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Lose Big, Win Big, Sum Big: An Exploration of Ranked Voting Systems

Erin Else Stuckenbruck
Bard College, es8455@bard.edu

Lose Big, Win Big, and Sum Big: An Exploration of Ranked Voting Systems

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by
Erin Stuckenbruck

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Abstract

This project focuses on the mathematical study of voting systems, known as voting theory. The procedure in which each voting system selects a winner is different for each one. There were some situations that we analyzed where the population generally liked an alternative more, but no pre-existing voting system chose them as a winner. The Lose Big method was created by the writer to choose this more *average* alternative, and through the creation of Lose Big, the Win Big and Sum Big method also formed. We analyze the various pre-existing ranked voting methods and compare them to the Win Big, Lose Big, and Sum Big method, exploring the fairness criteria that they satisfy or violate. We also explore some connections found between the Sum Big voting system and Borda Count.

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1
Introduction

1.1 Voting

In order to make a decision or express an opinion within a group where not all members agree, each individual preference needs to be accounted for. Voting is a very common method used for reaching a decision peacefully and **voting theory** or **social choice theory** is the mathematical treatment and study of these social choice procedures (voting systems). A formal definition is given:

Definition 1.1.1. A **social choice procedure** is a function that takes as input a sequence of lists (without ties) of some set A (the set of alternatives) and the corresponding output is either an element of A, a subset of A, or NW (no winner) [9]. $\triangle$

There are over 50 different social choice procedures, and since democracy in ancient Athens, these methods have been developed and heavily debated. For which social choice procedure will select a winner that best represents the collective body? At the heart of social choice theory is the study of preferential aggregation, which is the aggregation of several individual preference rankings of two or more alternatives into a single, collective preference ranking over these alternatives [7]. These types of social choice procedures, known as *ranked voting systems*, will be the focus of this project.

In this years American Presidential election (2016), we have seen that it is possible for a candidate who is greatly disliked by many, even by his/her own party, to still have a chance at winning the general election. For example, while Donald Trump has a massive following, his opposition is probably just as big. Since the American voting system only accounts for voter's first preference, the voters who dislike Trump can only represent their dislike for an alternative by voting against them.

By using preferential voting systems to select a winner, voters preferences are more accurately accounted for. It is common though, for many ranked systems to still put a heavier weight on a voter's first choice. This is of course more favorable, since the main purpose of an election is to select an alternative that is most liked by the group. But there are some situations where perhaps selecting an alternative that is least disliked would be more suitable.

For example, let's say three friends were trying to decide on what to eat and came up with two options: pizza or a burger. Two of the friends want pizza but would settle for a burger, while the third friend hates pizza but loves burgers. In this situation, it seems that a burger would be the better option since everyone would be able to enjoy the meal; but if they were to merely take into account their first choice, they would chose pizza, since two out of three prefer pizza, and one friend would not eat.

The **positional voting system**, defined below, that I created, the **Lose Big** method, was designed to place more importance on a voter's dislikes, which was influenced by this years election. From this, we observed when flipping/reversing the Lose Big voting system, alternatives who were placed higher in a ballot would receive a greater score. We called this system **Win Big**. Then, when adding Win Big and Lose Big's scoring procedure, another voting system formed. We call this the **Sum Big** voting system.

Definition 1.1.2. A **positional voting system** is a ranked voting method in which the alternatives receive points based on their rank position in each ballot and the alternative with the most points overall wins [11]. △

While there are other voting systems that exist based off the same idea as the Lose Big method, there were certain situations where our voting system would select a unique set of plausible winners that no other voting system which we analyzed selected.

In order to get a general idea of how different ranked voting systems produce various outcomes, including ours, a more formal example is shown below:

Example 1.1.3. The Bard Mathematics department is looking to hire a new faculty member for the tenure-track position. The math committee, made up of 3 faculty members, interview 3 candidates: Olivia, Larry, and Yang. After each candidate is interviewed, each faculty member votes by ranking the candidates in descending order of preference, from most to least preferred. Below are the faculty's ballots:

$$
\begin{array}{ccc}
Olivia & Yang & Olivia \\
| & | & | \\
Yang & Larry & Yang \\
| & | & | \\
Larry & Olivia & Larry \\
 & & \\
B_1 & B_2 & B_3
\end{array}
$$

Depending on which voting system is used, different results will occur. Voting systems such as the Plurality method, Instant Run-Off, Coombs, and Win Big method would chose Olivia; Sum Big and Borda Count would tie Yang and Olivia; and Anti-Plurality and Lose Big would chose Yang. These voting systems will be further explained later in the project. ◊

We see in the example above, that two out of the three of the candidates have the potential to win. So, who do you think the Bard Math Committee should hire?

All voting systems make assumptions on what aspects of an election are important when aggregating the votes based on the certain *fairness criteria* it satisfies or violates. It turns out that there is a theorem, **Arrow's Impossibility Theorem**, that tells us that no voting system satisfies a certain set of fairness criteria, which will be defined later. In other words, it is impossible a voting system will chose a winner that satisfies the desires of an entire population. Refer to [7] or [9] for more information on Arrow's Impossibility Theorem. Therefore, it is important that there be a voting system in place that satisfies and violates different sets of fairness criteria.

In this project we will explore different positional voting systems including the Win Big, Lose Big, and Sum Big method, and see which fairness criteria are dismissed or highlighted, and compare their properties to other systems such as **Borda Count**, **Instant Runoff**, **Coombs**, **Plurality**, and **Anti-Plurality**.

In Chapter 2, we introduce some basic voting theory terminology as well as theorems that will be used throughout this project. We also define the ranked voting systems analyzed in this project. Chapter 3 is where certain fairness criteria are defined and we see which voting systems violate or satisfy which criteria. In Chapter 4, we introduce the Lose Big, Win Big, and Sum Big method and explain how they work. We then show which fairness criteria they satisfy or violate and compare them with the pre-existing systems defined in Chapter 2. In Chapter 5, we ground our voting systems by showing that the Sum Big method is in fact an affine transformation of the Borda Count method.

2
Preliminaries

In this chapter we will define the basics of voting theory, introduce some existing voting systems, and explore a few characteristics needed for our voting system.

2.1 Basics of Voting Theory

In this section we will define some basic voting theory terminology as well as some definitions and theorems that will be needed for later in this project. In an election that uses a preferential voting system to select a winner, a set of *voters* rank a set of *alternatives*, A, in order of each voters' personal preference. Each voters' preferences are arranged in a *preferential ballot*. The set of all preferential ballots within an election is called a*profile*. We formalize this below.

Definition 2.1.1. A **preferential ballot**, B, is a *total order* on a set of alternatives $A = \{a_1, a_2, \cdots, a_n\}$, where $n \in \mathbb{N}$ [4]. $\triangle$

Since we will only be working with preferential ballots, we will refer to them as merely ballots.

Definition 2.1.2. A **profile**, P, is a set of ballots. $\triangle$

Example 2.1.3. From Example 1.1.3, $A = \{\text{Nora, Ying, Mario, Stephan}\}$ is the set of alternatives and the profile is $P = \{B_1, B_2, B_3\}$. $\diamondsuit$

Notation: Since each ballot is a linearly ordered set, we use the notation $x > y$ to mean x is preferred over y. Also if $x, y, z \in A$ are ranked in a ballot as such: $x > y > z$, we say x is in row 1, first place, y is in row 2, second place, and z is in row 3, last place.

Example 2.1.4. From Example 1.1.3, B_1's linearly ordered set could be notated as such: Nora $>$ Mario $>$ Ying $>$ Stephan. $\diamondsuit$

All positional voting systems rely on some *weighted vector*, defined below, where the first weight corresponds to the alternative in row 1, the second weight to the alternative in row 2, and so on.

Definition 2.1.5. Let A be a set of n alternatives and P be a profile. Let V be a voting system. Then the **weighted vector**, $\vec{w}_V = [w_1, w_2, \ldots, w_n]^t$, corresponding to V , is a column vector in $\mathbb{R}^n$, where $w_1 \geq w_2 \geq \ldots \geq w_n$ and for each ballots $B \in P$ and $a \in A$, $w_B(a) = w_i$, where a is in the i^{th} row of B [1]. $\triangle$

Example 2.1.6. Consider the ballot, B, below where $A = \{a, b, c, d\}$.

$$\begin{array}{c} a \\ | \\ b \\ | \\ c \\ | \\ d \end{array}$$

B

Suppose a positional voting, V, was used on this ballot. This means $\vec{w}_V = [w_1, w_2, w_3, w_4]^t$ since $n = |A| = 4$. This means

- $w_B(a) = w_1$
- $w_B(b) = w_2$
- $w_B(c) = w_3$
- $w_B(d) = w_4$.

$\Diamond$

Definition 2.1.7. Let $a \in A$ and $P = \{B_1, B_2, \ldots, B_k\}$ where $k \in \mathbb{N}$. The total score of a in an election is $s(a) = \sum_{i=1}^{k} \vec{w}_{B_i}(a)$ where $k = |P|$. $\triangle$

Notation:

- The winners of a social choice procedure are the set $f(P) = \{a \in A |\ s(a) \text{ is maximal}\}$ [5].

Example 2.1.8. Suppose we conduct an election with a set of ballots where $P = \{B_1, B_2, B_3\}$ and a set of alternatives $A = \{a, b, c, d\}$. In the ballots below, each row is assigned a weight, the Borda Count, BC, weighted vector $\vec{w}_{BC} = [(n-1), (n-2), \ldots, 1, 0]^t$, where $n = |A|$, which will be explained further in the next section.

[3]	a	b	c
[2]	c	c	b
[1]	d	a	d
[0]	b	d	a
$\vec{w}_{BC}$	B_1	Bv_2	B_3

So in a single ballot, an alternative will receive $(n - i)$ points, where i is the row corresponding to the alternative. For example, the resulting weights for a are:

- $\vec{w}_{B_1}(a) = (4-1) = 3$

- $\vec{w}_{B_2}(a) = (4-3) = 1$

- $\vec{w}_{B_3}(a) = (4-4) = 0$

Now calculating the score for each candidate using the corresponding weights we get:

- $s(a) = \sum_{i=1}^{3} w_{B_i}(a) = 3+1+0 = 4$

- $s(b) = \sum_{i=1}^{3} w_{B_i}(b) = 0+3+2 = 5$

- $s(c) = \sum_{i=1}^{3} w_{B_i}(c) = 2+2+3 = 7$

- $s(d) = \sum_{i=1}^{3} w_{B_i}(d) = 1+0+1 = 2$

Since $s(c) > s(b) > s(a) > s(d)$, we see $f(P) = \{c\}$. Therefore, c is the winner. $\Diamond$

Each preferential voting system will produce a final ranking, in the form of an ordered list (with possible ties), of the alternatives that according to the used voting system, best represents the will of the voters. This is known as the *societal preference order*.

Definition 2.1.9. The **societal preference order** of a ranked voting system is the aggregate of single preferential ballots [4]. $\triangle$

Example 2.1.10. Using Example 2.1.8., the societal preference order would be

$$c > b > a > d \quad \text{or} \quad \begin{array}{c} c \\ | \\ b \\ | \\ a \\ | \\ d \end{array}$$

$\Diamond$

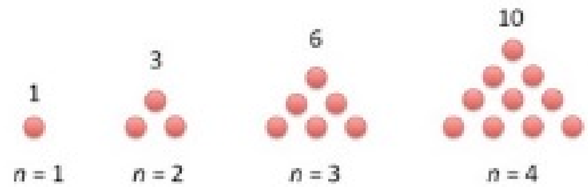

Figure 2.2.1. Triangle Numbers, T_n[12].

2.2 Triangle Numbers

In this section, we introduce *triangle numbers*, which will be used throughout this project. First, a formal definition is given below:

Definition 2.2.1. Let $n \in \mathbb{N} \cup \{0\}$. A **triangle number**, T_n, is a number obtained by adding all positive integers less than or equal to a given positive integer n i.e. $T_n = 1 + 2 + 3 + \ldots + n = \sum_{k=1}^{n} k$. △

Lemma 2.2.2. *The n^{th} triangle number is,*

$$T_n = \sum_{k=0}^{n} k = \tfrac{n(n+1)}{2} \text{ [12].}$$

The reason $\{T\}$ are called triangle numbers is because they have a geometric interpretation: T_n is equal to the number of nodes that complete an equilateral triangle with side-length n [12]. Figure 2.1.1. shows the first four triangle numbers.

For our purposes, we introduce a slight variation of the triangle numbers, t_n. Instead of starting at $T_0 = 0$, we will denote the first triangle number as $t_1 = 0$. For the rest of this project we will be referring to $\{t\}$ as the *t-numbers*.

Definition 2.2.3. Let $n \in \mathbb{N}$. Then

$$t_n = T_{(n-1)}$$

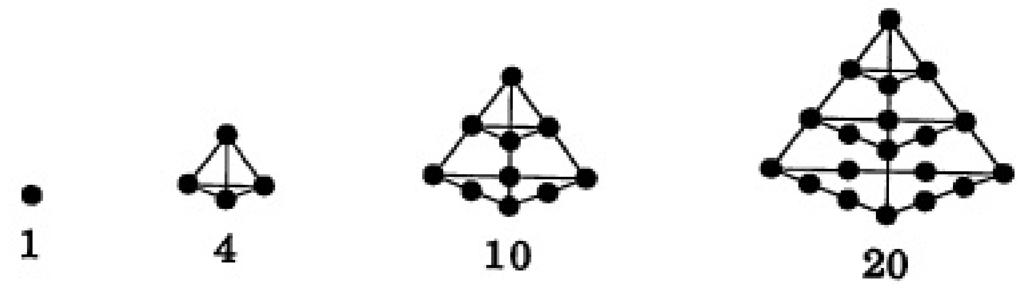

Figure 2.2.2. Tetrahedral Numbers [12].

is called the n^{th} triangle number. △

The first few t-numbers are listed below:

$$t_1 = 0,\ t_2 = 1,\ t_3 = 3,\ t_4 = 6,\ t_5 = 10.$$

Lemma 2.2.4. *Let $n \in \mathbb{N}$. Then*

$$t_n = \sum_{k=0}^{n-1} k = \tfrac{n(n-1)}{2}.$$

Proof. By Definition 2.2.3, we know $t_n = T_{n-1}$. Then by Lemma 2.2.2, we see that $T_{n-1} = \frac{(n-1)n}{2}$. Thus, $t_n = \frac{n(n-1)}{2}$. □

Definition 2.2.5. The sum of the first n t-numbers, $0 + 1 + 3 + \ldots + \frac{n(n-1)}{2}$, is called the n^{th} tetrahedral number or triangle pyramidal number, denoted S_n [12]. △

These numbers correspond to placing discrete points in the configuration of a tetrahedron (triangle base pyramid) [12]. Figure 2.2.2. shows the first few tetrahedral numbers.

Theorem 2.2.6. *Let S_n be the n^{th} tetrahedral number. Then*

$$S_n = \sum_{k=1}^{n} t_k = \tfrac{n(n-1)(n+1)}{6}.$$

Proof. We can prove this by induction.

Let $n \in \mathbb{N}$.

Case 1: Let $n = 1$

Then $S_1 = \frac{1(0)(2)}{6} = \frac{0}{6} = 0$.

Case 2: When $n = k$,

assume $S_k = t_1 + t_2 + \ldots + t_k = \frac{k(k-1)(k+1)}{6}$ is true.

Let $n = k + 1$.

Then $S_{(k+1)} = t_1 + t_2 + \ldots + t_k + t_{(k+1)}$. Since $S_k = t_1 + t_2 + \ldots + t_k = \frac{k(k-1)(k+1)}{6}$ we can deduce that $S_{(k+1)} = t_1 + t_2 + \ldots + t_k + t_{(k+1)} = S_k + t_{(k+1)}$. Using the hypothesis, $S_n = \frac{n(n-1)(n+1)}{6}$ and the fact that $t_{(k+1)} = \frac{(k+1)((k+1)-1)}{2} = \frac{k(k+1)}{2}$, we see that

$$\begin{aligned} S_{(k+1)} &= \frac{k(k-1)(k+1)}{6} + \frac{(k+1)(k)}{2} \\ &= \frac{k(k-1)(k+1)}{6} + \frac{3(k+1)(k)}{6} \\ &= \frac{k(k-1)(k+1) + 3(k+1)(k)}{6} \\ &= \frac{k^3 + 3k^2 + 2k}{6} \\ &= \frac{(k+1)k(k+2)}{6} \\ &= \frac{(k+1)((k+1)-1)((k+1)+1)}{6}. \end{aligned}$$

Thus, by induction, $S_n = \frac{n(n-1)(n+1)}{6}$ for all $n > 0$.

□

2.3 Voting Systems

In this section, we will introduce several different types of ranked voting systems: Plurality, Anti-Plurality, Instant Runoff, Coombs, and Borda Count.

First we look at the *Plurality* method. According to [6], the plurality voting system is the oldest voting system and is most often used in the United States.

Definition 2.3.1. The **Plurality voting system** selects the alternative(s) with the most first place votes, i.e. $\vec{w}_P = [1, 0, 0, \ldots, 0]^t$ [9]. △

Note that the Plurality method only takes into account the first row of information in a preferential ballot, resulting in the weighted vector $\vec{w}_P = [1, 0, \cdots, 0]^t$. So a voters first choice is all that matters.

Example 2.3.2. Let C be the set of Republican candidates running in the 2016 presidential election. In the 2016 Georgia GOP primary, the three leading competitors were Donald Trump, Ted Cruz, and Marc Rubio. In a study, in [6], they were able to use polls to get an idea of how people might have voted if the election used preferential ballots. Candidate *Other* will represent any candidate in $C - \{$Trump, Cruz, Rubio$\}$. The results are below:

38.8%	20.5%	24.4%	6.0%	3.8%	3.2%	3.1%
Trump	*Cruz*	*Rubio*	*Other*	*Other*	*Other*	*Cruz*
Rubio	*Other*	*Cruz*	*Rubio*	*Trump*	*Cruz*	*Other*
Cruz	*Trump*	*Other*	*Cruz*	*Rubio*	*Rubio*	*Trump*
Other	*Rubio*	*Trump*	*Trump*	*Cruz*	*Trump*	*Rubio*

Since the Plurality method only assigns 1 point for each time a candidate is ranked in first place, the end result is:

Candidate	**Score**
Trump	38.8
Rubio	24.4
Cruz	23.6
Other	13.0

From this, we see that the Plurality method choses Trump as the winner. So $f(P) = \{\text{Trump}\}$. $\Diamond$

The next voting system we will look at is the *Anti-Plurality* method, also known as the Inverse Plurality Rule. This method is the opposite of Plurality such that the candidate in last place receives one point and the candidate with the lowest score is selected to be the winner. So voters vote against against a single candidate instead of for.

Definition 2.3.3. The **Anti-Plurality voting system** selects the alternative with the least last place vote, i.e. $\vec{w}_A = [0, 0, \ldots, 1]^t$ [8]. △

Note when using the Anti-Plurality method, $f(P) = \{a \in A | s(a) \text{ is minimal}\}$.

Example 2.3.4. If we apply the Anti-Plurality method to Example 2.3.2, then we get:

Candidate	**Score**
Cruz	3.8
Rubio	23.5
Trump	33.6
Other	38.8

From this, we can see that Cruz has the least amount of points, therefore he would be selected as the winner. ♢

Although this voting system is rarely used, it proposes an interesting method by trying to avoid selecting a winner who the majority of voters dislike, rather than trying to select a winner who is most preferred.

The Coombs method, used more frequently than Anti-Plurality, is based off the same idea. It could be thought of as Anti-Plurality with Elimination– the alternative bottom-ranked by the most voters is eliminated and the ballots are re-counted until an alternative receives a majority, meaning that an alternative gets more than 50% of the first place votes. Note that this is not a positional voting system but uses elimination instead of weights.

Definition 2.3.5. The **Coombs voting system** eliminates the alternative with the most last place votes and recounts the ballots until one alternative receives the majority of first place votes [8]. △

Example 2.3.6. Using Coombs method on Example 2.3.2, since there is no majority, we see that Other has the most last place votes. When we eliminate Other, the resulting ballots are:

38.8%	20.5%	24.4%	6.0%	3.8%	3.2%	3.1%
Trump	*Cruz*	*Rubio*	*Rubio*	*Trump*	*Cruz*	*Cruz*
Rubio	*Rubio*	*Cruz*	*Cruz*	*Rubio*	*Trump*	*Trump*
Cruz	*Trump*	*Trump*	*Trump*	*Cruz*	*Rubio*	*Rubio*

Notice now that Trump is ranked last the most. So we eliminate him and get the resulting ballots:

72.4%	26.8%
Rubio	*Cruz*
Cruz	*Rubio*

From this, we see that $f(P) = \{\text{Rubio}\}$, resulting in the societal preference order

Rubio

Cruz

Trump

Other

◊

The Coombs method was actually based off of the Instant Runoff voting system, also known as Plurality with Elimination. Instant Runoff, like Plurality is to Anti-Plurality, can be thought of as the reversal of the Coombs method. Instead of eliminating the alternative with the most last place votes, it eliminates the alternative with the least first place votes.

Definition 2.3.7. The **Instant Runoff voting system** eliminates the alternative with the least first-place votes from the election. Any votes for that alternative are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%) [8]. △

Example 2.3.8. Again, using Example 2.3.2, we see that Other has the least first-place votes. When Other is eliminated we get the resulting ballots (which happens to be the same as Example 2.3.6 when using Coombs method):

38.8%	20.5%	24.4%	6.0%	3.8%	3.2%	3.1%
Trump	*Cruz*	*Rubio*	*Rubio*	*Trump*	*Cruz*	*Cruz*
Rubio	*Rubio*	*Cruz*	*Cruz*	*Rubio*	*Trump*	*Trump*
Cruz	*Trump*	*Trump*	*Trump*	*Cruz*	*Rubio*	*Rubio*

Since there is still no majority, we eliminate Cruz since he 26.8% of the first-place votes while Trump has 42.6% and Rubio has 30.4%. So,

50.9%	48.9%
Rubio	*Trump*
Trump	*Rubio*

Therefore, since Rubio has more than 50% of the first-place votes, $f(P) = \{\text{Rubio}\}$. The societal preference order is

Rubio
|
Trump
|
Cruz
|
Other

◊

Next, we will look at Borda Count– briefly mentioned in Example 2.1.8. Borda Count is used extensively; for example it is used for the NBA's Most Valuable Player award, the Eurovision Song Contest, Presidential elections in Kiribati, and more [8]. The Borda Count method determines a winner by assigning each alternative, for each ballot, a number of points corresponding to the number of alternatives ranked below them. A formal definition is given below.

Definition 2.3.9. Suppose $n = |A|$. The **Borda Count voting system** gives each alternative $(n-i)$ points for each ballot that ranks the alternative(s) in the i^{th} row, i.e. the $\vec{w}_{BC} = [(n-1), (n-2), \dots, 1, 0]^t$. These points are totaled for each alternative and the alternative(s) with the most points wins [8]. △

Example 2.3.10. Using the Borda Count method on Example 2.3.2 we get,

$\vec{w}_{BC}$	38.8%	20.5%	24.4%	6.0%	3.8%	3.2%	3.1%
[3]	*Trump*	*Cruz*	*Rubio*	*Other*	*Other*	*Other*	*Cruz*
[2]	*Rubio*	*Other*	*Cruz*	*Rubio*	*Trump*	*Cruz*	*Other*
[1]	*Cruz*	*Trump*	*Other*	*Cruz*	*Rubio*	*Rubio*	*Trump*
[0]	*Other*	*Rubio*	*Trump*	*Trump*	*Cruz*	*Trump*	*Rubio*

This means

$$
\begin{aligned}
\textbf{s(Trump)} &= (3\cdot 38.8)+(1\cdot 20.5)+(0\cdot 24.4)+(0\cdot 6.0)+(2\cdot 3.8)+(0\cdot 3.2)+(1\cdot 3.1) = 147.6\\
\textbf{s(Rubio)} &= (2\cdot 38.8)+(0\cdot 20.5)+(3\cdot 24.4)+(2\cdot 6.0)+(1\cdot 3.8)+(1\cdot 3.2)+(0\cdot 3.1) = 169.8\\
\textbf{s(Cruz)} &= (1\cdot 38.8)+(3\cdot 20.5)+(2\cdot 24.4)+(1\cdot 6.0)+(0\cdot 3.8)+(2\cdot 3.2)+(3\cdot 3.1) = 170.8\\
\textbf{s(Other)} &= (0\cdot 38.8)+(2\cdot 20.5)+(1\cdot 24.4)+(3\cdot 6.0)+(3\cdot 3.8)+(3\cdot 3.2)+(2\cdot 3.1) = 110.6.
\end{aligned}
$$

From this we can see that s(Cruz) > s(Rubio) > s(Trump) > s(Other). Therefore, $f(P) = \{Cruz\}$ and the societal preference order is

$$
\begin{array}{c}
Cruz\\ | \\ Rubio\\ | \\ Trump\\ | \\ Other
\end{array}
$$

◊

When comparing all the different ways a winner was found and the different resulting societal preference orders, using the ballots from Example 2.3.2, one can notice the many possible outcomes.

$$
\begin{array}{ccccc}
\textbf{Plurality} & \textbf{Anti-Plurality} & \textbf{Coombs} & \textbf{Instant RunOff} & \textbf{Borda Count}\\
Trump & Cruz & Rubio & Rubio & Cruz\\
| & | & | & | & |\\
Rubio & Rubio & Cruz & Trump & Rubio\\
| & | & | & | & |\\
Cruz & Trump & Trump & Cruz & Trump\\
| & | & | & | & |\\
Other & Other & Other & Other & Other
\end{array}
$$

An interesting thing to note is that Plurality, the voting system the United States uses, is the only method in which Trump was selected as the winner, while the other voting systems chose either Cruz or Rubio. This is due to Trump being preferred less by many voters. Which voting system an electorate or group will use is usually based off of which fairness criteria the voting system satisfies or fails. This will be discussed more in the next chapter.

3
Properties of Voting Systems

In order to understand what kind of winner a voting system selects, voting theorists have evaluated social choice procedures using certain *voting system fairness criteria*, which define potentially desirable properties of voting systems mathematically [11]. In this chapter, we will be looking at several of these fairness criteria and seeing which voting systems satisfy or violate criteria.

3.1 Fairness Criteria

In this section, we will define several of the main fairness criteria discussed in voting theory: the **Majority Criterion**, the **Condorcet Criterion**, the **Monotonicity Criterion**,and the **Independence of Irrelevant Alternatives (IIA) Criterion**. We will also be looking at the **Majority Loser Criterion** as well as the **Condorcet Loser Criterion** since we are dealing with reversed systems, i.e. Anti-Plurality and Coombs. The first criteria we will look at is the Majority criterion.

Definition 3.1.1. A voting system satisfies the **Majority Criterion** if it choses the alternative who receives more than 50% of the first place votes as the winner [11]. We call this winner the Majority winner. $\triangle$

Example 3.1.2. The United States Department of Agriculture started a campaign "You Control the School Menu", where middle school students sample and vote on a variety of school menu options [10]. Suppose 60 8^{th} grade students used preferential voting and $P =$ {Plate A, Plate B, Plate C}. Suppose the final results were the following:

12	18	20	10
$PlateA$	$PlateB$	$PlateA$	$PlateC$
$PlateB$	$PlateC$	$PlateB$	$PlateB$
$PlateC$	$PlateA$	$PlateC$	$PlateA$

Since there are an even number of votes, where $v = 60$, a plate must have $\frac{v}{2} + 1$ votes, which would be $\frac{60}{2} + 1 = 31$ votes in order to have a majority. From the preference table we see that Plate A has 32 first place votes, Plate B has 18, and Plate C has 10. Since $32 \geq 31$, Plate A is the Majority winner.

Using the Plurality method, Plate A would win, satisfying the Majority criterion. Using Borda Count, with the weighted vector $\vec{w} = \{2, 1, 0\}$, Plate A receives 64 points, Plate B 68 points, and Plate C 38 points. This means Plate B would win, violating the Majority criterion. $\lozenge$

The Plurality, Instant RunOff, and the Coombs method satisfy the Majority criterion, while Borda Count and Anti-Plurality violate the Majority Criterion [11].

Related to the Majority criterion is the Condorcet criterion, which we formally define below.

Definition 3.1.3. Let $a \in A$. A voting system satisfies the **Condorcet Criterion** if $a \in f(P)$, such that $a > b$ for all $b \in A - \{a\}$ by more voters than $b > a$ [11]. We call a the Condorcet winner. △

Informally, a voting system that satisfies the Condorcet Criterion would select the alternative who would win a two-candidate election against every other candidate [11].

Example 3.1.4. Using Example 3.1.2, when comparing each alternative head-to-head we get

	{Plate A, Plate B}
32	Plate A > Plate B
28	Plate B > Plate A

	{Plate A, Plate C}
32	Plate A > Plate C
28	Plate C > Plate A

	{Plate B, Plate C}
50	Plate B > Plate C
10	Plate C > Plate B

From this we see that Plate A is preferred by more voters over every other Plate. Therefore a voting system that selects Plate A as the winner satisfies the Condorcet Criterion. ◊

Notice in that the Majority winner and Condorcet winner are the same when using the ballots from Example 3.1.2. In fact, the Condorcet criterion implies the Majority criterion. We will prove this below.

Theorem 3.1.5. *Every Majority winner is a Condorcet winner.*

Proof. Let P be a profile and let $a \in A$ and $b \in A - \{a\}$. Suppose a is a Majority winner of P. This means a is in the first row in at least $\frac{k+1}{2}$ ballots. This also means that $a > b$ in at least $\frac{k+1}{2}$ ballots, meaning that a would win in a head-to-head competition with every other alternative. Thus, by Definition Condorcet, a is also a Condorcet winner. □

Theorem 3.1.6. *If a voting system, V, satisfies the Condorcet criterion, then it also satisfies the Majority criterion.*

Proof. We will prove this by contrapositive. Let V be a voting system that fails the Majority criterion. This means there exists a profile, P, such that $a \in A$ is the Majority winner and $a \notin f(P)$. By Theorem 3.1.5, since a is a Majority winner, a is also a Condorcet winner. Since $a \notin f(P)$, we can conclude that V fails the Condorcet criterion. □

Although the Condorcet criterion is very well known, none of the voting systems we are looking at satisfy it. The Condorcet and Majority criterion both have "loser" versions of them. We will first look at the Majority Loser criterion, which ensures that an alternative least preferred is never selected as a winner.

Definition 3.1.7. A voting system satisfies the **Majority Loser Criterion** if it never selects an alternative who the majority prefers every other alternative over as the winner [11]. △

Example 3.1.8. Referencing Example 3.1.2, we see that Plate C would be the majority loser and that $f(P) \neq \{Plate\ C\}$ when using the Plurality and the Borda Count method. Therefore, in this case, Plurality and Borda Count satisfy the Majority Loser Criterion. ♢

In general Borda Count, Coombs, Instant RunOff, and Anti-Plurality satisfy the Majority Loser Criterion while Plurality does not.

Example 3.1.9. We will show that Plurality violates the Majority Loser criterion. Consider an election where $A = \{a, b, c, d\}$ and $P = \{B_1, B_2, B_3, B_4, B_5\}$, forming the below ballots.

a	a	d	c	b
b	c	c	d	c
d	d	b	b	d
c	b	a	a	a
B_1	B_2	B_3	B_4	B_5

From the ballots above we can see that a is the majority loser but the Plurality would choose a as its winner since $s(a) > s(b) = s(c) = s(d)$. Thus, the Plurality method violates the Majority Loser criterion.

$\Diamond$

Similar to the Majority Loser criterion is the Condorcet Loser criterion, and this is satisfied by Anti-Plurality, Coombs, Instant Runoff and Borda Count method while the Plurality method violates it [11].

Definition 3.1.10. Let $a \in A$. A voting system satisfies the **Condorcet Loser Criterion** if $a \notin f(P)$, such that for all $b \in A - \{a\}$, $b > a$ by more voters than $a > b$. We call a the Condorcet Loser [11]. $\triangle$

Example 3.1.11. Using Example 3.1.2 and comparing each alternative to one another, we get

	{Plate A, Plate B}
32	Plate A > Plate B
28	Plate B > Plate A

	{Plate A, Plate C}
32	Plate A > Plate C
28	Plate C > Plate A

	{Plate B, Plate C}
50	Plate B > Plate C
10	Plate C > Plate B

From these tables we can see that Plate C is least preferred by more voters than any other plate. This means any voting system that never selects Plate C as a winner would satisfy the Condorcet Loser Criterion. $\Diamond$

Just like the Condorcet criterion is a stronger version of the Majority criterion, the Condorcet Loser criterion is a stronger version of the Majority Loser criterion. A similar proof is given below.

Theorem 3.1.12. *Every Majority Loser is a Condorcet Loser.*

Proof. Let P be a profile and let $a \in A$ and $b \in A - \{a\}$. Suppose a is a Majority Loser of P. This means a is in the last row in at least $\frac{k+1}{2}$ ballots. This also means that $b > a$ in at least $\frac{k+1}{2}$ ballots, meaning that a would lose in a head-to-head competition with every other alternative. Thus, by the Definition 3.1.10, a is also a Condorcet Loser. $\square$

Theorem 3.1.13. *If a voting system satisfies the Condorcet Loser criterion, then it also satisfies the Majority Loser criterion.*

Proof. We will prove this using the contrapositive. Let V be a voting system that fails the Majority Loser criterion. Then there exists a profile, P, such that $a \in A$ is the Majority loser and $a \in f(P)$. By Theorem 3.1.12, since a is the Majority Loser, a is also the Condorcet Loser and since $a \in f(P)$, V also fails the Condorcet Loser criterion. $\square$

The next fairness criterion we will look at is the Monotonicity Criterion.

Definition 3.1.14. Let $a \in A$. Suppose $a \in f(P)$. A voting system satisfies the **Monotonicity Criterion** if when voters move a higher in a new set of ballots, P', then $a \in f(P')$ also [3]. $\triangle$

Example 3.1.15. Using the Plurality method with Example 3.1.2, we see that Plate A would be the winner of profile P. Now suppose the original ballots were changed, P', in favor of Plate A as below,

$$\begin{array}{ccccc} & 12 & 18 & 20 & 10 \\ & PlateA & PlateB & PlateA & PlateC \\ & \vert & \vert & \vert & \vert \\ \mathbf{P} = & PlateB & PlateC & PlateB & PlateB \\ & \vert & \vert & \vert & \vert \\ & PlateC & \textbf{Plate A} & PlateC & \textbf{Plate A} \end{array}$$

$$\begin{array}{ccccc} & Plate\ A & \textbf{Plate A} & PlateA & PlateC \\ & \vert & \vert & \vert & \vert \\ \mathbf{P'} = & PlateB & PlateC & PlateB & \textbf{Plate A} \\ & \vert & \vert & \vert & \vert \\ & PlateC & PlateB & PlateC & PlateB \end{array}$$

Using the Plurality method on the altered ballots, we see that Plate A is still the winner, thus satisfying the Monotonicity Criterion. $\Diamond$

The voting systems which satisfy the Monotonicity Criterion are Plurality, Ant-Plurality and Borda Count while Coombs and Instant Runoff fail.

The final criterion we will look at is the Independence of Irrelevant Alternatives (IIA) Criterion. This Criterion basically states that if an election is held and a winner declared, this winner should remain the winner if one or more of the losing alternatives drops out [11]. A more formal definition is given below:

Definition 3.1.16. Let $a \in A$. A voting system satisfies the **Independence of Irrelevant Alternatives Criterion** if $a \in f(P)$ and a revote is taken where losing alternative(s) drop out, forming a new set of ballots, P', then $a \in f(P')$ also [3]. $\triangle$

Example 3.1.17. Using the Anti-Plurality method with the ballots from Example 3.1.2, we see that $f(P) = \{\text{Plate B}\}$. Suppose we remove Plate A from P and recalculate the ballots, P'. We get

12	18	20	10
$PlateB$	$PlateB$	$PlateB$	$PlateB$
$PlateC$	$PlateC$	$PlateC$	$PlateC$

From this we can see that, using the Anti-Plurality method on the ballots above, s(Plate B) $= 0$ and s(Plate C) $= 60$, so $f(P) = \{$Plate B$\}$. Recall $f(P) = \{a \in A|$ s(a) is minimal$\}$.

If we remove Plate C instead of Plate A, P' will be:

12	18	20	10
$PlateA$	$PlateB$	$PlateA$	$PlateB$
$PlateB$	$PlateA$	$PlateB$	$PlateA$

Still using Anti-Plurality, we see that Plate A and Plate B tie, showing that Plate B is also in $f(P')$. Thus Anti-Plurality satisfies the IIA criterion in this case. $\Diamond$

The IIA criterion is violated by all the voting systems we analyze in this project: Borda Count, Plurality, Anti-Plurality, Instant Runoff, Coombs and Instant Runoff [11].

4
Win Big, Lose Big, Sum Big

In this chapter, we will explain how the Win Big, Lose Big, and Sum Big voting systems work and see what fairness criteria they satisfy or violate.

4.1 Win Big, Lose Big, Sum Big

In this section will show how the voting systems Win Big, Lose Big, and Sum Big work.

First, we will look at the Win Big method. Similar to Borda Count, the Win Big method assigns points to an alternative based on their position in a preferential ballot. The difference is that the Win Big method gives an alternative a greater reward for being more preferred by a voter than the Borda Count method, selecting a winner that is generally liked.

Instead of counting how many alternatives are below, an alternative's score is calculated by counting the number of positions they are above *each* alternative; the last place alternative receives 0 points, the second-to-last place alternative receives 1 point, for being 1 row above an alternative, the third-to-last place alternative receives 3 points, 1 for being 1 row above the second-to-last place alternative and 2 for being 2 rows above the last place alternative and so on. The alternative with the highest score wins.

Example 4.1.1. Let $P = \{B_1, B_2, B_3\}$ and $A = \{a, b, c, d\}$. Consider the following ballots:

$$\begin{array}{ccc} a & c & a \\ | & | & | \\ b & b & b \\ | & | & | \\ c & d & d \\ | & | & | \\ d & a & c \\ & & \\ B_1 & B_2 & B_3 \end{array}$$

Using the Win Big method, the weights distributed in B_1 are

- $w_{B_1}(a) = 0 + 1 + 2 + 3 = 6$
- $w_{B_1}(b) = 0 + 1 + 2 = 3$
- $w_{B_1}(c) = 0 + 1 = 1$
- $w_{B_1}(d) = 0.$

So the final score for each alternative would be

- $s(a) = 6 + 0 + 6 = 12$
- $s(b) = 3 + 3 + 3 = 9$
- $s(c) = 1 + 6 + 0 = 7$
- $s(d) = 0 + 1 + 1 = 2.$

Resulting in $s(a) > s(b) > s(c) > s(d)$. So $f(P) = \{a\}$. $\Diamond$

From the Example above, we see that the weighted vector would be $\vec{w} = [6, 3, 1, 0]^t$. Note that these are the first 4 terms of t-numbers, i.e. $\vec{w} = [t_4, t_3, t_2, t_1]$. We now define this formally,

Definition 4.1.2. Let $n = |A|$. The **Win Big** method is a voting system with weighted vector $\vec{w}_W = [t_n, t_{(n-1)}, \ldots, t_2, t_1]^t$, i.e. if $a \in A$ is in row i in $B \in P$ then $w_B(a) = t_{n-i+1}$. △

Example 4.1.3. If $n = 4$, then $\vec{w}_W = [t_4, t_3, t_2, t_1]^t = [6, 3, 1, 0]^t$. ♢

The *Lose Big* voting system also uses the t-numbers when assigning points to alternatives. Similar to how Anti-Plurality is the reversal of Plurality, Lose Big is the *negative* reversal of Win Big. The idea is that the least preferred an alternative is, the more they are penalized, selecting a winner that is generally not disliked.

Instead of counting the number of positions the alternative is above every other alternative, we count the number of positions the alternative is below each alternative, assigning a negative value. This means the first place alternative receives 0 points, the second place alternative receives -1 points, for being 1 row below the first place alternative, the third place alternative receives -3 points, -1 for being 1 row below the second place alternative and -2 for being 2 rows below the first place alternative, and so on. The alternative with the highest score wins.

Like in the Win Big method, notice that the weights are the t-numbers, merely negated.

Definition 4.1.4. Let $n = |A|$. The **Lose Big** method is a voting system with weighted vector $\vec{w}_L = -[t_1, t_2, \ldots, t_{(n-1)}, t_n]^t$, i.e. if $a \in A$ is in row i in $B \in P$ then $w_B(a) = -t_i$. △

Example 4.1.5. If $n = 5$, then $\vec{w}_L = -[t_1, t_2, t_3, t_4, t_5]^t = [0, -1, -3, -6, -10]^t$. ♢

Example 4.1.6. Using the ballots from Example 4.1.1 and the Lose Big weighted vector, $\vec{w}_L = [0, -1, -3, -6]^t$, we see

$$\begin{array}{cccc}
\textbf{[0]} & a & c & a \\
 & | & | & | \\
\textbf{[-1]} & b & b & b \\
 & | & | & | \\
\textbf{[-3]} & c & d & d \\
 & | & | & | \\
\textbf{[-6]} & d & a & c \\
 & & & \\
\vec{w}_L & B_1 & B_2 & B_3
\end{array}$$

Therefore, the final score for each alternative is

- $s(a) = 0 + -6 + 0 = -6$
- $s(b) = -1 + -1 + -1 = -3$
- $s(c) = -3 + 0 + -6 = -9$
- $s(d) = -6 + -3 + -3 = -12.$

This means $s(b) > s(a) > s(c) > s(d)$, resulting in $f(P) = \{b\}$. $\Diamond$

Referring back to Example 4.1.1, notice that Win Big chose a as the winner since a was first in two out of three of the ballots, therefore receiving a greater reward. In Example 4.1.6, b was chosen as the winner instead of a. This is because a was put last in B_2, therefore being penalized more, while b was never placed below second place, since b is not truly disliked by any of the voters.

The *Sum Big* voting system is a combination of the Lose Big and Win Big method. It scores alternatives by adding the weights that an alternative would receive from using the Lose Big and Win Big method.

Definition 4.1.7. Let $n = |A|$. The **Sum Big** method is a voting system with weighted vector

$$\vec{w}_{SB} = \vec{w}_{WB} + \vec{w}_{LB} = \begin{bmatrix} t_n \\ t_{(n-1)} \\ t_{(n-2)} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{bmatrix} + \begin{bmatrix} -t_1 \\ -t_2 \\ -t_3 \\ \vdots \\ -t_{(n-2)} \\ -t_{(n-1)} \\ -t_n \end{bmatrix} = \begin{bmatrix} t_n - t_1 \\ t_{(n-1)} - t_2 \\ t_{(n-2)} - t_3 \\ \vdots \\ t_3 - t_{(n-2)} \\ t_2 - t_{(n-1)} \\ t_1 - t_n \end{bmatrix}$$

,

i.e. if $a \in A$ is in the i^{th} row in $B \in P$, then $w_B(a) = t_{n-i+1} - t_i$. △

Example 4.1.8. Let $n = 4$, then using Example 4.1.1, we see that

$$\vec{w}_{SB} = [t_4, t_3, t_2, t_1]^t + -[t_1, t_2, t_3, t_4]^t = [6, 3, 1, 0]^t + -[0, 1, 3, 6]^t = [6, 2, -2, -6]^t.$$

So,

$$\begin{array}{cccc}
\textbf{[6]} & a & c & a \\
 & | & | & | \\
\textbf{[2]} & b & b & b \\
 & | & | & | \\
\textbf{[-2]} & c & d & d \\
 & | & | & | \\
\textbf{[-6]} & d & a & c \\
 & & & \\
\vec{w}_S & B_1 & B_2 & B_3
\end{array}$$

The resulting scores for each alternative is

- $s(a) = 6 + -6 + 6 = 6$
- $s(b) = 2 + 2 + 2 = 6$
- $s(c) = -2 + 6 + -6 = -2$
- $s(d) = -6 + -2 + -6 = -14$,

which shows $s(a) = s(b) > s(c) > s(d)$. Therefore, $f(P) = \{a, b\}$. $\Diamond$

Proposition 4.1.9. *Let B be a ballot and $a \in A$. Let a be in the i^{th} row of B, then using the Sum Big method, $w_B(a) = t_{(n+1)} - ni$.*

Proof. Let B be a ballot and $a \in A$. Suppose a is in row i in ballot B.

By Definition 4.3.1, we know $\vec{w}_S = \vec{w}_W + \vec{w}_L$. Since $w_B(a) = t_{n-i+1}$ using the Win Big method and $w_B(a) = -t_i$ using the Lose Big method, $w_B(a) = t_{n-i+1} - t_i$ when using the Sum Big method. Therefore,

$$\begin{aligned} w_B(a) &= t_{n-i+1} - t_i \\ &= \frac{(n-i+1)(n-i)}{2} - \frac{i(i-1)}{2} \\ &= \frac{n^2 + n - 2ni}{2} \\ &= \frac{n(n+1)}{2} - ni. \end{aligned}$$

Since $t_{n+1} = \frac{n(n+1)}{2}$, we see that $w_B(a) = t_{(n+1)} - ni$ using the Sum Big method. $\square$

Notation: Let P be a profile and let $a \in A$.

- We will denote the winners of a Win Big election as $f_W(P)$, the winners of a Lose Big election $f_L(P)$, and the winners of a Sum Big election as $f_S(P)$.
- We will denote the score of an alternative in a similar way. The score of a in a Win Big election will be denoted as $s_W(a)$, for Lose Big, $s_L(a)$, and Sum Big, $s_S(a)$.

Theorem 4.1.10. *Let P be a profile and let $a \in A$. If $a \in f_W(P)$ and $a \in f_L(P)$, then $a \in f_S(P)$.*

Proof. Let P be a profile and A be a set of alternative. Let $a \in A$. Suppose the Win Big method was used on P such that $a \in f_W(P)$. Now suppose that on the same set of ballots, P, the Lose Big method was used and resulted in the same winner(s), $a \in f_L(P)$. This means

$$s_W(a) \geq s_W(b) \text{ and } s_L(a) \geq s_L(b) \text{ for all } b \in A - \{a\}.$$

From this, we can deduce that $s_W(a) + s_L(a) \geq s_W(b) + s_L(b)$ for all $b \in A - \{a\}$. Then, using the Sum Big method on P, we get

$$s_S(a) = s_W(a) + s_L(a) \text{ and } s_S(b) = s_W(b) + s_L(b) \text{ for all } b \in A - \{a\}.$$

Since $s_W(a) + s_L(a) \geq s_W(b) + s_L(b)$ for all $b \in A - \{a\}$ we can conclude that $s_S(a) \geq s_S(b)$ for all $b \in A - \{a\}$. Thus, $a \in f_S(P)$ when $a \in f_W(P)$ and $a \in f_L(P)$. □

4.2 Properties of Win Big

In the following sections, we will see which fairness criteria the Win Big and Lose Big method satisfies or violates and then compare them with existing voting systems. Figure 4.2.1. summarizes the results from Chapter 3.

Theorem 4.2.1. *The Win Big Voting System violates the*

1. *Condorcet criterion.*
2. *Majority criterion.*
3. *Condorcet Loser criterion.*
4. *Majority Loser criterion.*
5. *Independence of Irrelevant Alternatives criterion.*

	Majority	Majority Loser	Condorcet	Condorcet Loser	Monotonicity	IIA
Plurality	Yes	—	—	—	Yes	—
Anti-Plurality	—	Yes	—	Yes	Yes	—
Instant Runoff	Yes	—	—	—	—	—
Coombs	Yes	Yes	—	—	—	—
Borda Count	—	Yes	—	Yes	Yes	—

Figure 4.2.1. Voting System's and their Fairness Criteria

Proof. (1)

We will show that the Win Big voting system violates the Condorcet criterion by a counterexample. Suppose the Win Big method is used on a set of ballots $\{B_1, B_2, B_3, B_4\}$ with a set of alternatives $\{A, B, C, D\}$.

A	A	A	B	B
B	B	B	D	C
D	D	D	C	D
C	C	C	A	A
B_1	B_2	B_3	B_4	B_5

Suppose the Win Big method satisfies the Condorcet Criterion. Comparing each alternative head-to-head in the table below we see that

	{A, B}
3	A > B
2	B > A

	{A, C}
3	A > C
2	C > A

	{A, D}
3	A > D
2	D > A

	{B, C}
5	B > C
0	C > B

	{B, D}
5	B > D
0	D > B

	{D, C}
4	D > C
1	C > D

.

From the tables above, we can deduce that A is the Condorcet winner. Using the Win Big method, since $\vec{w}_W = [6, 3, 1, 0]^t$, the final scores for each alternative are

- $s(A) = 6 + 6 + 0 + 6 + 0 = 18$
- $s(B) = 3 + 3 + 6 + 3 + 6 = 21$
- $s(C) = 0 + 0 + 1 + 0 + 6 = 7$
- $s(D) = 1 + 1 + 3 + 1 + 1 = 6$

Therefore, $f(P) = \{B\} \neq \{A\}$. Thus, the Win Big method violates the Condorcet Criterion.

(2)

By Theorem 3.1.6, since Win Big violates the Condorcet criterion it also violates the Majority criterion.

(3)

We will show that the Win Big voting system violates the Condorcer Loser criterion by a counterexample. Suppose a Win Big election was held with 7 voters voting for the set of alternatives $\{a, b, c, d, e\}$. Suppose the ballots below occurred:

e	e	e	a	d	b	c
a	d	a	b	c	c	d
b	c	b	c	b	d	a
c	b	d	d	a	a	b
d	a	c	e	e	e	e
B_1	B_2	B_3	B_4	B_5	B_6	B_7

From these ballots we see that e is the Condorcet loser since e loses in a head-to-head competition with every other alternative at least 4 out of 7 times. This means e must not win if the Win Big method satisfies the Condorcet Loser criterion. Using the Win Big weighted vector, $\vec{w}_W = [t_5, t_4, t_3, t_2, t_1]^t = [10, 6, 3, 1, 0]^t$, we get the resulting scores:

- s(a) = 6 + 0 + 6 + 10 + 1 + 1 + 3 = 27
- s(b) = 3 + 1 + 3 + 6 + 3 + 10 + 1 = 27
- s(c) = 1 + 3 + 0 + 3 + 6 + 6 + 10 = 29
- s(d) = 0 + 6 + 1 + 1 + 10 + 3 + 6 = 27
- s(e) = 10 + 10 + 10 + 0 + 0 + 0 + 0 = 30

Thus we see that $f(P) = \{e\}$, the Majority Loser, when using the Win Big method. Therefore, the Win Big method violates the Majority Loser criterion.

(4)

By Theorem refconlos-majlos, since the Lose Big method violates the Condorcet Loser criterion by Part (3) of this theorem, the Lose Big method also violates the Majority Loser criterion.

(5)

We will show that the Win Big voting system violates the Independence of Irrelevant Alternatives criterion by a counterexample. Let $A = \{a, b, c, d\}$ and let $P = \{B_1, B_2, B_3, B_4\}$. Suppose the preference rankings were made and the Win Big method, with $\vec{w}_{WB} = [6, 3, 1, 0]^t$, is used

$$
\begin{array}{cccc}
a & b & d & a \\
| & | & | & | \\
d & c & b & d \\
| & | & | & | \\
b & a & a & b \\
| & | & | & | \\
c & d & c & c \\
 & & & \\
B_1 & B_2 & B_3 & B_4
\end{array}
$$

This means

- $s(a) = 14$
- $s(b) = 11$
- $s(c) = 3$
- $s(d) = 12$

resulting in $f(P) = \{a\}$. Since $d \notin f(P)$, lets remove d from the above ballots. This would result in the ballots below,

$$
\begin{array}{cccc}
a & b & b & a \\
| & | & | & | \\
b & c & a & b \\
| & | & | & | \\
c & a & c & c \\
 & & & \\
B_1 & B_2 & B_3 & B_4
\end{array}
$$

resulting in $f(P) = \{b\} \neq \{a\}$. Thus, the Win Big method fails the IIA criterion since the removal of a non-winning alternative changes the original winner.

□

It makes sense that the Win Big method fails the Majority, Majority Loser, Condorcet, and Condorcet Loser criterion. While the more preferred alternatives do receive more points than the least preferred, unlike Borda Count the points attributed to the alternatives in higher rows increase exponentially. This means that when an alternative is preferred less, proportionally they are more penalized than, and the reverse for if an alternative is preferred more. By creating a larger range for points to be distributed it gives alternatives who are rarely least preferred a greater chance at winning.

For the IIA criterion, if we look at the counterexample in Theorem 4.2.1 (5), when d is removed from the first set of ballots, we see that b is placed first twice and second twice while a also placed first twice, a is placed second only once and last once. This difference of a being placed last should result in b being the favorable alternative. Borda count and Anti-Plurality would also fail the IIA criterion in the counterexample for Theorem 4.2.1 (5) while Plurality, Instant Runoff, and Coombs tie a and b in the second round. In fact, as seen in Figure 4.2.1., all the voting systems we analyzed also fail this criterion.

Theorem 4.2.2. *The Win Big method satisfies the Monotonicity criterion.*

Proof. Let P be a profile and A be a set of alternatives in an election using the Win Big method. Let $a \in A$ and $f(P) = \{a\}$. This means $s(a) > s(b)$ for all $b \in A - \{a\}$. Let $B \in P$ and let a be in row i in B. This means $w_B(a) = t_{n-i+1}$. Let $b \in A$ and assume b is in row $(i-1)$ in B_1, the row above a, such that $w_B(b) =\mathrm{t}_{n-(i-1)+1} = t_{n-i+2}$. Note that in this ballot, $w_B(a) < w_B(b)$. Let $s(a)$ be the score of a in P and let $s'(a)$ be the score of a in P'. In a revote, forming the profile P', assume a and b swap rows, i.e. change ballot B to $B' \in P'$.This means a would receive an additional

$$\mathrm{t}_{n-(i+1)+1} - t_{n-i+1} = \tfrac{(n-i+2)(n-i+1)}{2} - \tfrac{(n-i+1)(n-i)}{2} = \tfrac{2n-2i+2}{2} = n - i + 1$$

points while b will lose $n - i + 1$ points. So,

$$s'(a) = s(a) + (n - i + 1) \text{ and } s'(b) = s(b) - (n - i + 1).$$

Since $s(a) > s(b)$ and $i < n$, we can deduce that $s(a)+(n-i+1) > s(b) > s(b)-(n-i+1)$. Therefore $s'(a) > s'(b)$. Thus, $f(P) = \{a\}$ remains. □

4.3 Properties of Lose Big

In this section we will see what fairness criteria Lose Big satisfies or violates. First, we look at the fairness criteria the Lose Big method violates.

Theorem 4.3.1. *The Lose Big method violates the*

1. *Condorcet criterion.*
2. *Majority criterion.*
3. *Condorcet Loser criterion.*
4. *Independence of Irrelevant Alternatives criterion.*

Proof. (1)

We will show that the Lose Big voting system violates the Condorcet criterion by a counterexample. Consider the ballots below:

$$
\begin{array}{ccccc}
A & A & B & A & B \\
| & | & | & | & | \\
B & B & D & B & C \\
| & | & | & | & | \\
D & D & C & D & D \\
| & | & | & | & | \\
C & C & A & C & A \\
 & & & & \\
B_1 & B_2 & B_3 & B_4 & B_5
\end{array}
$$

Suppose the Lose Big method satisfies the Condorcet criterion. From the above ballots, we see that A is above every other alternative 3 out of 5, therefore, by Definition 3.1.3, A is the Condorcet winner. This means $f(P) = \{A\}$. Since we know $\vec{w}_L = [0, -1, -3, -6]^t$, the final scores for each alternative are

- $s(A) = -12$
- $s(B) = -3$
- $s(C) = -22$
- $s(D) = -13$

Notice that $s(B) > s(A) > s(D) > s(C)$. This means $f(P) = \{B\}$, which is a contradiction. Thus, the Lose Big method violates the Condorcet criterion.

(2)

By Theorem 3.1.6, since the Win Big voting system violates the Condorcet criterion it also violates the Majority criterion.

This can also be seen by using the ballots in Part (1) of this theorem. Since A is placed first three out of five of the ballots, A is the Majority winner. But we saw from Part (1) that the Lose Big method chose B as it's winner. Therefore, the Lose Big method violates the Majority criterion.

(3)

We will show that the Lose Big voting system violates the Condorcet Loser criterion by a counterexample. Let $A = \{a, b, c\}$ and let $|P| = 7$. Consider the following ballots:

$$\begin{array}{ccccccc} a & a & a & a & b & b & b \\ | & | & | & | & | & | & | \\ b & c & c & c & c & c & c \\ | & | & | & | & | & | & | \\ c & b & b & b & a & a & a \end{array}$$

Since a is preferred to c four out of seven times and b is also preferred to c four out of seven times, c loses each head-to-head competition. Therefore, c is the Condorcet loser. Using the Lose Big method, we see that

- s(a) = 4(0) + 0(-1) + 3(-3) = -9
- s(b) = 3(0) + 1(-1) + 3(-3) = -10
- s(c) = 0(0) + 6(-1) + 1(-3) = -9

This means $f(P) = \{a, c\}$ which means the Condorcet loser, c, is a winner. Therefore, the Win Big method violates the Condorcet Loser Criterion.

(4)

We will show that the Lose Big voting system violates the Independence of Irrelevant Alternatives criterion by a counterexample. Using the Lose Big method on the same ballots from the counterexample for Theorem 4.2.1 (5),

$$\begin{array}{cccc}
a & b & d & a \\
\mid & \mid & \mid & \mid \\
d & c & b & d \\
\mid & \mid & \mid & \mid \\
b & a & a & b \\
\mid & \mid & \mid & \mid \\
c & d & c & c \\
\\
B_1 & B_2 & B_3 & B_4
\end{array}$$

we see that

- $s(a) = -6$
- $s(b) = -7$
- $s(c) = -19$
- $s(d) = -8$

resulting in $f(P) = \{a\}$. Since $d \notin f(P)$, lets remove d from the above ballots. This would result in the below ballots

$$\begin{array}{cccc}
a & b & b & a \\
\mid & \mid & \mid & \mid \\
b & c & a & b \\
\mid & \mid & \mid & \mid \\
c & a & c & c \\
\\
B_1 & B_2 & B_3 & B_4
\end{array}$$

resulting in $f(P) = \{b\}$. Thus, the Lose Big method fails the IIA criterion since the removal of a non-winning alternative changed the original winner. $\square$

Just like the Win Big method, the Lose Big method tries to avoid selecting an alternative who is preferred less, giving a chance for alternatives who are barely ranked low in a profile the chance to win. In the counterexample for Theore 4.3.1, we see that while A is preferred by three out of five voters (B_1, B_2, B_4) and B is preferred only two out of five (B_3, B_5), B

is pretty liked by the remaining three voters (B_1, B_2, B_4) while A is hated by two (B_3, B_5). The Lose Big method was designed to avoid selecting an alternative like A so it makes sense that Lose Big would fail the Majority and Condorcet criterion.

One would think that the Condorcet Loser criterion would be satisfied by the Lose Big voting system. But in the counterexample of Theorem 4.3.1 (3) we see again the same issue that arose in the counterexample for the Majority and Condorcet criterion. Since a and b are both ranked last three out of seven times and c only once, a and b appropriate more negative points, resulting in c winning.

Notice that if in the first ballot of Theorem 4.3.1 (3)'s counterexample, if b and c swapped places, b would now be the Condorcet Loser as well as the Majority Loser and in this case, the Lose Big theorem would still select c, not the Majority Loser. We will now prove the fairness criteria the Lose Big method satisfies.

Theorem 4.3.2. *The Lose Big method satisfies the*

1. *Majority Loser criterion.*

2. *Monotonicity criterion.*

Proof. (1) We want to prove that the Lose Big method will never select the alternative who is least preferred by a majority of voters. Meaning that the majority loser's final score will be less then the average amount of points in a Lose Big election. Recall the Lose Big method uses negative weights.

Let A be a set of alternatives, where $n = |A|$, and $P = \{B_1, B_2, \ldots, B_k\}$ be a set of ballots, where $k = |P|$. Since each ballot has a total of $S_n = \frac{-n(n-1)(n+1)}{6}$ points, by Corollary 2.2.6, we can deduce that the average number of points per alternative would be

$$M = \frac{k\left(\frac{-n(n-1)(n+1)}{6}\right)}{n}$$
$$= \frac{-k(n-1)(n+1)}{6}.$$

Let $a \in A$ and let a be the majority loser. In order for a to lose, $s(a) \leq M$.

Not that since $2 \geq n$, we can deduce that

$$nk \geq 2k$$

$$nk + 3n > 2k$$

$$2(nk + 3n) > 4k$$

$$2(nk + 3n) + 4nk > 4k + 4nk$$

$$6nk + 6n > 4k + 4nk$$

$$-(6nk + 6n) < -(4k + 4nk)$$

$$-6n(k + 1) < -4k(n + 1)$$
$$\frac{-n(k+1)(n-1)}{4} < \frac{-k(n+1)(n-1)}{6}$$

.

This will be used in the following two cases.

Case 1: Suppose k is odd.

Then the majority of ballots would be $\frac{k+1}{2}$. Let $X = \{B_1, B_2, \ldots, B_{\frac{k+1}{2}}\}$ and let a be last in all ballots in X. This means a would be in the n^{th} row and

$$w_X(a) = w_{B_1}(a) + w_{B_2}(a) + \ldots + w_{B_{\frac{k+1}{2}}}$$
$$= -t_n + -t_n + \ldots + -t_n.$$

Since $|X| = \frac{k+1}{2}$, we see that

$$w_X(a) = \left(\frac{k+1}{2}\right) \cdot -t_n.$$

This means the most points a could acquire is by being first in the remaining ballots. Let $Y = P - X$, represent the remaining ballots. Let a be in row 1 in all the ballots in Y. This means, $s_Y(a) = 0$ when $a \in Y$ since the the weighted vector for Lose Big maps $t_1 = 0$ to the alternative in the first row. This means

$$\begin{aligned} s(a) &\leq \left(\frac{k+1}{2}\right) \cdot -t_n \\ &= -\left(\frac{k+1}{2}\right)\left(\frac{(n)(n-1)}{2}\right) \\ &= -\frac{n(k+1)(n-1)}{4}. \end{aligned}$$

Since we know $\frac{-n(k+1)(n-1)}{4} < \frac{-k(n+1)(n-1)}{6}$, we see that $s(a) < M$, when k is odd.

Case 2: Suppose k is even.

Then the majoring of the ballots would be $\frac{k}{2} + 1$. Let $X = \{B_1, B_2, \ldots, B_{\frac{k}{2}+1}\}$. Suppose a is preferred last in all $B \in X$ and first in all $B \in (P - X)$. This means,

$$\begin{aligned} s(a) &\leq \left(\frac{k}{2} + 1\right) \cdot -t_n \\ &= -\left(\frac{k}{2} + 1\right)\left(\frac{n(n-1)}{2}\right) \\ &= -\frac{(k+2)(n)(n-1)}{4}. \end{aligned}$$

Since $-(k+2) < -(k+1)$ we can deduce that

$$\frac{-n(k+2)(n-1)}{4} < -\frac{(k+1)(n)(n-1)}{4}$$

By Case 1, since $\frac{-n(k+1)(n-1)}{4} < \frac{-k(n+1)(n-1)}{6}$, by transitivity

$$\frac{-n(k+2)(n-1)}{4} < \frac{-k(n+1)(n-1)}{6}.$$

This means $s(a) < M$ when k is also even. Therefore we can conclude that if a is preferred last is more than half of the ballots, a will never win. So the Lose Big method satisfies the Majority Lose Criterion.

(2)

Let P be a profile and A be a set of alternative in an election using the Lose Big method.

Let $a \in A$. Assume $f(P) = \{a\}$. This means $s(a) > s(b)$ for all $b \in A - \{a\}$. Let $B \in P$ and let a be in row i in B. This means $w_B(a) = -t_i$.

Now let $b \in A$ and assume b is in row $(i-1)$ in B, the row above a, such that $w_{B_1}(b) = -t_{i-1}$. Thus, $w_{B_1}(b) > w_{B_1}(a)$, which implies $-t_{i-1} > -t_i$.

Let $s(a)$ be the score of a in P and $s'(a)$ be the score of a in P'. In a revote, forming P', assume a and b swap rows, i.e. change ballots from B to B'. This means a would receive an additional

$$(-t_{i-1}) - (-t_i) = \tfrac{-(i-1)(i-2)}{2} - \tfrac{-i(i-1)}{2} = \tfrac{2i-2}{2} = i - 1$$

points while b will lose $i+1$ points. So

$$s'(a) = s(a) + (i-1) \text{ and } s'(b) = s(b) - (i-1).$$

Since $s(a) > s(b)$, we can deduce that $s(a) + (i-1) > s(b) > s(b) - (i-1)$. Therefore, $s'(a) > s'(b)$. Thus, $f(P) = f(P') = \{a\}$ showing that the Lose Big method satisfies the Monotonicity criterion.

□

4.4 Relating Win Big and Lose Big to Other Voting Systems

From figure 4.4.1, we see that just like the other voting systems both Win Big and Lose Big violate the Condorcet and Independence of Irrelevant Alternatives criterion and satisfy a unique set. We now show, in the example below, a situation where the Lose Big method selects a unique plausible winner:

Example 4.4.1. Let $A = \{a, b, c, d\}$ and $P = \{B_1, B_2, B_3, B_4, B_5\}$. Suppose an election was run and the following ballots formed.

	Majority	Majority Loser	Condorcet	Condorcet Loser	Monotonicity	IIA
Plurality	Yes	—	—	—	Yes	—
Anti-Plurality	—	Yes	—	Yes	Yes	—
Instant Runoff	Yes	Yes	—	Yes	—	—
Coombs	Yes	Yes	—	—	—	—
Borda Count	—	Yes	—	Yes	Yes	—
Win Big	—	—	—	—	Yes	—
Lose Big	—	Yes	—	—	Yes	—

Figure 4.4.1. Voting System's and their Fairness Criteria

$$
\begin{array}{ccccc}
a & a & a & d & d \\
| & | & | & | & | \\
b & b & b & b & b \\
| & | & | & | & | \\
c & c & c & c & c \\
| & | & | & | & | \\
d & d & d & a & a \\
 & & & & \\
B_1 & B_2 & B_3 & B_4 & B_5
\end{array}
$$

When using the different voting systems, the resulting winner occur: For Borda Count, Sum Big, Win Big, Plurality, Coombs, and Instant Runoff $f(P) = \{a\}$ and for Anti-Plurality $f(P) = \{b, c\}$. The Lose Big method is the only system that selects b as its winner since b is the least disliked alternative in profile P. $\Diamond$

The Win Big method can be used in situations similar to when Plurality,

5
Sum Big's Relationship to Borda Count

In working with the Sum Big voting system, we noticed some connections with the Borda Count voting system. In this chapter, we will prove some of these connections.

5.1 Affine Transformation of Borda Count

In this section, we will show that the Sum Big's weighted vector is actually an affine transformation of Borda Count's weighted vector.

Notation: Let $a \in A$ and let $B \in P$.

- Let $w_B(a)$ be the weight of a in B when using the Sum Big method and $w'_B(a)$ be the weight of a in B when using the Borda Count method.
- Let $s(a)$ be the total score of a in an election using the Sum Big method and let $s'(a)$ be the total score of a in an election using the Borda Count method.
- Let $f(P)$ be the set of winners in a Sum Big election and $f'(P)$ be the set of winners in a Borda Count election.
- Let $\vec{1}_n$ be a vector with n rows of 1.

Example 5.1.1. Let $A = \{a, b, c, d\}$ where $n = |A| = 4$. Consider profile, P, with the single ballot shown below:

$$\begin{array}{c} a \\ | \\ b \\ | \\ c \\ | \\ d \end{array}$$

The weighted vectors for Sum Big and Borda Count for this profile are shown below:

$$\text{Sum Big: } \begin{bmatrix} 6 \\ 2 \\ -2 \\ -6 \end{bmatrix} \text{ Borda Count: } \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

We noticed some correlation between Sum Big and Borda Count. Note that

$$\begin{aligned} n \cdot \vec{w}_B - t_n \vec{1}_n &= 4 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \cdot \frac{3(4)}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ 8 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 2 \\ -2 \\ -6 \end{bmatrix} \\ &= \vec{w}_S \end{aligned}$$

Therefore, from this example we see that the Sum Big method's weighted vector is an affine transformation of the Borda Count method's weighted vector. $\Diamond$

Theorem 5.1.2. *Let P be a profile and A be a set of alternatives. Let $n = |A|$. Then*

$$\vec{w}_S = n \cdot \vec{w}_B - t_n \cdot \vec{1}_n.$$

Proof. Let $n = |A|$ and let $a \in A$. This means the weight of a in row i using the Sum Big method and the Borda Count method is, $w(a) = (t_{n+1} - ni)$ and $w'(a) = (n - i)$. Expanding $w(a)$ we see that,

$$w(a) = t_{n+1} - ni = \tfrac{n(n+1)}{2} - ni = \tfrac{n^2+n-2ni}{2}$$

and since

$$\begin{aligned} n \cdot w'(a) - t_n \vec{1}_n &= n \cdot (n-i) - \frac{n(n-1)}{2}\vec{1}_n \\ &= \frac{2n^2 - 2ni - n^2 + n}{2} \\ &= \frac{n^2 + n - 2ni}{2} \end{aligned}$$

we can deduce that

$$w(a) = n \cdot w'(a) - t_n \vec{1}_n.$$

when a is in row i. This holds for all weights in $\vec{w}_S$ and $\vec{w}_B$. Thus, we can conclude that

$$\vec{w}_S = n \cdot \vec{w}_B - t_n \cdot \vec{1}_n.$$

$\square$

Theorem 5.1.3. *Let $a, b \in A$. Then $s(a) > s(b)$ if and only if $s'(a) > s'(b)$.*

Proof. Let P be a profile where $k = |P|$. Let $a, b \in A$ and let $n = |A|$. Let a_i be the number of times alternative a is in row i in a ballot.

By Theorem 5.1.2, we know that $\vec{w}_S = n \cdot \vec{w}_B - t_n \vec{1}_n$. Then we know for the i^{th} row in $\vec{w}_S$ and $\vec{w}_B$ we get $t_{n+1} - ni = n \cdot (n-i) - t_n \vec{1}_n$. Since s(a) $= \sum\limits_{i=1}^{n} a_i(t_{n+1} - ni)$ and $s'(a) = \sum\limits_{i=1}^{n} a_i(n-i)$, we can deduce that

$$\begin{aligned}\sum_{i=1}^{n} a_i(t_{n+1} - ni) &= \sum_{i=1}^{n} a_i[n(n-i) - t_n] \\ &= n\sum_{i=1}^{n} a_i(n-i) - t_n\sum_{i=1}^{n} a_i \\ &= n \cdot s'(a) - k \cdot t_n.\end{aligned}$$

Therefore, $s(a) = n \cdot s'(a) - k \cdot t_n$.

Assume $s(a) > s(b)$.

Since $s(a) = n \cdot s'(a) - k \cdot t_n$ and $s(b) = n \cdot s'(b) - k \cdot t_n$, this means

$$n \cdot s'(a) - k \cdot t_n > n \cdot s'(b) - k \cdot t_n so,$$

$$s'(a) > s'(b)$$

Assume $s'(a) > s'(b)$.

Then $n \cdot s'(a) - k \cdot t_n > n \cdot s'(b) - k \cdot t_n$. Since $s(a) = n \cdot s'(a) - k \cdot t_n$ and $s(b) = n \cdot s'(b) - k \cdot t_n$ we can conclude that $s(a) > s(b)$.

Therefore, $s(a) > s(b)$ if and only if $s'(a) > s'(b)$. □

Corollary 5.1.4. *The Sum Big voting system and the Borda Count voting system will always produce the same societal preference order and $f(P) = f'(P)$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ where $n = |A|$. Assume in an election using the Sum Big method with profile P, the societal preference order was

$$a_1 > a_2 > \ldots > a_n$$

such that $f(P) = \{a_1\}$. This means that $s(a_1) > s(a_2) > \ldots > s(a_n)$. By Theorem 5.1.3, this means when using the Borda Count method on P, $s'(a_1) > s'(a_2) > \ldots > s'(a_n)$. This means Borda Count would produce the societal preferential order below

$$a_1 > a_2 > \ldots > a_n,$$

the same as Sum Big, and $f'(P) = \{a_1\} = f(P)$. □

Corollary 5.1.5. *Let P be a profile and let $a \in A$. If $a \in f_W(P)$ and $a \in f_L(P)$, then $a \in f'(P)$.*

Proof. Let P be a profile and let $a \in A$. Suppose the Win Big and Lose Big method was used on P, resulting in $a \in f_W(P)$ and $a \in f_L(P)$. By Theorem 4.1.10, this mean $a \in f(P)$. By Corollary 5.1.4, since $f(P) = f'(P)$, $a \in f'(P)$. □

Thus, we can conclude that since the Sum Big method aggregates and assigns points proportionally to how the Borda Count method does, producing the same societal preference order, that the Sum Big method satisfies the same fairness criteria as the Borda Count voting system. See Figure 5.1.1. for a complete summary on which voting systems satisfies which unique set of fairness criteria.

	Majority	Majority Loser	Condorcet	Condorcet Loser	Monotonicity	IIA
Plurality	Yes	—	—	—	Yes	—
Anti-Plurality	—	Yes	—	Yes	Yes	—
Instant Runoff	Yes	Yes	—	Yes	—	—
Coombs	Yes	Yes	—	—	—	—
Borda Count	—	Yes	—	Yes	Yes	—
Sum Big	—	Yes	—	Yes	Yes	—
Win Big	—	—	—	—	Yes	—
Lose Big	—	Yes	—	—	Yes	—

Figure 5.1.1. Voting System's and their Fairness Criteria

6
Future Work

After finding this connection between the Sum Big method and the Borda Count method, we wanted to see if any sequence of weights, when reversed and added, would be some affine transformation of Borda Count. In some quick examples, one can see that this is not true. So what is it about the triangle numbers that makes this connection.

We also wanted to come up with a method to use the Lose Big, Win Big, and Sum Big weighted vector on bucket posets and see if the Sum Big's weighted vector was an affine transformation of the partial Borda Count method. Refer to [2] for more information on partial Borda.

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