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Thermal Bending, Buckling and Post-Buckling Analysis of Unsymmetrically Laminated Composite Beams with the Effects of Moisture and Geometric Imperfections

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THERMAL BENDING, BUCKLING AND POST-BUCKLING ANALYSIS OF UNSYMMETRICALLY LAMINATED COMPOSITE BEAMS WITH THE EFFECTS OF MOISTURE AND GEOMETRIC IMPERFECTIONS

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by

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LIST OF SYMBOLS

N	Resultant force matrix
M	Resultant moment matrix
A	Extensional stiffness matrix
B	Bending-extension coupling stiffness matrix
D	Bending stiffness matrix
N_{ij}^{T}	Resultant thermal force matrix
N_{ij}^{H}	Resultant hygro force matrix
M_{ij}^{T}	Resultant thermal moment matrix
K_s	Shear factor
Q_x	Shear resultant force in the x-direction
Q_y	Shear resultant force in the y-direction
$\left(\bar{Q}_{ij}\right)_k$	Conjugate stiffness matrix of each ply k
L	Length of the beam
P_{cr}	Critical buckling load
T_{cr}	Critical buckling temperature
h_k	Thickness of each ply
n	Number of plies
b	Width of the beam which is assumed to be equal to unity
h	Thickness of the beam
u_0	In-plane displacement in the x-direction
v_0	In-plane displacement in the y-direction
w	In-plane displacement in the z-direction
w^*	Initial deformation or the geometric imperfection
q	Transverse applied load
ε_0	Mid-plane strain matrix
κ	Curvature matrix
σ_x	Normal stress in the x-direction
τ_{xy}	Shear stress in the xy-direction

τ_{yz}	Shear stress in the yz-direction
τ_{xz}	Shear stress in the xz-direction
ϕ	Shear deformation
ΔT_0	Constant temperature increment
ΔT_h	Temperature increment through the thickness of the beam
ΔT_{cr}	Critical buckling temperature increment
ΔH	Constant moisture increment
α_x	Coefficient of thermal expansion
β_x	Coefficient of moisture expansion
μ	Coefficient of geometric imperfection

ABBREVIATIONS

CBT	Classical beam theory
CC	Clamped-clamped
CS	Clamped at the left end and Simply supported at the right end
FSDT	First-order shear deformation theory
SS	Simply supported

ABSTRACT

This thesis is concerned with the analytical study of the thermal bending, buckling, and post-buckling of unsymmetrically laminated composite beams with imperfection under hygrothermal effects. Three different boundary conditions will be considered on this study. The non-linear governing partial differential equations are derived by taking into account the von-Karman geometrical nonlinearity for an imperfect unsymmetrical laminated composite beam. Classical beam theory (CBT) as well as first order shear deformation theory (FSDT) will be used. The effects of temperature, angle of orientation, moisture variations, imperfection, and geometrical parameters, will be evaluated and discussed. Two different laminated composite laminates will be considered: unsymmetrical cross-ply and unsymmetrical angle-ply.

1. Introduction

Composite materials are widely used due to their high strength-to-weight ratio and the controllability of their properties with the variation of the fiber angle. Because of their applications in harsh environments, composite materials are often exposed to variation of temperatures and changes in moisture. It is important to study the effects of moisture on the thermal buckling behavior of laminated beams because even if they were designed to be symmetric, a beam could deform if subjected to a critical thermal load.

It is well known that laminated composite materials are broadly used for aerospace applications mainly because of their light weight yet strong and extremely stiff (Eslami, 2018). Usually, symmetrically laminated composite materials are more desirable since they do not have bending-extension coupling (Pompei, 1994). However, in some cases, such as for jet turbine fan blades with a pre-twist, the non-symmetric laminated composites are desired to achieve the design requirements (Pompei, 1994). The main characteristic of unsymmetrically laminated composites is that they have a bending-extension coupling which complicates the analysis. Geometric imperfections are common among composite structures due to manufacturing and environmental factors (Emam, 2009) and particularly unsymmetrically laminated composite structures. Additionally, no structure can be perfectly flat and straight (Brush & Almroth, 1975). Therefore, even symmetrically laminated composites can become unsymmetrical after manufacturing processes. In the case of unsymmetrical laminates, geometric imperfections are more common because of their bending-extension coupling.

Many aerospace structures, including turbine blades, can be exposed to very high and drastic temperature changes as well as to environmental conditions like moisture, and that is the motive to research the buckling behavior of unsymmetrical laminated composites with temperatures and moisture effects. When composite structures are exposed to hygrothermal environment conditions, structural failure can occur due to problems like dimensional stability, residual stresses, material degradation, or delamination (Emam, 2016). Buckling is one common mode of failure when structures are exposed to hygrothermal effects (Emam, 2016).

In the past decades, the study of the elastic stability (or buckling) of plate structures has become very important because plates tend to buckle at very low applied stress, causing large deformations. This behavior can be very dangerous for the structures. Buckling can be defined as a sudden large deformation resulting when a structure is critically loaded in compression. Thus, there is a critical load which causes a structure to deform drastically and lose its ability to carry the load (Wang, Wang & Reddy, 2004). For example, if a rod is subjected to an axially compressive force, it will contract slightly at first, however, when it reaches a critical buckling load, it will bow out or buckle (Wang, Wang & Reddy, 2004). The deflection path that occurs before reaching a critical buckling load or bifurcation buckling is called the primary path. Additionally, the path that exists after the bifurcation point is called the secondary path or post-buckling. The post-buckling path can be symmetric or asymmetric depending on the structural properties and its loading.

1.1. Literature Survey

Since the structure considered in this study is a beam, a literature survey on the nonlinear behavior of beams was conducted. It should be noted that the literature survey was conducted in three parts: bending, buckling, and post-buckling.

For cylindrical bending, the nonlinear governing equations for cross-ply laminates simply-supported beam were reduced to linear differential equations with nonlinear boundary conditions (Sun & Chin, 1998). As expected, Sun and Chin (1998) concluded that the large deflection theory could not be neglected when studying asymmetric composites. Park (2000) presented a nonlinear analysis of unsymmetrical long and narrow laminated beams under cylindrical bending. In this paper, the researcher derived the nonlinear equations of motion considering the classical theory and the first-order shear deformation theory for angle ply laminates. Park (2000) also concluded that for unsymmetrical composites, the nonlinearity could not be neglected even when subjected to small loads. It was determined that the fiber angle is directly proportional to the maximum deflection and inversely proportional to the in-plane load.

In the book by E. A. Thornton (1996), the author discusses thermally induced deformations and stresses of isotropic beams and rods used in aerospace structures in chapter 6. The author employed linear theory to derive the governing equations of a beam, and later, the author also used this equation in chapter 10 to study the thermal buckling behavior of isotropic beams under uniform temperature rise and linear temperature variation. The same method developed by Thornton (1996) in chapter 10, will be extended for angle-ply laminated composite beams in section 5 of this thesis. Majeed (2005) presented a thesis that focuses on the response of flat unsymmetrical laminated laminates subjected to in-plane compressive loading. The purpose of the study was to determine if unsymmetrical laminated composites can undergo bifurcation buckling. For antisymmetric angle-ply composites, the nonlinear theory predicted that there would be a deflection once the load reaches the critical compressive load which is called the postbuckling deflection.

Some authors have achieved a close-form solution for the post-buckling analysis of composite beams. Gupta, Gunda, Janardhan, and Rao (2009) used the Rayleigh-Ritz method to obtain simple expressions for the critical buckling load, and post-buckling axial load of composite beams considering several types of boundary conditions like hinged guided and conventional supports. Gunda and Rao (2013) continued the same study performed by Gupta et al. (2009), and concluded that there is a slight deviation of the previous results which is due to the assumed mode shape used in the study. Other authors have researched the nonlinear vibration of unsymmetrically laminated composite beams. Pompei (1994) studied the forced vibration of angle-ply and cross-ply laminated beams, whereas Emam and Nayfeh (2009) studied the free vibration and post-buckling of different ply configurations.

Khdeir (1999) presented a thermal buckling analysis of symmetric cross-ply beams with different boundary conditions. The analysis was based on a three-degree-of-freedom shear deformable beam theory and used a shape function to account for the continuity of symmetrically laminated beams. Khdeir (1999) concluded that some cross-ply beams buckle upon cooling instead of heating. Fu, Wang and Hu (2014) derived the governing equations for thermal buckling and post-buckling of symmetric cross-ply laminated beams using the von-Karman nonlinearity and the first-order shear deformation theory. Three different methods of solution were performed to find the critical buckling and post-buckling amplitudes of composite beams with general boundary conditions and mixed boundary conditions (Fu et al., 2014). Aydogdu (2007) presented an analysis of the thermal buckling of cross-ply laminates. The author used the energy method to derive the governing equations and the Ritz method to develop a solution for the critical buckling load and the critical buckling temperature. Thivend, Eslami and Zhao, (2008) analyzed the thermal post-buckling of functionally graded materials, which are a sophisticated form of asymmetric composites. Thivend et al. (2008) noted that the effective length of a clamped-clamped beam in post-buckling would be affected by the temperature.

Emam and Eltaher (2016) conducted a buckling and post-buckling study of composite beams with the effects of temperature and moisture. The classical beam theory and higher-order shear deformation theory were applied to calculate the critical buckling load and the post-buckling amplitude varying the temperature, moisture, and fiber volume. Emam (2009) also presented a study on the static and dynamic behavior of geometrically imperfect laminated composite beams with fixed supports at both ends. The amplitude of the imperfection was a function of the material properties, which means that the critical buckling could be enhanced by manipulating the imperfection amplitude. Also, the imperfection was found to have a significant effect on the vibrations (Emam, 2009).

1.2. Scope and Motivation

The purpose of this thesis is to perform an extensive analysis of the nonlinear bending, buckling, and post-buckling of unsymmetrically laminated composite beams. Even though previous authors have also conducted a nonlinear thermal analysis of unsymmetrically composite beams, the present study takes into account different types of temperature variations in conjunction with the effects of moisture and imperfections. Indeed, based on the literature survey presented above and extensive research performed by the author, in the case of the nonlinear bending, there are no published works that consider a temperature variation through the thickness and the length of the beam.

Two types of asymmetric laminate configurations are considered in this study. The primary purpose is to note the difference between the buckling behavior of angle-ply and cross-ply laminates. It is expected that angle-ply laminates will present a behavior similar to the bifurcation buckling when subjected to a constant temperature rise.

The nonlinear governing equations of motion will be derived applying both the classical theory and the first-order shear deformation theory in conjunction with the von-Karman geometric nonlinearity. A solution method is to be developed for different boundary conditions: SS; CS; and CS. For simplicity, the solution methods will be presented for the classical theory only since the same procedure will apply for the shear theory. However, the effect of the shear will be shown in the numerical examples presented in section 6.

2. Formulation of the Governing Equations of Motion

This section is concerned with the derivation of the nonlinear differential equations of an unsymmetrically laminated beam under the effects of temperature, moisture, and geometric imperfections. The beam considered in this thesis was undergoing large deflections and subjected to a transverse load. First, the governing equations for a thick beam will be derived by using the classical theory assumptions. For thick beams, the effects of shear can be neglected. However, the effects of the shear are much significant when the beam is thin. Therefore, the governing equations of motion will also be derived by following the assumptions of the first-order shear deformation theory.

Figure 2.1 shows the dimension of the beam where:

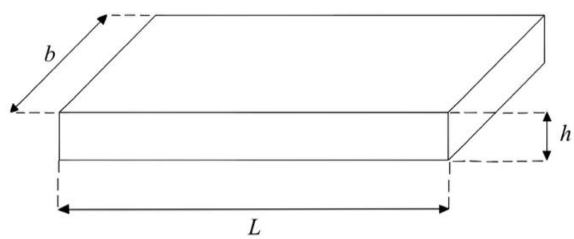

Figure 2.1. Dimensions of the beam.

2.1. Classical Theory

The beam type considered here was a Bernoulli-Euler beam. Thus, the following assumptions are considered to derive the governing equations of motion for a thick beam.

- The beam length is much larger than its width and thickness.
- The in-plane stresses ε_x, ε_z, and γ_{xz} are small compared to unity.
- The effects of shear and rotary inertia are neglected.
- The problem is a plane stress type problem.
- The transverse deflection w is a function of □ only

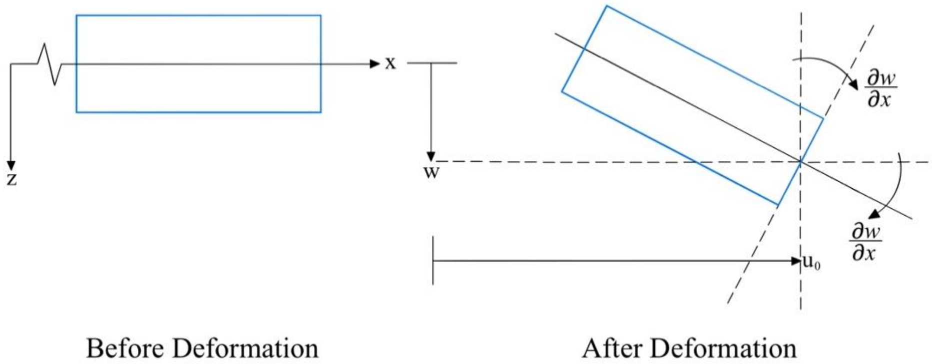

Figure 2.2. Bernoulli-Euler cross-section undergoing bending.

Composite Beam Constitutive Equations

The governing equations for a composite beam can be derived from the constitutive equations of a composite laminate. Consider an unsymmetrical laminate subjected to a uniform in-plane to thermal loads with the effect of moisture:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} - \begin{Bmatrix} N_{ij}^T \\ M_{ij}^T \end{Bmatrix} - \begin{Bmatrix} N_{ij}^H \\ M_{ij}^H \end{Bmatrix} \tag{2.1}$$

where

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \tag{2.2.1}$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \tag{2.2.2}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{2.2.3}$$

$$N_{ij}^T = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \left(\bar{Q}_{ij}\right)_k \left(\alpha_{ij}\right)_k \Delta T dz \tag{2.3.1}$$

$$M_{ij}^T = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \left(\bar{Q}_{ij}\right)_k \left(\alpha_{ij}\right)_k \Delta T z dz \tag{2.3.2}$$

$$N_{ij}^{H} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \left(\bar{Q}_{ij}\right)_k \left(\beta_{ij}\right)_k \Delta H dz \tag{2.3.3}$$

$$M_{ij}^{H} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \left(\bar{Q}_{ij}\right)_k \left(\beta_{ij}\right)_k \Delta H z dz \tag{2.3.4}$$

where,

$$i, j = 1, 2, 6.$$

For a composite beam with geometrical imperfections, the non-linear strain-displacement and curvature-displacement can be expressed using von-Karman nonlinearity as:

$$\{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{\partial w}{\partial x}\dfrac{\partial w^*}{\partial x} \\ 0 \\ \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \tag{2.4}$$

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w^*}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} \tag{2.5}$$

Therefore, the resultant in-plane loads and moments are defined as shown below:

$$N_x = A_{11}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + A_{16}\frac{\partial v_0}{\partial x} - B_{11}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) - N_x^T - N_x^H \tag{2.6}$$

$$N_{xy} = A_{16}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + A_{66}\frac{\partial v_0}{\partial x} - B_{16}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) \tag{2.7}$$

$$M_x = B_{11}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + B_{16}\frac{\partial v_0}{\partial x} - D_{11}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) - M_x^T - M_x^H \tag{2.8}$$

From the equilibrium equations for a composite plate, ignoring inertial loads

In the x-direction,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{2.9}$$

In the y-direction,

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_x}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{2.10}$$

In the z-direction,

$$\begin{gathered}\frac{\partial}{\partial x}\left(\tau_{xz} + \sigma_x \frac{\partial w}{\partial x} + \tau_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(\tau_{yz} + \tau_{xy}\frac{\partial w}{\partial x} + \sigma_y \frac{\partial w}{\partial y}\right) \\ + \frac{\partial}{\partial z}\left(\sigma_z + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y}\right) = 0\end{gathered} \tag{2.11}$$

Multiplying both sides of the equilibrium equations by dz and integrating:

$$\int_{h_{k-1}}^{h_k} \frac{\partial \sigma_x}{\partial x} dz + \int_{h_{k-1}}^{h_k} \frac{\partial \tau_{xy}}{\partial y} dz + \int_{h_{k-1}}^{h_k} \frac{\partial \tau_{xz}}{\partial z} dz = 0$$

$$\int_{h_{k-1}}^{h_k} \frac{\partial \tau_{xy}}{\partial x} dz + \int_{h_{k-1}}^{h_k} \frac{\partial \sigma_x}{\partial y} dz + \int_{h_{k-1}}^{h_k} \frac{\partial \tau_{yz}}{\partial z} dz = 0$$

$$\begin{gathered}\int_{h_{k-1}}^{h_k} \frac{\partial}{\partial x}\left(\tau_{xz} + \sigma_x \frac{\partial w}{\partial x} + \tau_{xy}\frac{\partial w}{\partial y}\right) dz + \int_{h_{k-1}}^{h_k} \frac{\partial}{\partial y}\left(\tau_{yz} + \tau_{xy}\frac{\partial w}{\partial x} + \sigma_y \frac{\partial w}{\partial y}\right) dz \\ + \int_{h_{k-1}}^{h_k} \frac{\partial}{\partial z}\left(\sigma_z + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y}\right) dz = 0\end{gathered}$$

Thus,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{2.12}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{2.13}$$

$$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P = 0 \tag{2.14}$$

where,

$$P = \sigma_z|_{h_{k-1}}^{h_k} \tag{2.15}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \tag{2.16}$$

Similarly, multiplying both sides of the equilibrium equations by zdz and integrating,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{2.17}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{2.18}$$

differentiating with respect to x and y respectively,

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} \tag{2.19}$$

$$\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \tag{2.20}$$

Substituting these last two equations in the equation (2.14).

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = 0 \tag{2.21}$$

Since the length of the beam is larger than its cross-sectional area, the dy terms in equations (2.12), (2.13) and (2.21) are neglected. Thus, the equilibrium equations for a composite beam will be as shown below.

$$\frac{\partial N_x}{\partial x} = 0 \tag{2.22}$$

$$\frac{\partial N_{xy}}{\partial x} = 0 \tag{2.23}$$

$$\frac{\partial^2 M_x}{\partial x^2} + N_x \frac{\partial^2 w}{\partial x^2} + q = 0 \tag{2.24}$$

To obtain the governing equations for a composite beam, equations (2.6), (2.7) and (2.8) are substituted in the equilibrium equations found above.

$$\begin{aligned} A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} - B_{11}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) + A_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} \\ + A_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + A_{11}\frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x} - \frac{\partial N_x^T}{\partial x} - \frac{\partial N_x^H}{\partial x} = 0 \end{aligned} \tag{2.25}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 v_0}{\partial x^2}-B_{16}\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w^*}{\partial x^3}\right)+A_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + A_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x}+A_{16}\frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x}=0 \tag{2.26}$$

$$-D_{11}\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w^*}{\partial x^4}\right) + B_{11}\left[\frac{\partial^3 u_0}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial x}+\frac{\partial^3 w}{\partial x^3}\frac{\partial w^*}{\partial x} + \frac{\partial^3 w^*}{\partial x^3}\frac{\partial w}{\partial x}+2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}\right]+B_{16}\frac{\partial^3 v_0}{\partial x^3}+N_x^0\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 M_x^T}{\partial x^2}-\frac{\partial^2 M_x^H}{\partial x^2}+q=0 \tag{2.27}$$

To find the governing equations in terms of w, several algebraic manipulations are done to equations (2.25) and (2.26). These simplifications are shown as:

$$\begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 u_0}{\partial x^2} \\ \dfrac{\partial^2 v_0}{\partial x^2} \end{Bmatrix}=\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}=\begin{Bmatrix} B_{11}\left(\dfrac{\partial^3 w}{\partial x^3}+\dfrac{\partial^3 w^*}{\partial x^3}\right)-A_{11}\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial x}+\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w_0}{\partial x}+\dfrac{\partial^2 w_0}{\partial x^2}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial N_x^T}{\partial x}+\dfrac{\partial N_x^H}{\partial x} \\ B_{16}\left(\dfrac{\partial^3 w}{\partial x^3}+\dfrac{\partial^3 w^*}{\partial x^3}\right)-A_{16}\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial x}+\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w_0}{\partial x}+\dfrac{\partial^2 w_0}{\partial x^2}\dfrac{\partial w}{\partial x}\right) \end{Bmatrix}$$

solving for $\frac{\partial^2 u_0}{\partial x^2}$ and $\frac{\partial^2 v_0}{\partial x^2}$,

$$\begin{Bmatrix} \dfrac{\partial^2 u_0}{\partial x^2} \\ \dfrac{\partial^2 v_0}{\partial x^2} \end{Bmatrix}=\begin{Bmatrix} \dfrac{A_{16}\eta_2-A_{66}\eta_1}{A_{16}^2-A_{11}A_{66}} \\ \dfrac{A_{16}\eta_1-A_{11}\eta_2}{A_{16}^2-A_{11}A_{66}} \end{Bmatrix}$$

rearranging,

$$\frac{\partial^2 u_0}{\partial x^2} = \left(\frac{B_{16}A_{16} - B_{11}A_{66}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) + \left(\frac{A_{11}A_{66} - A_{16}^2}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x}\right) - \left(\frac{A_{66}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right)$$

$$\frac{\partial^2 v_0}{\partial x^2} = \left(\frac{B_{11}A_{16} - B_{16}A_{11}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) + \left(\frac{A_{16}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right)$$

Hence, $\frac{\partial^2 u_0}{\partial x^2}$ and $\frac{\partial^2 v_0}{\partial x^2}$ and their derivatives can be expressed as a function of w only.

These equations are shown below.

$$\frac{\partial^2 u_0}{\partial x^2} = K_1\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) - \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x}\right) - K_1^T\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right) \tag{2.28}$$

$$\frac{\partial^2 v_0}{\partial x^2} = K_2\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) + K_2^T\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right) \tag{2.29}$$

$$\frac{\partial^3 u_0}{\partial x^3} = K_1\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) - \left[\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial^3 w}{\partial x^3}\frac{\partial w^*}{\partial x} + \frac{\partial^3 w^*}{\partial x^3}\frac{\partial w}{\partial x} + 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}\right] - K_1^T\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) \tag{2.30}$$

$$\frac{\partial^3 v_0}{\partial x^3} = K_2\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) + K_2^T\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) \tag{2.31}$$

$$K_1 = \frac{B_{16}A_{16} - B_{11}A_{66}}{A_{16}^2 - A_{11}A_{66}} \tag{2.32}$$

$$K_2 = \frac{B_{11}A_{16} - B_{16}A_{11}}{A_{16}^2 - A_{11}A_{66}}$$

$$K_1^T = \frac{A_{66}}{A_{16}^2 - A_{11}A_{66}}$$

$$K_2^T = \frac{A_{16}}{A_{16}^2 - A_{11}A_{66}}$$

Finally, substituting equations (2.30) and (2.31) in equation (2.27), the non-linear differential equation can be expressed as follows:

$$-D_{11}\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w^*}{\partial x^4}\right)$$
$$+B_{11}\left[\frac{\partial^3 u_0}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial x}+\frac{\partial^3 w}{\partial x^3}\frac{\partial w^*}{\partial x}+\frac{\partial^3 w^*}{\partial x^3}\frac{\partial w}{\partial x}\right.$$
$$\left.+2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}\right]+B_{16}\frac{\partial^3 v_0}{\partial x^3}+N_x^0\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 M_x^T}{\partial x^2}-\frac{\partial^2 M_x^H}{\partial x^2}+P=0$$

$$-D_{11}\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w^*}{\partial x^4}\right)+B_{11}K_1\frac{\partial^4 w}{\partial x^4}-B_{11}\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial x}-B_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2-B_{11}\frac{\partial^3 w}{\partial x^3}\frac{\partial w^*}{\partial x}$$
$$-B_{11}\frac{\partial^3 w^*}{\partial x^3}\frac{\partial w}{\partial x}-2B_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}-B_{11}K_1^T\left(\frac{\partial^2 N_x^T}{\partial x^2}+\frac{\partial^2 N_x^H}{\partial x^2}\right)$$
$$+B_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+B_{11}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}+B_{11}\frac{\partial^3 w}{\partial x^3}\frac{\partial w^*}{\partial x}+B_{11}\frac{\partial^3 w^*}{\partial x^3}\frac{\partial w}{\partial x}$$
$$+2B_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial x^2}+B_{16}K_2\frac{\partial^4 w}{\partial x^4}+B_{16}K_2^T\left(\frac{\partial^2 N_x^T}{\partial x^2}+\frac{\partial^2 N_x^H}{\partial x^2}\right)+N_x^0\frac{\partial^2 w}{\partial x^2}$$
$$-\frac{\partial^2 M_x^T}{\partial x^2}-\frac{\partial^2 M_x^H}{\partial x^2}+P=0$$

In which N_x^0 can be defined by multiplying equation (6) by dx and integrating over the beam's length.

$$\int_0^L N_x dx=\int_0^L A_{11}\frac{\partial u_0}{\partial x}dx+\int_0^L\frac{A_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2dx+\int_0^L A_{11}\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}dx+\int_0^L A_{16}\frac{\partial v_0}{\partial x}dx$$
$$-\int_0^L B_{11}\frac{\partial^2 w}{\partial x^2}dx-\int_0^L N_x^T dx-\int_0^L N_x^H dx$$

$$N_x^0=\frac{A_{11}}{L}\big(u_0(L)-u_0(0)\big)+\frac{A_{16}}{L}\big(v_0(L)-v_0(0)\big)+\frac{A_{11}}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2dx$$
$$+\frac{A_{11}}{L}\int_0^L\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}dx-\frac{B_{11}}{L}\int_0^L\frac{\partial^2 w}{\partial x^2}dx-N_x^T-N_x^H$$

$$\big(u_0(L)-u_0(0)\big)=0$$
$$\big(v_0(L)-v_0(0)\big)=0$$

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{A_{11}}{L}\int_0^L \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} dx - \frac{B_{11}}{L}\int_0^L \frac{\partial^2 w}{\partial x^2} dx - N_x^T - N_x^H \tag{2.33}$$

$$EI_{eff}\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) + N_x^0 \frac{\partial^2 w}{\partial x^2} + (B_{16}K_2^T - B_{11}K_1^T)\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) - \frac{\partial^2 M_x^T}{\partial x^2} - \frac{\partial^2 M_x^H}{\partial x^2} + q = 0 \tag{2.34}$$

$$EI_{eff} = B_{11}K_1 + B_{16}K_2 - D_{11} \tag{2.35}$$

The moment M_x is defined as a function of w only. From equations (2.28) and (2.29), we obtain:

$$\frac{\partial u_0}{\partial x} = K_1\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + \frac{N_x^0 + N_x^T + N_x^H}{A_{11}} \tag{2.36}$$

$$\frac{\partial v_0}{\partial x} = K_2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) \tag{2.37}$$

Then, equations (2.36) and (2.37) are substituted in equations (2.8).

$$M_x = B_{11}K_1\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) - \frac{B_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2 - B_{11}\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + \frac{B_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2 + B_{11}\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + B_{16}K_2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} + \frac{B_{11}}{A_{11}}(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$M_x = EI_{eff}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2}\right) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H) \tag{2.38}$$

2.2. First-Order Shear Deformation Theory

The following assumptions are considering for the derivation of the governing equations of motion with the first-order shear deformation theory:

- The beam length is much larger than its width and thickness.
- The in-plane strains ε_x, ε_z, and γ_{xz} are small compared to unity.
- The effects of shear are considered, but the in-plane strain γ_{yz} is neglected

- The problem is a plane stress type problem.
- The transverse deflection w is a function in terms of x only

$$\gamma_{xy} = \frac{\partial w}{\partial x} + \frac{\partial w^*}{\partial x} - \phi \tag{2.39}$$

In equation (2.39), $\frac{\partial w}{\partial x}$ is the total deflection of the beam, and ϕ is the rotary deformation. Hence, considering the shear effects, curvature-displacement relations can be redefined.

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = -\begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ 0 \\ 0 \end{Bmatrix} \tag{2.40}$$

Therefore, the resultant in-plane loads and moments with the effects of shear are defined as bellow:

$$N_x = A_{11}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + A_{16}\frac{\partial v_0}{\partial x} - B_{11}\frac{\partial \phi}{\partial x} - N_x^T - N_x^H \tag{2.41}$$

$$N_{xy} = A_{16}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + A_{66}\frac{\partial v_0}{\partial x} - B_{16}\frac{\partial \phi}{\partial x} \tag{2.42}$$

$$M_x = B_{11}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right] + B_{16}\frac{\partial v_0}{\partial x} - D_{11}\frac{\partial \phi}{\partial x} - M_{xy}^T - M_{xy}^H \tag{2.43}$$

Equilibrium Equations for a Plate

From previous derivations, recall equations (2.22) and (2.23). However, two more equilibrium equations are needed to solve for the deflection in terms of w only. These two equilibrium equations can be obtained by rearranging equations (2.14) and (2.17) for a beam. Hence, the equilibrium equations will be as shown below.

$$\frac{\partial N_x}{\partial x} = 0 \tag{2.22}$$

$$\frac{\partial N_{xy}}{\partial x} = 0 \tag{2.23}$$

$$\frac{\partial M_x}{\partial x} - Q_x = 0 \tag{2.44}$$

$$N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_x}{\partial x} + q = 0 \tag{2.45}$$

where,

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

$$Q_x = K_s A_{55} \left(\frac{\partial w}{\partial x} + \frac{\partial w^*}{\partial x} - \phi \right) \tag{2.46}$$

$$\frac{\partial Q_x}{\partial x} = K_s A_{55} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) \tag{2.47}$$

K_s is a constant shear factor that depends on the cross-section shape of the beam.

To obtain the governing equations of motion for a composite beam with the shear effects, equations (2.41), (2.42) and (2.43) are substituted in the equilibrium equations found above.

$$\begin{gathered} A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} - B_{11} \frac{\partial^2 \phi}{\partial x^2} + A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} + A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w^*}{\partial x} \\ + A_{11} \frac{\partial^2 w^*}{\partial x^2} \frac{\partial w}{\partial x} - \frac{\partial N_x^T}{\partial x} - \frac{\partial N_x^H}{\partial x} = 0 \end{gathered} \tag{2.48}$$

$$\begin{gathered} A_{16} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{16} \frac{\partial^2 \phi}{\partial x^2} + A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} + A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w^*}{\partial x} \\ + A_{16} \frac{\partial^2 w^*}{\partial x^2} \frac{\partial w}{\partial x} = 0 \end{gathered} \tag{2.49}$$

$$\begin{gathered} B_{11} \frac{\partial^2 u_0}{\partial x^2} + B_{16} \frac{\partial^2 v_0}{\partial x^2} - D_{11} \frac{\partial^2 \phi}{\partial x^2} + B_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} + B_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w^*}{\partial x} \\ + B_{11} \frac{\partial^2 w^*}{\partial x^2} \frac{\partial w}{\partial x} - \frac{\partial M_x^T}{\partial x} - \frac{\partial M_x^H}{\partial x} - Q_x = 0 \end{gathered} \tag{2.50}$$

$$N_x^0 \frac{\partial^2 w}{\partial x^2} + K_s A_{55} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) + q = 0 \tag{2.51}$$

Where N_x^0 is obtained by multiplying N_x by dx and integrating over the beam's length.

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{A_{11}}{L}\int_0^L \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}dx - \frac{B_{11}}{L}\int_0^L \frac{\partial \phi}{\partial x}dx - N_x^T - N_x^M \tag{2.52}$$

To find the governing equations of motions in terms of w only several algebraic manipulations are done to equations (2.48), (2.49) and (2.50). These simplifications are shown below.

$$\begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 u_0}{\partial x^2} \\ \dfrac{\partial^2 v_0}{\partial x^2} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} B_{11}\dfrac{\partial^2 \phi}{\partial x^2} - A_{11}\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w^*}{\partial x} + \dfrac{\partial^2 w^*}{\partial x^2}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial N_x^T}{\partial x} + \dfrac{\partial N_x^H}{\partial x} \\ B_{16}\dfrac{\partial^2 \phi}{\partial x^2} - A_{16}\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial w^*}{\partial x} + \dfrac{\partial^2 w^*}{\partial x^2}\dfrac{\partial w}{\partial x}\right) \end{Bmatrix}$$

solving for $\frac{\partial^2 u_0}{\partial x^2}$ and $\frac{\partial^2 v_0}{\partial x^2}$,

$$\begin{Bmatrix} \dfrac{\partial^2 u_0}{\partial x^2} \\ \dfrac{\partial^2 v_0}{\partial x^2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{A_{16}\eta_2 - A_{66}\eta_1}{A_{16}^2 - A_{11}A_{66}} \\ \dfrac{A_{16}\eta_1 - A_{11}\eta_2}{A_{16}^2 - A_{11}A_{66}} \end{Bmatrix}$$

rearranging,

$$\frac{\partial^2 u_0}{\partial x^2} = \left(\frac{B_{16}A_{16} - B_{11}A_{66}}{A_{16}^2 - A_{11}A_{66}}\right)\frac{\partial^2 \phi}{\partial x^2} + \left(\frac{A_{11}A_{66} - A_{16}^2}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x}\right) - \left(\frac{A_{66}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right)$$

$$\frac{\partial^2 v_0}{\partial x^2} = \left(\frac{B_{11}A_{16} - B_{16}A_{11}}{A_{16}^2 - A_{11}A_{66}}\right)\frac{\partial^2 \phi}{\partial x^2} + \left(\frac{A_{16}}{A_{16}^2 - A_{11}A_{66}}\right)\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right)$$

Hence, $\frac{\partial^2 u_0}{\partial x^2}$ and $\frac{\partial^2 v_0}{\partial x^2}$ and can be expressed as a function of only w. These equations are shown below.

$$\frac{\partial^2 u_0}{\partial x^2} = K_1 \frac{\partial^2 \phi}{\partial x^2} - \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x} \right) - K_1^T \left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x} \right) \tag{2.53}$$

$$\frac{\partial^2 v_0}{\partial x^2} = K_2 \frac{\partial^2 \phi}{\partial x^2} + K_2^T \left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x} \right) \tag{2.54}$$

K_1 and K_2 were previously defined in equations (2.32.1) and (2.32.2).

Then, replacing equations (2.53) and (2.54) in equation (2.50), an expression for ϕ can be obtained as shown below.

$$B_{11}K_1 \frac{\partial^2 \phi}{\partial x^2} - B_{11} \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x} \right) - B_{11}K_1^T \left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x} \right) + B_{16}K_2 \frac{\partial^2 \phi}{\partial x^2} + B_{16}K_2^T \left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x} \right) - D_{11} \frac{\partial^2 \phi}{\partial x^2} + B_{11} \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x} + B_{11} \frac{\partial^2 w}{\partial x^2}\frac{\partial w^*}{\partial x} + B_{11} \frac{\partial^2 w^*}{\partial x^2}\frac{\partial w}{\partial x} - \frac{\partial M_x^T}{\partial x} - \frac{\partial M_x^H}{\partial x} - Q_x = 0$$

$$EI_{eff} \frac{\partial^2 \phi}{\partial x^2} - K_s A_{55} \frac{\partial w}{\partial x} + K_s A_{55}\phi + (B_{16}K_2^T - B_{11}K_1^T) \left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x} \right) - \left(\frac{\partial M_x^T}{\partial x} + \frac{\partial M_x^H}{\partial x} \right) = 0$$

Thus,

$$\phi = \frac{\partial w}{\partial x} - \frac{EI_{eff}}{K_s A_{55}} \frac{\partial^2 \phi}{\partial x^2} - \frac{(B_{16}K_2^T - B_{11}K_1^T)}{K_s A_{55}} \left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2} \right) + \frac{1}{K_s A_{55}} \left(\frac{\partial M_x^T}{\partial x} + \frac{\partial M_x^H}{\partial x} \right) \tag{2.55}$$

From equation (2.51), $\frac{\partial \phi}{\partial x}$ and $\frac{\partial^2 \phi}{\partial x^2}$ can be outlined as

$$\frac{\partial \phi}{\partial x} = \left(1 + \frac{N_x^0}{K_s A_{55}} \right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w^*}{\partial x^2} \right) + \frac{q}{K_s A_{55}} \tag{2.56}$$

$$\frac{\partial^2\phi}{\partial x^2} = \left(1 + \frac{N_x^0}{K_s A_{55}}\right)\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) \tag{2.57}$$

Thus, replacing equation (2.57) into equation (2.55) and deriving, a new expression for $\frac{\partial\phi}{\partial x}$ is found as shown in equation (2.58).

$$\phi = \frac{\partial w}{\partial x} - \frac{EI_{eff}}{K_s A_{55}}\left(1 + \frac{N_x^0}{K_s A_{55}}\right)\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w^*}{\partial x^3}\right) - \frac{(B_{16}K_2^T - B_{11}K_1^T)}{K_s A_{55}}\left(\frac{\partial N_x^T}{\partial x} + \frac{\partial N_x^H}{\partial x}\right) + \frac{1}{K_s A_{55}}\left(\frac{\partial M_x^T}{\partial x} + \frac{\partial M_x^H}{\partial x}\right)$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{EI_{eff}}{K_s A_{55}}\left(1 + \frac{N_x^0}{K_s A_{55}}\right)\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) - \frac{(B_{16}K_2^T - B_{11}K_1^T)}{K_s A_{55}}\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) + \frac{1}{K_s A_{55}}\left(\frac{\partial^2 M_x^T}{\partial x^2} + \frac{\partial^2 M_x^H}{\partial x^2}\right) \tag{2.58}$$

Equating equations (2.58) and (2.56), a non-linear differential equation of motion can be obtained.

$$\frac{\partial^2 w}{\partial x^2} - \frac{EI_{eff}}{K_s A_{55}}\left(1 + \frac{N_x^0}{K_s A_{55}}\right)\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) - \frac{(B_{16}K_2^T - B_{11}K_1^T)}{K_s A_{55}}\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) + \frac{1}{K_s A_{55}}\left(\frac{\partial^2 M_x^T}{\partial x^2} + \frac{\partial^2 M_x^H}{\partial x^2}\right) = \left(1 + \frac{N_x^0}{K_s A_{55}}\right)\frac{\partial^2 w}{\partial x^2} + \frac{q}{K_s A_{55}}$$

$$\frac{EI_{eff}}{K_s A_{55}}\left(1 + \frac{N_x^0}{K_s A_{55}}\right)\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w^*}{\partial x^4}\right) + \left(\frac{N_x^0}{K_s A_{55}}\right)\frac{\partial^2 w}{\partial x^2} + \frac{q}{K_s A_{55}} + \frac{(B_{16}K_2^T - B_{11}K_1^T)}{K_s A_{55}}\left(\frac{\partial^2 N_x^T}{\partial x^2} + \frac{\partial^2 N_x^H}{\partial x^2}\right) - \frac{1}{K_s A_{55}}\left(\frac{\partial^2 M_x^T}{\partial x^2} + \frac{\partial^2 M_x^H}{\partial x^2}\right) = 0$$

$$EI_{eff}\left(1+\frac{N_x^0}{K_sA_{55}}\right)\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w^*}{\partial x^4}\right)+N_x^0\frac{\partial^2 w}{\partial x^2} + (B_{16}K_2^T - B_{11}K_1^T)\left(\frac{\partial^2 N_x^T}{\partial x^2}+\frac{\partial^2 N_x^H}{\partial x^2}\right) - \left(\frac{\partial^2 M_x^T}{\partial x^2}+\frac{\partial^2 M_x^H}{\partial x^2}\right)+q=0 \tag{2.59}$$

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{A_{11}}{L}\int_0^L \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}dx - \frac{B_{11}}{L}\int_0^L\left[\left(1+\frac{N_x^0}{K_sA_{55}}\right)\frac{\partial^2 w}{\partial x^2}+\frac{q}{K_sA_{55}}\right]dx - N_x^T - N_x^H \tag{2.60}$$

The moment M_x is defined as a function of only w. From equations (2.53) and (2.54):

$$\frac{\partial u_0}{\partial x} = K_1\frac{\partial \phi}{\partial x} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + \frac{N_x^0 + N_x^T + N_x^H}{A_{11}} \tag{2.61}$$

$$\frac{\partial v_0}{\partial x} = K_2\frac{\partial \phi}{\partial x} \tag{2.62}$$

Then, equations (2.61) and (2.62) are substituted in equations (2.43) to give:

$$M_x = B_{11}K_1\frac{\partial \phi}{\partial x} - \frac{B_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2 - B_{11}\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + \frac{B_{11}}{2}\left(\frac{\partial w}{\partial x}\right)^2 + B_{11}\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x} + B_{16}K_2\frac{\partial^2 w}{\partial x^2} - D_{11}\frac{\partial \phi}{\partial x} + \frac{B_{11}}{A_{11}}(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$M_x = EI_{eff}\frac{\partial \phi}{\partial x} + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$M_x = EI_{eff}\left[\left(1+\frac{N_x^0}{K_sA_{55}}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w^*}{\partial x^2}\right)+\frac{q}{K_sA_{55}}\right] + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H) \tag{2.63}$$

2.3. Material Modeling

Two types of unsymmetrically laminated composites considered in this thesis: angle-ply and cross-ply laminated composite. It is important to note that some of the stiffness constants are equal to zero for angle-ply and cross-ply.

For angle-ply laminated composite:

$$A_{16} = A_{26} = 0 \tag{2.64}$$
$$B_{11} = B_{12} = B_{22} = B_{66} = B_{11}^{T} = 0 \tag{2.65}$$
$$D_{16} = D_{26} = 0 \tag{2.66}$$

Therefore,

$$K_1 = 0 \tag{2.67}$$
$$K_2 = \frac{B_{16}}{A_{66}} \tag{2.68}$$
$$K_1^T = -\frac{1}{A_{11}} \tag{2.69}$$
$$K_2^T = 0 \tag{2.70}$$
$$EI_{eff} = \frac{B_{16}^2}{A_{66}} - D_{11} \tag{2.71}$$
$$EI_{eff}^T = 0 \tag{2.72}$$

For cross-ply laminated composite:

$$A_{16} = A_{26} = 0 \tag{2.73}$$
$$B_{12} = B_{16} = B_{26} = B_{66} = 0 \tag{2.74}$$
$$D_{16} = D_{26} = 0 \tag{2.75}$$

Therefore,

$$K_1 = \frac{B_{11}}{A_{11}} \tag{2.76}$$
$$K_2 = 0 \tag{2.77}$$
$$K_1^T = -\frac{1}{A_{11}} \tag{2.78}$$
$$K_2^T = 0 \tag{2.79}$$
$$EI_{eff} = \frac{B_{11}^2}{A_{11}} - D_{11} \tag{2.80}$$
$$EI_{eff}^T = \frac{B_{11}}{A_{11}} \tag{2.81}$$

3. Method of Solution for Thermal Bending

This section deals with the nonlinear bending analysis for a beam subjected to two different types of loading. The first subsection takes care of the nonlinear bending of a beam subjected to a mechanical transverse load q with the effects of temperature and moisture. The second part of this section is concerned with the derivation and solution of the nonlinear bending of a beam subjected to a thermal load. For both cases, linear temperature rise and a uniform moisture variation are considered.

3.1. Mechanical Loading

To solve for bending of a beam subjected to a transverse loading q, equation (2.34) can be rearranged as follow:

$$\frac{\partial^4 w}{\partial w^4} - \zeta^2 \frac{\partial^2 w}{\partial x^2} = \varphi \tag{3.1}$$

$$\zeta^2 = -\frac{N_x^0}{EI_{eff}} \tag{3.2}$$

$$\varphi = -\frac{q}{EI_{eff}} \tag{3.3}$$

The 4th order ODE shown in equation (3.1) has a solution with two parts: homogeneous solution and a particular solution.

The homogeneous solution can be obtained as follow:

$$m^4 - \zeta^2 m^2 = 0$$

$$m^2(m^2 - \zeta^2) = 0$$

$$m = \pm 0$$

$$m = \pm \zeta$$

Thus, assuming a solution of the form $w(x) = Ce^{mx}$

$$w_h(x) = c_1 e^{0x} + c_2 e^{-0x} + c_3 e^{\zeta x} + c_4 e^{-\zeta x}$$

rearranging,

$$w_h(x) = C_1 + C_2 x + C_3 \sinh \lambda x + C_4 \cosh \lambda x \tag{3.4}$$

Additionally, the particular solution is obtained by assuming a solution of the form:

$$w_p(x) = Ax^2 + Bx + C$$

$$\frac{\partial w_p(x)}{\partial x} = 2Ax + B$$

$$\frac{\partial^2 w_p(x)}{\partial x^2} = 2A$$

To find the constant A, the above equations are substituted in equation (3.1).

$$0 - \zeta^2 (2A) = \varphi$$

$$A = -\frac{\varphi}{2\zeta^2}$$

Therefore,

$$w_p(x) = -\frac{\varphi}{2\zeta^2} x^2 \tag{3.5}$$

The total solution to the differential equation is shown in equation (3.6).

$$w(x) = C_1 + C_2 x + C_3 \sinh \zeta x + C_4 \cosh \zeta x - \frac{\varphi}{2\zeta^2} x^2 \tag{3.6.1}$$

$$\frac{\partial w(x)}{\partial x} = C_2 + C_3 \zeta \cosh \zeta x + C_4 \zeta \sinh \zeta x - \frac{\varphi}{\zeta^2} x \tag{3.6.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = C_3 \zeta^2 \sinh \zeta x + C_4 \zeta^2 \cosh \zeta x - \frac{\varphi}{\zeta^2} \tag{3.6.3}$$

$$\frac{\partial^3 w(x)}{\partial x^3} = C_3 \zeta^3 \cosh \zeta x + C_4 \zeta^3 \sinh \zeta x \tag{3.6.4}$$

Considering a linear variation of temperature as shown in equation (3.7), and a constant moisture gradient, $\Delta M = constant$, the hygrothermal moments and loads for an unsymmetrical laminated beam can be defined as in equation (3.8).

$$\Delta T = \Delta T_0 \left(1 + r\frac{z}{h}\right) \tag{3.7.1}$$

$$r = \frac{\Delta T_h}{\Delta T_0} \tag{3.7.2}$$

$$N_x^T = \sum_{k=1}^{N} \int_{h_{k-1}}^{k_k} (\overline{Q_{11}})_k (\alpha_x)_k \Delta T_0 \left(1 + r\frac{z}{h}\right) dz = \Delta T_0 \left(A_{11}^T + B_{11}^T \frac{r}{h}\right) \tag{3.8.1}$$

$$N_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\beta_x)_k \Delta H dz = A_{11}^H \beta_x \Delta H \tag{3.8.2}$$

$$M_x^T = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\alpha_x)_k \Delta T_0 \left(1 + r\frac{z}{h}\right) zdz = \Delta T_0 \left(B_{11}^T + D_{11}^T \frac{r}{h}\right) \tag{3.8.3}$$

$$M_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\beta_x)_k \Delta H zdz = B_{11}^H \beta_x \Delta H \tag{3.8.4}$$

$$A_{11}^T = \sum_{k=1}^{N} (\bar{Q}_{11})_k h_k (\alpha_x)_k \tag{3.8.5}$$

$$A_{11}^H = \sum_{k=1}^{N} (\bar{Q}_{11})_k h_k (\alpha_x)_k \tag{3.8.6}$$

$$B_{11}^T = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{11})_k h_k^2 (\alpha_x)_k \tag{3.8.7}$$

$$B_{11}^H = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{11})_k h_k^2 (\alpha_x)_k \tag{3.8.8}$$

3.1.1. Simply Supported Beam

For a simply-supported beam, the boundary conditions are as shown below in equation (3.9). For this case, the origin has been placed at mid-span of the beam. Since the origin is in the middle, due to symmetry the second and third constant of deflection will be zero.

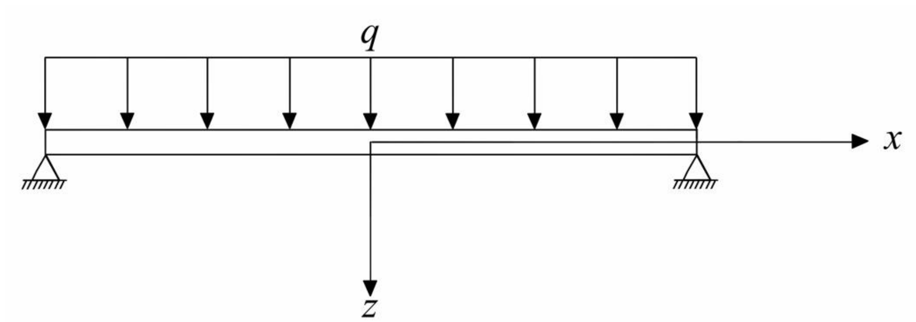

Figure 3.1. Simply-supported beam subjected to mechanical loading.

$$w(-l) = w(l) = 0 \tag{3.9.1}$$

$$\frac{\partial^2 w(-l)}{\partial x^2} = \frac{\partial^2 w(l)}{\partial x^2} = 0 \tag{3.9.2}$$

$$M_x(-l) = M_x(l) = 0 \tag{3.9.3}$$

When $x = -l$, $w(-l) = 0$ and $M_x(-l) = 0$.

$$C_1 + C_2(-l) + C_3 \, sinh \, \zeta(-l) + C_4 \, cosh \, \zeta(-l) - \frac{\varphi}{2\zeta^2}(-l)^2 = 0$$

$$C_1 - C_2 l - C_3 \, sinh \, \zeta l + C_4 \, cosh \, \zeta l - \frac{\varphi}{2\zeta^2} l^2 = 0$$

$$0 = EI_{eff}\left(C_3\zeta^2 \, sinh \, \zeta(-l) + C_4\zeta^2 \, cosh \, \zeta(-l) - \frac{\varphi}{\zeta^2}\right) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$0 = EI_{eff}\left(-C_3\zeta^2 \, sinh \, \zeta l + C_4\zeta^2 \, cosh \, \zeta l - \frac{\varphi}{\zeta^2}\right) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

When $x = l$, $w(l) = 0$ and $M_x(l) = 0$,

$$C_1 + C_2(l) + C_3 \, sinh \, \zeta(l) + C_4 \, cosh \, \zeta(l) - \frac{\varphi}{2\zeta^2}(l)^2 = 0$$

$$C_1 + C_2 l + C_3 \, sinh \, \zeta l + C_4 \, cosh \, \zeta l - \frac{\varphi}{2\zeta^2} l^2 = 0$$

$$0 = EI_{eff}\left(C_3\zeta^2 \, sinh \, \zeta(l) + C_4\zeta^2 \, cosh \, \zeta(l) - \frac{\varphi}{\zeta^2}\right) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$0 = EI_{eff}\left(C_3\zeta^2 \, sinh \, \zeta l + C_4\zeta^2 \, cosh \, \zeta l - \frac{\varphi}{\zeta^2}\right) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

Therefore, the constants of equation (3.6) for a simply supported beam are equal to:

$$C_1 = \frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4} + \frac{\varphi l^2}{2\zeta^2} \tag{3.10.1}$$

$$C_2 = 0 \tag{3.10.2}$$

$$C_3 = 0 \tag{3.10.3}$$

$$C_4 = -\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2 \cosh \zeta l} + \frac{\varphi}{\zeta^4 \cosh \zeta l} \tag{3.10.4}$$

Thus, the function for the deflection and the geometric imperfection deformation can be expressed as follows:

$$w(x) = \frac{\varphi}{2\zeta^2}(l^2 - x^2) + \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4}\right]\left(1 - \frac{\cosh\zeta x}{\cosh\zeta l}\right) \quad (3.11.1)$$

$$\frac{\partial w(x)}{\partial x} = -\frac{\varphi}{\zeta^2}x - \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4}\right]\left(\frac{\zeta\sinh\zeta x}{\cosh\zeta l}\right) \quad (3.11.2)$$

$$\frac{\partial^2 w(x)}{\partial x^2} = -\frac{\varphi}{\zeta^2} - \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4}\right]\left(\frac{\zeta^2\cosh\zeta x}{\cosh\zeta l}\right) \quad (3.11.3)$$

Equations (3.11) is substituted in equation (2.33) to account for the nonlinearity and to solve for the in-plane load.

$$N_x^0 = \frac{A_{11}}{4l}\int_{-l}^{l}\left(-\frac{\varphi}{\zeta^2}x - \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4}\right]\left(\frac{\zeta\sinh\zeta x}{\cosh\zeta l}\right)\right)^2 dx - \frac{B_{11}}{2l}\int_{-l}^{l}\left(-\frac{\varphi}{\zeta^2} - \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} - \frac{\varphi}{\zeta^4}\right]\left(\frac{\zeta^2\cosh\zeta x}{\cosh\zeta l}\right)\right) dx - N_x^T - N_x^H \quad (3.12)$$

In equation (3.12), λ depend on N_x^0, so it should be solved numerically. Using Matlab, the Newton method is implemented to solve for N_x^0.

3.1.2. Clamped-Clamped Beam

The boundary conditions for a clamped-clamped beam assuming the origin to be as shown in Figure 3.2 are described in equation (3.13).

$$w(0) = w(L) = 0 \quad (3.13.1)$$

$$\frac{\partial w(0)}{\partial x} = \frac{\partial w(L)}{\partial x} = 0 \quad (3.13.2)$$

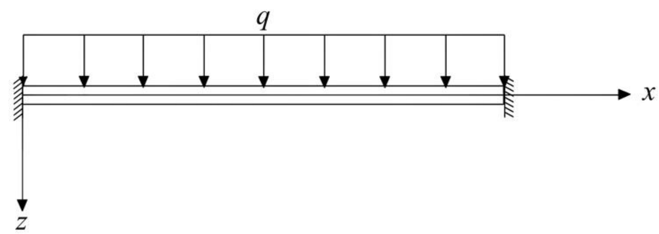

Figure 3.2. Clamped-clamped beam subjected to mechanical loading.

Due to the nature of the boundary conditions, a different method is applied to find the deformation amplitude. Instead of solving for the constants in equation (3.4), a solution that satisfy the boundary conditions is assumed. This solution is shown in equation (3.14).

$$EI_{eff}\frac{\partial^4 w}{\partial x^4} + N_x^0\frac{\partial^2 w}{\partial x^2} = -q \tag{3.13}$$

$$w(x) = a\left(1 - \cos\frac{2\pi x}{L}\right) \tag{3.14.1}$$

$$\frac{\partial w(x)}{\partial x} = a\left(\frac{2\pi}{L}\right)\sin\frac{2\pi x}{L} \tag{3.14.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = a\left(\frac{2\pi}{L}\right)^2\cos\frac{2\pi x}{L} \tag{3.14.3}$$

$$\frac{\partial^4 w(x)}{\partial x^4} = -a\left(\frac{2\pi}{L}\right)^4\cos\frac{2\pi x}{L} \tag{3.14.4}$$

By replacing equation (3.14) in equation (3.13) and equation (2.33), a complex expression in terms of x and the amplitude is obtained.

$$-a\left(\frac{2\pi}{L}\right)^4\cos\frac{2\pi x}{L} - \zeta^2 a\left(\frac{2\pi}{L}\right)^2\cos\frac{2\pi x}{L} = -\frac{q}{EI_{eff}}$$

$$\zeta^2 = -\frac{A_{11}}{2EI_{eff}L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{B_{11}}{EI_{eff}L}\int_0^L\frac{\partial^2 w}{\partial x^2}dx + \frac{N_x^T + N_x^H}{EI_{eff}}$$

$$\zeta^2 = -\frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^2 + \frac{N_x^T + N_x^H}{EI_{eff}}$$

$$\left[-\frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{2\pi}{L}\right)^2 a - \left(\frac{2\pi}{L}\right)^4 a + \frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^3\right]\cos\frac{2\pi x}{L} + \frac{q}{EI_{eff}} = 0 \tag{3.15}$$

In order to solve equation (3.15), the Galerkin method is applied as follow:

$$R(x) = (X_1 a + X_2 a^3)\cos\frac{2\pi x}{L} + \frac{q}{EI_{eff}}$$

$$X_1 = -\left(\frac{2\pi}{L}\right)^4 - \frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{2\pi}{L}\right)^2$$

$$X_2 = \frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2$$

$$\phi(x) = \left(1 - \cos\frac{2\pi x}{L}\right)$$

$$\int_0^L \phi(x)R(x)dx = 0$$

$$\int_0^L \left(1 - \cos\frac{2\pi x}{L}\right)\left[(X_1 a - X_2 a^3)\cos\frac{2\pi x}{L} + \frac{q}{EI_{eff}}\right]dx = 0$$

$$\frac{qL}{EI_{eff}} - \frac{(X_1 a - X_2 a^3)L}{2} = 0$$

$$\frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^3 + \left[\left(\frac{2\pi}{L}\right)^4 + \frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{2\pi}{L}\right)^2\right]a - \frac{2q}{EI_{eff}} = 0 \tag{3.16}$$

Equation (3.16) is polynomial function; thus, to find the amplitude, the roots of equation (3.16) are obtained.

3.1.3. Mixed Boundary Conditions

Assuming the origin to be in the middle of the beam and considering a beam that is simply supported at one end, and fixed at the other end, the boundary conditions are as shown below in equation (3.17).

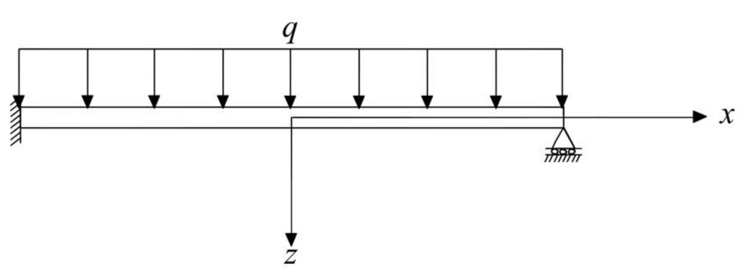

Figure 3.3. Beam with mixed boundary conditions subjected to a mechanical load.

$$w(l) = \frac{\partial^2 w(l)}{\partial x^2} = M_x(l) = 0 \tag{3.17.1}$$

$$w(-l) = \frac{\partial w(-l)}{\partial x} = 0 \tag{3.17.2}$$

When $x = -l$, $w(-l) = 0$ and $\frac{\partial w(-l)}{\partial x} = 0$

$$C_1 - C_2 l - C_3 \, sinh\, \zeta l + C_4 \, cosh\, \zeta l - \frac{\varphi}{2\zeta^2} l^2 = 0$$

$$C_2 + C_3 \zeta \cosh \zeta l - C_4 \zeta \sinh \zeta l - \frac{\varphi}{\zeta^2} l = 0$$

When $x = l$, $w(l) = 0$, and $M_x(l) = 0$

$$C_1 + C_2 l + C_3 \, sinh\, \zeta l + C_4 \, cosh\, \zeta l - \frac{\varphi}{2\zeta^2} l^2 = 0$$

$$0 = EI_{eff}\left(C_3 \zeta^2 \, sinh\, \zeta l + C_4 \zeta^2 \, cosh\, \zeta l - \frac{\varphi}{\zeta^2}\right) + K_1(N_x^0 + N_x^T + N_x^H)$$

Therefore, the constants of equation (3.6) for a CS beam are equal to:

$$\begin{aligned} C_1 &= \frac{\varphi \cosh \zeta l \,(2 \sinh \zeta l - 2\zeta l \cosh \zeta l - 3\zeta^2 l^2 \sinh \zeta l + 1)}{2\zeta^4 (1 - \cosh \zeta l \sinh \zeta l)} \\ &- \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right] \frac{\cosh \zeta l \,(\sinh \zeta l - \zeta l \cosh \zeta l)}{(1 - \cosh \zeta l \sinh \zeta l)} \end{aligned} \tag{3.18.1}$$

$$\begin{aligned} C_2 &= \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta}\right] \frac{\sinh^2 \zeta l}{(1 - \cosh \zeta l \sinh \zeta l)} \\ &- \frac{\varphi \sinh^2 \zeta l}{\zeta^3 (1 - \cosh \zeta l \sinh \zeta l)} + \frac{\varphi l \cosh \zeta l \sinh \zeta l}{\zeta^2 (1 - \cosh \zeta l \sinh \zeta l)} \end{aligned} \tag{3.18.2}$$

$$\begin{aligned} C_3 &= \frac{\varphi l}{\zeta^3 (1 - \cosh \zeta l \sinh \zeta l)} (\sinh \zeta l - \zeta l \cosh \zeta l) \\ &- \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta}\right] \frac{l \sinh \zeta l}{(1 - \cosh \zeta l \sinh \zeta l)} \end{aligned} \tag{3.18.3}$$

$$\begin{aligned} C_4 &= \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right] \frac{\sinh \zeta l - \zeta l \cosh \zeta l}{1 - \cosh \zeta l \sinh \zeta l} \\ &+ \frac{\varphi(\zeta^2 l^2 \sinh \zeta l - \sinh \zeta l + \zeta l \cosh \zeta l)}{\zeta^4 (1 - \cosh \zeta l \sinh \zeta l)} \end{aligned} \tag{3.18.4}$$

Note that in the case of a CS beam, the expressions for the constants are much longer than in the previous cases. However, the process to follow is the same. These constants in equations (3.18) should be substituted in equation (3.6) to find deflection solution. Then, the deflection function should be evaluated in equation (2.33), and the newton method is applied to solve for the in-plane force N_x^0.

3.1.4. Shear Deformation Effects

To account for the shear deformation effects in the nonlinear bending analysis of unsymmetrically laminated beams with different boundary conditions, a process similar to the ones described in sections 3.1.1, 3.1.2 and 3.1.3 should be performed. Therefore, the 4th order differential equation with the shear deformation effects is described as follow.

$$\frac{\partial^4 w}{\partial w^4} - \zeta_S^2 \frac{\partial^2 w}{\partial x^2} = \varphi \tag{3.19}$$

$$\beta_S = \frac{N_x^0}{K_s A_{55}} \tag{3.20}$$

$$\zeta_S^2 = -\frac{N_x^0}{EI_{eff}(1+\beta_S)} \tag{3.21}$$

$$\varphi_S = -\frac{q}{EI_{eff}(1+\beta_S)} \tag{3.22}$$

The solution for equation (3.19) is described in equation (3.23), and it has the same shape as the solution found for classical theory.

$$w(x) = C_1 + C_2 x + C_3 \sinh \zeta_S x + C_4 \cosh \zeta_S x - \frac{\varphi_S}{2\zeta_S^2} x^2 \tag{3.23}$$

The solution procedure to follow is the same as in the previous sections, but in this case equation (2.63) should be used in the boundary conditions for simply-supported and mixed boundary conditions. Additionally, equation (2.60) will be used to obtain N_x^0.

3.2. Thermal Loading

In order to study the bending behavior of a beam subjected to thermal loading, the following temperature function is assumed. It should be noted that the moisture is still considered to be constant.

$$\Delta T = \Delta T_0 \left(1 + r\frac{z}{h}\right) \sin\frac{\pi x}{L} \tag{3.24}$$

$$\begin{aligned} N_x^T &= \sum_{k=1}^{N} \int_{h_{k-1}}^{k_k} (\overline{Q_{11}})_k (\alpha_x)_k \Delta T_0 \left(1 + r\frac{z}{h}\right) \sin\frac{\pi x}{L} dz \\ &= \Delta T_0 \left(A_{11}^T + B_{11}^T \frac{r}{h}\right) \sin\frac{\pi x}{L} \end{aligned} \tag{3.25.1}$$

$$N_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\beta_x)_k \Delta H dz = A_{11}^H \beta_x \Delta H \tag{3.25.2}$$

$$\begin{aligned} M_x^T &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\alpha_x)_k \Delta T_0 \left(1 + r\frac{z}{h}\right) \sin\frac{\pi x}{L} z dz \\ &= \Delta T_0 \left(B_{11}^T + D_{11}^T \frac{r}{h}\right) \sin\frac{\pi x}{L} \end{aligned} \tag{3.25.3}$$

$$M_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\overline{Q_{11}})_k (\beta_x)_k \Delta H z dz = B_{11}^H \beta_x \Delta H \tag{3.25.4}$$

The 4th order ODE equation for the case of thermal loading is as follows:

$$\frac{\partial^4 w}{\partial w^4} - \zeta^2 \frac{\partial^2 w}{\partial x^2} = \varphi^T \tag{3.26}$$

$$\zeta^2 = -\frac{N_x^0}{EI_{eff}} \tag{3.27}$$

$$\varphi^T = \frac{1}{EI_{eff}} \left(\frac{\partial^2 M_x^T}{\partial x^2} + \frac{B_{11}}{A_{11}} \frac{\partial^2 N_x^T}{\partial x^2}\right) \tag{3.28}$$

Equations (3.25) are substituted in equation (3.28). Thus, a more convenient expression for the thermal load is expressed in equation (3.30).

$$\varphi^T = C_T \sin\frac{\pi}{L} x \tag{3.29}$$

$$C_T = \frac{\Delta T_0}{EI_{eff}} \left(\frac{\pi}{L}\right)^2 \left[\frac{B_{11}}{A_{11}} A_{11}^T + \left(\frac{r}{h} - 1\right) B_{11}^T - \frac{r}{h} D_{11}^T\right] \tag{3.30}$$

The 4th order ODE shown in equation (3.26) has a solution composed of two parts: homogeneous solution and a particular solution.

The homogeneous solution can be obtained as follow:

$$m^4 - \zeta^2 m^2 = 0$$

$$m^2(m^2 - \zeta^2) = 0$$

$$m = \pm 0$$

$$m = \pm \zeta$$

Thus, assuming a solution of the form $w(x) = Ce^{mx}$

$$w_h(x) = c_1 e^{0x} + c_2 e^{-0x} + c_3 e^{\zeta x} + c_4 e^{-\zeta x}$$

rearranging,

$$w_h(x) = C_1 + C_2 x + C_3 \sinh \lambda x + C_4 \cosh \lambda x \tag{3.31}$$

Additionally, the particular solution is derived by assuming a solution of the form:

$$w_p(x) = A_T C_T \sin \frac{\pi}{L} x \tag{3.32}$$

$$\frac{\partial w_p(x)}{\partial x} = A_T \left(\frac{\pi}{L}\right) C_T \cos \frac{\pi}{L} x$$

$$\frac{\partial^2 w_p(x)}{\partial x^2} = -A_T \left(\frac{\pi}{L}\right)^2 C_T \sin \frac{\pi}{L} x$$

$$\frac{\partial^3 w_p(x)}{\partial x^3} = -A_T \left(\frac{\pi}{L}\right)^3 C_T \cos \frac{\pi}{L} x$$

$$\frac{\partial^4 w_p(x)}{\partial x^4} = A_T \left(\frac{\pi}{L}\right)^4 C_T \sin \frac{\pi}{L} x$$

To find the constant A, the equations above are substituted in equation (3.26).

$$A_T \left(\frac{\pi}{L}\right)^4 C_T \sin \frac{\pi}{L} x + \zeta^2 A_T \left(\frac{\pi}{L}\right)^2 C_T \sin \frac{\pi}{L} x = C_T \sin \frac{\pi}{L} x$$

$$A_T = \frac{L^4}{\pi^2(\pi^2 + \zeta^2 L^2)} \tag{3.33}$$

Therefore,

$$w_p(x) = \frac{L^4}{\pi^2(\pi^2 + \zeta^2 L^2)} C_T \sin \frac{\pi}{L} x \tag{3.34}$$

The total solution to the differential equation is shown in equation (3.23).

$$\begin{aligned} w(x) = {} & C_1 + C_2 x + C_3 \sinh \zeta x + C_4 \cosh \zeta x \\ & + \frac{L^4}{\pi^2(\pi^2 + \zeta^2 L^2)} C_T \sin \frac{\pi}{L} x \end{aligned} \tag{3.35.1}$$

$$\frac{\partial w(x)}{\partial x} = C_2 + C_3\zeta \cosh \zeta x + C_4 \zeta \sinh \zeta x \\ + \frac{L^4}{\pi^2(\pi^2 + \zeta^2 L^2)}\left(\frac{\pi}{L}\right) C_T \cos \frac{\pi}{L} x \tag{3.35.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = C_3 \zeta^2 \sinh \zeta x + C_4 \zeta^2 \cosh \zeta x \\ - \frac{L^4}{\pi^2(\pi^2 + \zeta^2 L^2)}\left(\frac{\pi}{L}\right)^2 C_T \sin \frac{\pi}{L} x \tag{3.35.3}$$

3.2.1. Simply Supported Beam

For simply supported beams, two different methods are developed here. The first one consist of finding the constant in equation (4.35) by applying the boundary conditions. The constants will be a function of ζ, which is a function of the in-plane load. Thus, the nonlinearity is considered there. For simplicity and symmetry of the problem, the origin is moved to the mid-span of the beam. The second method consists of assuming a solution that satisfies the boundary conditions that has an amplitude. The amplitude can be found by replacing the expected solution in equation (3.26) to find the amplitude.

Method 1

For a simply supported beam, the boundary conditions are as shown below in equation (3.36).

$$w(-l) = w(l) = 0 \tag{3.36.1}$$

$$\frac{\partial^2 w(-l)}{\partial x^2} = \frac{\partial^2 w(l)}{\partial x^2} = 0 \tag{3.36.2}$$

$$M_x(-l) = M_x(l) = 0 \tag{3.36.3}$$

When $x = -l$, $w(-l) = 0$ and $M_x(-l) = 0$.

$$C_1 + C_2(-l) + C_3\, sinh\, \zeta(-l) + C_4\, cosh\, \zeta(-l) + A_T C_T \sin\left(\frac{\pi}{2l}(-l) + \frac{\pi}{2}\right) = 0$$

$$C_1 - C_2 l - C_3\, sinh\, \zeta l + C_4\, cosh\, \zeta l = 0$$

$$0 = EI_{eff}\left(C_3\zeta^2\, sinh\,\zeta(-l) + C_4\zeta^2\, cosh\,\zeta(-l) - A_T\left(\frac{\pi}{2l}\right)^2 C_T \sin\left(\frac{\pi}{2l}(-l) + \frac{\pi}{2}\right)\right)$$
$$+ K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$
$$0 = EI_{eff}(-C_3\zeta^2\, sinh\,\zeta l + C_4\zeta^2\, cosh\,\zeta l) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

When $x = l$, $w(l) = 0$ and $M_x(l) = 0$,

$$C_1 + C_2(l) + C_3\, sinh\,\zeta(l) + C_4\, cosh\,\zeta(l) + A_T C_T \sin\left(\frac{\pi}{2l}(l) + \frac{\pi}{2}\right) = 0$$
$$C_1 + C_2 l + C_3\, sinh\,\zeta l + C_4\, cosh\,\zeta l = 0$$
$$0 = EI_{eff}\left(C_3\zeta^2\, sinh\,\zeta(l) + C_4\zeta^2\, cosh\,\zeta(l) - A_T\left(\frac{\pi}{2l}\right)^2 C_T \sin\left(\frac{\pi}{2l}(l) + \frac{\pi}{2}\right)\right)$$
$$+ K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$
$$0 = EI_{eff}(C_3\zeta^2\, sinh\,\zeta l + C_4\zeta^2\, cosh\,\zeta l) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

Therefore, the constants of equation (3.35) for a simply supported beam are equal to:

$$C_1 = \frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2} \tag{3.37.1}$$

$$C_2 = 0 \tag{3.37.2}$$

$$C_3 = 0 \tag{3.37.3}$$

$$C_4 = -\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2 \cosh\zeta l} \tag{3.37.4}$$

Thus, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right]\left(1 - \frac{\cosh\zeta x}{\cosh\zeta l}\right) + \frac{16l^4}{\pi^2(\pi^2 + 4\zeta^2 l^2)} C_T \sin\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right) \tag{3.38.1}$$

$$\frac{\partial w(x)}{\partial x} = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right]\left(\frac{\zeta \sinh\zeta x}{\cosh\zeta l}\right) + \frac{16l^4}{\pi^2(\pi^2 + 4\zeta^2 l^2)} C_T \left(\frac{\pi}{2l}\right)\cos\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right) \tag{3.38.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right]\left(\frac{\zeta^2 \cosh \zeta x}{\cosh \zeta l}\right) - \frac{16l^4}{\pi^2(\pi^2 + 4\zeta^2 l^2)} C_T \left(\frac{\pi}{2l}\right)^2 \sin\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right) \tag{3.38.3}$$

Equations (3.38) are substituted in equation (2.33).

$$\begin{aligned} N_x^0 &= \frac{A_{11}}{4l}\int_{-l}^{l}\left\{\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right]\left(\frac{\zeta \sinh \zeta x}{\cosh \zeta l}\right) + \frac{16l^4}{\pi^2(\pi^2 + 4\zeta^2 l^2)} C_T \left(\frac{\pi}{2l}\right)\cos\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right)\right\}^2 dx \\ &- \frac{B_{11}}{2l}\int_{-l}^{l}\left\{\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)}{EI_{eff}\zeta^2}\right]\left(\frac{\zeta^2 \cosh \zeta x}{\cosh \zeta l}\right) - \frac{16l^4}{\pi^2(\pi^2 + 4\zeta^2 l^2)} C_T \left(\frac{\pi}{2l}\right)^2 \sin\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right)\right\} dx - N_x^T - N_x^H \end{aligned} \tag{3.39}$$

In equation (3.39), λ depend on N_x^0, so it should be solved numerically. The Newton method is implemented using Matlab to solve for N_x^0.

Method 2

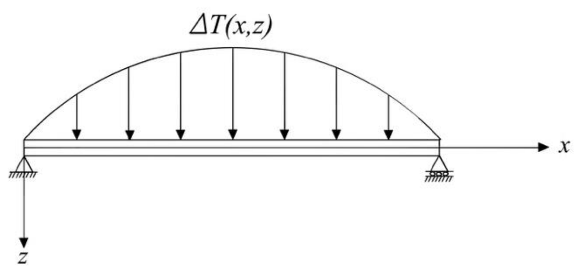

Figure 3.4. Simply supported beam subjected to thermal loading.

A sinusoidal solution that satisfies the boundary conditions in equations for a simply-supported beam is assumed. This solution is shown in equation (3.40)

$$w(x) = a \sin\frac{\pi x}{L} \tag{3.40.1}$$

$$\frac{\partial w(x)}{\partial x} = a\left(\frac{\pi}{L}\right)\cos\frac{\pi x}{L} \tag{3.40.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = -a\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L} \tag{3.40.3}$$

$$\frac{\partial^4 w(x)}{\partial x^4} = a\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} \tag{3.40.4}$$

$$a\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} + \zeta^2 a\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L} = C_T \sin\frac{\pi x}{L}$$

$$\zeta^2 = -\frac{A_{11}}{2EI_{eff}L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{B_{11}}{EI_{eff}L}\int_0^L \frac{\partial^2 w}{\partial x^2} dx + \frac{N_x^T + N_x^H}{EI_{eff}}$$

$$\zeta^2 = -\frac{A_{11}}{4EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^2 - \frac{2B_{11}\pi}{EI_{eff}L^2} a + \frac{N_x^T + N_x^H}{EI_{eff}}$$

$$a\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} + \left(-\frac{A_{11}}{4EI_{eff}}\left(\frac{\pi}{L}\right)^4 a^3 - \frac{2B_{11}\pi}{EI_{eff}L^2}\left(\frac{\pi}{L}\right)^2 a^2 + \frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a\right)\sin\frac{\pi x}{L} = C_T \sin\frac{\pi x}{L}$$

$$-\frac{A_{11}}{4EI_{eff}}\left(\frac{\pi}{L}\right)^4 a^3 - \frac{2B_{11}\pi}{EI_{eff}L^2}\left(\frac{\pi}{L}\right)^2 a^2 + \left[\left(\frac{\pi}{L}\right)^4 + \frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{\pi}{L}\right)^2\right] a - C_T = 0$$

3.2.2. Clamped-Clamped Beam

The second method applied to the simply-supported is also adopted for a clamped-clamped beam. The main difference will be that a different function for the solution is assumed. Thus, the Galerkin method should be applied to solve for the amplitude.

Galerkin Method

A solution that satisfies the boundary conditions for a clamped-clamped is assumed.

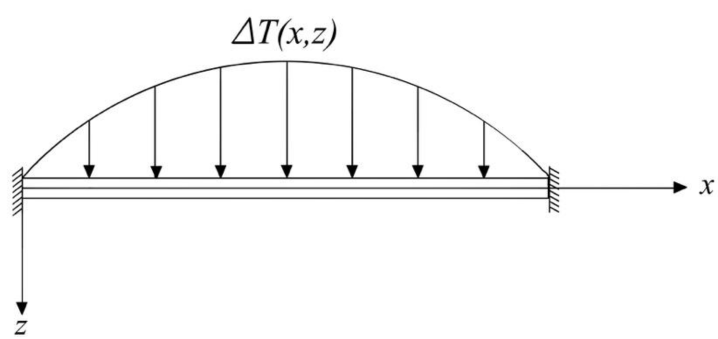

Figure 3.5. Clamped-clamped beam subjected to thermal loading.

This solution is shown in equation (3.41):

$$w(x) = a\left(1 - \cos\frac{2\pi x}{L}\right) \tag{3.41.1}$$

$$\frac{\partial w(x)}{\partial x} = a\left(\frac{2\pi}{L}\right)\sin\frac{2\pi x}{L} \tag{3.41.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = a\left(\frac{2\pi}{L}\right)^2 \cos\frac{2\pi x}{L} \tag{3.41.3}$$

$$\frac{\partial^4 w(x)}{\partial x^4} = -a\left(\frac{2\pi}{L}\right)^4 \cos\frac{2\pi x}{L} \tag{3.41.4}$$

Substituting equation (3.41) in equation (3.26) and (3.27);

$$-a\left(\frac{2\pi}{L}\right)^4 \cos\frac{2\pi x}{L} - \zeta^2 a\left(\frac{2\pi}{L}\right)^2 \cos\frac{2\pi x}{L} = C_T \sin\frac{\pi x}{L}$$

$$\zeta^2 = -\frac{A_{11}}{2EI_{eff}L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{B_{11}}{EI_{eff}L}\int_0^L \frac{\partial^2 w}{\partial x^2} dx + \frac{N_x^T + N_x^H}{EI_{eff}}$$

$$\zeta^2 = -\frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^2 + \frac{N_x^T + N_x^H}{EI_{eff}}$$

Therefore, equation (3.42) is obtained. The goal is to solve for the amplitude; however, the expression in equation (3.42) it is very complex. The Galerkin method should be applied to obtain a polynomial expression for the amplitude a.

$$\begin{aligned}&\left[-\frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{2\pi}{L}\right)^2 a - \left(\frac{2\pi}{L}\right)^4 a + \frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2 a^3\right]\cos\frac{2\pi x}{L} \\ &- C_T \sin\frac{\pi x}{L} = 0\end{aligned} \tag{3.42}$$

$$R(x) = (X_1 a + X_2 a^3)\cos\frac{2\pi x}{L} - C_T \sin\frac{\pi x}{L}$$

$$X_1 = -\left(\frac{2\pi}{L}\right)^4 - \frac{N_x^T + N_x^H}{EI_{eff}}\left(\frac{2\pi}{L}\right)^2$$

$$X_2 = \frac{A_{11}}{EI_{eff}}\left(\frac{\pi}{L}\right)^2$$

$$\phi(x) = \left(1 - \cos\frac{2\pi x}{L}\right)$$

$$\int_0^L \phi(x)R(x)dx = 0$$

$$\int_0^L \left(1-\cos\frac{2\pi x}{L}\right)\left[(X_1 a - X_2 a^3)\cos\frac{2\pi x}{L} - C_T \sin\frac{\pi x}{L}\right] dx = 0$$

$$\frac{L(X_1 a + X_2 a^3)}{2} + \frac{8C_T L}{3\pi} = 0 \tag{3.43}$$

From equation (3.43), the roots for the amplitude can be obtained. The positive root should be picked as the amplitude of the deflection.

3.2.3. Mixed Boundary Conditions

Considering a beam that is fixed supported at the left end and simply supported at the right end, the boundary conditions are as shown below in equation (3.44).

$$w(l) = \frac{\partial^2 w(l)}{\partial x^2} = M_x(l) = 0 \tag{3.44.1}$$

$$w(-l) = \frac{\partial w(-l)}{\partial x} = 0 \tag{3.44.2}$$

When $x = -l$, $w(-l) = 0$ and $\frac{\partial w(-l)}{\partial x} = 0$

$$C_1 - C_2 l - C_3\, sinh\, \zeta l + C_4\, cosh\, \zeta l = 0$$

$$C_2 + C_3 \zeta \cosh \zeta l - C_4 \zeta \sinh \zeta l + \left(\frac{\pi}{2l}\right) A_T C_T \cos\left(\frac{\pi x}{2l} + \frac{\pi}{2}\right) = 0$$

When $x = l$, $w(l) = 0$, and $M_x(l) = 0$

$$C_1 + C_2 l + C_3\, sinh\, \zeta l + C_4\, cosh\, \zeta l = 0$$

$$0 = EI_{eff}(C_3 \zeta^2\, sinh\, \zeta l + C_4 \zeta^2\, cosh\, \zeta l) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

It should be noted that the expression for the constants of the deflection in the case of CS beam is more complex. Therefore, they are not presented in the report. However, the same process should be followed. The constants are substituted in equation (3.35) and then, in equation (2.33) where λ will depend on N_x^0, so it should be solved numerically. Newton's method is implemented using Matlab to solve for N_x^0.

4. Method of Solution for Thermal Buckling and Post-Buckling

This section is subdivided into two parts. The first part is concerned with the effects of applying a constant temperature to an angle-ply beam. The purpose of this is to observe a bifurcation buckling behavior in unsymmetrical laminates. The second part of this section is to study the behavior of cross-ply laminates when a linear temperature variation function is applied.

4.1. Uniform Temperature Rise

First, we consider a constant temperature gradient, $(\Delta T = constant)$, and a constant moisture gradient, $(\Delta H = constant)$, then the resultant hygrothermal and thermal moments and resultant hygrothermal and thermal normal in-plane loads for an unsymmetrical laminated beam can be defined as in equation (4.1).

$$N_x^T = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\bar{Q}_{11})_k (\alpha_x)_k \Delta T dz = A_{11}^T \Delta T \tag{4.1.1}$$

$$N_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\bar{Q}_{11})_k (\beta_x)_k \Delta H dz = A_{11}^H \Delta H \tag{4.1.2}$$

$$M_x^T = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\bar{Q}_{11})_k (\alpha_x)_k \Delta T z dz = B_{11}^T \Delta T \tag{4.1.3}$$

$$M_x^H = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\bar{Q}_{11})_k (\beta_x)_k \Delta H z dz = B_{11}^T \Delta H \tag{4.1.4}$$

In some cases, like angle-ply laminates, the bending-extension stiffness B_{11} is zero, thus, the hygrothermal moments will also be zero. For that reason, if no external force is applied, the beam will remain straight as the temperature gradient reaches its critical point ΔT_{cr}. If the temperature gradient keeps increasing beyond ΔT_{cr}., the beam will be in the post-buckling domain.

When $\Delta T < \Delta T_{cr}$, the load P that beam is experiencing can be determined by neglecting the deflection and non-linear terms on equation (2.33). It is assumed that $P = N_x^0$.

$$-P = N_x^T + N_x^H \tag{4.2}$$

When $\Delta T = \Delta T_{cr}$, equation (2.34) can be rearranged to solve for the critical buckling load and its corresponding temperature considering different boundary conditions.

$$\frac{\partial^4 w}{\partial w^4} + \lambda^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{4.3}$$

$$\lambda^2 = \frac{N_x^0}{EI_{eff}} \tag{4.4}$$

Solving the 4th order ODE:

$$m^4 + \lambda^2 m^2 = 0$$

$$m^2(m^2 + \lambda^2) = 0$$

$$m = \pm 0$$

$$m = \pm \lambda i$$

Thus, assuming a solution of the form $w(x) = Ce^{mx}$

$$w(x) = c_1 e^{0x} + c_2 e^{-0x} + c_3 e^{\lambda i x} + c_4 e^{-\lambda i x}$$

rearranging,

$$w(x) = C_1 + C_2 x + C_3 \sin \lambda x + C_4 \cos \lambda x \tag{4.5.1}$$

$$\frac{\partial w(x)}{\partial x} = C_2 + C_3 \lambda \cos \lambda x - C_4 \lambda \sin \lambda x \tag{4.5.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = -C_3 \lambda^2 \sin \lambda x - C_4 \lambda^2 \cos \lambda x \tag{4.5.3}$$

$$\frac{\partial^3 w(x)}{\partial x^3} = -C_3 \lambda^3 \cos \lambda x + C_4 \lambda^3 \sin \lambda x \tag{4.5.4}$$

For post-buckling, when $\Delta T > \Delta T_{cr}$, equation (2.33) and the respective solution for equations (4.5) are utilized to find the post-buckling amplitude.

4.1.1. Simply Supported Beam

For a simply-supported beam, the boundary conditions are as shown below in equation (4.6).

$$w(0) = w(L) = 0 \tag{4.6.1}$$

$$\frac{\partial^2 w(0)}{\partial x^2} = \frac{\partial^2 w(L)}{\partial x^2} = 0 \tag{4.6.2}$$

$$M_x(0) = M_x(L) = 0 \tag{4.6.3}$$

To find the critical buckling load, equations (4.6.1) and (4.6.2) are applied to equations (4.5).

When $x = 0$, $w(0) = 0$, and $\frac{\partial^2 w(0)}{\partial x^2} = 0$,

$$0 = C_1 + C_2(0) + C_3 \sin \lambda(0) + C_4 \cos \lambda(0)$$

$$C_1 + C_4 = 0$$

$$0 = -C_3\lambda^2 \sin \lambda(0) - C_4\lambda^2 \cos \lambda(0)$$

$$0 = -C_4\lambda^2$$

When $x = L$, $w(L) = 0$, and $\frac{\partial^2 w(L)}{\partial x^2} = 0$,

$$0 = C_1 + C_2 L + C_3 \sin \lambda L + C_4 \cos \lambda L$$

$$0 = -C_3\lambda^2 \sin \lambda L - C_4\lambda^2 \cos \lambda L$$

or

$$C_2 = 0$$

$$\lambda L = n\pi \tag{4.7}$$

$$\lambda^2 = \left(\frac{\pi}{L}\right)^2 = \frac{P_{cr}}{EI_{eff}} \tag{4.8}$$

Therefore, the critical buckling load is expressed as:

$$P_{cr} = EI_{eff}\left(\frac{\pi}{L}\right)^2 \tag{4.9}$$

From equation (4.9) and equation (4.2) when the critical buckling is reached, an expression for the critical buckling temperature can be found, as shown below in equation (4.11).

$$-P_{cr} = A_{11}(\alpha_x \Delta T_{cr} + \beta_x \Delta H) \tag{4.10}$$

$$\Delta T_{cr} = -\frac{EI_{eff}}{A_{11}\alpha_x}\left(\frac{\pi}{L}\right)^2 - \frac{\beta_x \Delta H}{\alpha_x} \tag{4.11}$$

To find the post-buckling amplitude, it is assumed that in equation (4.7) $n = 1$. Therefore, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = a\sin\frac{\pi}{L}x \tag{4.12.1}$$

$$\frac{\partial w(x)}{\partial x} = a\frac{\pi}{L}\cos\frac{\pi}{L}x \tag{4.12.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = -a\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi}{L}x \tag{4.12.3}$$

Equations (4.12) are substituted in equation (2.33). Then, the post-buckling amplitude can be found by getting the roots of equation (4.13).

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L \left(a\frac{\pi}{L}\cos\frac{\pi}{L}x\right)^2 dx - N_x^T - N_x^M = -P$$

$$\frac{A_{11}a^2\pi^2}{4L^2} + (P - N_x^T - N_x^H) = 0 \tag{4.13}$$

$$a_1 = -\frac{2L}{\pi}\sqrt{\frac{P - N_x^T - N_x^H}{A_{11}}} \tag{4.14.1}$$

$$a_2 = \frac{2L}{\pi}\sqrt{\frac{P - N_x^T - N_x^H}{A_{11}}} \tag{4.14.2}$$

4.1.2. Clamped-Clamped Beam

For a clamped-clamped beam, the boundary conditions are as shown below in equation (4.15).

$$w(0) = w(L) = 0 \tag{4.15.1}$$

$$\frac{\partial w(0)}{\partial x} = \frac{\partial w(L)}{\partial x} = 0 \tag{4.15.2}$$

To find the critical buckling load, equations (4.15) are applied to equations (4.5).

When $x = 0$, $w(0) = 0$, and $\frac{\partial w(0)}{\partial x} = 0$,

$$0 = C_1 + C_2(0) + C_3 \sin \lambda(0) + C_4 \cos \lambda(0)$$

$$C_1 + C_4 = 0$$

$$0 = C_2 + C_3 \lambda \cos \lambda(0) - C_4 \lambda \sin \lambda(0)$$

$$0 = C_2 + C_3 \lambda$$

When $x = L$, $w(L) = 0$, and $\frac{\partial w(L)}{\partial x} = 0$,

$$0 = C_1 + C_2 L + C_3 \sin \lambda L + C_4 \cos \lambda L$$

$$0 = C_2 + C_3 \lambda \cos \lambda L - C_4 \lambda \sin \lambda L$$

Writing the above equations in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \lambda & 0 \\ 1 & L & \sin \lambda L & \cos \lambda L \\ 0 & 1 & \lambda \cos \lambda L & -\lambda \sin \lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$2\lambda \cos \lambda L - \lambda - \lambda \cos^2 \lambda L - \lambda \sin^2 \lambda L + \lambda^2 L \sin \lambda L = 0$$

$$2\lambda \cos \lambda L - 2\lambda + \lambda^2 L \sin \lambda L = 0$$

$$\lambda(2 \cos \lambda L - 2 + \lambda L \sin \lambda L) = 0$$

$$(1 - \cos \lambda L) - \frac{\lambda L}{2} \sin \lambda L = 0$$

$$\sin^2 \left(\frac{\lambda L}{2}\right) - \frac{\lambda L}{2} \sin \left(\frac{\lambda L}{2}\right) \cos \left(\frac{\lambda L}{2}\right) = 0$$

$$\sin \left(\frac{\lambda L}{2}\right) = 0$$

$$\tan \left(\frac{\lambda L}{2}\right) = \frac{\lambda L}{2}$$

Therefore,

$$C_1 = C_2 = C_4 = 0$$

$$\lambda L = 2n\pi \tag{4.16}$$

$$\lambda^2 = 4\left(\frac{\pi}{L}\right)^2 = \frac{P_{cr}}{EI_{eff}} \tag{4.17}$$

Therefore, the critical buckling load is expressed in equation (4.18).

$$P_{cr} = 4EI_{eff}\left(\frac{\pi}{L}\right)^2 \tag{4.18}$$

From equation (4.18) and equation (4.2) when the critical buckling is reached, an expression for the critical buckling temperature can be found, as shown below in equation (4.20).

$$P_{cr} = A_{11}(\alpha_x \Delta T_{cr} + \beta_x \Delta M) \tag{4.19}$$

$$\Delta T_{cr} = \frac{4EI_{eff}}{A_{11}\alpha_x}\left(\frac{\pi}{L}\right)^2 - \frac{\beta_x \Delta M}{\alpha_x} \tag{4.20}$$

To find the post-buckling amplitude, it is assumed that in equation (4.16) $n = 1$. Therefore, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = a\left(1 - \cos\frac{2\pi}{L}x\right) \tag{4.21.1}$$

$$\frac{\partial w(x)}{\partial x} = a\frac{2\pi}{L}\cos\frac{2\pi}{L}x \tag{4.21.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = -a\left(\frac{2\pi}{L}\right)^2 \sin\frac{2\pi}{L}x \tag{4.21.3}$$

Equations (4.21) are substituted in equation (2.33). Then, the post-buckling amplitude can be found by getting the roots of equation (4.22).

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L \left(a\frac{2\pi}{L}\cos\frac{2\pi}{L}x\right)^2 dx - N_x^T - N_x^M = -P$$

$$\frac{A_{11}\pi^2}{L^2}a^2 + (P_{cr} - N_x^T - N_x^M) = 0 \tag{4.22}$$

$$a_1 = -\frac{L}{\pi}\sqrt{\frac{P - N_x^T - N_x^H}{A_{11}}} \tag{4.23.1}$$

$$a_2 = \frac{L}{\pi}\sqrt{\frac{P - N_x^T - N_x^H}{A_{11}}} \tag{4.23.2}$$

4.1.3. Mixed Boundary Conditions

Considering a beam that is simply supported at one end, and fixed at the other end, the boundary conditions are as shown below in equation (4.24).

$$w(0) = \frac{\partial w(0)}{\partial x} = 0 \tag{4.24.1}$$

$$w(L) = \frac{\partial^2 w(L)}{\partial x^2} = M_x(L) = 0 \tag{4.24.2}$$

To find the critical buckling load, equations (4.24) are applied to equations (4.5).

When $x = 0$, $w(0) = 0$, and $\frac{\partial w(0)}{\partial x} = 0$,

$$0 = C_1 + C_2(0) + C_3 \sin\lambda(0) + C_4 \cos\lambda(0)$$

$$C_1 + C_4 = 0$$

$$0 = C_2 + C_3\lambda\cos\lambda(0) - C_4\lambda\sin\lambda(0)$$

$$0 = C_2 + C_3\lambda$$

When $x = l$, $w(L) = 0$, and $\frac{\partial^2 w(L)}{\partial x^2} = 0$,

$$0 = C_1 + C_2 L + C_3 \sin\lambda L + C_4 \cos\lambda L$$

$$0 = -C_3\lambda^2 \sin\lambda L - C_4\lambda^2 \cos\lambda L$$

Writing the above equations in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \lambda & 0 \\ 1 & L & \sin\lambda L & \cos\lambda L \\ 0 & 0 & -\lambda^2 \sin\lambda L & -\lambda^2\cos\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-\lambda^2(\sin\lambda L - \lambda L\cos\lambda L) = 0$$

$$\tan\lambda L = \lambda L \tag{4.25}$$

Therefore, the solution for equation (3.25) can be obtained numerically as follow:

$$\lambda l = 2.2467$$

knowing that $l = L/2$,

$$\lambda L = 1.4303n\pi \tag{4.26}$$

$$\lambda^2 = \left(\frac{\pi}{0.7L}\right)^2 = \frac{P_{cr}}{EI_{eff}} \tag{4.27}$$

Therefore, the critical buckling load is expressed in equation (4.28).

$$P_{cr} = EI_{eff}\left(\frac{\pi}{0.7L}\right)^2 \tag{4.28}$$

From equation (4.9) and equation (4.2) when the critical buckling is reached, an expression for the critical buckling temperature can be found, as shown below in equation (4.30).

$$P_{cr} = A_{11}(\alpha_x \Delta T_{cr} + \beta_x \Delta M) \tag{4.29}$$

$$\Delta T_{cr} = \frac{EI_{eff}}{A_{11}\alpha_x}\left(\frac{\pi}{0.7L}\right)^2 - \frac{\beta_x \Delta M}{\alpha_x} \tag{4.30}$$

To find the post-buckling amplitude, it is assumed that in equation (4.26) $n = 1$. Therefore, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = a\left[\sin\frac{1.43\pi}{L}x - 1.43\pi\cos\frac{1.43\pi}{L}x + \frac{1.43\pi}{L}(L-x)\right] \tag{4.3.1}$$

$$\frac{\partial w(x)}{\partial x} = a\left[\frac{1.43\pi}{L}\cos\frac{1.43\pi}{L}x + 1.43\pi\frac{1.43\pi}{L}\sin\frac{1.43\pi}{L}x - \frac{1.43\pi}{L}\right] \tag{4.31.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = a\left[-\left(\frac{1.43\pi}{L}\right)^2\sin\frac{1.43\pi}{L}x + 1.43\pi\left(\frac{1.43\pi}{L}\right)^2\cos\frac{1.43\pi}{L}x\right] \tag{4.31.3}$$

Equations (4.31) are substituted in equation (2.33). Then, the post-buckling amplitude can be found by getting the roots of equation (4.32).

$$N_x^0 = \frac{A_{11}}{2L}\int_0^L a^2\left[\frac{1.43\pi}{L}\cos\frac{1.43\pi}{L}x + 1.43\pi\frac{1.43\pi}{L}\sin\frac{1.43\pi}{L}x - \frac{1.43\pi}{L}\right]^2 dx - N_x^T - N_x^M = -P \tag{4.32}$$

4.1.4. Shear Effects

To account for the shear deformation effects in the nonlinear bending analysis of unsymmetrically laminated beams with different boundary conditions, a process like the ones described in sections 4.1.1, 4.1.2 and 4.1.3 must be performed. Therefore, the 4th order differential equation with the shear deformation effects is described as follow.

$$\frac{\partial^4 w}{\partial w^4} + \lambda_S^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{4.33}$$

$$\beta_S = \frac{N_x^0}{K_s A_{55}} \tag{4.34}$$

$$\lambda_S^2 = \frac{N_x^0}{EI_{eff}(1+\beta_S)} \tag{4.35}$$

The solution for equation (4.33) is described in equation (4.36), and it has the same shape as the solution found for classical theory.

$$w(x) = C_1 + C_2 x + C_3 \sin \lambda_S x + C_4 \cos \lambda_S x \tag{4.36}$$

The solution procedure to follow is the same as in the previous sections, but in this case, the effects of shear are considering. Below are shown the solutions for critical buckling load and critical buckling temperature with different boundary conditions.

For simply supported beams:

$$P_{cr} = \frac{EI_{eff} K_s A_{55} \pi^2}{K_s A_{55} L^2 - EI_{eff} \pi^2} \tag{4.37}$$

$$\Delta T_{cr} = \frac{EI_{eff} K_s A_{55} \pi^2}{A_{11} \alpha_x \left(K_s A_{55} L^2 - EI_{eff} \pi^2\right)} - \frac{\beta_x \Delta H}{\alpha_x} \tag{4.38}$$

For clamped-clamped beams:

$$P_{cr} = \frac{4EI_{eff} K_s A_{55} \pi^2}{K_s A_{55} L^2 - 4EI_{eff} \pi^2} \tag{4.39}$$

$$\Delta T_{cr} = \frac{4EI_{eff} K_s A_{55} \pi^2}{A_{11} \alpha_x \left(K_s A_{55} L^2 - 4EI_{eff} \pi^2\right)} - \frac{\beta_x \Delta H}{\alpha_x} \tag{4.40}$$

For beams with mixed boundary conditions:

$$P_{cr} = \frac{2.04EI_{eff}K_sA_{55}\pi^2}{K_sA_{55}L^2 - 2.04EI_{eff}\pi^2} \tag{4.41}$$

$$\Delta T_{cr} = \frac{2.04EI_{eff}K_sA_{55}\pi^2}{A_{11}\alpha_x\left(K_sA_{55}L^2 - 2.04EI_{eff}\pi^2\right)} - \frac{\beta_x\Delta H}{\alpha_x} \tag{4.42}$$

4.2. Linear Temperature Variation

Considering a linear variation of temperature as shown in equation (4.7), and a constant moisture gradient, $\Delta M = constant$, the hygrothermal moments and loads for an unsymmetrical laminated beam can be defined as in equation (4.8).

When the beam is considered to have a linear temperature variation through its thickness, there will be a thermal moment, M_x^T, which in the previous case was simply zero. This moment will cause the beam to deflect as soon as the temperature starts to increase. Thus, the nature of the problem is not a bifurcation buckling anymore, but it is a bending-buckling type of problem which occurs before, during, and after the critical buckling load.

4.2.1. Simply Supported Beam

To find the critical buckling load, equations (4.6.1) and (4.6.3) are applied to equations (4.5).

When $x = 0$, $w(0) = 0$ and $M_x(0) = 0$.

$$0 = C_1 + C_2(0) + C_3\sin\lambda(0) + C_4\cos\lambda(0)$$

$$C_1 + C_4 = 0$$

$$0 = EI_{eff}(-C_3\lambda^2\sin\lambda(0) - C_4\lambda^2\cos\lambda(0)) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

$$0 = -EI_{eff}C_4\lambda^2 + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

When $x = L$, $w(L) = 0$ and $M_x(L) = 0$,

$$0 = C_1 + C_2L + C_3\sin\lambda L + C_4\cos\lambda L$$

$$0 = -EI_{eff}(C_3\lambda^2 \sin\lambda L + C_4\lambda^2 \cos\lambda L) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

Therefore, the constants of equation (4.5) are equal to:

$$C_1 = -\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2} \tag{4.43.1}$$

$$C_2 = 0 \tag{4.43.2}$$

$$C_3 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\right]\frac{1-\cos\lambda L}{\sin\lambda L} \tag{4.43.3}$$

$$C_4 = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2} \tag{4.43.4}$$

Thus, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[\cos\lambda x - 1 + \left(\frac{1-\cos\lambda L}{\sin\lambda L}\right)\sin\lambda x\right] \tag{4.44.1}$$

$$\frac{\partial w(x)}{\partial x} = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[-\lambda\sin\lambda x + \left(\frac{1-\cos\lambda L}{\sin\lambda L}\right)\lambda\cos\lambda x\right] \tag{4.44.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[-\lambda^2\cos\lambda x - \left(\frac{1-\cos\lambda L}{\sin\lambda L}\right)\lambda^2\sin\lambda x\right] \tag{4.44.3}$$

Equations (4.44) are substituted in equation (2.33). In equation (4.44), λ depend on N_x^0, so it should be solved numerically. The Newton method is implemented using Matlab to solve for N_x^0.

4.2.2. Clamped-Clamped Beam

For a clamped-clamped beam, there are two inflection points separated by an effective length L_e. At these two points, it can be shown that $M_x(x) = 0$. Therefore, it can be assumed that the beam behaves as a simply supported beam along its effective length. Using the solution in equation (3.47), the value for L_e can be obtained.

$$w(x) = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[\cos\lambda x - 1 + \left(\frac{1-\cos\lambda L_e}{\sin\lambda L_e}\right)\sin\lambda x\right] \tag{4.45.1}$$

$$\frac{\partial w(x)}{\partial x} = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[-\lambda\sin\lambda x + \left(\frac{1-\cos\lambda L_e}{\sin\lambda L_e}\right)\lambda\cos\lambda x\right] \tag{4.45.2}$$

$$\frac{\partial^2 w(x)}{\partial x^2} = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[-\lambda^2\cos\lambda x - \left(\frac{1-\cos\lambda L_e}{\sin\lambda L_e}\right)\lambda^2\sin\lambda x\right] \tag{4.45.3}$$

Applying the boundary conditions on equation (3.16.2).

$$\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2}\left[-\lambda\sin\lambda x + \left(\frac{1-\cos\lambda L_e}{\sin\lambda L_e}\right)\lambda\cos\lambda x\right] = 0$$

$$\tan\lambda x - \left(\frac{1-\cos\lambda L_e}{\sin\lambda L_e}\right) = 0 \tag{4.46}$$

It is noted that equation (4.46) is $\frac{\pi}{\lambda}$ periodic; therefore:

$$0 = \frac{L_e}{2} - \frac{\pi}{\lambda} \tag{4.47}$$

$$L_e = \frac{2\pi}{\lambda} \tag{4.48}$$

4.2.3. Mixed Boundary Conditions

To find the critical buckling load, equations (4.24) are applied to equations (4.5).

When $x = 0$, $w(0) = 0$ and $\frac{\partial w(0)}{\partial x} = 0$.

$$0 = C_1 + C_2(0) + C_3\sin\lambda(0) + C_4\cos\lambda(0)$$

$$C_1 + C_4 = 0$$

$$0 = C_2 + C_3\lambda\cos\lambda(0) - C_4\lambda\sin\lambda(0)$$

$$0 = C_2 + C_3\lambda$$

When $x = L$, $w(L) = 0$ and $M_x(L) = 0$,

$$0 = C_1 + C_2L + C_3 \sin \lambda L + C_4 \cos \lambda L$$

$$0 = -EI_{eff}(C_3\lambda^2 \sin \lambda L + C_4\lambda^2 \cos \lambda L) + K_1(N_x^0 + N_x^T + N_x^H) - (M_x^T + M_x^H)$$

Therefore, the constants of equation (4.5) are equal to:

$$C_1 = -\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right](\sin \lambda L - \lambda L) \qquad (4.49.1)$$

$$C_2 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right]\lambda(\cos \lambda L - 1) \qquad (4.49.2)$$

$$C_3 = -\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right](\cos \lambda L - 1) \qquad (4.49.3)$$

$$C_4 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right](\sin \lambda L - \lambda L) \qquad (4.49.4)$$

Thus, the function for the deflection and the geometric imperfection deformation can be expressed as below.

$$w(x) = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right][(\sin \lambda L - \lambda L)(\cos \lambda x - 1) + (\cos \lambda L - 1)(\lambda x - \sin \lambda x)] \qquad (4.50.1)$$

$$\frac{\partial w(x)}{\partial x} = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right][(\sin \lambda L - \lambda L)(-\lambda \sin \lambda x) + (\cos \lambda L - 1)(\lambda - \lambda \cos \lambda x)] \qquad (4.50.2)$$

$$\frac{\partial^2 w(x)}{\partial x^2} = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin \lambda L - \lambda L \cos \lambda L)}\right][(\sin \lambda L - \lambda L)(-\lambda^2 \cos \lambda x) + (\cos \lambda L - 1)(\lambda^2 \sin \lambda x)] \qquad (4.50.3)$$

Equations (4.50) are substituted in equation (2.33). Since λ depend on N_x^0, the problem cannot be solved analytically. Therefore, a Matlab code is implemented to solve for N_x^0 using Newton's methods.

4.2.4. Shear Effects

If the same solution procedure as the sections above is conducted, the constants of the displacement function with the effects of shear can be obtained as follow.

For a simply supported beam:

$$C_1 = -\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)} \tag{4.51.1}$$

$$C_2 = 0 \tag{4.51.2}$$

$$C_3 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)}\right]\frac{1 - \cos\lambda L}{\sin\lambda L} \tag{4.51.3}$$

$$C_4 = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)} \tag{4.51.4}$$

For a clamped-clamped beam:

$$C_1 = -\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)} \tag{4.52.1}$$

$$C_2 = 0 \tag{4.52.2}$$

$$C_3 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)}\right]\frac{1 - \cos\lambda L_e}{\sin\lambda L_e} \tag{4.52.3}$$

$$C_4 = \frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\beta_S + 1)} \tag{4.52.4}$$

For a beam with mixed boundary conditions:

$$C_1 = -\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin\lambda L - \lambda L\cos\lambda L)(\beta_S + 1)}\right](\sin\lambda L - \lambda L) \tag{4.53.1}$$

$$C_2 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin\lambda L - \lambda L\cos\lambda L)(\beta_S + 1)}\right]\lambda(\cos\lambda L - 1) \tag{4.53.2}$$

$$C_3 = -\left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin\lambda L - \lambda L\cos\lambda L)(\beta_S + 1)}\right](\cos\lambda L - 1) \tag{4.53.3}$$

$$C_4 = \left[\frac{K_1(N_x^0 + N_x^T + N_x^H) - M_x^T + M_x^H}{EI_{eff}\lambda^2(\sin\lambda L - \lambda L\cos\lambda L)(\beta_S + 1)}\right](\sin\lambda L - \lambda L) \tag{4.53.4}$$

4.2.5. Geometric Imperfection Effects

To account for the imperfection effects, an initial displacement function is defined as in equation (4.54). It should be noted that the geometric imperfection amplitude μ is defined as a percentage of the thickness of the beam (Brush & Almroth, 1975).

$$w^*(x) = I \sin\frac{\pi x}{L} \tag{4.54.1}$$

$$\frac{\partial w^*(x)}{\partial x} = I\left(\frac{\pi}{L}\right)\cos\frac{\pi x}{L} \tag{4.54.2}$$

$$\frac{\partial^2 w^*(x)}{\partial x^2} = -I\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L} \tag{4.54.3}$$

$$\frac{\partial^3 w^*(x)}{\partial x^3} = -I\left(\frac{\pi}{L}\right)^3 \cos\frac{\pi x}{L} \tag{4.54.4}$$

$$\frac{\partial^4 w^*(x)}{\partial x^4} = I\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} \tag{4.54.5}$$

$$I = \mu h \tag{4.54.6}$$

The governing equation of motion can be rewritten as follow:

$$EI_{eff}\frac{\partial^4 w}{\partial x^4} + N_x^0\frac{\partial^2 w}{\partial x^2} = -EI_{eff}\frac{\partial^4 w^*}{\partial x^4} \tag{4.55}$$

To solve for the equation (4.56), a solution that satisfies the different boundary conditions is assumed.

For a simply-supported beam, the solution assumed was:

$$w(x) = a \sin\frac{\pi x}{L} \tag{4.56}$$

Where a is the amplitude of the deflection. To find the amplitude, the solution equation needs to be evaluated in equation (4.55) and in the nonlinear equation (2.33) which is the equation that defines N_x^0. That leads to equation (4.57).

$$\frac{a\pi^4}{L^4} - \frac{a\pi^2}{EI_{eff}L^2}\left(\frac{A_{11}a^2\pi^2}{4L^2} - N_x^T - N_x^H + \frac{A_{11}Ia\pi^2}{2L^2}\right) + \frac{I\pi^4}{L^4} = 0 \tag{4.57}$$

Since the above equation has a polynomial shape, a can be solved by finding the roots.

For a clamped-clamped beam the solution assumed was:

$$w(x) = a\left(1 - \cos\frac{2\pi x}{L}\right) \tag{4.58}$$

In this case, the problem needs to be solved by applying the Galerkin method as shown below

$$R(w) = EI_{eff}\frac{\partial^4 w}{\partial x^4} + N_x^0\frac{\partial^2 w}{\partial x^2} + EI_{eff}\frac{\partial^4 w^*}{\partial x^4} \tag{4.59}$$

$$\phi(x) = \left(1 - \cos\frac{2\pi x}{L}\right) \tag{4.60}$$

$$\int_0^L R(w)\phi(x)dx = 0 \tag{4.61}$$

A similar procedure should be conducted for beams with mixed boundary conditions. For which the solution was assumed to be:

$$w(x) = a\left(\sin\frac{1.43\pi x}{L} - 1.43\pi\cos\frac{1.43\pi x}{L} + 1.43\pi - \frac{1.43\pi x}{L}\right) \tag{4.62}$$

5. Comparison of Results

The results of the methods developed in this thesis have been compared with previously published works. A good agreement was found between the present work and the references. For the method developed for bending with mechanical load, a comparison with Sun & Chin (1998) was made and shown in Figure 5.1 and Figure 5.2. The material properties and the composite layup used by Sun & Wang (1998) were adopted for comparison purposes. It should be noted that there is a slight deviation from the present results and the ones found by Sun & Wang (1998). The reason for this difference is that the previous authors neglected the effects of N_{xy} on the derivations of the governing equations of motion and that the mathematical software used nowadays is more sophisticated.

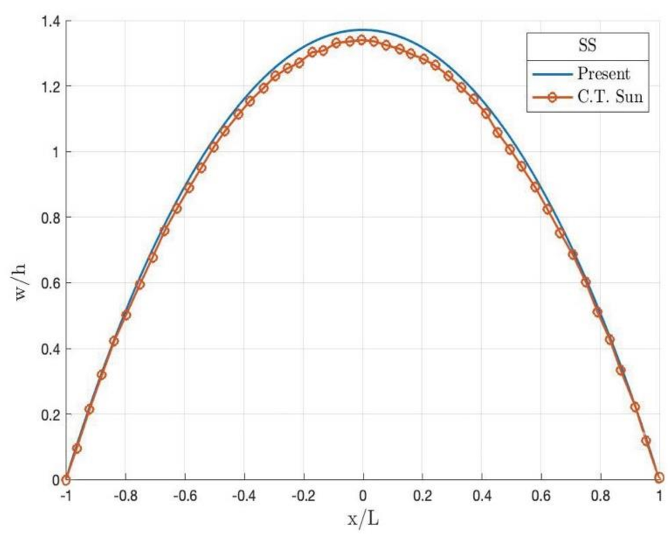

Figure 5.1. Validation of the nonlinear bending deflection formulation (Sun & Chin, 1998).

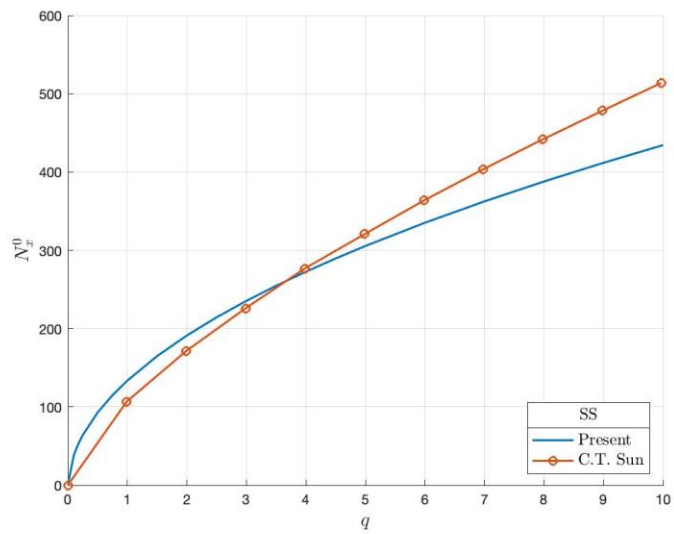

Figure 5.2. Validation of nonlinear bending mid-span rise for different loadings (Sun & Chin, 1998).

A comparison with the results for the dimensionless critical buckling load obtained by Gupta et al. (2009) was made. They developed a general closed-form solution that works for any kind of composite material by assuming a mode shape function. This closed-form solution was plotted in Figure 5.3, together with the solution developed in this thesis. The plots were made for graphite-epoxy angle-ply laminated beams. It should be noted that there is a slight deviation from the reference work. This is due to an underestimation in the mode shape function assumed by the reference work (Gunda & Rao 2013).

The critical buckling is compared with the numerical results obtained by Khdeir (1999). Khdeir (1999) obtained dimensionless numerical values for cross-ply graphite-epoxy beams subjected to a temperature using both the classical theory and the shear deformation theory. The same material properties and geometric parameters as the reference were adopted to compare the results, and a perfect agreement with the reference was found for the dimensionless critical buckling temperature, as shown in

Table 5.1.

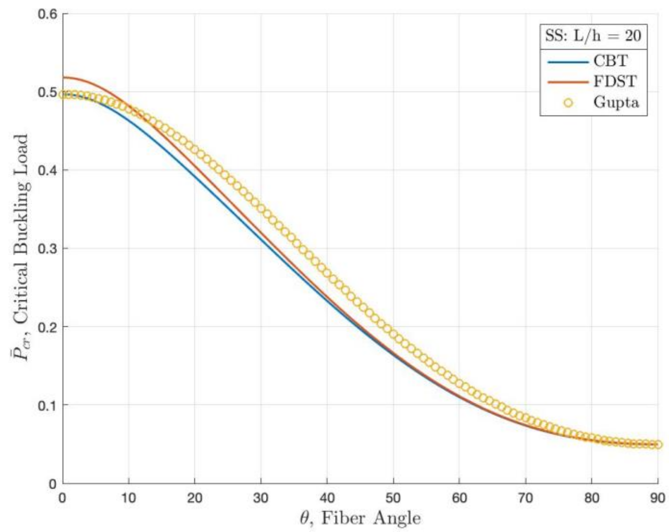

Figure 5.3. Comparison of the dimensionless critical buckling load (Gupta, Gunda, Janardhan, & Rao, 2009).

Table 5.1

Critical buckling temperature for different boundary conditions (Khdeir, 1999).

L/h		Beam theories	Boundary conditions		
			S-S	C-S	C-C
5	Reference	FSDT	0.4715	0.5667	0.6927
	Present	FSDT	0.4715	0.6022	0.6927
10	Reference	FSDT	0.8281	1.2896	1.8859
	Present	FSDT	0.8281	1.3383	1.8858
20	Reference	FSDT	1.0212	1.9044	3.3123
	Present	FSDT	1.0212	1.9274	3.3123
50	Reference	FSDT	1.0925	2.1985	4.2023
		CBT	1.1072	2.2652	4.4290
	Present	FSDT	1.0925	2.1983	4.2023
		CBT	1.1072	2.2588	4.4290

The thermal post-buckling results were compared with Fu, Wang and Hu (2014). Fu et al. (2014) obtained the solutions for the thermal post-buckling of cross-ply laminated composite beams using a different method of solution. A perfect agreement was achieved by adopting the same geometry and material properties as the reference in the method developed in this thesis. Figure 5.4, Figure 5.5 and Figure 5.6 depict the comparison results for the mid-span rise of a cross-ply beam subjected to different temperatures. Figure 5.7, Figure 5.8 and Figure 5.9 show the comparison for the post-buckling deflection of a beam subjected to a $T = 1350\ K$.

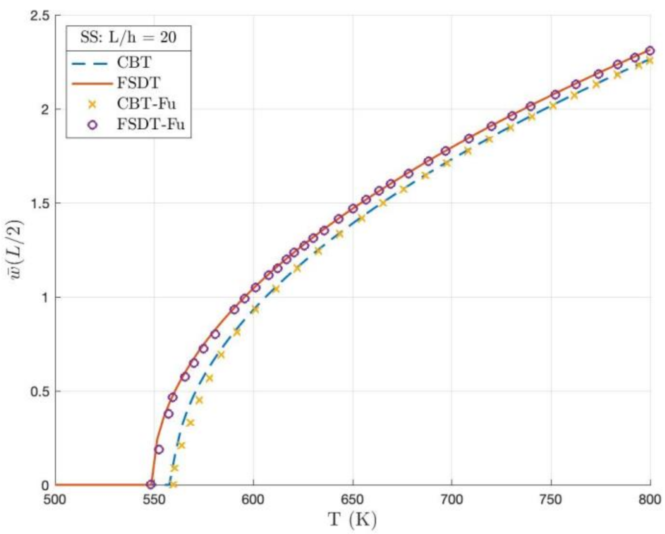

Figure 5.4. Post-buckling validation for SS beams (Fu, Wang & Hu, 2014).

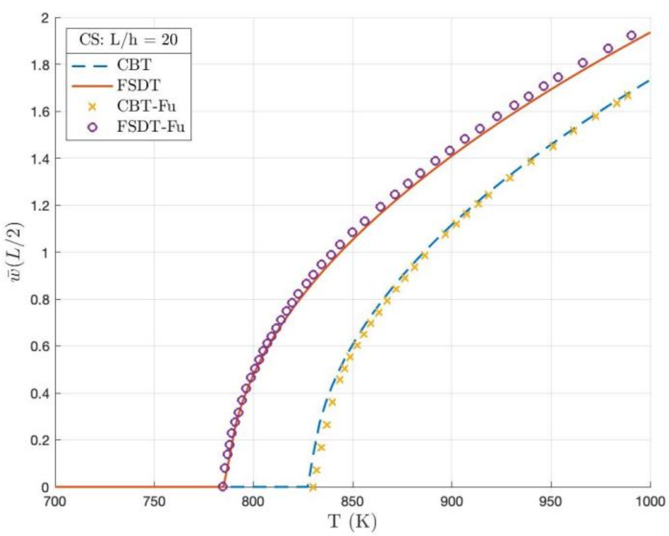

Figure 5.5. Post-buckling validation for CS beams (Fu, Wang & Hu, 2014).

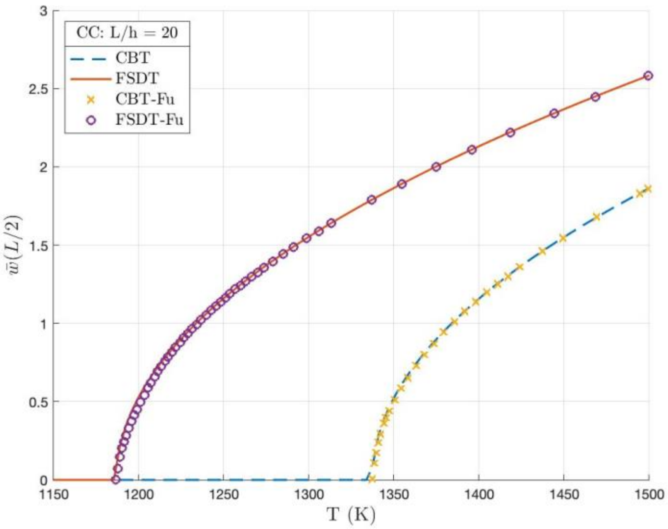

Figure 5.6. Post-buckling validation for CC beams (Fu, Wang & Hu, 2014).

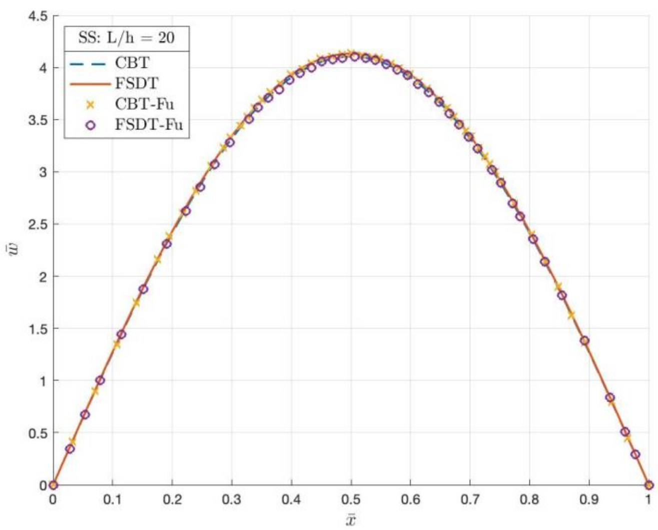

Figure 5.7. Post-buckling validation for SS beams (Fu, Wang & Hu, 2014).

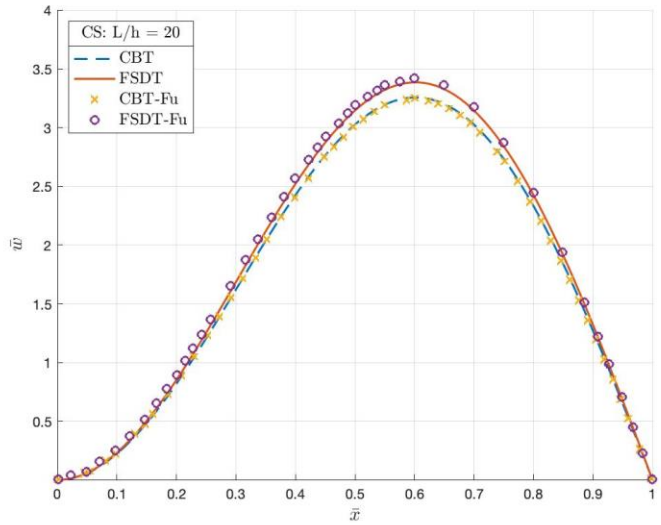

Figure 5.8. Post-buckling validation for CS beams (Fu, Wang & Hu, 2014).

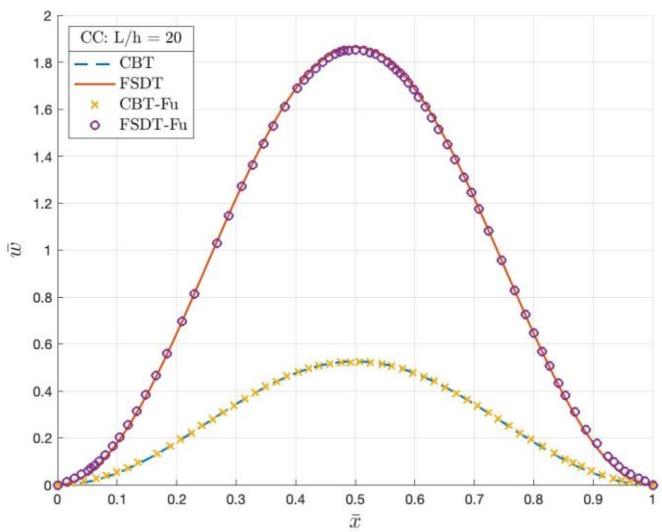

Figure 5.9. Post-buckling validation for CC beams (Fu, Wang &Hu, 2014)

6. Numerical Examples and Discussions

The numerical results of section 3 and 4 are presented here. Since two different types of laminates were studied, first the results of bending, buckling, and post-buckling of angle-ply laminated beams are presented. Then, the results for cross-ply will be presented in the next subsection. The beams in this study are made of graphite-epoxy whose materials properties are listed in Table 6.1. Material 2 was used for the solution of mechanical bending (Sun & Chin, 1998). In the rest of examples, Material 1 was utilized (Fu, Wang & Hu, 2014).

Table 6.1

Material and Geometric Properties of the Graphite-Epoxy beams

	Material 1	Material 2
Length, L	12 m	9 in
Thickness, h	0.6 m	0.02 in
E_1	189 GPa	20 msi
E_2	18.9 GPa	1.4 msi
G_{12}	11.34 GPa	0.7 msi
G_{13}	11.34 GPa	0.7 msi
G_{23}	9.45 GPa	0.7 msi
ν_{12}	0.25	0.3
ν_{23}	0.25	0.3
K_s	5/6	5/6
α_1	10 μm/m/°C	5.5 μin/in/°F
α_2	30 μm/m/°C	16.5 μin/in/°F
β_1	0 m/m/kg/kg	0 in/in/lb/lb
β_2	0.6 m/m/kg/kg	0.6 in/in/lb/lb

6.1. Angle-Ply

In this section, the numerical examples of angle-ply laminates are presented. The angle-ply considered is a [θ/-θ/θ/-θ] layup. Here, the effects of shear, geometric parameter, fiber angle, moisture, and imperfections on thermal bending, buckling, and post-buckling are discussed.

It should be noted that the results shown for the mechanical bending were obtained using material 2. Material 2 describes a very thin beam, therefore, only the results from classical theory are shown. Several trends can be observed in the nonlinear behavior of angle-ply laminates. When a mechanical load is applied, the in-plane load of the beam is increased as the mechanical load is increased, as seen in Figure 6.1 through Figure 6.3. This behavior agrees with the results found by Sun and Chin (1998). On the other hand, when the temperature is increased, the in-plane load is decreased. This could have been predicted from equation (2.33) where it is seen that the thermal load reduced the value of the in-plane load.

When an angle-ply beam is subjected to a thermal loading as described in section 4.2, the beam shows a bending behavior similar to the bifurcation buckling, which means that when the beam reaches a critical buckling temperature, it starts to deflect positively. It should also be noted that simply-supported beams will deflect more than the other two configurations, as seen in Figure 6.7, Figure 6.8 and Figure 6.9. Material 1 was used to model the thermal bending of angle-ply laminates. Also, the dimensionless deflection is defined as $\bar{w}(x) = \sqrt{12}\,\frac{w(x)}{h}$ (Fu, Wang & Hu, 2014).

The effects of the shear deformation are analyzed in Table 6.2 for SS beams, Table 6.3 for CS beams, and Table 6.4 for CC beams. If the percentage of the difference between FSDT and CBT of the three types of beam configurations are compared, it can be noted that the effects of shear deformation are more significant for CC beams. Also, it should be noted that for the fiber angle of 90 degrees, the effects of shear deformation are reduced. From Table 6.5, it can be concluded that the effects of shear cannot be neglected for thick beams since the CBT overestimates the values. Figure 6.12 and Figure *6.13* show the dimensionless critical buckling load and temperature for different fiber angles, respectively where $\bar{P}_{cr} = \frac{P_{cr}}{E_1}\left(\frac{L}{h}\right)^2$, and $\bar{T}_{cr} = \frac{T_{cr}}{\alpha_1}\left(\frac{L}{h}\right)^2$(Fu, Wang & Hu, 2014). From these figures, it can be concluded that an increment in the fiber angle will cause a reduction on the critical buckling load and temperature which makes sense since at 90 degrees the laminate strength will be the same as the matrix.

The effects of moisture in the critical buckling temperature are shown in Figure 6.14 through Figure 6.16. As expected from the equations derived in section 4.2, an increment on the moisture percentage will reduce the critical buckling temperature, and will increase the post-buckling deflection.

The effects of imperfection in the post-buckling are presented in Figure 6.29, and Figure 6.30 for a simply-supported beam, and for a clamped-clamped beam. The geometric imperfections make the beam start deflecting after the bifurcation point.

Mechanical Bending

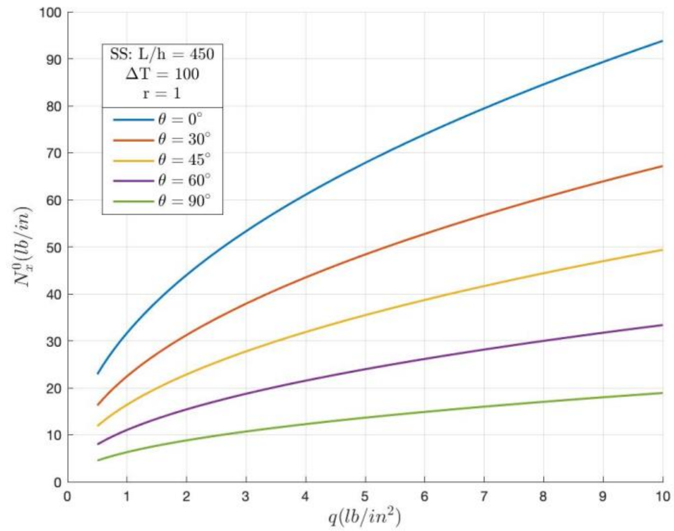

Figure 6.1. In-plane load for different transverse loads of a simply supported beam.

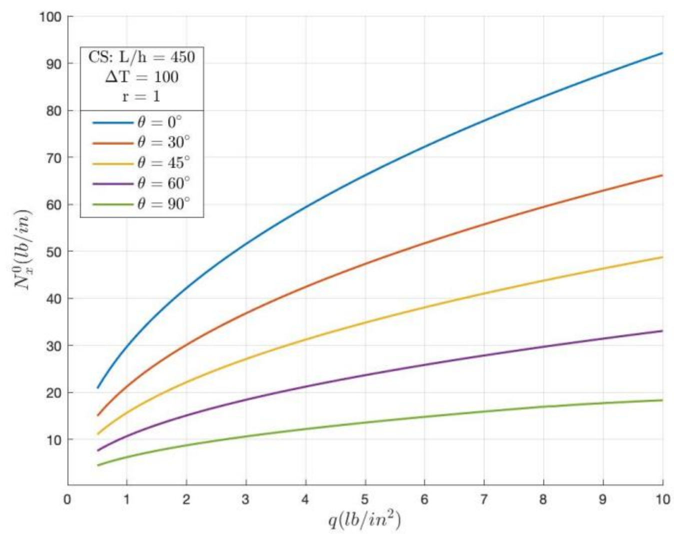

Figure 6.2. In-plane load for different transverse loads of a beam with mixed boundary conditions.

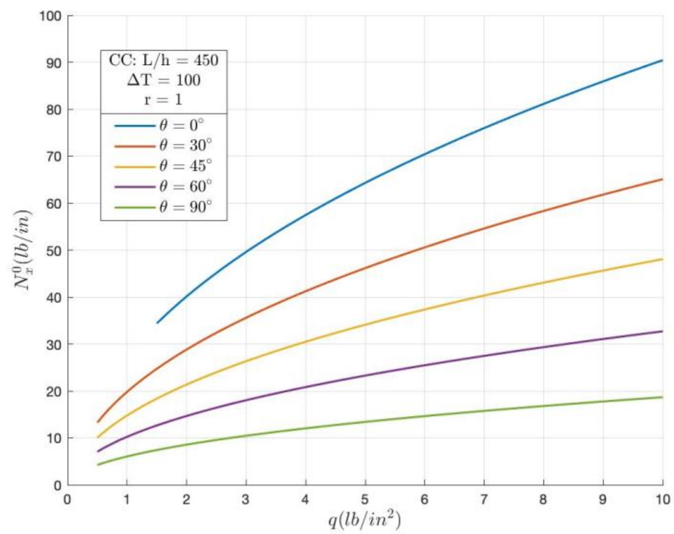

Figure 6.3. In-plane load for different transverse loads of a clamped-clamped beam.

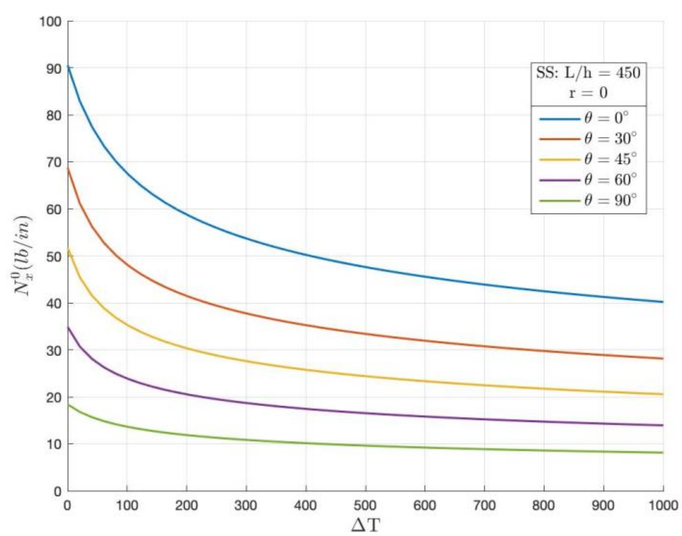

Figure 6.4. In-plane load for different temperature variations of a simply supported beam.

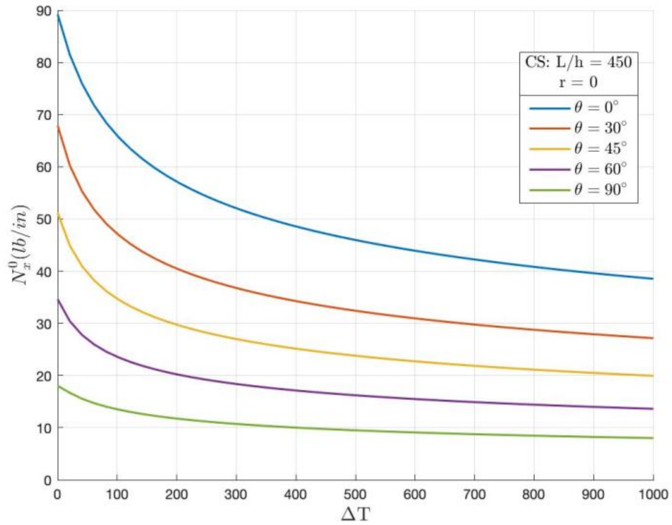

Figure 6.5. In-plane load for different transverse loads of a beam with mixed boundary conditions.

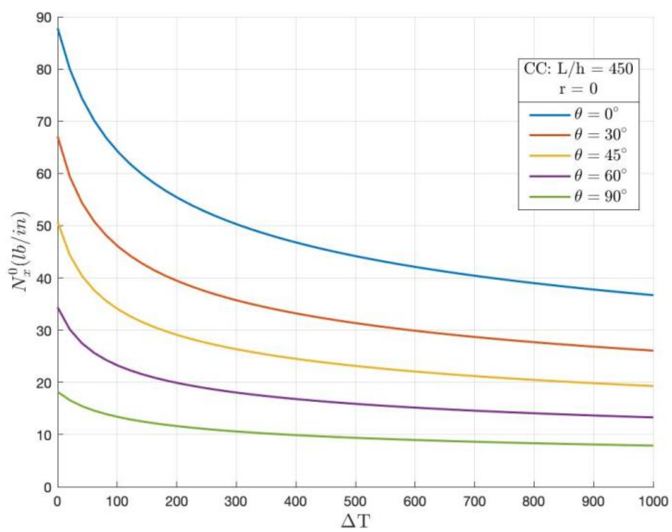

Figure 6.6. In-plane load for different transverse loads of a clamped-clamped beam.

Thermal Bending

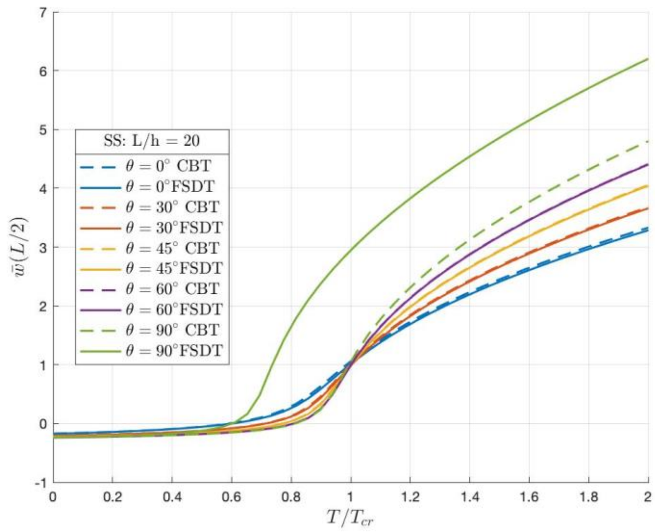

Figure 6.7. Mid-span rise for a simply-supported beam subjected to thermal loading.

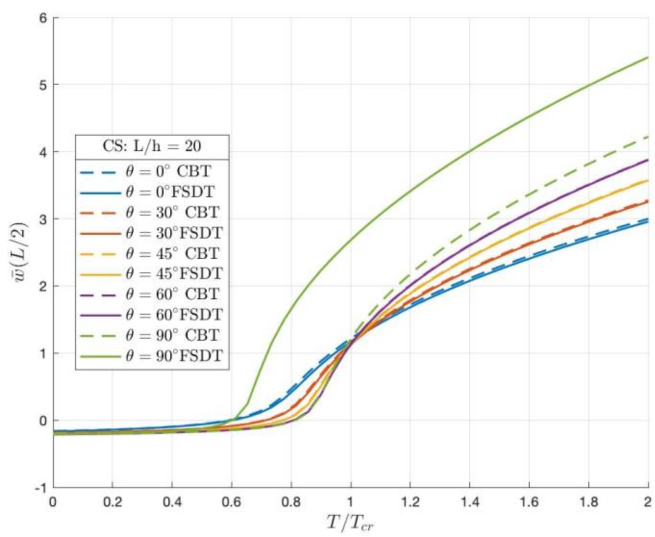

Figure 6.8. Mid-span rise for a beam with mixed boundary conditions subjected to thermal loading.

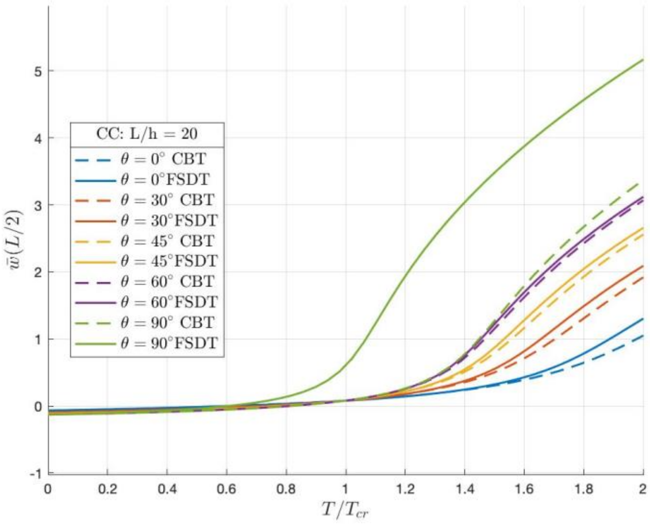

Figure 6.9. Mid-span rise for a clamped-clamped beam subjected to thermal loading.

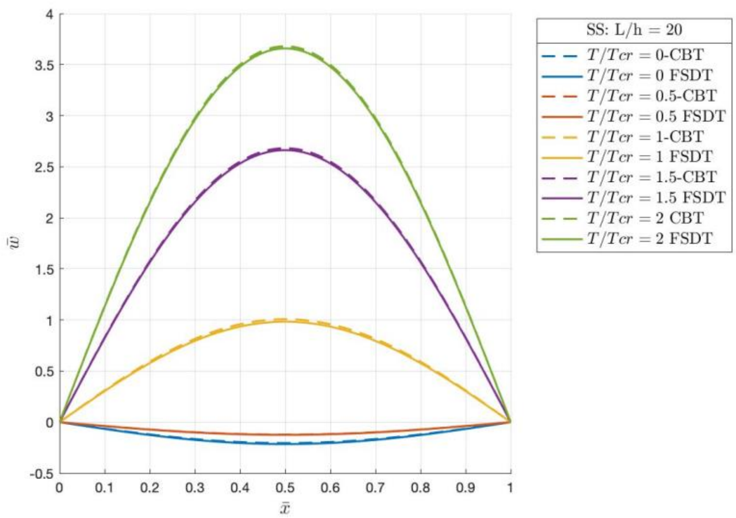

Figure 6.10. Deformation of a 30° angle-ply simply support beam.

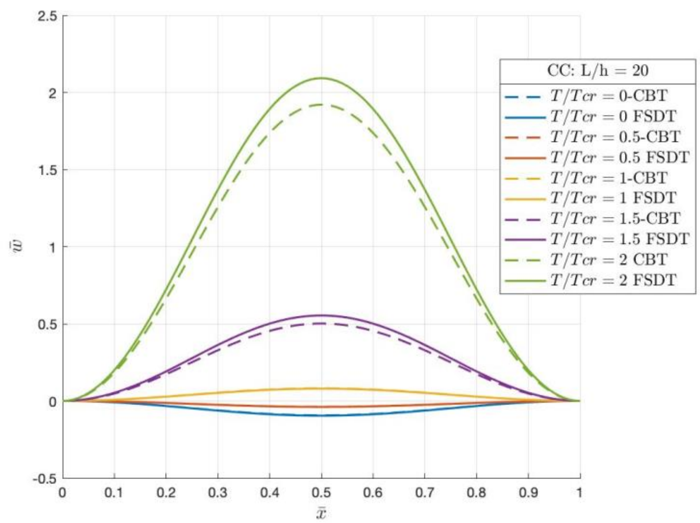

Figure 6.11. Deformation of a 30° angle-ply clamped-clamped beam.

Buckling Results

Table 6.2

Critical Buckling Load T_{cr} (K) for simply supported beams

θ (°)	CBT	FSDT	% diff
0	505.6168	497.4460	1.6425
30	421.4602	418.2602	0.7651
45	388.6537	387.0953	0.4626
60	370.8356	370.1012	0.1984
90	368.5389	368.2002	0.0920

Table 6.3

Critical Buckling Load T_{cr} (K) for beams with mixed boundary conditions

θ (°)	CBT	FSDT	% diff
0	719.6227	686.9443	4.7500
30	574.8759	534.9040	2.4251
45	480.9254	474.5507	1.3441
60	444.5613	441.5351	0.6854
90	439.8742	438.4709	0.3200

Table 6.4

Critical Buckling Load T_{cr} (K) for clamped-clamped beams

θ (°)	CBT	FSDT	% diff
0	1122.5111	1005.7000	11.6149
30	785.8407	738.3907	6.4261
45	654.6148	630.9301	3.7539
60	583.3424	571.9457	1.9926
90	574.1557	568.8161	0.9387

Table 6.5

Dimensionless critical buckling loads for different length to thickness ratios for a 30° angle-ply laminated beam

L/h	Beam theories	Boundary conditions		
		S-S	C-S	C-C
5	FSDT	0.5616	1.0403	1.7373
	CBT	0.6224	1.2701	2.4894
10	FSDT	0.3030	0.6018	1.1232
	CBT	0.3112	0.6351	1.2447
20	FSDT	0.1239	0.2518	0.4894
	CBT	0.1245	0.2540	0.4979
50	FSDT	0.0124	0.0254	0.0498
	CBT	0.0124	0.0254	0.0498s

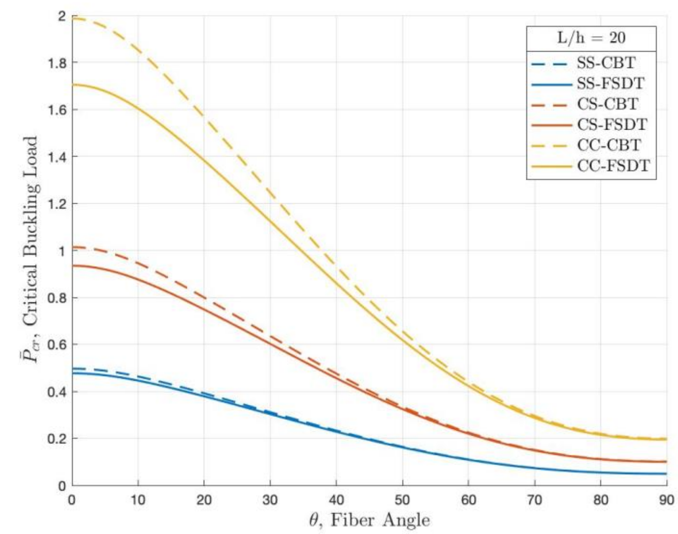

Figure 6.12. Dimensionless critical buckling load for different fiber angles.

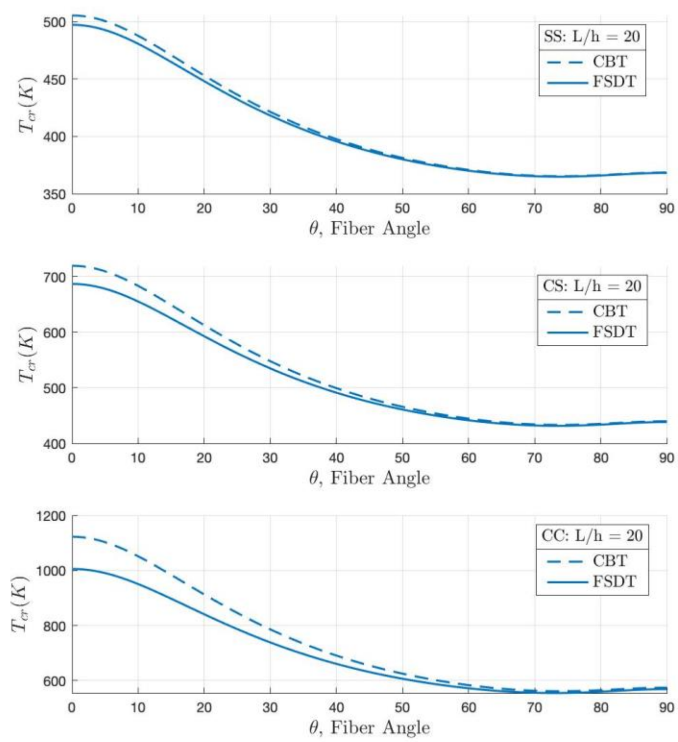

Figure 6.13. Dimensionless critical temperature for different fiber angles.

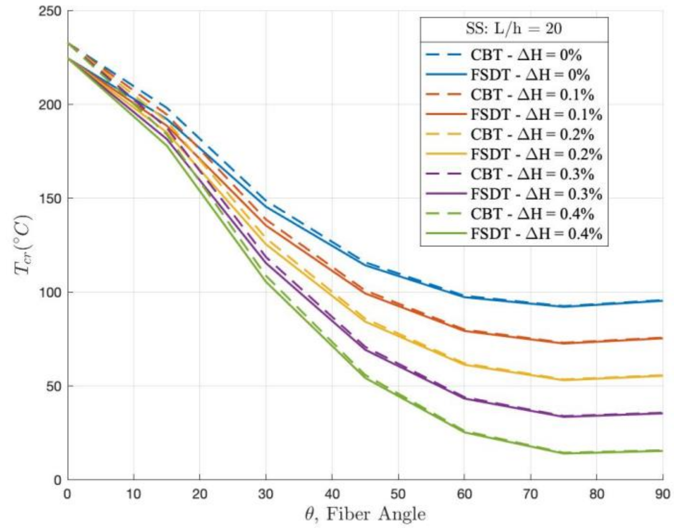

Figure 6.14: Effects of moisture in the critical buckling load of simply supported beams.

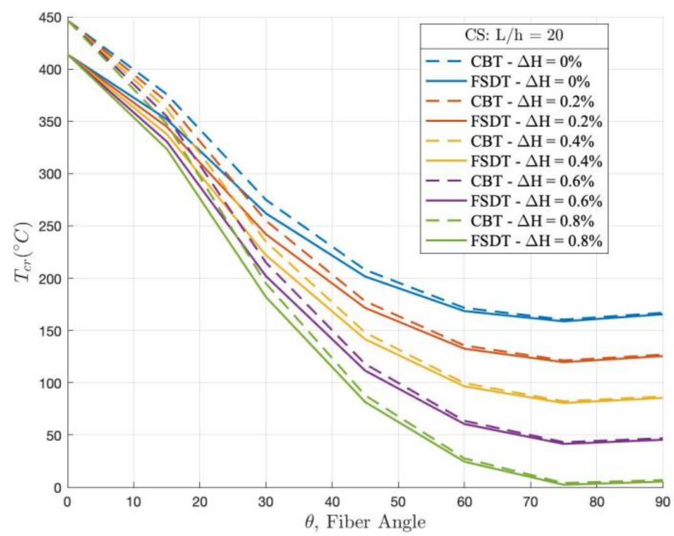

Figure 6.15. Effects of moisture in the critical buckling load of beams with mixed boundary conditions.

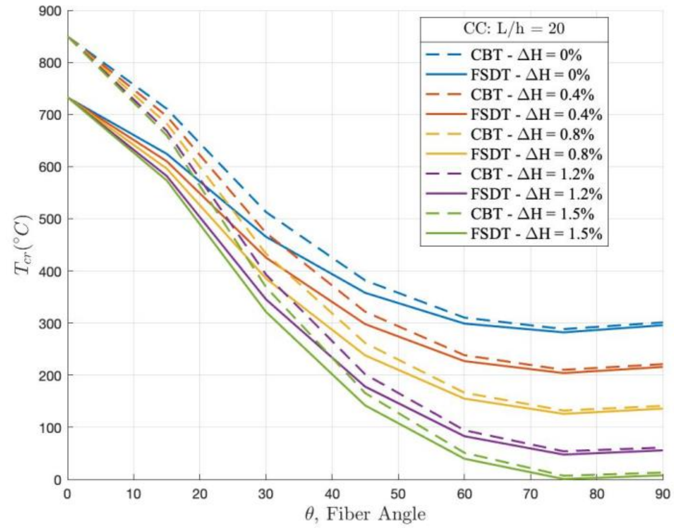

Figure 6.16. Effects of moisture in the critical buckling load of clamped-clamped beams.

Post-Buckling Results

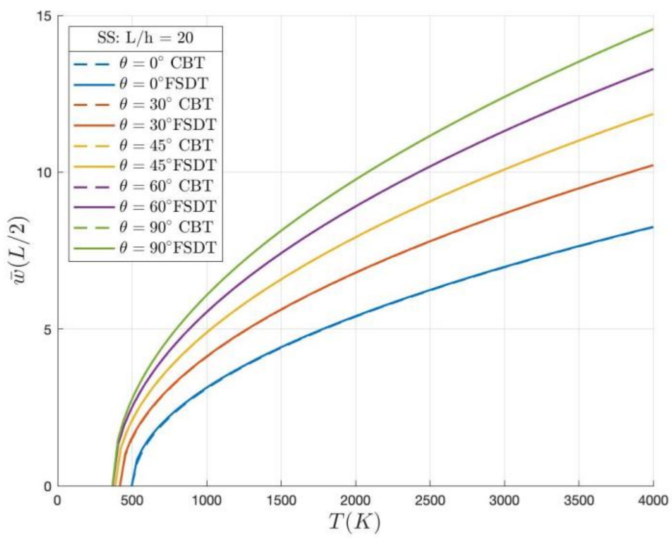

Figure 6.17. Dimensionless mid-span deflection for simply-supported beams with different angle-ply configurations.

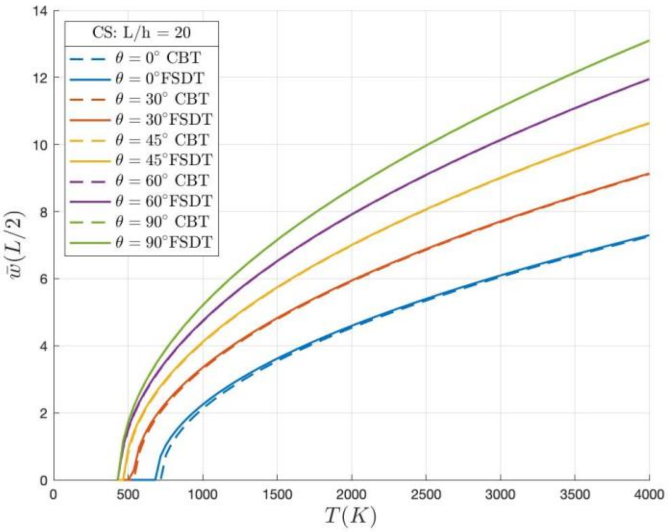

Figure 6.18. Dimensionless mid-span deflection for beams with mixed boundary conditions for different angle-ply configurations.

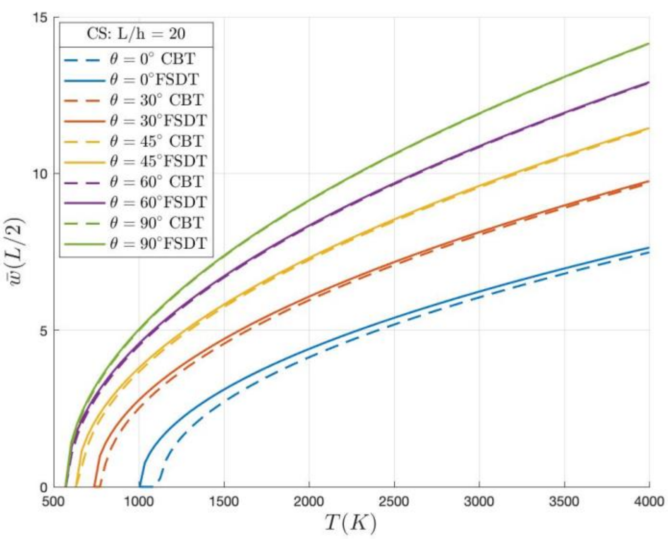

Figure 6.19. Dimensionless mid-span deflection for clamped-clamped beams with different angle-ply configurations.

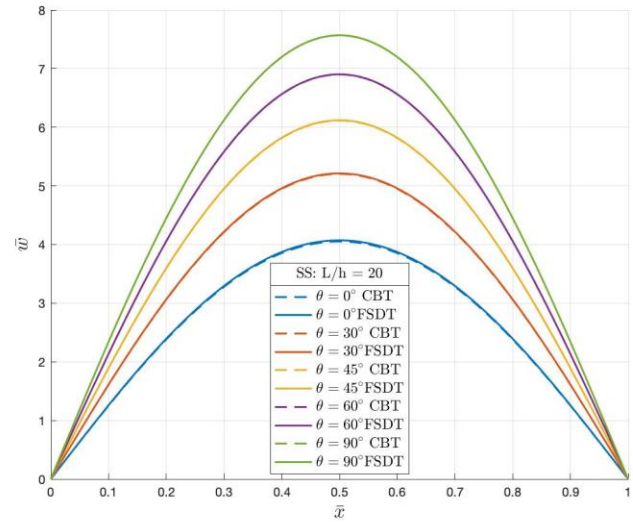

Figure 6.20. Post-buckling dimensionless deflection for simply supported beams for different angle-ply configurations.

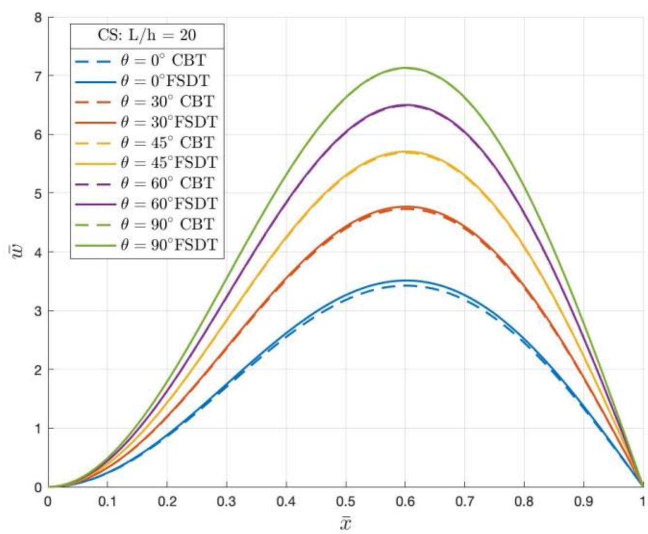

Figure 6.21. Post-buckling dimensionless deflection for beams with mixed boundary conditions with different angle-ply configurations.

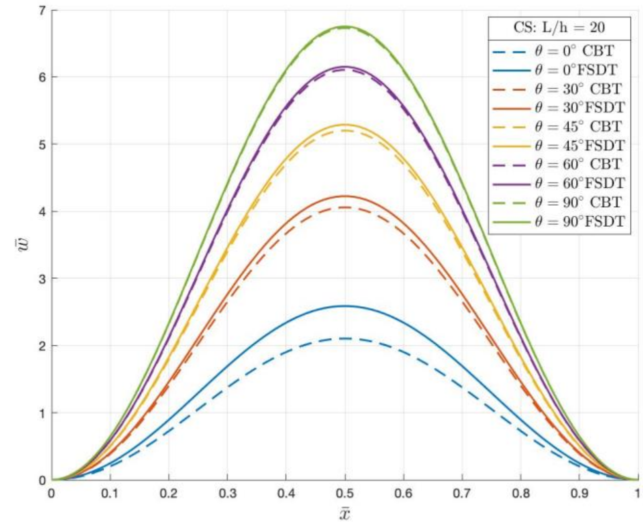

Figure 6.22. Post-buckling dimensionless deflection for clamped-clamped beams with different angle-ply configurations.

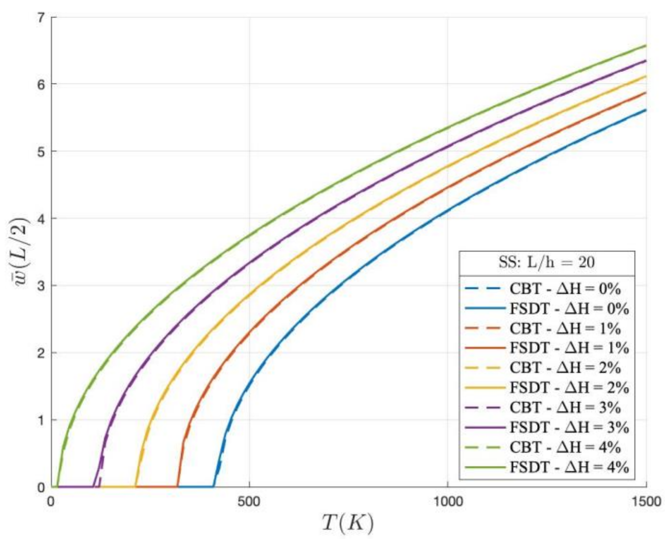

Figure 6.23. Effects of moisture on the dimensionless mid-span rise for a 30° angle-ply simply supported beam.

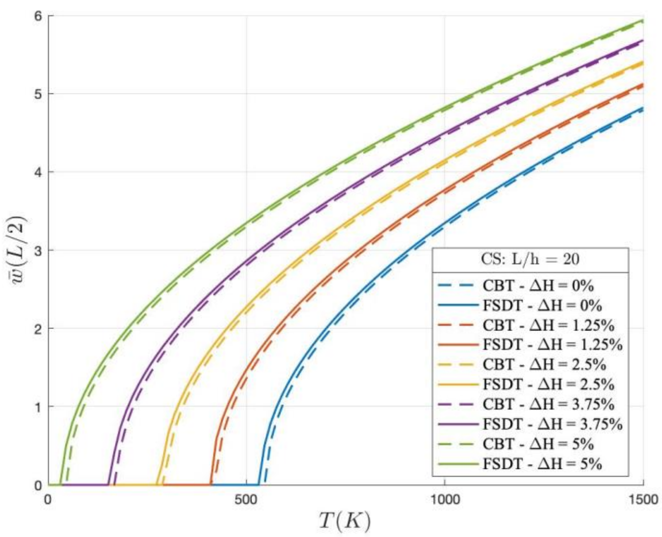

Figure 6.24. Effects of moisture on the dimensionless mid-span rise for a 30° angle-ply beam with mixed boundary conditions.

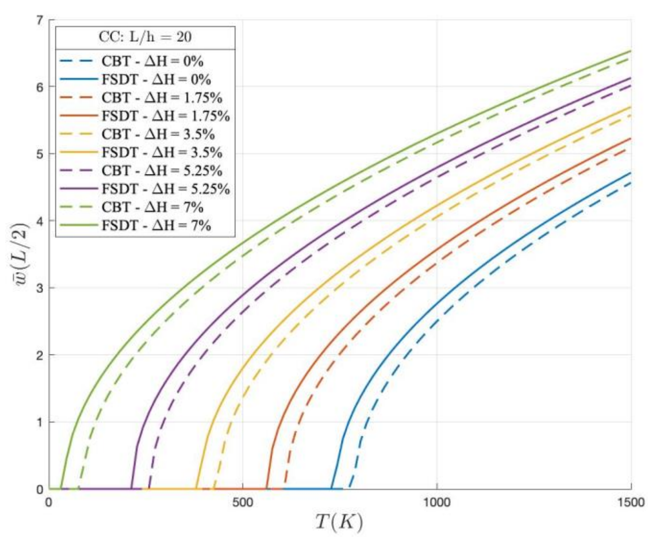

Figure 6.25. Effects of moisture on the dimensionless mid-span rise for a 30° angle-ply or clamped-clamped beam.

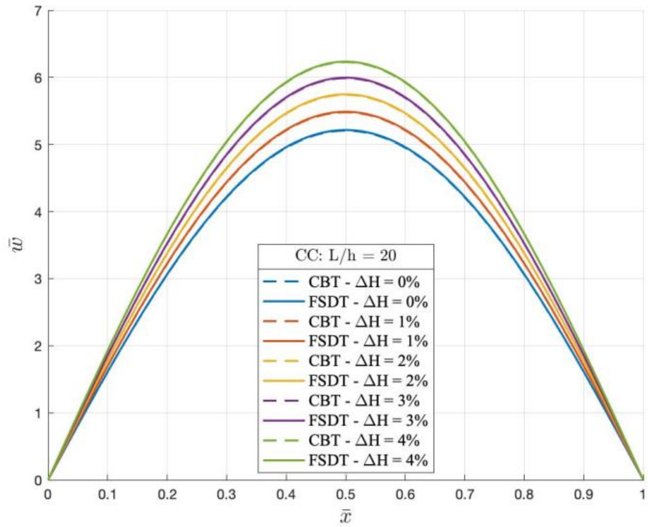

Figure 6.26. Moisture effects on the post-buckling dimensionless deflection for a 30° angle-ply simply-supported beam.

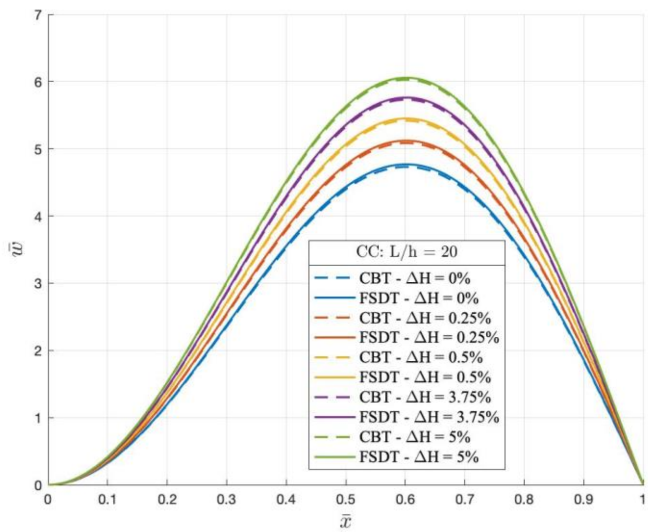

Figure 6.27. Moisture effects on the post-buckling dimensionless deflection for a 30° angle-ply beam with mixed boundary conditions.

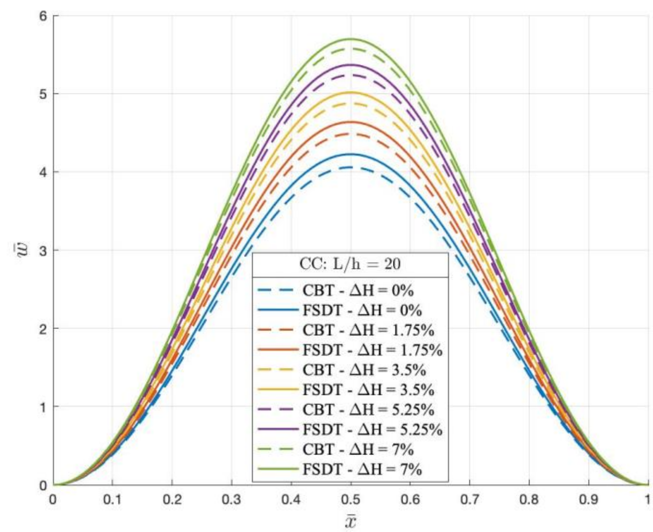

Figure 6.28. Moisture effects on the post-buckling dimensionless deflection for a 30° angle-ply clamped-clamped beam.

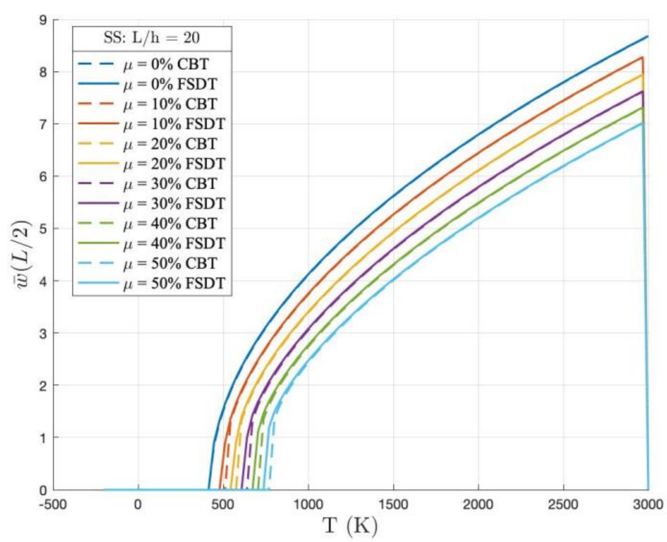

Figure 6.29. Imperfection effects on the dimensionless mid-span deflection for a 30° angle-ply simply supported beam.

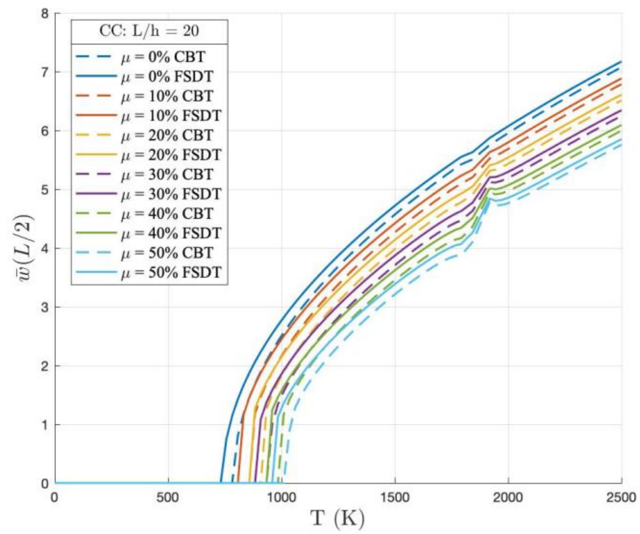

Figure 6.30. Imperfection effects on the dimensionless mid-span rise for a 30° angle-ply clamped-clamped beam.

6.2. Cross-Ply

In this section, the numerical examples of cross-ply laminates are presented. The cross-ply considered is a $[90_2/0_2]$ layup. The mechanical bending and post-buckling of cross-ply laminates are presented here. Figure 6.31 shows that in the case of cross-ply laminates, the boundary conditions do not cause a big difference, which is the same case for angle-ply. Also, an increment on the mechanical load causes an increment in the in-plane load. However, as in Figure 6.32, an increment in the temperature rise will cause a reduction in the in-plane load like the case of angle-ply laminates. From Figure 6.33, it should be noted that cross-ply laminates do not present a bifurcation buckling behavior since they start to deflect as soon as an increment in the temperature is applied.

Bending

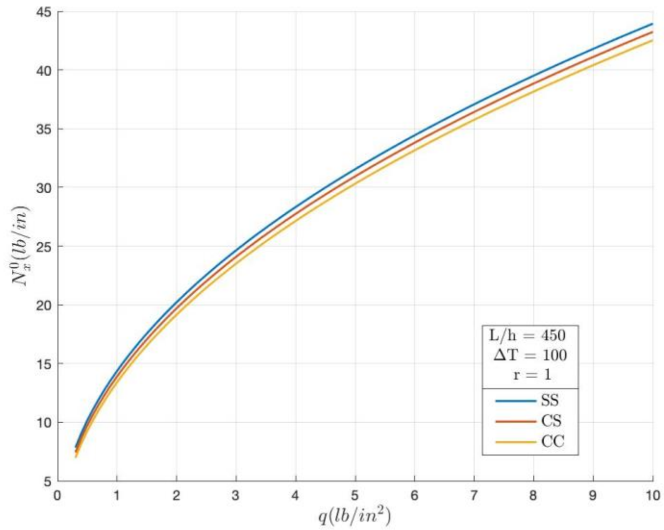

Figure 6.31. In-plane load for different transverse loads of cross-ply laminated beam subjected to linear temperature variation.

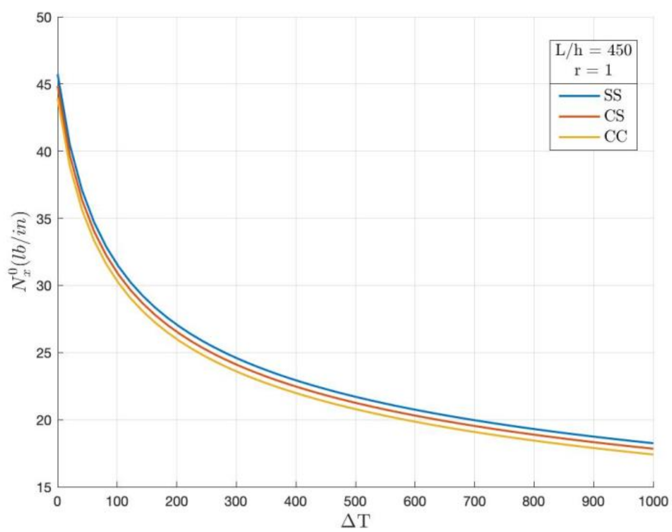

Figure 6.32. In-plane load for beam subjected to a transverse load $q = 5\ lb/in^2$.

Post-Buckling

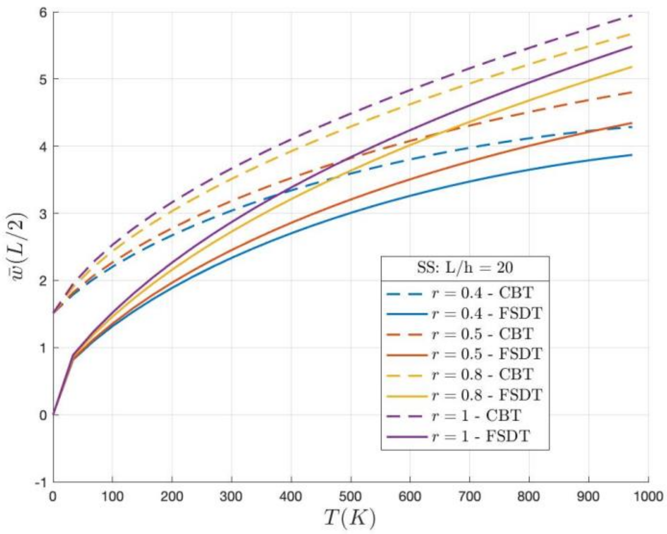

Figure 6.33. Mid-span rise for a simply supported cross-ply beam subjected to a linear temperature variation.

7. Conclusions and Future Recommendations

This section describes the conclusion obtained from the analysis of the thermal bending, buckling and post-blucking of angle-ply and cross-ply laminated beams under the effects of moisture and geometric imperfections. Additionally, this section presents examples for future recommendations.

7.1. Conclusions

Nonlinear analysis of the thermal bending, buckling, and post-buckling of unsymmetrically laminated beams with the effects of moisture and imperfections was performed in this thesis. The nonlinear equations of motion were derived with both classical beam theory and first-order shear deformation theory. The von-Karman geometrical nonlinearity is considered in the derivations. Analytical expressions for the bending, buckling, and post-buckling were derived. Two types of unsymmetrically laminated composites beams were analyzed here: angle-ply and cross-ply.

The following conclusions can be drawn from this research:

- When the length to thickness ratio is less than 30, the effects of shear cannot be neglected when performing a nonlinear analysis of unsymmetrical composites. The classical theory overestimates the values of critical buckling load and critical buckling temperature.
- The effects of the shear deformation are more significant in the case of CC beams.
- The buckling behavior of angle-ply and cross-ply laminates are different. Therefore, different methods have been presented in this thesis to obtain the solution for these two laminates.
- Angle-ply laminates show a bifurcation buckling, thus, a critical buckling point is

observed due to their B_{11} being equal to zero.

- For bending, when the applied transverse load is increased, an increase in the in-plane load is observed, whereas when the temperature is increased, the in-plane load is reduced.
- When a thermal load is applied to an angle-ply laminate, a critical point is observed when at the critical temperature.
- An increase in the fiber angle will cause a reduction in the in-plane load, the critical buckling load, and the critical buckling temperature, but an increase in the post-buckling deflection.
- As expected, a rise of moisture percentage in angle ply laminates will cause a reduction in the critical buckling temperature. It was also observed that CC beams tend to sustain higher moisture percentages than SS and CS beams.
- Cross-ply laminates do not have a bifurcation buckling; hence, as soon as a change in temperature is applied, the beam starts to extend and bend.
- The effects of shear deformation are more significant in the case of cross-ply due to their B_{11}.

7.2. Future Work

The study performed in this thesis is extensive; however, there are still more aspects to be analyzed. For example, analyzing the effects of imperfections and moisture for cross-ply, and doing experimental validation of all the results obtained above. Extending the problem to plates or shells. Also, doing the analysis of thermally induced vibration of composite laminates. In addition, one can work on aerodynamic or thermoelastic flutter analysis of composite laminates, and so on.

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