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Integer Generalized Splines on the Diamond Graph

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A Senior Project submitted to
The Division of Science, Mathematics, and Computing
of
Bard College

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Annandale-on-Hudson, New York
May, 2016

Abstract

In this project we extend previous research on integer splines on graphs, and we use the methods developed on n-cycles to characterize integer splines on the diamond graph. First, we find an explicit module basis consisting of flow-up classes. Then we develop a determinantal criterion for when a given set of splines forms a basis.

Contents

Dedication

For my grandfather, who would not have read this but by whom this project would be most valued.

Acknowledgments

I would like to thank my advisor, Lauren Rose for the direction she gave me throughout this research. She, along with the other professors in the mathematics department, have had a profound impact on my education and therefore my project. My parents and sisters offered more than I could accept and cared more than I could comprehend. I am eternally indebted to Janet Barrow, whose patience inspired a portion of this paper. I also owe many thanks to Talia Eshel, for keeping me from being consumed by the project. So much love to all family and friends who have been so supportive.

1
Introduction

Splines are a topic with many applications, from their origin in the construction of model ships to smooth piecewise polynomial functions to graphs with integer labels. The majority of the research on splines has been on piecewise functions, because of their use in computer graphics and data interpolation. However, the principles of polynomial splines have been translated to integer splines, which in recent years have begun to be studied more extensively.

In this project, we study the module of splines of the ring of integers. Since not all modules have a basis, we aim to prove that the set of all splines on a given edge labeled has a basis. With our focus on the diamond graph, shown in Figure 1.0.1, we determine a generalized method of finding a basis for the set of splines, no matter what the edge labels are. Finding one basis enables us to make conjectures on the nature of bases of modules of splines on the diamond graph. We develop a theorem that easily verifies whether or not a set of splines is a basis, without needing to follow the typical procedure of showing that the span the module and that they are linearly independent.

The previous research we refer to on integer splines has all been on $n-$cycles. We chose to study the diamond graph because it has characteristics that are not present in cycles, thus complicating the topic. Our hope in extending the previous research is to open up the possibility of these principles being able to be applied to more complicated graphs.

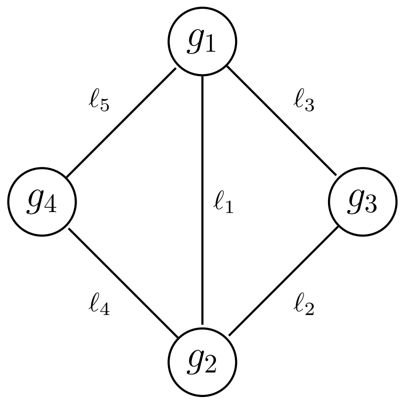

Figure 1.0.1. $G = (g_1, g_2, g_3, g_4)$ is an integer spline on the diamond graph, D.

In Chapter 2, we introduce some basic number theory to prepare the reader for the techniques used in later chapters. A large component of splines is the satisfaction of a set of congruences, thus our preliminary chapter involves a reworking of the Chinese Remainder Theorem that better fits our uses as well as defining the different operations used later on.

In Chapter 3, we summarize the previous work done by Handschy, Melnick, and Reinders [3, Handschy et al.] on the development of flow-up classes on cycles, and Ester Gjoni's [2, Gjoni] work on the determinantal criterion for cycles. We introduce their theorems and rework their notation to better fit the topic. A theorem by Handschy et al. states that for any integer spline on an $n-$cycle, flow-up classes form a basis for the module of splines over the integers. In her senior project, Gjoni provides a determinantal criterion for when a set of splines forms a basis.

In Chapter 4, we present our findings on splines on the diamond graph. In Section 4.1, we reconstruct the arguments used to build the flow-up classes on the $3-$cycle for our purposes on the diamond graph, considering the added restraints of the changed form of the graph. In Section 4.2, we use the same practice of reconstructing an argument on the diamond graph with Gjoni's work, which unlike the flow-up classes, proves to be more difficult to recreate.

In Chapter 5, we present conjectures developed from research done on the topic.

2
Preliminaries

2.1 Background Number Theory

Before introducing the main concepts of the paper, we must first establish the required background knowledge. This section primarily consists of basic number theory, with definitions and theorems. The more complicated theorems are accompanied by proofs to aid the reader.

Definition 2.1.1. [4, Section 1.4, p. 31] If a and b are integers with $a \neq 0$, we say that a *divides* b if there is an integer c such that $b = ac$. If a divides b, we also say that a is a *divisor* or *factor* of b and that b is a *multiple* of a.

Note: If a divides b we write $a|b$, if a does not divide b we write $a \nmid b$.

Definition 2.1.2. [4, Section 3.2, p. 80] The *greatest common divisor* of integers a and b, that are not both zero, is the largest integer which divides both a and b.

Note: The greatest common divisor of a and b is written as (a,b).

Definition 2.1.3. [4, Section 3.4, p. 100] The *least common multiple* of two positive integers a and b is the smallest positive integer that is divisible by a and b.

Note: The least common multiple of a and b is written as $[a,b]$.

Lemma 2.1.4. [4, Section 3.4, p. 100] *Let a and b be integers. Then $[a,b] = \frac{ab}{(a,b)}$.*

The following corollary is a direct result from Lemma 2.1.4.

Corollary 2.1.5. *Let a and b be integers. Then $(a,b)[a,b] = ab$.*

We can generalize this corollary to n integers. However, first we must show that the greatest common divisor and least common multiple can be calculated for more than two integers at once, and introduce some notation.

Definition 2.1.6. [4, Section 3.2, p. 83] Let $a_1,\ a_2, \ldots,\ a_n$ be integers, not all 0. The greatest common divisor of these integers is the largest integer that is a divisor of all of the integers in the set.

Definition 2.1.7. [4, Section 3.4, p. 107] The least common multiple of the integers $a_1,\ a_2, \ldots,\ a_n$, which are not all zero, is the smallest positive integer that is divisible by all the integers in the set.

Definition 2.1.8. Given $a_1,\ a_2, \ldots,\ a_n$,

$$\hat{a}_1 = a_2 a_3 \cdots a_n$$

$$\hat{a}_j = a_1 \cdots a_{j-1} a_{j+1} \cdots a_n$$

$$\hat{a}_n = a_1 a_2 \cdots a_{n-1}$$

for all j where $1 < j < n$.

Theorem 2.1.9. *Let $a_1, a_2, \ldots a_n$ be integers. Then* $[a_1, a_2, \ldots, a_n] = \frac{a_1 a_2 \ldots a_n}{(\hat{a_1}, \hat{a_2}, \ldots, \hat{a_n})}$.

Proof. Let $x = [a_1, a_2, \ldots, a_n]$. Therefore, we know that $a_1 \mid x$, $a_2 \mid x, \ldots,$ $a_n \mid x$. This collection of statements is equivalent to

$$a_1(a_2 a_3 \ldots a_n) \mid x \cdot a_2 a_3 \ldots a_n \tag{1}$$

$$a_2(a_1 a_3 a_4 \ldots a_n) \mid x \cdot a_1 a_3 a_4 \ldots a_n \tag{2}$$

$$\vdots$$

$$a_n(a_1 a_2 \ldots a_{n-1}) \mid x \cdot a_1 a_2 \ldots a_{n-1}. \tag{3}$$

Note that the right side of (1) can be rewritten as $x \cdot \hat{a_1}$, (2) as $x \cdot \hat{a_2}$, and (3) as $x \cdot \hat{a_n}$. This implies that

$$a_1 a_2 a_3 \ldots a_n \mid (x \cdot \hat{a_1}, x \cdot \hat{a_2}, \ldots, x \cdot \hat{a_n})$$

$$a_1 a_2 a_3 \ldots a_n \mid x \cdot (\hat{a_1}, \hat{a_2}, \ldots, \hat{a_n})$$

$$\frac{a_1 a_2 a_3 \ldots a_n}{(\hat{a_1}, \hat{a_2}, \ldots, \hat{a_n})} \mid x$$

Since we see that $\frac{a_1 a_2 a_3 \ldots a_n}{(\hat{a_1}, \hat{a_2}, \ldots, \hat{a_n})} \in \mathbb{Z}$, and it divides $x = [a_1, a_2, \ldots, a_n]$, then $\frac{a_1 a_2 a_3 \ldots a_n}{(\hat{a_1}, \hat{a_2}, \ldots, \hat{a_n})} = [a_1, a_2, \ldots, a_n]$. □

We must introduce several traits of greates common divisors for the sake of later proofs.

Lemma 2.1.10. *If $a_1,\ a_2, \ldots,\ a_n$ are integers, not all 0, the* $(a_1, a_2, \ldots, a_{n-1}, a_n) = (a_1, a_2, \ldots, (a_{n-1}, a_n))$.

Lemma 2.1.11. *If $a_1,\ a_2, \ldots,\ a_n,\ c$ are integers, where none are 0, then* $(ca_1, ca_2, \ldots, ca_n) = c(a_1, a_2, \ldots, a_n)$.

Definition 2.1.12. [4, Section 4.1, p. 128] Let m be a positive integer. If a and b are integers, we say that a is *congruent to b modulo m* if $m|(a-b)$.

Note: If a is congruent to b modulo m, we write $a \equiv b \bmod m$.
The integer m is called the *modulus* of the congruence.

Here we introduce the Chinese Remainder Theorem, which we will ultimately use in another form.

Theorem 2.1.13. [4, Theorem 4.12] *Let $m_1, m_2, \ldots, m_i$ be pairwise relatively prime positive integers. Then the system of congruences*

$$x \equiv a_1 \ mod\ m_1$$
$$x \equiv a_2 \ mod\ m_2$$
$$\vdots$$
$$x \equiv a_i \ mod\ m_i$$

has a unique solution modulo $\boldsymbol{M} = m_1 m_2 \cdots m_i$.

Theorem 2.1.14. [4, Theorem 4.8] *If $a \equiv b \ mod\ m_1$, $a \equiv b \ mod\ m_2, \ldots, a \equiv b \ mod\ m_k$, where $a, b, m_1, m_2, \ldots, m_k$ are integers with $m_1, m_2, \ldots, m_k$ positive, then*

$$a \equiv b \ mod\ [m_1, m_2, \ldots, m_k].$$

The following theorem is a generalization of the Chinese Remainder Theorem, where the moduli aren't coprime. The theorem afterwards is an extension of this new form.

Theorem 2.1.15. [2, Theorem 2.1.22] *The system of congruences*

$$x \equiv a_1 \ mod\ m_1$$
$$x \equiv a_2 \ mod\ m_2$$

has a solution if and only if $(m_1, m_2)|(a_1 - a_2)$. *When there is a solution, it is unique modulo* $[m_1, m_2]$.

Theorem 2.1.16. [2, Theorem 2.1.23] *The system of congruences*

$$\begin{aligned} x &\equiv a_1 \ \mathit{mod}\ m_1 \\ x &\equiv a_2 \ \mathit{mod}\ m_2 \\ &\vdots \\ x &\equiv a_r \ \mathit{mod}\ m_r \end{aligned}$$

has a solution if and only if $(m_i, m_j)|(a_i - a_j)$ *for all pairs of integers* (i, j) *where* $1 \leq i < j \leq r$. *If a solution exists, it is unique modulo* $[m_1, m_2, \ldots, m_r]$.

3
Generalized Integer Splines

In this chapter, we introduce the reader to integer splines on graphs, and show that the set of all integer splines on an edge labeled graph form a $\mathbb{Z}-$module. We define the flow-up classes on a 3-cycle, and that the flow-up classes on a 3-cycle form a basis for the spline module. Finally, we provide a proof of the basis criterion developed by Ester Gjoni for 3-cycles.

3.1 An Introduction to Splines

Before introducing generalized integer splines, we must first look at an edge labeled graph.

Definition 3.1.1. [3, Definition 2.1] Let G be a graph with k edges ordered $e_1, e_2, \ldots, e_k$ and n vertices ordered $v_1, v_2, \ldots, v_n$. Let ℓ_i be a positive integer label on edge e_i and let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be the set of all edge labels. Then (G, L) is an edge labeled graph.

A generalized integer spline, then, is an assignment of integers to vertices of an edge labeled graph satisfying a system of congruences, as seen in the following definition.

Definition 3.1.2. [3, Definition 2.2] A generalized spine on the edge labeled graph (G, L) is a vertex labeling satisfying the following: if $e = (v_i, v_j)$ is an edge with label ℓ, then $x_i \equiv x_j \bmod \ell$. We denote a generalized spline $X = (x_1, x_2, \ldots, x_n)$ where x_i is the label on vertex v_i for $1 \leq i \leq n$. The set of generalized integer splines is denoted $\mathcal{S}(G, L)$.

Note: For simplicity's sake, generalized integer splines will be referred to as splines for the duration of this paper.

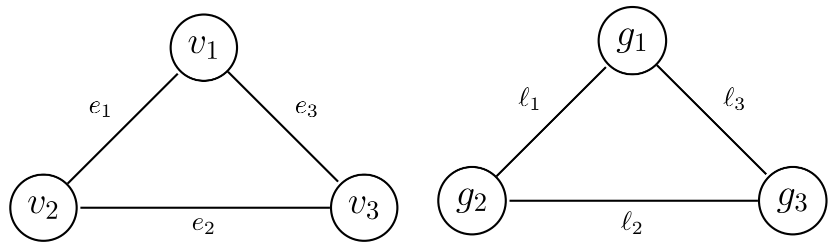

Figure 3.1.1. An edge labeled graph on the left, and on the right is a generalized spline on an edge labeled graph.

Observe that the image on the right in Figure 3.1.1 is a graphical representation of a spline, thus we can write the spline shown as $X = (x_1, x_2, x_3)$. At times in this paper we will use a third form of representation by presenting splines in the transposed form, particularly when referring to flow-up classes, with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

As it is a spline, that means it has the property

$$x_1 \equiv x_2 \bmod \ell_1$$

$$x_2 \equiv x_3 \bmod \ell_2$$

$$x_3 \equiv x_1 \bmod \ell_3.$$

Splines can exist on any edge labeled graph. For example, consider the following graphs.

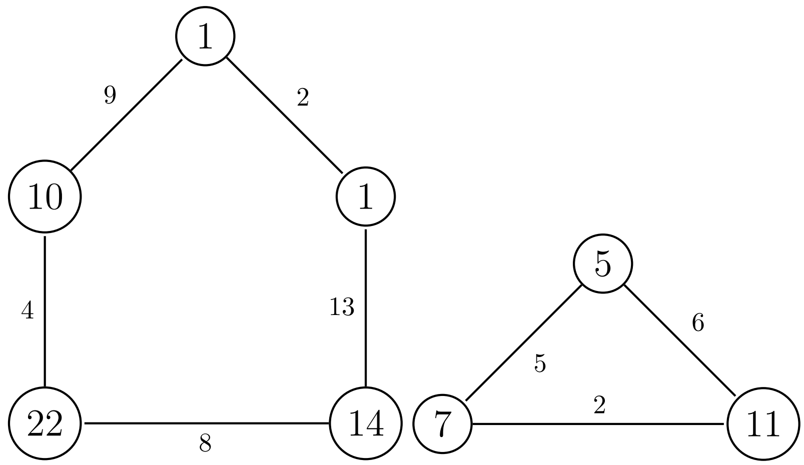

Figure 3.1.2.

The left graph is a spline because the congruences below are satisfied, but the right graph is not because $7 \not\equiv 5 \bmod 5$.

Left Graph Congruences	Right Graph Congruences
$1 \equiv 1 \bmod 2$	$5 \equiv 11 \bmod 6$
$1 \equiv 14 \bmod 13$	$11 \equiv 7 \bmod 2$
$14 \equiv 22 \bmod 8$	$7 \not\equiv 5 \bmod 5$
$22 \equiv 10 \bmod 4$	
$10 \equiv 1 \bmod 9$	

Nearly all of the previous research done on Generalized Integer Splines has been limited to splines on n-cycles. Figure 3.1.2 shows a $5-$cycle on the left and a $3-$cycle on the right. An n-cycle graph is a cycle with n edges, denoted C_n.

Let C_n be an $n-$cycle, and $L = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{Z}^n$ be an ordered set of n edge labels. Then the spline $X = (x_1, x_2, \ldots, x_n)$ is an element of $\mathcal{S}(C_n, L)$ if and only if the following conditions are satisfied

$$x_1 \equiv x_2 \bmod \ell_1$$
$$x_2 \equiv x_3 \bmod \ell_2$$
$$\vdots$$
$$x_{n-1} \equiv x_n \bmod \ell_{n-1}$$
$$x_n \equiv x_1 \bmod \ell_n.$$

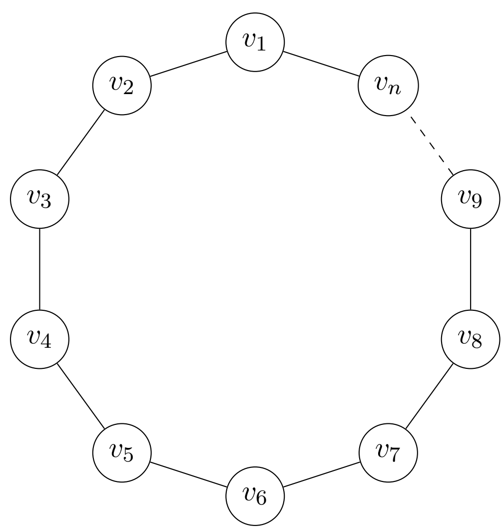

Figure 3.1.3. A representation of C_n, a cycle with n edges and without edge labels.

Every graph contains at least one type of spline, the trivial spline and its multiples.

Definition 3.1.3. Given an edge-labeled graph (G, L), with k vertices, $X = (1, 1, \ldots, 1)$ with length k is called the Trivial Spline. Note that X satisfies the congruences, because $1 \equiv 1 \bmod \ell$, for all $\ell \in \mathbb{Z}$.

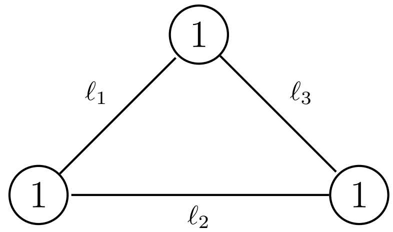

Figure 3.1.4. An edge labeled 3−cycle, or (C_3, L), where the node labels are trivial.

The graph in Figure 3.1.4 easily satisfies the requirements of a spline, since

$$1 \equiv 1 \bmod \ell_1$$

$$1 \equiv 1 \bmod \ell_2$$

$$1 \equiv 1 \bmod \ell_3.$$

The trivial splines turn out to be a key element of the flow-up classes introduced in Section 3.3. However, before expanding on that, we must define more characteristics of the set of all integer splines on an edge labeled graph, $\mathcal{S}(G, L)$. More specifically, we show that the set of all integer splines on a given edge labeled graph forms a $\mathbb{Z}-$module.

3.2 $\mathbb{Z}-$Modules

We now define a $\mathbb{Z}-$module and show that $\mathcal{S}(G, L)$ is a $\mathbb{Z}-$module.

Definition 3.2.1. [1, Section 0.3] If R is a ring, then an $R-$module M is an abelian group with an action of R, that is, a map $R \times M \to M$, written $(r, m) \mapsto rm$, satisfying for all

$r, s \in R$ and $m, n \in M$:

$$r(sm) = (rs)m \qquad (associativity)$$

$$r(m+n) = rm + rn$$

$$(r+s)m = rm + sm \qquad (distributivity,\ or\ bilnearity)$$

$$1m = m \qquad (identity).$$

Note: $\mathbb{Z}$ is an abelian group, and any abelian group is an $R-$module.

A module, therefore, shares many traits with a vector space. What is important to differentiate between the two, is that scalars for a module come from a ring R, while scalars in a vector space are from a field F. One difference between the two is tthat vector spaces always have a basis, while modules may or may not have a basis. Fortunately, it will turn out that modules of integer splines always have bases.

Theorem 3.2.2. *Fix the edge labels on (G, L) where G is any graph with m nodes and $L = (\ell_1, \ell_2, \ldots, \ell_n)$. Then $\mathcal{S}(G, L)$ is a subgroup of $\mathbb{Z}^m$, hence a $\mathbb{Z}-$module.*

Proof. To show that $\mathcal{S}(G, L)$ is a subgroup of $\mathbb{Z}^m$, we must show

1. $I \in \mathcal{S}(G, L),$ where $I = (0, 0, \ldots, 0)$ is the identity of $\mathbb{Z}^m$
2. $\mathcal{S}(G, L)$ closed under addition
3. $\forall X \in \mathcal{S}(G, L),\ \exists -X \in \mathcal{S}(G, L)$.

First, we see that $I = (0, \ldots, 0)$ satisfies the congruences since $0 \equiv 0 \bmod \ell$ for all $\ell \in \mathbb{Z}$, and thus $I \in \mathcal{S}(G, L)$.

Now, let $X, Y \in \mathcal{S}(G, L)$, with $X = (x_1, x_2 \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_m)$. Note that there are m node labels in each spline, since the number of edges on the graph can be less than, equal to, or greater than the number of nodes.

Since $X \in \mathcal{S}(G, L)$, we know that for every edge $e = (v_i, v_j)$ with edge label ℓ, that

$$x_i \equiv x_j \bmod \ell$$

$$y_i \equiv y_j \bmod \ell.$$

Then by the rules of modular arithmetic,

$$x_i + y_i \equiv x_j + y_j \bmod \ell.$$

Thus $X + Y \in \mathcal{S}(G, L)$, so $\mathcal{S}(G, L)$ is closed under addition.

Now let $Z \in \mathcal{S}(G, L)$ where $Z = (z_1, z_2, \ldots, z_m)$. Then for all $e = (v_i, v_j)$ with edge label ℓ_{ij},

$$z_i \equiv z_j \bmod \ell_{ij} \Rightarrow -z_i \equiv -z_j \bmod \ell_{ij}$$

Therefore, $-Z = (-z_1, -z_2, \ldots, -z_m) \in \mathcal{S}(G, L)$.

Satisfying the three requirements, $\mathcal{S}(G, L)$ is a subgroup of $\mathbb{Z}^m$, and a $\mathbb{Z}-$module. □

The following theorem shows that any $\mathcal{S}(G, L)$ has a basis.

Definition 3.2.3. An $R-$module M is called *free* if it has a basis. The rank of M is the number of elements in any basis.

Theorem 3.2.4. [7, Theorem 6.1] *Let F be a free module over a principal ideal domain R and G a submodule of F. Then, G is a free $R-$module and rank $G \leq$ rank F.*

Lemma 3.2.5. [6, Chapter 4, p. 243] *The ring of integers $\mathbb{Z}$ is a principal ideal domain.*

Corollary 3.2.6. *Given any graph G with integer edge labels L, the set of all splines $\mathcal{S}(G, L)$ is a free $\mathbb{Z}-$module.*

Proof. $\mathbb{Z}$ is a principal ideal domain, and $\mathbb{Z}^m$ is a free $\mathbb{Z}-$module using Theorem 3.2.4. Since $\mathcal{S}(G, L)$ is a submodule of $\mathbb{Z}^m$, we see that it is a free $\mathbb{Z}-$module of rank $\leq m$, hence it has a basis. □

3.3 Introduction to Flow-up Classes

Now we will look at flow-up classes. Flow-up classes were developed by Madeline Handschy, Julie Melnick, and Stephanie Reinders in their paper titled *Integer Generalized Splines on Cycles* [3]. While their findings can be generalized for any $n-$cycle, their base case is a $3-$cycle, which we reprove in order to develop and apply their techniques to a other graphs. They prove that there exists a bisis consisting of flow-up classes.

Definition 3.3.1. [3, Section 2.3, p. 5] Fix the edge labels on (G, L) and fix k with $1 \leq k \leq n$, where n is the one less than the total number of vertices of the graph. A flow-up class $\mathcal{F}_k$ is the set of splines in $\mathcal{S}(G, L)$ with k-leading zeroes. More precisely, $\mathcal{F}_k = \{F \in \mathcal{S}(G, L) \mid F \text{ has } k \text{ leading zeroes}\}$.

Part of what makes flow-up classes so useful is that a set of splines where each spline has a different number of leading zeroes is linearly independent. This is because the splines in the set can be viewed as columns of a matrix, ordered by number of leading zeroes, resulting in a lower triangle matrix. Lower triangle matrices have non-zero determinants, therefore making the splines linearly independent.

Another feature of flow-up classes is that they can be categorized by size. This measurement is based on the on the size of the leading term, or first non-zero value of the flow-up spline.

Definition 3.3.2. [3, Definition 2.3] Fix a graph with edge labels (G, L). The smallest element F_k of flow-up class $\mathcal{F}_k$, has form $F_k = (0, \ldots, f_{k+1}, \ldots, f_n)$, and if $H_k = (0, \ldots, h_{k+1}, \ldots, h_n)$ is another flow-up class element, then $h_i \geq f_i$ for all entries. By convention, we consider the trivial spline, $b_0 = (1, 1, \ldots, 1)$, to be the smallest flow-up element in $\mathcal{F}_0$.

To introduce techniques used in this paper, we will first look at the work done by Handschy, Melnick, and Reinders on flow-up classes on triangle graphs. They introduced flow-up classes as a method of finding bases for generalized integer splines on n-cycles.

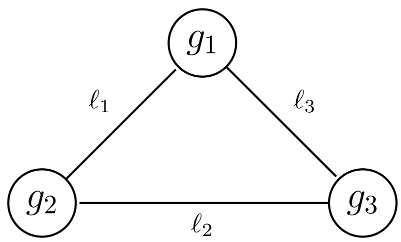

Figure 3.3.1. Generalized Integer Spline on C_3.

Lemma 3.3.3. *Fix a cycle with edge labels (C_3, L). The trivial spline $b_0 = (1, 1, 1)$ is the smallest element in $\mathcal{F}_0$.*

Proof. Let $b_0 = (1, 1, 1)$. Assume there exists $X \in \mathcal{F}_0$, where $X = (x_1, x_2, x_3)$, $x_1, x_2, x_3 \in \mathbb{Z}^+$, and $X \leq b_0$. Then

$$x_1 \leq 1$$

$$x_2 \leq 1$$

$$x_3 \leq 1.$$

However, given that $x_1, x_2, x_3 \in \mathbb{Z}^+$, this implies $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$. Therefore, $X = (1, 1, 1) = b_0$, so b_0 is the smallest element of flow-up class $\mathcal{F}_0$. $\square$

Lemma 3.3.4. [3, Theorem 3.1] *Fix a cycle with edge labels* (C_3, L). *All elements of flow-up class* $\mathcal{F}_1$ *have the form* $m_1 = (0, g_2, g_3)$, *and any* $m_1 = (0, g_2, g_3) \in (\mathbb{Z}^3)^+$ *lies in* $\mathcal{F}_1$ *if and only if* $[\ell_1, (\ell_2, \ell_3)] | g_2$.

Similar to $\mathcal{F}_0$, there exists a smallest possible leading term for the elements of $\mathcal{F}_1$.

Lemma 3.3.5. [3, Theorem 3.2] *Fix a cycle with edge labels* (C_3, L). *Let* $m_1 = (0, g_2, g_3)$, *be a spline on* (C_3, L). *If* $g_2 = [\ell_1, (\ell_2, \ell_3)]$, *then it is the smallest positive leading term.*

Proof. Let $m_1 = (0, g_2, g_3)$, be a spline on (C_3, L), and $g_2 = [\ell_1, (\ell_2, \ell_3)]$. First, notice that m_1 has one leading zero, and is a spline on (C_3, L), then by Lemma 3.3.4, its leading term must be a multiple of $[\ell_1, (\ell_2, \ell_3)]$. However, we already know $g_2 = [\ell_1, (\ell_2, \ell_3)]$, so it is the smallest positive leading term. □

The idea of categorizing by size can be extended further into elements of flow-up classes as a whole. This means each term of the spline is as small as it can possibly be while still being positive and a spline.

Lemma 3.3.6. [3, Theorem 3.3] *Fix a cycle with edge labels* (C_3, L). *The smallest element* b_1, *of flow-up class* $\mathcal{F}_1$, *exists on* (C_3, L).

Proof. We want to construct $b_1 \in \mathcal{F}_1$ on (C_3, L), such that b_1 is the smallest element. So it has the form $b_1 = (0, g_2, g_3)$, and by Lemma 3.3.5, $g_2 = [\ell_1, (\ell_2, \ell_3)]$. Then all possible positive integers that satisfy the congruence restrictions set by the spline for g_3 can be well ordered, and we choose the smallest one to be equal to g_3. Therefore b_1 is the smallest element of $\mathcal{F}_1$. □

Now smallest elements of flow-up classes $\mathcal{F}_0$ and $\mathcal{F}_1$ have been defined, leaving the definition of the smallest element of $\mathcal{F}_2$. Handschy et al. include this in a larger lemma.

Lemma 3.3.7. [3, Proposition 2.6] *Fix a cycle with edge labels (C_n, L). The flow-up class $\mathcal{F}_{n-1}$ consists of splines on (C_n, L) of the form $m_{n-1} = (0, \ldots, 0, g_n)$, and m_{n-1} is a spline if and only if g_n is a multiple of $[\ell_{n-1}, \ell_n]$. If $g_n = [\ell_{n-1}, \ell_n]$, then m_{n-1} is the smallest element of the flow-up class.*

Proof. $\Rightarrow$ Let $m_{n-1} \in \mathcal{S}(C_n, L)$, where $m_{n-1} = (0, \ldots, 0, g_n)$. Then we see that the first $n-1$ elements satisfy the congruences trivially, thus we are concerned with the two congruences

$$g_{n-1} \equiv 0 \bmod \ell_{n-1}$$
$$g_{n-1} \equiv 0 \bmod \ell_n.$$

Using Theorem 2.1.15, we see that the solution g_{n-1} is unique modulo $[\ell_{n-1}, \ell_n]$. Thus $[\ell_{n-1}, \ell_n] \mid g_{n-1}$, which implies $g_{n-1} = a[\ell_{n-1}, \ell_n]$, for some $a \in \mathbb{Z}^+$.

$\Leftarrow$ Let $a \in \mathbb{Z}^+$. Suppose $m_{n-1} = (0, \ldots, 0, a[\ell_{n-1}, \ell_n])$, then looking at the system of congruences

$$0 \equiv 0 \bmod \ell_1$$
$$\vdots$$
$$x \equiv 0 \bmod \ell_{n-1}$$
$$x \equiv 0 \bmod \ell_n,$$

we see that m_{n-1} is a spline on (C_n, L).

Now we would like to see that the smallest positive value for x that satisfies the system is $[\ell_{n-1}, \ell_n]$. By definition, x must be a multiple of ℓ_{n-1} and ℓ_n. It follows that $x = [\ell_{n-1}, \ell_n]$ is the smallest positive solution. $\square$

Using Lemma 3.3.7, we can define b_2 as the smallest element of $\mathcal{F}_2$ on the graph with edge labels (C_3, L) with the form $b_2 = (0, 0, [\ell_2, \ell_3])$.

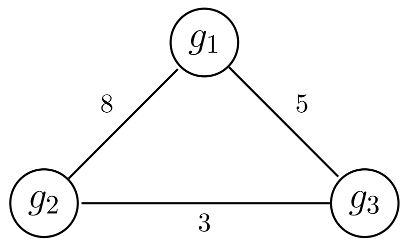

Figure 3.3.2.

Example 3.3.8. Fix the edges on (C_3, L) where $L = (8, 3, 5)$, as shown in Figure 3.3.2. We find the smallest elements of each flow-up class $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{S}(C_3, L)$. By Theorem 3.3.3, we define $b_0 = (1, 1, 1)$. By Theorem 3.3.5, we know it has leading term $[8, (3, 5)] = 8$. Now we must calculate for the third term of b_1. We know it must satisfy

$$x \equiv 8 \text{ mod } 3$$

$$x \equiv 0 \text{ mod } 5$$

By Theorem 2.1.15, we know the solution exists and is unqiue modulo $[3, 5] = 15$. Thus we find the smallest possible solution is $x = 5$. So $b_1 = (0, 8, 5)$, and by Lemma 3.3.7, $b_2 = (0, 0, 15)$.

Having shown the existence of flow-up classes and described the form of the leading term of each one, we are now able to define the smallest elements of the flow-up classes of splines on triangles as

$$b_0 = (1, 1, 1)$$

$$b_1 = (0, [\ell_1, (\ell_2, \ell_3)], g_3)$$

$$b_2 = (0, 0, [\ell_2, \ell_3])$$

we now are equipped to show that these form a basis for the module of splines on a triangle graph over the integers. We include the proof to help the reader understand the final outcome of their paper, as well as a reference for a similar theorem on the diamond graph.

Theorem 3.3.9. [3, Theorem 3.4] *Fix a cycle with edge labels* (C_3, L)*. Let* b_0*,* b_1*, and* b_2 *be the smallest elements of the corresponding flow-up classes on* (C_3, L)*. Then* $\{b_0, b_1, b_2\}$ *is a basis for the module of splines over the integers.*

Proof. Let b_0, b_1, and b_2 be the smalles elements of $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, respectively, on edge labeled graph (C_3, L). Since each of the three have different numbers of leading zeroes, they are linearly independent.

Now we check that every spline on (C_3, L) is in the span of $\{b_0, b_1, b_2\}$. Let $Y = (y_1, y_2, y_3)$ be a spline on (C_3, L), and then define Y' as

$$Y' = Y - y_1 b_0 = \begin{pmatrix} 0 \\ y_2 - y_1 \\ y_3 - y_1 \end{pmatrix}$$

Since Y is a linear combination of splines, Y and b_0, and the set of splines is a module, the vector Y' is a spline as well. We also note that it has one leading zero, and so we have $Y' \in \mathcal{F}_1$. Then by Lemma 3.3.4, the leading term $y_2 - y_1 = s[\ell_1, (\ell_2, \ell_3)]$, for some $s \in \mathbb{Z}$. By Lemma 3.3.5 we know the leading term of b_1 is $[\ell_1, (\ell_2, \ell_3)]$, which leads us to defining Y'' as

$$Y'' = Y' - sb_1 = \begin{pmatrix} 0 \\ 0 \\ y_3 - y_1 - sg_3 \end{pmatrix}$$

Again, this is a spline, as it is the result of a linear combination of splines. So Y'' is a spline and $Y'' \in \mathcal{F}_2$. Then by Lemma 3.3.7, $y_3 - y_1 - sg_3$ must be a multiple of $[\ell_2, \ell_3]$, implying $y_3 - y_1 - sg_3 = t[\ell_2, \ell_3]$ for some $t \in \mathbb{Z}$. By Lemma 3.3.7 the leading term of b_2 is $[\ell_2, \ell_3]$, so it follows that

$$Y'' - tb_2 = \begin{pmatrix} 0 \\ 0 \\ y_3 - y_1 - sg_3 \end{pmatrix} - t\begin{pmatrix} 0 \\ 0 \\ [\ell_2, \ell_3] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore we can rewrite Y as

$$Y = y_1 b_0 + sb_1 + tb_2$$

for $y_1, s, t \in \mathbb{Z}$. So we have shown that Y is an integer linear combination of b_0, b_1, and b_2. Thus $\{b_0, b_1, b_2\}$ forms a basis over the integers for the splines on (C_3, L). □

Example 3.3.10. We use the edge labeled graph (C_3, L) from Example 3.3.8 to show that the smallest flow-up class elements form a basis for the module. Thus $b_0 = (1, 1, 1)$, $b_1 = (0, 8, 5)$, and $b_2 = (0, 0, 15)$, and we want to see that a spline on (C_3, L) can be represented as a linear combination as the three. Let $X = (19, 83, 44)$. Then we see

$$X = 19b_0 + 8b_1 - b_2 = \begin{pmatrix} 19(1) + 8(0) - (0) \\ 19(1) + 8(8) - (0) \\ 19(1) + 8(5) - (15) \end{pmatrix} = \begin{pmatrix} 19 \\ 83 \\ 44 \end{pmatrix}.$$

Example 3.3.11. Here we will show that a linear combination of basis elements results in a spline. Let us again refer to Example 3.3.8. As shown in Example 3.3.10, a basis for the module of splines is $\{b_0, b_1, b_2\}$, where $b_0 = (1, 1, 1)$, $b_1 = (0, 8, 5)$, and $b_2 = (0, 0, 15)$. Then an example of a linear combination of the three is

$$X = 23b_0 - 14b_1 + 39b_2 = 23\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 14\begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} + 39\begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} = \begin{pmatrix} 23 \\ -89 \\ 538 \end{pmatrix}$$

.

Checking if it is a spline on (C_3, L), we see that $23 \equiv -89 \mod 8$, $-89 \equiv 538 \mod 3$, and $538 \equiv 23 \mod 5$. Thus this linear combination of the three elements gives a spline in $\mathcal{S}(C_3, L)$. We offer a depiction of X on (C_3, L) in Figure 3.3.3.

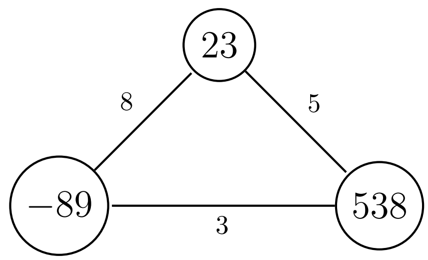

Figure 3.3.3.

3.4 Basis Criterion for Splines on 3-Cycles

In this section we introduce research done by Ester Gjoni in her senior project *Basis Criteria for n-Cycle Splines*[2]. She develops a quick method to verify whether or not a set of splines is a basis for a module of splines. We state most of her results without proofs, as very similar proofs will be given later for splines on the diamond graph.

Theorem 3.4.1. [2, Theorem 4.2.3] *Fix the edge labels on (C_n, L), where $L = (\ell_1, \ell_2, \ldots, \ell_n)$. Let $m_0, m_1, \ldots, m_{n-1}$ be elements of their respective flow-up classes in $\mathcal{S}(C_n, L)$. Then, $|m_0, m_1, \ldots, m_{n-1}| = c \cdot \frac{\ell_1 \ell_2 \ldots \ell_n}{(\ell_1, \ell_2, \ldots, \ell_n)}$, where $c \in \mathbb{N}$.*

Example 3.4.2. Fix the edge labels on (C_3, L) where $L = (4, 5, 7)$. Let $m_0 = (23, 7, 2)$, $m_1 = (0, 32, 7)$, and $m_2 = (0, 0, 70)$. These three are easily verified to be splines in $\mathcal{S}(C_3, L)$, and furthermore, we see $m_0 \in \mathcal{F}_0$, $m_1 \in \mathcal{F}_1$, and $m_2 \in \mathcal{F}_2$. Now we compute the determi-

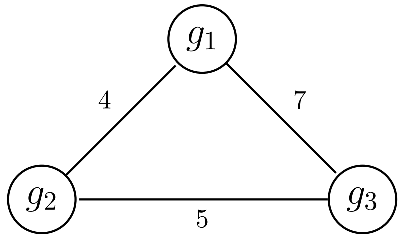

Figure 3.4.1.

nant of their matrix in transposed form

$$\begin{aligned} |m_0, m_1, m_2| &= \begin{vmatrix} 23 & 0 & 0 \\ 7 & 32 & 0 \\ 2 & 7 & 70 \end{vmatrix} && (1) \\ &= 23 \cdot 32 \cdot 70 && (2) \\ &= 368 \cdot 4 \cdot 5 \cdot 7 && (3) \\ &= 368 \cdot \frac{4 \cdot 5 \cdot 7}{(4,5,7)} && (4) \end{aligned}$$

We are able to compute the determinant so easily due to the fact that it is a lower triangle matrix, thus as shown in step (2), it is a matter of multiplying the elements in the diagonal. In step (3) we factor out values equal to the edge labels, leaving $c = 368$. Since the edge labels are coprime, $(4, 5, 7) = 1$, and so in step (4) we are able to rewrite the expression in the form described by Theorem 3.4.1, $c \cdot \frac{\ell_1 \ell_2 \ldots \ell_n}{(\ell_1, \ell_2, \ldots, \ell_n)}$.

This leads to a much simpler corollary, which serves as a hint to the basis criterion.

Corollary 3.4.3. [2, Corollary 4.2.4] *Fix the edge labels on* (C_n, L)*, where* $L = (\ell_1, \ell_2, \ldots, \ell_n)$*. Let* $b_0, b_1, \ldots, b_{n-1}$ *be the smallest elements of the corresponding flow-up classes in* $\mathcal{S}(C_n, L)$*. Then,* $|b_0, b_1, \ldots, b_{n-1}| = \frac{\ell_1 \ell_2 \ldots \ell_n}{(\ell_1, \ell_2, \ldots, \ell_n)}$.

Lemma 3.4.4. [2, Lemma 4.3.1] *Fix the edge labels on* (C_3, L)*, where* $L = (\ell_1, \ell_2, \ell_3)$*. Let* $Q = \frac{\ell_1 \ell_2 \ell_3}{(\ell_1, \ell_2, \ell_3)}$ *and* $X, Y, Z, D \in \mathcal{S}(C_3, L)$*. Suppose* $|X, Y, Z| = \pm Q$*. Then* QD *is in the linear span of* $\{X, Y, Z\}$.

Example 3.4.5. Fix the edge labels on (C_3, L) where $L = (4,5,7)$ and let $Q = \frac{4\cdot5\cdot7}{(4,5,7)} = 140$. Let $X = (4,0,25)$, $Y = (5,1,26)$, $Z = (0,0,35)$, and $D = (41,17,62)$, so we have $X, Y, Z, D \in \mathcal{S}(C_3, L)$. Taking the determinant of $[X, Y, Z]$, we have

$$\begin{aligned}
|X,Y,Z| &= \begin{vmatrix} 4 & 5 & 0 \\ 0 & 1 & 0 \\ 25 & 26 & 35 \end{vmatrix} \\
&= 4\cdot(1\cdot35 - 0\cdot26) - 5\cdot(0\cdot35 - 0\cdot25) + 0\cdot(0\cdot26 - 1\cdot25) \\
&= 4\cdot(35-0) - 0 + 0 \\
&= 4\cdot5\cdot7 \\
&= \frac{4\cdot5\cdot7}{(4,5,7)} \\
&= Q.
\end{aligned}$$

Since $|X,Y,Z| = Q$, then by Lemma 3.4.4, we should be able to show that QD is in the linear span of $\{X,Y,Z\}$. To do so, we must solve the following equality to prove the existence of a_1, a_2, and a_3

$$\begin{bmatrix} 4 & 5 & 0 \\ 0 & 1 & 0 \\ 25 & 26 & 35 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 140 \cdot \begin{bmatrix} 41 \\ 17 \\ 62 \end{bmatrix}$$

Solving the system of equations we find $a_1 = -1,540$, $a_2 = 2,380$ and $a_3 = -420$. Therefore, $QD \in span\{X,Y,Z\}$.

Lemma 3.4.6. [2, Lemma 4.3.3] *Fix the edge labels on (C_3, L), where $L = (\ell_1, \ell_2, \ell_3)$. Let $X, Y, Z \in \mathcal{S}(C_3, L)$. Then $\ell_1 \mid |X,Y,Z|$, $\ell_2 \mid |X,Y,Z|$, and $\ell_3 \mid |X,Y,Z|$.*

Lemma 3.4.7. [2, Lemma 4.3.4] *Fix the edge labels on (C_3, L), where $L = (\ell_1, \ell_2, \ell_3)$. Let $X, Y, Z \in \mathcal{S}(C_3, L)$. Then $\ell_1\ell_2 \mid |X,Y,Z|$, $\ell_2\ell_3 \mid |X,Y,Z|$, and $\ell_3\ell_1 \mid |X,Y,Z|$.*

Example 3.4.8. Fix the edge labels on (C_3, L) where $L = (4,5,7)$. Let $X = (12,8,22)$, $Y = (19,27,47)$, and $Z = (6,18,13)$, which are all splines in $\mathcal{S}(C_3, L)$. We want to check that the product of any two edge labels divides the determinant of the three elements, X, Y, and Z. First, let us compute the determinant

$$\begin{aligned}|X, Y, Z| &= \begin{vmatrix} 8 & 19 & 6 \\ 12 & 27 & 18 \\ 22 & 47 & 13 \end{vmatrix} \\ &= 8 \cdot (27 \cdot 13 - 18 \cdot 47) - 19 \cdot (12 \cdot 13 - 18 \cdot 22) + 6 \cdot (12 \cdot 47 - 27 \cdot 22) \\ &= 420.\end{aligned}$$

Now that we have found $|X, Y, Z| = 420$, we check that the product of any two edge labels divides this value. We see that the three statements $4 \cdot 5 \mid 420$, $4 \cdot 7 \mid 420$, and $5 \cdot 7 \mid 420$ all hold true.

Lemma 3.4.9. [2, Theorem 4.3.5] *Fix the edge labels on* (C_3, L) *where* $L = (\ell_1 1, \ell_2, \ell_3)$. *Let* $Q = \frac{\ell_1\ell_2\ell_3}{(\ell_1,\ell_2,\ell_3)}$. *If* $X, Y, Z \in \mathcal{S}(C_3, L)$, *then* $Q \mid |X, Y, Z|$.

Lemma 3.4.10. [2, Lemma 4.3.7] *Fix the edge labels on* (C_3, L) *where* $L = (\ell_1 1, \ell_2, \ell_3)$. *If* X, Y, *and* Z *form a basis for* $\mathcal{S}(C_3, L)$, *and* J, K, *and* M *are linear combinations of* X, Y, *and* Z, *then* $|X, Y, Z| \mid |J, K, M|$.

Corollary 3.4.11. [2, Lemma 4.3.8] *Fix the edge labels on* (C_3, L) *where* $L = (\ell_1 1, \ell_2, \ell_3)$. *If* $\{X, Y, Z\}$ *is a basis for* $\mathcal{S}(C_3, L)$ *and* $\{J, K, M\}$ *is another basis, then* $|X, Y, Z| = \pm|J, K, M|$.

Here we have one of the major results from Gjoni's work. We include a proof to highlight the importance of the preceding lemmas, corollaries, and theorems.

Theorem 3.4.12. [2, Theorem 4.3.9] *Fix the edge labels on* (C_3, L), *where* $L = (\ell_1, \ell_2, \ell_3)$. *Let* $Q = \frac{\ell_1\ell_2\ell_3}{(\ell_1,\ell_2,\ell_3)}$ *and let* $X, Y, Z \in \mathcal{S}(C_3, L)$. *Then,* $\{X, Y, Z\}$ *form a module basis for* $\mathcal{S}(C_3, L)$ *if and only if* $|X, Y, Z| = \pm Q$.

Proof. $\Rightarrow$ As shown in Theorem 3.3.9, we know that the smallest elements, $\{b_0, b_1, b_2\}$ of each flow-up class for $\mathcal{S}(C_3, L)$. By Corollary 3.4.3 we know that $|b_0, b_1, b_2| = \frac{\ell_1\ell_2\ell_3}{(\ell_1,\ell_2,\ell_3)}$.

Using Lemma 3.4.11, we know that $|b_0, b_1, b_2| = \frac{\ell_1\ell_2\ell_3}{(\ell_1,\ell_2,\ell_3)} = \pm|X, Y, Z|$, where $\{X, Y, Z\}$ is another module basis for $\mathcal{S}(C_3, L)$. Therefore, $|X, Y, Z| = \pm Q$.

$\Leftarrow$ Suppose $|X, Y, Z| = \pm Q$. We want to see that this implies that $\{X, Y, Z\}$ is linearly independent and spans $\mathcal{S}(C_3, L)$. Since the determinant of the three is $Q \neq 0$, we know they are linearly independent. Let $D \in \mathcal{S}(C_3, L)$. From Lemma 3.4.4 we know

$$QD = a_1X + a_2Y + a_3Z$$

for some $a_1, a_2, a_3 \in \mathbb{Z}$. Then by the properties of determinants,

$$\begin{aligned}
\pm a_1Q &= a_1|X, Y, Z| \\
&= |a_1X, Y, Z| \\
&= |(a_1X + a_2Y + a_3Z), Y, Z| \\
&= |QD, Y, Z| \\
&= Q|D, Y, Z|
\end{aligned}$$

This implies $a_1 = \pm|D, Y, Z|$, and by Lemma 3.4.9 we know that $Q \mid |D, Y, Z|$, so for some $s_1 \in \mathbb{Z}$, $s_1Q = |D, Y, Z| \implies a_1 = \pm s_1Q$. Using a similar argument we find $a_2 = \pm s_2Q$ and $a_3 = \pm s_3Q$, for some $s_2, s_3 \in \mathbb{Z}$. Finally we have

$$\begin{aligned}
QD &= a_1X + a_2Y + a_3Z \\
&= \pm(s_1Q)X \pm (s_2Q)Y \pm (s_3Q)Z \\
&= Q(\pm s_1X \pm s_2Y \pm s_3Z) \\
D &= \pm s_1X \pm s_2Y \pm s_3Z.
\end{aligned}$$

Therefore, D is a linear combination of X, Y, and Z, meaning that $\{X, Y, Z\}$ spans $\mathcal{S}(C_3, L)$. $\square$

The most significant result of Theorem 3.4.12 is that should you have any three splines X, Y, and Z in $\mathcal{S}(C_3, L)$ with edge labels $L = (\ell_1, \ell_2, \ell_3)$, if their determinant is equal to $\pm Q = \pm\frac{ell_1\ell_2\ell_3}{(\ell_1,\ell_2,\ell_3)}$, then they form a module basis for $\mathcal{S}(C_3, L)$.

Example 3.4.13. Fix the edge labels on (C_3, L) where $L = (4, 5, 7)$. Let $X = (4, 0, 25)$, $Y = (5, 1, 26)$, and $Z = (0, 0, 35)$. As shown in Example 3.4.5, $|X, Y, Z| = Q$. Then by Theorem 3.4.12, $\{X, Y, Z\}$ is a module basis, so any spline $D \in \mathcal{S}(C_3, L)$ can be written as a linear combination of X, Y, and Z. Let $D = (41, 17, 62)$. Then we want to find $a_1, a_2, a_3 \in \mathbb{Z}$ such that satisfy the following equation

$$\begin{bmatrix} 4 & 5 & 0 \\ 0 & 1 & 0 \\ 25 & 26 & 35 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 41 \\ 17 \\ 62 \end{bmatrix}.$$

We solve this system and get $a_1 = -11$, $a_2 = 17$, and $a_3 = -3$. Therefore we have

$$D = -11X + 17Y - 3Z.$$

Thus $D \in span\{X, Y, Z\}$.

4
Splines on the Diamond Graph

In this chapter we prove two important theorems on the diamond graph. The first proves that the flow-up classes form a basis for the module of splines, and the second theorem is a basis criterion.

4.1 The Flow-up Classes on the Diamond Graph

Let the spline on the diamond graph be defined as shown below. Observe that it consists of two 3-cycles, sharing two nodes and one edge.

We must now see that flow-up classes exist on the diamond graph. We cannot assume it to be true, because unlike the cycle graphs, g_1 and g_2 are connected to three other nodes. We exclue a proof for the flow-up class $\mathcal{F}_0$, as the trivial spline serves as a satisfactory example.

Lemma 4.1.1. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. There exists a flow-up class $\mathcal{F}_1$ in $\mathcal{S}(D, L)$.*

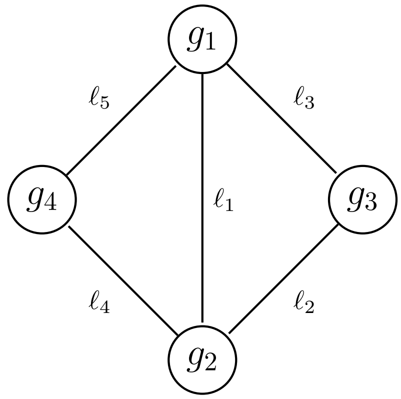

Figure 4.1.1. A generalized diamond spline on (D, L).

Proof. For the flow-up class $\mathcal{F}_1$ to exist, we want to see that there exists a F_1 in the set $\mathcal{F}_1$. Let $F_1 = (0, g_2, g_3, g_4)$. Then we want the congruences determined by the graph to hold true for g_2, g_3, g_4, that is

$$0 \equiv g_2 \bmod \ell_1 \qquad (1)$$

$$g_2 \equiv g_3 \bmod \ell_2 \qquad (2)$$

$$g_2 \equiv g_4 \bmod \ell_4 \qquad (3)$$

$$g_3 \equiv 0 \bmod \ell_3 \qquad (4)$$

$$g_4 \equiv 0 \bmod \ell_5 \qquad (5)$$

For the first congruence, we have to use two applications of Theorem 2.1.15. Using equations (2) and (4), we see that there exists g_3 satisfying the conditions if and only if $g_2 \equiv 0 \bmod (\ell_2, \ell_3)$. However, before finding a value for g_2, we must also take into account its relation with g_4. Using equations (3) and (5), we see that g_4 exists if and only if $g_2 \equiv 0 \bmod (\ell_4, \ell_5)$. This means g_2 must satisfy $\ell_1 \mid g_2$, $(\ell_2, \ell_3) \mid g_2$, and $(\ell_4, \ell_5) \mid g_2$. Then by the definition of least common multiples, we see that if $g_2 = a[\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$ for any $a \in \mathbb{N}$, g_2 satisfies the congruences, and by Theorem 2.1.15, there exist values for g_3 and g_4 satisfying the system. $\square$

Now we would like to see that the other two flow-up classes, $\mathcal{F}_2$ and $\mathcal{F}_3$, exist on the diamond graph as well.

Lemma 4.1.2. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. The flow-up classes $\mathcal{F}_2$ and $\mathcal{F}_3$ both exist in $\mathcal{S}(D, L)$.*

Proof. First, let us look at $\mathcal{F}_2$. We want to find $F_2 \in \mathcal{F}_2$, where $F_2 = (0, 0, g_3, g_4)$. For F_2 to exist in $\mathcal{F}_\in$, we are essentially faced with two pairs of congruences. Those being

$$g_3 \equiv 0 \bmod \ell_2 \qquad (1.1)$$

$$g_3 \equiv 0 \bmod \ell_3 \qquad (1.2)$$

$$g_4 \equiv 0 \bmod \ell_4 \qquad (2.1)$$

$$g_4 \equiv 0 \bmod \ell_5 \qquad (2.2)$$

Using Theorem 2.1.15, we see that $g_3 = a_1[\ell_2, \ell_3]$, for any $a_1 \in \mathbb{N}$. Similarly, for g_4 to exist, we must have $g_4 = a_2[\ell_4, \ell_5]$, for some $a_2 \in \mathbb{N}$. Therefore, g_3 and g_4 exist, and so $F_2 \in \mathcal{F}_2$. Now let us look at $\mathcal{F}_3$. We want to find $F_3 \in \mathcal{F}_3$, where $F_3 = (0, 0, 0, g_4)$. We only need to find a value for g_4. As just shown above, solving the equations for g_4, we find that $g_4 = a_3[\ell_4, \ell_5]$ for any $a_3 \in \mathbb{N}$. Therefore $\mathcal{F}_3$ exists on the diamond graph as well. $\square$

Example 4.1.3. Let D be a diamond graph, with $L = (4, 3, 7, 4, 5)$. (D, L) is shown in Figure 4.1.2. We find an element of each flow-up class in $\mathcal{S}(D, L)$. Let $m_0 = (6, 2, 20, 6)$.

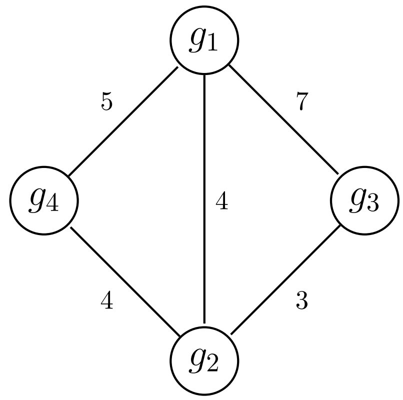

Figure 4.1.2.

Since it has no leading zeroes, and it satisfies the congruences

$$6 \equiv 2 \bmod 4$$
$$2 \equiv 20 \bmod 3$$
$$20 \equiv 6 \bmod 7$$
$$2 \equiv 6 \bmod 4$$
$$6 \equiv 6 \bmod 5,$$

therefore $m_0 \in \mathcal{F}_0$. By Lemma 4.1.1, the leading term of m_1 is a multiple of $[4,(3,7),(4,5)] = 4$. With values found for the first two vertices, we calculate for values that fit for the other two, and see that letting $m_1 = (0, 12, 21, 20)$, then $m_1 \in \mathcal{F}_1 \subset \mathcal{S}(D, L)$. In a similar manner, using Lemma 4.1.2, the leading term of m_2 must be a multiple of $[3, 7] = 21$, and we find the fourth element as satisfies the congruences. Lettin $m_2 = (0, 0, 42, 60)$, results in $m_2 \in \mathcal{F}_2$. Lastly, define the leading term of m_3 as a multiple of $[4, 5] = 20$. Setting $m_3 = (0, 0, 0, 40)$, it is clearly in $\mathcal{F}_3$. So we have examples of elements of the four flow-up classes on an edge-labeled diamond graph.

Lemma 4.1.4. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $m_1 = (0, g_2, g_3, g_4)$ be in the flow-up class $\mathcal{F}_1$. The leading element g_2 is a multiple of $[\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$, and $g_2 = [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$ is the smallest possible positive value such that m_1 is a spline.*

Proof. Let $m_1 = (0, g_2, g_3, g_4)$ be an element of the flow-up class $\mathcal{F}_1$ on (D, L). By Lemma 4.1.1, we know that if $m_1 \in \mathcal{F}_1$, then its leading term g_2 must be a multiple of $[\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$. So setting $g_2 = [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$, we see that g_2 is the smallest possible value that still satisfies the Chinese Remainder Theorem as used in Lemma 4.1.1. Thus there exist g_3 and g_4 that conform to the conditions. □

Similarly we can show that there exist smallest leading terms in flow-up classes $\mathcal{F}_2$ and $\mathcal{F}_3$ on the diamond graph.

Lemma 4.1.5. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $m_2 = (0, 0, g_3, g_4)$ be in the flow-up class $\mathcal{F}_2$ on (D, L). The leading element g_3 is a multiple of $[\ell_2, \ell_3]$, and $g_3 = [\ell_2, \ell_3]$ is the smallest possible positive value such that m_2 is a spline.*

Proof. Let $m_2 = (0, 0, g_3, g_4)$ be an element of the flow-up class $\mathcal{F}_2$ on (D, L). Then by Lemma 4.1.2, we know that if $m_2 \in \mathcal{F}_2$, then the leading term g_3 must be a multiple of $[\ell_2, \ell_3]$, and so the smallest possible leading term for m_2 is $g_3 = [\ell_2, \ell_3]$. □

Lemma 4.1.6. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $m_3 = (0, 0, 0, g_4)$ be in the flow-up class $\mathcal{F}_3$ on (D, L). The leading element g_4 is a multiple of $[\ell_4, \ell_5]$, and $g_2 = [\ell_4, \ell_5]$ is the smallest possible positive value such that m_3 is a spline.*

Proof. Let $m_3 = (0, 0, 0, g_4)$ be an element of the flow-up class $\mathcal{F}_3$ on (D, L). Then by Lemma 4.1.2, we know that if $m_3 \in \mathcal{F}_3$, then the leading term g_4 must be a multiple of

$[\ell_4, \ell_5]$, and so the smallest possible leading term for m_3 is $g_4 = [\ell_4, \ell_5]$. $\square$

Now we have $m_0 \in \mathcal{F}_0, \ldots, m_3 \in \mathcal{F}_3$, with the smallest possible leading terms. We are only looking at splines with positive integer labels, so through well-ordering there exists a smallest value for each non-leading term of the $m_0, \ldots, m_3$.

Establishing an order to the vertices of the diamond graph allows the creation of the flow-up classes. Now that the flow-up classes are defined with smallest elements, we are equipped to show that the set of smallest flow up classes form a basis for the module of splines on the edge labeled diamond graph (D, L).

Here we present our main theorem on the flow-up classes. We prove that the smallest elements of the flow-up classes form a basis for $\mathcal{S}(D, L)$.

Theorem 4.1.7. *Fix the edges on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let b_0, b_1, b_2, and b_3 be the smallest elements of the corresponding flow-up classes in $\mathcal{S}(D, L)$. These four splines $\{b_0, b_1, b_2, b_3\}$ are a basis for the module of splines over the integers on the graph.*

Proof. Let $b_0, \ldots, b_3$ be the smallest elements of their flow-up classes. Since each has a different number of leading zeroes, they are linearly independent.

Now we want to see that these splines span $\mathcal{S}(D, L)$. Let $Y \in \mathcal{S}(D, L)$ with $Y = (y_1, y_2, y_3, y_4)$. Then we define Y' as

$$Y' = Y - y_1 b_0 = \begin{pmatrix} 0 \\ y_2 - y_1 \\ y_3 - y_1 \\ y_4 - y_1 \end{pmatrix}$$

Notice that y_1 is an integer, this is a linear combination of splines, Y and b_0. Since $\mathcal{S}(D, L)$ is a module, we therefore know that $Y' \in \mathcal{S}(D, L)$. With a leading zero, Y' is an element of the flow-up class $\mathcal{F}_1$, and by Lemma 4.1.4, its leading term, $y_2-y_1 = a_1[\ell_1,(\ell_2,\ell_3),(\ell_4,\ell_5)]$, for some $a_1 \in \mathbb{Z}$. Recall that $b_1 = (0, g_2, g_3, g_4)$ with $g_2 = [\ell_1,(\ell_2,\ell_3),(\ell_4,\ell_5)]$, so $y_2 - y_1 = a_1 g_2$. Then define Y'' as

$$Y'' = Y' - a_1 b_1 = \begin{pmatrix} 0 \\ y_2 - y_1 \\ y_3 - y_1 \\ y_4 - y_1 \end{pmatrix} - a_1 \begin{pmatrix} 0 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_3 - y_1 - a_1 g_3 \\ y_4 - y_1 - a_1 g_4 \end{pmatrix}$$

Therefore $Y'' \in \mathcal{S}(D, L)$, and $Y'' \in \mathcal{F}_2$. By Lemma 4.1.5, the leading term of Y'', $y_3 - y_1 - a_1 g_3 = a_2 h_3$, for some $a_2 \in \mathbb{Z}$, where $h_3 = [\ell_2, \ell_3]$ is the leading term of $b_2 = (0, 0, h_3, h_4)$. Now we define Y''' to be

$$Y''' = Y'' - a_2 b_2 = \begin{pmatrix} 0 \\ 0 \\ y_3 - y_1 - a_1 g_3 \\ y_4 - y_1 - a_1 g_4 \end{pmatrix} - a_2 \begin{pmatrix} 0 \\ 0 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_4 - y_1 - a_1 g_4 - a_2 h_4 \end{pmatrix}$$

This linear combination of splines results in $Y''' \in \mathcal{S}(D, L)$, and $Y''' \in \mathcal{F}_3$. Thus its leading term must be a multiple of the leading term of $b_3 = (0, 0, 0, j_4)$, where $j_4 = [\ell_4, \ell_5]$ is the smallest possible leading term. By Lemma 4.1.6, choose $a_3 \in \mathbb{Z}$ such that

$$Y''' - a_3 b_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_4 - y_1 - a_1 g_4 - a_2 h_4 \end{pmatrix} - a_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ j_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we see that

$$Y = y_1 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

for $y_1, a_1, a_2, a_3 \in \mathbb{Z}$. This means that Y is a linear combination of the four splines b_0, b_1, b_2, b_3, and so $\{b_0, b_1, b_2, b_3\}$ forms a basis over the integers for the set of splines on (D, L). $\square$

Now let us look at an example of how the smallest flow-up class elements form a basis by choosing a spline on an edge labeled graph and rewriting it as the linear combination of basis elements.

Example 4.1.8. Let us refer back to Example 4.1.3, where $L = (4, 3, 7, 4, 5)$. Calculating the smallest element of each flow-up class, we have

$$b_0 = (1, 1, 1, 1)$$
$$b_1 = (0, 4, 7, 20)$$
$$b_2 = (0, 0, 21, 20)$$
$$b_3 = (0, 0, 0, 20).$$

Let $m = (6, 2, 20, 26)$, which can easily be shown to be in $\mathcal{S}(D, L)$. Then we see

$$m - 6b_0 = (0, -4, 14, 20)$$
$$m - 6b_0 + b_1 = (0, 0, 21, 40)$$
$$m - 6b_0 + b_1 - b_2 = (0, 0, 0, 20)$$
$$m - 6b_0 + b_1 - b_2 - b_3 = (0, 0, 0, 0).$$

Thus we can rewrite m as

$$m = 6b_0 - b_1 + b_2 + b_3.$$

And so $m \in span\{b_0, b_1, b_2, b_3\}$.

4.2 Basis Criterion for Splines on a Diamond Graph

In this section we use the techniques developed by Gjoni for $3-$cycles and apply them to the diamond graph.

Theorem 4.2.1. *Fix the edge labels on (D, L), where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let m_0, m_1, m_2, and m_3 be elements from each corresponding flow-up class in $\mathcal{S}(D, L)$. Then $det(m_0, m_1, m_2, m_3) = |m_0, m_1, m_2, m_3, m_4| = c \cdot \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3))}$, where $c \in \mathbb{N}$.*

Proof. Let $m_0 \in \mathcal{F}_0, \ldots, m_3 \in \mathcal{F}_3$, and all be splines in $\mathcal{S}(D, L)$. Looking at the structures of these four splines, we have

$$m_0 = (g_1, g_2, g_3, g_4)$$
$$m_1 = (0, h_2, h_3, h_4)$$
$$m_2 = (0, 0, j_3, j_4)$$
$$m_3 = (0, 0, 0, k_4)$$

Observing their leading elements, we see $g_1 = c_1 \cdot 1$, for some $c_1 \in \mathbb{N}$. By Theorem 4.1.4, $h_2 = c_2 \cdot [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)]$, for some $c_2 \in \mathbb{N}$. By Theorem 4.1.5, $j_3 = c_3 \cdot [\ell_2, \ell_3]$, and by Theorem 4.1.6, $k_4 = c_4 \cdot [\ell_4, \ell_5]$, for $c_3, c_4 \in \mathbb{N}$.

Transposing the four splines and viewing them as columns of a matrix, we have

$$M = [m_0, m_1, m_2, m_3] = \begin{bmatrix} c_1 \cdot 1 & 0 & 0 & 0 \\ g_2 & c_2 \cdot [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)] & 0 & 0 \\ g_3 & h_3 & c_3 \cdot [\ell_2, \ell_3] & 0 \\ g_4 & h_4 & j_4 & c_4 \cdot [\ell_4, \ell_5] \end{bmatrix}$$

Note that M is a lower triangle matrix, so taking the determinant we have

$$|M| = c_1 \cdot 1 \cdot c_2 \cdot [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)] \cdot c_3 \cdot [\ell_2, \ell_3] \cdot c_4 \cdot [\ell_4, \ell_5]$$

Let $c = c_1 \cdot c_2 \cdot c_3 \cdot c_4$. Then,

$$\begin{aligned} |M| &= c \cdot 1 \cdot [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)] \cdot [\ell_2, \ell_3] \cdot [\ell_4, \ell_5] && (1) \\ &= c \cdot \frac{\ell_1(\ell_2, \ell_3)(\ell_4, \ell_5)}{((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3))} \cdot \frac{\ell_2\ell_3}{(\ell_2, \ell_3)} \cdot \frac{\ell_4\ell_5}{(\ell_4, \ell_5)} && (2) \\ &= c \cdot \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3))}. && (3) \end{aligned}$$

To get from (1) to (2), we use Theorem 2.1.9, and from (2) to (3) is simple distribution. Thus $|M|$ is a multiple of $\frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3))}$. $\square$

Corollary 4.2.2. *Fix the edge labels on (D, L), where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let m_0, m_1, m_2, and m_3 be elements from each corresponding flow-up class in $\mathcal{S}(D, L)$. Then $|m_0, m_1, m_2, m_3, m_4| = c \cdot \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$, where $c \in \mathbb{N}$.*

Proof. Let M be the lower triangle matrix formed by combining m_0, m_1, m_2, and m_3 in transposed form into one matrix. As shown in Theorem 4.2.1, $|M| \mid \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3))}$. Through properties of greatest common divisors we see the following are equivalent

$$\begin{aligned}
((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3)) &= ((\ell_2, \ell_3)(\ell_4, \ell_5), (\ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3))) && (1)\\
&= ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1((\ell_4, \ell_5), (\ell_2, \ell_3))) && (2)\\
&= ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5, (\ell_2, \ell_3))) && (3)\\
&= ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5, \ell_2, \ell_3)) && (4)\\
&= ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1((\ell_2, \ell_3, \ell_4, \ell_5)). && (5)
\end{aligned}$$

In step (1), we use Lemma 2.1.10, in step (2) we use Lemma 2.1.11, in steps (3) and (4) we use Lemma 2.1.10 again, and in step (5) we simply reorder the elements. Therefore, $|M| \mid \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$. □

The inclusion of Corollary 4.2.2 may appear to be arbitrary, but it is valuable as a representation for the diamond graph. The denominator $((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1((\ell_2, \ell_3, \ell_4, \ell_5)$ has the structure of

((edges of cycle 1)(edges of cycle 2),center edge(outer edges))

Much like with Gjoni's work on the three cycle, we would like to prove that any set of splines form a module basis for the set of splines on an edge labeled graph if and only if their determinant is equal to some value. For the diamond graph, the value is $Q = \pm\frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$. We manage to prove this given certain restrictions placed on the values for ℓ_1, ℓ_2, ℓ_3, ℓ_4, and ℓ_5.

Corollary 4.2.3. *Fix the edge labels on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $Q = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$ and b_0, b_1, b_2, and b_3 be the smallest elements of the respective flow-up classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset \mathcal{S}(D, L)$. Then $|b_0, b_1, b_2, b_3| = Q$.*

Proof. Since b_0, b_1, b_2, and b_3 are the smallest elements of the flow-up classes, we already know their forms to be

$$\begin{aligned}
b_0 &= (1, 1, 1, 1)\\
b_1 &= (0, [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)], g_3, g_4)\\
b_2 &= (0, 0, [\ell_2, \ell_3], h_4)\\
b_3 &= (0, 0, 0, [\ell_4, \ell_5]).
\end{aligned}$$

Then taking the determinant of their matrix in transposed form,

$$\begin{aligned}
|b_0, b_1, b_2, b_3| &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)] & 0 & 0 \\ 1 & g_3 & [\ell_2, \ell_3] & 0 \\ 1 & g_4 & h_4 & [\ell_4, \ell_5] \end{vmatrix}\\
&= 1 \cdot [\ell_1, (\ell_2, \ell_3), (\ell_4, \ell_5)] \cdot [\ell_2, \ell_3] \cdot [\ell_4, \ell_5]\\
&= \frac{\ell_1(\ell_2, \ell_3)(\ell_4, \ell_5)}{((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3))} \cdot \frac{\ell_2\ell_3}{(\ell_2, \ell_3)} \cdot \frac{\ell_4\ell_5}{(\ell_4, \ell_5)}\\
&= \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3))}\\
&= \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3, \ell_4, \ell_5))}
\end{aligned}$$

Therefore, $|b_0, b_1, b_2, b_3| = Q$. □

The following lemma is crucial in proving our final proof, but it is only possible with the restrictions stated. While reading the proof, it may help the reader to refer back to Figure 4.1.1.

Lemma 4.2.4. *Fix the edges on (D, L), where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $(\ell_2, \ell_3, \ell_4, \ell_5) = (\ell_1, \ell_2) = (\ell_1, \ell_3) = (\ell_1, \ell_4) = (\ell_1, \ell_5) = 1$, and $Q = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$. If $W, X, Y, Z \in \mathcal{S}(D, L)$, then $Q \mid |W, X, Y, Z|$.*

Proof. Based on the restrictions set on ℓ_1, ℓ_2, ℓ_3, ℓ_4, and ℓ_5, we see that

$$\begin{aligned} ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1(\ell_2, \ell_3, \ell_4, \ell_5)) &= ((\ell_2, \ell_3)(\ell_4, \ell_5), \ell_1) \qquad (1) \\ &= 1. \qquad (2) \end{aligned}$$

We get to step (1) by the given restraints, and we get to (2), because ℓ_1 is coprime with all other edges, and therefore coprime with the product of their greatest common divisors. Therefore,

$$Q = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))} = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{1} = \ell_1\ell_2\ell_3\ell_4\ell_5.$$

Since $W, X, Y, Z \in \mathcal{S}(D, L)$, we know $\ell_2 \mid (w_2 - w_3)$, $\ell_2 \mid (x_2 - x_3)$, $\ell_2 \mid (y_2 - y_3)$, and $\ell_2 \mid (z_2 - z_3)$. Let $M = |W, X, Y, Z|$. Thus

$$M = \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} = \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 - w_3 & x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} = \ell_2 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix}$$

for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}$. Similarly, $\ell_3 \mid (w_3 - w_1)$, $\ell_3 \mid (x_3 - x_1)$, $\ell_3 \mid (y_3 - y_1)$, and $\ell_3 \mid (z_3 - z_1)$, so

$$M = \ell_2 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ w_3 - w_1 & x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} = \ell_2\ell_3 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix},$$

for some $b_1, b_2, b_3, b_4 \in \mathbb{Z}$. Finally, $\ell_5 \mid (w_4 - w_1)$, $\ell_5 \mid (x_4 - x_1)$, $\ell_5 \mid (y_4 - y_1)$, and $\ell_5 \mid (z_4 - z_1)$

$$M = \ell_2\ell_3 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ w_4 - w_1 & x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = \ell_2\ell_3\ell_5 \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{Z}$. Since the above matrices all have integer entries, we know $\ell_2\ell_3\ell_5 \mid M$. Using the same technique we can show $\ell_2\ell_3\ell_4 \mid M$, $\ell_2\ell_4\ell_5 \mid M$, and $\ell_3\ell_4\ell_5 \mid M$. Then by the definition of least common multiples, if these four products divide M, then their least common multiple does too, or $[\ell_2\ell_3\ell_4, \ell_2\ell_3\ell_5, \ell_2\ell_4\ell_5, \ell_3\ell_4\ell_5] \mid M$. By Theorem 2.1.9 this implies $\frac{\ell_2\ell_3\ell_4\ell_5}{(\ell_2,\ell_3,\ell_4,\ell_5)} \mid M$, and since we know $(\ell_2, \ell_3, \ell_4, \ell_5) = 1$, we get $\ell_2\ell_3\ell_4\ell_5 \mid M$. Using the same method, we can see that $\ell_1 \mid M$. Moreover, we know that ℓ_1 is pairwise coprime with ℓ_2, ℓ_3, ℓ_4, and ℓ_5, thus $[\ell_1, \ell_2\ell_3\ell_4\ell_5] = \ell_1\ell_2\ell_3\ell_4\ell_5$. So we have $\ell_1\ell_2\ell_3\ell_4\ell_5 \mid M$. Since $Q = \ell_1\ell_2\ell_3\ell_4\ell_5$, we conclude that $Q \mid |W, X, Y, Z|$. □

Lemma 4.2.5. *Fix the edge labels on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. Let $Q = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$ and $W, X, Y, Z, H \in \mathcal{S}(D, L)$. Suppose $|W, X, Y, Z| = \pm Q$, then QH is in the span of $\{W, X, Y, Z\}$.*

Proof. Let $W = (w_1, w_2, w_3, w_4)$, $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$, $Z = (z_1, z_2, z_3, z_4)$, and $H = (h_1, h_2, h_3, h_4)$. Let

$$M = \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix}$$

and suppose $|M| = \pm Q$. To show that $QH \in span\{W, X, Y, Z\}$, we must show that QH is a linear combination of W, X, Y, and Z. We will do this by showing that there exists $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ such the following equation has a solution

$$\begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} Qh_1 \\ Qh_2 \\ Qh_3 \\ Qh_4 \end{bmatrix}.$$

Since $|M| \neq 0$, we know the system has a solution in $\mathbb{Q}$, and since $\mathbb{Q}$ is a field, we use Cramer's Rule over $\mathbb{Q}$, to compute

$$a_1 = \frac{\begin{vmatrix} Qh_1 & x_1 & y_1 & z_1 \\ Qh_2 & x_2 & y_2 & z_2 \\ Qh_3 & x_3 & y_3 & z_3 \\ Qh_4 & x_4 & y_4 & z_4 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} h_1 & x_1 & y_1 & z_1 \\ h_2 & x_2 & y_2 & z_2 \\ h_3 & x_3 & y_3 & z_3 \\ h_4 & x_4 & y_4 & z_4 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} h_1 & x_1 & y_1 & z_1 \\ h_2 & x_2 & y_2 & z_2 \\ h_3 & x_3 & y_3 & z_3 \\ h_4 & x_4 & y_4 & z_4 \end{vmatrix}.$$

Using the same technique, we compute a_2, a_3, and a_4

$$a_2 = \frac{\begin{vmatrix} w_1 & Qh_1 & y_1 & z_1 \\ w_2 & Qh_2 & y_2 & z_2 \\ w_3 & Qh_3 & y_3 & z_3 \\ w_4 & Qh_4 & y_4 & z_4 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} w_1 & h_1 & y_1 & z_1 \\ w_2 & h_2 & y_2 & z_2 \\ w_3 & h_3 & y_3 & z_3 \\ w_4 & h_4 & y_4 & z_4 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} w_1 & h_1 & y_1 & z_1 \\ w_2 & h_2 & y_2 & z_2 \\ w_3 & h_3 & y_3 & z_3 \\ w_4 & h_4 & y_4 & z_4 \end{vmatrix},$$

$$a_3 = \frac{\begin{vmatrix} w_1 & x_1 & Qh_1 & z_1 \\ w_2 & x_2 & Qh_2 & z_2 \\ w_3 & x_3 & Qh_3 & z_3 \\ w_4 & x_4 & Qh_4 & z_4 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} w_1 & x_1 & h_1 & z_1 \\ w_2 & x_2 & h_2 & z_2 \\ w_3 & x_3 & h_3 & z_3 \\ w_4 & x_4 & h_4 & z_4 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} w_1 & x_1 & h_1 & z_1 \\ w_2 & x_2 & h_2 & z_2 \\ w_3 & x_3 & h_3 & z_3 \\ w_4 & x_4 & h_4 & z_4 \end{vmatrix},$$

and lastly

$$a_4 = \frac{\begin{vmatrix} w_1 & x_1 & y_1 & Qh_1 \\ w_2 & x_2 & y_2 & Qh_2 \\ w_3 & x_3 & y_3 & Qh_3 \\ w_4 & x_4 & y_4 & Qh_4 \end{vmatrix}}{|M|} = \frac{Q\begin{vmatrix} w_1 & x_1 & y_1 & h_1 \\ w_2 & x_2 & y_2 & h_2 \\ w_3 & x_3 & y_3 & h_3 \\ w_4 & x_4 & y_4 & h_4 \end{vmatrix}}{\pm Q} = \pm \begin{vmatrix} w_1 & x_1 & y_1 & h_1 \\ w_2 & x_2 & y_2 & h_2 \\ w_3 & x_3 & y_3 & h_3 \\ w_4 & x_4 & y_4 & h_4 \end{vmatrix}.$$

Since the entries of the matrices are all in $\mathbb{Z}$, which means then by the properties of determinants that a_1, a_2, a_3, a_4 are in $\mathbb{Z}$. Therefore $QH \in span_{\mathbb{Z}}\{W, X, Y, Z\}$. $\square$

Example 4.2.6. Fix the edges on (D, L) where $L = (2, 4, 3, 6, 5)$. Doing the computation we find $Q = \frac{2 \cdot 4 \cdot 3 \cdot 6 \cdot 5}{((4,3)(6,5), 2(4,3,6,5))} = 720$. Let

$$W = (0, 0, 12, 30)$$

$$X = (2, 0, 8, 12)$$

$$Y = (3, 1, 9, 13)$$

$$Z = (0, 0, 12, 0)$$

which are easily verified to be elements of $\mathcal{S}(D, L)$. Taking their determinant, we get $|W, X, Y, Z| = -720 = -Q$. Then by Lemma 4.2.5 we should be able to let H be any spline on (D, L), and see that $QH \in span\{W, X, Y, Z\}$. Let $H = (19, 33, 25, 39)$. Then $QH = (13680, 23760, 18000, 28080)$. We can rewrite this as

$$QH = 2160W - 28800X + 23760Y + 720Z.$$

Therefore, $QH \in span\{W, X, Y, Z\}$.

Lemma 4.2.7. *Fix the edge labels on (D, L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4)$. If W, X, Y, Z form a basis for $\mathcal{S}(D, L)$, and $J, K, M, N \in \mathcal{S}(D, L)$, then $|W, X, Y, Z|$ divides $|J, K, M, N|$.*

Proof. Since W, X, Y, Z are basis elements for $\mathcal{S}(D, L)$ and J, K, M, N are splines on (D, L), then they can be represented as linear combinations of the four basis elements

$$J = a_1W + a_2X + a_3Y + a_4Z \qquad \text{for some } a_1, a_2, a_3, a_4 \in \mathbb{Z},$$

$$K = b_1W + b_2X + b_3Y + b_4Z \qquad \text{for some } b_1, b_2, b_3, b_4 \in \mathbb{Z},$$

$$M = c_1W + c_2X + c_3Y + c_4Z \qquad \text{for some } c_1, c_2, c_3, c_4 \in \mathbb{Z},$$

$$N = d_1W + d_2X + d_3Y + d_4Z \qquad \text{for some } d_1, d_2, d_3, d_4 \in \mathbb{Z}.$$

Putting those four into matrix form, we get

$$[J,K,M,N] = \begin{bmatrix} a_1w_1 + a_2x_1 + a_3y_1 + a_4z_1 & b_1w_1 + b_2x_1 + b_3y_1 + b_4z_1 & c_1w_1 + c_2x_1 + c_3y_1 + c_4z_1 & d_1w_1 + d_2x_1 + d_3y_1 + d_4z_1 \\ a_1w_2 + a_2x_2 + a_3y_2 + a_4z_2 & b_1w_2 + b_2x_2 + b_3y_2 + b_4z_2 & c_1w_2 + c_2x_2 + c_3y_2 + c_4z_2 & d_1w_2 + d_2x_2 + d_3y_2 + d_4z_2 \\ a_1w_3 + a_2x_3 + a_3y_3 + a_4z_3 & b_1w_3 + b_2x_3 + b_3y_3 + b_4z_3 & c_1w_3 + c_2x_3 + c_3y_3 + c_4z_3 & d_1w_3 + d_2x_3 + d_3y_3 + d_4z_3 \\ a_1w_4 + a_3x_4 + a_3y_4 + a_4z_4 & b_1w_4 + b_3x_4 + b_3y_4 + b_4z_4 & c_1w_4 + c_3x_4 + c_3y_4 + c_4z_4 & d_1w_4 + d_3x_4 + d_3y_4 + d_4z_4 \end{bmatrix}$$

$$= \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}.$$

By the properties of determinants, we know $|AB| = |A| \cdot |B|$. Therefore,

$$\begin{aligned} |J,K,M,N| &= \begin{vmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \\ w_4 & x_4 & y_4 & z_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \\ &= |W,X,Y,Z| \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \end{aligned}$$

We already know that $a_1, b_1, c_1, d_1, a_2, \ldots, c_4, d_4 \in \mathbb{Z}$, so $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \in \mathbb{Z}$. Therefore we see that $|W,X,Y,Z|$ divides $|J,K,M,N|$. □

Lemma 4.2.8. *Fix the edge labels on (D,L) where $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. If $\{W,X,Y,Z\}$ is a basis for $\mathcal{S}(D,L)$ and $\{J,K,M,N\}$ is another basis for $\mathcal{S}(D,L)$, then $|W,X,Y,Z| = \pm|J,K,M,N|$.*

Proof. Let $|W,X,Y,Z| = h \neq 0$. From Lemma 4.2.7, we know that $h \mid |J,K,M,N|$. Hence, for some $a \in \mathbb{Z}$, we have $ah = |J,K,M,N|$. Since $\{J,K,M,N\}$ is a basis as well, then by Lemma 4.2.7, $|J,K,M,N| \mid |W,X,Y,Z|$. Then there exists some $b \in \mathbb{Z}$ such that

$$|J,K,M,N|\cdot b=|W,X,Y,Z| \implies a\cdot b\cdot H=H \implies a\cdot b=1 \implies b=\pm 1.$$

Since $a\cdot b=\pm 1$, and $a,b\in\mathbb{Z}$, we get $|W,X,Y,Z|=\pm|J,K,M,N|$. □

Theorem 4.2.9. *Fix the edge labels on (D,L) where $L=(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$. Let $Q=\frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$ and let $W,X,Y,Z\in\mathcal{S}(D,L)$. If $\{W,X,Y,Z\}$ form a module basis for $\mathcal{S}(D,L)$, then $|W,X,Y,Z|=\pm Q$.*

Proof. By Theorem 4.1.7 we know that the smallest element of the four flow-up classes of the diamond spline, $\{b_0,b_1,b_2,b_3\}$ form a module basis for $\mathcal{S}(D,L)$. Then by Corollary 4.2.3 we know that $|b_0,b_1,b_2,b_3|=\pm Q$. Since both $\{b_0,b_1,b_2,b_3\}$ and $\{W,X,Y,Z\}$ are module bases for $\mathcal{S}(D,L)$, we use Lemma 4.2.8 to see that $|W,X,Y,Z|=|b_0,b_1,b_2,b_3|=\pm Q$, and therefore $|W,X,Y,Z|=\pm Q$. □

In Section 3.4 we showed that Gjoni was able to prove the converse this result for $3-$cycle splines. However, the conditions imposed by the diamond spline complicate the proof immensely, thus we are only able to prove in a specialized case of the converse of Theorem 4.2.9.

Theorem 4.2.10. *Fix the edges on (D,L) where $L=(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$. Let $Q=\frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$, and set $(\ell_2,\ell_3,\ell_4,\ell_5)=(\ell_1,\ell_2)=(\ell_1,\ell_3)=(\ell_1,\ell_4)=(\ell_1,\ell_5)=1$. Suppose $W,X,Y,Z\in\mathcal{S}(D,L)$, with $|W,X,Y,Z|=\pm Q$, then $\{W,X,Y,Z\}$ is a module basis of $\mathcal{S}(D,L)$.*

Proof. First, we see that $|W,X,Y,Z|=\pm Q\neq 0$, and thus $\{W,X,Y,Z\}$ is linearly independent.

Let $H\in\mathcal{S}(D,L)$. To show that $\{W,X,Y,Z\}$ spans $\mathcal{S}(D,L)$, we must show that H is a linear combination of the proposed basis elements. By Lemma 4.2.5 we know $QH\in span\{W,X,Y,Z\}$, i.e.

$$QH = a_1W + a_2X + a_3Y + a_4Z$$

for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}$. Now by the properties of determinants,

$$\begin{aligned}
\pm a_1 Q &= a_1|W, X, Y, Z| \\
&= |a_1W, X, Y, Z| \\
&= |(a_1W + a_2X + a_3Y + a_4Z), X, Y, Z| \\
&= |QH, X, Y, Z| \\
&= Q|H, X, Y, Z|,
\end{aligned}$$

so we have

$$a_1 = \pm|H, X, Y, Z|.$$

The same method can be used to show

$$\begin{aligned}
a_2 &= \pm|W, H, Y, Z| \\
a_3 &= \pm|W, X, H, Z| \\
a_4 &= \pm|W, X, Y, H|.
\end{aligned}$$

By Theorem 4.2.4, we know that $Q \mid \pm|H, X, Y, Z|$, so for some $k_1 \in \mathbb{Z}$, $k_1Q = |H, X, Y, Z|$, implying that $a_1 = k_1Q$. Similarly, $a_2 = k_2Q$, $a_3 = k_3Q$, and $a_4 = k_4Q$, for some $k_2, k_3, k_4 \in \mathbb{Z}$. Thus, returning to our initial equation,

$$\begin{aligned}
QH &= a_1W + a_2X + a_3Y + a_4Z \\
&= k_1QW + k_2QX + k_3QY + k_4QZ \\
&= Q(k_1W + k_2X + k_3Y + k_4Z) \\
H &= k_1W + k_2X + k_3Y + k_4Z.
\end{aligned}$$

Therefore, H is a linear combination of W, X, Y, and Z, so W, X, Y, and Z span $\mathcal{S}(D, L)$ since they are also linearly independent, $\{W, X, Y, Z\}$ is a module basis for $\mathcal{S}(D, L)$. $\square$

Example 4.2.11. Fix the edge labels on (D, L) where $L = (5, 2, 4, 7, 3)$. Note that $(2, 4, 7, 3) = (5, 2) = (5, 4) = (5, 7) = (5, 3) = 1$, and so $Q = \frac{5\cdot 2\cdot 4\cdot 7\cdot 3}{((2,4)(7,3),5(2,4,7,3))} = 5 \cdot 2 \cdot 4 \cdot 7 \cdot 3 = 840$. Now, consider the four splines on (D, L)

$$W = (12, 2, 0, 9)$$

$$X = (13, 3, 1, 10)$$

$$Y = (60, 0, 0, 0)$$

$$Z = (40, 0, 0, 7).$$

Taking their determinant we see

$$|W, X, Y, Z| = \begin{vmatrix} 12 & 13 & 60 & 40 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 9 & 10 & 0 & 7 \end{vmatrix} = 840 = Q.$$

Then by Theorem 4.2.10, since $|W, X, Y, Z| = Q$, we should be able to show that $\{W, X, Y, Z\}$ forms a module basis for $\mathcal{S}(D, L)$. To illustrate this, let us choose a spline $H \in \mathcal{S}(D, L)$, and show that it can be rewritten as a linear combination of these four splines. Let $H = (26, 46, 54, 74)$, then we can compute

$$H = (26, 46, 54, 74) = -58W + 54X - 5Y + 8Z.$$

And so H is in $span\{W, X, Y, Z\}$.

5
Future Work

In this section we will look at some conjectures developed on the diamond graph as well as $(m,n)-$Cycles that hopefully follow from the work shown in Chapter 4. These conjectures were formed by increasing the number of outer edges of the diamond graph and observing the impact it has on the flow-up classes and the determinantal criterion.

While we were unable to prove the following conjecture, we found it to be true in all of the example we have computed.

Conjecture 5.0.12. *Fix the edge labels on* (D,L) *where* $L = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$*. Let* $Q = \frac{\ell_1\ell_2\ell_3\ell_4\ell_5}{((\ell_2,\ell_3)(\ell_4,\ell_5),\ell_1(\ell_2,\ell_3,\ell_4,\ell_5))}$ *and let* $W, X, Y, Z \in \mathcal{S}(D,L)$*. If* $|W, X, Y, Z| = \pm Q$*, then* W, X, Y, Z *are a basis for* $\mathcal{S}(D,L)$*.*

Here we offer an example to strengthen the conjecture.

Example 5.0.13. Fix the edges on (D, L) where $L = (8, 5, 3, 4, 6)$. We compute $Q = \frac{8 \cdot 5 \cdot 3 \cdot 4 \cdot 6}{((5,3)(4,6), 8(5,3,4,6))} = \frac{2880}{2} = 1440$. Let $W, X, Y, Z \in \mathcal{S}(D, L)$ where

$$W = (16, 0, 10, 4)$$

$$X = (0, 0, 15, 0)$$

$$Y = (17, 1, 11, 5)$$

$$Z = (24, 0, 15, 0).$$

Computing the determinant of W, X, Y, Z when put in matrix form we find

$$|W, X, Y, Z| = \begin{vmatrix} 16 & 0 & 17 & 24 \\ 0 & 0 & 1 & 0 \\ 10 & 15 & 11 & 15 \\ 4 & 0 & 5 & 0 \end{vmatrix} = 1440 = Q.$$

Since $|W, X, Y, Z| = Q$, then by Conjecture 5.0.12 we should be able to choose any $H \in \mathcal{S}(D, L)$ and be able to show that $H \in span\{W, X, Y, Z\}$. Suppose $H = (19, 43, 58, 31)$, then we compute

$$H = (19, 43, 58, 31) = -46W + 2X + 43Y + Z,$$

and therefore $H \in span\{W, X, Y, Z\}$.

5.1 Conjectures of Traits of $(m, n)-$Cycles

Let us first define an $(m, n)-$cycle with a figure.

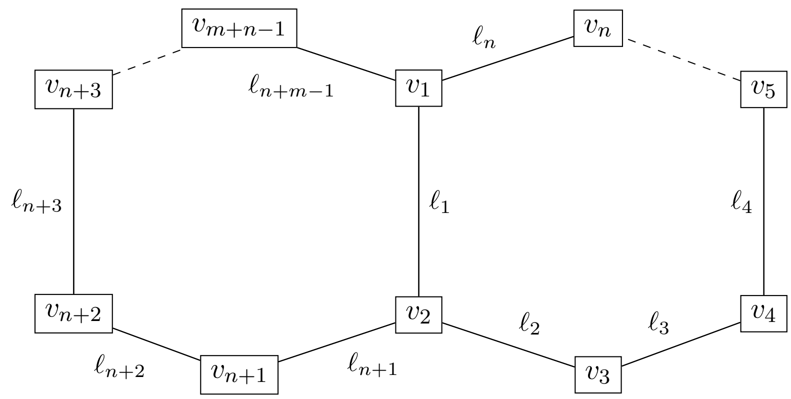

Figure 5.1.1. An $(m, n)-$cycle with edge labels.

Conjecture 5.1.1. *Fix the edges on* $(C_(m, n), L)$ *where* $L = (\ell_1, \ell_2, \dots, \ell_n, \ell_{n+1}, \dots, \ell_{n+m-1})$. *Then the flow-up classes exist, and the smallest elements of each are of the form*

$$
\begin{aligned}
b_0 &= (1, 1, \dots, 1)\\
b_1 &= (0, [\ell_1, (\ell_2, \ell_3, \dots, \ell_n), (\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m-1})], g_3, g_4, \dots, g_n, g_{n+1}, \dots, g_{n+m-1})\\
b_2 &= (0, 0, [\ell_2, (\ell_3, \ell_4, \dots, \ell_n)], h_4, h_5, \dots, h_n, h_{n+1}, \dots, h_{n+m-1})\\
b_3 &= (0, 0, 0, [\ell_3, (\ell_4, \ell_5, \dots, \ell_n)], h_5, h_6, \dots, h_n, h_{n+1}, \dots, h_{n+m-1})\\
&\vdots\\
b_{n-1} &= (0, \dots, 0, [\ell_{n-1}, \ell_n], j_{n+1}, j_{n+2}, \dots, j_{n+m-1})\\
b_n &= (0, \dots, 0, [\ell_{n+1}, (\ell_{n+2}, \ell_{n+3}, \dots, \ell_{n+m-1})\\
&\vdots\\
b_{n+m-2} &= (0, \dots, 0, [\ell_{n+m-2}, \ell_{n+m-1}])
\end{aligned}
$$

After showing that they exist, we believe the smallest elements form a basis.

Conjecture 5.1.2. *Fix the edge on* $(C_(m, n), L)$, *where* $L = (\ell_1, \ell_2, \dots, \ell_n, \ell_{n+1}, \dots, \ell_{n+m-1})$. *The smallest elements of the flow-up classes* $b_0, b_1, \dots, b_{n+m-2}$ *form a basis for* $\mathcal{S}(C_(m, n), L)$.

Similar to the determinantal criterion for the 3−cycle and the diamond graph, we would hope to show there is one for the $(m,n)-$cycle, where the Q value follows the structural pattern observed in the Q value of the diamond graph. That is $\frac{\text{product of all edges}}{\text{((edges cycle 1)(edges cycle 2),center edge(outer edges))}}$

Conjecture 5.1.3. *Fix the edges on* $(C_(m,n),L)$ *where* $L = (\ell_1, \ell_2, \ldots, \ell_n, \ell_{n+1}, \ldots, \ell_{n+m-1})$. *Let* $X_1, X_2, \ldots, X_{n+m-1} \in \mathcal{S}(C_(m,n),L)$, *and* $Q = \frac{\ell_1\ell_2\cdots\ell_n\ell_{n+1}\cdots\ell_{n+m-1}}{((\ell_2,\ell_3,\ldots,\ell_n),(\ell_{n+1},\ldots,\ell_{n+m-1}),\ell_1(\ell_2,\ell_3,\ldots,\ell_n,\ell_{n+1},\ldots,\ell_{n+m-1}))}$. *Then* $\{X_1, X_2, \ldots, X_{n+m-1}\}$ *is a basis for* $\mathcal{S}(C_(m,n),L)$ *if and only if* $|X_1, X_2, \ldots, X_{n+m-1}| = \pm Q$.

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