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Filtering Irreducible Clifford Supermodules

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Abstract

A Clifford algebra is an associative algebra that generalizes the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, etc. Filtrations are increasing chains of subspaces that respect the structure of the object they are filtering. In this paper, we filter ideals in Clifford algebras. These filtrations must also satisfy a "Clifford condition", making them compatible with the algebra structure. We define a notion of equivalence between these filtered ideals and proceed to analyze the space of equivalence classes. We focus our attention on a specific class of filtrations, which we call principal filtrations. Principal filtrations are described by a single element in complex projective space and their equivalence classes are orbits of a group action inside complex projective space. In this paper, we identify when the space of equivalence classes of principal filtrations has a discrete topology or not. We find one example where the space of equivalence classes is not discrete, and is instead homeomorphic to S^2.

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Dedication

To my classmates in the Bard mathematics department, whose support and friendship have been invaluable. To my first research collaborator and loving boyfriend, David Cochran.

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I'd like to acknowledge my adviser, Gregory Landweber, for his guidance, encouragement, and incredible flexibility. He has been instrumental in nurturing my mathematical interests as I have developed into the student that I am today. I'd also like to thank the entire faculty in the Bard mathematics department, each of whom has been a valuable resource throughout the past four years. This includes a special thanks to John Cullinan and Robert McGrail, who were both wonderful senior project board members but also valuable mentors as I struggled with my senior year. Also, to Lauren Rose, whose persistence encouraged me to dedicate myself to my academic interests. Finally, I'd like to acknowledge Brad Safnuk, my first research advisor, who introduced me to advanced mathematics and supported me as I struggled to keep up.

1
Introduction

A Clifford algebra, denoted $\mathrm{Cl}(n, K)$, is a unital associative algebra that generalizes the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, etc. Specifically, $\mathrm{Cl}(0, \mathbb{R}) \cong \mathbb{R}$, $\mathrm{Cl}(1, \mathbb{R}) \cong \mathbb{C}$ and $\mathrm{Cl}(2, \mathbb{R}) \cong \mathbb{H}$. Clifford algebras were initially discovered in physics as matrix algebras, it was not until the 1960's that mathematicians began working with Clifford algebras abstractly. Given their origin as matrix algebras, it makes sense that Clifford algebras are noncommutative. Clifford algebras contain and are generated by a vector space equipped with a quadratic form. A detailed construction of Clifford algebras can be found in [1] and Section 2.2 of this paper. This paper focuses on Clifford supermodules, which are modules over Clifford algebras that can be decomposed into even and odd parts. A filtration over a module is an increasing chain of subspaces, and a filtration over a Clifford supermodule is a filtration satisfying the "Clifford Condition". The Clifford condition requires a filtration be compatible with the module structure of a Clifford supermodule. An comprehensive discussion on filtered Clifford supermodules can be found in [3].

This paper seeks to analyze equivalences between filtrations on a specific family of minimal complex Clifford supermodules, which come from minimal ideals of the Clifford algebra. Two filtrations on a Clifford supermodule are equivalent if they differ only by a permutation or rotation of the generators of the Clifford algebra. As a means of analyzing equivalences, we construct a moduli space of filtrations to be the space of their equivalence classes. This paper seeks to understand the structure of moduli spaces of filtrations over this family of minimal Clifford supermodules. This problem is motivated by one in physics about off-shell supersymmetry. More information about the relevance of filtered Clifford supermodules to off-shell supersymmetry is given in [3].

The research in this paper builds from the research done at a 2010 REU led by Gregory Landweber. In this REU, Landweber's students analyzed filtrations on real Clifford supermodules. Their work resulted in a theorem that served as the launching point for this paper. As described in [11], they found that equivalence functions between filtrations satisfying the Clifford condition are uniquely determined by an action of the Lie group $\mathrm{Spin}(n)$. A Lie group is a group that is a differentiable topological space, where the group product and inverse maps are smooth. The Lie group $\mathrm{SO}(n)$ is the group of rotations on $\mathbb{R}^n$, and the Lie group $\mathrm{Spin}(n)$ is the double cover of $\mathrm{SO}(n)$ contained in $\mathrm{Cl}(n, \mathbb{C})$. We use the results from [11] to analyze equivalences between filtrations on complex Clifford supermodules via the action of $\mathrm{Spin}(n)$.

Chapter 2 builds preliminary background material. This chapter follows [4] to introduce algebras, modules, and group actions, and then pulls material from [3] to present filtered Clifford supermodules. We begin Chapter 3 by closely following [6] to develop basic Lie theory and representation theory, focusing on the Lie group $\mathrm{Spin}(n)$ and its Lie algebra $\mathfrak{spin}(n)$. We proceed to use the treatment found in [7] to construct $\mathrm{Spin}(n)$ as a subgroup of the invertible elements of a Clifford algebra. Finally, we describe how the results from [11] allow us to analyze the action of $\mathrm{Spin}(n)$ on complex projective space to determine the structure of our moduli spaces. Chapter 4 explicitly computes the action of $\mathrm{Spin}(n)$ on filtrations over low-dimensional minimal Clifford supermodules.

Our final two chapters analyze equivalences between a specific class of filtrations, which we call principal filtrations. Principal filtrations are uniquely determined by a single element in our minimal submodule, and they turn out to provide interesting examples of non-discrete moduli spaces. Chapter 5 determines when a moduli space of principal filtrations is discrete or not, and then analyzes specific elements of non-discrete moduli spaces. We use tools from representation theory to accomplish this.

It is not until Chapter 6, however, that we are successful in identifying the structure of a non-discrete moduli space. It is shown in [5] that Spin(8) has a useful property called "triality". We are interested in the moduli space of principal filtrations on the submodule of $\mathrm{Cl}(8,\mathbb{C})$ in our family of minimal complex Clifford submodules. Complex projective space, denoted $\mathbb{C}P^n$, is the space of lines through the origin in $\mathbb{C}^{n+1}$. Using this property of Spin(8), we are able to conclude that this moduli space is the space of orbits of $\mathbb{C}P^7$ under the action of the Lie group SO(8). We can first extend the action of SO(8) to $\mathbb{C}^8$ by viewing matrices in SO(8) as complex matrices, and then descend to consider the action of SO(8) on $\mathbb{C}P^7$. We are able to find a unique representative for each orbit in $\mathbb{C}P^7$ under the action of SO(8), thus describing the elements of the moduli space of principal filtrations. This allows us to determine that the moduli space of principal filtrations on our minimal Clifford supermodule is homeomorphic to S^2.

2
Algebraic Preliminaries

This chapter seeks to build an understanding of modules, algebras, and group actions, and then proceeds to specifically develop an intuition for the behavior of filtrations on Clifford supermodules. We assume the reader is familiar with a first course in abstract algebra and linear algebra.

2.1 Algebras, Modules, and Group Actions

Our goal for the first half of this section is to develop an understanding of how an algebra behaves as a module over itself. The material presented in this section is largely pulled from [4]. We assume the reader is comfortable with both group theory and ring theory.

We begin by introducing algebras. For our purposes, it is sufficient to define an algebra over a field. Note that there is a more general definition of an algebra over a ring that is not presented here.

Definition 2.1.1. An *algebra A over a field F* is both a ring with a scalar product and a vector space over F equipped with a bilinear vector product. If the bilinear vector product is associative, then the algebra is an *associative algebra*. If the algebra contains a unit, then the algebra is *unital*. △

For the remainder of this paper, all algebras are implicitly defined over fields. As expected, ideals of algebras behave almost identically to ideals of rings.

Definition 2.1.2. Let A be an algebra, and let I be a linear subspace of A. Then I is a *left ideal* of A if I is closed under addition, multiplication by scalars, and left multiplication by arbitrary elements of A (i.e. $AI \subset I$). A *right ideal* is defined analogously. △

Observe that an ideal of an algebra differs from that of a ring only in the additional requirement that it be closed under multiplication by scalars. This is because algebras have a vector space structure that is not present in rings.

We conclude our discussion of algebras with two examples. We assume that the reader is familiar with vector spaces.

Example 2.1.3. The complex numbers are an algebra over the real numbers. Elements of the complex numbers are of the form $a + bi$, where $a, b \in \mathbb{R}$. We can therefore express the complex numbers as

$$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

It follows that the complex numbers are a 2-dimensional vector space over $\mathbb{R}$ with the expected definition of scalar multiplication and vector addition. The natural definition of multiplication between complex numbers yields a bilinear vector product. Explicitly this product is defined as,

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

It is easy to check that this product is bilinear. ♢

Example 2.1.4. A matrix ring over a field K, denoted $M_n(K)$ is the set of all $n \times n$ matrices over K. It is clear that $M_n(K)$ is a vector space over K with an associative bilinear vector product. It follows that $M_n(K)$ is an associative algebra. $\Diamond$

Next, we turn to a presentation of modules. As with algebras, modules are defined over a separate algebraic structure.

Definition 2.1.5. Let R be a a ring with identity. A *left R-module* is a set M together with

1. a binary operation $+$ on M under which M is an abelian group, and
2. an action of R on M (that is, a map $R \times M \longrightarrow M$) denoted by rm, which satisfies
   (a) $(r+s)m = rm + rs, \quad \text{for all } r, s \in R, m \in M,$
   (b) $(rs)m = r(sm), \quad \text{for all } r, s \in R, m \in M,$
   (c) $r(m+n) = rm + rn, \quad \text{for all } r \in R, m, n \in M$, and
   (d) $1m = m, \quad \text{for all } m \in M.$

A *right R-module* is defined analogously. $\triangle$

When not specified, a module is assumed to be a left R-module. Though this construction may seem abstract, many familiar structures are modules. For example, a module over a field is a vector space. Another familiar example comes from group theory. It turns out that every abelian group is a $\mathbb{Z}$-module. We present this construction in the next example.

Example 2.1.6. Let $(G, +)$ be an abelian group. We define the action of $\mathbb{Z}$ on M as,

$$ng = \begin{cases} g + g + \cdots + g \ (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -g - g - \cdots - g \ (-n \text{ times}) & \text{if } n < 0 \end{cases}.$$

It is straightforward to check that this action satisfies the necessary axioms. Under this construction, G is a $\mathbb{Z}$-module. $\Diamond$

The discussion of modules is rich, but we will limit the remainder of our investigation to rings as modules over themselves. The following example explicitly gives this construction.

Example 2.1.7. Let R be a ring, and let $M = R$. Then $(M, +)$ is an abelian group, by the definition of a ring. Let the associated action $R \times M \longrightarrow M$ be given by $(r, m) \mapsto rm$, i.e. left multiplication by ring elements. It is clear that this actions satisfies the necessary axioms. Thus, M is an R-module. $\Diamond$

Because an algebra is also a ring, we can take an algebra to be a module over itself in this same manner.

We proceed to define a submodule. Submodules behave similarly to the more familiar structure of ideals, and this relationship is both interesting and useful.

Definition 2.1.8. Let R be a ring and let M be an R-module. An *R-submodule* of M is a subgroup of M that is closed under the action of ring elements. $\triangle$

It is instructive to understand exactly how this definition relates to that of ideals of rings. A subring is a left ideal if it is closed under left multiplication by ring elements. Similarly, a submodule is a subgroup that is closed under the action of ring elements. It follows that every left ideal of a ring is also a submodule of that ring as a module over itself. Identically, a left ideal of an algebra is also a submodule of that algebra as a module over itself. To this end, we present the following two propositions.

Proposition 2.1.9. *Let R be a ring and I be a left ideal of R. If $M = R$ is an R-module, then $N = I$ is a submodule of M.*

Proposition 2.1.10. *Let A be an algebra and I be a left ideal of A. If $M = A$ is an A-module, then $N = I$ is a submodule of M.*

A detailed investigation of this construction can be found in [4, 10.1].

This paper will primarily focus on the study of irreducible submodules and minimal principal ideals. We present these constructions now.

Definition 2.1.11. Let R be a ring and $A \subset R$ be a set. The ideal *generated* by A, denoted $\langle A \rangle$, is the intersection of all ideals containing A. An ideal that is generated by a single element a is called a *principal ideal*, and is denoted $\langle a \rangle$. △

Definition 2.1.12. An ideal I of a ring R is *minimal* if its only subideals are 0 and I. △

Definition 2.1.13. An R-module N is *irreducible* if its only submodules are N and 0. △

From our previous discussion, we can conclude that minimal ideals of an algebra are irreducible submodules of that algebra as a module over itself.

Proposition 2.1.14. *Let A be an algebra and I be a minimal ideal of A. If $M = A$ is an A-module, then $N = I$ is an irreducible submodule of M.*

Consult [4] for a more comprehensive presentation of minimal ideals and irreducible submodules.

We complete this discussion with the construction of superalgebras.

Definition 2.1.15. A *superalgebra* A is an algebra with the algebra decomposition

$$A = A_0 \oplus A_1$$

such that

$$A_i \cdot A_j \subset A_{i+j \pmod 2},$$

where $\cdot$ denotes the bilinear product associated to A. We say that a superalgebra has a product that respects the $\mathbb{Z}_2$-grading. △

We often refer to A_0 as odd and A_1 as even. We return to the complex numbers for an example of a superalgebra.

Example 2.1.16. Recall from Example 2.1.3 that the complex numbers are an algebra over the real numbres. The complex numbers are also a superalgebra with a $\mathbb{Z}_2$-decomposition given by

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R},$$

where vector multiplication respect the $\mathbb{Z}_2$-grading. $\Diamond$

We devote the remainder of this section to the presentation of group actions. While exploring modules earlier in this section, we witnessed the behavior of a ring acting on a set. Similarly, we can define the action of a group on set. Group actions are a useful tool in discovering additional information about a group, in the same way that a ring acting on a set produces the interesting structure encoded in a module.

Definition 2.1.17. A *group action* of a group G on a set A is a map $G \times A \longrightarrow A$ (written as $g \cdot a$, for all $g \in G$ and $a \in A$) satisfying the following properties:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, for all $g_1, g_2 \in G$, $a \in A$, and
2. $1 \cdot a = a$, for all $a \in A$.

We say that G is a group acting on a set A. $\triangle$

This definition is suprising at first, as we are defining "multiplication" between a group and set whose elements may have no relation to one another. However, we must be careful to note that an action is not a binary operation. This is clearly seen in the following three examples.

Example 2.1.18. Let $G = D_4$ be the dihedral group a square, and let A be the set of vertices on a square. Then we can define the action of D_4 on A in the obvious manner. $\Diamond$

Example 2.1.19. The Lie group SO(3), defined later in Section 3.1, is the group of rotations of $\mathbb{R}^3$. Then SO(3) acts on S^2 by rotating S^2. Note that this action is continuous. $\Diamond$

Example 2.1.20. Let $G = S_n$ be the symmetric group of degree n, and let $A = \{1, 2, \ldots, n\}$. Then G acts on A by permuting the elements of A, i.e. for $\sigma \in G, a \in A$ the action $\sigma \cdot a$ is defined as the $\sigma(a)$. $\Diamond$

There is additional terminology associated with group actions that allow us to better describe the behavior of a group action.

Definition 2.1.21. Let G be a group acting on a set A.

1. For each $a \in A$ the *stabilizer* of a in G, denoted G_a, is the set of elements of G that fix the element a, i.e.

$$G_a = \{g \in G \mid g \cdot a = a\}.$$

2. The equivalence class $\{g \cdot a \mid g \in G\}$ is the *orbit* of G containing a, denoted by $\mathcal{O}_a$.

3. The action of G on A is called *transitive* if there is only one orbit, i.e. given any two elements $a, b \in A$ there is some $g \in G$ such that $a = g \cdot b$.

$\triangle$

Later chapters of this paper will rely heavily on the use of orbits and stabilizers to determine if a group action is transitive. We build from a previous example to witness the way these tools describe the properties of a group action.

Example 2.1.22. Let $G = D_4$ be the dihedral group a square, and let A be the set of vertices on a square. Define the action of G on A in the obvious manner. Then the stabilizer of $a \in A$ is a two element set consisting of the identity and the reflection about the line passing through a. This action has only one orbit, as the set of rotations contained in G maps any $a \in A$ to all three remaining elements of A. It follows that the action of G on A is transitive. $\Diamond$

Using this new terminology, we can present a helpful theorem about group actions.

Theorem 2.1.23. *Let G be a group acting on a set A. For $a, b \in A$, if a is in the orbit of b then G_a is isomorphic to G_b.*

Proof. Let $g_0 \in G$ such that $a = g_0 \cdot b$. Define $\varphi : G_a \longrightarrow G_b$ such that $\varphi(g) = g_0^{-1} g g_0$. Observe that for any $g \in G_a$,

$$\varphi(g) = (g_0^{-1} g g_0) \cdot b = (g_0^{-1} g) \cdot (g_0 \cdot b) = (g_0^{-1} g) \cdot a = g_0^{-1} \cdot (g \cdot a) = g_0^{-1} \cdot a = b,$$

implying that $\varphi(g) \in G_b$. It follows that φ is well-defined. It is clear that φ is a homomorphism because it is a conjugation map. Define $\psi : G_b \longrightarrow G_a$ such that $\psi(g) = g_0 g g_0^{-1}$. Given any $h \in G_b$, observe that

$$\psi(h) = (g_0 h g_0^{-1}) \cdot a = (g_0 h) \cdot (g_0^{-1} \cdot a) = (g_0 h) \cdot b = g_0 \cdot (h \cdot b) = g_0 \cdot b = a,$$

implying that $\psi(h) \in G_a$. It follows that ψ is well-defined. It is clear that ψ is the two-sided inverse of φ, implying that φ is bijective. Therefore φ is an isomorphism and $G_a \cong G_b$. $\square$

We will rely on this theorem in later chapters. This concludes our discussion of group actions for the moment.

We have completed our presentation of algebras, modules, and group actions. Note that this section was extremely selective in pulling material that directly relates to the goals of this paper. For this reason, we in no way provided a thorough investigation into these algebraic structures. We invite the curious reader to consult [4] for a more detailed exploration of these concepts.

2.2 Filtrations on Clifford Supermodules

In this section, we build from our understanding of algebras and modules to introduce a Clifford supermodule and its relevant properties.

We begin by introducing Clifford algebras. Note that there are multiple equivalent definitions of a Clifford algebra, but the definition presented here is the most relevant to our computations. Consult [1] and [9] for additional presentations of Clifford algebras.

Definition 2.2.1. Let K be a field with $\mathrm{char}(K) \neq 2$. A *Clifford algebra* $\mathrm{Cl}(n, K)$ is a unital associative algebra over K with a basis generated multiplicatively by $\gamma_1, \gamma_2, \ldots, \gamma_n$ with relations

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

A Clifford algebra over $\mathbb{C}$, denoted $\mathrm{Cl}(n, \mathbb{C})$, is called a complex Clifford algebra. △

This paper is primarily concerned with complex Clifford algebras. In the following example, we show the construction of a complex Clifford algebra.

Example 2.2.2. Let γ_1 and γ_2 be a basis for $\mathbb{C}^2$. Define multiplication subject to the relations

$$\gamma_1\gamma_2 = -\gamma_2\gamma_1 \text{ and } \gamma_1^2 = \gamma_2^2 = 1.$$

Then multiplication also satisfies the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

Under this construction, the set $\{\gamma_1, \gamma_2\}$ multiplicatively generates the algebra $\mathrm{Cl}(2, \mathbb{C})$. It follows that

$$\mathrm{Cl}(2, \mathbb{C}) = \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\}.$$

In general, we can generate a basis for $\mathrm{Cl}(n, \mathbb{C})$ by the basis elements of $\mathbb{C}^n$ with multiplication subject to these two relations. ◊

To help the reader build a solid understanding of the basic structure of complex Clifford algebras, we explicitly describe a few low-dimensional cases.

Example 2.2.3. Using $\gamma_1, \ldots, \gamma_n$ to multiplicatively generate a basis for $\mathrm{Cl}(n, \mathbb{C})$, we have the following constructions of low-dimensional complex Clifford algebras:

$$
\begin{aligned}
\mathrm{Cl}(0, \mathbb{C}) &= \mathrm{span}\{1\} = \mathbb{C}, \\
\mathrm{Cl}(1, \mathbb{C}) &= \mathrm{span}\{1, \gamma_1\} \cong \mathbb{C} \oplus \mathbb{C}, \\
\mathrm{Cl}(2, \mathbb{C}) &= \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\} \cong M_2(\mathbb{C}), \\
\mathrm{Cl}(3, \mathbb{C}) &= \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3, \gamma_1\gamma_2\gamma_3\} \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).
\end{aligned}
$$

We simply note the above isomorphisms as an aside, as they are a general property of Clifford algebras. In general, $\mathrm{Cl}(2n, \mathbb{C}) \cong M_{2^n}(\mathbb{C})$ and $\mathrm{Cl}(2n+1, \mathbb{C}) \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$. See [9] for an explicit construction of these isomorphisms. $\lozenge$

From the previous example, we observe that the dimension of $\mathrm{Cl}(n, \mathbb{C})$ is 2^n. This makes sense because a basis of $\mathrm{Cl}(n, \mathbb{C})$ is multiplicatively generated by $\gamma_1, \ldots, \gamma_n$.

Note that every Clifford algebra is also a superalgebra when the generators are taken to be odd. In this construction, even products of generators are even and odd products of generators are odd. For this reason, we refer to Clifford algebras and Clifford superalgebras interchangeably. This is demonstrated in the following example.

Example 2.2.4. We provide the $\mathbb{Z}_2$-decomposition of low-dimensional complex Clifford superalgebras:

$$
\begin{aligned}
\mathrm{Cl}(0, \mathbb{C}) &= \mathbb{C} \oplus \{0\}, \\
\mathrm{Cl}(1, \mathbb{C}) &= \mathrm{span}\{1\} \oplus \mathrm{span}\{\gamma_1\}, \\
\mathrm{Cl}(2, \mathbb{C}) &= \mathrm{span}\{1, \gamma_1\gamma_2\} \oplus \mathrm{span}\{\gamma_1, \gamma_2\}, \\
\mathrm{Cl}(3, \mathbb{C}) &= \mathrm{span}\{1, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3\} \oplus \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2\gamma_3\}.
\end{aligned}
$$

It is easily verified that multiplication respects each $\mathbb{Z}_2$-grading. Notice that the generators $\gamma_1, \ldots, \gamma_n$ are taken to be odd, and this uniquely determines the entire decomposition. $\lozenge$

Next, we will introduce Clifford supermodules. We are interested in this construction because the module structure provides additional tools that are not available when working exclusively with algebras.

Definition 2.2.5. A *supermodule* is a module M with a $\mathbb{Z}_2$-decomposition whose product respects this $\mathbb{Z}_2$-decomposition, i.e. there exists submodules M_0, M_1 such that $M = M_0 \oplus M_1$ and $M_i M_j \subset M_{i+j (\text{mod } 2)}$. △

Definition 2.2.6. A *Clifford supermodule* is a supermodule over a Clifford superalgebra. △

We will be interested primarily in Clifford superalgebras as modules over themselves, and their irreducible submodules.

From the definition of a vector space, one can check that a module defined over an algebra is a vector space if the algebra is defined over a field. It follows that Clifford supermodules are vector spaces. We are interested in filtrations, and there is a nice definition of filtrations over vector spaces. This is one reason that we prefer the supermodule structure over the superalgebra structure.

Definition 2.2.7. A *filtration* on a vector space V is a $\mathbb{Z}$-indexed increasing sequence of subspaces such that $F_i \subset F_{i+1}$, and both

$$\bigcup F_i = V \quad \text{and} \quad \bigcap F_i = 0.$$

The *type* of a filtration F is a non-decreasing function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $i \mapsto \dim(F_i)$. △

All filtrations presented in this paper begin at F_0 and have finite length, allowing us to denote filtration type by the list

$$\dim(F_0) - \dim(F_1) - \cdots - \dim(F_n).$$

Using our construction of Clifford supermodules, we can now nicely define the filtrations that we would like to work with.

Definition 2.2.8. A *filtered Clifford supermodule* is a Clifford supermodule together with a filtration F such that

1. $\gamma_i F_k \subset F_{k+1}$ for every γ_i, and
2. $F_k - F_{k-1} \subset A_0$ or $F_k - F_{k-1} \subset A_1$.

We refer to the first condition as the "Clifford condition".

$\triangle$

We proceed with an example of a filtration on a familiar complex Clifford supermodule.

Example 2.2.9. Recall from the previous two examples that

$$\mathrm{Cl}(3,\mathbb{C}) = \mathrm{span}\{1, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3\} \oplus \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2\gamma_3\}.$$

An example of a filtration on $\mathrm{Cl}(3,\mathbb{C})$ as a Clifford supermodule over itself is

$$0 \subset \mathrm{span}\{1\} \subset \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_3\} \subset \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3\} \subset \mathrm{Cl}(3,\mathbb{C}).$$

Observe that this sequence of subspaces satisfies the requirements of a filtration, and also respects the Clifford condition. We say that this filtration has type $0-1-4-7-8$. $\diamondsuit$

We will commonly describe filtrations by their type. For every filtrations of type $d_0 - \cdots - d_n$ in which $F_0 \subset A_0$, there is a corresponding filtration of type $0 - d_0 - \cdots - d_n$ in which $F_1 \subset A_1$. For the purposes of this paper, we will consider only filtrations in which F_0 is nontrivial.

Given this construction, we are interested in the notion of equivalence between filtrations. Most generally, we allow two filtrations to be equivalent if they differ simply by permuting the generators.

Definition 2.2.10. Let V be a Clifford supermodule over $\mathrm{Cl}(n, K)$, and let F and F' be filtrations on V. An equivalence between F and F' is a pair of vector space isomorphisms $f : V \longrightarrow V$ and $g : \mathrm{span}\{\gamma_1, \ldots, \gamma_n\} \longrightarrow \mathrm{span}\{\gamma_1, \ldots, \gamma_n\}$ such that for all $v \in V$ and all $\gamma_i \in \{\gamma_1, \ldots, \gamma_n\}$,

1. $f(\gamma_i v) = g(\gamma_i) f(v)$, and
2. $f(F_k(V)) = F'_k(V)$.

A pair of equivalence functions f, g satisfying these two conditions is said to be a twisted strict isomorphism. $\triangle$

The g in this definition permutes or rotates the generators, and the f requires that filtration degrees are mapped to each other. Note that Condition (2) requires equivalent filtration to be of the same type, but this is not sufficient for two filtrations to be equivalent. We return to $\mathrm{Cl}(3, \mathbb{C})$ for an example of equivalent filtrations.

Example 2.2.11. Again, recall that

$$\mathrm{Cl}(3, \mathbb{C}) = \mathrm{span}\{1, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3\} \oplus \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2\gamma_3\}.$$

Let V be $\mathrm{Cl}(3, \mathbb{C})$ as a Clifford supermodule over itself. We have two examples of filtrations on V given by,

$$0 \subset \mathrm{span}\{1, \gamma_1\gamma_2\} \subset \mathrm{span}\{1, \gamma_1, \gamma_2, \gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_2\gamma_3\} \subset \mathrm{Cl}(3, \mathbb{C})$$

and

$$0 \subset \mathrm{span}\{\gamma_2\gamma_3, \gamma_1\gamma_3\} \subset \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_1\gamma_2\gamma_3\} \subset \mathrm{Cl}(3, \mathbb{C}).$$

Define $f : V \longrightarrow V$ by $f(v) = \gamma_2\gamma_3 v$, and define $g : \mathrm{span}\{\gamma_1, \ldots, \gamma_n\} \longrightarrow \mathrm{span}\{\gamma_1, \ldots, \gamma_n\}$ subject to

$$g(\gamma_1) = \gamma_1, \quad g(\gamma_2) = -\gamma_2, \text{ and } \quad g(\gamma_3) = -\gamma_3.$$

It is clear that f, g is a pair of isomorphisms. Observe that,

$$f(\gamma_1 v) = \gamma_2\gamma_3\gamma_1 v = \gamma_1\gamma_2\gamma_3 v = g(\gamma_1)f(v),$$

$$f(\gamma_2 v) = \gamma_2\gamma_3\gamma_2 = -\gamma_2\gamma_2\gamma_3 v = g(\gamma_2)f(v), \text{ and}$$

$$f(\gamma_3 v) = \gamma_2\gamma_3\gamma_3 v = -\gamma_3\gamma_2\gamma_3 v = g(\gamma_3)f(v).$$

So the pair f, g satisfies Condition (1). Similarly, it is straightforward to check that f, g also satisfy Condition (2). It follows that f, g is an equivalence between these two filtrations, implying that they are equivalent.

$\diamondsuit$

Building from the tools presented in this section, we can now introduce a moduli space of filtrations on Clifford supermodules.

Definition 2.2.12. Let V be a Clifford supermodule. The *moduli space of filtrations* on V is a set of filtrations on V satisfying the Clifford condition, modulo equivalence. $\triangle$

This definition is specific to filtrations on Clifford supermodules, and is not a general definition of a moduli space. In fact, we constructed this specific type of moduli space for the purposes of this research. The remainder of this paper is largely spent analyzing the structure of such moduli spaces.

This concludes our preliminary introduction to Clifford supermodules.

3
Lie Theory and Representation Theory

This chapter seeks to develop background material from Lie theory and representation theory. We will define a Lie group and Lie algebra in general, but quickly focus our attention on the pieces of Lie theory relevant to our discussion. Our presentation of representation theory will be limited to representation theory of Lie groups and Lie algebras, and is specifically focused around the Lie group $\mathrm{Spin}(n)$ and its Lie algebra $\mathfrak{spin}(n)$.

3.1 Basic Lie Theory

The discussion of filtrations on Clifford supermodules turns out to rely heavily on a few pieces of Lie theory. This section does not present a comprehensive overview of Lie theory, but rather pulls together just the pieces necessary for this paper. Our treatment of Lie theory focuses primarily on matrix Lie groups, as they are easier to study without extensive knowledge of topology. The material presented in this section is roughly based on [6].

We are not assuming the reader has any familiarity with matrix groups. For this reason, we begin with the most basic matrix group (which happens to also be a Lie group).

Definition 3.1.1. The *general linear group* over the real numbers, denoted $\mathrm{GL}(n;\mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. The general linear group over the complex numbers, denoted $\mathrm{GL}(n;\mathbb{C})$, is the group of all $n \times n$ invertible matrices with complex entries. $\triangle$

We proceed with the introduction of a Lie group. We define convergence such that a sequence of matrices A_m converges to a matrix A if each entry of A_m converges to the corresponding entry of A, as shown in [6].

Definition 3.1.2. A *matrix Lie group* is any subgroup of $\mathrm{GL}(n;\mathbb{C})$ with the following property: If A_m is any sequence of matrices in G, and A_m converges to some matrix A then either $A \in G$ or A is not invertible. $\triangle$

Matrix Lie groups are closed subgroups of $\mathrm{GL}(n;\mathbb{C})$ or $\mathrm{GL}(n;\mathbb{R})$. The best way to develop an understanding of Lie groups is to work through examples.

The easiest example of a Lie group is the matrix group $\mathrm{GL}(n;\mathbb{R})$, given earlier. We present two additional examples of Lie groups that are important for later computation, but also interesting in their own right. Note that these examples are taken from [6].

Example 3.1.3. A real matrix A is said to be orthogonal if the column vectors of A are orthonormal, i.e.

$$\sum_{l=1}^{n} A_{lj} A_{lk} = \delta_{jk}.$$

Equivalently, a matrix A is orthogonal if and only if $A^T A = I$. The Orthogonal matrix Lie Group, denoted $\mathrm{O}(n)$, is the set of real $n \times n$ orthogonal matrices. The Special Orthogonal matrix Lie group, denoted $\mathrm{SO}(n)$, is the set of real $n \times n$ orthogonal matrices with determinant 1. It can be checked that $\mathrm{O}(n)$ has two connected components, one consisting of matrices with determinant 1 and the other with determinant -1, each of which look like $\mathrm{SO}(n)$. We leave this as an exercise to the reader. $\Diamond$

Example 3.1.4. A complex matrix A is said to be unitary if the column vectors of A are orthonormal, i.e.

$$\sum_{l=1}^{n} \overline{A_{lj}} A_{lk} = \delta_{jk}.$$

The Unitary matrix Lie group, denoted $\mathrm{U}(n)$, is the set of all complex $n \times n$ unitary matrices. The Special Unitary matrix Lie group, denoted $\mathrm{SU}(n)$, is the set of all complex $n \times n$ unitary matrices with determinant 1. It is shown in [6] that a matrix A is unitary if and only if $A^*A = I$, where A^* denotes the conjugate transpose of A. $\diamondsuit$

Since we are restricting our attention to matrix Lie groups, we must investigate the matrix exponential before we can introduce Lie algebras.

Definition 3.1.5. For any $n \times n$ matrix X, the *matrix exponential* is defined by its Taylor series:

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

$\triangle$

It is shown in [6] that the power series for e^X converges for any $n \times n$ real or complex matrix X. We will use the matrix exponential to construct Lie algebras. In the following theorem, we state a few selected properties of the matrix exponential that will be useful in our computations.

Theorem 3.1.6. *Let X and Y be arbitrary $n \times n$ matrices. Then, we have the following:*

1. $det(e^X) = e^{trace(x)}$
2. $E^0 = I$,
3. $(E^X)^* = E^{X^*}$,
4. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$,
5. $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ *for all* $\alpha, \beta \in \mathbb{C}$,

6. *If* $XY = YX$, *then* $e^{X+Y} = e^X e^Y = e^Y e^X$, *and*

7. *if* C *is invertible, then* $e^{CXC^{-1}} = Ce^X C^{-1}$.

Consult [6] for a proof of this theorem. We will use these properties extensively in the examples of Lie algebras to follow.

We now proceed to a presentation of Lie algebras. Though we introduce a Lie algebra in general, we are almost exclusively concerned with the Lie algebra $\mathfrak{spin}(n)$ defined in the next section.

Definition 3.1.7. Let G be a matrix Lie group. The *Lie algebra of* G, denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all real numbers t. △

Lie algebras are a useful tool in studying Lie groups, as they encode much of the information about their corresponding Lie groups. Lie algebras are nicer to work with because they are linear spaces. Specifically, a Lie algebra is the tangent space to a Lie group at the identity. In practice, we will compute a Lie algebra by differentiating paths of the form e^{tX} at $t = 0$. Though we will not delve further into the properties of Lie algebras, it is important to note that the discussion of Lie algebras is quite rich. Consult [6] for a more developed exposition on Lie algebras of matrix Lie groups.

We begin with two standard examples of Lie algebras. In each example, we determine the Lie algebras of given Lie groups.

Example 3.1.8. In this example, we find the Lie algebras of $\mathrm{GL}(n;\mathbb{R})$ and $\mathrm{GL}(n;\mathbb{C})$. If X is any $n \times n$ real matrix, then e^{tX} will be invertible and real by Proposition 3.1.6. Similarly, if e^{tX} is a real $n \times n$ invertible matrix, then $X = \frac{d}{dt}e^{tX}|_{t=0}$ will also be real. Therefore, the Lie algebra of $\mathrm{GL}(n;\mathbb{R})$, denoted $\mathfrak{gl}(n;\mathbb{R})$, is the space of all $n \times n$ real matrices. Identically, the Lie algebra of $\mathrm{GL}(n;\mathbb{C})$, denoted $\mathfrak{gl}(n;\mathbb{C})$ is the space of all $n \times n$ complex matrices. ◊

Example 3.1.9. Recall that an invertible $n \times n$ matrix A is an element of $\mathrm{U}(n)$ if and only if $A^*A = I$. Then e^{tX} is unitary if and only if

$$(e^{tX})^* = (e^{tX})^{-1}.$$

Applying Proposition 3.1.6, we see that e^{tX} is unitary if and only if

$$e^{tX^*} = e^{-tX}.$$

By differentiating at $t = 0$, we can conclude that this holds if and only if $X^* = -X$. It follows that the Lie algebra of $\mathrm{U}(n)$, denoted $\mathfrak{u}(n)$, is the space of $n \times n$ complex matrices X such that $X^* = -X$. Matrices satisfying this property are said to be skew-Hermitian. Applying Proposition 3.1.6, we can further conclude that the Lie algebra of $\mathrm{SU}(n)$, denoted $\mathfrak{su}(n)$, is the space of $n \times n$ complex matrices X such that $X^* = -X$ and $\mathrm{trace}(X) = 0$. $\diamondsuit$

This paper is primarily concerned with the Lie algebra of the Lie group $\mathrm{Spin}(n)$. It turns out that $\mathrm{Spin}(n)$ shares its Lie algebra with $\mathrm{SO}(n)$, which will be shown in the following section. It for this reason that we now present an example that identifies the Lie algebra of $\mathrm{SO}(n)$.

Example 3.1.10. Recall that an $n \times n$ matrix A is orthogonal if and only if $A^T = A^{-1}$. Given an $n \times n$ real matrix X, it follows from Proposition 3.1.6 that e^{tX} is orthogonal if and only if

$$e^{tX^T} = e^{-tX}.$$

Differentiating at $t = 0$, we see that this holds if and only if $X^T = -X$. So the Lie algebra of $\mathrm{O}(n)$, denoted $\mathfrak{o}(n)$, is the space of $n \times n$ real matrices X with $X^T = -X$. Matrices satisfying this property are said to be skew-symmetric. Observe that $X^T = -X$ implies that,

$$\mathrm{trace}(X) = \mathrm{trace}(X^T) = \mathrm{trace}(-X) = -\mathrm{trace}(X).$$

It follows that the condition $X^T = -X$ forces the trace of X to be 0. By Proposition 3.1.6, this implies that the lie algebra of $\mathrm{O}(n)$ is also the Lie algebra of $\mathrm{SO}(n)$. Then both $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$ refer to the space of $n \times n$ real matrices X such that $X^T = -X$. $\Diamond$

The "bracket" operation is naturally defined on Lie algebras and is used extensively in their study. In fact, in alternate definitions a Lie algebra is defined as a vector space equipped with a bracket operation.

Definition 3.1.11. Given two $n \times n$ matrices A and B, the *bracket* of A and B is defined to be

$$[A, B] = AB - BA.$$

Two matrices commute with each other if and only if their bracket is zero. $\triangle$

An important property of Lie algebras is that they are closed under this operation, i.e. if $X, Y \in \mathfrak{g}$ then $XY - YX \in \mathfrak{g}$. This is seen in the following two examples.

Example 3.1.12. Recall from Example 3.1.9 that

$$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}.$$

Let $X, Y \in \mathfrak{u}(n)$. Observe that

$$([X, Y])^* = (XY - YX)^* = (XY)^* - (YX)^* = Y^*X^* - X^*Y^* = YX - XY = -[X, Y].$$

It follows that $[X, Y] \in \mathfrak{u}(n)$. $\Diamond$

Example 3.1.13. Recall from Example 3.1.10 that

$$\mathfrak{o}(n) = \{X \in M_n(\mathbb{C}) \mid X^T = -X\}.$$

Let $X, Y \in \mathfrak{o}(n)$. Observe that

$$([X, Y])^T = (XY - YX)^T = (XY)^T - (YX)^T = Y^T X^T - X^T Y^T = YX - XY = -[X, Y].$$

It follows that $[X, Y] \in \mathfrak{o}(n)$. $\Diamond$

We proceed to the discussion of Lie algebra homomorphisms. Note that a Lie group homomorphism is simply a group homomorphism between Lie groups. Using our understanding of the bracket operation, Lie algebra homomorphisms are easy to define.

Definition 3.1.14. Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism if it is a linear map such that

$$\phi([X, Y]) = [\phi(X), \phi(Y)],$$

for all $X, Y \in \mathfrak{g}$. $\triangle$

It is a very useful property of Lie algebra homomorphisms that they uniquely descend from Lie group homomorphisms. This is shown in the following theorem.

Theorem 3.1.15. [6] *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose that $\Phi : G \longrightarrow H$ is a Lie group homomorphism. Then there exists a unique real linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that*

$$\Phi(e^X) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$. The map ϕ has the following additional properties:

1. $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$, *for all* $X \in \mathfrak{g}, A \in G$,
2. $\phi([X, Y]) = [\phi(X), \phi(Y)]$, *for all* $X, Y \in \mathfrak{g}$, *and*
3. $\phi(X) = \frac{d}{dt}\Phi(e^{tX})|_{t=0}$, *for all* $X \in \mathfrak{g}$.

The implication is that every Lie group homomorphism gives rise to a Lie algebra homomorphism. The converse turns out to be true only under special circumstances. One can use Property 3 to compute a Lie algebra homomorphism from a Lie group homomorphism. Consult [6] for a proof of this theorem. Our next example nicely illustrates this property of Lie algebra homomorphisms.

Example 3.1.16. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then for each $A \in G$, define a linear map $\mathrm{Ad}_A : \mathfrak{g} \longrightarrow \mathfrak{g}$ by the formula

$$\mathrm{Ad}_A(X) = AXA^{-1}.$$

If $\mathfrak{g}$ is a vector space of dimension k, let $\mathrm{GL}(\mathfrak{g})$ denote the Lie group $\mathrm{GL}(k, \mathbb{R})$. Define $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ by $\mathrm{Ad}(A) = \mathrm{Ad}_A$. It can be checked that Ad is a Lie group homomorphism. By Theorem 3.1.15, there exists a unique Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ such that

$$e^{\mathrm{ad}_X} = \mathrm{Ad}(e^X).$$

Denote $\mathrm{ad}(X)$ by ad_X. Applying Theorem 3.1.15 again, we observe that

$$\mathrm{ad}_X(Y) = \frac{d}{dt}\mathrm{Ad}(e^{tX})(Y)|_{t=0} = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} = [X, Y].$$

This is a nice property of ad that will be referenced later in this paper. $\Diamond$

Next, we introduce Cartan subalgebras of Lie algebras. Consult [6] for a more in-depth discussion of Cartan subalgebras.

Definition 3.1.17. If $\mathfrak{g}$ is a Lie algebra, then a *Cartan subalgebra* of $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that

1. $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$,
2. if $[H, X] = 0$ for all $H \in \mathfrak{h}$ then $X \in \mathfrak{h}$, and
3. ad_H is diagonalizable for all $H \in \mathfrak{h}$.

$\triangle$

Conditions 1 and 2 imply that a Cartan subalgebra is a maximal commuting subalgebra. The implication of Condition 3 is less obvious. Recall from linear algebra that a set of matrices is simultaneously diagonalizable if it has a basis $v_1, \ldots, v_n$ such that each v_i is an eigenvector for every matrix in the set. It is a standard result that any commuting family of diagonalizable matrices is simultaneously diagonalizable. It follows that Condition 3 guarantees that a Cartan subalgebra is simultaneously diagonalizable. For a semisimple Lie algebra, Conditions 1 and 2 imply Condition 3. Though we do not define semisimple Lie algebras, all Lie algebras considered in this paper are semisimple. Further explanation on semisimple Lie algebras can be found in [6]. The Cartan subalgebras of semisimple Lie algebras all have the same dimension. The rank of a semisimple Lie algebra is defined as the dimension of its Cartan subalgebras.

We conclude this section with a presentation of a useful theorem regarding group actions and a discussion of its implications to Lie theory. Recall that $\mathcal{O}_x$ and G_x denote the orbit and stabilizer of x, respectively, under the action of G.

Theorem 3.1.18. *Let G be a group acting on a set X. For any $x \in X$, there is a one-to-one correspondence between $\mathcal{O}_x$ and G/G_x.*

Proof. Define $\varphi : G/G_x \longrightarrow \mathcal{O}_x$ as $\varphi(gG_x) = g \cdot x$. Observe that

$$\begin{aligned} gG_x = hG_x &\iff h^{-1}gG_x = G_x \\ &\iff h^{-1}g \in G_x \\ &\iff h^{-1}g \cdot x = x \\ &\iff g \cdot x = h \cdot x \\ &\iff \varphi(gG_x) = \varphi(hG_x) \end{aligned}$$

It follows that φ is well-defined and injective. It is clear that φ is surjective. Therefore, there is a one-to-one correspondence between $\mathcal{O}_x$ and G/G_x. $\square$

We get a nice corollary from the proof of this theorem because the map defined between $\mathcal{O}_x$ and G/G_x turns out to be a diffeomorphism. In particular, we can identify the dimension of orbits under Lie group actions by working at the Lie algebra level.

Corollary 3.1.19. *Let G be a Lie group acting on a set X, and let $\mathfrak{g}$ be the Lie algebra of G. If $\mathcal{O}_x$ denotes the orbit of x under G and $\mathfrak{g}_x$ denotes the stabilizer of x under $\mathfrak{g}$, then*

$$\dim(\mathcal{O}_x) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_x).$$

We will use this corollary heavily in later computations.

While this section focused on Lie algebras in a general context, the next section discusses the Lie group $\mathrm{Spin}(n)$ and its relationship to Clifford algebras.

3.2 Constructing $\mathrm{Spin}(n)$ as a subgroup of $\mathrm{Cl}(n, \mathbb{C})$

In this section, we outline the construction of $\mathrm{Spin}(n)$ as a subgroup of the invertible elements in $\mathrm{Cl}(n, \mathbb{C})$. In doing so, we demonstrate that $\mathrm{Spin}(n)$ shares its Lie algebra with $\mathrm{SO}(n)$. We will find that the Lie algebra $\mathfrak{spin}(n)$ is a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$. This section roughly follows the notes from [7] but also pulls material from [1].

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis for $\mathbb{C}^n$ as a complex vector space. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ multiplicatively generates a basis for $\mathrm{Cl}(n, \mathbb{C})$ with multiplication subject to $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ and $\mathbf{e}_i^2 = 1$. Let $\mathrm{Cl}^\times(\mathbb{C}, n)$ denote the subgroup of invertible elements in $\mathrm{Cl}(n, \mathbb{C})$. Recall from Example 3.1.16 that $\mathrm{Ad} : \mathrm{Cl}^\times(\mathbb{C}, n) \to \mathrm{Aut}(\mathrm{Cl}(n, \mathbb{C}))$ is defined such that $\mathrm{Ad}_x(y) = xyx^{-1}$. We are interested in identifying the image of Ad acting on unit vectors in $\mathbb{C}^n$. As an example, observe that

$$\begin{aligned}
\mathrm{Ad}_{\mathbf{e}_i}(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) =& \mathbf{e}_i(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n)\mathbf{e}_i \\
=& a_1\mathbf{e}_i\mathbf{e}_1\mathbf{e}_i + \cdots + a_n\mathbf{e}_i\mathbf{e}_n\mathbf{e}_i \\
=& -a_1\mathbf{e}_1 - \cdots - a_{i-1}\mathbf{e}_{i-1} + a_i\mathbf{e}_i - a_{i+1}\mathbf{e}_{i+1} - \cdots - a_n\mathbf{e}_n.
\end{aligned}$$

It follows that $\mathrm{Ad}_{\mathbf{e}_i}$ can be restricted to a reflection on $\mathbb{C}^n$, up to a sign. It is shown in [7] that this holds for any unit vector in $\mathbb{C}^n$. In particular, if $\mathbf{u} \in \mathbb{C}^n$ is a unit vector then $\mathrm{Ad}_{\mathbf{u}}$ is a reflection through a single hyperplane. Because Ad is a group homomorphism, we can conclude that for unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$,

$$\mathrm{Ad}_{\mathbf{u}_1 \cdots \mathbf{u}_n} = \mathrm{Ad}_{\mathbf{u}_1} \circ \cdots \circ \mathrm{Ad}_{\mathbf{u}_n}.$$

This means $\mathrm{Ad}_{\mathbf{u}_1 \cdots \mathbf{u}_n}$ is the composition of n reflections through hyperplanes.

Before proceeding, we define $\mathrm{Spin}(n)$ as a subgroup of $\mathrm{Cl}^{\times}(n, \mathbb{C})$. This will allow us to consider the image of Ad on $\mathrm{Spin}(n)$.

Definition 3.2.1. The group $\mathrm{Pin}(n) \subset \mathrm{Cl}^{\times}(\mathbb{C}, n)$ is the subgroup generated by unit vectors in $\mathbb{C}^n$. The group $\mathrm{Spin}(n)$ is defined as

$$\mathrm{Spin}(n) = \mathrm{Pin(n)} \cap A_0,$$

which consists of even products. △

It follows from our previous discussion that the image of $\mathrm{Pin}(n)$ is the space of reflections on $\mathbb{C}^n$ consisting of reflections through hyperplanes. Furthermore, the image of $\mathrm{Spin}(n)$ is the space of reflections on $\mathbb{C}^n$ consisting of an even number of reflections through hyperplanes. The Cartan-Dieudonné Theorem allows us to identify this space.

Theorem 3.2.2 (Cartan-Dieudonné Theorem)**.** *Every element of $O(n)$ can be expressed as the product of at most n reflections through hyperplanes, and $\mathrm{SO}(n)$ consists of all elements of $O(n)$ that are a product of an even number of such reflections.*

This means that the image of Ad restricted $\mathrm{Spin}(n)$ is $\mathrm{SO}(n)$. Then we can consider the restriction $\mathrm{Ad} : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. We can conclude that this restriction is onto from our previous discussion. It is shown in [7] that the kernel of this restriction is $\{-1, 1\}$, implying that it is two-to-one. This means that $\mathrm{SO}(n)$ is a double cover of $\mathrm{Spin}(n)$. This is presented in the following theorem.

Theorem 3.2.3. *The spin group $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ with a covering map given by $\mathrm{Ad} : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ defined as $\mathrm{Ad}_x(y) = xyx^{-1}$.*

A rigorous proof of this theorem can be found in [1] and [7].

Since Ad is a double covering map and a group homomorphism, the induced map ad : $\mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ is a Lie algebra isomorphism. It follows that $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$. In the following theorem, we show that $\mathfrak{spin}(n)$ is a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$.

Theorem 3.2.4. *The Lie algebra $\mathfrak{spin}(n)$ is a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$ defined as $\mathfrak{spin}(n) = \mathrm{span}\{\boldsymbol{e}_i \boldsymbol{e}_j\}_{i<j\leq n}$.*

Proof. First, we demonstrate that $\mathbf{e}_i\mathbf{e}_j$ is an element of $\mathfrak{spin}(n)$. Observe that

$$\begin{aligned} e^{t\mathbf{e}_i\mathbf{e}_j} = \sum_{n=0}^{\infty} \frac{t^n(\mathbf{e}_i\mathbf{e}_j)^n}{n!} &= 1 + t\mathbf{e}_i\mathbf{e}_j - \frac{t^2}{2!} - \frac{t^3\mathbf{e}_i\mathbf{e}_j}{3!} + \frac{t^4}{4!} + \frac{t^5\mathbf{e}_i\mathbf{e}_j}{5!} \cdots \\ &= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots) + (t\mathbf{e}_i\mathbf{e}_j - \frac{t^3\mathbf{e}_i\mathbf{e}_j}{3!} + \frac{t^5\mathbf{e}_i\mathbf{e}_j}{5!} \cdots) \\ &= \cos(t) + \sin(t)\mathbf{e}_i\mathbf{e}_j. \end{aligned}$$

For all t, the curve $\cos(t) + \sin(t)\mathbf{e}_i\mathbf{e}_j$ is a unit vector in $\mathbb{C}^n$ contained in the even part of $\mathrm{Cl}(n, \mathbb{C})$. This means that $\cos(t) + \sin(t)\mathbf{e}_i\mathbf{e}_j$ is an element of $\mathrm{Spin}(n)$, implying that $\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n} \subset \mathfrak{spin}(n)$. The set $\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n}$ is linearly independent and has $\binom{n}{2} = \frac{n(n-1)}{2}$ elements. The dimension of $\mathfrak{so}(n)$ is $\frac{n(n-1)}{2}$ because it is the set of skew-symmetric matrices, which are determined by the entries above or below the diagonal. Then the dimension of $\mathfrak{spin}(n)$ is also $\frac{n(n-1)}{2}$, implying that $\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n}$ is a basis for $\mathfrak{spin}(n)$. Therefore, $\mathfrak{spin}(n)$ is a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$ defined as $\mathfrak{spin}(n) = \mathrm{span}\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n}$. □

In the previous section, we defined the bracket operation on Lie algebras. In the following example, we show that the bracket operation is closed on $\mathfrak{spin}(n) = \mathrm{span}\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n}$.

Example 3.2.5. Consider $[\mathbf{e}_i\mathbf{e}_j, \mathbf{e}_k\mathbf{e}_l] = \mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l - \mathbf{e}_k\mathbf{e}_l\mathbf{e}_i\mathbf{e}_j$. There are multiple cases to consider. If $i \neq k, l$ and $j \neq k, l$, then

$$\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l - \mathbf{e}_k\mathbf{e}_l\mathbf{e}_i\mathbf{e}_j = 0.$$

If $i = k$ and $j \neq l$, without loss of generality, then

$$\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l - \mathbf{e}_k\mathbf{e}_l\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_l - \mathbf{e}_j\mathbf{e}_l = -2\mathbf{e}_j\mathbf{e}_l.$$

If $i = k$ and $j = l$, without loss of generality, then

$$\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l - \mathbf{e}_k\mathbf{e}_l\mathbf{e}_i\mathbf{e}_j = -1 + 1 = 0.$$

It follows that $\mathfrak{spin}(n) = \mathrm{span}\{\mathbf{e}_i\mathbf{e}_j\}_{i<j\leq n}$ is closed under the bracket operation. $\Diamond$

This paper will primarily treat $\mathfrak{spin}(n)$ as a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$, but will frequently use the fact that $\mathfrak{spin}(n)$ is also the Lie algebra of $\mathrm{Spin}(n)$.

Now that we have constructed the group $\mathrm{Spin}(n)$, we are ready to give seven useful and well-known isomorphisms:

$$\begin{aligned}
&\mathrm{Spin}(1) \cong \{\pm 1\},\\
&\mathrm{Spin}(2) \cong \mathrm{U}(n) \cong S^1,\\
&\mathrm{Spin}(3) \cong \mathrm{SU}(2),\\
&\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2),\\
&\mathrm{Spin}(5) \cong \mathrm{Sp}(2),\\
&\mathrm{Spin}(6) \cong \mathrm{SU}(4),\\
&S^7 \cong \mathrm{Spin}(7)/G_2.
\end{aligned}$$

A construction of these isomorphisms can be found in [8]. Though not defined in this paper, $\mathrm{Sp}(2)$ is the compact symplectic group and G_2 is an exceptional Lie group. Consult [6] for a definition of these two Lie groups.

The results of this section allow us to treat $\mathrm{Spin}(n)$ as both a subgroup of $\mathrm{Cl}(n, \mathbb{C})$ and also the double cover of $\mathrm{SO}(n)$. Additionally, we now have an explicit basis for $\mathfrak{spin}(n)$ as a vector subspace of $\mathrm{Cl}(n, \mathbb{C})$. These properties will be quite useful in later chapters.

3.3 A Bit of Representation Theory

In this section, we present background material from representation theory. We focus solely on representation theory of Lie groups and Lie algebras. A rigorous investigation into the material from this section can be found in [5] and [6].

We begin with an introduction to representations in general, which will loosely follow material from Chapter 4 of [6].

Definition 3.3.1. Let G be a matrix Lie group and let $\mathfrak{g}$ be a Lie algebra. Then, a *finite-dimensional complex representation* of G is Lie group homomorphism

$$\Pi : G \longrightarrow \mathrm{GL}(n; \mathbb{C}).$$

Similarly, a *finite-dimensional complex representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(n; \mathbb{C}).$$

Abusing notation, we often refer to the vector space $\mathbb{C}^n$ as the representation. △

This definition is specific to Lie groups and Lie algebras, but can easily be generalized. We have actually already seen an example of a representation, as described in the following example.

Example 3.3.2. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let Ad and ad be defined as in Example 3.1.16. In this example, we showed that Ad and ad are Lie group and Lie algebra homomorphisms, respectively. Consequently, Ad is a representation of G on $\mathfrak{g}$ and ad is a representation of $\mathfrak{g}$ on itself. We call ad the adjoint representation of $\mathfrak{g}$. ◊

We proceed to define the weights of a representation. The weights of a Lie algebra representation are defined in reference to a Cartan subalgebra.

Definition 3.3.3. Let π be an finite-dimensional complex representation of a Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A *weight* of π is a map $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ such that there exists a nonzero $v \in \pi(\mathfrak{g})$ with

$$\pi(H)v = \lambda(H)v$$

for all $H \in \mathfrak{h}$. △

If v is an eigenvector for each element of $\mathfrak{h}$, then a weight encodes the eigenvalues associated to v for all of $\mathfrak{h}$. Recall that such eigenvectors exist because Cartan subalgebra are defined to be simultaneously diagonalizable. When there is no risk of confusion, we refer to weights as n-tuples whose entries are these eigenvalues corresponding to each basis element of $\mathfrak{h}$. Building from our understanding of the weights of Lie algebra representations, we can easily define the roots of a Lie algebra.

Definition 3.3.4. The *roots* of a Lie algebra $\mathfrak{g}$ are the nonzero weights of the adjoint representation of $\mathfrak{g}$. △

There is a useful interaction between weights and roots. Let π be a representation of $\mathfrak{g}$ into a vector space V, implicitly defining an action of $\mathfrak{g}$ on V. If $X \in \mathfrak{g}$ has root k and $v \in V$ has weight l, then either: 1) Xv has weight $k + l$ if $k + l$ is a weight of π, or 2) $Xv = 0$ if $k + l$ is not a weight of π.

We are primarily concerned with weights and roots of the Lie algebra $\mathfrak{spin}(n)$. We turn to $\mathfrak{spin}(3)$ as our next example.

Example 3.3.5. The Lie algebra $\mathfrak{spin}(3)$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$, which is the set of 2×2 complex matrices X such that $X^* = -X$ and $\operatorname{trace}(X) = 0$. We begin with a convenient basis for $\mathfrak{su}(2)$, given by

$$H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Note that H spans a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(2)$. To find the roots of $\mathfrak{su}(2)$, we must identify the weights of the adjoint representation. Recall from Example 3.1.16 that $\mathrm{ad}_A(B) = [A, B]$ for all $A, B \in \mathfrak{g}$. Using this fact, it can be checked that

$$\mathrm{ad}_H(X) = [H, X] = X \quad \text{and} \quad \mathrm{ad}_H(Y) = [H, Y] = -Y.$$

It follows that the weights of the adjoint representation of $\mathfrak{su}(2)$ are 1 and -1. Consequently, the roots of both $\mathfrak{su}(2)$ and $\mathfrak{spin}(3)$ are 1 and -1. $\diamondsuit$

In the preceding example, we analyzed the roots of $\mathfrak{spin}(3)$. In higher dimension, the computation is similar but gets complicated quickly.

We proceed to describe the roots of the Lie algebras $\mathfrak{spin}(2n)$ and $\mathfrak{spin}(2n + 1)$ in general. As the computation is quite involved, we simply state the roots and weights as given. Consult [5] for a more thorough explanation. First, we need to identify the Cartan subalgebra. Recall that $\mathfrak{spin}(2n) \subset \mathrm{Cl}(2n, \mathbb{C})$ and $\mathfrak{spin}(2n + 1) \subset \mathrm{Cl}(2n + 1, \mathbb{C})$. Then for both $\mathfrak{spin}(2n)$ and $\mathfrak{spin}(2n + 1)$, the Cartan subalgebra $\mathfrak{h}$ is defined as

$$\mathfrak{h} = \mathrm{span}\{\gamma_1\gamma_2, \gamma_3\gamma_4, \ldots, \gamma_{2n-1}\gamma_{2n}\}.$$

We leave it to the reader to verify that $\mathfrak{h}$ is a maximal commuting subalgebra. Note that $\mathfrak{h}$ is n-dimensional in both cases. Consequently, we list the roots of both $\mathfrak{spin}(2n)$ and $\mathfrak{spin}(2n + 1)$ as n-tuples in which the ith entry corresponds to an eigenvalue of the ith basis element of $\mathfrak{h}$. An n-tuple is a root of $\mathfrak{spin}(2n)$ if only two entries are nonzero and each nonzero entry is either 1 or -1. An n-tuple is a root of $\mathfrak{spin}(2n + 1)$ if either 1) it is a root of $\mathfrak{spin}_{\mathbb{C}}(2n)$, or 2) one entry is either a 1 or -1 and all other entries are zero. This can be seen in the following example.

Example 3.3.6. The roots of $\mathfrak{spin}(4)$ are

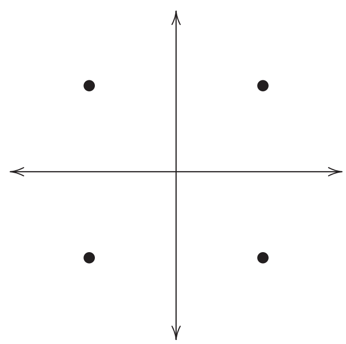
$$(1, -1), \quad (1, 1), \quad (-1, 1), \quad \text{and} \quad (-1, -1).$$

The roots of $\mathfrak{spin}(5)$ are

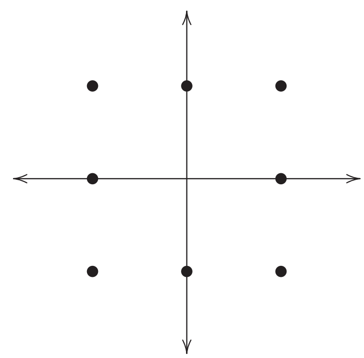
$$(1,-1),\ \ (1,1),\ \ (-1,1),\ \ (-1,-1),\ \ (0,1),\ \ (0,-1),\ \ (1,0),\ \ \text{and}\ \ (-1,0).$$

Observe that the roots of $\mathfrak{spin}(5)$ include the roots of $\mathfrak{spin}(4)$.

Roots of $\mathfrak{spin}(4)$



Roots of $\mathfrak{spin}(5)$



Frequently, we use diagrams that display roots as n-tuples in $\mathbb{R}^n$ to help visualize their behavior. This is shown in the preceding figures. ◊

Though we are stating the roots of complex Lie algebras, we can use these roots to define the corresponding real Lie algebra in a useful way. Suppose we know the roots of the complex Lie algebra $\mathfrak{spin}(2n)$ or $\mathfrak{spin}(2n+1)$. Let $E_{ij}^{\pm\pm}$ be an element of $\mathfrak{spin}(2n)$ or $\mathfrak{spin}(2n+1)$ whose root has nonzero entries of ± 1 in the ith entry and ± 1 in the jth entry, as dictated by the superscript. Additionally, let $E_i^{\pm}$ be the element of $\mathfrak{spin}(2n+1)$ whose ith entry is either ± 1, as dictated by the superscript. Finally, let $\mathfrak{h} = \operatorname{span}\{H_1, \ldots, H_n\}$ be the Cartan subalgebra of $\mathfrak{spin}(2n)$ and $\mathfrak{spin}(2n+1)$. Then we can choose roots such that the real Lie algebra $\mathfrak{spin}(2n)$ is defined as

$$\begin{aligned}\mathfrak{spin}(2n) = {} & \operatorname{span}\{iH_1, \ldots, iH_n\} \\ & + \operatorname{span}\{E_{ij}^{++} - E_{ij}^{--}, iE_{ij}^{++} + iE_{ij}^{--}, E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq n}.\end{aligned}$$

Similarly, we can choose roots such that the real Lie algebra $\mathfrak{spin}(2n+1)$ is defined as

$$\begin{aligned}\mathfrak{spin}(2n+1) = &\operatorname{span}\{iH_1, \ldots, iH_n\} + \operatorname{span}\{E_i^+ - E_i^-, iE_i^+ + iE_i^-\}_{i \leq n} \\ &+ \operatorname{span}\{E_{ij}^{++} - E_{ij}^{--}, iE_{ij}^{++} + iE_{ij}^{--}, E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq n}.\end{aligned}$$

Building from a previous example, we see that it is fairly easy to construct $\mathfrak{spin}(n)$ in this manner.

Example 3.3.7. Recall from Example 3.3.5 that $\mathfrak{spin}(3)$ is isomorphic to $\mathfrak{su}(2)$, which has a basis given by

$$H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In this example, we saw that X has root 1 and Y and root -1. Also, we are given that H spans the Cartan subalgebra. Using this information, we can define the real Lie algebra $\mathfrak{spin}(3)$ as

$$\mathfrak{spin}(3) = \operatorname{span}\{iH, X - Y, iX + iY\}.$$

$\diamondsuit$

This definition of $\mathfrak{spin}(n)$ is helpful because it preserves the information given from explicitly knowing the roots of a Lie algebra.

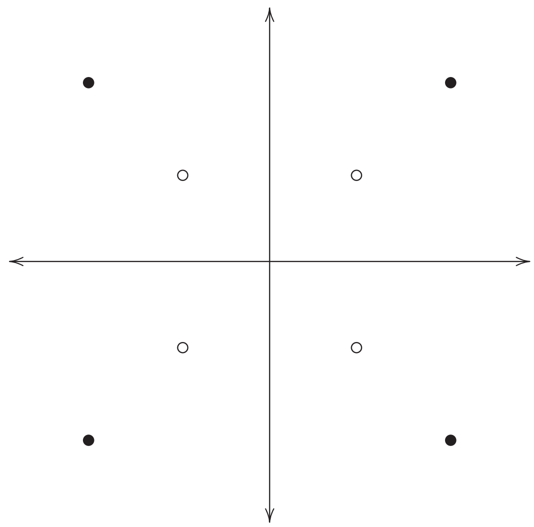
We proceed to an introduction of the spin representation. The spin representation S is a representation of the complex Lie algebra $\mathfrak{spin}(n)$. The construction of S is beyond the scope of this paper, so we simply describe S and invite the curious reader to consult [5] for its construction. It is easiest to describe S by its weight decomposition. If S is a representation of either $\mathfrak{spin}(2n)$ or $\mathfrak{spin}(2n+1)$, then the weights of S are n-tuples in which each entry is either $\frac{1}{2}$ or $-\frac{1}{2}$. This is easily seen in the following example.

Example 3.3.8. The weights of the spin representation of $\mathfrak{spin}(4)$ are

$$\left(\frac{1}{2}, \frac{1}{2}\right) \ \left(\frac{1}{2}, -\frac{1}{2}\right), \ \left(-\frac{1}{2}, \frac{1}{2}\right), \ \text{and} \ \left(-\frac{1}{2}, -\frac{1}{2}\right).$$

Recall that the roots of $\mathfrak{spin}(4)$ are $(1,1),(1,-1),(-1,1)$, and $(-1,-1)$.

Roots of $\mathfrak{spin}(4)$ and Weights of its Spin Representation



It is often instructive to display the weights and roots in the same diagram as n-tuples in $\mathbb{R}^n$, as done above. $\Diamond$

Recall that each weight is a list of eigenvalues. Let $\mathbb{C}^{\pm\pm\cdots\pm}$ denote the eigenspace over $\mathbb{C}$ of the weight with either $\frac{1}{2}$ or $-\frac{1}{2}$ in each entry, as determined by the list of signs in the superscript. For example, $\mathbb{C}^{+-++}$ is the eigenspace of the weight $(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Then S is the vector space direct sum of all possible eigenspaces corresponding to the weights of S. To demonstrate, we provide an example.

Example 3.3.9. As shown above, the weights of the spin representation of $\mathfrak{spin}(4)$ are

$$\left(\frac{1}{2},\frac{1}{2}\right)\ \left(\frac{1}{2},-\frac{1}{2}\right),\ \left(-\frac{1}{2},\frac{1}{2}\right),\quad\text{and}\quad\left(-\frac{1}{2},-\frac{1}{2}\right).$$

Then $S=\mathbb{C}^{++}\oplus\mathbb{C}^{+-}\oplus\mathbb{C}^{-+}\oplus\mathbb{C}^{--}$. . $\Diamond$

For $\mathfrak{spin}(2n)$, we define the positive half-spin representations S^+ as the direct sum of all eigenspaces of weights with an even number of $+$ signs. We similarly define the negative half-spin representation S^- as the direct sum of all eigenspaces of weights with an odd number of $+$ signs. This is seen in the example below.

Example 3.3.10. In the previous example, we found that the spin representation of $\mathfrak{spin}(4)$ is given by $S = \mathbb{C}^{++} \oplus \mathbb{C}^{+-} \oplus \mathbb{C}^{-+} \oplus \mathbb{C}^{--}$. Then $S^+ = \mathbb{C}^{++} \oplus \mathbb{C}^{--}$ and $S^- = \mathbb{C}^{+-} \oplus \mathbb{C}^{-+}$. $\lozenge$

If S is the spin representation of $\mathfrak{spin}(2n)$, then S can be decomposed in the module direct sum

$$S = S^+ \oplus S^-.$$

If S is the spin representation of $\mathfrak{spin}(2n+1)$, then S is irreducible.

In [9], we find that S is closely related to the minimal ideals that will be presented in Section 3.4. If I is the minimal submodule defined as

$$I = \langle (1 + \gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n+1}) \rangle$$

of $\mathrm{Cl}(2n, \mathbb{C})$, then S is isomorphic to I. Furthermore, for $\mathfrak{spin}(2n)$, S^+ is isomorphic to the even part of I and S^- is isomorphic to the odd part of I. If I is the minimal submodule defined as

$$I = \langle (1 + \gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n+1}) \rangle$$

of $\mathrm{Cl}(2n+1, \mathbb{C})$, then S is isomorphic to both the even and odd part of I.

We conclude this section with a theorem from algebraic topology that will allow us to use the tools from representation theory presented in this section.

Theorem 3.3.11. *Let X be a connected n-dimensional manifold. The only closed n-dimensional submanifold of X is X itself.*

The relevant implication to this theorem lies in determining when an action is transitive. Specifically, if the orbit of any element of A_0 under the action of $\mathrm{Spin}(n)$ is maximal then this action is transitive.

3.4 Equivalences from the Action of Spin(n)

Pulling together the extensive background developed in this chapter, we are ready to introduce the primary method we will use to analyze filtrations. The following theorem served as the inspiration for the topic of this paper.

Theorem 3.4.1 (Landweber, et. al [11]). *Let V be an irreducible complex Clifford supermodule. If $f : V \longrightarrow V$ and $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are a pair of equivalence functions on V, then there exists some $\varphi \in \mathrm{Spin}(n)$ and $c \in \mathbb{C}$ such that*

$$f(v) = c\varphi v \quad \textit{and} \quad g(\gamma) = \varphi\gamma\varphi^{-1}.$$

We note that the proof of this theorem rests on Schur's Lemma, for those familiar with it. We get the following corollary.

Corollary 3.4.2. *Let V be an irreducible Clifford supermodule over $\mathrm{Cl}(n, \mathbb{C})$, and let $\mathrm{Fil}(V)$ be the space of filtrations of a fixed type defined on V. Then an action of $\mathrm{Spin}(n)$ on $\mathrm{Fil}(V)$ maps an element of $\mathrm{Fil}(V)$ to another element of $\mathrm{Fil}(V)$ if and only if these two elements correspond to equivalent filtrations.*

Proof. Let $F, F' \in \mathrm{Fil}(V)$. Then F and F' are equivalent if and only if there exists a pair of equivalence functions between them. By the previous theorem, it follows that F and F' are equivalent if and only if there is an element of $\mathrm{Spin}(n)$ mapping F to F'. $\square$

Filtration degrees are described by spanning over the elements of a vector space, making constants irrelevant. It if for this reason that we briefly turn our attention to complex projective spaces.

Definition 3.4.3. The complex projective space $\mathbb{C}P^n$ is the space of lines in $\mathbb{C}^{n+1}$ passing through the origin. $\triangle$

It follows that elements of $\mathbb{C}P^n$ can be described as ordered $n+1$-tuples, often denoted as $[x_0 : x_1 : \cdots : x_n]$, such that $[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$ for any $\lambda \in \mathbb{C}$. It turns out that $\mathrm{Fil}(V)$ is often (if not always) a complex projective space.

We will be concerned exclusively with a specific family of irreducible submodules. It is shown in [9] that the principal ideal

$$\langle (1+\gamma_1\gamma_2)(1+\gamma_3\gamma_4)\cdots(1+\gamma_{2n-1}\gamma_{2n})\rangle$$

is minimal in both $\mathrm{Cl}(2n,\mathbb{C})$ and $\mathrm{Cl}(2n+1,\mathbb{C})$. From our previous discussion in Section 2.1, it follows that ideals of this form are also irreducible submodules of $\mathrm{Cl}(2n,\mathbb{C})$ and $\mathrm{Cl}(2n+1,\mathbb{C})$.

This paper seeks to understand the structure of the moduli space of filtrations on this family of irreducible submodules. This amounts to fixing our submodule and then determining the equivalence classes of filtrations on it. Using the tools presented in this chapter, we can rephrase this problem. To analyze these filtrations, we will look at the orbit of each filtration type on a fixed submodule under the action of $\mathrm{Spin}(n)$. These orbits will be our equivalence classes, and the space of these orbits will form our moduli space.

4

Low-Dimensional Cases

This chapter uses a somewhat brute-force approach to explore filtrations on minimal submodules of low-dimensional Clifford supermodules. This is only possible because the corresponding moduli spaces turn out to be discrete. Much of the computation in this chapter will be revisited in later chapters, primarily to juxtapose this early approach with more theoretical approaches presented in later chapters. The material presented in this section is of significance because it investigates our moduli spaces at the ground level, providing a concrete basis for the possibly more abstract computations to come later.

4.1 Trivial Moduli Spaces

We begin by considering $\mathrm{Cl}(0, \mathbb{C}) \cong \mathbb{C}$. Since $\mathbb{C}$ is a field, it follows that the only nontrivial ideal of $\mathrm{Cl}(0, \mathbb{C})$ is itself. Therefore, $\mathrm{Cl}(0, \mathbb{C})$ is an irreducible submodule of itself. The $\mathbb{Z}_2$-decomposition associated to $\mathrm{Cl}(0, \mathbb{C})$ is

$$\mathrm{Cl}(0, \mathbb{C}) = \mathbb{C} \oplus \{0\}.$$

Given this decomposition, it is easy to see that $\mathrm{Cl}(0, \mathbb{C})$ has no nontrivial filtrations.

Next, we proceed to $\mathrm{Cl}(1, \mathbb{C})$. In [9], we are given that $\mathrm{Cl}(1, \mathbb{C})$ is a minimal submodule of itself. Note that $\mathrm{Cl}(1, \mathbb{C})$ is equipped with the following $\mathbb{Z}_2$-decomposition:

$$\mathrm{Cl}(1, \mathbb{C}) = \mathrm{span}\{1\} \oplus \mathrm{span}\{\gamma_1\}.$$

It follows that the only nontrivial filtration type on this ideal is $1-2$, and there is exactly one filtration of this type. This means that there is exactly one filtration on this submodule.

We proceed to consider the $\mathrm{Cl}(2, \mathbb{C})$ case, which is similar to the previous case. From Section 3.4, we can conclude that $\langle 1 + i\gamma_1\gamma_2 \rangle$ is an irreducible submodule of $\mathrm{Cl}(2, \mathbb{C})$. The $\mathbb{Z}_2$-decomposition associated to this submodule is

$$\langle 1 + i\gamma_1\gamma_2 \rangle = \mathrm{span}\{1 + i\gamma_1\gamma_2\} \oplus \mathrm{span}\{\gamma_1 - i\gamma_2\}.$$

As before, the only nontrivial filtration type on this ideal is $1-2$, and there is exactly one filtration of this type. So there is exactly one filtration on this submodule.

Consequently, it is clear that the moduli space of filtrations is trivial on each of the submodules presented in this section.

4.2 Computing Discrete Moduli Spaces

Next, we consider the Clifford supermodule $\mathrm{Cl}(3, \mathbb{C})$. Recall from Section 3.4 that $\langle 1 + i\gamma_1\gamma_2 \rangle$ is a minimal submodule of $\mathrm{Cl}(3, \mathbb{C})$. This submodule is equipped with the following $\mathbb{Z}_2$-decomposition,

$$\langle 1 + i\gamma_1\gamma_2 \rangle = \mathrm{span}\{1 + i\gamma_1\gamma_2, \gamma_1\gamma_3 + i\gamma_2\gamma_3\} \oplus \mathrm{span}\{\gamma_1 + i\gamma_2, \gamma_3 + i\gamma_1\gamma_2\gamma_3\}.$$

The possible filtration types on this ideal are $1-3-4$ and $2-4$. We note that there is exactly one possible filtration of type $2-4$. Therefore, we need only consider equivalences between filtrations of type $1-3-4$.

Before proceeding, we enter into a brief discussion about the different methods to show that elements in a set of filtrations are all equivalent. At this stage, there are two ways to do this: 1) explicitly define equivalence functions between two arbitrary filtrations in the set, and 2) show that the action of $\mathrm{Spin}(n)$ on the set of filtrations is transitive. Because we are working in low dimensions, the first method is possible. However, we will present both methods because the second is helpful in higher dimensions.

We return to consider filtrations of type $1-3-4$. Filtrations of type $1-3-4$ are called principal filtrations, which will be defined in the following chapter. We claim that all $1-3-4$ filtrations on this submodule are equivalent, and proceed to show this in two ways. First, we will find equivalence functions between two arbitrary filtrations of this type.

Lemma 4.2.1. *Let $I = \langle 1 + i\gamma_1\gamma_2\rangle$ be the submodule of the Clifford supermodule* $\mathrm{Cl}(3, \mathbb{C})$. *All filtrations of type $1-3-4$ on I are equivalent.*

Proof. Filtrations of type $1-3-4$ are uniquely described by choosing F_0. Let F be a filtration of type $1-3-4$ with $F_0 = \mathrm{span}\{1 + i\gamma_1\gamma_2\}$. Let F' be an arbitrary filtration of type $1-3-4$. We can conclude that

$$F_0' = \mathrm{span}\{b(1 + i\gamma_1\gamma_2)\},$$

for some b in the even part of $\mathrm{Cl}(3, \mathbb{C})$. Then $b = c_1 + c_2\gamma_1\gamma_2 + c_3\gamma_1\gamma_3 + c_4\gamma_2\gamma_3$, for some $c_1, c_2, c_3, c_4 \in \mathbb{C}$. Observe that

$$\begin{aligned} b(1 + i\gamma_1\gamma_2) =& c_1(1 + i\gamma_1\gamma_2) + c_2\gamma_1\gamma_2(1 + i\gamma_1\gamma_2) + c_3\gamma_1\gamma_3(1 + i\gamma_1\gamma_2) + c_4\gamma_2\gamma_3(1 + i\gamma_1\gamma_2) \\ =& c_1(1 + i\gamma_1\gamma_2) - ic_2(1 + i\gamma_1\gamma_2) + c_3 i\gamma_1\gamma_3(1 + i\gamma_1\gamma_2) + c_4\gamma_1\gamma_3(1 + i\gamma_1\gamma_2) \\ =& (c_1 - ic_2)(1 + i\gamma_1\gamma_2) + (c_3 i + c_4)(\gamma_1\gamma_3)(1 + i\gamma_1\gamma_2). \end{aligned}$$

Define new constants $k_1 = c_1 - ic_2$ and $k_2 = c_3 i + c_4$. Let $b' = k_1 + k_2\gamma_1\gamma_3$. Then $b(1 + i\gamma_1\gamma_2) = b'(1 + i\gamma_1\gamma_2)$. It follows that

$$F_0' = \mathrm{span}\{b'(1 + i\gamma_1\gamma_2)\}.$$

Observe that $b' = (k_1\gamma_3 + k_2\gamma_1)(\gamma_3)$. From the construction of Spin(n) is Section 3.2, it follows that $b' \in$ Spin(3). Because Spin(3) is a subgroup of the invertible elements of Cl$(3, \mathbb{C})$, it follows that b' is invertible. We now explicitly define a pair of equivalence functions between F and F'. Let $f : \langle 1 + i\gamma_1\gamma_2\rangle \longrightarrow \langle 1 + i\gamma_1\gamma_2\rangle$ be given by $f(v) = b'v$, and let $g : \text{span}\{\gamma_1, \gamma_2, \gamma_3\} \longrightarrow \text{span}\{\gamma_1, \gamma_2, \gamma_3\}$ be given by $g(v) = b'vb'^{-1}$. Observe that

$$f(\gamma_i v) = b'\gamma_i v = b'\gamma_i b'^{-1} b' v = g(\gamma_i) f(v).$$

It follows that this pair of functions satisfies the first condition necessary to be a pair of equivalence functions. Because both F and F' are uniquely determined by the choice of F_0, it suffices to check that $f(F_0) = F_0'$ to verify the second condition. Observe that

$$f(F_0) = f(\text{span}\{1 + \gamma_1\gamma_2\}) = \text{span}\{b'(1 + \gamma_1\gamma_2)\} = \text{span}\{b(1 + \gamma_1\gamma_2)\} = F_0'.$$

Therefore, f, g is a pair of equivalence functions between F and F'. It follows that all filtrations of type $1 - 3 - 4$ are equivalent to F. □

As explained previously, this is only one method to arrive at this conclusion. This method was cumbersome, and not entirely enlightening. As we move to higher dimensions, such computations becomes nearly impossible. For this reason, we will rely on the second method for the remainder of this chapter.

Since filtrations of type $1-3-4$ are uniquely determined by the choice of a 1-dimensional subspace from a 2-dimensional space, it follows that the space of filtrations of type $1-3-4$ is isomorphic to $\mathbb{C}P^1$. Consequently, we would like to show that the action of Spin(3) on $\mathbb{C}P^1$ is transitive. Recall from Section 3.1 that Spin(3) is isomorphic to SU(2). It shown in [5] that there is exactly one nontrivial representation of SU(2) on $\mathbb{C}P^1$. It therefore suffices to show that there is a nontrivial action of SU(2) on $\mathbb{C}P^1$ that is transitive. We proceed to do exactly this in the following lemma.

Lemma 4.2.2. *Let $\varphi : \mathrm{SU}(2) \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be defined by the projective action on $\mathbb{C}P^1$ induced by the following action on $\mathbb{C}^2$:*

$$\left(A, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \mapsto A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Then φ is transitive.

Proof. Let $(w_1 : w_2)$ and $(z_1 : z_2)$ be arbitrary elements of $\mathbb{C}P^1$. Because we are working over projective space, we can assume that $|w_1|^2 + |w_2|^2 = 1$ and $|z_1|^2 + |z_2|^2 = 1$. Define A and B as

$$A = \begin{pmatrix} \overline{z_2} & z_1 \\ -\overline{z_1} & z_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} w_2 & -w_1 \\ \overline{w_1} & \overline{w_2} \end{pmatrix}.$$

Notice that

$$\begin{pmatrix} \overline{z_2} & z_1 \\ -\overline{z_1} & z_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_2 & -w_1 \\ \overline{w_1} & \overline{w_2} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that

$$AB \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A\left(B \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

To show that this action is transitive, it therefore suffices to check that AB is an element of $\mathrm{SU}(2)$. To do this, we demonstrate that both A and B are elements of $\mathrm{SU}(2)$. First, observe that

$$\det(A) = \overline{z_1} z_1 + \overline{z_2} z_2 = 1 \quad \text{and} \quad \det(B) = \overline{w_1} w_1 + \overline{w_2} w_2 = 1.$$

Next, we check that each matrix is unitary. We provide the following computation,

$$A^* A = \begin{pmatrix} z_2 & -z_1 \\ \overline{z_1} & \overline{z_2} \end{pmatrix} \begin{pmatrix} \overline{z_2} & z_1 \\ -\overline{z_1} & z_2 \end{pmatrix} = \begin{pmatrix} \overline{z_1} z_1 + \overline{z_2} z_2 & 0 \\ 0 & \overline{z_1} z_1 + \overline{z_2} z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly,

$$B^* B = \begin{pmatrix} \overline{w_2} & w_1 \\ -\overline{w_1} & w_2 \end{pmatrix} \begin{pmatrix} w_2 & -w_1 \\ \overline{w_1} & \overline{w_2} \end{pmatrix} = \begin{pmatrix} \overline{w_1} w_1 + \overline{w_2} w_2 & 0 \\ 0 & \overline{w_1} w_1 + \overline{w_2} w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that both A and B are elements of $\mathrm{SU}(2)$, and also AB is an element of $\mathrm{SU}(2)$. Therefore, $\varphi(AB, (w_1 : w_2)) = (z_1 : z_2)$. Consequently, this action is transitive. □

Applying this lemma, we can entirely classify filtrations of type $1-3-4$. This means that we fully understand the moduli space of filtrations on our minimal submodule.

Theorem 4.2.3. *Let* $\mathrm{Cl}(3,\mathbb{C})$ *be a module over itself. The moduli space of filtrations on the minimal submodule* $\langle 1+i\gamma_1\gamma_2\rangle$ *of* $\mathrm{Cl}(3,\mathbb{C})$ *consists of two discrete points.*

Proof. The only possible filtrations on this submodule are of type $2-4$ or type $1-3-4$. There is exactly one filtration of type $2-4$. Filtrations of type $1-3-4$ are uniquely determined by the choice of F_0. In turn, F_0 is uniquely described by choosing a 1-dimensional subspace from a 2-dimensional space. It follows that the space of possible filtrations of type $1-3-4$ is $\mathbb{C}P^1$. Lemma 4.2.2 states that there is a transitive action of $\mathrm{SU}(2)$ on $\mathbb{C}P^1$. Since there is exactly one representation of $\mathrm{SU}(2)$ on $\mathbb{C}P^1$ and $\mathrm{Spin}(3)$ is isomorphic to $\mathrm{SU}(2)$, we can conclude the action of $\mathrm{Spin}(3)$ on $\mathbb{C}P^1$ is transitive. It follows from Corollary 3.4.2 that all filtrations of type $1-3-4$ are equivalent.

This means that there are exactly two distinct nontrivial filtrations on the submodule $\langle 1+i\gamma_1\gamma_2\rangle$. Therefore, the moduli space of filtrations on this submodule consists of two discrete points. □

We now consider the Clifford supermodule $\mathrm{Cl}(4,\mathbb{C})$. Our computation is simplified by our results over $\mathrm{Cl}(3,\mathbb{C})$. The submodule $\langle(1+\gamma_1\gamma_2)(1+\gamma_3\gamma_4)\rangle$ is minimal in $\mathrm{Cl}(4,\mathbb{C})$, and has the associated $\mathbb{Z}_2$-decomposition

$$\begin{aligned}\langle(1+\gamma_1\gamma_2)(1+\gamma_3\gamma_4)\rangle =\ &\mathrm{span}\{1+i\gamma_1\gamma_2+i\gamma_3\gamma_4-\gamma_1\gamma_2\gamma_3\gamma_4, \gamma_1\gamma_3+i\gamma_1\gamma_4+i\gamma_2\gamma_3-\gamma_2\gamma_4\}\\ &\oplus \mathrm{span}\{\gamma_1+i\gamma_1\gamma_3\gamma_4+i\gamma_2-\gamma_2\gamma_3\gamma_4, \gamma_3+i\gamma_4+i\gamma_1\gamma_2\gamma_3-\gamma_1\gamma_2\gamma_4\}.\end{aligned}$$

As before, the only possible filtrations on this submodule are of type $1-3-4$ and $2-4$. There is exactly one filtration of type $2-4$, so we have left to consider filtrations of type $1-3-4$.

Filtrations of type $1-3-4$ on this submodule are uniquely determined by our choice of F_0. As in the previous case, this implies that the space of possible filtrations of this type is $\mathbb{C}P^1$. Therefore, we would like to show that the action Spin(4) on $\mathbb{C}P^1$ is transitive. In Section 3.1, we are given that Spin(4) is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. We can adapt our previous computation to show that the action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}P^1$ is transitive.

Lemma 4.2.4. *Let $\varphi : (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be the projective action on $\mathbb{C}P^1$ induced by the following action on $\mathbb{C}^2$:*

$$\left((A, B), \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \mapsto A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Then φ is transitive.

Proof. Let $(w_1 : w_2)$ and $(z_1 : z_2)$ be arbitrary elements of $\mathbb{C}P^1$. By Lemma 4.2.2, there exists an $A \in \mathrm{SU}(2)$ such that

$$A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Then for any $B \in SU(2)$, we can conclude $\varphi((A, B), (w_1 : w_2)) = (z_1 : z_2)$. Therefore, φ is transitive. □

As before, we can now conclude that all filtrations of type $1-3-4$ are equivalent. This means we understand the structure of the moduli space of filtrations on our minimal submodule.

Theorem 4.2.5. *Let*

$$I = \langle (1 + i\gamma_1\gamma_2)(1 + i\gamma_3\gamma_4) \rangle$$

be the minimal submodule of the Clifford supermodule $\mathrm{Cl}(4, \mathbb{C})$*. The moduli space of filtrations on I consists of two discrete points.*

Proof. The only possible filtrations on this submodule are of type $2-4$ or type $1-3-4$. There is exactly one filtration of type $2-4$. Filtrations of type $1-3-4$

are uniquely determined by the choice of F_0. Because F_0 is uniquely described by choosing a 1-dimensional subspace from a 2-dimensional space, it follows that the space of possible filtrations of type $1-3-4$ is $\mathbb{C}P^1$. In [8], we find that the action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}P^1$ is defined as in Lemma 4.2.4. Since Spin(4) is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$, Lemma 4.2.4 coupled with Corollary 3.4.2 allows us to conclude that all filtrations of type $1-3-4$ are equivalent. It follows that there are exactly two distinct nontrivial filtrations on the submodule $\langle 1 + i\gamma_1\gamma_2 \rangle$. Therefore, the moduli space of filtrations on this submodule consists of two discrete points. $\square$

In this section we saw our first examples of nontrivial moduli spaces. However, these moduli spaces still turned out to be discrete. The computation used in this section to discover the structure of moduli spaces was quite cumbersome. We would like to develop a nicer way to investigate these moduli spaces, and hopefully find one that is not discrete.

5

Equivalence Classes of Principal Filtrations

This chapter seeks to investigate a special class of filtrations, which we will call principal filtrations. We find that in lower-dimensional cases, principal filtrations yield discrete moduli spaces. In higher-dimensional cases, the structure of the moduli space of principal filtrations turns out to be much less obvious. As before, this chapter investigates moduli spaces by analyzing the orbit of principal filtrations under the action of $\mathrm{Spin}(n)$. However, we take a new approach to studying these orbits, as we turn to representation theory for the bulk of our computations.

5.1 Why Principal Filtrations?

The most basic filtration type we have seen is one in which F_0 is a 1-dimensional subspace of A_0 and the remaining filtration degrees are determined only by the Clifford condition. Such filtrations are nice because they are uniquely described by the single basis element of F_0. It is for this reason that we now turn our attention to principal filtrations.

Definition 5.1.1. A principal filtration is a filtrations F satisfying the Clifford condition such that

1. F_0 is 1-dimensional, and

2. $F_k = F_{k-1} \cup \text{span}\{\gamma_1, \ldots, \gamma_n\} \cdot F_{k-1}$.

$\triangle$

It is clear that principal filtrations are uniquely determined by choosing a basis element for F_0 from A_0. Every submodule has exactly one principal filtration type. This is shown in the following example.

Example 5.1.2. The submodule $\langle 1 + i\gamma_1\gamma_2 \rangle$ is a minimal submodule of $\text{Cl}(3, \mathbb{C})$ with the associated $\mathbb{Z}_2$ decomposition given by

$$\langle 1 + i\gamma_1\gamma_2 \rangle = \text{span}\{1 + i\gamma_1\gamma_2, \gamma_1\gamma_3 + i\gamma_2\gamma_3\} \oplus \text{span}\{\gamma_1 + i\gamma_2, \gamma_3 + i\gamma_1\gamma_2\gamma_3\}.$$

The possible filtration types on this ideal are $1-3-4$ and $2-4$. It is easy to see that $1-3-4$ is a principal filtration type but $2-4$ is not a principal filtration type. $\diamondsuit$

The nice properties of principal filtrations simplify computation used to determine equivalences between filtrations.

Suppose we have a minimal submodule $I = A_0 \oplus A_1$ of $\text{Cl}(n, \mathbb{C})$. Two principal filtrations F and F' are equivalent if and only if the action of $\text{Spin}(n)$ maps the basis element of F_0 to a scalar multiple of the basis element of F'_0. Consequently, the equivalence classes of principal filtrations on I are precisely the orbits of elements in A_0 under the action of $\text{Spin}(n)$. The space of these orbits is the moduli space of principal filtrations on I.

To identify the orbits of elements in A_0 under the action of $\text{Spin}(n)$, we turn to representation theory. Recall from Section 3.3 that A_0 is isomorphic to either S or S^+. Consequently, the equivalence classes of principal filtrations on I are the orbits of either S or S^+ under the action of $\text{Spin}(n)$. It is relatively easy to analyze the action of $\text{Spin}(n)$ on either S or S^+ because we can use the additional information from the weight decomposition of either of these spaces.

5.2 Discrete Moduli Spaces of Principal Filtrations

We are interested in determining when all principal filtrations on a minimal submodule will be equivalent, and also when they will not all be equivalent. To do so, we analyze the action of $\mathrm{Spin}(n)$ on filtrations that correspond to elements in the highest weight space of S.

We begin by presenting an example that computes the equivalence class of a principal filtration that corresponds to an element in the highest weight space of S. Consider filtrations of type $1-3-4$ on the minimal submodule I of $\mathrm{Cl}(4,\mathbb{C})$ defined as

$$I = \langle (1+i\gamma_1\gamma_2)(1+i\gamma_2\gamma_4)\rangle \,.$$

It is easy to check that $1-3-4$ is a principal filtration type on I. Recall that I has an associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Because $1-3-4$ filtrations are principal, $\mathrm{Fil}_{1-3-4}(I)$ is the set of possible ways to choose a 1-dimensional subspace from A_0. Let S^+ be the half-spin representation of the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(4)$, as presented in Section 3.3. Then A_0 is isomorphic to S^+, where $S^+ = \mathbb{C}^{++} \oplus \mathbb{C}^{--}$. It follows that filtrations of type $1-3-4$ can be uniquely described by choosing a single basis element for F_0 from the basis elements of S^+.

From Section 3.3, we know that the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(4)$ has four roots: $(1,1),(1,-1),(-1,1)$, and $(-1,-1)$. Then we have pairs E_{12}^{++}, E_{12}^{--} and E_{12}^{+-}, E_{12}^{-+}, as defined in Section 3.3. Let $\mathfrak{h} = \mathrm{span}\{H_1, H_2\}$ be the Cartan subalgebra of $\mathfrak{spin}_{\mathbb{C}}(4)$. It follows that the real Lie algebra $\mathfrak{spin}(4)$ is defined as

$$\mathfrak{spin}(4) = \mathrm{span}\{iH_1, iH_2, E_{12}^{++} - E_{12}^{--}, iE_{12}^{++} + iE_{12}^{--}, E_{12}^{+-} - E_{12}^{-+}, iE_{12}^{+-} + iE_{12}^{-+}\}.$$

Note that this implies the real dimension of $\mathfrak{spin}(4)$ is 6.

Let $x \in S^+$ have weight $(\frac{1}{2}, \frac{1}{2})$, and let F be the corresponding filtration of type $1-3-4$. We are interested in finding the dimension of the stabilizer of F under the action of

$\mathfrak{g} = \mathfrak{spin}(4)$, which we will denote $\mathfrak{g}_F$. Observe that $\mathrm{Fil}_{1-3-4}(I)$ is the space of ways to choose a 1-dimensional complex subspace from a 2-dimensional complex subspace. We note that this implies $\mathrm{Fil}_{1-3-4}(I)$ is isomorphic to $\mathbb{C}P^1$. It follows that two elements of S^+ correspond to the same filtration if and only if they are linearly dependent over the complex numbers. Therefore, elements of $\mathfrak{spin}(4)$ are in $\mathfrak{g}_F$ if and only if they map x to a complex multiple of itself.

Recall that if $E \in \mathfrak{spin}(4)$ has root (k_1, k_2), then either: 1) Ex has weight $(k_1, k_2)+(\frac{1}{2}, \frac{1}{2})$ if $(k_1, k_2) + (\frac{1}{2}, \frac{1}{2})$ is a weight of S^+, or 2) $Ex = 0$ if $(k_1, k_2) + (\frac{1}{2}, \frac{1}{2})$ is not a weight of S^+. Let $y = E_{12}^{--}x$. By the definition of a weight, $(iH_1)x = (iH_2)x = \frac{1}{2}ix$. Observe that

$$E_{12}^{++}x = E_{12}^{+-}x = E_{12}^{-+}x = 0.$$

This implies that

$$(E_{12}^{++} - E_{12}^{--})x = -y,\ (iE_{12}^{++} + iE_{12}^{--})x = iy,\ (E_{12}^{+-} - E_{12}^{-+})x = 0,\ \text{and}\ (iE_{12}^{+-} + iE_{12}^{-+})x = 0.$$

Note that 0 and $\frac{1}{2}ix$ are both complex multiples of x, but $-y$ and iy are linearly independent over $\mathbb{R}$ and not complex multiples of x. Therefore,

$$\mathfrak{g}_F = \mathrm{span}\{iH_1, iH_2, E_{12}^{++} - E_{12}^{--}, iE_{12}^{++} + iE_{12}^{--}\}.$$

It follows that $\dim_{\mathbb{R}}(\mathfrak{g}_F) = 4$, implying that $\dim_{\mathbb{R}}(\mathfrak{spin}(4)/\mathfrak{g}_F) = 2$.

Let $\mathcal{O}_F$ denote the orbit of F in $\mathrm{Fil}_{1-3-4}(I)$ under the action of $\mathrm{Spin}(4)$. Applying Corollary 3.1.19, we can conclude that the real dimension of $\mathcal{O}_F$ is 2. Since $\mathrm{Fil}_{1-3-4}(I)$ is isomorphic to $\mathbb{C}P^1$ and the real dimension of $\mathbb{C}P^1$ is 2, it follows that the real dimension of $\mathrm{Fil}_{1-3-4}(I)$ is 2. This means that $\mathcal{O}_F$ is maximal. By Theorem 3.3.11, it follows the action of $\mathrm{Spin}(4)$ on $\mathrm{Fil}_{1-3-4}(I)$ is transitive. Therefore, all $1-3-4$ filtrations on I are equivalent.

We now present this computation in general, producing a very nice result. The proof of this theorem largely relies on a dimension counting argument, and closely follows the computation presented in the previous example.

Theorem 5.2.1. *Principal filtrations on the submodule*

$$\langle (1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n}) \rangle$$

of $\mathrm{Cl}(2n, \mathbb{C})$ *are all equivalent for* $n = 2$ *and* $n = 3$*, but are not all equivalent for* $n \geq 4$*. Principal filtrations of a fixed type on the submodule*

$$\langle (1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n}) \rangle$$

of $\mathrm{Cl}(2n+1, \mathbb{C})$ *are all equivalent for* $n = 1$ *and* $n = 2$*, but are not all equivalent for* $n \geq 3$*.*

Proof. Let I be the minimal submodule $\langle (1 + i\gamma_1\gamma_1) \cdots (1 + i\gamma_{2n-1}\gamma_{2n}) \rangle$ of $\mathrm{Cl}(2n, \mathbb{C})$. Then I has an associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Let T be a principal filtration type on I. Then $\mathrm{Fil}_T(I)$ is the set of possible ways to choose a 1-dimensional subspace from A_0. Let S^+ be the half-spin representation of the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(2n)$, as presented in Section 3.3. Then A_0 is isomorphic to S^+. It follows that filtrations of type T can be uniquely described by choosing a single basis element for F_0 from the basis elements of S^+.

Before proceeding, we look closely at $\mathrm{Fil}_T(I)$. Recall that the complex dimension of I is 2^n, implying that the complex dimension of A_0 is 2^{n-1}. Then $\mathrm{Fil}_T(I)$ is the space of possible ways to choose a 1-dimensional complex subspace from a 2^{n-1}-dimensional complex subspace. This means that $\mathrm{Fil}_T(I)$ is $\mathbb{C}P^{2^{n-1}-1}$, implying that the real dimension of $\mathrm{Fil}_T(I)$ is $2^n - 2$.

We now use a counting argument to determine the number of roots of $\mathfrak{spin}_{\mathbb{C}}(2n)$. Roots of $\mathfrak{spin}_{\mathbb{C}}(2n)$ are n-tuples with two nonzero entries of value ± 1. There are $\binom{n}{k}$ ways to choose the location of these two nonzero entries. Because the nonzero entries are restricted to

± 1, there are 4 ways to choose their values at each location. It follows that the number of roots of $\mathfrak{spin}_{\mathbb{C}}(2n)$ is

$$4\binom{n}{2} = \frac{4(n!)}{2!(n-2)!} = 2n(n-1) = 2n^2 - 2n.$$

So $\mathfrak{spin}_{\mathbb{C}}(2n)$ has $2n^2 - 2n$ roots. Then we can find $n^2 - n$ distinct pairs $E_{ij}^{\pm\pm}, E_{ij}^{\pm\pm}$ with $i < j$, as defined in Section 3.3, such that the superscript entries in the first are the opposite of the superscript entires in the second. Let the Cartan subalgebra of $\mathfrak{spin}_{\mathbb{C}}(2n)$ be $\mathfrak{h} = \operatorname{span}\{H_1, \ldots, H_n\}$. It follows that the real Lie algebra $\mathfrak{spin}(2n)$ is defined as

$$\begin{aligned}\mathfrak{spin}(2n) = & \operatorname{span}\{iH_1, \ldots, iH_n\} \\ & + \operatorname{span}\{E_{ij}^{++} - E_{ij}^{--}, iE_{ij}^{++} + iE_{ij}^{--}, E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq n}.\end{aligned}$$

Note that this implies the dimension of $\mathfrak{spin}(2n)$ is $2n^2 - n$.

Let $x \in S^+$ with weight $(\frac{1}{2}, \ldots, \frac{1}{2})$, and let F be the principal filtration corresponding to x. We are interested in finding the dimension of the stabilizer of F under the action of $\mathfrak{g} = \mathfrak{spin}(2n)$, which we will denote $\mathfrak{g}_F$. Recall again that $\mathrm{Fil}_T(I)$ is the space of ways to choose a 1-dimensional complex subspace from S^+. It follows that two elements of S^+ correspond to the same filtration if and only if they are linearly dependent over the complex numbers. Therefore, elements of $\mathfrak{spin}(2n)$ are in $\mathfrak{g}_F$ if and only if they map x to a complex multiple of itself.

We proceed to explicitly analyze the action of $\mathfrak{spin}(2n)$ on x. By the definition of a weight, $(iH_j)x = \frac{1}{2}ix$ for all $j \leq n$. Since $\frac{1}{2}ix$ is a complex multiple of x, we know that $iH_j \in \mathfrak{g}_F$ for all $j \leq n$. For all $i < j \leq n$, we have

$$E_{ij}^{+-}x = E_{ij}^{-+}x = E_{ij}^{++}x = 0.$$

This implies that for all i, j with $i < j \leq n$,

$$(E_{ij}^{+-} - E_{ij}^{-+})x = 0, \text{ and } (iE_{ij}^{+-} + iE_{ij}^{-+})x = 0.$$

Because 0 is a complex multiple of x, it follows $E_{ij}^{+-} - E_{ij}^{-+}$ and $iE_{ij}^{+-} + iE_{ij}^{-+}$ are elements of $\mathfrak{g}_F$ for all $i < j \leq n$. Additionally, observe that for all i, j with $i < j \leq n$,

$$(E_{ij}^{++} - E_{ij}^{--})x = -y, \text{ and } (iE_j^{++} + iE_j^{--})x = iy,$$

where $y = E_{ij}^{--}x$. Note that $-y$ and iy are linearly independent over $\mathbb{R}$. Then for all $i < j \leq n$, we can conclude that $E_{ij}^{++} - E_{ij}^{--}$ and $iE_j^{++} + iE_j^{--}$ are not elements of $\mathfrak{g}_F$. It follows that,

$$\mathfrak{g}_F = \operatorname{span}\{iH_1, \ldots, iH_n\} + \operatorname{span}\{E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq n}.$$

Therefore, $\dim_{\mathbb{R}}(\mathfrak{g}_F) = n + (n^2 - n) = n^2$. This means $\dim_{\mathbb{R}}(\mathfrak{spin}(2n)/\mathfrak{g}_F) = n^2 - n$.

Let $\mathcal{O}_F$ denote the orbit of F under the action of $\mathrm{Spin}(2n)$. It follows from Corollary 3.1.19 that $\dim_{\mathbb{R}}(\mathcal{O}_F) = n^2 - n$. Applying Theorem 3.3.11, we can therefore conclude that the action of $\mathrm{Spin}(2n)$ on $\mathrm{Fil}_T(I)$ is transitive if and only if $n^2 - n = 2^n - 2$. It follows that principal filtrations on I are all equivalent when $n = 2$ and $n = 3$, but are not all equivalent when $n \geq 4$.

The computation proceeds identically over $\mathrm{Cl}(2n+1, \mathbb{C})$. We leave this as an exercise to the reader. If done correctly, one should find $\dim_{\mathbb{R}}(\mathfrak{spin}(2n)) = 2n^2 + n$ and $\dim_{\mathbb{R}}(\mathfrak{g}_F) = n^2$. It follows that $\dim_{\mathbb{R}}(\mathcal{O}_F) = n^2 + n$. Since $\dim_{\mathbb{R}}(\mathrm{Fil}_T(I)) = 2^n - 2$, we can conclude that all principal filtrations on I are equivalent when $n = 1$ and $n = 2$, but are not all equivalent when $n \geq 4$. □

We have now determined the structure of the moduli space of principal filtrations on our minimal submodules of $\mathrm{Cl}(3, \mathbb{C}), \mathrm{Cl}(4, \mathbb{C}), \mathrm{Cl}(5, \mathbb{C})$, and $\mathrm{Cl}(6, \mathbb{C})$.

Corollary 5.2.2. *The moduli space of principal filtrations on the submodule*

$$\langle (1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n}) \rangle$$

of $\mathrm{Cl}(2n, \mathbb{C})$ *is discrete for* $n = 2$ *and* $n = 3$. *The moduli space of principal filtrations on the submodule*

$$\langle(1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n})\rangle$$

of $\mathrm{Cl}(2n+1, \mathbb{C})$ *is discrete for* $n = 1$ *and* $n = 2$.

While this result is certainly of interest, we still do not know the structure of these moduli spaces in higher dimensions. In higher dimensions, we can conclude only that these moduli spaces will be non-discrete.

The proof of this theorem calculates the orbit and stabilizer of a specific filtration F, and we can use this information to draw additional conclusions about the equivalence class of F. Note that a Lie algebra is completely classified by its root system. We explicitly computed the roots of $\mathfrak{g}_F$ as a subset of the roots of $\mathfrak{g}$. By an analysis of this root system, it follows that $\mathfrak{g}_F \cong \mathfrak{u}(n)$. Therefore, the orbit of F is isomorphic to $\mathrm{Spin}(2n)/\mathrm{U}(n)$ in the even case and $\mathrm{Spin}(2n+1)/\mathrm{U}(n)$ in the odd case. Each of these quotients is an element of the corresponding moduli space of principal filtrations. Though not defined in this paper, $\mathrm{Spin}(2n)/\mathrm{U}(n)$ is isomorphic to the homogeneous space of complex structures on $\mathbb{R}^{2n}$, denoted $J(\mathbb{R}^{2n})$. Consult [10] for further information on both this isomorphism and the space $J(\mathbb{R}^{2n})$. We were unable to identify the space $\mathrm{Spin}(2n+1)/\mathrm{U}(n)$. So we have the following additional corollary.

Corollary 5.2.3. *There exists an equivalence class in the moduli space of principal filtrations on the submodule*

$$\langle(1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n})\rangle$$

of $\mathrm{Cl}(2n, \mathbb{C})$ *that is isomorphic to* $J(\mathbb{R}^{2n})$. *There exists an equivalence class in the moduli space of principal filtrations on the submodule*

$$\langle(1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n})\rangle$$

of $\mathrm{Cl}(2n+1, \mathbb{C})$ *that is isomorphic to* $\mathrm{Spin}(2n+1)/U(n)$.

In this corollary, we identified a specific element in the moduli space of principal filtrations. In the next section we will identify additional elements in this moduli space, in hopes that it will allow us to better understand its structure.

5.3 Elements of Non-Discrete Moduli Spaces

In the previous section, we investigated the equivalence classes of principal filtrations. We did this by associating principal filtrations with an element from the highest weight space of the spin representation. In this section, we consider principal filtrations associated to elements of S that do not lie in a single weight space. In particular, we analyze filtrations that correspond to $x + y \in S$ such that x is in the highest weight space and y is in the lowest weight space. Our goal is to find the dimension of all equivalence classes of principal filtrations corresponding to elements in S of this form. Such equivalence classes are of interest because they are elements of the moduli space of principal filtrations.

We have determined that all principal filtration are equivalent on the minimal submodules of $\mathrm{Cl}(2, \mathbb{C})$ through $\mathrm{Cl}(6, \mathbb{C})$, and therefore have exactly one equivalence class. It is for this reason that we begin by working over $\mathrm{Cl}(7, \mathbb{C})$. The computation over $\mathrm{Cl}(7, \mathbb{C})$ and $\mathrm{Cl}(8, \mathbb{C})$ works out differently than in higher dimensions, so we present these cases as separate results.

Theorem 5.3.1. *There exists an equivalence class that is isomorphic to S^7 in the moduli space of principal filtrations on the submodule*

$$I = \langle (1 + i\gamma_1\gamma_2)(1 + i\gamma_3\gamma_4)(1 + i\gamma_5\gamma_6) \rangle$$

of $\mathrm{Cl}(7, \mathbb{C})$.

Proof. Let I have associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Let S be the spin representation of the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(7)$, as presented in Section 3.3. Recall that

A_0 is isomorphic to S. Let T be a principal filtration type on I. It follows that filtrations of type T can be uniquely described by choosing a basis element for F_0 from the basis elements of S.

We first look closely at the real Lie algebra $\mathfrak{spin}(7)$. The roots of the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(7)$ are

$$(\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1), (0, \pm 1, \pm 1), (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1).$$

Let $\mathfrak{h} = \operatorname{span}\{H_1, H_2, H_3\}$ be the Cartan subalgebra of $\mathfrak{spin}_{\mathbb{C}}(7)$. Then the real Lie algebra $\mathfrak{spin}(7)$ is defined as

$$\begin{aligned}\mathfrak{spin}(7) = \operatorname{span}\{iH_1, iH_2, iH_3\} &+ \operatorname{span}\{E_i^+ - E_i^-, iE_i^+ + iE_i^{-0}\}_{i\leq 3} \\ &+ \operatorname{span}\{E_{ij}^{++} - E_{ij}^{--}, iE_{ij}^{++} + iE_{ij}^{--}, E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq 3}.\end{aligned}$$

It follows that the dimension of $\mathfrak{spin}(7)$ is 21

Let $x, y \in S$ such that x has weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and y has weight $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Let F be the filtration of type T on I corresponding to $x + y$. We are interested in finding the dimension of the stabilizer of F under the action of $\mathfrak{g} = \mathfrak{spin}(7)$, which we will denote $\mathfrak{g}_F$. Note that $\mathrm{Fil}_T(I)$ is the space of ways to choose a 1-dimensional complex subspace from S. Then two elements of S correspond to the same filtration if and only if they are linearly dependent over the complex numbers. It follows that that elements of $\mathfrak{spin}(7)$ are in $\mathfrak{g}_F$ if and only if they map x to a complex multiple of itself.

Consider the action of $\mathfrak{spin}(7)$ on $x + y$. Observe first that

$$iH_1(x+y) = iH_2(x+y) = iH_3(x+y) = \frac{1}{2}ix - \frac{1}{2}iy.$$

It follows that $iH_1 - iH_2$ and $iH_1 - iH_3$ are elements of $\mathfrak{g}_F$. This means the Cartan subalgebra contributes 2 dimensions to $\mathfrak{g}_F$. Next, we notice that for all $i < j \leq 3$,

$$(E_{ij}^{+-} - E_{ij}^{-+})(x+y) = (iE_{ij}^{+-} + iE_{ij}^{-+})(x+y) = 0.$$

It follows that elements in this form contribute 6 dimensions to $\mathfrak{g}_F$.

We proceed to compute the action of the remaining elements, which is slightly more complicated. For all $i < j \leq n$,

$$(E_{ij}^{++} - E_{ij}^{--})(x+y) = -E_{ij}^{--}x + E_{ij}^{++}y, \quad (iE_{ij}^{++} + iE_{ij}^{--})(x+y) = iE_{ij}^{--}x + iE_{ij}^{++}y.$$

Next, for all $i \leq n$, observe that

$$(E_i^{+} - E_i^{-})(x+y) = -E_i^{-}x + E_i^{+}y, \quad (E_i^{+} - E_i^{-})(x+y) = -E_i^{-}x + E_i^{+}y.$$

None of the above elements are complex multiples of $x+y$. It follows that these elements can only contribute to $\mathfrak{g}_F$ if some linear combination is equal to 0. Therefore, the sets of linearly dependent elements resulting from the above computation will contribute to $\mathfrak{g}_F$. To compute these sets, we list the weights that resulted from this computation:

$$\begin{gathered}
-(--+) + (++-), \quad i(--+) + i(++-) \\
-(++-) + (--+), \quad i(++-) + i(--+) \\
-(-+-) + (+-+), \quad i(-+-) + i(+-+) \\
-(+-+) + (-+-), \quad i(+-+) + i(-+-) \\
-(+--) + (-++), \quad i(+--) + i(-++) \\
-(-++) + (+--), \quad i(-++) + i(+--)
\end{gathered}$$

Organized in this suggestive manner, it is easy to see that we have 6 pairs of linearly dependent elements over $\mathbb{R}$. It follows that these 12 elements contribute 6 dimensions to $\mathfrak{g}_F$.

Adding it all up, we have shown that $\mathfrak{g}_F$ has dimension 14 and rank 2. Moreover, we completely identified the roots of $\mathfrak{g}_F$ as a subset of the roots of $\mathfrak{g}$. By an analysis of the root system of $\mathfrak{g}_F$, it follows that $\mathfrak{g}_F$ is isomorphic to the Lie algebra of the exceptional Lie group G_2. This means that the equivalence class of F is isomorphic to $\mathrm{Spin}(7)/G_2$. In [8], we find that $\mathrm{Spin}(7)/G_2$ is isomorphic to S^7. $\square$

We proceed to present a similar result over $\mathrm{Cl}(8,\mathbb{C})$. The proof of this theorem is almost identical to the previous. Is it for the reason that we only sketch an outline, and leave the details as an exercise for the reader.

Theorem 5.3.2. *There exists an equivalence class that is isomorphic to S^7 in the moduli space of principal filtrations on the submodule*

$$I = \langle (1+i\gamma_1\gamma_2)(1+i\gamma_3\gamma_4)(1+i\gamma_5\gamma_6)(1+i\gamma_7\gamma_8)\rangle$$

of $\mathrm{Cl}(8,\mathbb{C})$.

Proof. Let I have associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Let S^+ be the half-spin representation of the complex Lie algebra $\mathfrak{spin}_{\mathbb{C}}(8)$. From Section 3.3, we are given that A_0 is isomorphic to S. Let T be a principal filtration type on I. It follows that filtrations of type T are uniquely described by choosing a basis element for F_0 from the basis elements of S.

Let $x, y \in S$ such that x has weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and y has weight $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Let F be the filtration of type T associated to $x+y$. We are interested in finding the dimension of the stabilizer of F under the action of $\mathfrak{g} = \mathfrak{spin}(8)$, which we will denote $\mathfrak{g}_F$. As before, elements of $\mathfrak{spin}(8)$ are in $\mathfrak{g}_F$ if and only if they map $x+y$ to a complex multiple of itself.

We leave it to the reader to explicitly compute the basis elements of $\mathfrak{spin}(8)$ and the action of $\mathfrak{spin}(8)$ on $x+y$, following the computation in the previous proof. If done correctly, one should find that the Cartan subalgebra contributes 3 dimensions to $\mathfrak{g}_F$, and elements of the form $E_{ij}^{+-} - E_{ij}^{-+}$ and $iE_{ij}^{+-} + iE_{ij}^{-+}$ contribute 12 dimensions to $\mathfrak{g}_F$. Elements of the form $E_{ij}^{++} - E_{ij}^{--}$ and $iE_{ij}^{++} + iE_{ij}^{--}$ act on $x+y$ to produce elements with weights:

$$-(--++) + (++--), \quad i(--++) + i(++--)$$

$$-(++--) + (--++), \quad i(++--) + i(--++)$$

$$-(-++-) + (+--+), \quad i(-++-) + i(+--+)$$

$$-(+--+)+(-++-),\quad i(+--+)+i(-++-)$$

$$-(-+-+)+(+-+-),\quad i(-+-+)+i(+-+-)$$

$$-(+-+-)+(-+-+),\quad i(+-+-)+i(-+-+)$$

It follows that elements of this form contribute 6 dimensions to $\mathfrak{g}_F$.

Adding it all up, we have found that $\mathfrak{g}_F$ is 21-dimensional and has rank 3. Moreover, we completly identified the roots of $\mathfrak{g}_F$ as a subset of the roots of $\mathfrak{g}$. By an analysis of the root system of $\mathfrak{g}_F$, it follows that $\mathfrak{g}_F$ is isomorphic to $\mathfrak{spin}(7)$. Then the equivalence class of F is isomorphic to $\mathrm{Spin}(8)/\mathrm{Spin}(7)$. In [8], it is shown that $\mathrm{Spin}(8)/\mathrm{Spin}(7)$ isomorphic to S^7. □

In [11], the authors considered principal filtration over $\mathrm{Cl}(7,\mathbb{R})$ and $\mathrm{Cl}(8,\mathbb{R})$. The previous two theorems reproduced their results using an entirely different method.

To generalize in higher dimensions, we follow the computation from the previous two proofs closely. The computation will differ in the final step, as we will be unable to find linear combinations of elements in $\mathfrak{spin}(n)$ or $\mathfrak{spin}(n+1)$ that contribute to $\mathfrak{g}_F$.

Theorem 5.3.3. *There exists an equivalence class of dimension n^2-n+1 in the moduli space of principal filtrations on the submodule*

$$\langle(1+i\gamma_1\gamma_2)\cdots(1+i\gamma_{2n-1}\gamma_{2n})\rangle$$

of $\mathrm{Cl}(2n,\mathbb{C})$ for $n\geq 5$. There exists an equivalence class of dimension n^2+n+1 in the moduli space of principal filtrations on the submodule

$$\langle(1+i\gamma_1\gamma_2)\cdots(1+i\gamma_{2n-1}\gamma_{2n})\rangle$$

of $\mathrm{Cl}(2n+1,\mathbb{C})$ for $n\geq 4$.

Proof. First, we consider the even case. Let $n \geq 5$, and let I be the submodule of $\mathrm{Cl}(2n, \mathbb{C})$ given by

$$I = \langle (1 + i\gamma_1\gamma_2) \cdots (1 + i\gamma_{2n-1}\gamma_{2n}) \rangle$$

such that I has associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Let S^+ be the half-spin representation of $\mathfrak{spin}(2n)$. Recall from Section 3.3 that A_0 is isomorphic to S^+. Let T be a principal filtration type of I. It follows that filtrations of type T can be uniquely described by choosing a basis element of F_0 from the basis of S^+.

Let the Cartan subalgebra of $\mathfrak{spin}(2n)$ be $\mathfrak{h} = \{H_1, \ldots, H_n\}$. As in Section 3.3, we can define $\mathfrak{spin}(2n)$ as

$$\begin{aligned} \mathfrak{spin}(2n) =& \operatorname{span}\{iH_1, \ldots, iH_n\} \\ &+ \operatorname{span}\{E_{ij}^{++} - E_{ij}^{--}, iE_{ij}^{++} + iE_{ij}^{--}, E_{ij}^{+-} - E_{ij}^{-+}, iE_{ij}^{+-} + iE_{ij}^{-+}\}_{i<j\leq n}. \end{aligned}$$

This means that the dimension of $\mathfrak{spin}(2n)$ is $2n^2 - n$, as found in the proof of Theorem 5.2.1

Let $x, y \in S^+$ such that x has weight $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and y has weight $(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$. Let F be the filtration of type T associated to $x + y$. We are interested in finding the stabilizer of F under the action of $\mathfrak{g} = \mathfrak{spin}(2n)$, which we will denote $\mathfrak{g}_F$. As before, elements of $\mathfrak{spin}(2n)$ lie in $\mathfrak{g}_F$ if and only if they map $x + y$ to a complex multiple of itself.

We now explicitly compute the action of $\mathfrak{spin}(2n)$ on $x + y$. By the definition of a weight,

$$iH_1(x + y) = iH_2(x + y) = \cdots = iH_n(x + y).$$

It follows that $iH_1 - iH_j \in \operatorname{stab}(x + y)$ for all $1 < j \leq n$. Then elements of this form contribute $n - 1$ dimension to $\mathfrak{g}_F$. Next, we observe that for all $i < j \leq n$,

$$(E_{ij}^{+-} - E_{ij}^{-+})(x + y) = (iE_{ij}^{+-} + iE_{ij}^{-+})(x + y) = 0.$$

As there are $n^2 - n$ elements of this form, such elements contribute $n^2 - n$ dimension to $\mathfrak{g}_F$.

Observe that for all $i < j \leq n$,

$$(E_{ij}^{++} - E_{ij}^{--})(x + y) = x' + y'$$

such that y' has the opposite weight as x', and x' has a weight with $-\frac{1}{2}$ in the ith and jth entries but $\frac{1}{2}$ in all other entries. It follows that acting by such elements on $x+y$ produces elements that are not complex multiples of $x + y$. Moreover, acting by such elements on $x + y$ will not produce the elements of the same weight twice. Then the set produced by acting on $x + y$ by elements of this form will be linearly independent. It follows that elements of this form do not contribute to the dimension of $\mathfrak{g}_F$.

Therefore,

$$\dim_{\mathbb{R}}(\mathfrak{g}_F) = (n - 1) + (n^2 - n) = n^2 - 1.$$

It follows that $\dim_{\mathbb{R}}(\mathfrak{spin}(2n)/\mathfrak{g}_F) = n^2 - n + 1$. Let $\mathcal{O}_F$ denote the orbit of F under the action of $\mathrm{Spin}(2n)$. Applying Corollary 3.1.19, the dimension of $\mathcal{O}_F$ is $n^2 - n + 1$.

The computation proceeds identically over $\mathrm{Cl}(2n + 1, \mathbb{C})$ for $n \geq 4$. We leave this as an exercise to the reader. The dimension of $\mathfrak{g}_F$ is still $n^2 - 1$. Since the dimension of $\mathfrak{spin}(2n + 1)$ is $2n^2 + n$, it follows that the dimension of $\mathcal{O}_F$ is $n^2 + n + 1$. $\square$

Observe that in both the even and odd cases, the proof of this theorem finds that $\mathfrak{g}_F$ is a Lie algebra of dimension $n^2 - 1$ and rank $n - 1$. We also identified the root system of $\mathfrak{g}_F$ as a subset of the roots of $\mathfrak{g}$. By an analysis of this root system, it follows that $\mathfrak{g}_F$ is isomorphic to $\mathfrak{su}(n)$. Unlike before, we were unable to identify the equivalence class of F.

The results in this section do not explicitly contribute to understanding the structure of moduli spaces of principal filtrations. However, we have identified the dimension of an equivalence class in every moduli space of principal filtrations.

It is clear from this section that analyzing principal filtrations in this manner will not easily produce the structure of a moduli space. We must find an approach that does not treat principal filtrations as independent elements in S or S^+, as this seems inefficient.

6

The Structure of a Non-Discrete Moduli Spaces

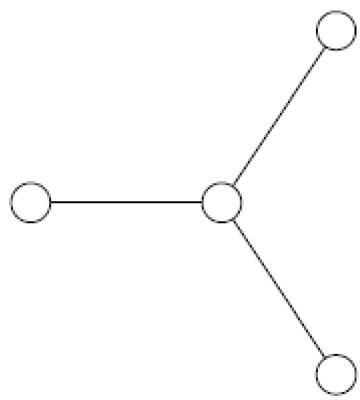
In this chapter, we continue our discussion of principal filtrations but slightly alter our focus. We have previously identified when moduli spaces of principal filtrations will be discrete or not, but we have yet to determine the structure of any non-discrete moduli space. We will use this chapter as an opportunity to explore the structure of the moduli space of principal filtration on a minimal submodule of $\mathrm{Cl}(8, \mathbb{C})$. It is here that we find our first example of an interesting, non-discrete moduli space. We will use a nice property of $\mathrm{Spin}(8)$ to reframe the problem, allowing us to analyze $\mathbb{C}^8$ instead of S^+.

6.1 Using Triality to Reframe the Problem

Let I be the usual minimal submodule of $\mathrm{Cl}(8, \mathbb{C})$ defined as

$$I = \langle (1 + i\gamma_1\gamma_2)(1 + i\gamma_3\gamma_4)(1 + i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8) \rangle \,,$$

with associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. We are interested in determining equivalences between filtrations on I. In Chapter 5.1, we approached this problem by analyzing the action of $\mathrm{Spin}(8)$ on A_0. This worked nicely, as A_0 is isomorphic to the half-spin representation S^+. In this chapter, we use a property specific to $\mathrm{Spin}(8)$ known as "triality". Due to certain nice geometry properties of Spin(8), triality dictates that the three representations of $\mathfrak{spin}_{\mathbb{C}}(8)$ given by $\mathbb{C}^8, S^+$, and S^- are isomorphic up to automorphisms of Spin(8). This means that Spin(8) acts isomorphically on $\mathbb{C}^8$, S^+, and S^-. A Dynkin diagram is a tool that describes a Lie group by its roots. The Dynkin diagram of Spin(8) looks like:



We can see that there are three ways to rotate this diagram without disrupting its fundamental structure, and it is for this reason that triality exists for Spin(8). For a detailed explanation of this phenomenon, please see [5]. It follows that instead of analyzing the action of Spin(8) on S^+, we can simply analyze the action of Spin(8) on $\mathbb{C}^8$.

Before proceeding, we need to build some intuition about the Lie group SO(n) and its action on $\mathbb{C}^n$. We start by considering the Lie group O(3). We can describe O(3) as the group of rotations and reflection about the x-, y-, and z-axes. Then the Lie group SO(3) contained in O(3) is the group of rotations about these three axes. Using this description, it is relatively easy to visualize the action of SO(3) on $\mathbb{R}^3$. However, the existence of these three axes is just feature of $\mathbb{R}^3$. When generalizing to higher dimensions, SO(n) can be described as the group of rotations about 2-planes through the origin. To help understand the action of SO(n) on $\mathbb{C}^n$, we can analyze SO(n) acting simultaneously on two copies of

$\mathbb{R}^n$ and then extend in the obvious manner. We will rely on this intuition to understand the action of $\mathrm{SO}(n)$ on $\mathbb{C}P^{n-1}$.

The moduli space of principal filtrations on I is the space of orbits of filtrations under the action of Spin(8). In Section 3.2, we defined the covering map $\mathrm{Ad} : \mathrm{Spin}(8) \longrightarrow \mathrm{SO}(8)$. We can pullback via this map to instead look at the action of SO(8) on $\mathbb{C}^8$. Filtrations correspond to points in A_0, which we can now describe as points in $\mathbb{C}^8$ using the notion of triality. Recall from previous examples that the space of principal filtrations is always complex projective space. It follows that we are actually interested in the action of SO(8) on $\mathbb{C}P^7$. The following sections investigate this action.

6.2 The Action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$.

In this section, we explore the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ as a means to later understand the action of $\mathrm{SO}(n)$ on $\mathbb{C}^n$. To begin, we would like to show that the action of $\mathrm{SO}(n)$ on $\mathbb{R}P^{n-1}$ is transitive. From our previous discussion of the geometric properties of SO(3), it is relatively easy to understand why SO(3) maps any line through the origin to the x-axis via rotations about the other two axes. This implies that SO(3) acts transitively on $\mathbb{R}P^2$. This same idea applies in any dimension, as seen in the following lemma.

Lemma 6.2.1. *The Lie group* $\mathrm{SO}(n)$ *acts transitively on* $\mathbb{R}P^{n-1}$.

Proof. Let $\mathbf{a}$ be an arbitrary vector in $\mathbb{R}^n$, and let $\mathbf{u}_1 \in \mathbb{R}^n$ be the unit vector in the direction of $\mathbf{a}$. It is a standard result from linear algebra that any linearly independent set in a vector space can be extended to a basis for that vector space. It follows that we can extend $\mathbf{u}_1$ to a basis of $\mathbb{R}^n$, given by $\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$. Applying the Gram-Schmidt process to $\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, we can find a orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\mathbb{R}^n$. Let

$$Q = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}.$$

It follows that Q is an element of SO(8). Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis for $\mathbb{R}^n$. Then $Q\mathbf{e}_1 = \mathbf{u}_1$. Note that $\mathbf{u}_1 = c\mathbf{a}$ for some $c \in \mathbb{R}$. This implies that $\mathrm{SO}(n)$ can map the 1-axis in $\mathbb{R}P^{n-1}$ to any line in $\mathbb{R}P^{n-1}$. Because elements of $\mathrm{SO}(n)$ are invertible, it follows that $\mathrm{SO}(n)$ acts transitively on $\mathbb{R}P^{n-1}$. □

Though presented in general, this lemma specifically allows us to map any line in $\mathbb{R}P^{n-1}$ to the 1-axis. This is of particular significance when trying to find a standard representation for orbits under the action of $\mathrm{SO}(n)$.

We are eventually interested in the action of $\mathrm{SO}(n)$ on $\mathbb{C}^n$. As presented earlier, this is almost equivalent to $\mathrm{SO}(n)$ acting simultaneously on two copies of $\mathbb{R}^n$. It is for this reason that we now consider the behavior of $\mathrm{SO}(n)$ acting simultaneously on two vectors in $\mathbb{R}^n$, as explored in the following two lemmas.

Lemma 6.2.2. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent, then there exists some $g \in \mathrm{SO}(n)$ such that $g\boldsymbol{a} = \alpha\boldsymbol{e}_1$ and $g\boldsymbol{b} = \beta\boldsymbol{e}_1$ for $\alpha, \beta \in \mathbb{R}$.*

Proof. Applying Lemma 6.2.1, there exists some $g \in \mathrm{SO}(n)$ such that $g\mathbf{a} = \alpha\mathbf{e}_1$ for some $\alpha \in \mathbb{R}$. Because $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, it follows that $g\mathbf{b}$ is a multiple of $g\mathbf{a}$. This means that $g\mathbf{b} = \beta\mathbf{e}_1$ for some $\beta \in \mathbb{R}$. □

Lemma 6.2.3. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then there exists some $g \in \mathrm{SO}(n)$ such that $g\boldsymbol{a} = \alpha\boldsymbol{e}_1$ and $g\boldsymbol{b} = \beta_1\boldsymbol{e}_1 + \beta_2\boldsymbol{e}_2$ for $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ with α, β_2 nonzero.*

Proof. Applying Lemma 6.2.1, there exist some $g_1 \in \mathrm{SO}(n)$ such that $g_1\mathbf{a} = \alpha\mathbf{e}_1$. Note that α is nonzero because $\mathbf{a}$ must be nonzero. Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, we can conclude that $g_1\mathbf{b}$ is not a multiple of $g_1\mathbf{a}$. This means that $g_1\mathbf{b}$ is not a multiple of $\mathbf{e}_1$, implying that the vector in $\mathbb{R}^{n-1}$ consisting of the final $n-1$ entries of $g\mathbf{b}$ is nonzero. It again follows from Lemma 6.2.1 that there exists some block diagonal matrix $g_2 \in \mathrm{SO}(n)$ with $[1]$ as the first block and an element in $\mathrm{SO}(n-1)$ as the second block such that

$g_2 g_1 \mathbf{b} = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2$ for $\beta_1, \beta_2 \in \mathbb{R}$ with β_2 nonzero, which looks like

$$g_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \in \mathrm{SO}(n-1) & \\ 0 & & & \end{bmatrix}.$$

Observe that g_2 leaves $g_1\mathbf{a}$ unchanged. Note that $g_2 g_1 \in \mathrm{SO}(8)$. Our proof is therefore complete. □

6.3 The Orbits and Stabilizers of $\mathbb{C}P^{n-1}$.

To understand the moduli space of principal filtrations on I, we need to identify the orbits of $\mathbb{C}P^7$ under the action of SO(8). Before doing so, it is instructive to identify the stabilizers of elements in $\mathbb{C}P^7$ under the action of SO(8).

In the preceding section, we found a way to simultaneously map two arbitrary vectors in $\mathbb{R}^n$ to a "standard" form via the action of $\mathrm{SO}(n)$. Then we can extend this action in the obvious manner to map vectors in $\mathbb{C}^n$ to a standard form via the action of $\mathrm{SO}(n)$. It is easy to identify the stabilizer of vectors in this form. We follow exactly this model in the proof of the following theorem.

Theorem 6.3.1. *Consider the action of* $\mathrm{SO}(n)$ *on* $\mathbb{C}P^{n-1}$. *Let* $\boldsymbol{v} \in \mathbb{C}^n$ *and let* l *be the line in* $\mathbb{C}P^{n-1}$ *passing through* v. *Suppose that* $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}i$ *for* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$. *If* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are linearly dependent, then the stabilizer of* l *under the action of* $\mathrm{SO}(n)$ *is isomorphic to* $O(n-1)$. *If* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are linearly independent, then the stabilizer of* l *under* $\mathrm{SO}(n)$ *is isomorphic to* $\mathrm{SO}(n-2) \times \mathbb{Z}_2$.

Proof. Suppose first that $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. It follows from Lemma 6.2.2 that there exists some $g \in \mathrm{SO}(n)$ such that $g\mathbf{a} = \alpha\mathbf{e}_1$ and $g\mathbf{b} = \beta\mathbf{e}_1$ for $\alpha, \beta \in \mathbb{R}$. It follows that

$$g\mathbf{v} = g\mathbf{a} + i(g\mathbf{b}) = \alpha\mathbf{e}_1 + i\beta\mathbf{e}_1.$$

Let l' be the line in $\mathbb{C}^n$ through $g\mathbf{v}$ and the origin. An element of $\mathrm{SO}(n)$ stabilizes l' if and only if it maps l' to a complex multiple of itself. Since the columns of elements of $\mathrm{SO}(n)$ must be orthonormal, it follows that elements of $\mathrm{SO}(n)$ stabilize l' if and only if their first column is $\pm\mathbf{e}_1$. This means that the stabilizer of l' under $\mathrm{SO}(n)$ is precisely the set of block diagonal matrices such that either: 1) $[1]$ is the first block and an element of $\mathrm{SO}(n-1)$ is the second block, or 2) $[-1]$ is the first block and an element of $\mathrm{O}(n-1)$ is the second block. Equivalently, the stabilizer of l' under $\mathrm{SO}(n)$ is precisely the set of matrices in either of the following two forms:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \in \mathrm{SO}(n-1) & \\ 0 & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \in \mathrm{O}(n-1) - \mathrm{SO}(n-1) & \\ 0 & & & \end{bmatrix}.$$

Consequently, elements of $\mathrm{O}(n-1)$ uniquely extend to elements of $\mathrm{SO}(n)$ to form the stabilizer of l' under $\mathrm{SO}(n)$. Therefore, the stabilizer of l' under $\mathrm{SO}(n)$ is isomorphic to $\mathrm{O}(n-1)$. Since l' is in the orbit of l under $\mathrm{SO}(n)$, it follows that the stabilizer of l under $\mathrm{SO}(n)$ is also isomorphic to $\mathrm{O}(n-1)$.

Next, suppose that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Applying Lemma 6.2.3, there exists some $g \in \mathrm{SO}(n)$ such that $g\mathbf{a} = \alpha\mathbf{e}_1$ and $g\mathbf{b} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$ for $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ with α, β_2 nonzreo. This means that

$$g\mathbf{v} = g\mathbf{a} + i(g\mathbf{b}) = \alpha\mathbf{e}_1 + i\beta_1\mathbf{e}_1 + i\beta_2\mathbf{e}_2 = (\alpha + \beta_1)\mathbf{e}_1 + i\beta_2\mathbf{e}_2.$$

Observe that $\alpha + i\beta_1$ is linearly independent from β_2 over the real numbers. Let l' be the line in $\mathbb{C}^n$ through $g\mathbf{v}$ and the origin. An element of $\mathrm{SO}(n)$ stabilizes l' if and only if it maps l' to a complex multiple of itself. Since the columns of elements of $\mathrm{SO}(n)$ must be orthonormal and the first two entries in $\mathbf{v}$ are linearly independent, it follows that elements of $\mathrm{SO}(n)$ stabilize l' if and only if they are block diagonal matrices whose first block is $\pm I_2$. This means the stabilizer of l' is precisely the block diagonal elements of

$\mathrm{SO}(n)$ whose first block is $\pm I$ and whose second block is an element of $\mathrm{SO}(n-2)$, which look like

$$\begin{bmatrix} \pm I & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \in \mathrm{SO}(n-2) & \\ 0 & & & \end{bmatrix}.$$

Consequently, elements of $\mathrm{SO}(n-2) \times \mathbb{Z}_2$ uniquely extend to form the stabilizer of l' under $\mathrm{SO}(n)$. So the stabilizer of l' under $\mathrm{SO}(n)$ is isomorphic to $\mathrm{SO}(n-2) \times \mathbb{Z}_2$. Since l' is in the orbit of l under $\mathrm{SO}(n)$, it follows that the stabilizer of l under l' is isomorphic to $\mathrm{SO}(n-2) \times \mathbb{Z}_2$. $\square$

We presented this proof in general because it is interesting in its own right. However, we are interested only in the action of $\mathrm{SO}(8)$. Though understanding the stabilizers of elements in $\mathbb{C}P^7$ does not explicitly identify the orbits of these elements, it does give insight into the structure of these orbits.

Building from the preceding proof, we can identify a standard representative of the orbit of each element in $\mathbb{C}P^7$ under the action of $\mathrm{SO}(8)$.

Theorem 6.3.2. *The space of orbits of elements in* $\mathbb{C}P^7$ *under the action of* $\mathrm{SO}(8)$ *is homeomorphic to* S^2.

Proof. Let

$$U = \{\mathbf{u} = (u_1, u_2, u_3) \in S^2 \mid \mathbf{u} = (1,0,0), \text{ or } u_1, u_3 > 0\}.$$

Let $\mathbf{u}$ describe the orbit under the action of $\mathrm{SO}(8)$ of the line passing through

$$(u_1 + iu_2)\mathbf{e}_1 + u_3\mathbf{e}_2.$$

We are tasked with showing that orbits of $\mathbb{C}P^{n-1}$ are represented by exactly one element in U. Let $\mathbf{v} \in \mathbb{C}^8$ and let l be the line in $\mathbb{C}P^7$ passing through $\mathbf{v}$. Suppose that $\mathbf{v} = \mathbf{a} + \mathbf{b}i$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^8$. There are two cases to consider.

First, suppose that **a** and **b** are linearly dependent. Then there exists exists some $g \in \mathrm{SO}(8)$ such that $g\mathbf{v} = (\alpha + i\beta_1)\mathbf{e}_1$. It follows that the line in $\mathbb{C}P^7$ passing through $\mathbf{e}_1$ is in the orbit of l under the action of SO(8). Therefore, the orbit of l under the action of SO(8) can be uniquely described by the point $(1, 0, 0)$ in U.

Suppose next that **a** and **b** are linearly independent. It follows from Lemma 6.2.3 that there exists exists some $g \in \mathrm{SO}(8)$ such that $g\mathbf{v} = (\alpha + i\beta_1)\mathbf{e}_1 + \beta_2\mathbf{e}_2$ for α, β_2 nonzero. Because elements of SO(8) preserve distance in $\mathbb{R}^8$, it follows that $g\mathbf{a} = \pm||\mathbf{a}||\mathbf{e}_1$. This means that α is fixed up to multiplication by -1. Also because elements of SO(8) preserve distance in $\mathbb{R}^8$, we can conclude that $g\mathbf{b}$ lies on two circles: 1) centered at the origin of radius $||\mathbf{b}||$, and 2) centered at $(\alpha, 0)$ of radius $||\mathbf{b} - \mathbf{a}||$. This means that β_1, β_2 satisfy two equations:

$$||\mathbf{b}|| = \beta_1^2 + \beta_2^2 \quad \text{and} \quad ||\mathbf{b} - \mathbf{a}|| = (\beta_1 - \alpha)^2 + \beta_2^2.$$

These two equations uniquely fix β_1 and fix β_2 up to multiplication by -1. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the unit vector in the direction $(|\alpha|, \beta_1, |\beta_2|)$. Then the line corresponding to **u** is in the orbit of l under the action of SO(8). Furthermore, **u** is the only element of U in this orbit.

In either case, we find that the orbits of elements in $\mathbb{C}P^7$ under the action of SO(8) can be uniquely described by elements in U. Notice that U describes a quarter of S^2 with only one boundary point defined. The above argument implies that the space of orbits of $\mathbb{C}P^7$ under the action of SO(8) is a quarter of S^2 with both boundary semicircles identified to a single point. Then this space of orbits is a 2-dimensional region whose boundary is identified to a point, implying that is it homeomorphic to S^2. Therefore, our proof is complete. $\square$

It follows from this theorem that we understand the structure of the moduli space of principal filtrations on our minimal ideal. This is exciting because it is the first example we've seen of a non-discrete moduli space.

Corollary 6.3.3. *Let I be the minimal submodule of* $\mathrm{Cl}(8, \mathbb{C})$ *given by*

$$I = \langle (1 + i\gamma_1\gamma_2)(1 + i\gamma_3\gamma_4)(1 + i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8) \rangle .$$

The moduli space of principal filtrations on I is homeomorphic to S^2.

Proof. Suppose that I has an associated $\mathbb{Z}_2$-decomposition given by $I = A_0 \oplus A_1$. Applying Corollary 3.4.2, the moduli space of principal filtrations is the space of orbits of $\mathrm{Spin}(n)$ acting on A_0. Recall that A_0 is isomorphic to S^+, and triality dictates that $\mathrm{Spin}(8)$ acts isomorphically on $\mathbb{C}^8$ and S^+. Consequently, our corollary follows from the preceding theorem. $\square$

This theorem suggests that non-discrete moduli spaces of principal filtrations are likely to have interesting topological structure, which might be a fruitful place to begin future research. In addition, it would be of interest to analyze the topology of a moduli space of filtrations that are not all principal. However, this is considerably more difficult because the space of such filtrations is likely to be some sort of Grassmannian manifold, rather than complex projective space. Proceeding in a different direction, it might also be constructive to consider applying the techniques from this paper to filtrations on real Clifford supermodules.

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