1-15-2017

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Abstract

Let $\mathfrak{A}$ be a C*-algebra with identity and real rank zero. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of $\mathfrak{A}$ and the identity component of the set of invertible elements of $\mathfrak{A}$. We give a short transparent proof that the function has a holomorphic extension to the entire open unit ball of $\mathfrak{A}$. The author previously deduced this from a more general fact about Banach algebras.

Keywords: infinite dimensional holomorphy, weak (FU)
2010 MSC: 46G20, 46L05

1. Preliminary definitions and theorems.

Recall [1] that a C*-algebra is a closed complex subalgebra $\mathfrak{A}$ of the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space with the operator norm such that $\mathfrak{A}$ contains the adjoints of each of its elements. All our C*-algebras contain the identity operator I.

To give a basic example, let S be a compact Hausdorff space and let $C(S)$ be the algebra of all continuous complex-valued functions on S with the sup norm. Then there exist a Hilbert space H, a C*-algebra $\mathfrak{A}$ in $\mathcal{B}(H)$ and an isomorphism $\rho : C(S) \to \mathfrak{A}$ that preserves norms and adjoints. To see this, let H be the Hilbert space having the same dimension as the cardinality of S and let $\{e_s : \ s \in S\}$ be an orthonormal basis for H. Then we may take $\rho(f)$ to be the multiplication operator defined by $\rho(f)(e_s) = f(s)e_s$ for all $s \in S$ and $f \in C(S)$.

More generally, one can define a Banach algebra that is an abstraction of a C*-algebra and show that an isomorphism like the above exists. Specifically, a B*-algebra is a complex Banach algebra A with an involution * such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Then a norm and adjoint preserving isomorphism ρ of A onto a C*-algebra exists by the Gelfand-Naimark theorem [1, p. 209].

☆Dedicated to Richard Aron with gratitude

We now turn to some basic facts about complex-valued holomorphic functions defined on a domain D in a complex Banach space X. We say that a function $f : D \to \mathbb{C}$ is holomorphic if for each $x \in D$ there exists a continuous complex-linear functional $\ell \in X^*$ such that

$$\lim_{y \to 0} \frac{f(x+y) - f(x) - \ell(y)}{\|y\|} = 0.$$

Clearly, if f is holomorphic in D then the function $\phi(\lambda) = f(x + \lambda y)$ is holomorphic (in the usual sense) in a neighborhood of the origin for each $x \in D$ and $y \in X$. It is well known [7, Theorem 3.17.1] that this property also implies holomorphy when f is locally bounded in D. One can extend many classical results about holomorphic functions by applying the above property. For example, this is true for the following elementary form of the identity theorem [7, Theorem 3.16.4].

Proposition 1. *Let D be a domain in a complex Banach space X and let $f : D \to \mathbb{C}$ be holomorphic in D. If f vanishes on a ball in D then f vanishes everywhere in D.*

By definition, a ball is a set of the form

$$B_r(x_0) = \{x \in X : \ \|x - x_0\| < r\},$$

where $x_0 \in X$ and $r > 0$.

We will need the following elementary version of Taylor's theorem, which can be proved as in [7, Theorem 3.17.1], and a simple converse, which can be obtained from the Weierstrass M-test and [7, Theorem 3.18.1].

Proposition 2. *Let X be a complex Banach space and let $x_0 \in X$ and $r > 0$. If $f : B_r(x_0) \to \mathbb{C}$ is a bounded holomorphic function, then for each n there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree n such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x - x_0) \quad \text{for } x \in B_r(x_0). \tag{1}$$

Conversely, if for each n there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree n and if

$$\|P_n\| \le \frac{M}{r^n}, \quad n = 0, 1, \ldots \tag{2}$$

for some positive constants r and M, then the function f given by (1) is holomorphic in $B_r(x_0)$.

For example, if (1) holds then

$$P_n(y) = \frac{1}{n!} \, \frac{d^n}{dt^n} f(x_0 + ty)\bigg|_{t=0}, \quad n = 0, 1, \ldots \tag{3}$$

for all $y \in X$. If f is holomorphic on $B_r(x_0)$ and M is a bound for f, then (2) is a consequence of the classical Cauchy estimates. As usual,

$$\|P_n\| = \sup\{|P_n(x)| : \|x\| \le 1, x \in X\}.$$

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2. Real rank zero.

Definition 1. *(See [2].) Let $\mathfrak{A}$ be a C*-algebra and let $\mathcal{S}$ be the set of self-adjoint elements of $\mathfrak{A}$. Then $\mathfrak{A}$ has real rank zero if the elements of $\mathcal{S}$ with finite spectra are dense in $\mathcal{S}$.*

As shown by Brown and Pedersen [2], many interesting C*-algebras have real rank zero. For example, the C*-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H has real rank zero. More generally, any von Neumann algebra has real rank zero. The space $C(S)$ of all continuous functions on a compact Hausdorff space S has real rank zero if and only if S is totally disconnected. (It is a von Neumann algebra only if S is extremely disconnected.) Also, any AF-algebra has real rank zero. If $\mathcal{BC}(H)$ is the C*-algebra of all compact operators on H, then $\mathbb{C}I + \mathcal{BC}(H)$ has real rank zero as does the Calkin algebra $\mathcal{B}(H)/\mathcal{BC}(H)$. Note that the set of invertible elements of the Calkin algebra has a different component for each value of the Fredholm index and thus is not connected. See [3] for further details and references.

Let $\mathfrak{A}$ be a C*-algebra with identity, let

$$\mathfrak{A}_0 = \{A \in \mathfrak{A} : \|A\| < 1\}$$

be the open unit ball of $\mathfrak{A}$ and let $\mathfrak{A}^e_{\text{inv}}$ be the identity component of the set of invertible elements of $\mathfrak{A}$. Our main result is the following:

Theorem 1. *Suppose $\mathfrak{A}$ has real rank zero and let f be a complex-valued function that is holomorphic and bounded on the intersection of the domains $\mathfrak{A}_0$ and $\mathfrak{A}^e_{\text{inv}}$. Then f has a holomorphic extension to $\mathfrak{A}_0$.*

The author does not know even in the commutative case whether the removable singularity property of Theorem 1 characterizes C*-algebras of real rank zero. However, it is shown in [4] that $C(S)$ does not have this property when S contains the homeomorphic image of an interval.

The proof given below of the previous theorem depends on two important facts about the identity component $\mathcal{U}$ of the set of unitary operators in $\mathfrak{A}$. The first is a maximum principle that is a special case of [6, Theorem 8] and [5, Theorem 9] and the second is a density theorem due to Huaxin Lin [8].

Proposition 3. *Let $f : \mathfrak{A}_0 \to \mathbb{C}$ be a holomorphic function having a continuous extension to the closed unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$. If $|f(U)| \le 1$ for all $U \in \mathcal{U}$ then $|f(A)| \le 1$ for all $A \in \mathfrak{A}_1$.*

Proposition 4. *If $\mathfrak{A}$ has real rank zero then the set of unitaries in $\mathcal{U}$ with finite spectrum is dense in $\mathcal{U}$.*

Proof of Theorem 1. Given any ϵ with $0 < \epsilon < 1/2$, let $r = 1-\epsilon$. The set $D = B_r(\epsilon I)\cap\mathfrak{A}^e_{\text{inv}}$ is open since $\mathfrak{A}^e_{\text{inv}}$ is open and one can deduce that D is connected from the fact that $B_r(\epsilon I)$ contains a neighborhood of 0. By Proposition 1, it suffices to show that there exists a function f_ϵ that is holomorphic in the ball $B_r(\epsilon I)$ and satisfies $f_\epsilon(A) = f(A)$ for all $A \in D$. Since the function f is holomorphic in a ball with center at $x_0 = \epsilon I$, it follows from Proposition 2 that

$$f(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I) \tag{4}$$

for all A in this ball. Thus by the converse part of Proposition 2, it suffices to show that

$$\|P_n\| \le \frac{M}{r^n}, \quad n = 0, 1, \dots, \tag{5}$$

where M satisfies $|f| \le M$ on $\mathfrak{A}_0 \cap \mathfrak{A}^e_{\text{inv}}$, since then the function

$$f_\epsilon(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I)$$

is holomorphic on $B_r(\epsilon I)$ and agrees with f on D by Proposition 1.

Let $B \in \mathfrak{A}$ with $\|B\| \le 1$ and suppose the spectrum $\sigma(B)$ is finite. Define $\phi(\lambda) = f(\epsilon I + \lambda B)$. If $|\lambda| < r$ then $\epsilon I + \lambda B \in \mathfrak{A}_0$, $\epsilon I + \lambda B \in \mathfrak{A}^e_{\text{inv}}$ and $|\phi(\lambda)| \le M$ for all but finitely many λ. By the classical Riemann removable singularity theorem, the function ϕ has a holomorphic extension to the disc $|\lambda| < r$ with $|\phi| \le M$. Hence $|\phi^{(n)}(0)| \le n!M/r^n$ by the Cauchy estimates so

$$|P_n(B)| \le \frac{M}{r^n} \tag{6}$$

by (3).

By Proposition 4, inequality (6) holds whenever B is in the identity component of the set of unitary elements of $\mathfrak{A}$ and hence for all $B \in \mathfrak{A}$ with $\|B\| \le 1$ by Proposition 3. This establishes (5) and completes the proof.

The proof of Theorem 1 given in [4] does not require Proposition 4 but the argument is less straightforward. See [4] for further results, examples and references.

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