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Abstractions and Analyses of Grid Games

Taylor Rowan Boone
Bard College, tb3382@bard.edu

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by
Taylor Boone

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Abstract

In this paper, we define various combinatorial games derived from the NQueens Puzzle and scrutinize them, particularly the Knights Game, using combinatorial game theory and graph theory. The major result of the paper is an original method for determining who wins the Knights Game merely from the board's dimensions. We also inspect the Knights Game's structural similarities to the Knight's Tour and the Bishops Game, and provide some historical background and real-world applications of the material.

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For Thistle

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A Game of Knights

1.1 Introduction to Knights

Consider the following game, played on an empty chessboard: two players alternate turns, where a turn consists of placing a chess knight on a square of the board that is neither occupied nor threatened by any knight that was placed in a previous move; if a player can't find a suitable square on his turn, then he loses and the other player wins.

We will call this the *Knights Game*, and as it will be the primary subject of the paper, some clarifications should be made.

- Unlike chess, where every knight is loyal to one player or the other (i.e. each knight is either white or black), those in Knights do not discriminate between the players. Think of them as being colored gray, if you like.

- Once a knight has been placed on the board, it remains on its initial square for the rest of the game. The distinctive L-shaped movement matters only in determining squares that are threatened in all future moves.

Informally we can categorize the squares of the board at each stage of the Knights Game:

Definition 1.1.1. Let x be an arbitrary square of the board. Then x is either *occupied* by a knight, or it is not occupied. If x is exactly a knight's move away from an occupied square, we say x is *threatened.* If x is occupied or threatened, then x is *infeasible*; otherwise, it is *feasible.* △

There's no reason to restrict ourselves to the standard 8×8 chessboard; instead, assume Knights can be played on any $m \times n$ rectangle ($m, n \in \mathbb{N}$). If the board's dimensions are finite, then the associated Knights game will end after a finite sequence of moves, since every addition of a knight to the board reduces the number of squares available for future turns by at least 1. For any instance of the game, exactly one of the two players, whom we will denote P1 and P2 (with P1 always making the first move), will win and the other will lose; a draw is not possible. We shall prove this later. As in chess, the gameplay of Knights is determined entirely by the choices made by its players; there are no random elements.

Assuming each of the players has full knowledge of the game and plays optimally, it is natural to ask: which player will be victorious? To answer this and similar questions, we need some initial definitions. The next two sections outline the major mathematical machinery that will be used throughout, each representing the Knights Game in a different manner; keep in mind that these are far from the only approaches one could take.

1.2 First Interpretation: Combinatorial Game Theory

One of the many mathematical lenses through which we can analyze a game is *combinatorial game theory* (CGT).

CGT concerns itself with games that are fundamentally different from those encountered in the more widely-known field of economic game theory. If G is an arbitrary combinatorial game, then G must obey each of the following:

1. G has exactly two players, whom we shall call Player 1 and Player 2, or simply P1 and P2. They may also be referred to as Max and Min, or (as in chess) White and Black, and in this paper we will follow the convention that P1 is male and P2 is female. The players alternate turns, with P1 making the first move.

2. G is a *game of no chance*; that is, the progression of the game from beginning to end is determined entirely by the decisions made by the two players. A game of no chance can't have its course altered by, say, a roll of dice or a shuffling of decks.

3. G is a *game of perfect information*; that is, any aspect of the game that is visible to one player is visible to that player's opponent as well.

For our purposes, there are several other properties that a game should have. A game G can be represented as a *set of games* $\{G_1, G_2, \ldots, G_n\}$, where each G_i is an *option* of G, that is, a game that can be moved to by the next player in G. For instance, if G_0 is the standard starting position of chess, then G_0 is equivalently the collection of all board states that could result from White's first turn. Given a game G, the progression of the game to its conclusion corresponds to some choice of option at each turn until the game's end, which is represented by an empty set of options– if the next player to move must do so in the empty set, that player has no available moves and thus loses the game. Given G, we can construct a tree, called the *game tree* for G, which has G as the root. The options of G are the children of the root node of the game tree; similarly, the options of each option G_i of G are the children of the node corresponding to G_i. Each leaf of the game tree is an endgame position. A path from root to leaf in any game tree represents

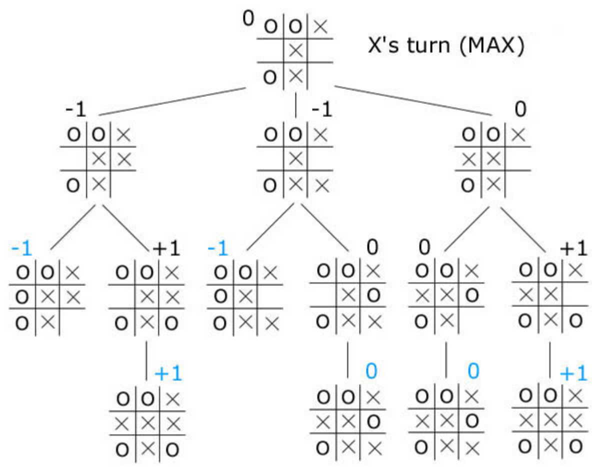

Figure 1.2.1. A sample Tic Tac Toe game tree.

the exact sequence of moves the players take over the course of that game. We will refer to all the nodes of the game tree as the *positions* of G. We require that each game G will end after a finite number of moves, which implies that no position of G has G as an option.

All games discussed here are assumed to be short, and to obey the normal play convention. A game is called *short* if it has only finitely many positions. The *normal play convention* is the assumption that if it is a player's turn to move but no legal move can be made, then that player loses the game; this is in contrast to the less common *misere play convention*, under which that player would win the game instead. Further, we restrict our attention to impartial games. A combinatorial game G is called *impartial* if at each stage of the game, the moves available to $P1$ are precisely those available to $P2$.

The Knights Game fits all of the criteria for impartial games.

Definition 1.2.1. Given a grid of arbitrary size, let K_0 denote the game's *initial position*, on which no moves have yet been made. A *playthrough* of the Knights Game on this board is a finite sequence $K = (K_0, K_1, K_2, \dots, K_\alpha)$ with the properties that

1. With the exception of K_0, each term of K is an option of the term preceding it, i.e., $K_{i+1} \in K_i$.

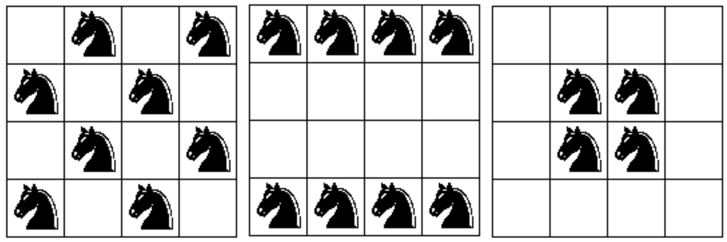

Figure 1.2.2. Three possible terminal positions of 4×4 Knights. Note that they do not all feature the same number of knights.

2. The last term, K_α, is an endgame position with no feasible squares. It will be referred to as a *terminal position*.

Equivalently, a playthrough K is a path through the game tree starting at the root K_0 and ending at some leaf K_α. The ith term, K_i, of K is the ith turn; if i is odd, then K_i is P1's turn, and if i is even, then K_i is P2's turn. △

Then K_0 is identical for all playthroughs of the Knights Game on that board. However, as illustrated in Figure 1.2.2, there tend to be many possible endgame positions K_α, and many possible playthrough lengths.

The remainder of this section will be somewhat more abstract, and is not stated in terms of the Knights Game. Rather, it pertains to the class of Surreal Numbers, a number system devised by John H. Conway (see [4], [39]) containing the real and ordinal numbers. It is a beautiful theory in which numbers are defined in terms of games, with impartial games corresponding to ordinals and partizan (i.e., non-impartial) games corresponding to reals; I had hoped to cover the Surreals in an appendix, but since time forbids it, I will instead emphatically suggest that the reader explore that universe on his or her own.

It turns out that short, impartial combinatorial games all have some deep similarities that allow us, at least in theory, to determine which player will win. Before formalizing this, one vital game needs defining. In the game of Nim, two players alternate turns, and on each player's turn he must remove one or more objects from a heap. The game, which can involve any finite number of heaps, ends when all heaps are empty; the player who empties the last heap wins (under normal play, at least). In the 1930s two mathematicians, R.P. Sprague and P.M. Grundy [61], independently discovered the following–

Theorem 1.2.2 (Sprague-Grundy Theorem)**.** *Every short impartial combinatorial game is equivalent in play to some single-heap game of Nim.*

By 'equivalent in play' we mean that the optimal strategies are identical between the two games. This theorem is immensely useful in determining the outcome of a game; Nim is simple and well understood, and it tends to be far easier to determine directly a winning strategy for Nim than for whatever other game one wishes to study.

Each impartial game G has an associated value called a *nimber*, or *nim-value*, which is the size of the Nim heap to which G is equivalent. If $G = *3$ (conventionally, nonzero nimbers are prefaced with a star), then to win G, one should turn to the optimal strategy for the Nim game consisting of a single heap of size 3. If $G = 0$, then G is a zero game, and the next player to move will lose. To compute the nimber of G, it is necessary to know the nimbers of each option $G' \in G$, which can be done by an exhaustive recursion through the game tree. Once those values are known, we define the nimber of G to be the smallest nimber that is not the nim-value of any option of G. Thus, the nimber of G is the *minimal excluded value* of the set of option-nimbers.

By assigning a value to each game, we establish an algebraic structure for the theory, complete with notions of arithmetic and ordering. Given games $G = \{G_1, G_2, \ldots, G_\alpha\}$ and $H = \{H_1, H_2, \ldots, H_\beta\}$, their sum $G + H$ refers to the game

$$G + H = \{G + H_1, G + H_2, \ldots, G + H_\beta, H + G_1, H + G_2, \ldots, G_\alpha + H\}.$$

While this may not be notationally intuitive, it simply means that to move in $G + H$, a player must choose either G or H and then make a standard move in the chosen game. If the move is made to some position of H then it corresponds to some option $G + H_j$

of $G + H$, and if the move is in G then the corresponding option is of the form $G_i + H$. Nim provides a simple illustration of this. If G and H are two Nim heaps, then $G + H$ is the two-heap Nim game where a turn consists of reducing the size of precisely one of the two heaps. Some caution should be taken when using the equals sign in the context of a combinatorial game. If G and H are both games, the statement '$G = H$' indicates equality of sets: it means that G has precisely the same options as H has. However, if G is a game and $*x$ is a nimber, then '$G = *x$' is an abstract declaration of the value of the game. Many possible games can share the same nimber $*x$. For instance, all three terminal positions shown in Figure 1.2.2 have nimber 0, but they are clearly distinct.

Lemma 1.2.3. *Every nimber is its own additive inverse; that is,* $*x + *x = 0$ *for each nimber* $*x$.

Suppose the Nim heaps G and H mentioned above are both the same size, namely $*x$ for some positive x. In the game $G + H$, we may assume without loss of generality that P1 moves to some option $G \cup H_j$ by removing part or all of heap H. Then P2 can move to the option $G_j \cup H_j$, where $G_j = H_j$; by making this move, she shrinks heap G by the same number as her opponent just shrunk H, reestablishing the equality of the heaps. If she responds to each move from P1 in this manner, then *she is guaranteed the last move* and thus victory in $G + H$. Because it is a losing position, $G + H$ must have nim-value 0; since $G + H$ also equals $*x + *x$, it follows that $*x + *x = 0$.

Theorem 1.2.4. *Let* $\mathcal{N}$ *be the class of all games that will be won by the next player (assuming optimal play), and let* $\mathcal{P}$ *be the class of all games that can be won by the previous player. If* G *is an impartial game, then* G *is in exactly one of* $\mathcal{N}, \mathcal{P}$.

Proof. Let $Q(G)$ denote 'either $G \in \mathcal{N}$ or $G \in \mathcal{P}$, but not both.' Aligning ourselves arbitrarily with P1, we can view $\mathcal{N}$ as the class of all winning positions, and $\mathcal{P}$ as the class of all losing positions. Clearly $Q(\emptyset)$ holds; the next player loses, since no more moves

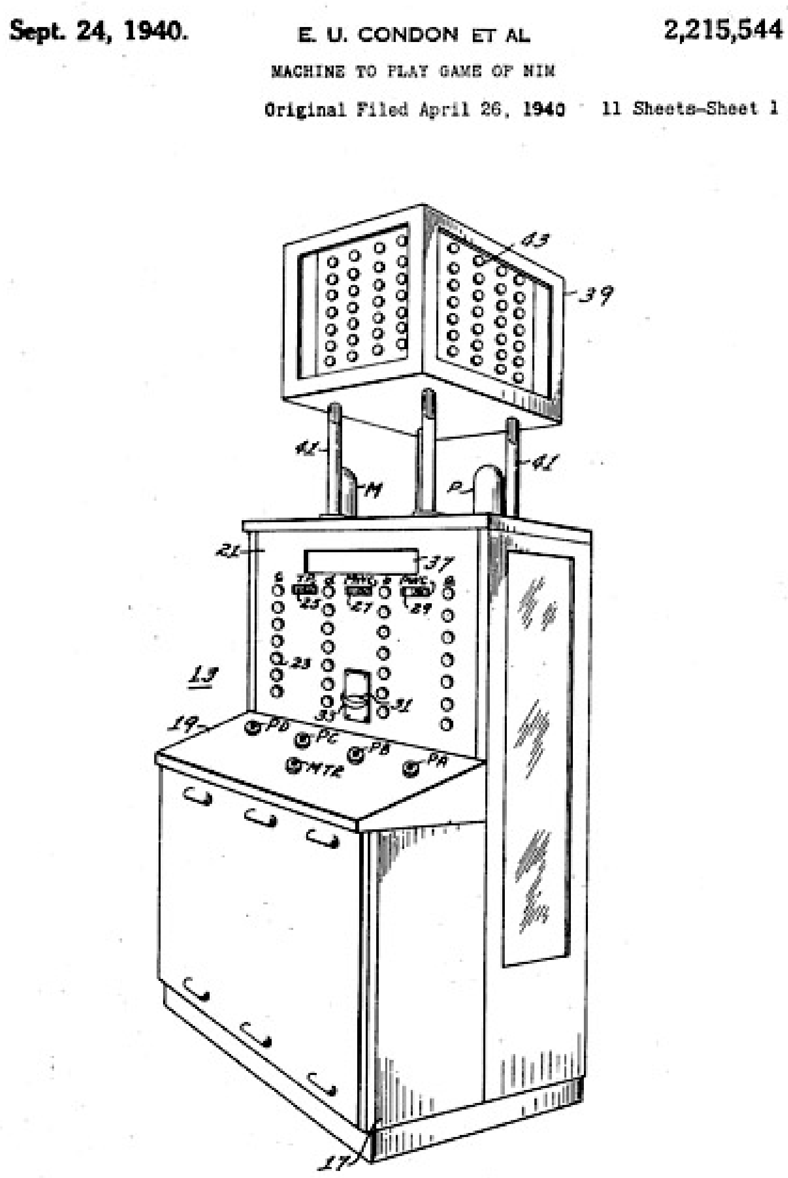

Figure 1.2.3. A sketch of the *Nimatron*, which was built for display at the 1940 World's Fair. It weighed about one ton, and could play a mighty fine game of Nim (but do little else).

are available, so $\emptyset \notin \mathcal{N}$, and whichever player is not the next player (that is, the previous player) wins, so $\emptyset \in \mathcal{P}$. We shall prove that $Q(G)$ holds for an arbitrary impartial game G, using induction with the empty game $\emptyset$ as the base case. Suppose $Q(G')$ holds for each $G' \in G$. Then either

(i) Every option G' of G is in class $\mathcal{N}$, or

(ii) At least one option of G is in class $\mathcal{P}$.

If (i), then no matter what the current player does, he hands his opponent a winning position; thus $G \in \mathcal{P}$. If (ii), then the current player has access to a strategic move that hands his opponent a losing position; thus $G \in \mathcal{N}$.

Therefore $Q(G)$ holds, so every impartial game is in exactly one of $\mathcal{N}, \mathcal{P}$.

□

1.3 Second Interpretation: Graph Theory

Combinatorial game theory provides the tools to analyze 'the big picture' of Knights, the abstract space of all the game's possible positions and how they relate to one another. But we also want to delve into the details of those positions: how do knights on grids really behave?

Definition 1.3.1. A *graph* is an ordered pair $G = (V, E)$, where V is a set whose elements are called *vertices* (or *nodes*) and E is a set of *edges* (or *arcs*) joining vertices to one another. An edge $e \in E$ is an unordered pair $e = (u, v)$ (for some distinct nodes $u, v \in V$). If $e = (u, v)$ is an edge, then u and v are said to be *adjacent*. These vertices u and v may be called the *ends* of edge e, and we say that e is *incident with* its ends, or equivalently, that it *joins* them. A vertex is *isolated* if it is adjacent to no other vertices.

△

Graphs are ubiquitous in mathematics and computer science, because they are a simple and easily-visualized means of conveying relationships between objects. Graph theory will be central to this paper. First we will provide a graph-theoretic definition of the grid that hosts the game:

Definition 1.3.2. Describe the $m \times n$ grid by the graph $\mathcal{G}_{m,n} = (V, E)$, called the *Knights Graph*, where a node $v \in V$ is a square of the board, and an edge $(u, v) \in E$ is a possible knight's move between nodes u and v. More formally, letting (x, y) denote the vertex in row x and column y, we define

$$V = \{(i, j) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\},$$

$$E = \{((i, j), (k, l)) \mid \{|i - k|, |j - l|\} = \{1, 2\}\}. \qquad \triangle$$

We will describe vertices of the Knights Graph using their coordinates (row and column number) where it is useful to do so, but typically this will not be necessary. Consider an arbitrary board on which there is knight. Interpreting the board to be a Knights Graph $\mathcal{G}_{m,n} = (V, E)$, the knight occupies some square $g \in V$. What is of interest to us here is how to characterize the implication of this knight's existence for the Knights Game, and further, how to do so in graph-theoretic terms. That is, we want a convenient way to represent the notion that if g is occupied, then g and all squares one knight's move away from g are off limits.

This section will involve a cornucopia of definitions from graph theory. To motivate them, let's informally introduce a few types of graphs based on what we know so far.

Consider an arbitrary Knights Graph $\mathcal{G}_{m,n} = (V, E)$, and suppose node $g \in V$ is occupied. Let $I(g)$ denote the graph whose vertex set contains g and all nodes adjacent to g, and has an edge joining g to each of its adjacent nodes. The graph $I(g)$ will be referred to as a *pointed infeasible set*, or *piset*; a turn in a playthrough of the Knights Game can be identified by the unique $g \in V$ that the active player chooses to occupy, hence threatening the nodes adjacent to g, so g 'points' to $I(g)$. It would seem that any piset $I(g)$ resembles a star when drawn, with 'hub' node g and between 0 and 8 'spoke' nodes adjacent to g.

Let $K = (K_0, K_1, K_2, \ldots, K_\alpha)$ be a playthrough on $\mathcal{G}_{m,n} = (V, E)$. For each index i, the position K_i can be regarded as the unordered set of its options, as in Section 1.2, providing a glimpse into the immediate future and its possibilities. What we will do now is to present K_i in a manner that displays the whole of its *past*. If $0 < i \leq \alpha$, then some pointer $g \in V$ is moved to during the ith turn of K; call it g_i. Then $V_i = \{g_1, g_2, \ldots, g_i\}$ is the set of pointers corresponding to the moves made during each of the first i turns, and $\bigcup_{0<i\leq\alpha} I(g_i)$ is the set of all nodes that are infeasible after the ith turn. In addition to the interpretation of K_i as the set of its options, we will describe K_i as a graph (V_i, E_i); at this point we'll keep our mouths shut about the edge sets E_i. Of special interest is the set $V_\alpha \subseteq V$ of vertices of K_α, which we will call a *terminal set*.

With these graphs in mind, let's proceed with the definitions.

Definition 1.3.3. An edge that joins a vertex to itself is a *loop*. A finite graph is called *simple* if it contains no loops and allows for no more than one edge between any pair of distinct points. $\triangle$

Intuitively, a loop (v, v) in $\mathcal{G}_{m,n} = (V, E)$ would imply that a knight occupying v is a threat to itself. That doesn't seem realistic, nor does the notion that the occupant of square x threatens square y in more than one way. Any Knights Graph, and indeed every graph that will show up in this paper, is simple.

Definition 1.3.4. Let $G = (V, E)$ is a graph. Then a graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph $G' = (V', E')$ is called an *induced subgraph* of G if every edge joining two points of V' that is present in E is also an element of E'. $\triangle$

Pisets, as we have described them, are subgraphs of $\mathcal{G}_{m,n}$, as are sets of the form V_i, because they cannot contain any vertices or edges not present in $\mathcal{G}_{m,n}$.

Definition 1.3.5. Let $G = (V, E)$ be a graph. A *walk* in G is a sequence of pairwise adjacent edges $e \in E$. A *path* is a walk in which no vertex is visited more than once. The number of edges in a path is the *length* of the path. A path of length at least three whose final vertex is the same as its initial vertex is a *cycle*. A cycle is called *Eulerian* if it contains every edge of G, and a cycle is called *Hamiltonian* if it contains every vertex of G. The graph G is said to be *cyclic* if it contains a cycle, and *acyclic* otherwise. △

We will demonstrate that any every graph of the form $I(g)$ or K_α is acyclic. The cyclicity of $\mathcal{G}_{m,n} = (V, E)$ is more involved, and will be reserved for an appendix. It's worth mentioning that in general, it is much easier to determine whether a graph is Eulerian than whether it is Hamiltonian.

Definition 1.3.6. The *complete graph* on n vertices is the simple graph consisting of all $\binom{n}{2} = \frac{n(n-1)}{2}$ possible edges. Somewhat unfortunately for us, the standard notation for the complete graph on n vertices is K_n. Hopefully it will be clear from context whether K_n refers to a complete graph or to a Knights position. △

Definition 1.3.7. Let $G = (V, E)$ be a graph. Then G is said to be *connected* if for any two vertices $u, v \in V$, there exists a path from u to v in G. If G is not connected, then it must be the union of some number of disjoint connected subgraphs; these are the *components* of G. A subset $V' \subseteq V$ is called an *independent set* (or *stable set*) if no pair of distinct points $u, v \in V'$ are adjacent. A subset $V' \subseteq V$ is called a *clique* if every pair of distinct points $u, v \in V'$ is adjacent. The *complement* of G, denoted $\overline{G}$, is the graph whose vertex set is V and whose edge set E' is defined by $(u, v) \in E'$ if and only if $(u, v) \notin E$ $(\forall u, v \in V : u \neq v)$. △

Depending on its dimensions, a Knights Graph $\mathcal{G}_{m,n}$ may or may not be connected. Consider the 3×3 case, where a knight on the center square can't threaten anything.

That vertex is isolated, and so $\mathcal{G}_{3,3}$ is not a connected graph. However, $\mathcal{G}_{3,4}$ is connected, as are larger Knights Graphs in general.

Let $I(g)$ be a piset. It may be the case that g is the isolated center square of $\mathcal{G}_{3,3}$, in which case $I(g)$ is trivially connected. If $I(g)$ has spokes, then a path with two edges can be forged between any two of them simply by going through the hub g. We see that any piset is connected.

Let V_α be a terminal set. Suppose there are distinct nodes $g_i, g_j \in V_\alpha$ that are joined by an edge in E. Because players in the Knights Game alternate turns, one of g_i, g_j became occupied earlier in the playthrough than the other; without loss of generality, assume it was g_i. Then $I(g_i)$ contains g_j, so during the later turn when g_j was chosen, the square was already threatened, so the move was not a legal one according to the rules of Knights. Therefore no two vertices of V_α are adjacent, meaning it is an independent set. Further, identifying the ith turn K_i with a graph (V_i, E_i), the edge set E_i is empty.

Lemma 1.3.8. *Let V_α be a terminal set in $\mathcal{G}_{m,n}$. Then V_α is a clique in $\overline{\mathcal{G}_{m,n}}$.*

Proof. We know that V_α is an independent set, and that $\overline{\mathcal{G}_{m,n}}$ has edges precisely where $\mathcal{G}_{m,n}$ does not. Hence every element of V_α is adjacent to every other element of V_α, meaning V_α meets the criteria to be a clique in $\overline{\mathcal{G}_{m,n}}$.

□

The lemma above is a special case of a property that will be useful:

Theorem 1.3.9. *Let $G = (V, E)$ be a graph, and $S \subseteq V$ an independent set. Then S is a clique in $\overline{G}$.*

Definition 1.3.10. Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$ is the number of distinct vertices adjacent to v, and is denoted $\deg(v)$. If all vertices in G have the same degree k, we say G is *regular of degree k*.

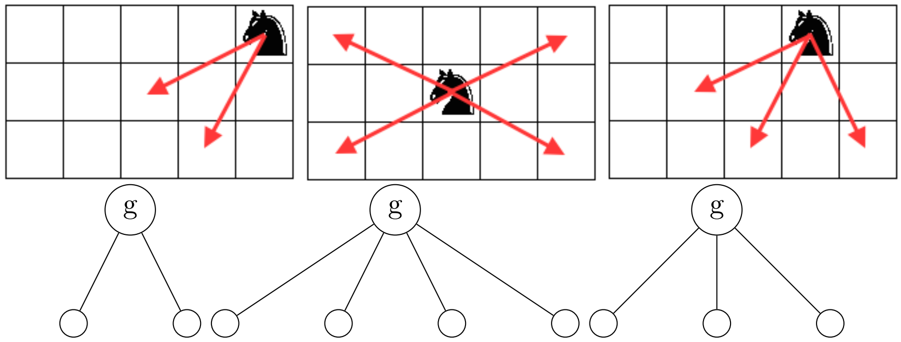

Figure 1.3.1. Three positions of Knights, and their associated pisets

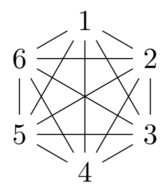

Figure 1.3.2. The complete graph on 6 vertices, K_6.

For $v \in V$, define

$N(v) = \{u : u \in V, u \neq v \text{ and } (u,v) \in E\}$ to be the *open neighborhood* of v, and $N[v] = N(v) \cup \{v\}$ to be the *closed neighborhood* of v. Observe that $\deg(v) = |N(v)|$ for all $v \in V$. If $S \subseteq V$, the open and closed neighborhoods $N(S)$ and $N[S]$ are defined to be the unions of all such neighborhoods for elements $s \in S$.

$\triangle$

Definition 1.3.11. Let $G = (V, E)$ be a graph, and let $C = \{1, 2, \ldots, k\}$ be a set of k 'colors.' A *coloring* of G is a function $f \colon V \to C$ such that any two adjacent vertices have different colors. If such a coloring exists, we say that G is *k-colored.* The *chromatic number* of G, denoted $\chi(G)$, is the minimal number of colors for which a coloring of G exists. Formally,

$\chi(G) = \min \{k : \exists \text{ a partition of } V \text{ into } k \text{ disjoint independent sets}\}$. $\triangle$

A graph $G = (V, E)$ with a chromatic number of 2 is called a *bipartite graph*, or simply *bigraph*; if X and Y are the subsets of V containing all the nodes of one color and all the nodes of the other color, respectively, then we may represent the graph instead as $G = (X \cup Y, E)$. Bigraphs have lots of nice properties, and will be of great importance. Observe that any Knights Graph $\mathcal{G}_{m,n} = (V, E)$ is bipartite– if we take X and Y to be the subsets of V consisting of all the white squares and all the black squares, respectively, then any edge $(u, v) \in E$ has one end in X and the other in Y, since a knight can only ever move from a white to a black square or vice versa.

Definition 1.3.12. Let $G = (X \cup Y, E)$ be a bigraph. We say that G is a *complete bipartite graph* if for each pair of nodes $x \in X, y \in Y$, the edge (x, y) is in E. That is, a complete bipartite graph has the maximum number of edges it could have while remaining bipartite. The complete bipartite graph with i nodes in X and j nodes in Y is denoted $K_{i.j}$. △

Every piset is complete bipartite, with one node (the hub) belonging to one color, and the other 0-8 nodes belonging to the other color.

Definition 1.3.13. Let $G = (V, E)$ be a graph. The subset $S \subseteq V$ is a *dominating set* of G if every vertex outside S has a neighbor in S, i.e., $N(S) = V$. The *domination number* $\gamma(G)$ is the smallest size of a dominating set of G. △

So if V_α is any terminal set for Knights on an arbitrary grid, then V_α must be a dominating set of $\mathcal{G}_{m,n}$. But not just any dominating set will do; if $u, v \in V_\alpha$, then one of u, v was occupied before the other was, so $(u, v) \neq E$; this implies that V_α is both a dominating set and an independent set of $\mathcal{G}_{m,n}$. The set of occupied squares shown in Figure 1.3.3 is dominating but not independent, and the position can't legally occur in a playthrough of Knights.

Definition 1.3.14. Let $G = (V, E)$ be a graph.

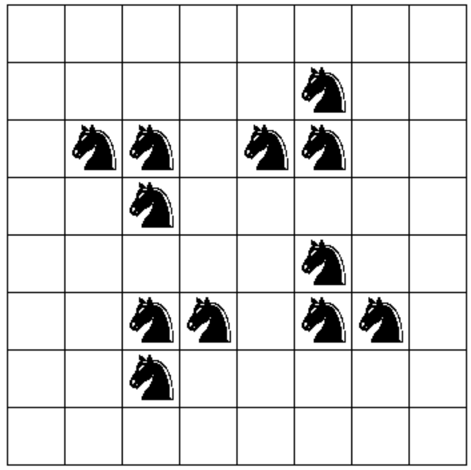

Figure 1.3.3. A board state that threatens the entire 8×8 board using 12 knights and is unique up to reflection. While there is no known formula or efficient algorithm for computing $\gamma(\mathcal{G}_{m,n})$ for arbitrary m, n, it is accepted [24] that $\gamma(\mathcal{G}_{8,8}) = 12$.

The *independence number* (or *stability number*) of G, denoted $\alpha(G)$, is the greatest cardinality among independent sets of G:

$\alpha(G) = \max \{|S| : S \subseteq V \text{ and } S \text{ is an independent set in } G\}$.

Similarly, the *clique number* of G, denoted $\omega(G)$, is the greatest cardinality among cliques of G:

$\omega(G) = \max \{|S| : S \subseteq V \text{ and } S \text{ is a clique in } G\}$. $\triangle$

Let $\mathcal{G}_{m,n}$ be a Knights Graph with a nonempty edge set. Then $\omega(\mathcal{G}_{m,n}) = 2$; the only cliques are pairs of adjacent vertices. As seen evidenced by the leftmost position in Figure 1.2.2, the largest independent set that can be obtained in $\mathcal{G}_{m,n}$ is the set of all white squares or of all black squares, whichever is more numerous (for a board of even area, they are the same), and so $\alpha(\mathcal{G}_{m,n}) = \lceil \frac{mn}{2} \rceil$. Note that any independent set $S \subseteq V$ with cardinality $\lceil \frac{mn}{2} \rceil$ is *maximal*, as any attempt to append another knight to it will create an edge and therefore strip S of its independence.

Definition 1.3.15. Let $G = (V, E)$ be a graph. Suppose $S \subseteq V$ is a dominating set of G. If no proper subset of S is dominating, then S is a *minimal dominating set*. Every minimal

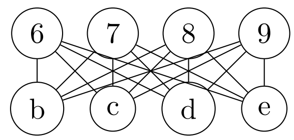

Figure 1.3.4. The complete bipartite graph $K_{4,4}$.

dominating set is independent, and such sets may also be called *independent dominating sets*.

The minimum cardinality of a minimal dominating set of G is the *independent domination number*, denoted $i(G)$.

(A word on terminology: a dominating set S is *minimal* if no proper subset of S is dominating; a minimal dominating set is *minimum* if there is no minimal dominating set of G of lesser cardinality [59]. The words are far from interchangeable.)

We may define *maximal dominating sets* analogously: they are dominating sets that are not proper subsets of any other dominating set.

$\triangle$

Let V_α be any terminal set for a Knights graph $\mathcal{G}_{m,n}$. Then $|V_\alpha| \geq i(\mathcal{G}_{m,n})$, as V_α is an independent dominating set, and $|V_\alpha| \leq \lceil \frac{mn}{2} \rceil$, as $\alpha(\mathcal{G}_{m,n}) = \lceil \frac{mn}{2} \rceil$. Hence we have bounded the cardinality of V_α:

$$\gamma(\mathcal{G}_{m,n}) \leq i(\mathcal{G}_{m,n}) \leq |V_\alpha| \leq \lceil \frac{mn}{2} \rceil.$$

Definition 1.3.16. Let $\mathcal{G}_{m,n} = (V, E)$ be a Knights Graph, and let $g \in V$. Let $I(g)$ denote the subgraph of $\mathcal{G}_{m,n}$ whose vertex set is the closed neighborhood $N[g]$ and whose edge set is that induced by the set $N[g] \subseteq V$. The graph $I(g)$ is the *piset* of g. Define the *co-piset* $\overline{I(g)}$ to be the complement of $I(g)$. $\triangle$

A fun observation is that $I(g)$ is connected, but the vertex set $N[g] - \{g\}$ is an independent set in $\mathcal{G}_{m,n}$. Since the co-piset $\overline{I(g)}$ has an edge (u, v) precisely when (u, v) is not an edge of $I(g)$, we find that every co-piset consists of two components: the independent set $\{g\}$, and the complete graph K_i of the i spokes of $I(g)$. That is, $\overline{I(g)} = K_i \cup (\{g\}, \emptyset)$ for any piset $I(g)$ with i spokes.

2

Victory and Chivalrous Defeat

This chapter presents the main original result of this paper: for any game of Knights, the parity of the board's dimensions determine whether P1 or P2 has a winning strategy. Thus we have 'solved' the Knights Game. The strategies are proven to work, but the players certainly don't have to use them; it could be very interesting to investigate what happens when strategies are discarded, too.

2.1 How to Lose at Knights

> The knight is the irresponsible low comedian of the chessboard.
>
> –Dudeney, *Amusements in Mathematics* [12]

We have shown that K_0 belongs to either $\mathcal{N}$ or $\mathcal{P}$. Because neither player has yet moved, the next player is P1 and the 'previous' player is P2, which means if $K_0 \in \mathcal{N}$ then P1 wins (assuming optimal play), and if $K_0 \in \mathcal{P}$ then P2 wins (again, AOP). It is possible to determine the outcome class of K_0 by recursively determining the outcome class of each of its options, but the amount of computation such techniques require is unacceptable on large boards.

In this section we show that P2 has a winning strategy for Knights if the board is even. This certainly does not mean that P2 is guaranteed victory on such a board, but she can always win the game if she chooses to use the strategy.

Let K_0 be the starting position of a game of Knights on an $m \times n$ board. Denote a square on this board by an ordered pair (i, j), with $1 \leq i \leq m$ and $1 \leq j \leq n$. Suppose m is even. Then $\frac{m}{2}$ is an integer. We can partition the game board into two equal-sized disjoint sets X and Y, where

$$X = \left\{(a, b) \mid a \leq \frac{m}{2}\right\} \text{ and}$$
$$Y = \left\{(c, d) \mid c > \frac{m}{2}\right\}.$$

That is, X is the board's top half and Y its bottom half. Both X and Y contain exactly $\frac{mn}{2}$ squares, since mn is even (strictly speaking, one of X, Y is of size $\lceil\frac{mn}{2}\rceil$ and the other is of size $\lfloor\frac{mn}{2}\rfloor$, so their sizes sum to mn; the parity of mn alone determines whether the floor and ceiling are equal or instead differ by 1). For each square (a, b) of the board $X \cup Y$, since either $a \leq \frac{m}{2}$ or $a > \frac{m}{2}$, exactly one of the following must hold:

1. $(a, b) \in X$, or

2. $(a, b) \in Y$.

In either case, due to the symmetry of the board, the reflection of (a, b) about the $\frac{m}{2}$ axis exists; denote it by $(a, b)'$. We can observe

- If $(a, b) \in X$ then $(a, b)' \in Y$, and if $(a, b) \in Y$ then $(a, b)' \in X$.

- $(a, b)'$ is unique; it is the only other square that shares with (a, b) both its row and its distance from the vertical origin, $\frac{m}{2}$.

- It will never be the case that the move to (a, b) threatens $(a, b)'$. Observe that a knight can only threaten squares exactly one or two rows away, but $(a, b)'$ is always zero rows away from (a, b).

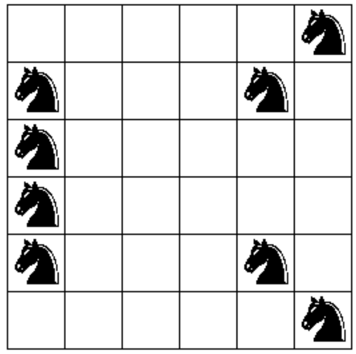

Figure 2.1.1. A symmetric Knights position on the 6×6 grid.

- If the grid is tiled with black and white squares as chessboards are, then (a, b) and $(a, b)'$ will always occupy opposite-colored squares, like white and black queens in the standard starting position of chess. This is because (a, b) and $(a, b)'$ are separated by a linear segment of the grid consisting of an even number of squares.

Definition 2.1.1. Let K_0 be the starting position of a game of Knights on an $m \times n$ board, and let K' be a position of K_0. We say that K' is *symmetric* if it has the property that whenever (a, b) is occupied, $(a, b)'$ is occupied as well. $\triangle$

Lemma 2.1.2. *Suppose K' is a symmetric position of K_0, where K_0 is the starting position of Knights on an $m \times n$ board and m is even. If the next player can move to some square (a, b), then his opponent can respond by moving to $(a, b)'$ and thus restoring symmetry.*

Proof. The proof of this lemma is by contradiction: Suppose the next player moves to (a, b) and his opponent cannot respond by moving to $(a, b)'$; that is, the square $(a, b)'$ is already either occupied or threatened.

Case 1: Suppose $(a, b)'$ is occupied. We know that the board was symmetric until the recent move to (a, b), and also that $(a, b)'$ must be distinct from (a, b). Thus the move to (a, b) must also have been made in K', immediately before or after the other player moved to $(a,b)'$. But there were knights at both (a,b) and $(a,b)'$ already, so the hypothetical symmetry-breaking move to (a,b) was not a legal one.

Case 2: Suppose $(a,b)'$ is threatened. Naturally the threatening piece is a knight (for simplicity we will assume there is only one knight on the board that threatens $(a,b)'$; our argument also holds if there are multiple threateners); suppose said knight is located at square (i,j). The symmetry of K' implies there is also a knight at $(i,j)'$. Further, since an edge (i.e. a knight's move) joins $(a,b)'$ to (i,j), there must also be an edge joining (a,b) to $(i,j)'$, so as in case 1, we have shown that the assumed move to (a,b) was illicit.

By contradiction, if (a,b) is available then it can be followed immediately by the move to $(a,b)'$. □

Lemma 2.1.2 has significant implications for the outcome class of an arbitrary starting position of the Knights Game.

2.2 How to Win at Knights

Let K_0 be the starting position of the Knights game on an $m \times n$ board, and assume that mn is odd. By defining symmetry for such boards, we will show that P1 has access to a symmetry-based winning strategy comparable to the one his opponent has on even boards.

Note that $x = \left(\lceil\frac{m}{2}\rceil, \lceil\frac{n}{2}\rceil\right)$ is the center square of K_0 when m and n are odd. If P1's first move is to that square, then he has established a 180° symmetry about the point x. From there, if P2 can move to some square (a,b), then P1 can respond by moving to $(a,b)'$, interpreted here as the 180° reflection of (a,b) about x, and restoring the symmetry of the board. Having defined symmetry for odd-area boards, we can use the odd analogue to Lemma 2.1.2:

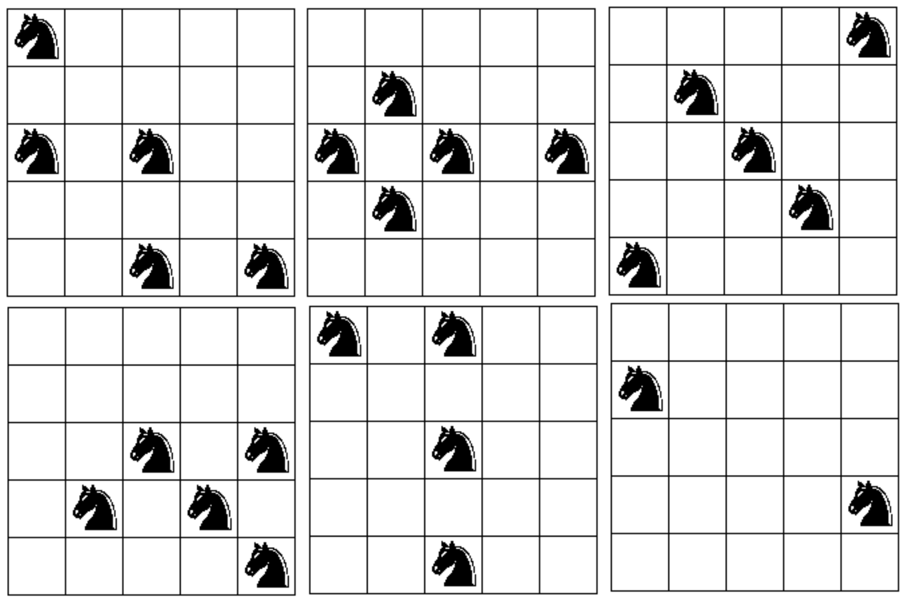

Figure 2.2.1. Six positions of Knights position on the 5×5 board. The top three are symmetric, and the bottom three are not.

Lemma 2.2.1. *Suppose K' is a symmetric position of K_0, where K_0 is the starting position of Knights on an $m \times n$ board and m and n are odd. If the next player can move to some square $(a, b) \neq \left(\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\right)$, then her opponent can respond by moving to $(a, b)'$ and thus restoring symmetry.*

This lemma's proof is nearly identical to the proof of Lemma 2.1.2, and shall be omitted.

Theorem 2.2.2 below is an original result, combining the symmetry strategies shown to exist in this section and its predecessor.

Theorem 2.2.2. *Let K_0 be the starting position of Knights on an $m \times n$ board. Then*

(a) *If m is even, then $K_0 = 0$, and*

(b) *If m and n are both odd, then $K_0 > 0$.*

Proof. Theorem 2.2.2 equivalently states that $mn \mid 2$ implies $\mathcal{G}_{m,n} \in \mathcal{P}$ and that $mn \nmid 2$ implies $\mathcal{G}_{m,n} \in \mathcal{N}$.

To prove part (a), take K_0 to be the K' described in Lemma 2.1.2; since no knights have yet been placed and at least one side of the board has even length, K_0 is trivially symmetric. We see that with his opening move, P1 will break this symmetry, and by Lemma 2.1.2, P2 can respond by restoring symmetry; indeed, there exists a playthrough K in which each move by P1 is symmetry-breaking and each move by P2 is symmetry-restoring. The lemma tells us that P2 can force K; assume she does so. Now, the game ends when the current position is some terminal position K_α; let's prove that the terminal position of K will be moved to by P2.

Suppose K_α is the terminal position of K, and suppose that K_α is not symmetric. Then, as a consequence of how K was defined, the penultimate position $K_{\alpha-1}$ must be symmetric; hence it is P1 who breaks this symmetry by moving to the position K_α. But Lemma 2.1.2 implies that when P2 is then presented with K_α, she has a symmetry-restoring move available to her, which contradicts the assumption that K_α is a terminal position. Therefore K_α is symmetric, so it must have been P2 who moved to it. This establishes K_0 as a game in which the second player to move can always force a win, so $K_0 = 0$.

Now to prove (b). For this part of the proof, we depart from the proof of (a) by using the 'odd version' of symmetry introduced earlier this section– in other respects the proof is quite similar. Suppose that m and n are both odd, and let K_0 be the initial position of Knights on $\mathcal{G}_{m,n}$. Let K be a playthough of the game in question, and suppose that in the first turn, K_1, P1 chooses to occupy the center square $x = \left(\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\right)$. Then P1 has established symmetry. Assume that for the rest of the playthrough, during each turn by P1 he chooses the reflection of his opponent's most recently-chosen square to occupy. Suppose K_α is the terminal position of K, and suppose that K_α is not symmetric. Then $K_{\alpha-1}$ must be symmetric, and it must have been P2 who moved to the asymmetric K_α. Yet Lemma 2.2.1 implies that P1 should have a symmetry-restoring move available to him, contradicting the assumption that K_α is terminal. Therefore K_α is symmetric, so it must have been P1 who moved to it. This establishes K_0 as a game in which the first player to move can always force a win, so $K_0 > 0$.

$\square$

It is worth noting that if $K_i = (V_i, \emptyset)$ is a symmetric Knights position, then $|V_i|$ has the same parity as mn; that is, $|V_i| \equiv mn$ mod 2.

Proposition 2.2.3. *If V_α is a Knights terminal set on an arbitrary $\mathcal{G}_{m,n}$, then*

$$|V_\alpha| \equiv mn \ \mathit{mod}\ 2.$$

I have not proven Proposition 2.2.3, but every terminal set I have encountered in my research satisfies the property.

3
History and Applications

Chess has attracted the attention of recreational mathematicians for as long as it has been played. In fact, many graph-theoretic definitions emerged from the study of chessboard problems [54], particularly definitions related to independence and domination.

3.1 NPiece Puzzles

The game I call Knights has not been heavily studied. It and its kin, the games introduced in Section 3.2, can trace their origins to a common source: the Eight Queens Puzzle.

In the mid-1800s, a problem known as the *Eight Queens Puzzle* was published ([22], [29], [12]); it is most commonly attributed to German chess player Max Bezzel, and it soon drew the attention of the mathematical community. It concerned the placement of eight chess queens on a standard 8×8 chessboard in such a way that none of the queens is threatened by any of the others, that is, no queen shares a row, column, or diagonal with any other queen. A *solution* to the problem is any specific arrangement of eight queens that satisfies this property. Observe that any attempt to place a ninth queen on the board must result in a board state where a queen is threatened, since inevitably there will be

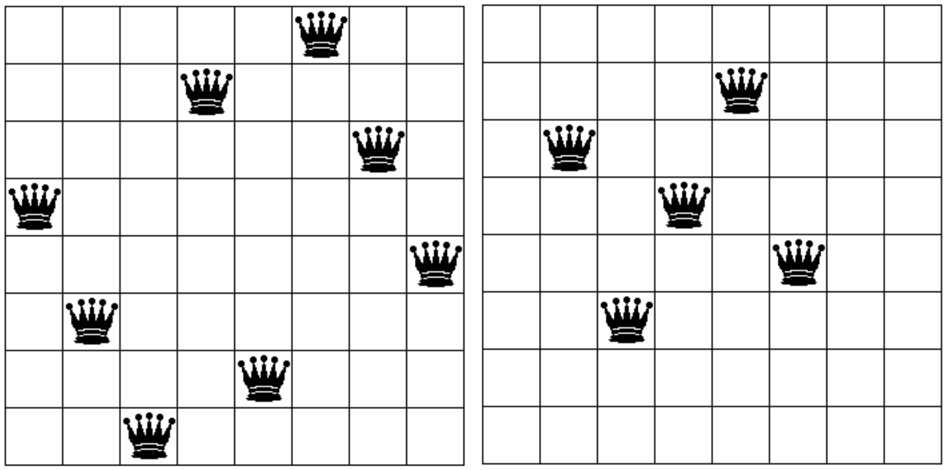

Figure 3.1.1. Solutions for the Eight Queens and Five Queens puzzles, respectively.

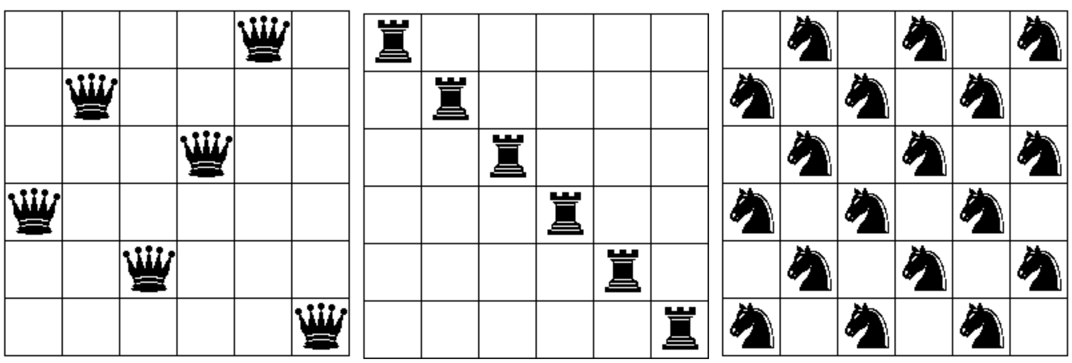

Figure 3.1.2. Sample solutions for the NQueens, NRooks, and NKnights Puzzles on the 6×6 board.

two queens that share a row. Recalling the graph parlence of Section 1.3, this means that if Q_α is a solution to the Eight Queens Puzzle, then Q_α is a maximal dominating set.

The Eight Queens Puzzle is an instance of the *NQueens Puzzle*, for which any natural number N can be chosen instead of 8. That is, the Eight Queens Puzzle is the NQueens Puzzle with $N = 8$. When $N = 1$, the NQueens Puzzle has as many solutions as there are ways to arrange a single queen on a single square. It is easy to see that this problem has exactly one solution. When $N = 2$ or $N = 3$, the puzzle has no solutions, but solutions exist when N is any natural number other than 2 or 3. As N increases, the number of solutions grows exponentially. The puzzle can be generalized to other chess pieces, but compared to the Queens version they tend to be trivial, as illustrated in Figure 3.1.2.

Eight Queens was just one of many of the chessboard puzzles to get attention from mathematicians of the day. A similar problem was the *Five Queens Puzzle*, to which a solution was a dominating set of five queens on the 8×8 board. Though it had not yet been proven, 19th century mathematicians were correct in believing five to be the minimum number of queens necessary for such a task. This puzzle is merely that of providing a minimal dominating set on a very specific graph.

These sorts of chessboard puzzles can be rephrased for general graphs:

Definition 3.1.1. Given a graph G and a number k, does G contain an independent set of size at least k? This is the *independent set problem.*

$\triangle$

Definition 3.1.2. Given a graph G and a number k, does G contain a dominating set of size at most k? This is the *dominating set problem.*

$\triangle$

In the two definitions above, there is a major departure from the phrasing of the chessboard puzzles: the puzzles requested specific board positions, whereas these new problems ask only if such positions exist. The former approach is called the *optimization version* of the problem, and the latter is the *decision version.* Why distinguish between the two?

For one thing, this is a mathematics paper and not a computer science paper, and in math we tend to be more interested in whether something is possible than in viewing (or worse yet, enumerating) instances of that something.

This is the time to bring up complexity theory. Just about every interesting chessboard graph problem is *NP-complete*, which basically means that while a computer can theoretically solve these problems, doing so could be quite computationally intense and timeconsuming. A possible novel benefit of restricting these problems to the Knights Graph (or the Bishops, Rooks, or Queens Graphs, etc.) is *polynomial time reduction.* Often an NP-complete problem can be simplified in a manner that reduces its computational complexity; for instance, the Hamiltonian Path Problem is NP-complete, but it can be reduced to the Knight's Tour Problem, which is less computationally demanding (see Appendix B). There is no significant difference in computational complexity between the decision and optimization versions [30].

3.2 Compare to the Bishop

There is nothing stopping us from generalizing the Knights Game to a host of similar impartial games. The Queens Game in particular has received much more attention than Knights. In this section, the Knights and Bishops games will be considered in relation to one another.

In 2002, Noon [31] demonstrated the following:

Lemma 3.2.1. *Let B_0 denote the starting position of Bishops on an arbitrary $m \times n$ grid. Then*

1. *If mn is even, then $B_0 \in \mathcal{P}$,*
2. *If mn is odd, then $B_0 \in \mathcal{N}$.*

For the even case, Noon gave a symmetry strategy very similar to the one I used for mn-even Knights: the gist of the argument is that because there are equally many white and black squares, any time P1 moves to some square b, his opponent can respond by moving to a corresponding square b' of the opposite color. When mn is odd, there are more squares of one color than the other, and P2 can't get away with her copycat strategy.

The following theorem arises naturally from Noon's and my findings, but is nonetheless a curious result:

	1	2	3	4	5	6
1	(1,1)	(0,0)	(1,1)	(0,0)	(1,1)	(0,0)
2	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
3	(1,1)	(0,0)	(1,2)	(0,0)	(4,1)	(0,0)
4	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
5	(1,1)	(0,0)	(4,1)	(0,0)	(1,1)	(0,0)
6	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Figure 3.2.1. The ordered pair in cell (m, n) shows the nim-values for starting positions of Knights and Bishops, respectively, on the $m \times n$ grid. I'm not sure why $K_0 = *4$ on the 3×5 grid, but a likely culprit is that that is the only odd-area rectangular board whose center square threatens every corner.

Theorem 3.2.2. *For any rectangular grid that hosts starting positions K_0 and B_0 of the Knights and Bishops game, respectively, the games K_0 and B_0 belong to the same outcome class, though their nimbers might not agree.*

Knights and bishops behave in totally different ways– for one thing, the knight is the only chess piece whose movement is independent of all activity on squares other than its destination, whereas bishops can't 'jump over' other pieces. An observation that makes Theorem 3.2.2 much more surprising is that no matter how large the board, a knight can threaten at most eight squares, but no such upper bound holds for bishops: as m and n each approach infinity, the number of squares a bishop can threaten similarly approaches infinity.

A chess notion that will return later in this paper is that of *chess piece relative value*. Because chess is not an impartial game (though it *is* combinatorial), it is not subject to the Sprague-Grundy Theorem or any other hard-and-fast rules for determining the winner. The chess-playing community has various heuristics on hand to gauge whether White or Black has a superior board state; one is to assign each piece a number. This way, a crude estimate of who is winning on an arbitrary stage of the game is to sum up and compare the white and black piece values on the board. Piece values exist entirely in the realm of meta-chess and have no direct bearing on the game. The most widely-recognized relative value assignment is 1 for a pawn, 5 for a rook, 9 for a queen, and 3 *for both a knight and a bishop.* The king is not given a value, since it can't be captured. There is some controversy when it comes to optimal piece values, but by convention the pawn is pinned to unity, and most of the relative value assignments out there differ only slightly from the standard. Presumably if one were to play chess on a board much larger than 8×8, the value of the bishop relative to the knight would have to be raised accordingly.

Let $\mathcal{H}_{m,n}$ denote the Bishops Graph on the $m \times n$ grid. We know the Knights Graph is bipartite, but $\mathcal{H}_{m,n}$ is clearly not– it is something else entirely. The Bishops Graph is the union of two disjoint subgraphs, one of the white squares and of for the black. Further, the Bishops Game is actually two separate games (white and black) played simultaneously, which gives an alternate demonstration that $B_0 = 0$ when $mn|2$: the even board has identical white and black component games, and recall from Lemma 1.2.3 that every game is its own nim-inverse.

The following definition, from Brandstadt's *Graph Classes: a survey* [28], may shed more light on the elusive relationship between Knights and Bishops:

Definition 3.2.3. A graph G is *perfect* if for all induced subgraphs H of G,

$\chi(H) = \omega(H)$. △

Recall from Section 1.3 that χ denotes chromatic number and ω denotes clique number (maximum cardinality among the graph's cliques).

Theorem 3.2.4 (The Perfect Graph Theorem)**.** *The complement of a perfect graph is perfect.*

I bring this up because of Lemma 1.3.8, which states that if V_α is terminal in $\mathcal{G}_{m,n}$ then V_α is a clique in $\overline{\mathcal{G}_{m,n}}$. In practice [22], there are efficient standard algorithms for procuring cliques (if they exist).

Lemma 3.2.5. *The only canonical chess pieces whose graphs are perfect on all $m \times n$ boards are the knight and the bishop.*

Lemma 3.2.6. *If C is a cycle in a Knights Graph or Bishops Graph, then the length of C is at least 4.*

3.3 Artificial Intelligence

Board games have had a natural place in the field we now know as computer science since its fledgeling years. Some of their features make them particularly alluring to practitioners of artificial intelligence; to name but a handful,

- They make for a simple and easily-visualized simulation of conflict. Making an artificially intelligent agent interact with an outside force is a fine way to facilitate learning.
- They have relatively simple rules and operate in discrete space, and hence are not too hard to model and to test on machines.
- The simplicity of the rules for chess and similar games masks the finesse necessary to play them well. Rather than being determined by rigid winning strategies, they follow vague heuristics (e.g. relative piece values) and require enough psychological sophistication to be fun for adults to play and to force computer players to 'think.'
- Perhaps most importantly, they are well-known to the general public. Could any arcane theoretical breakthrough have done as much for the image of AI as did the victory of IBM's Deep Blue over grandmaster Gary Kasparov?

As we saw in Section 3.1, the amount of computation required to analyze a game can be tremendous; the number of legal chess positions is roughly 10^{43}, while some other board games are even more computationally demanding (Bouzy and Cazenave [45] claim that the popular Asian board game of Go is PSPACE-complete; at the time of writing, nearly 20 years after Blue vs. Kasparov, advancements in Computer Go are major feats). In 1950, established mathematician Claude Shannon published a theoretical article [43] outlining how a humanlike computer chess program would realistically work. This was before anyone had written a successful chess program; the computers of the day had little storage capacity, and it would take a series of innovations before anyone could write a program efficient enough for chess.

By the end of the decade, some of those innovations had come around, and the pieces were in place for computer programs to approach board game play by searching the relevant game tree and seeking out the 'best' position available. The *minimax algorithm* was developed for this purpose, and it was improved upon by a technique that came to be known as *alpha-beta pruning*; the improved algorithm would eliminate branches of the game tree from consideration if they were deemed likely to lead to failure.

Compared to chess, the game of checkers is significantly less complex, while still satisfying the relevant features listed at the start of this section. That was why IBM researcher Arthur Samuel, who spent the '50s developing a gaming program capable of rudimentary supervised learning, chose checkers as the program's dedicated game. Samuel's program took advantage of minimax with alpha-beta pruning, and was implemented on a mass-produced IBM 704 computer. Once the program was capable of playing the game and learning from its experience, Samuel had it play against human opponents. This brought modest success, but was plagued by quirks of human-AI interaction, some of which are still significant impediments to such interactions. For instance, an impish human opponent could use strategies that would be considered bad play in human-on-human games, and thereby trick the naïve computer into teaching itself to play badly.

In trying to remedy this sort of problem, Samuel had a stroke of insight. After pitting the program against a handful of human opponents, he made it play many games against a distinct instance of the same program that had learned from different experiences. The program began to learn, comparatively quite quickly, how to play a good game of checkers. It wasn't long before it could routinely beat its creator, and after a few more revisions it could stand its ground against world checker masters. This may serve as a reminder that AI has a hefty philosophical component; consider the musically-untrained programmer who writes X, an entity that trains itself and then composes a score of lovely symphonies. Which party deserves credit for the resultant art– X or its creator?

Today, Arthur Samuel is remembered as a pioneer of Machine Learning, the AI-like discipline behind self-driving cars, personalized advertisements and recommendations, and countless other burgeoning facets of the digital quotidian. Claude Shannon, for his part, launched what has come to be called Information Theory with his 1948 paper *A Mathematical Theory of Communication* [44]. The theory is broad and interdisciplinary, and is concerned with rigorous treatment of the use and spread of information.

3.4 Knights in Real Life

What's that? You're still not convinced that the knight can be gainfully applied beyond the realms of recreation and academia? Fair enough; the previous section treated chess as the holistic game it is, without implying anything in particular about the knight. I hope to remedy the matter by giving the knight a privileged position in this section; alas, once again, we first turn our attention to the knight's popular big sister, the queen.

In their article *A survey of known results and research areas for n-queens*, [15] Bell and Stevens mention some applications of the NQueens Puzzle, among them: parallel memory storage schemes, VLSI testing, traffic control, neural networks, and constraint satisfaction problems.

While most of this lies outside of my field, I have reason to think that knights are as applicable in industry as queens. The hub-and-spoke model has many practical uses in e.g.

supply chain management and airline administration; logistically it is often pragmatic to structure distribution and transportation in such a manner [47]. This can also be observed in the *star network topology* [19], a parallel processing dynamic based on the star graph.

A notable difference between a knight and a queen is that the knight threatens in radial manner. This might make it more attractive for modeling something like the optimization of radio broadcasting center distribution (an independent dominating set of broadcasting center locations would be one that provides maximum coverage with minimal redundancy). The length of a knight's 'leap' can be generalized as much as one likes, as can the number of spokes in its piset.

The knight may also have applications in the theory of error-correcting codes [5] [37], and in information theory, where perfect graphs have desirable properties for channeling information [55] [56] [60].

4
Future Directions

4.1 Multipiece Games

Consider a game like Knights or Bishops, except that once a player has chosen the square he wishes to move to on a given turn, he then selects one of two chess pieces and places it on that square. In this way we construct a slew of new games: BishopsRooks, KnightsQueens, etc. It is natural to assume that each of these *two-piece games* is much harder to analyze than either of the one-piece games that inspired it, and this does seem to be the case. Another natural assumption is that for every such two-piece game, its nim-analysis and winning strategy have some similarities to those of its one-piece 'component' games; this too appears to hold, but in subtler ways than one might expect.

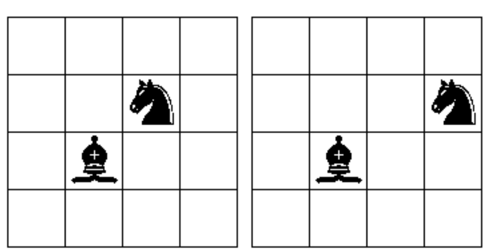

Figure 4.1.1. Two positions of BishopsKnights.

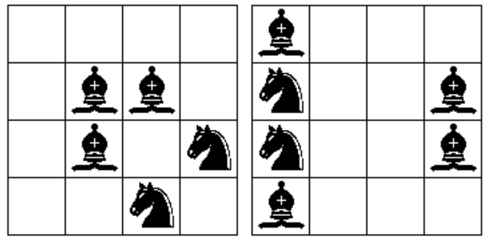

Figure 4.1.2.

Examination of multipiece games reveals some properties of single-piece games that were previously underappreciated. For instance, an implicit assumption we made when discussing the Knights Game was that for all nodes a and b, a threatens b if and only if b threatens a. When dealing with multipiece games, we might not be so lucky. For instance, Figure 4.1.1 shows two positions of BishopsKnights, both of which can occur legally in the game. In the position on the left, the bishop threatens the knight and is not threatened by it. Hence, the knight must have been placed first. The position to the right is near-identical but with roles reversed: the bishop must have been placed before the knight was.

In Figure 4.1.2 are pictured two terminal positions of BishopsKnights. Both can be reached legally in the game, but only for very specific sequences of moves.

It is certainly possible for a player to win a game of BishopsKnights using a symmetry strategy: if the board size is even and P1 uses knights exclusively, then P2 can win by the strategy described in Section 2.1. If, as is more likely, the players take advantage of both allowed pieces, then there may or may not be symmetry strategies available. I have not been able to find one, and as far as I know, the subject remains open for investigation. There are all kinds of fun to be had when it comes to multipiece games: one could add a third piece to the mix, or allow for manuevers resembling castling in chess, or introduce any number of more complicated rules.

4.2 Alternative Approaches

Early on we made it clear that we would only be considering rectangular generalized chessboards. But objects like the Knights Game and the Knights Tour can be studied on surfaces of many types:

- The game board can be treated as a torus or a Möbius strip, on which piece movement may wrap around the boundary. See the relevant articles by Bell and Stevens ([14], [15]). The Knight's Tour has been considered on three-dimensional surfaces; see DeMaio's paper [40] for an example.

- Other relevant material overlooked in this paper: rook polynomials, combinatorial design theory, linear programming, comparable graph games like Chomp and Hackenbush (see *Winning Ways* [8]).

Appendices

Appendix A
CGSuite

I used an open-source program called CGSuite (Combinatorial Game Suite) for modeling games and gathering data about them; hypothesis based on patterns in these data were instrumental in finding some major results, e.g. Theorem 2.2.2. The code below defines the Knights Game class, here called NKnights, and is very similar to the code for similar grid games.

```
class NKnights extends ImpartialGame, GridGame

    method NKnights(grid)
        this.GridGame(grid);
    end

    override method Options(Player player)
        options := {};
        for m from 1 to grid.RowCount do
            for n from 1 to grid.ColumnCount do
                if grid[m,n] == 0 then
                // found a non-threatened square- add to options
                    copy := grid;
                    copy[m,n] := 1;
                    for d1 in Direction.Vertical do
                        for d2 in Direction.Horizontal do
                            if grid[m + d1.RowShift, n + 2*d2.ColumnShift] != nil then
                                if grid[m + d1.RowShift, n + 2*d2.ColumnShift] == 0 then
                                    copy[m + d1.RowShift, n + 2*d2.ColumnShift] := 2;
```

```
                                end
                            end
                            if grid[m + 2*d1.RowShift, n + d2.ColumnShift] != nil then
                                if grid[m + 2*d1.RowShift, n + d2.ColumnShift] == 0 then
                                    copy[m + 2*d1.RowShift, n + d2.ColumnShift] := 2;
                                end
                            end
                        end
                    end
                    options.Add(NKnights(copy));
                end
            end
        end
        return options;
    end



    override property CharMap.get
        return "oxy";
        // maps o -> 0, x -> 1, y -> 2 (resp. available, occupied, threatened)
    end


    override property Icons.get
        return
        [
            GridIcon.Blank,
            GridIcon.BlackKnight,
            GridIcon.GrayStone
        ];
    end

end
```

A.1 Nimber Data

Figure 3.2.1 showed nimbers for starting positions of Knights and Bishops on all rectangular boards up to 6×6. Below are similar nimber tables for other games, all computed using CGSuite. Some of the tables have absent entries, signifying that the nimber computation was too intense for my computer to tackle.

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	2	0	0	0
3	1	2	2	3	1	1
4	1	0	3	1	2	0
5	1	0	1	2	3	0
6	1	0	1	0	0	1

Figure A.1.1. Queens

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	0	0	0	0	0
3	1	0	1	1	1	1
4	1	0	1	0	0	0
5	1	0	1	0	1	1
6	1	0	1	0	1	0

Figure A.1.2. Rooks

	1	2	3	4	5	6
1	1	1	2	0	3	1
2	1	1	2	0	3	1
3	2	2	1	0	3	2
4	0	0	0	0	0	0
5	3	3	3	0	4	3
6	1	1	2	0	3	1

Figure A.1.3. Kings

	1	2	3	4	5	6
1	1	0	1	0	1	0
2	0	0	0	0	0	0
3	1	0	3	0	1	
4	0	0	0	0		
5	1	0	1			
6	0	0				

Figure A.1.4. BishopsKnights

	1	2	3	4	5	6
1	1	2	1	2	1	2
2	2	0	0	0	0	0
3	1	0	2	3	2	1
4	2	0	3	3	0	
5	1	0	2	0		
6	2	0	1			

Figure A.1.5. BishopsRooks

	1	2	3	4	5	6
1	1	2	1	2	1	2
2	2	2	4	0	0	0
3	1	4	4	1	1	2
4	2	0	1	1	4	2
5	1	0	1	4	2	
6	2	0	2	2		

Figure A.1.6. BishopsQueens

	1	2	3	4	5	6
1	1	2	3	0	1	2
2	2	2	3	0	1	1
3	3	3	1	0	2	
4	0	0	0	0		
5	1	1	2			
6	2	1				

Figure A.1.7. KingsKnights

	1	2	3	4	5	6
1	1	2	1	2	1	2
2	2	2	3	0	0	0
3	1	3	3	1	6	2
4	2	0	1	2	0	
5	1	0	6	0		
6	2	0	2			

Figure A.1.8. KnightsQueens

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	0	0	0
3	1	2	3	3	1	1
4	1	0	3	2	0	0
5	1	0	1	0	3	1
6	1	0	1	0	1	1

Figure A.1.9. QueensRooks

Appendix B
On Knightly Journeys

B.1 The Knight's Tour

A *knight's tour* of an $m \times n$ chessboard is a Hamiltonian cycle on $\mathcal{G}_{m,n}$, whereby a solitary chess knight sets out from some initial square x and returns to x after reaching each intermediate square exactly once. A little thought reveals that the knight is the only chess piece whose tour is nontrivial (good luck trying to tour the grid with a bishop!).

If the board is small enough, it might not have any edges at all– if we take the dimensions (m, n) to be $(1, 2)$, for instance, then there is no way for a knight to occupy this board so that it threatens another square. All the boards we are considering contain at least one vertex (in an arbitrary graph, there is no requirement that a vertex have any edges).

In earlier chapters we hinted at the idea that knights behave differently on small and large boards; here we must actually address it. Let $\mathcal{G}_{m,n} = (V, E)$ be a tourable Knights Graph, and let $x \in V$ be a corner square of the board. Then $\deg(x) = 2$, while every node that is not a corner has degree strictly greater than 2. If $T = (t_0, t_1, \ldots, t_{mn-1}, t_0)$ is a tour of $\mathcal{G}_{m,n}$, then $x = t_i$ for some $i \in \{0, 1, 2, \ldots, mn - 1\}$, since by definition T includes every square of the board. Since T is a cycle, it is regular of degree 2. Hence, if the two edges of

x are (x, y_1) and (x, y_2), then T must contain either $(\ldots y_1, x, y_2, \ldots)$ or $(\ldots y_2, x, y_1, \ldots)$. This is a fancy way of saying that a Knights Tour has no 'wiggle room' when it comes to corners, and so extra care is taken with corners, in practice.

At first glance, even when both are considered with regard to the same $m \times n$ board, there is no strong connection between the concepts of the Knights Game and the knight's tour– besides the chess context, the most obvious parallel is that both are iterative processes defined to terminate precisely when the board has been dominated. But subtler similarities reveal themselves when the structures of the two are compared.

In general, it is extremely difficult to determine whether a given graph is Hamiltonian (i.e. is host to some Hamiltonian cycle). A simpler problem is to determine which chess-boards allow for a knight's tour; recall from Section 3.1 that the Knight's Tour Problem is a polynomial time reduction of the Hamiltonian Path Problem.

In a 1991 article [13], Allen J. Schwenk proved the following:

Theorem B.1.1 (Schwenk's Theorem)**.** *An $m \times n$ chessboard with $m \leq n$ has a knight's tour unless one or more of these three conditions holds:*

(a) m *and* n *are both odd;*

(b) $m = 1, 2,$ *or* 4*; or*

(c) $m = 3$ *and* $n = 4, 6,$ *or* 8.

Condition (a) of this theorem is particularly exciting: it holds precisely for those boards on which the starting position of the Knights Game is a winning position.

Here are the outlines of a few tour-constructing algorithms and their runtimes.

- Parberry [26] proposes a divide-and-conquer algorithm for $n \times n$ boards with $n \geq 6$ even. The board is partitioned into four quadrants of equal size (or as close to equal as possible), and each quadrant is toured independently according to some criteria that

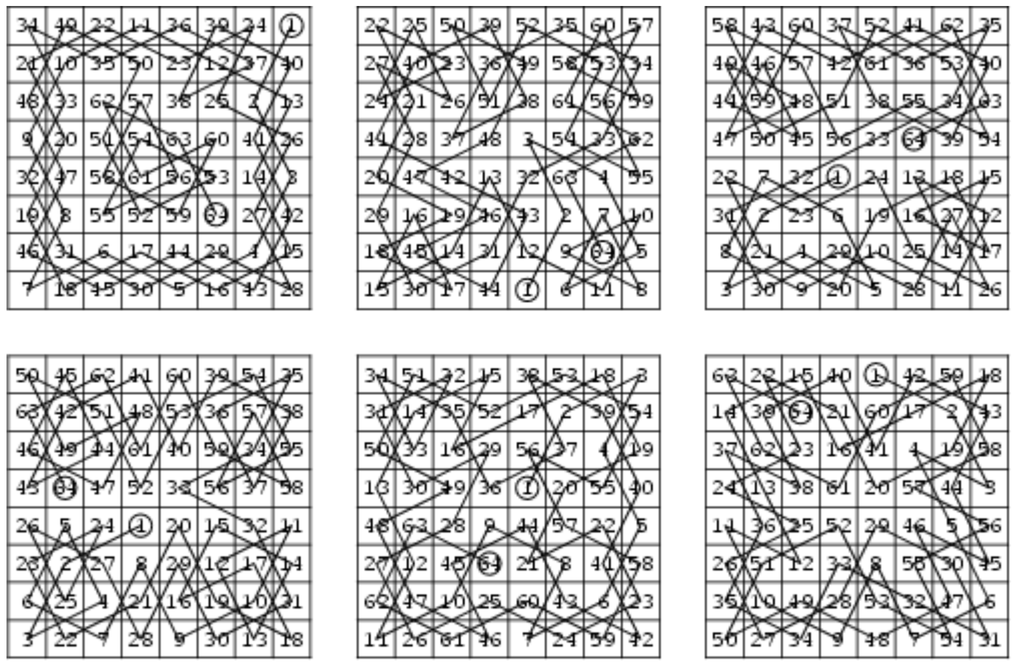

Figure B.1.1. Various open tours of the 8×8 board.

allow the four tours to be stitched together into a tour of the full board. Parberry's algorithm has a runtime of $O(n^2)$.

- Lin and Wei [25] draw inspiration from Parberry; they modify his algorithm so that it can be applied to non-square boards as well. They claim it runs in time $O(mn)$.

- While not exactly an algorithm, a common heuristic for constructing a knight's tour is *Warnsdorff's Rule*, which stipulates that at any stage of tour construction, one should always move to the adjacent unvisited square that has the fewest unvisited neighbors (ties can be broken arbitrarily). Several analyses and algorithms featuring this rule exist, and the curious reader is directed to Ganzfried's paper [16] for more on the matter.

One of the first to attempt an algorithmic knight's tour generator was Euler [29], whose approach was to bisect the chessboard, tour each resulting half separately, and then make a few minor adjustments to obtain a complete tour. This bears a strong resemblance to the symmetry-based Knights strategy presented in Section 2.1.

I have not been able to draw a significant conclusion tying the tour to the Knights Game, but the two have lots of overlap, and further investigation is warranted.

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