Instantaneous blowup, old and new results

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Consider

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + \frac{c}{|x|^2} u + f(x,t), \\ x &\in \mathbb{R}^N, \ t \geq 0, \\ u(x,0) &= u_0(x).\end{aligned}$$

where $f \geq 0$, $u_0 \geq 0$ and $(f, u_0) \neq 0$. Assume u_0 and f are such that a unique positive solution exists for all $t \geq 0$ if $c = 0$. Then for $c \leq C^*(N) = \left(\frac{N-2}{2}\right)^2$,many global (in time) solutions exist, but for $c > C^*(N)$, no positive solution in the sense of distributions exists. In fact, if $V(x) = c/|x|^2$ is replaced by $V_n(x) = \min\{V(x), n\}$ and if $f(x,t)$ is replaced by $f_n(x,t) = \min\{f(x,t), n\}$, and if u_n is the corresponding unique positive solution, then $u_n(x,t) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}^N$, $t > 0$. This is instantaneous blowup (IBU). This 1984 theorem of Pierre Baras and Jerry Goldstein led to much additional research on singular linear and nonlinear parabolic PDE. We will discuss new proofs and new theorems, including IBU when $\mathbb{R}^N$ is replaced by the Heisenberg group $\mathbb{H}^N$, nonexistence of positive solutions (even locally in time) for certain nonlinear parabolic problems, and related topics, involving the Ornstein-Uhlenbeck equation and others. . This is joint work with many coauthors.