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The Schur Factorization Property as it Applies to Subsets of the General Laguerre Polynomials

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Abstract

A family of polynomials, $\{F_n(x)\}_{n=0}^{\infty}$ indexed by the degree n of the polynomials, exhibits the Schur Factorization Property if, when writing n in base p as

$$n = a_0 + a_1 p + ... + a_r p^r,$$

one can factor $F_n(x)$ as follows

$$F_n(x) \equiv F_{a_0}(x) \cdot (F_{a_1}(x))^p \cdot ... \cdot (F_{a_r}(x))^{p^r} \pmod{p}.$$

Some families of orthogonal polynomials are known to exhibit this property. For example, the Legendre polynomials exhibit this property for any odd prime p and any n. The Gegenbauer polynomials, a generalization of the Legendre polynomials, exhibit the property for some p and some n, but not others. However, for many families of orthogonal polynomials, it is unknown whether there is any subfamily of the family such that the subfamily exhibits the Schur Factorization Property. In this project, we give a criteria for determining which polynomials exhibit the Schur Factorization Property for a certain specialization of the General Laguerre polynomials.

Contents

Dedication

To Lenny and Marcie Gunnell for their love and continued support.

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1
Introduction

Let $f(x)$ be a polynomial defined over the integers $\mathbb{Z}$. That is, let $f(x)$ be an integral polynomial. Now let $f(x)$ be irreducible over the integers. That is, $f(x)$ cannot be factored into two non-constant polynomials $g(x)$ and $h(x)$ such that $g(x)$ and $h(x)$ are both integral polynomials. Now consider some prime p. Since $f(x)$ is integral, the coefficients can be reduced mod p, defining a new polynomial over the field $\mathbb{Z}/p$. Is $f(x)$ irreducible over $\mathbb{Z}/p$? In general, this question is difficult to answer. However, some sequences of polynomials exhibit a special property, called here the Schur Factorization Property such that, if $f(x)$ exhibits the property, $f(x)$ has a nice factorization in $\mathbb{Z}/p$. More specifically, when a sequence of polynomials exhibit the Schur Factorization Property, we can factor a polynomial in the sequence mod p in the same way we can rewrite a number mod p. While the Schur Factorization Property is known to be exhibited by certain sequences of polynomials, it is in general very difficult to determine whether or not a sequence of functions exhibit the property. Specifically, we would like to know which polynomials exhibit the Schur Factorization Property in the collection of sequences of polynomials known as the classical orthogonal polynomials. This collection of sequences of polynomials can be broken into three general families: The Jacobi polynomials, the General Laguerre polynomials, and the Hermite polynomials. Each of these three families consist of sequences of orthogonal polynomials, that is, sequences of polynomials where any two polynomials in the sequence result in an inner product of 0, where the inner product is some integral defined for each specific family in the classical orthogonal polynomials.

Specifically, the classical orthogonal polynomials are important because of how frequently they come up in physics. Specifically, the Hermite polynomials are the solution to the Schrodinger equation for the harmonic oscillator while a specialization of the General Laguerre polynomials (referred to as simply the Laguerre polynomials) are the solution to the Schrodinger equation for a one-electron atom.

While some of the classical orthogonal polynomials have been explored for their potential to exhibit the Schur Factorization Property, and other congruence properties (see [2] and [3]) there is still much that is unknown. Specifically, both Carlitz and Allouche focus on the Jacobi polynomials and their specializations (such as the Legendre polynomials and the Gegenbauer polynomials). This leaves both the General Laguerre polynomials and the Hermite polynomials relatively unexplored for their potential to exhibit the Schur Factorization Property. In this paper, then, we focus on a specialization of the General Laguerre polynomials, giving a criteria by which one can determine whether or not some specific General Laguerre polynomial exhibits the Schur Factorization Property.

2
Orthogonality and Orthogonal Polynomials

2.1 Orthogonality

This section will give a rigorous definition of orthogonality, through material on vector spaces and the inner product. A basic understanding of vector spaces is assumed on the part of the reader, but all other concepts will be presented within this section. This project's main focus is orthogonal polynomials and thus an understanding of what is meant by orthogonality is critical. We now give a definition of orthogonality.

Definition 2.1.1. Definition: Let V be a vector space over some field F. Then an inner product $\langle\ ,\ \rangle$ is a function

$$\langle\ ,\ \rangle : V \times V \to F$$

satisfying, for all vectors $x, y, z \in V$ and all scalars $a \in F$ the following axioms:

$$1. \langle x, y\rangle = \langle y, x\rangle$$

$$2. \langle ax, y\rangle = a\langle x, y\rangle$$

$$3.\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$$

$$4.\langle x,x\rangle \geq 0 \text{ and equals 0 only when } x=0$$

If $\langle x,y\rangle = 0$, then x and y are said to be orthogonal in V.

Consider the following example.

Example 2.1.2. Let $V = \mathbb{R}^2$ with $F = \mathbb{R}$. Then vectors take the form of ordered pairs (x,y).

Recall from linear algebra that the dot product of two vectors, $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ...b_n)$ is the sum

$$\sum_{i=1}^{n} a_j b_j.$$

Then the dot product in our space V for some two vectors (x_1, y_1) and (x_2, y_2) is the sum

$$x_1 \cdot x_2 + y_1 \cdot y_2$$

Lemma 2.1.3. *The dot product of two vectors in V is an inner product.*

Proof. Consider some three vectors, $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in V.

In order to prove our lemma, it suffices to show that, for each of the four axioms in Definition 2.1.1., the dot product satisfies the axiom.

Axiom 1 : $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1 \cdot x_2 + y_1 \cdot y_2 = x_2 \cdot x_1 + y_2 \cdot y_1 = \langle (x_2, y_2), (x_1, y_1)\rangle$.

Axiom 2 : $\langle a(x_1, y_1), (x_2, y_2)\rangle = ax_1 \cdot x_2 + ay_1 \cdot y_2 = a(x_1 \cdot x_2 + y_1 \cdot y_2 = a\langle (x_1, y_1), (x_2, y_2)\rangle$.

Axiom 3:

$$\begin{aligned}
\langle (x_1, y_1) + (x_2, y_2), (x_3, y_3) \rangle &= \langle (x_1 + x_2, y_1 + y_2), (x_3, y_3) \rangle \\
&= (x_1 + x_2)x_3 + (y_1 + y_2)y_3 \\
&= x_1 \cdot x_3 + x_2 \cdot x_3 + y_1 \cdot y_3 + y_2 \cdot y_3 \\
&= x_1 \cdot x_3 + y_1 \cdot y_3 + x_2 \cdot x_3 + y_2 \cdot y_3 \\
&= \langle (x_1, y_1), (x_3, y_3) \rangle + \langle (x_2, y_2), (x_3, y_3) \rangle.
\end{aligned}$$

Axiom 4 : $\langle (x_1, y_1), (x_1, y_1) \rangle = (x_1)^2 + (y_1)^2$ Since x_1 and y_1 are real numbers, $(x_1)^2 + (y_1)^2$ must be nonnegative and can only equal 0 if $(x_1, y_1) = (0, 0)$.

Thus, since the dot product satisfies all four of our conditions, the dot product is an inner product. □

Recall from linear algebra that, since $V = \mathbb{R}^2$ is a Euclidean space, the dot product for V can be rewritten as as follows

Corollary 2.1.4. *For any two vectors, (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$,*

$(x_1, y_1) \cdot (x_2, y_2) = \|(x_1, y_1)\| \|(x_2, y_2)\| \cos\theta$ *where θ is the angle between (x_1, y_1) and (x_2, y_2)*

Lemma 2.1.5. *For any nonzero vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, (x_1, y_1)$ and (x_2, y_2) are orthogonal in $\mathbb{R}^2$ if and only if (x_1, y_1) and (x_2, y_2) are perpendicular.*

Proof. Consider some $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that (x_1, y_1) and (x_2, y_2) are perpendicular. That is, $\theta = \frac{(2k+1)\pi}{2}$ for some $k \in \mathbb{Z}_+$. Then

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \|(x_1, y_1)\| \|(x_2, y_2)\| cos(\frac{(2k+1)\pi}{2}) = 0.$$

Then (x_1, y_1) and (x_2, y_2) are orthogonal in $\mathbb{R}^2$.

Now let (x_1, y_1) and (x_2, y_2) be orthogonal in $\mathbb{R}^2$. That means $\|(x_1, y_1)\| \|(x_2, y_2)\| cos\theta = 0$.

However, since neither (x_1, y_1) nor (x_2, y_2) are zero vectors, their magnitude cannot be zero. Thus, it must be the case that $cos\theta = 0$. This is only the case when $\theta = \frac{(2k+1)\pi}{2}$, meaning (x_1, y_1) and (x_2, y_2) are perpendicular. □

2.2 Orthogonal Functions

In this section, we expand on the concept of orthogonality, applying it to functions. Trigonometric functions are used as examples and, thus, trigonometric identities and knowledge of trigonometric integrals is assumed on the part of the reader.

As a more complicated example of orthogonality consider $S = \{\sin(nx), \cos(nx)\}_{n=0}^{\infty}$ on the interval $[-\pi, \pi]$. Now let V be the vector space generated by the set S. That is, V is the set S closed under addition and closed under scalar multiplication for all $r \in \mathbb{R}$. Thus V is a vector space over $\mathbb{R}$ under addition.

Definition 2.2.1. For V as it is defined above, define the inner product, $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ to be

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

First, let us check that our proposed inner product satisfies the axioms of an inner product as defined in Definition 2.1.1.

Lemma 2.2.2. *For V, $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ is an inner product.*

Proof. Let $f(x), g(x), h(x) \in V$ and let $r \in \mathbb{R}$.

Axiom 1:

Consider

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx = \int_{-\pi}^{\pi} g(x)f(x)dx = \langle g(x), f(x) \rangle.$$

Axiom 2:

Also note

$$\langle rf(x), g(x)\rangle = \int_{-\pi}^{\pi} rf(x)g(x)dx = r\int_{-\pi}^{\pi} f(x)g(x)dx = r\langle f(x), g(x)\rangle.$$

Axiom 3:

Also

$$\begin{aligned}\langle f(x)+g(x), h(x)\rangle &= \int_{-\pi}^{\pi} (f(x)+g(x))h(x)dx \\ &= \int_{0\pi}^{\pi} (f(x)h(x)+g(x)h(x))dx \\ &= \int_{0\pi}^{\pi} f(x)h(x)dx + \int_{0\pi}^{\pi} g(x)h(x)d \\ &= \langle f(x), h(x)\rangle + \langle g(x), h(x)\rangle\end{aligned}$$

Axiom 4: Furthermore, consider $\langle f(x), f(x)\rangle$ where $f(x) \neq 0$. Then either $f(x) = \sin(nx)$ for some n such that $\sin(nx) \neq 0$ or $f(x) = \cos(nx)$ for some n such that $\cos(nx) \neq 0$.

Case 1: $f(x) = \sin(nx)$. Then

$$\begin{aligned}\langle f(x), f(x)\rangle &= \int_{-\pi}^{\pi} \sin(nx)^2 dx \\ &= \frac{x}{2} - \frac{1}{4n}\sin(2x)\Big|_{-\pi}^{\pi} \\ &= \frac{\pi}{2} - 0 - \frac{-\pi}{2} - 0 \\ &= \pi\end{aligned}$$

Case 2: $f(x) = \cos(nx)$. Then

$$\begin{aligned}
\langle f(x), f(x)\rangle &= \int_{-\pi}^{\pi} \cos(nx)^2 dx \\
&= \frac{x}{2} + \frac{1}{4n}\sin(2x)\Big|_{-\pi}^{\pi} \\
&= \frac{\pi}{2} + 0 - \frac{-\pi}{2} + 0 \\
&= \pi
\end{aligned}$$

Thus, for any $f(x) \in V$ such that $f(x) \neq 0$, $\langle f(x), f(x)\rangle > 0$.

We have now shown that our proposed inner product satisfies all the axioms for an inner product. □

Lemma 2.2.3. *For any two distinct functions, $f(x), g(x) \in V$, $f(x)$ and $g(x)$ are orthogonal.*

Proof. In order to check orthogonality for every $f(x)$ and $g(x)$, we need to take each of the following inner products:

$$\begin{aligned}
1 &: \int_{-\pi}^{\pi} (1)(\sin(nx))dx \\
2 &: \int_{-\pi}^{\pi} (1)(\cos(nx))dx \\
3 &: \int_{-\pi}^{\pi} (\sin(nx))(\sin(mx))dx \\
4 &: \int_{-\pi}^{\pi} (\cos(nx))(\cos(mx))dx \\
5 &: \int_{-\pi}^{\pi} (\cos(nx))(\sin(nx))dx \\
6 &: \int_{-\pi}^{\pi} (\cos(nx))(\sin(mx))dx
\end{aligned}$$

While we have left out $f(x) = 0$ and $g(x) = 0$, it is clear that, if either $f(x)$ or $g(x)$ are 0, the whole inner product becomes the integral of 0, which is also 0 and thus the two functions will be orthogonal trivially. Thus, it suffices to check each of the 6 cases we have listed.

$$1: \int_{-\pi}^{\pi}(1)(\sin(nx))dx = -\frac{1}{n}\cos(nx)\Big|_{-\pi}^{\pi} = -\frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(-n\pi).$$

If n is odd, $\cos(n\pi) = -1$. Then $\frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(-n\pi) = \frac{1}{n} - \frac{1}{n} = 0$.

If n is even, $\cos(n\pi) = 1$. Then $\frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(-n\pi) = -\frac{1}{n} + \frac{1}{n} = 0$.

In any case, 1 and $\sin(nx)$ are orthogonal in V.

$$2: \int_{-\pi}^{\pi}(1)(\cos(nx))dx = \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} = \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(-n\pi).$$

Since n is an integer, $\sin(n\pi) = 0$ and thus, our entire inner product equals 0. Thus, 1 and $\cos(nx)$ are orthogonal in V.

$$3: \int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx = \int_{-\pi}^{\pi}\frac{1}{2}(\cos(m-n)x - \cos(m+n)x)dx$$
$$= \left(\frac{\sin((m-n)x)}{2(m-n)} - \frac{\sin((m+n)x)}{2(m+n)}\right)\Big|_{-\pi}^{\pi}$$
$$= \left(\frac{\sin((m-n)\pi)}{2(m-n)} - \frac{\sin((m+n)\pi)}{2(m+n)}\right) - \left(\frac{\sin((m-n)(-\pi))}{2(m-n)} - \frac{\sin((m+n)(-\pi))}{2(m+n)}\right)$$

Since m and n are integers,

$$\sin(m+n)\pi = \sin(m-n)\pi = \sin(m+n)(-\pi) = \sin(m-n)(-\pi) = 0$$

and thus the inner product of $\sin(nx$ and $\sin(mx)$ is zero. Thus, $\sin(nx$ and $\sin(mx)$ are orthogonal in V.

$$4: \int_{-\pi}^{\pi} (\cos(nx))(\cos(mx))dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)dx$$
$$= \left(\frac{\sin((m-n)x)}{2(m-n)} + \frac{\sin((m+n)x)}{2(m+n)}\right)\Big|_{-\pi}^{\pi}$$
$$= \left(\frac{\sin((m-n)\pi)}{2(m-n)} - \frac{\sin((m+n)\pi)}{2(m+n)}\right) - \left(\frac{\sin((m-n)(-\pi))}{2(m-n)} - \frac{\sin((m+n)(-\pi))}{2(m+n)}\right)$$

Since m and n are integers,

$$\sin(m+n)\pi = \sin(m-n)\pi = \sin(m+n)(-\pi) = \sin(m-n)(-\pi) = 0$$

and thus the inner product of $\cos(nx)$ and $\cos(mx)$ is zero. Thus, $\cos(nx)$ and $\cos(mx)$ are orthogonal in V.

$$5: \int_{-\pi}^{\pi} (\cos(nx))(\sin(nx))dx = \int_{-\pi}^{\pi} \frac{\sin^2(nx)}{2n}\Big|_{-\pi}^{\pi}$$
$$= \frac{\sin^2(n\pi)}{2n} - \frac{\sin^2(-n\pi)}{2n}$$

Since n is an integer, $\sin^2(n\pi) = 0$ and thus the inner product for $\cos(nx)$ and $\sin(nx)$ is zero. Thus, $\cos(nx)$ and $\sin(nx)$ are orthogonal in V.

$$6: \int_{-\pi}^{\pi} (\cos(nx))(\sin(mx))dx = \int_{-\pi}^{\pi} \frac{1}{2}(\sin((m-n)x) + \sin((m+n)x))dx$$
$$= \left(-\frac{\cos((m-n)x)}{2(m-n)} - \frac{\cos((m+n)x)}{2(m+n)}\right)\Big|_{-\pi}^{\pi}$$
$$= \left(-\frac{\cos((m-n)\pi)}{2(m-n)} - \frac{\cos((m+n)\pi)}{2(m+n)}\right) - \left(-\frac{\cos((m-n)(-\pi))}{2(m-n)} - \frac{\cos((m+n)(-\pi))}{2(m+n)}\right)$$

If m and n are even, then $m+n$ and $m-n$ will be even. If m and n are odd, then $m+n$ and $m-n$ will be even. If m is even and n is odd or if m is odd and n is even, then $m+n$ and $m-n$ will be odd. Thus, our inner product has two cases: one where $m+n$ and $m-n$

are even and one where $m+n$ and $m-n$ are odd.

Case 1: $m+n$ and $m-n$ are odd.

Then $\cos((m+n)\pi) = \cos((m-n)\pi) = \cos((m+n)(-\pi)) = \cos((m-n)(-\pi)) = -1$.

Thus our inner product equals

$$\left(-\frac{-1}{2(m-n)} - \frac{-1}{2(m+n)}\right) - \left(-\frac{-1}{2(m-n)} - \frac{-1}{2(m+n)}\right) = 0.$$

Case 2: $m+n$ and $m-n$ are even.

Then $\cos((m+n)\pi) = \cos((m-n)\pi) = \cos((m+n)(-\pi)) = \cos((m-n)(-\pi)) = 1$.

Thus our inner product equals

$$\left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) - \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) = 0.$$

Thus, the inner product for $\cos(nx)$ and $\sin(mx)$ is 0 in for any m and n and, thus, $\cos(nx)$ and $\sin(mx)$ are orthogonal in V.

Having show that all six of our cases result in an inner product of 0, we can conclude that, for any functions $f(x)$ and $g(x)$ in S, $f(x)$ and $g(x)$ are orthogonal. $\square$

This conception of the inner product for our vector space V being an integral holds for more general vector spaces of functions. Now we will move on to developing a notion of orthogonal sequences polynomials.

2.3 Orthogonal Polynomials

In this section, we develop the idea of a sequence of orthogonal polynomials in general, leaving more specific examples of orthogonal polynomials to be introduced and worked with in later sections.

Definition 2.3.1. Let $\{F_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials, defined on some interval $[a,b]$ where, for any $n \in \mathbb{N}$, $F_n(x)$ is the unique polynomial of degree n in F.

Definition 2.3.2. The sequence $\{F_n(x)\}$ defined on the interval $[a,b]$ is said to be orthogonal with respect to $w(x)$ if we set the inner product to be

$$\langle F_n(x), F_m(x)\rangle = \int_a^b F_n(x)F_m(x)w(x)dx$$

and if

$$\langle F_n(x), F_m(x)\rangle = 0 \text{ unless } n = m.$$

The function $w(x)$ is called a weight function and examples of them follow.

2.3.1 Weight Functions

In our previous example, where V was the vector space formed by $S = \{\sin(nx), \cos(nx)\}_{n=0}^{\infty}$ being made closed under addition, $w(x) = 1$ and was thus trivial. However, with more complicated sets of polynomials such as the ones we will be working with later in this project, $w(x)$ becomes a necessary component to insuring that our sequence of polynomials is orthogonal.

For an example, consider the Laguerre polynomials. While this sequence of orthogonal polynomials will be introduced in more detail later, for now it suffices to say that these polynomials are defined on the interval $[0,\infty)$ and that the Laguerre polynomial of degree n can be written as

$$L_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{n-k}\frac{x^k}{k!}$$

and that these polynomials are orthogonal with respect to $w(x) = e^{-x}$.

First, let us attempt to take their inner product, excluding the weight function (that is, setting the weight function to the trivial function $w(x) = 1$).

Example 2.3.3. Consider

$$\int_0^\infty L_2(x)L_3(x)dx = \int_0^\infty \left(\frac{1}{2}x^2 - 2x + 1\right)\left(-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1\right)dx$$

$$= \int_0^\infty \left(-\frac{1}{12}x^5 + \frac{3}{4}x^4 - \frac{3}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x^4 - 3x^3 + 6x^2 - 2x - \frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1\right)dx$$

$$= \int_0^\infty \left(-\frac{1}{12}x^5 + \frac{13}{12}x^4 - \frac{28}{6}x^3 + 8x^2 - 5x + 1\right)dx$$

$$= -\frac{1}{72}x^6 + \frac{13}{60}x^5 - \frac{7}{6}x^4 + \frac{8}{3}x^3 - \frac{5}{2}x^2 + x + C\Big|_0^\infty$$

$$= \lim_{h\to\infty} -\frac{1}{72}h^6 + \frac{13}{60}h^5 - \frac{7}{6}h^4 + \frac{8}{3}h^3 - \frac{5}{2}h^2 + h \neq 0$$

Thus, the Laguerre polynomials are not orthogonal with respect to $w(x) = 1$. However, now consider the Laguerre polynomials with respect to the weight function $w(x) = e^{-x}$.

Returning to our previous inner product of $L_2(x)$ and $L_3(x)$, this time including our new weight function, we have

$$\langle L_2(x), L_3(x)\rangle = \int_0^\infty L_2(x)L_3(x)e^{-x}dx$$

$$= \int_0^\infty \left(-\frac{1}{12}x^5 + \frac{13}{12}x^4 - \frac{28}{6}x^3 + 8x^2 - 5x + 1\right)e^{-x}dx$$

$$= \left(\frac{1}{12}x^5 - \frac{2}{3}x^4 + 2x^3 - 2x^2 + x\right)e^{-x}\Big|_0^\infty$$

$$= \lim_{h\to\infty} \frac{\left(\frac{1}{12}h^5 - \frac{2}{3}h^4 + 2h^3 - 2h^2 + h\right)}{e^h} - \left(\frac{1}{12}0^5 - \frac{2}{3}0^4 + 0^3 - 0^2 + 0\right)e^{-0}$$

$$= 0 - 0 = 0$$

Thus, we can see that $L_2(x)$ and $L_3(x)$ are orthogonal with respect to the weight function e^{-x}. In general, each of the sequences of orthogonal polynomials we will introduce in this project will be defined as orthogonal with respect to its own specific weight function, $w(x)$.

2.3.2 Generating Functions

Another way of describing a sequence of orthogonal polynomials is to describe them by their generating function.

Definition 2.3.4. Let $\{F_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials, indexed by degree n of F. Then a generating function is the sum

$$\sum_{i=0}^{\infty} F_i(x)t^i.$$

For many complicated sequences of polynomials, it becomes much easier to simply work with their generating functions.

As an important example consider the sequence of polynomials $F_n(x) = 1$ Then the generating function for this sequence of polynomials is

$$\sum_{i=0}^{\infty} 1^i t^i = \sum_{i=0}^{\infty} t^i$$

Recall from calculus that $\sum_{i=0}^{\infty} t^i = 1 + t + t^2 + ...$ is the Maclaurin series for the $\frac{1}{1-t}$. Thus, the generating function for our sequence of polynomials $\{F_n(x)\}_{n=0}^{\infty}$ is

$$\sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$$

For a slightly more complicated example, consider the sequence of polynomials $\{G_n(x)\}_{n=0}^{\infty}$ where $G_n(x) = x^n$. Then the generating function for this sequence of polynomials is

$$\sum_{i=0}^{\infty} x^i t^i = \sum_{i=0}^{\infty} (xt)^i$$

But then, similarly to how $\sum_{i=0}^{\infty} t^i = 1 + t + t^2 + ...$ was the Maclaurin series for $\frac{1}{1-t}$, our new series $\sum_{i=0}^{\infty} (xt)^i = 1 + xt + (xt)^2 + ...$ is the Maclaurin series for $\frac{1}{1-xt}$. Thus the generating function for our sequence of polynomials $\{G_n(x)\}_{n=0}^{\infty}$ is

$$\sum_{i=0}^{\infty}(xt)^i = \frac{1}{1-xt}$$

In both of our examples, and with many other sequences of polynomials, the generating functions when expressed as a sum can be simplified into much more simple expressions.

For many of the more complicated sequences of orthogonal polynomials we will be introducing later, this is the case. Thus, as we introduce sequences of orthogonal polynomials in the following section, we will also introduce their generating functions.

2.4 The Classical Orthogonal Polynomials

In this section, we introduce the three families of Classical Orthogonal Polynomials, giving some of their characteristics and also introducing important specializations of each family as necessary. Each of the three families are then dealt with in depth in their own subsection.

While many families of orthogonal polynomials exist, for our purposes, we will be focusing on the Classical Orthogonal Polynomials. These are some of the best-known and best studied of the orthogonal polynomials and have a wide range of uses. In general, each of the three families of classical orthogonal polynomials is found as the solution to some differential equation. Thus, as we introduce each of the three families of the classical orthogonal solutions, we will begin by giving the differential equation that family of orthogonal polynomials is a solution for.

2.4.1 The Jacobi Polynomials

The Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ are the largest of the three families of the classical orthogonal polynomials. In order to find one specific Jacobi polynomial, it is necessary to specify one index n such that $n \in \mathbb{N}$, as well as two parameters α and β such that $\alpha, \beta > -1$. They are defined over the range $[-1, 1]$.

In general, the Jacobi polynomials are the polynomial solutions to the differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0.$$

Any two distinct Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ and $P_m^{(\alpha,\beta)}(x)$ are orthogonal to each other with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ and thus, the following is true for any two Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_m^{(\alpha,\beta)}(x)$ for any fixed α and β:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0.$$

One important representation of the Jacobi polynomials is given by Rodrigues' formula, which simplifies the Jacobi polynomials into the following:

Definition 2.4.1. For $\alpha, \beta > -1$ and for $n \in \mathbb{Z}_+$,

$$P_n^{(\alpha,\beta)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v}\binom{n+\beta}{v}\left(\frac{x-1}{2}\right)^v\left(\frac{x+1}{2}\right)^{n-v}.$$

Definition 2.4.2. The generating function for the Jacobi polynomials is

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)} t^n = 2^{\alpha+\beta}(1-2xt+t^2)^{-\frac{1}{2}}(1-t+(1-2xt+t^2)^{\frac{1}{2}})^{-\alpha}(1+t+(1-2xt+t^2)^{\frac{1}{2}})^{-\beta}$$

While the Jacobi polynomials represent an entire family of the classical orthogonal polynomials onto themselves, there are some specializations of the Jacobi polynomials that are important enough to be mentioned on their own here.

The Legendre Polynomials

Let $\alpha = \beta = 0$. This specialization of the Jacobi polynomials is known as the Legendre polynomials. Setting both α and β equal to 0 in our formula for the Jacobi polynomials, we get

Definition 2.4.3. For $n \in \mathbb{Z}_+$,

$$P_n(x) = \sum_{v=0}^{n} \binom{n}{n-v}\binom{n}{v}\left(\frac{x-1}{2}\right)^v\left(\frac{x+1}{2}\right)^{n-v}.$$

But

$$\binom{n}{n-v}\binom{n}{v} = \frac{n!}{(n-v)!v!}\frac{n!}{v!(n-v)!} = \binom{n}{v}^2$$

Furthermore,

$$\left(\frac{x-1}{2}\right)^v\left(\frac{x+1}{2}\right)^{n-v} = (x-1)^v(x+1)^{n-v}\left(\frac{1}{2}^v\frac{1}{2}^{n-v}\right) = (x-1)^v(x+1)^{n=v}\left(\frac{1}{2}\right)^n$$

Thus, we can rewrite the Legendre polynomials as follows:

Definition 2.4.4. Our simplified explicit formula for the Legendre polynomials is

$$P_n(x) = \left(\frac{1}{2}\right)^n \sum_{v=0}^{n} \binom{n}{v}^2 (x-1)^v(x+1)^{n-v}.$$

Notice that, since $\alpha = \beta = 0$, the weight function for the Legendre polynomials is $(1-x)^0(1+x)^0 = 1$. Thus, any two distinct Legendre polynomials, $L_n(x)$ and $L_m(x)$,

$$\int_{-1}^{1} L_n(x)L_m(x)dx = 0.$$

Also notice that, since $\alpha = \beta = 0$, the generating function for the Jacobi polynomials simplifies to the following:

Definition 2.4.5. The generating function for the Legendre polynomials is

$$\sum_{n=0}^{\infty} L_n(x)t^n = (1-2xt+t^2)^{-\frac{1}{2}}.$$

The Gegenbauer Polynomials

A generalization of the Legendre polynomials, known as the Gegenbauer (or Ultraspherical) polynomials is a specialization of the Jacobi polynomials where $\alpha = \beta = \lambda - \frac{1}{2}$.

Consider the weight function, $w(x)$ for these polynomials. Since the Gegenbauer polynomials are a specialization of the Jacobi polynomials, we can simply plug in $\lambda - \frac{1}{2}$ for α and β. Doing this gives us

$$\begin{aligned} w(x) &= (1-x)^{\lambda-\frac{1}{2}}(1+x)^{\lambda-\frac{1}{2}} \\ &= ((1-x)(1+x))^{\lambda-\frac{1}{2}} \\ &= (1-x^2)^{\lambda-\frac{1}{2}} \end{aligned}$$

Definition 2.4.6. The generating function for the Gegenbauer polynomials, a simplification of the generating function for the Jacobi polynomials is

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = \frac{1}{(1-2xt+t^2)^{\lambda}}.$$

Definition 2.4.7. In terms of the Jacobi polynomials, the Gegenbauer polynomials are given as

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},(\lambda-\frac{1}{2})}$$

where $(a)_n$ is a rising factorial, equal to $a \cdot (a-1) \cdot ... \cdot (a-(n-1))$.

The Chebyshev Polynomials

Another two specializations of the Gegenbauer polynomials (in addition to the Legendre polynomials, a special case of the Gegenbauer polynomials where $\lambda = \frac{1}{2}$) are the Chebyshev polynomials of the first and second kind. As the name suggests, these two specializations of the Gegenbauer polynomials are related to one another. In general, we can say the Chebyshev polynomials of the first kind (notated $T_n(x)$) are a specialization of the Gegenbauer polynomials where $\lambda = 0$.

Now consider the weight function for these polynomials.

Setting λ equal to 0 in our weight function for the Gegenbauer polynomials, we have

$$w(x) = (1 - x^2)^{0-\frac{1}{2}} = \frac{1}{\sqrt{1 - x^2}}$$

Definition 2.4.8. The generating function for the Chebyshev polynomials of the first kind is

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1 - xt}{1 - 2xt + t^2}$$

Since the Gegenbauer polynomials are a specialization of the Jacobi polynomials where $\alpha = \beta = \lambda - \frac{1}{2}$ and the Chebyshev polynomials of the first kind are a specialization of the Gegenbauer polynomials where $\lambda = 0$, we can determine a explicit formula of the Chebyshev polynomials of the first kind by setting $\alpha = \beta = -\frac{1}{2}$. Doing so, we get

Definition 2.4.9. The Chebyshev polynomials of the first kind can be written as

$$T_n(x) = \sum_{v=0}^{n} \binom{n - \frac{1}{2}}{n - v} \binom{n - \frac{1}{2}}{v} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v}.$$

In general, we can say the Chebyshev polynomials of the second kind (notated $U_n(x)$) are a specialization of the Gegenbauer polynomials where $\lambda = 1$. Now consider the weight function for these polynomials. Setting λ equal to 1 in our weight function for the Gegenbauer polynomials, we have

$$w(x) = (1 - x^2)^{1-\frac{1}{2}} = \sqrt{1 - x^2}$$

Similarly, setting λ equal to 1for the generating function of the Gegenbauer polynomials, we get the following

Definition 2.4.10. The generating function for the Chebyshev polynomials of the second kind is

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{(1 - 2xt + t^2)^1} = \frac{1}{1 - 2xt + t^2}$$

Since the Gegenbauer polynomials are a specialization of the Jacobi polynomials where $\alpha = \beta = \lambda - \frac{1}{2}$ and the Chebyshev polynomials of the second kind are a specialization of the Gegenbauer polynomials where $\lambda = 1$, we can determine a explicit formula for the Chebyshev polynomials of the second kind by setting $\alpha = \beta = \frac{1}{2}$. Doing so, we get

Definition 2.4.11. One explicit formula for the Chebyshev polynomials of the second kind is

$$U_n(x) = \sum_{v=0}^{n} \binom{n+\frac{1}{2}}{n-v} \binom{n+\frac{1}{2}}{v} \left(\frac{x-1}{2}\right)^{v} \left(\frac{x+1}{2}\right)^{n-v}.$$

2.4.2 The General Laguerre Polynomials

While not as large as the Jacobi polynomials, a specific General Laguerre Polynomial is determined by setting the parameter $\alpha > -1$ and some index $n \in \mathbb{N}$.

In general, the General Laguerre polynomials are the solution to the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0.$$

Any two distinct Laguerre polynomials $L_n^\alpha(x)$ and $L_m^\alpha(x)$ with $n, m \in \mathbb{N}$ are orthogonal with respect to the inner product

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) e^{-x} x^\alpha dx = 0$$

One important representation of the General Laguerre polynomials is given by Rodrigues' formula, giving us the following formula:

Definition 2.4.12. For $\alpha \in \mathbb{R}$ and for $n \in \mathbb{Z}_+$

$$L_n^\alpha(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}$$

Definition 2.4.13. The generating function for the General Laguerre polynomials is as follows:

$$\sum_{n}^{\infty} L_n^\alpha(x) t^n = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{tx}{1-t}}$$

The Laguerre Polynomials

One important specialization of the General Laguerre polynomials, simply referred to as the Laguerre polynomials, is found by setting $\alpha = 0$.

To avoid confusion, the General Laguerre polynomials, notated $L_n^\alpha(x)$ will always be referred to by their full name in the remaineder of this paper. Thus, any mention of the Laguerre polynomials, notated $L_n(x)$ can be taken to mean the specialization of the General Laguerre polynomials such that $\alpha = 0$. Setting α equal to 0 in the Rodrigues' formula for the General Laguerre polynomials, we get the following explicit representation of the Laguerre polynomials.

Definition 2.4.14. The Laguerre polynomials can be written as follows

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} \frac{x^k}{k!}.$$

Notice that since $\alpha = 0$, the weight function for the General Laguerre polynomials, $w(x) = e^{-x}x^\alpha$ simplifies to $w(x) = e^{-x}$.

Furthermore, the generating function for the General Laguerre polynomials simplifies, when one takes α to be zero, to the following:

Definition 2.4.15. The generating function for the Laguerre polynomials is

$$\sum_{n}^{\infty} L_n(x)t^n = \frac{1}{1-t} e^{-\frac{tx}{1-t}}$$

2.4.3 The Hermite Polynomials

The Hermite polynomials, unlike both the Jacobi polynomials and the General Laguerre polynomials, are a single sequence of orthogonal polynomials. Thus, each Hermite polynomial, $H_n(x)$ is specified simply by setting some index $n \in \mathbb{N}$.

Definition 2.4.16. The Hermite polynomials can be expressed as a special case of the General Laguerre polynomials, where the Hermite polynomials of an even order can be expressed as follows:

$$H_{2n}(x) = (-4)^n n! L_n^{(-\frac{1}{2})}(x^2) = (-4)^n n! \sum_{k=0}^{n} (-1)^k \binom{n-\frac{1}{2}}{n-k} \frac{x^{2k}}{k!}$$

Similarly, the Hermite polynomials of an odd order can be expressed as follows:

$$H_{2n+1}(x) = 2(-4)^n n! x L_n^{(\frac{1}{2})}(x^2) = 2(-4)^n n! x \sum_{k=0}^{n} (-1)^k \binom{n+\frac{1}{2}}{n-k} \frac{x^{2k}}{k!}$$

Any two distinct Hermite polynomials, $H_n(X)$ and $H_m(x)$ for $n, m \in \mathbb{N}$ are orthogonal with respect to the inner product

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 0$$

Definition 2.4.17. The generating function for the Hermite polynomials is as follows:

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{xt - \frac{1}{2}t^2}$$

Now, having summed up all three families of the classical orthogonal polynomials, we will delve deeper into one specific property of sequences of polynomials, known as the Schur Factorization Property. Investigating which, if any, of the classical orthogonal polynomials exhibit this property will be the main topic of this paper and, thus we will need to fully develop the Schur Factorization Property before beginning to answer that question.

3

The Schur Factorization Property

In this chapter, we introduce and rigorously define the Schur Factorization Property, the main topic of this paper. We introduce examples of sequences of polynomials that have the property, as well as methods by which previous researchers have shown the property to hold for more complicated sequences of polynomials.

3.1 The Schur Factorization Property

In this section, we introduce the Schur Factorization Property, giving its definition. We go on to give simple examples of sequences of polynomials that exhibit the property, proving for each example that the sequence exhibits the property.

Definition 3.1.1. Consider some sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ indexed by degree n. Note that this sequence of polynomials need not be orthogonal. Then the sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ is said to exhibit the Schur Factorization Property if, for any $n \in \mathbb{N}$ and any odd prime $p \in \mathbb{N}$ it is the case that

$$A_n(x) \equiv (A_{a_0}(x))(A_{a_1}(x))^p ... (A_{a_r}(x))^{p^r} \pmod{p}$$

where

$$n = a_0 + a_1 p + ... + a_r p^r \text{ where } 0 \leq a_i < p$$

Let us consider a simple example of a sequence of polynomials that exhibit the Schur Factorization Property.

Example 3.1.2. Consider the sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ where $A_n(x) = x^n$. First, let $p = 5$ and let $n = 21$. Writing 21 in base 5, we have

$$21 = 1 + 4(5)^1.$$

Now consider $A_{21}(x) = x^{21}$. In order to show that $\{A_n(x)\}_{n=0}^{\infty}$ has the Schur property when $p = 5$ and $n = 21$, we would need to show that

$$A_{21}(x) \equiv A_1(x) \cdot (A_4(x))^5 \pmod{5}.$$

But then

$$\begin{aligned} A_{21}(x) &= x^{21} \\ &= x \cdot x^{20} \\ &= x \cdot (x^4)^5 \\ &= A_1(x) \cdot (A_4(x))^5. \end{aligned}$$

And since $A_{21}(x) = A_1(x) \cdot (A_4(x))^5$, it is clear they are also equivalent mod 5.

Now we will prove that the sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property for all $n \in \mathbb{N}$ and for all $p \in \mathbb{N}$ such that p is an odd prime.

Lemma 3.1.3. *The sequence of polynomials $\{A_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property.*

Proof. Let $p \in \mathbb{N}$ be some odd prime. Now consider some $n \in \mathbb{N}$.

Then we can rewrite n such that

$$n = a_0 + a_1 p + ... + a_r p^r$$

where $0 \leq a_i < p$.

Now consider $(A_{a_0}(x))(A_{a_1}(x))^p ... (A_{a_r}(x))^{p^r}$. We can rewrite this expression as follows

$$\begin{aligned}(A_{a_0}(x))(A_{a_1}(x))^p ... (A_{a_r}(x))^{p^r} &= x^{a_0} \cdot x^{a_1^p} \cdot ... \cdot x^{a_r^{p^r}} \\ &= x^{a_0 + pa_1 + ... + p^r a_r} \\ &= x^n \\ &= A_n(x).\end{aligned}$$

Thus, for any n and any odd prime p, $A_n(x) = A_{a_0} \cdot (A_{a_1})^p \cdot ... \cdot (A_{a_r})^r$ which means they are equivalent mod p trivially. □

A slightly more complicated example follows.

Example 3.1.4. Consider the sequence of polynomials $\{B_n(x)\}_{n=0}^{\infty}$ where $B_n(x) = (x + k)^n$. Once again, let $p = 5$ and let $n = 21$. Then

$$21 = 1 + 4(5)^1.$$

In order to show that $\{B_n(x)\}_{n=0}^{\infty}$ has the Schur Factorization property when $p = 5$ and $n = 21$, we need to show that

$$B_{21}(x) = B_1(x) \cdot (B_4(x))^5$$

But

$$\begin{aligned}B_{21}(x) &= (x + k)^{21} \\ &= (x + k) \cdot (x + k)^{20} \\ &= (x + k) \cdot ((x + k)^4)^5 \\ &= B_1(x) \cdot (B_4(x))^5.\end{aligned}$$

And since $B_{21}(x) = B_1(x) \cdot (B_4(x))^5$, they must also be equivalent mod 5. So $\{B_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property when $n = 21$ and $p = 5$.

Now we will show that the sequence $\{B_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property for all $n \in \mathbb{N}$ and for every odd prime $p \in \mathbb{N}$.

Lemma 3.1.5. *The sequence $\{B_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property*

Proof. Let p be some odd prime and let n be some natural number. Then n, written in base p is as follows

$$n = a_0 + a_1 p + ... + a_r p^r$$

Now consider $B_n(x)$. It is clear that

$$\begin{aligned} B_n(x) &= (x+k)^n \\ &= (x+k)^{a_0 \cdot p a_1 \cdot ... \cdot p^r a_r} \\ &= (x+k)^{a_0} \cdot (x+k)^{p a_1} \cdot ... \cdot (x+k)^{p^r a_r} \\ &= (x+k)^{a_0} \cdot ((x+k)^{a_1})^p \cdot ... \cdot ((x+k)^{a_r})^{p^r} \\ &= B_{a_0}(x) \cdot (B_{a_1}(x))^p \cdot ... \cdot (B_{a_r}(x))^{p^r} \end{aligned}$$

But if $B_n(x) = B_{a_0}(x) \cdot (B_{a_1}(x))^p \cdot ... \cdot (B_{a_r}(x))^{p^r}$, then they are also equivalent mod p trivially. Thus, for any prime p and any natural number n, the sequence of polynomials $\{B_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property.

□

In fact, this demonstrates a wider lemma.

Lemma 3.1.6. *If the sequence of polynomials $\{F_n(x)\}_{n=0}^{\infty}$ has the Schur Factorization Property, then so does $\{F_n(ax+b)\}_{n=0}^{\infty}$ where $ax+b$ is any linear polynomial.*

Proof. Let n be some natural number and let p be some odd prime. Then

$$n = a_0 + a_1 p + ... + a_r p^r$$

where $0 \leq a_i < p$

We know that

$$F_n(x) \equiv F_{a_0}(x) \cdot F_{a_1}(x)^p \cdot \ldots \cdot F_{a_r}(x)^{p^r} \pmod{p}.$$

But we know that, for any factorization of the polynomials $F(x)$ such that $F(x) = a(x)b(x)$ it must also be the case that $F(p(x)) = a(p(x))b(p(x))$. Now let $p(x) = ax + b$. Then since $F_n(x) \equiv F_{a_0}(x) \cdot F_{a_1}(x)^p \cdot \ldots \cdot F_{a_r}(x)^{p^r} \pmod{p}$, it must also be the case that

$$F_n(ax+b) \equiv F_{a_0}(ax+b) \cdot F_{a_1}(ax+b)^p \cdot \ldots \cdot F_{a_r}(ax+b)^{p^r} \pmod{p}.$$

But since n can be any natural number and p can be any odd prime, this means that $\{F_n(ax+b)\}_{n=0}^{\infty}$ must exhibit the Schur Factorization Property where $ax+b$ can be any linear polynomial.

□

A natural question that may arise from this introduction of the Schur Factorization Property is which of the classical orthogonal polynomials introduced in the previous chapter of this paper exhibit the property. In the following section, we discuss different specializations of the Jacobi polynomials as they relate to the Schur Factorization Property.

3.2 The Legendre Polynomials, The Jacobi Polynomials and the Schur Factorization Property

In this section, we characterize various specializations of the Jacobi polynomials, showing that some specializations exhibit the Schur Factorization Property while others do not. To begin, consider the Legendre polynomials, the specialization of the Jacobi polynomials where $\alpha = \beta = 0$.

Recall from Definition 2.4.4 that the Legendre polynomials are defined by the following formula:

$$P_n(x) = \left(\frac{1}{2}\right)^n \sum_{v=0}^{n} \binom{n}{v}^2 (x-1)^v (x+1)^{n-v}.$$

Example 3.2.1. Now let $p = 3$ and $n = 8$. Writing 8 in base 3, we have

$$8 = 2 + 2(3)^1.$$

Now, taking the Legendre polynomial when $n = 8$, we have

$$P_8(x) = \left(\frac{1}{2}\right)^8 \sum_{v=0}^{8} \binom{8}{v}^2 (x-1)^v (x+1)^{8-v}.$$

Written out, we have

$$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

However, since we only need to show that

$$P_8(x) \equiv P_2(x) \cdot (P_2(x))^3 \pmod 3,$$

we can simplify $P_8(x)$ even further by taking the polynomial mod 3. This gets us

$$\begin{aligned} P_8(x) &\equiv \frac{1}{128}(2) \pmod 3 \\ &\equiv \frac{2}{128} \pmod 3 \\ &\equiv \frac{1}{64} \pmod 3 \\ &\equiv 1 \pmod 3. \end{aligned}$$

Now consider $P_2(x) \cdot (P_2(x))^3 = P_2(x)^4$. Written out fully, this is equal to

$$\begin{aligned} P_2(x)^4 &= \left(\frac{1}{2}(3x^2 - 1)\right)^4 \\ &= \frac{(3x^2-1)^4}{2^4} \\ &= \frac{81x^8 - 108x^6 + 54x^4 - 12x^2 + 1}{2^4}. \end{aligned}$$

Taking this mod 3, we have

$$\begin{aligned} P_2(x)^4 &= \frac{1}{16} \pmod 3 \\ &= 1 \pmod 3. \end{aligned}$$

Thus, as we can see, the Legendre polynomials exhibit the Schur Factorization Property when $n = 8$ and when $p = 3$.

In this previous example, $P_8(x) \pmod 3$ and $P_2(x)^4 \pmod 3$ were the constant function 1. Now let us consider a slightly more complicated example to demonstrate how the Schur Factorization Property works when the resulting functions are polynomials.

Example 3.2.2. Let $p = 5$ and $n = 8$. Writing 8 in base 5, we have

$$8 = 3 + 1 \cdot (5^1).$$

Thus, we are attempting to show that the Legendre polynomials exhibit the Schur Factorization Property when $p = 5$ and $n = 8$. That is, we are attempting to show that

$$P_8(x) \equiv P_3(x) \cdot P_1(x)^5 \pmod 5.$$

Recall from our previous example that

$$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35).$$

Simplified mod 5, we have

$$\begin{aligned} P_8(x) &\equiv \frac{1}{128}(3x^6) \pmod 5 \\ &\equiv \frac{3}{128}x^6 \pmod 5 \\ &\equiv x^6 \pmod 5 \end{aligned}$$

Now consider $P_3(x) \cdot P_1(x)^5$. Written out, this is

$$\begin{aligned} P_3(x) \cdot P_1(x)^5 &= \frac{1}{2}(5x^3 - 3x) \cdot (x)^5 \\ &= \frac{1}{2}(5x^8 - 3x^6). \end{aligned}$$

Simplified mod 5, this is

$$\begin{aligned} P_3(x) \cdot P_1(x)^5 &\equiv \frac{2x^6}{2} \pmod 5 \\ &\equiv x^6 \pmod 5 \end{aligned}$$

Thus, we have shown that $P_8(x) \equiv P_3(x) \cdot P_1(x)^5 \pmod 5$ meaning the Legendre polynomials exhibit the Schur Factorization Property when $p = 5$ and $n = 8$.

In fact, the Legendre polynomials exhibit the Schur property for every odd prime p and every natural number n.

While a rigorous proof of this is beyond the scope of this paper, we will now show that the Legendre polynomials have the Schur Factorization Property for every odd prime p when $n = p$, providing a reference for the reader who wishes to investigate a proof of the Schur Factorization Property as it applies to the Legendre polynomials as a whole.

Definition 3.2.3. Let $\{J_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ be a sequence of polynomials such that

$$J_n^{(\alpha,\beta)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v} \binom{n+\alpha+\beta+v}{v} x^v$$

Lemma 3.2.4. $J_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x+1)$ *where* $P_n^{(\alpha,\beta)}$ *are the Jacobi polynomials.*

To see that this lemma is true, see [1, Theorem 4.21.2.].

But since $J_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x+1)$, by Lemma 3.1.6. we can say that if $J_n^{(\alpha,\beta)}(x)$ exhibits the Schur property for some α and β, then $P_n^{(\alpha,\beta)}(x)$ exhibits the Schur Factorization Property for the same α and β. Then, since the Legendre polynomials are a specialization of $P_n^{(\alpha,\beta)}(x)$ where $\alpha = \beta = 0$, in order to show that the Legendre polynomials exhibit the Schur Factorization Property it suffices to show that $J_n^{(0,0)}(x)$ exhibits the Schur Factorization Property.

Lemma 3.2.5. $J_n^{(0,0)}(x)$ *exhibits the Schur Factorization Property when* $n = p$.

Proof. Setting α and β equal to 0 we have

$$J_n^{(0,0)}(x) = \sum_{v=0}^{n} \binom{n}{n-v}\binom{n+v}{v} x^v.$$

But since $n = p$, in order to prove that $J_n^{(0,0)}(x)$ exhibits the Schur Factorization Property when $n = p$, we must show that

$$J_p^{0,0}(x) \equiv (J_1^{0,0}(x))^p \pmod p.$$

Since $n = p$ and since $\binom{p}{p-v} = \binom{p}{v}$, we have

$$J_n^{(0,0)}(x) = \sum_{v=0}^{n} \binom{p}{v}\binom{p+v}{v} x^v.$$

But $\binom{p}{v} \equiv 0 \pmod p$ unless $j = 0$ or p, in which case, $\binom{p}{v} \equiv 1 \pmod p$. Also

$$\begin{aligned}\binom{p+v}{v} &= \frac{(p+v)(p+v-1)...(p+1)}{v!}\\ &\equiv \frac{(v)(v-1)...(1)}{v!} \pmod p\\ &\equiv 1 \pmod p\end{aligned}$$

unless $v = p$. When $v = p$, we have $\binom{2p}{p}$.

In that case,

$$\begin{aligned}\binom{2p}{p} &= \frac{2p(2p-1)(2p-2)...(p+1)}{p!}\\ &= \frac{2(2p-1)...(p+1)}{(p-1)!}.\end{aligned}$$

But, by Wilson's Theorem, $(p-1)! \equiv -1 \pmod p$. Making this substitution, we have

$$\begin{aligned}\binom{2p}{p} &\equiv -2(2p-1)....(p+1) \pmod p\\ &\equiv -2(p+1)(p+2)...(p+p-1) \pmod p\\ &\equiv -2((p-1)!) \pmod p\\ &\equiv (-2)(-1) \pmod p\\ &\equiv 2 \pmod p\end{aligned}$$

Then $J_p^{(0,0)}(x) \equiv 1 + 2x^p \pmod{p}$. But since $2^p = 2 \pmod{p}$ and since we are working mod p, we have

$$\begin{aligned} J_p^{(0,0)}(x) &\equiv 1 + 2x^p \pmod{p} \\ &\equiv 1^p + 2^p x^p \pmod{p} \\ &\equiv 1^p + (2x)^p \pmod{p} \\ &\equiv (1+2x)^p \pmod{p}. \end{aligned}$$

And since $J_1^{(0,0)}(x) = 1 + 2x$ we have shown what we wish to show, namely that the Legendre polynomials exhibit the Schur Factorization Property when $n = p$ where p is some odd prime. □

For a proof that the Legendre polynomials exhibit the Schur Factorization Property for all natural numbers n and all odd primes p, see [4].

Now consider another specialization of the Jacobi polynomials in order to draw a comparison. Recall from Definition 2.4.9. that the Chebyshev polynomials of the first kind are defined as follows:

$$T_n(x) = \sum_{v=0}^{n} \binom{n-\frac{1}{2}}{n-v} \binom{n-\frac{1}{2}}{v} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{n-v}.$$

Example 3.2.6. Now, once again, let $p = 3$ and let $n = 8$. Then $8 = 2 + 2(3)^1$. Setting p equal to 3 and n equal to 8 our expression for the Chebyshev polynomials of the first kind is as follows,

$$T_8(x) = \sum_{v=0}^{8} \binom{8-\frac{1}{2}}{8-v} \binom{8-\frac{1}{2}}{v} \left(\frac{x-1}{2}\right)^v \left(\frac{x+1}{2}\right)^{8-v}.$$

Written out, this can be simplified as follows

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1.$$

Once again, we can simplify this for our purposes by taking it mod 3. This gives us

$$T_8(x) \equiv 2x^8 + 2x^6 + x^4 + x^2 + 1 \pmod 3.$$

Now consider $(T_2(x))^4$. Written out, we have

$$\begin{aligned}(T_2(x))^4 &= (2x^2 - 1)^4 \\ &= 16x^8 - 32x^6 + 24x^4 - 8x^2 + 1.\end{aligned}$$

Taking this mod 3, we have

$$(T_2(x))^4 \equiv x^8 + x^6 + x^2 + 1 \pmod 3.$$

Then clearly, $T_8(x) \not\equiv (T_2(x))^4 \pmod 3$ and, thus, the Chebyshev polynomials of the first kind do not exhibit the Schur Factorization Property when $p = 3$ and $n = 8$.

As this example shows, it is not the case that Chebyshev polynomials of the first kind as a whole exhibit the Schur Factorization Property. But it might be the case that, for some specific odd prime p and some specific natural number n, the Chebyshev polynomials of the first kind do exhibit the Schur Factorization Property. Is there some criteria by which we can determine for which (if any) natural numbers n and odd primes p it is that case that $T_n(x)$ exhibits the Schur Factorization Property? In general, for any sequence of orthogonal polynomials that do not exhibit the Schur Factorization Property for all p and n, this is a question we would like to answer.

Thus, we have found that some specializations of the Jacobi polynomials (specifically, the Legendre polynomials) do exhibit the Schur Factorization Property while some (including, as we have seen, the Chebyshev polynomials of the first kind) do not.

Is there some criteria by which we can determine if any of the other classical orthogonal polynomials exhibit the Schur Factorization Property and, for those classical orthogonal polynomials that do not exhibit the Schur Factorization Property, is there some way to determine possible specializations of these sequences of polynomials that do exhibit the property? In general, this is a difficult question to answer. One way to simplify this problem is to work with the generating function of the sequence of orthogonal polynomials you wish to study.

In the next section, we will develop this course of study.

3.3 Allouche/Skordev, Generating Functions, and the Schur Factorization Property

In this chapter, we use the work of J.P. Allouche and G. Skordev to develop a method of determining whether or not a sequence of polynomials exhibits the Schur Factorization Property by working with their generating functions. For a more detailed explanation of the following method and its various uses, see [3].

Lemma 3.3.1 (The Allouche Skordev Criteria)**.** *Let p be an integer such that $p \geq 2$. Consider the sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with generating function in the field of p elements $F(X,t) = \sum_{n=0}^{\infty} P_n(x)t^n$. Then $\{P_n(x)\}_{n=0}^{\infty}$ exhibits the Schur Factorization Property if and only if there exists some polynomial $A(X,t)$ in the field of p elements such that $deg_t(A(X,t)) \leq p-1$ and $F(X,t) = A(X,t)F(X^p,t^p)$.*

In order to give an example of this method at work, let us consider the Legendre polynomials, since we know from previous experience that these polynomials do exhibit the Schur Factorization Property.

Example 3.3.2. Recall from Definition 2.4.5. that the generating function for the Legendre polynomials is as follows

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(x)t^n &= (1-2xt+t^2)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{1-2xt+t^2}}.\end{aligned}$$

That is, for the Legendre polynomials, $F(X,t) = \frac{1}{\sqrt{1-2xt+t^2}}$. Now consider $\frac{F(X,t)}{F(X^p,t^p)}$. Since we are working over the field of p elements, $\mathbb{Z}_p$, it is the case that $F(X^p,t^p) = F(X,t)^p$ and thus

$$\begin{aligned}\frac{F(X,t)}{F(X^p,t^p)} &= \frac{F(X,t)}{F(X,t)^p} \\ &= \frac{1}{F(X,t)^{p-1}} \\ &= (1-2xt+t^2)^{\frac{p-1}{2}}.\end{aligned}$$

Then let $A(X,t) = (1-2xt+t^2)^{\frac{p-1}{2}}$. Since p is an odd prime $p-1$ is even and, thus $\frac{p-1}{2}$ must be an integer. Thus $A(X,t)$ is a polynomial and can thus be analyzed with the Allouche/Skordev Criteria.

Now consider $\deg_t(A(X,t))$. The term in $A(X,t)$ with the highest degree in terms of t is $(t^2)^{\frac{p-1}{2}} = t^{p-1}$. Thus

$$\deg_t(A(X,t)) = p-1 < p.$$

And since $A(X,t) = \frac{F(X,t)}{F(X^p,t^p)}$ it must be the case that $F(X,t) = A(X,t)F(X^p,t^p)$.

Thus, the generating function for the Legendre polynomials meets the criteria set out in Lemma 3.3.1 and thus, by our criteria, the Legendre polynomials exhibit the Schur Factorization Property.

While this criteria gives us a new way to prove that the Legendre polynomials exhibit the Schur Factorization Property for all natural numbers n and all odd prime numbers p, we would like to be able to use this criteria to show that other specializations of the classical orthogonal polynomials exhibit the Schur Factorization Property. With this in, mind, consider the Gegenbauer polynomials.

Example 3.3.3. Recall that the generating function for the Gegenbauer polynomials is

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = \frac{1}{(1-2xt+t^2)^{\lambda}}.$$

That is, for the Gegenbauer polynomials, $F(X,t) = \frac{1}{(1-2xt+t^2)^{\lambda}}$.

Lemma 3.3.4. *Let $a, b \in \mathbb{N}$ such that neither a nor b is 0 with $a < b$, $gcd(a,b) = 1$ and $2 \le \frac{b}{a}$. Let p be a prime number such that $b|p-1$. Then $C_n^{\lambda}(x)$ exhibits the Schur Factorization Property when $\lambda = \frac{a}{b}$.*

Proof. Let $C_n^{\lambda}(x)$ be such a specialization of the Gegenbauer polynomials as was specified above. Recall that, for the Gegenbauer polynomials, $F(X,t) = \frac{1}{(1-2xt+t^2)^{\lambda}}$. Now consider $\frac{F(X,t)}{F(X^p,t^p)}$. As with the Legendre polynomials, since we are working in the field of p elements, it is the case that

$$\begin{aligned}\frac{F(X,t)}{F(X^p,t^p)} &= \frac{F(X,t)}{F(X,t)^p}\\ &= \frac{1}{F(X,t)^{p-1}}\\ &= (1-2xt+t^2)^{\lambda(p-1)}.\end{aligned}$$

Then let $A(X,t) = (1-2xt+t^2)^{\lambda(p-1)}$. Since $b|p-1$, it must be the case that $\frac{1}{b}\cdot(p-1)$ is some integer. But then $\alpha(p-1) = \frac{a}{b}(p-1)$ must be some integer and thus, $A(X,t)$ is a polynomial. And since $A(X,t) = \frac{F(X,t)}{F(X^p,t^p)}$ it must be the case that $F(X,t) = A(X,t)F(X^p,t^p)$. Now consider $\deg_t(A(X,t))$. The term with the highest power with respect to t in $A(X,t)$ is $(t^2)^{\lambda(p-1)}$. Thus, $\deg_t(A(X,t)) = 2\lambda(p-1)$. Now replace λ with $\frac{a}{b}$ such that $2 \le \frac{b}{a}$. This means that $\frac{1}{2} \ge \frac{a}{b}$. But then $2\lambda(p-1) \le 2\frac{1}{2}(p-1)$ meaning that $2\lambda(p-1) \le (p-1)$. Then $\deg_t(A(X,t))$ must be less than or equal to $p-1$.

We have thus show that our proposed $A(X,t)$ satisfies the criteria set in Lemma 3.3.1. and thus, the specializations of the Gegenbauer polynomials specified in Lemma 3.3.4. do, indeed, exhibit the Schur Factorization Property. □

Thus, we have used the Allouche Skordev criteria to demonstrate new specializations of the Jacobi (and, more specifically, the Gegenbauer) polynomials that exhibit the Schur Factorization Property. Can this criteria be used to explore other families of the classical orthogonal polynomials? Consider the General Laguerre polynomials.

Example 3.3.5. Recall that the generating function for the General Laguerre polynomials is as follows

$$\sum_{n}^{\infty} L_n^{\alpha}(x)t^n = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{tx}{1-t}}.$$

But recall from our criteria that, in order for the General Laguerre polynomials to exhibit the Schur Factorization Property, there must exist some *polynomial* $A(x,t)$ where $\deg_t(A(X,t)) \leq p-1$ and $F(X,t) = A(X,t)F(X^p,t^p)$. But if $F(X,t) = A(X,t)F(X^p,t^p)$, then $A(X,t)$ cannot be a polynomial and thus, we cannot use the Allouche Skordev criteria to determine whether or not the General Lagurre polynomials exhibit the Schur Factorization Property.

To determine which specializations of the General Laguerre polynomials exhibit the Schur Factorization Property, alternative methods will have to be used. In this next chapter of this paper, we develop such methods and give a criteria by which the General Laguerre polynomials can be studied for their potential to exhibit the Schur Factorization Property.

4

The General Laguerre Polynomials and the Schur Factorization Property

In this chapter, we rigorously develop a criteria by which it can be determined which polynomials $L_n^{\alpha}(x)$ in the integral monic General Laguerre polynomials exhibit the Schur Factorization Property. These monic General Laguerre polynomials, while closely related to the General Laguerre polynomials introduced earlier, drastically simplify the form of the polynomials. They also allow us to take the coefficients for each term of any particular polynomial and write them mod p as integers, since every coefficient for an integer polynomial must be some integer mod p.

Consider the Laguerre polynomials, recalling from Definition 2.3.14. that these are a specialization of the General Laguerre polynomials where $\alpha = 0$ defined by

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} \frac{x^k}{k!}.$$

Example 4.0.6. Now let $n = 5$ and let $p = 3$. Then $5 = 2 + 1(3^1)$. Then, if $L_5(x)$ exhibits the Schur Factorization Property when $p = 3$, it must be the case that

$$L_5(x) \equiv L_2(x) \cdot L_1(x)^3 \pmod 3.$$

But

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120).$$

Taking this mod 3, we get

$$L_5(x) \equiv \frac{1}{120}(2x^5 + x^4 + x^3) \pmod 3.$$

Now consider $L_2(x) \cdot L_1(x)^3$. But

$$\begin{aligned} L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\ &\equiv 2(x^2 + 2x + 2) \pmod 3 \end{aligned}$$

and

$$\begin{aligned} L_1(x)^3 &= (-x+1)^3 \\ &= -x^3 + 3x^2 - 3x + 1 \\ &\equiv 2x^3 + 1 \pmod 3 \end{aligned}$$

Then

$$\begin{aligned} L_2(x) \cdot L_1(x)^3 &\equiv (4x^3 + 2)(x^2 + 2x + 2) \pmod 3 \\ &\equiv 4x^5 + 8x^4 + 8x^3 + 2x^2 + 4x + 4 \pmod 3 \\ &\equiv x^5 + 2x^4 + 2x^3 + 2x^2 + x + 1 \pmod 3. \end{aligned}$$

But since $L_2(x) \cdot L_1(x)^3$ contains a term for x and $L_5(x)$ does not, clearly the General Laguerre polynomials do not exhibit the Schur Factorization Property when $\alpha = 0$, $n = 5$ and $p = 3$.

Thus we have shown that the General Laguerre polynomials in general do not exhibit the Schur Factorization Property. However, as with the Gegenbauer polynomials, there may be some criteria for α, n, and p by which we can determine whether or not the polynomial $L_n^\alpha(x)$ exhibits the Schur Factorization Property for odd prime p.

In order to make these polynomials monic (with a leading coefficient 1) we multiply this equation by $n!$ since when $k = n$, we have

$$(-1)^k \binom{n+\alpha}{n-n} \frac{x^n}{n!} = \frac{x^n}{n!}.$$

While this makes the General Laguerre polynomials monic, it also makes them integral since the coefficient for every term of the standard General Laguerre Polynomial of degree n has a denominator of $n!$. Then our expression for the integral monic General Laguerre polynomials are as follows:

Definition 4.0.7. The integral monic General Laguerre polynomials are defined by

$$L_n^\alpha(x) \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{n! x^k}{k!}.$$

For the remainder of this chapter, we will use the integral monic General Laguerre polynomials for all theorems and lemmas. We do this mainly to avoid the problem of reducing fractions $\pmod p$ when there is a factor of p in the denominator, thus making the fraction undefined. Thus, with the monic General Laguerre polynomials $\{L_n^\alpha(x)\}_{n=0}^\infty$, we needn't worry about an undefined polynomial for any odd prime p or natural number n.

For the same reason, we will only look at specializations of the monic General Laguerre polynomials when α is an integer. While in the General Laguerre polynomials, α can be any real number, setting α equal to some real number that isn't an integer makes $L_n^\alpha(x)$ not integral.

Lemma 4.0.8. *For the integral monic General Laguerre Polynomial $L_n^\alpha(x)$, it is the case that*

$$L_n^\alpha(x) \equiv r_n x^n + r_{n-1} x^{n-1} + ... + r_{n-p+1} x^{n-p+1} \pmod p$$

where r_i is the coefficient for x^i in $L_n^\alpha(x)$ and where $n > p$.

Proof. Recall from Definition 4.0.7. that, for $\alpha \in \mathbb{Z}$ and for $n \in \mathbb{Z}_+$,

$$L_n^\alpha(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{n!x^k}{k!}.$$

Then

$$\begin{aligned} r_i &= (-1)^i \binom{n+\alpha}{n-i} \frac{n!}{i!} \\ &= (-1)^i \frac{(n+\alpha)!(n)!}{(i)!(n-i)!(\alpha+i)!}. \end{aligned}$$

Consider r_{n-p}. We can rewrite this coeffecient as follows

$$\begin{aligned} r_{n-p} &= (-1)^{n-p} \frac{(n+\alpha)!(n!)}{(n-p)!(p)!(n+\alpha-p)!} \\ &= (-1)^{n-p} \frac{(n+\alpha)!}{(n+\alpha-p)!} \frac{n!}{(n-p)!} \frac{1}{p!} \end{aligned}$$

$$= (-1)^{n-p} \frac{(n+\alpha-p+1)(n+\alpha-p+2)...(n+\alpha-p+p)(n-p+1)(n-p+2)....(n-p+p)}{p!}$$

Since $(n+\alpha-p+1)(n+\alpha-p+2)...(n+\alpha-p+p)$ and $(n-p+1)(n-p+2)....(n-p+p)$ are two sets of p consecutive numbers, each set must contain at least one term divisible by p. Then there must be at least two factors of p in the numerator while there is only one factor of p in $p!$. Thus the entire coefficient for r_{n-p} is divisible by p and thus, is 0 (mod p).

Consider r_{n-p-h} where h is any natural number. Then

$$r_{n-p-h} = (-1)^{n-p-h} \frac{(n+\alpha)!}{(n+\alpha-(p+h))!} \frac{n!}{(n-(p+h))!} \frac{1}{(p+h)!}$$

$$= (-1)^{n-p} \cdot$$

$$\frac{(n+\alpha-(p+h)+1)...(n+\alpha-(p+h)+(p+h))(n-(p+h)+1)....(n-(p+h)+(p+h))}{p!}$$

And since $(n+\alpha-(p+h)+1)...(n+\alpha-(p+h)+(p+h))$ and $(n-(p+h)+1)....(n-(p+h)+(p+h))$ are two sequence of $p+h$ consecutive integers, one term from each sequence must be divisible by p, so there are at least two terms of p in the numerator while there is at most one term of p in the denominator. Thus, the entire coefficient r_{n-p-h} must be divisible by p and thus, is 0 (mod p). Thus, r_i is 0 (mod p) whenever $i \leq n-p$. This means that

$$L_n^\alpha(x) \equiv r_n x^n + r_{n-1}x^{n-1} + ... + r_{n-p+1}x^{n-p+1} \pmod p$$

□

Thus, we have shown that when determining a criteria for n and α to determine when $L_n^\alpha(x)$ exhibits the Schur Factorization Property for odd prime p, it suffices to simplify $L_n^\alpha(x)$, for any α and n, to the first p terms of highest power.

Thus,

$$L_n^\alpha(x) \equiv r_n x^n + r_{n-1}x^{n-1} + ... + r_{n-p+1}x^{n-p+1}. \pmod p$$

However, with our formula for the integral monic General Laguerre polynomials, we can give a more explicit formula for the coefficients of these first $n-p-1$ terms. Recall from Definition 4.0.7. that

$$L_n^\alpha(x) = \sum_{k=0}^{n}(-1)^k \binom{n+\alpha}{n-k}\frac{n!x^k}{k!}$$

where $L_n^\alpha(x)$ is the nth integral monic General Laguerre polynomial for $\alpha \in \mathbb{N}$.

Since we have just shown that only the first p terms of the sum are relevant when taking $L_n^\alpha(x)$ mod p, we can say that it is the case that

$$L_n^\alpha(x) \equiv (-1)^n\binom{n+\alpha}{0}x^n + ... + (-1)^{n-p+1}\binom{n+\alpha}{p-1}(n)...(n-p+1)x^{n-p+1} \pmod p$$

$$= (-1)^n x^n + (-1)^{n-1}(n+\alpha)(n)x^{n-1} + ... + (-1)^{n-p+1}\frac{(n+\alpha)...(n+\alpha-p+1)(n)...(n-p+1)}{(p-1)!}x^{n-p+1}$$

Theorem 4.0.9. *Consider the integral monic General Laguerre polynomial $L_n^\alpha(x)$. For any odd prime p and any $\alpha \in \mathbb{N}$, the polynomial $L_n^\alpha(x)$ is Schur if and only if*

$$n \equiv a_0 + a_1 p + ... + a_r p^r$$

where $0 \leq a_i < p$ *and where, for* $j > 0$ *either* $a_j = 0$ *or* $a_j = p - m$ *where* $m \equiv \alpha \pmod p$.

Proof. Since our theorem states that L_n^α exhibits the Schur Factorization Property if and only if our criteria holds, it is necessary to show that our criteria produces polynomials that exhibit the Schur Factorization Property and that any integral monic General Laguerre polynomial that exhibits the Schur Factorization Property satisfies our criteria. First, we will show that our criteria produces integral monic General Laguerre polynomials that exhibit the Schur Factorization Property.

Let p be an odd prime. Now let $\alpha \in \mathbb{N}$. Then $\alpha \equiv m \pmod p$ where m is some natural number. Now let n be a natural number such that

$$n = a_0 + a_1 p + ... + a_r p^r$$

where $0 \leq a_k < p$ and where $a_i = p - m$ or $a_i = 0$ when $i > 0$. Now consider $L_n^\alpha(x)$. By Lemma 4.0.7., we know that

$$L_n^\alpha(x) \equiv (-1)^n \tbinom{n+\alpha}{0} x^n + (-1)^{n-1} \tbinom{n+\alpha}{1} n x^{n-1} + ... \\ + (-1)^{n-p+1} \binom{n+\alpha}{p-1} (n)...(n-p+1) x^{n-p+1} \pmod p$$

$$\equiv (-1)^n x^n + (-1)^{n-1} (n+\alpha)(n) x^{n-1} + ... \\ + (-1)^{n-p+1} \frac{(n+\alpha)...(n+\alpha-p+1)(n)...(n-p+1)}{(p-1)!} x^{n-p+1} \pmod p$$

But since $\alpha \equiv m \pmod p$ we can rewrite this further

$$L_n^\alpha(x) \equiv (-1)^n x^n + (-1)^{n-1} (n+m)(n) x^{n-1} + ... \\ + (-1)^{n-p+1} \frac{(n+m)...(n+m-p+1)(n)...(n-p+1)}{(p-1)!} x^{n-p+1} \pmod p$$

Now consider $L_{a_0}^{\alpha}(x) \cdot (L_{a_1}^{\alpha}(x))^p \cdot ... \cdot (L_{a_r}^{\alpha}(x))^r$ However, we know that either $a_i = 0$ or $a_i = p - m$ when $i > 0$. Thus,

$$L_{a_i}^{\alpha}(x) = L_0^{\alpha}(x)$$

or

$$L_{a_i}^{\alpha}(x) = L_{p-m}^{\alpha}(x)$$

for every $L_{a_i}^{\alpha}(x)$ except $L_{a_0}^{\alpha}$.

But $L_0^{\alpha}(x) = \binom{\alpha}{0}1 = 1$.

Now consider $L_{p-m}^{\alpha}(x)$. Since $\alpha \equiv m \pmod{p}$, this is equivalent to

$$L_{p-m}^{\alpha}(x) \equiv (-1)^{p-m}x^{p-m} + (-1)^{p-m-1}(p-m+m)(n)x^{p-m-1} + ...$$
$$+(-1)^{p-m-p+1}\frac{(p-m+m)...(p-m+m-p+1)(p-m)...(p-m-p+1)}{(p-1)!}x^{p-m-p+1} \pmod{p}$$

$$\equiv (-1)^{p-m}x^{p-m} + (-1)^{p-m-1}(p)(p-m)x^{p-m-1} + ...$$
$$+ (-1)^{p-m-p+1}\frac{(p)...(1)(p-m)...(1+m)}{(p-1)!}x^{p-m-p+1} \pmod{p}.$$

But then every term except for $(-1)^{p-m}x^{p-m}$ contains a factor of p, and is, thus $0 \pmod{p}$. This leaves us with

$$L_{p-m}^{\alpha} = (-1)^{p-m}x^{p-m}.$$

Thus, every term from $L_{a_1}^{\alpha}(x)$ up to $L_{a_r}^{\alpha}(x)$ is either 1 or $(-1)^{p-m}x^{p-m}$.

Then

$$(L_{a_1}^{\alpha}(x))^p \cdot ... \cdot (L_{a_r}^{\alpha}(x))^r \equiv ((-1)^{p-m}x^{p-m})^w \equiv ((-1)x)^{(p-m)w} \pmod{p}.$$

where w is some integer, depending on how many of the General Laguerre polynomials in the product are $L_0^{\alpha}(x)$ and how many are $L_{p-m}^{\alpha}(x)$. In fact, since we know that

$$n = a_0 + a_1p + ... + a_rp^r,$$

we know that $L_n^\alpha(x)$ and $L_{a_0}^\alpha(x)\cdot(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r$ must be of the same power, since $L_n^\alpha(x)$ is a polynomial of degree n, while $L_{a_0}^\alpha(x)\cdot(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r$ is a product of polynomials of degree $a_0,\ a_1p,\ \ldots\ ,\ a_rp^r$, resulting in a single polynomial of degree $a_0+a_1p+\ldots+a_rp^r$. Then $w(p-m)=n-a_0$ since this is the degree of x in $(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r$. Thus, we know that

$$(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r\equiv((-1)^{n-a_0}x^{n-a_0})\equiv((-1)x)^{n-a_0}\pmod p.$$

Consider $L_{a_0}^\alpha(x)$. Setting a_0 equal to n in our previous representation for $L_n^\alpha(x)\pmod p$, we get

$$\begin{aligned}L_{a_0}^\alpha(x)&\equiv(-1)^{a_0}x^{a_0}+(-1)^{a_0-1}(a_0+m)(a_0)x^{a_0-1}+\ldots\\&+(-1)^{n-p+1}\frac{(a_0+m)\ldots(a_0+m-p+1)(a_0)\ldots(a_0-p+1)}{(p-1)!}x^{a_0-p+1}\pmod p.\end{aligned}$$

Distributing the term $((-1)x)^{n-a_0}$ over this expression for $L_{a_0}^\alpha(x)$, we get that

$$\begin{aligned}&L_{a_0}^\alpha(x)\cdot(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r\equiv(-1)^{a_0}x^{a_0}\cdot((-1)x)^{n-a_0}+\ldots\\&+(-1)^{a_0-p+1}\frac{(a_0+m)\ldots(a_0+m-p+1)(a_0)\ldots(a_0-p+1)}{(p-1)!}x^{a_0-p+1}\cdot((-1)x)^{n-a_0}\pmod p\\&\equiv(-1)^nx^n+\ldots+(-1)^{n-p+1}\frac{(a_0+m)\ldots(a_0+m-p+1)(a_0)\ldots(a_0-p+1)}{(p-1)!}x^{n-p+1}\pmod p\end{aligned}$$

Reconsider $L_n^\alpha(x)$. Since $n=a_0+a_1p+\ldots+a_rp^r$, it is clear that $n\equiv a_0\pmod p$. Then that means that

$$L_n^\alpha(x)\equiv(-1)^nx^n+\ldots+(-1)^{n-p+1}\frac{(a_0+m)\ldots(a_0+m-p+1)(a_0)\ldots(a_0-p+1)}{(p-1)!}x^{n-p+1}\pmod p$$

But then

$$L_{a_0}^\alpha(x)\cdot(L_{a_1}^\alpha(x))^p\cdot\ldots\cdot(L_{a_r}^\alpha(x))^r\equiv L_n^\alpha(x)\pmod p$$

which means that $L_n^\alpha(x)$ exhibits the Schur Factorization Property.

Thus, we have shown that our criteria produces integral monic General Laguerre polynomials that exhibit the Schur Factorization Property. Now, we must show that any integral monic General Laguerre Polynomial that exhibits the Schur Factorization Property satisfies our criteria.

Let $L_n^\alpha(x)$ exhibits the Schur Factorization Property. Furthermore, assume $\alpha \in \mathbb{Z}$. This means that, when $n = a_0 + a_1 p + ... + a_r p^r$, it is the case that

$$L_n^\alpha(x) \equiv L_{a_0}^\alpha \cdot (L_{a_1}^\alpha(x))^p \cdot ... \cdot (L_{a_r}^\alpha(x))^{p^r} \pmod p.$$

Also, since α is an integer, we can say that $\alpha \equiv m \pmod p$ where m is some integer. Recall that, since $n \equiv a_0 \pmod p$ and since $\alpha \equiv m \pmod p$, we can write $L_n^\alpha(x) \pmod p$ as

$$(-1)^n x^n + ... + (-1)^{n-p+1} \frac{(a_0+m)...(a_0+m-p+1)(a_0)...(a_0-p+1)}{(p-1)!} x^{n-p+1}.$$

Compare this to $L_{a_0}^\alpha(x) \pmod p$, which can be written as

$$(-1)^{a_0} x^{a_0} + ... + (-1)^{a_0-p+1} \frac{(a_0+m)...(a_0+m-p+1)(a_0)...(a_0-p+1)}{(p-1)!} x^{a_0-p+1},$$

and notice that the coeffecient for x^i in $L_n^\alpha(x)$ is the same as the coefficient for x^{i-n+a_0} for $n \leq i \leq (n-p+1)$. Thus, since $L_n^\alpha(x)$ exhibits the Schur Factorization Property, it must be the case that

$$\begin{aligned}(L_{a_1}^\alpha(x))^p \cdot ... \cdot (L_{a_r}^\alpha(x))^r &\equiv ((-1)x)^{n-a_0} \pmod p \\ &\equiv ((-1)^{n-a_0} x^{n-a_0}) \pmod p.\end{aligned}$$

Recall that

$$\begin{aligned}L_{a_i}^\alpha(x) = (-1)^{a_i} x^{a_i} + (-1)^{a_i-1}(a_i+\alpha)(a_i) x^{a_i-1} + ... \\ + (-1)^{a_i-p+1} \frac{(a_i+\alpha)...(a_i+\alpha-p+1)(a_i)...(a_i-p+1)}{(p-1)!} x^{a_i-p+1}\end{aligned}$$

But, since we know

$$\begin{aligned}(L^{\alpha}_{a_1}(x))^p \cdot ... \cdot (L^{\alpha}_{a_r}(x))^r &\equiv ((-1)x)^{n-a_0} \pmod p \\ &\equiv ((-1)^{n-a_0}x^{n-a_0}) \pmod p,\end{aligned}$$

it must be the case that

$$L^{\alpha}_{a_i}(x) = (-1)^{a_i}x^{a_i}$$

or

$$L^{\alpha}_{a_i}(x) = 1$$

Case 1: $L^{\alpha}_{a_i}(x) = (-1)^{a_i}x^{a_i}$

But then the term $(-1)^{a_i-1}(a_i+\alpha)(a_i)x^{a_i-1} \equiv 0 \pmod p$. Then either (a_i) or $(a_i+\alpha)$ must be 0 (mod p). And since $a_i < p$ it cannot be the case that $a_i \equiv 0 \pmod p$ and thus, it must be the case that $(a_i + \alpha) \equiv 0 \pmod p$. But this can only be the case when $a_i \equiv p - \alpha \mod p$ since otherwise a_i would be greater than p. And since $m \equiv \alpha \pmod p$ where $m < p$, we can conclude that $a_i = p - m$ in Case 1.

Case 2: $L^{\alpha}_{a_i}(x) = 1$ This is only the case when $a_i = 0$, since $L^{\alpha}_0(x) = 1$.

Thus, it is the case that $a_i = 1$ or $a_i = p - m$.

Then when $L^{\alpha}_n(x)$ exhibits the Schur Factorization Property, one can rewrite $n \pmod p$ in the form $n = a_0 + a_1p + ... + a_rp^r$ where a_i is either 0 or $p - m$ when $i > 0$. But this is exactly the criteria we set out to show was true. Thus, since we have shown that the Schur Factorization Property implies our criteria and that our criteria implies the Schur Factorization Property for the integral monic General Laguerre polynomials, we have proven what we wish to prove. □

To conclude this chapter, we give two examples, one showing our criteria effectively generating an integral monic General Laguerre Polynomial that exhibits the Schur Factorization Property and one showing how an integral monic General Laguerre Polynomial that doesn't satisfy our criteria does not exhibit the Schur Factorization Property.

Example 4.0.10. Let $p = 5$ and let $\alpha = 3$. Then $\alpha \equiv 3 \pmod{p}$. And let $n = 63$. Then

$$n = 3 + 2(5) + 2(5)^2.$$

Then, to show that $L_{63}^3(x)$ exhibits the Schur Factorization Property when $p = 5$, we must show that

$$L_{63}^3(x) \equiv L_3^3(x) \cdot (L_2^3(x))^5 \cdot (L_2^3(x))^{25} \pmod{5}.$$

Recall from Lemma 4.0.8. that we can write

$$L_{63}^3(x) \equiv (-1)^{63}x^{63} + (-1)^{62}(66)(63)x^{62} + (-1)^{61}\tfrac{(66)(65)(63)(62)}{2}x^{61} +$$
$$(-1)^{60}\frac{(66)(65)(64)(63)(62)(61)}{6}x^{60} + (-1)^{59}\frac{(66)(65)(64)(63)(63)(62)(61)(60)}{24}x^{59} \pmod{5}$$

$$\equiv -x^{63} + 3x^{62} \pmod{5}$$

Consider $L_3^3(x) \cdot (L_2^3(x))^5 \cdot (L_2^3(x))^{25} = L_3^3(x) \cdot (L_2^3(x))^{30}$. This is equivalent to

$$\left((-1)^3x^3 + (-1)^2\tfrac{(6)(3)}{1}x^2 + (-1)\tfrac{(6)(5)(3)(2)}{2}x + \tfrac{(6)(5)(4)(3)(2)(1)}{6}\right) \cdot$$
$$\left((-1)^2x^2 + (-1)(5)(2)x + \frac{(5)(4)(2)(1)}{2}\right)^{30}.$$

$$\equiv (-x^3 + 3x^2)(x^2)^{30} \equiv -x^{63} + 3x^{62} \pmod{5}.$$

Thus, as you can see, $L_{63}^3(x)$ exhibits the Schur Factorization Property when $p = 5$, just as our criteria predicted it would.

Now let us consider a similar example that shows how the integral monic General Laguerre polynomials that do not satisfy our criteria fail to exhibit the Schur Factorization Property.

Example 4.0.11. As in the the previous example, let $p = 5$ and let $\alpha = 3$. However, in this example, let $n = 68$. Then

$$n = 3 + 3(5) + 2(5)^2,$$

and since this does not meet our criteria, we should expect $L_{68}^3(x)$ to not exhibit the Schur Factorization Property when $p = 5$. In order for $L_{68}^3(x)$ to exhibit the Schur Factorization Property when $p = 5$, it would have to be the case that

$$L_{68}^3(x) \equiv L_3^3(x) \cdot (L_3^3(x))^5 \cdot (L_2^3(x))^{25} \pmod{5}.$$

We can write $L_{68}^3(x)$ as follows mod 5.

$$L_{68}^3(x) \pmod{5} \equiv (-1)^{68}x^{68} + (-1)^{67}(71)(68)x^{67} + (-1)^{66}\frac{(71)(70)(68)(67)}{2}x^{66}+$$

$$(-1)^{65}\frac{(71)(70)(69)(68)(67)(66)}{6}x^{65}+(-1)^{64}\frac{(71)(70)(69)(68)(68)(67)(66)(65)}{24}x^{64} \pmod{5}$$

$$\equiv x^{68} - 3x^{67} \pmod{5}.$$

But then $L_3^3(x) \cdot (L_3^3(x))^5 \cdot (L_2^3(x))^{25} = (L_3^3(x))6 \cdot (L_2^3(x))^{25}$ can be written mod 5 as follows:

$$\left((-1)^3x^3 + (-1)^2(6)(3)x^2 + (-1)\tfrac{(6)(5)(3)(2)}{1}x + \tfrac{(6)(5)(4)(3)(2)(1)}{2}\right)^6 \cdot$$

$$\left((-1)^2x^2 + (-1)(5)(2)x + (5)(4)(2)(1)\right)^{25}$$

$$\equiv (-x^3+3x^2)^6(x^2)^{25} \equiv (x^{18}-3x^{17}-3x^{13}+4x^{12})x^{50} \equiv x^{68}-3x^{67}-3x^{63}+4x^{62} \pmod{p}.$$

This is clearly not equivalent mod 5 to $L_{68}^3(x)$ and thus, we can say $L_{68}^3(x)$ does not exhibit the Schur Factorization Property when $p = 5$, just as we predicted.

Thus, as our examples suggest and as Theorem 4.0.9. makes explicit, our criteria for the integral monic General Laguerre polynomials gives every natural number n, odd prime p and integer α such that $L_n^\alpha(x)$ exhibits the Schur Factorization Property for prime p.

5 Conclusions

While in the previous chapter, we gave an exhaustive criteria by which we can determine which of the integral monic General Laguerre polynomials exhibit the Schur Factorization Property, we have by no means answered every question that can be asked about the Schur Factorization Property or even the Schur Factorization Property as it applies to the integral monic General Laguerre polynomials.

Definition 5.0.12. Let $L_n^\alpha(x)$ be some integral monic General Laguerre Polynomial such that

$$n = a_0 + a_1 p + ...a_r p^r.$$

Then define the Schur Factorization Difference to be the polynomial $S(x)$ such that

$$S(x) \equiv L_n^\alpha(x) - \left(L_{a_0}^\alpha(x) \cdot (L_{a_1}^\alpha)^p \cdot ... \cdot (L_{a_r}^\alpha)^{p^r}\right) \pmod{p}.$$

Notice that, if $L_n^\alpha(x)$ exhibits the Schur Factorization Property, then $S(x) = 0$. Thus, we can say that, using Theorem 4.0.9., we can determine for which $L_n^\alpha(x)$ it is the case that $S(x) = 0$, since these are only the polynomials $L_n^\alpha(x)$ such that $L_n^\alpha(x)$ exhibits the Schur Factorization Property. However, in researching the integral monic General Laguerre polynomials, we found evidence to suggest that $S(x)$ follows a more general pattern, of which the polynomials that exhibit the Schur Factorization Property are only a specialization.

Another area of study left open is the Schur Factorization Property as it applies to the General Laguerre polynomials where $\alpha \in \mathbb{R}$ such that α is not an integer. While our criteria as described in Theorem 4.0.9. gives every integral monic General Laugerre Polynomial that exhibits the Schur Factorization Property when α is an integer, it is unclear if there is a way to generalize this criteria to include possible General Laguerre polynomials that exhibit the Schur Factorization Property when α is some real number that is not an integer (such as when α is a non-integer rational number).

Outside of the General Laguerre polynomials, there is still much work to be done dealing with the Jacobi polynomials and finding, using the Allouche-Skordev criteria or other methods, specializations of the Jacobi polynomials that exhibit the Schur Factorization Property. Recall from Definition 2.4.2. that the generating function for the Jacobi polynomials is

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)} t^n = 2^{\alpha+\beta}(1-2xt+t^2)^{-\frac{1}{2}}(1-t+(1-2xt+t^2)^{\frac{1}{2}})^{-\alpha}(1+t+(1-2xt+t^2)^{\frac{1}{2}})^{-\beta}.$$

For what α and β can we apply the Allouche-Skordev criteria to determine whether or not $P_n^{(\alpha,\beta)}$ exhibits the Schur Factorization Property? While we have already answered this question for the Legendre polynomials and the Gegenbauer polynomials which we discussed earlier, this question has not been answered in general.

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