb{Z}_2$. From here, we can completely recover the polynomial over $\mathbb{Z}_2$ by factoring t our of the linear and quadratic terms to obtain that

$$P(t) = \left(x_i^3 \cdot \prod_{j \ne i} x_j\right) + x_1 x_2 x_3 x_4$$

Multiplying this with this, we obtain the equation

$$\begin{aligned} P(t) \cdot \left(\sum_{\sigma \in S_4} x_{\sigma(1)} x_{\sigma(2)}^2 x_{\sigma(3)}^3 x_{\sigma(4)}^4\right) &= \sum_{\sigma \in S_4} x_{\sigma(1)}^2 x_{\sigma(2)}^3 x_{\sigma(3)}^4 x_{\sigma(4)}^5 + \sum_{\sigma \in S_4} x_{\sigma(1)}^1 x_{\sigma(2)}^2 x_{\sigma(3)}^5 x_{\sigma(4)}^6 \\ &+ \sum_{\sigma \in S_4} x_{\sigma(1)}^1 x_{\sigma(2)}^2 x_{\sigma(3)}^4 x_{\sigma(4)}^6 + \sum_{\sigma \in S_4} x_{\sigma(1)}^1 x_{\sigma(2)}^2 x_{\sigma(3)}^4 x_{\sigma(4)}^7 \end{aligned}$$

which does not vanish in dimension 5, due to the first summand surviving. □

Corollary 4.3.2. *We can guarantee a equipartition of 2^k masses by any 3 of 4 hyperplanes in dimension $5 \cdot 2^k$.*

Proof. Taking the lowest term summand in 4.3.1 we get that the corresponding polynomial is

$$\left(\sum_{\sigma \in S_4} x_{\sigma(1)}^2 x_{\sigma(2)}^3 x_{\sigma(3)}^4 x_{\sigma(4)}^5\right)^{2^k},$$

where 2^k distributes inside by freshman's dream, and the polynomial clearly does not vanish in $d = 5 \cdot 2^k$ for just degree reasons. □

However, what I like is that the constant term $x_1 \ldots x_4$ *is unique* in the polynomial, and clearly of minimal degree in each x_i.

Note that if we do the same thing for the $\prod_{i<j}(x_i + x_j)$ part of the total polynomial, we get that the constant term is the same polynomial, since each term is $(t - x_i - x_j)$ up to some re-ordering. The constant term for $x_1 \cdots x_k$ is just the original polynomial for us. There is some kind of principle here that I've been exploring

But really, taking the constant term in the Viete expansion above, the total polynomial becomes

$$(x_1 \ldots x_4)^2 \left(\sum_{\sigma \in S_4} x_1^0 x_2^1 x_3^2 x_4^3 \right) = \sum_{\sigma \in S_4} x_1^2 x_2^3 x_3^4 x_4^5$$

which clearly does not vanish in dimension 5.

We note that the Viete expansion allows us to make the following claim, mostly due to the fact that the constant term in $t = \sum_{i=1}^n x_i$ is unique and minimal in degree.

Theorem 4.3.3. *If we know that the corresponding polynomial for any $(k-2)$ of k hyperplanes equiparting a mass m in dimension d is nonvanishing, then we can guarantee an equipartition by any $(k-1)$ of k hyperplanes in dimension $d+1$.*

Proof. Let J_{k-2} denote the polynomial for any $k-2$ of k. Then, the polynomial for any $k-1$ of k equiparting a mass is nothing but

$$J_{k-2} \cdot \prod_{1 \leq i_1 < \cdots < i_{k-1} < k} \left(\sum_{i=1}^{k-1} x_i \right),$$

but using the Viete expansion, and putting $t = (\sum_{i=1}^k x_i)$ allows us to rewrite this as

$$J_{k-2} \cdot \prod_{i=1}^n (t - x_i)$$

and taking the unique constant term (in t), we can deduce that $\prod_{i=1}^n x_i$ does nothing except to increase the dimension of J_{k-2} by one. □

We can actually use freshman's dream to generalize the former proof for 2^i masses:

Corollary 4.3.4. *Suppose that the corresponding polynomial to an equipartition by* $(k-2)$ *of* k *hyperplanes of* 2^i *masses* $\{m_i\}$ *in dimension* d *is nonvanishing. Then we can guarantee an equipartition of* $(k-1)$ *masses in dimension* $2^i(d+1)$.

Proof. Exactly as before, except we distribute the exponent to the full polynomial in t. □

We can use the Viete trick in 4.3.3 for the any 4 of 5 hyperplanes after calculating the following:

Theorem 4.3.5. *We can guarantee an equipartition by any* 3 *of* 5 *hyperplanes in* $d = 8$.

Proof. This was proved by direct computation in SageMath, and was shown to be minimal. □

Corollary 4.3.6. *we can guarantee that any* 4 *of* 5 *hyperplanes equipart a mass* m *in dimension* 9.

Returning to the 3 of 4 case, we provide our first "full cascade" computationally, and then offer some geometric refinements to tighten the upper bound (again computational), and offer a generalization that provides a suboptimal result.

Theorem 4.3.7. *Given* 4 *hyperplanes, we can guarantee an equipartition of two masses by any* 3 *of* 4 *of four of them, an equipartition of another mass by any* 2 *of the three remaining masses, and finally a bisection of one additional mass by any* 1 *of the remaining* 2 *masses in dimension* 10. *Additionally, we can also have* H_2, H_3, H_4 *orthogonal to* H_1.

Proof. Recall the polynomial given in 4.3.1. Since we are squaring this polynomial, and 2 distributes over each sum, we get that we need only consider the square of the lowest term summand

$$\sum_{\sigma \in S_4} x_{\sigma(1)}^4 x_{\sigma(2)}^6 x_{\sigma(3)}^8 x_{\sigma(4)}^{10}.$$

From this, we obtain the full polynomial corresponding to this polynomial:

$$\left(\sum_{\sigma \in S_4} x_{\sigma(1)}^4 x_{\sigma(2)}^6 x_{\sigma(3)}^8 x_{\sigma(4)}^{10}\right)\left(\sum_{\sigma \in S_3} x_{\sigma(1)}^1 x_{\sigma(2)}^2 x_{\sigma(3)}^3\right) x_3 x_4.$$

However, by our assumption on degree we know that all terms with $\sigma(4) \neq 1$ vanish, so we obtain the reduction to

$$x_1^{10}\left(\sum_{\sigma \in S_3} x_{\sigma(2)}^4 x_{\sigma(3)}^6 x_{\sigma(4)}^8\right)\left(\sum_{\sigma \in S_3} x_{\sigma(1)}^1 x_{\sigma(2)}^2 x_{\sigma(3)}^3\right) x_3 x_4.$$

One can check now that the polynomial does not vanish in dimension 10, and is in fact is precisely

```
x1^10*x2^10*x3^10*x4^6 + x1^10*x2^9*x3^9*x4^8 + x1^10*x2^8*x3^10*x4^8 +
x1^10*x2^9*x3^8*x4^9 + x1^10*x2^7*x3^10*x4^9 + x1^10*x2^10*x3^6*x4^10 +
x1^10*x2^8*x3^8*x4^10 + x1^10*x2^7*x3^9*x4^10.
```

Indeed, multiplying the previous polynomial by $(x_1+x_2)(x_1+x3)x_1+x_4)$ gives the polynomial $x_1^{10}x_2^{10}x_3^{10}x_4^9 + x_1^{10}x_2^{10}x_3^9x_4^{10}$ in dimension 10. □

Firstly, the number of "geometric conditions" in the above is 39, shy by 1 of making the topological result absolutely tight on the conjectured lower bound. Indeed, we can fit one more condition in, such as H_4 or H_3 bisecting another mass to get a stronger result.

Strictly speaking, we did not need to include the first few reductions in the previous computation, but we did this to motivate the methods used in the following theorem. As in the previous chapter, we obtain further control on the polynomial equations by increasing the number of masses we consider.

Theorem 4.3.8. *We can guarantee an equipartition of $m \geq 4$ masses by any 3 of 4 hyperplanes as well as an equipartition by any 2 of the 3 remaining hyperplanes of a single mass in $d = 5m$. Additionally, we can ask that all hyperplanes are orthogonal to the first in the same dimension. We can also obtain a full cascade, with both x_3, x_4 bisecting masses under the assumption that $m \geq 8$.*

Proof. We will prove the stronger result with orthogonality conditions, and this will imply the weaker result. As in the previous theorem, we factor out x_1^{5m} immediately from the following polynomial:

$$
\begin{aligned}
P_m &:= \left(\sum_{\sigma\in S_4} x^2_{\sigma(1)}x^3_{\sigma(2)}x^4_{\sigma(3)}x^5_{\sigma(4)}\right)^m \left(\sum_{\sigma\in S_3} x^1_{\sigma(1)}x^2_{\sigma(2)}x^3_{\sigma(3)}\right)(x_1+x_2)(x_1+x_3)(x_1+x_4)\\
&= x_1^{5m}\left(\sum_{\sigma\in S_3} x^2_{\sigma(2)}x^3_{\sigma(3)}x^4_{\sigma(4)}\right)^m \left(\sum_{\sigma\in S_3} x^1_{\sigma(1)}x^2_{\sigma(2)}x^3_{\sigma(3)}\right)(x_1+x_2)(x_1+x_3)(x_1+x_4)\\
&= x_1^{5m}\left(\sum_{\sigma\in S_3} x^2_{\sigma(2)}x^3_{\sigma(3)}x^4_{\sigma(4)}\right)^m \left(\sum_{\sigma\in S_3} x^2_{\sigma(1)}x^3_{\sigma(2)}x^4_{\sigma(3)}\right)\\
&= x_1^{5m}\left(\sum_{\sigma\in S_3} x^2_{\sigma(2)}x^3_{\sigma(3)}x^4_{\sigma(4)}\right)^{m+1}
\end{aligned}
$$

where the penultimate equality comes from the fact that any term multiplied by x_1 vanishes by our assumption on degree. Indeed, we can check that this fails to vanish whenver $4(m+1) \leq 5m$, or $m \geq 4$.

Multiplying by x_3x_4 will also not vanish, but we need that $m+2 \leq 5m$ (or $m \geq 8$) in order for the same method to work.

$\square$

5
Further Work

The most glaring deficiency in the above theorems is derived from the fact that the conjectured lower bound in cascading Makeev-type problems has not been proven, and hence many of the "tightness" results possible by way of geometric constraints are not in principle "as strong as can be" even when we strongly think that they are genuinely as good as possible. One possible further direction would be the proof of the geometric lower bound.

Other possible directions include further work on the any 3 of 4, and more generally any 3 of k equipartition-type problems. This is especially true for "orthogonality" conditions that can still fit into the problem. We have considered the decomposition for the three of 3 of 4 polynomial in the following way:

Define

$$P_n :\; = \prod_{1 \leq i < j < k \leq n} (x_i + x_j + x_k)$$

and note we may take

$$\prod_{1 \leq i < j < k \leq n} (x_i + x_j + x_k) = P_{n-1} \left(\prod_{1 \leq i < j \leq n-1} x_i + x_j + x_5 \right)$$

which may help with applying some kind of inductive upper bound.

It is also expected that the "Vandermonde Trick" may have further combinatorial advantages for special cases of the Makeev-problems due to the symmetric nature of the rewriting, although this is completely unknown.

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