2-1981

Mathematical Model for Water Quality in Streams Impacted by Point and Nonpoint Source Pollution

Digital Object Identifier: https://doi.org/10.13023/kwrri.rr.126

Michael E. Meadows
University of Kentucky

Research Report No. 126

MATHEMATICAL MODEL FOR WATER QUALITY IN STREAMS IMPACTED BY POINT AND NONPOINT SOURCE POLLUTION

by

Michael E. Meadows
Principal Investigator

Project Number: A-078-KY (Completion Report)

Agreement Numbers: 14-34-0001-9019 (FY 1979)
14-34-0001-0119 (FY 1980)

Period of Agreement: October 1978 - September 1980

University of Kentucky
Water Resources Research Institute
Lexington, Kentucky

The work upon which this report is based was supported in part by funds provided by the Office of Water Research and Technology, United States Department of the Interior, Washington, D.C., as authorized by the Water Research and Development Act of 1978. Public Law 95-467.

February 1981

DISCLAIMER

Research Report No. 126

MATHEMATICAL MODEL FOR WATER QUALITY IN STREAMS IMPACTED BY POINT AND NONPOINT SOURCE POLLUTION

Errata

* Page 12, paragraph 2, line 2: should read "Eq. 11" instead of "Eq. 12"
* Page 19, Figure 5: should show $Q + \frac{\partial Q}{\partial x} dx$ instead of $Q + \frac{\partial Q}{\partial x}$ and "dx" instead of "∂x"
* Page 29, Table 1; line 1; should read "Leopold, et al." instead of "Langbein, et al."
* Page 31, paragraph 2, line 3: should read "for Froude numbers" instead of "for Fronde numbers."
* Page 40, Eq. 61b: should be changed from $\theta = \frac{1}{2}\left[1 - \frac{Q(1-0.25F^2)}{2TS_0 \Delta x c(Q)}\right]$ to $\theta = \frac{1}{2}\left[1 - \frac{Q(1-0.25F^2)}{TS_0 \Delta x c(Q)}\right]$
* Page 43, paragraph 1, line 10: should read "by factors of 20 or more" instead of "by a factor of 720. The best prediction was still in error by a factor of 18."
* Page 51, reference 6: should read "Gburek, W. J. and D. E. Overton" instead of "Glurek, W. J. and D. G. Overton."
* Page 52, reference 1: should real "Ponce, V. M., and Vujica Yevjevich" instead of "Ponce, V. M. and Vujica Yerjerich."
* Page 52, reference 5: should read "Sonnen," instead of "Soonen."

ABSTRACT

Modeling the impacts of stormwater runoff on small streams requires that the prediction model has the capability of simulating the behavior of the hydrologic and water quality components of the stream system. Development of such a model involves coupling the equations for pollutant transport during unsteady flow with the appropriate flood routing equations. The decision on which equations to choose requires a full understanding of the pollutant transport and hydrograph dispersion processes.

This research was undertaken to develop a rigorous theoretical evaluation of the pollutant transport and hydrograph dispersion processes during unsteady flow, and to recommend a suitable model for simulating the impact of stormwater on small streams. It was determined that the one dimensional convective - dispersive equation for tracers (pollutants) coupled with a form of the diffusive wave model for unsteady streamflow would provide the basis for a simulation model that is both simple and consistent with the principal transport processes. Evaluation of the dynamic terms in the momentum equation yielded general estimators to model parameters and established that the Muskingum routing model is consistent with the modified diffusive wave model developed during this research.

The coefficient for hydrograph dispersion was tested on tracer dispersion data and was found to be a reasonable prediction

equation for channels with top widths less than 115 feet and bed slopes greater than 1.6 feet per mile. Most small streams satisfy these conditions.

Descriptors: Storm Water, Water Pollution, Streamflow, Model Studies, Water Quality

Identifiers: Storm Water Quality, Nonpoint Source Pollution, Water Quality Modeling, Unsteady Streamflow, Kinematic Streamflow

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

Introduction

Great emphasis continues at the national and state levels on establishing optimum local and regional water quality management policies. It is now recognized that the formulation of such policies must consider the impacts of point and nonpoint sources of pollution and include both large and small streams. Prior to the passage of Public Law 92-500, Federal Water Quality Act, Amendments of 1972, water quality studies and management decisions generally addressed only large streams impacted by municipal and industrial wastewater discharges. The feeling was that small streams were either not polluted or, if polluted, represented "open sewers" for transporting wastes to be discharged into larger receiving streams. This philosophy precluded acknowledging consumptive uses for small streams other than as sanitary and storm sewers, but has now changed as many small streams have been assigned water use classifications and in-stream quality standards.

The importance of small streams and the impact of nonpoint source pollution on basin water quality was documented nationwide by the recently completed 208 areawide water quality studies. In the case of small streams which receive no significant point source wastewater discharge, the hydraulic and water quality regimes are dominated respectively by distributed lateral inflows of water and pollution. During dry weather periods, this nonpoint source pollution arises from groundwater discharge, septic field drainage, sanitary sewer leakage, etc., i.e., those sources which are unknown and/or unidentifiable and whose flows cannot be concentrated at a point for proper control and treatment. During storm events, stormwater runoff is the single largest contributor of nonpoint source pollution. Pollutant materials, which can occur with runoff range from organics, nutrients, and sediments to toxic materials. The type and extent of material present typically correlates highly with land use and human activities.

To properly formulate suitable water quality management policies requires the capability for predicting changes in hydrologic and water quality response due to land use changes, implementation of best management practices, etc. One of the goals of the 208 studies was to provide a functional understanding of the relationship between nonpoint source pollution and land use to establish a basis for management decision making. Many of the studies produced less than anticipated results. A criticism of the data collection conducted during these studies was offered by Sonnen (1980) who contends that:

> We still have not designed, let alone implemented, even one monitoring program based on postulated mechanisms of fundamental physics and chemistry to demonstrate in one urban area, much less all of them, whether urban runoff poses a quality problem or not, and to the degree that it does, what could be done about it.

This criticism is provocative, but underscores the obvious need to better understand the linkage mechanisms which relate land use and drainage to instream water quality responses.

The primary objective of any stream water quality model development is to produce a tool that has the capability of simulating the behavior of the hydrologic and water quality components of a stream system. Available models such as QUAL II (Roesner, 1977) incorporate the various processes and reactions that control the water quality (in this case, stream dissolved oxygen) but are formulated in terms of steady hydraulics, i.e. streamflow is allowed to vary in space but not in time. This is unrealistic for simulating the effects of stormwater runoff when streamflow varies in both time and space requiring a model that incorporates unsteady streamflow equations. Though much work has been performed to develop and test models for unsteady streamflow, and the state-of-the-art in water quality modeling is well advanced, there have been few efforts to couple the two systems of models to provide a comprehensive simulation model for predicting impacts of stormwater runoff. Based on this research effort, it is evident that major factors discouraging this development have been the large data and computer requirements for the complete unsteady flow model, a lack of understanding of the transport processes during unsteady flow, overcoming an inherent mathematical problem in coupling the two systems of models, and the absence of simple yet well founded criteria for approximating key model parameters.

The objectives of this research were to develop a simple yet accurate model for simulating the impacts of stormwater runoff and quality on small receiving streams and to investigate the transport and flood routing processes during unsteady flow.

CHAPTER 2

Water Quality Model

Pollutant Transport Processes

The processes for pollutant transport in turbulent* streamflow are convection, diffusion, and dispersion. Convection is simply the mass transport of the pollutant by the fluid. Diffusion and dispersion are basically convective transport mechanisms in that diffusion refers to the transport in a given direction at a point in the flow due to the difference between the true convection in that direction and the time average of the convection in that direction, and dispersion refers to the transport in a given direction due to the difference between the true convection in that direction and the spatial average of the convection in that direction (Holley, 1969).

Diffusion: To illustrate the diffusive transport processes, consider a mass balance of some conservative pollutant (tracer) about the three dimensional control volume as shown in Fig. 1. The velocity components in the longitudinal, transverse, and vertical (x, y and z) directions of flow are u, v, and w, respectively, and c is the concentration of the conservative pollutant. If molecular diffusion is included, the following equation for the conservation of mass of the conservative pollutant in incompressible streamflow is obtained.

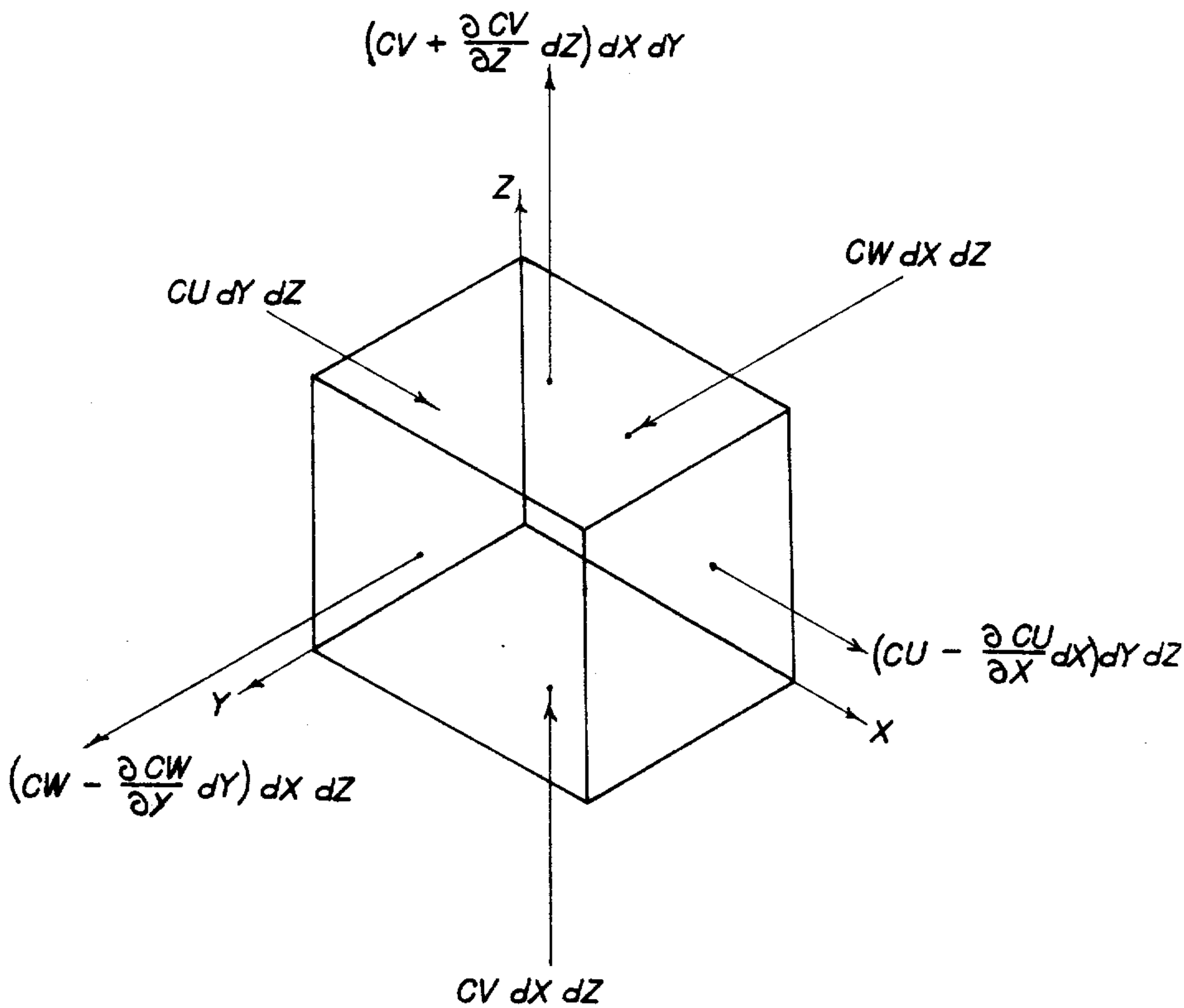
$$\frac{\partial c}{\partial t} + \frac{\partial (cu)}{\partial x} + \frac{\partial (cv)}{\partial y} + \frac{\partial (cw)}{\partial z} = \frac{\partial}{\partial x}\left(D_m \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_m \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_m \frac{\partial c}{\partial z}\right) \quad \text{(Eq. 1)}$$

where D_m is the molecular diffusion coefficient.

---

*Generally, all open channel flow is turbulent. During low flow, the effects of natural roughness, alignment, etc., create turbulence; and, during periods of stormwater runoff, channel characteristics, lateral inflow and changing stage produce turbulence.

FIGURE 1



Derivation of Conservation of Mass of a Pollutant

Eq. 1 states that the time rate of change of the concentration at a point is due to convection of the pollutant with the fluid and molecular diffusion. Molecular diffusion was included as a transport mechanism to represent part of the convection because true molecular motion is not correctly represented by the convection terms.

Even though it includes no turbulent diffusion terms, Eq. 1 is valid for turbulent flow since the velocity terms represent instantaneous point velocities that include the turbulent fluctuations.

The nature of turbulence is stochastic and it has been described by a deterministic component (the time average at a point $\overline{u}$, $\overline{v}$, $\overline{w}$) and the fluctuations around the mean (u', v', w'). This is characterized in Fig. 2 and is represented mathematically as

$$u = \overline{u} + u' \qquad \text{(Eq. 2a)}$$

$$v = \overline{v} + v' \qquad \text{(Eq. 2b)}$$

and $$w = \overline{w} + w' \qquad \text{(Eq. 2c)}$$

Since the pollutant is transported by the fluid, its concentration is also a stochastic process and is represented as

$$c = \overline{c} + c' \qquad \text{(Eq. 2d)}$$

Combining the expressions for point velocities and concentration with Eq. 1 and taking a time average of both sides of the equation results in

$$\frac{\partial \overline{c}}{\partial t} + \frac{\partial}{\partial x}(\overline{uc}) + \frac{\partial}{\partial x}(\overline{u'c'}) + \frac{\partial}{\partial y}(\overline{vc}) + \frac{\partial}{\partial y}(\overline{v'c'}) + \frac{\partial}{\partial z}(\overline{wc}) + \frac{\partial}{\partial z}(w'c') =$$

$$\frac{\partial}{\partial x}\left(D_m \frac{\partial \overline{c}}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_m \frac{\partial \overline{c}}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_m \frac{\partial \overline{c}}{\partial z}\right) \qquad \text{(Eq. 3)}$$

FIGURE 2

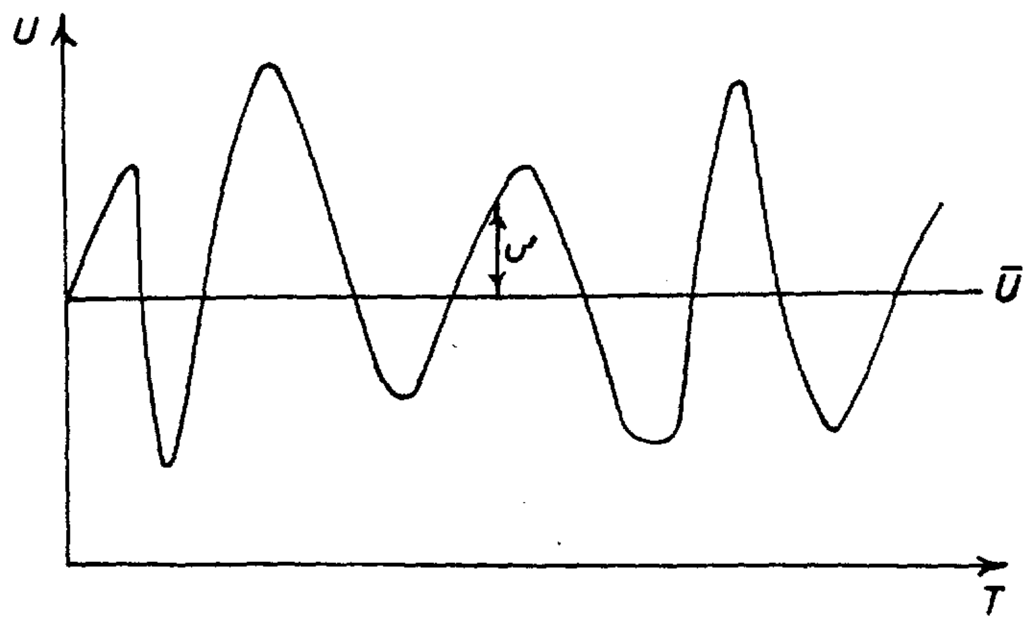


Figure 2: Trace of Point Velocity

since the time average of a fluctuating term or product of a fluctuating term is zero (e.g., $\overline{uc'} = 0$). Recognizing that the conservation of mass of the fluid itself is described by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Eq. 4)}$$

Eq. 3 reduces to

$$\frac{\partial \bar{c}}{\partial t} + u \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} + \bar{w} \frac{\partial \bar{c}}{\partial z} + \frac{\partial}{\partial x} (\overline{u'c'}) + \frac{\partial}{\partial y} (\overline{v'c'}) + \frac{\partial}{\partial z} (\overline{w'c'}) =$$

$$\frac{\partial}{\partial x} (D_m \frac{\partial \bar{c}}{\partial x}) + \frac{\partial}{\partial y} (D_m \frac{\partial \bar{c}}{\partial y}) + \frac{\partial}{\partial z} (D_m \frac{\partial \bar{c}}{\partial z}) \qquad \text{(Eq. 5)}$$

The bars indicate the time average of the quantity under the bar. The last three terms on the left hand side represent the net convection due to turbulent fluctuations.

It has been assumed and verified experimentally that the transport due to turbulent fluctuations follows a diffusive type law analogous to Fick's first law (Holley, 1969). That is, the transport is proportional to the concentration gradient, By analogy to molecular diffusion, turbulent diffusion coefficients can be introduced to represent part of the convection as follows:

$$\overline{u'c'} = - \varepsilon_x \frac{\partial \bar{c}}{\partial x} \qquad \text{(Eq. 6a)}$$

$$\overline{v'c'} = - \varepsilon_y \frac{\partial \bar{c}}{\partial y} \qquad \text{(Eq. 6b)}$$

$$\overline{w'c'} = - \varepsilon_z \frac{\partial \bar{c}}{\partial z} \qquad \text{(Eq. 6c)}$$

where ε_x, ε_y, and ε_z are the turbulent diffusion coefficients in the x, y, and z directions, respectively. Eq. 5 may now be rewritten as

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\,\frac{\partial \overline{c}}{\partial x} + \overline{v}\,\frac{\partial \overline{c}}{\partial y} + \overline{w}\,\frac{\partial \overline{c}}{\partial z} = \frac{\partial}{\partial x}\left[(D_m + \varepsilon_x)\,\frac{\partial \overline{c}}{\partial x}\right] + \left[(D_m + \varepsilon_y)\,\frac{\partial \overline{c}}{\partial y}\right] + \frac{\partial}{\partial z}\left[(D_m + \varepsilon_z)\,\frac{\partial \overline{c}}{\partial z}\right] \qquad \text{(Eq. 7)}$$

which is the pollutant mass balance equation for turbulent flow in terms of the time averaged quantities and a process called turbulent diffusion. The diffusion terms account for the difference between the true convective transport and the time averaged convective transport.

Longitudinal Dispersion: Eq. 7 describes the longitudinal, transverse, and vertical mixing of a pollutant discharged at a point in a stream as this pollutant is convected downstream with the flow. At some point, the pollutant concentration will become essentially completely mixed laterally and vertically so that there is only slight variation in the flow cross-section. When this occurs, the primary variation in the concentration is then only in one direction - the longitudinal direction of flow, and a one dimensional equation rather than the three dimensional Eq. 7 may be used to simulate the concentration profile. The desired one dimensional equation may be obtained by integrating or averaging Eq. 7 over the cross-sectional area after the following substitution:

$$\overline{u} = U + u'' \qquad \text{(Eq. 8)}$$

$$\overline{c} = C + c'' \qquad \text{(Eq. 8b)}$$

where U and C are the average values of the velocity and concentration in a cross section. Simply replacing $\overline{u}$ with U in Eq. 7 is not sufficient since this provides no means for accounting for the convection due to the difference between $\overline{u}$ and U as seen in Fig. 3.

The cross-sectional averaging of Eq. 7 gives

$$\frac{\partial C}{\partial t} + \frac{U\partial C}{\partial x} = \frac{\partial}{\partial x}\left[(D_m + \varepsilon_x)\frac{\partial C}{\partial x}\right] + \frac{\partial(\overline{\overline{-u''c''}})}{\partial x} \qquad \text{(Eq. 9)}$$

where the last term on the right hand side represents the net convection associated with the variation of $\overline{u}$ and $\overline{c}$ from their cross-sectional average values. The double bars indicate the average value for the cross section. The convection associated with u" is also proportional to the longitudinal concentration gradient so that the following replacement is possible.

$$E\frac{\partial C}{\partial x} = -\overline{\overline{u''c''}} + (D_m + \varepsilon_x)\frac{\partial C}{\partial x}; \qquad c'' < C \qquad \text{(Eq. 10)}$$

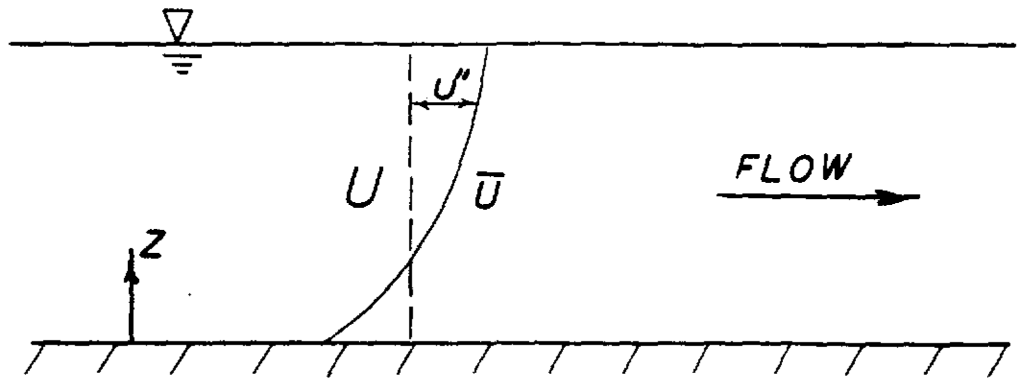
where E is called the coefficient of longitudinal dispersion.
Eq. 9 may now be written as:

$$\frac{\partial C}{\partial t} + \frac{U\partial C}{\partial x} = \frac{\partial}{\partial x}\left(E\frac{\partial C}{\partial x}\right) \qquad \text{(Eq. 11)}$$

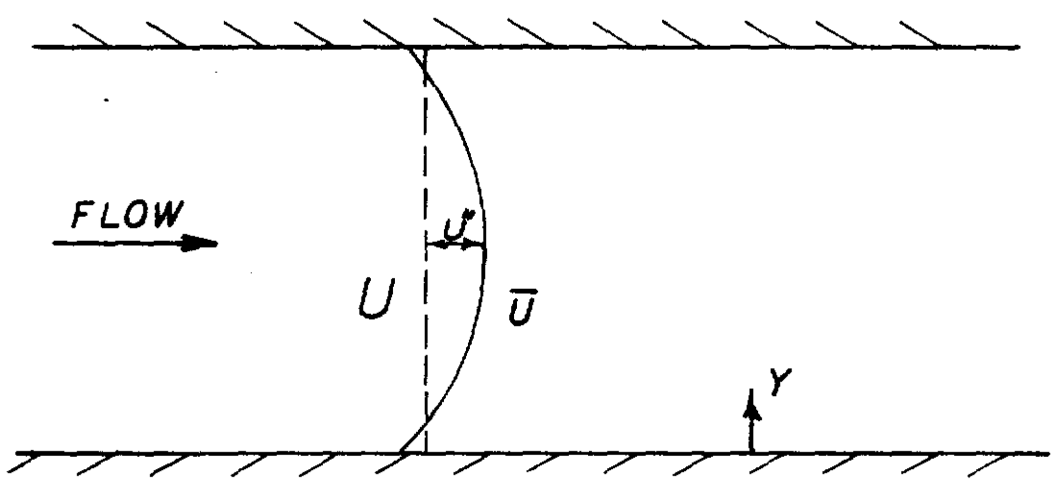
which is the one-dimensional mass balance equation for turbulent flow in terms of cross-sectional average quantities and a process called longitudinal dispersion. The dispersion term is used to account for the difference between the longitudinal convective mass transfer which is associated with the actual cross-sectional velocity distribution and that which is accounted for by the mean cross-sectional velocity distribution. When coupled with a model for unsteady streamflow, variations of velocity in time and space are incorporated.

Application of Eq. 11 for simulation purposes requires that either predictive estimates or field determinations be made of the

FIGURE 3



(a) Vertical Velocity Profile



(b) Transverse (Lateral) Velocity Profile value(s) for the longitudinal dispersion coefficient, E. For open channel flow, predictive models based on laboratory or steady flow field conditions have been published and are generally of the form:

$$E = \alpha R U_* = \alpha R \sqrt{\gamma R S_f} \qquad \text{(Eq. 12)}$$

where R is hydraulic radius, U_* is shear velocity, $\gamma = 62.4\ \mathrm{lb/ft^3}$, S_f is friction slope ($\approx S_0$, the channel slope) and α is some constant. The value of α varies with streamflow and channel characteristics. As seen from Eq. 12, as streamflow changes it is reasonable to assume that the dispersion characteristics also will change. Hence, it is anticipatory that unsteady streamflow will be characterized by unsteady longitudinal dispersion. As evidenced by the literature, to date, little research has been conducted to quantify and/or qualify the dispersion process during unsteady flow. Thus, if Eq. 11 is adopted for simulation purposes, a means of estimating the variation of longitudinal dispersion with varying streamflow is required.

Effects of Lateral Inflow

The state-of-the-art model for one-dimensional routing of pollutants discharged at a point is Eq. 12. In the case of small streams that receive no significant point source discharges, the hydraulic and water quality regimes are controlled respectively by the distributed lateral inflow of water and pollutants vectored by the lateral inflow. Thus, for the purposes of simulating water quality in small streams, a mathematical model is required that accepts both point and nonpoint sources of pollution and is descriptive of the wide range of streamflow conditions that are likely to occur, i.e., steady or unsteady and uniform or nonuniform. Additionally, the model should be easily coupled with stormwater runoff and quality models.

To derive the model governing equations, consider the mass balance of a pollutant during infinitesimal time dt in a volume A·dx bounded by two cross-sections dx apart, as shown in Fig. 4. Define C as the concentration of the pollutant in the stream in mg/l, C_L is the pollutant concentration in the lateral inflow, q is the lateral inflow in cfs/ft-length, and the other terms are as previously defined. Any difference between inflow and outflow (both convective and dispersive) and between natural assimilative addition and subtraction* must cause a change in mass of pollutant stored within the reach, dx. Thus, during dt:

$$QC + EA\,\frac{\partial c}{dx} - (QC + \frac{\partial QC}{\partial x})dx - EA\,\frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\,(EA\,\frac{\partial C}{\partial x})dx$$

$$+\ qC_L\ dx \pm \Sigma\ S\cdot Adx = \frac{\partial AC}{\partial t}\,dx \qquad \text{(Eq. 13)}$$

Multiplying the conservation of mass for open channel flow,

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \qquad \text{(Eq. 14)}$$

which is derived in the next chapter, by the pollutant concentration, C, and subtracting from Eq. 13 simplifies Eq. 13 to

$$\frac{\partial C}{\partial t} + \bar{U}\,\frac{\partial C}{\partial x} = \frac{1}{A}\,\frac{\partial}{\partial x}\,(EA\,\frac{\partial C}{\partial x}) + \frac{q}{A}\,(C_L - C) \pm \Sigma\ S \qquad \text{(Eq. 15)}$$

---

*In the case of non-conservative pollutants, reaction kinetics must be considered. That is, a term for natural sources and sinks is included in the model formulation.

FIGURE 4

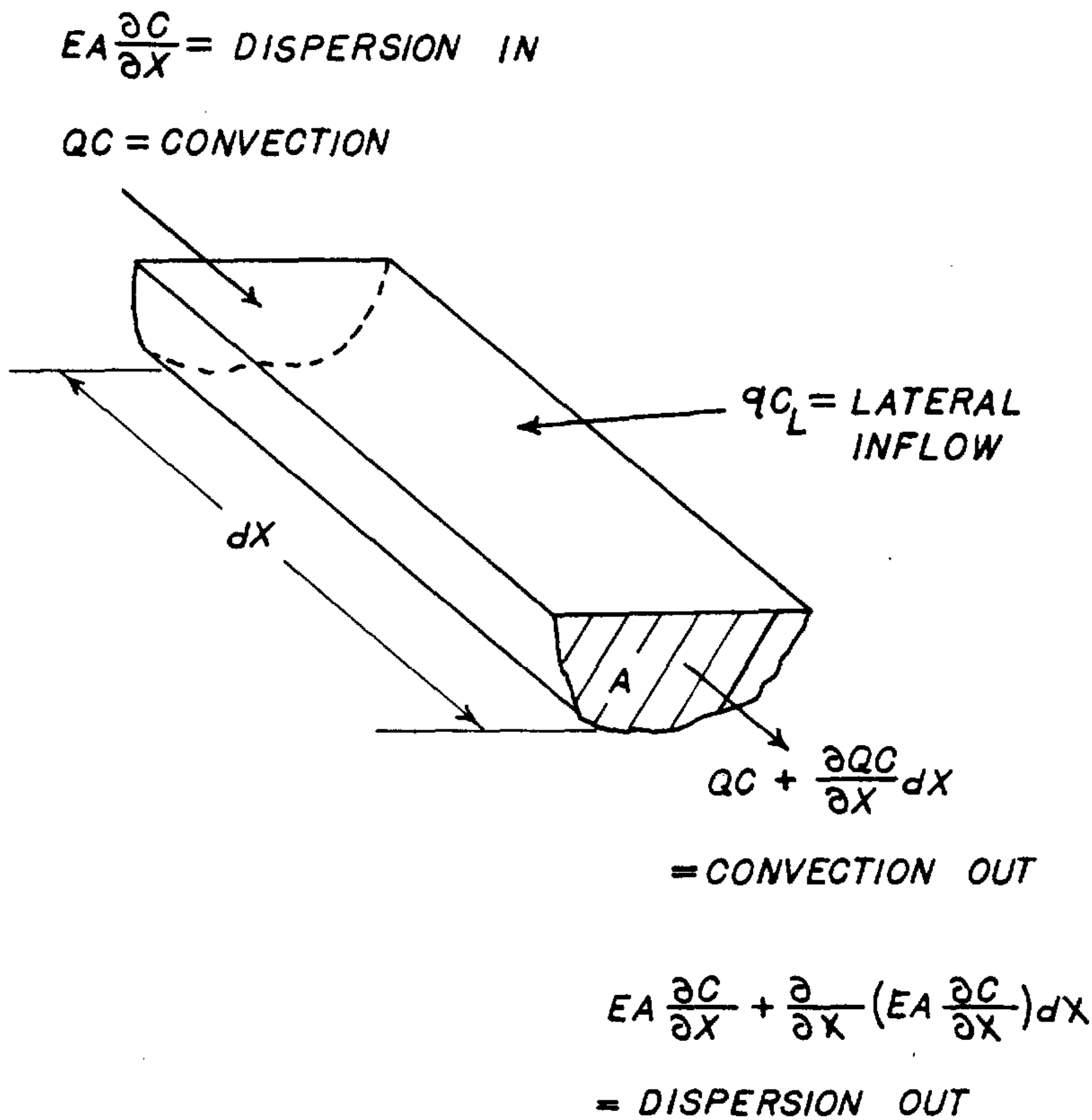


Figure 4: Elemental Channel Section with Nonpoint Source Inputs

Coupling this equation with one of the unsteady streamflow models yields a model descriptive of the water quality in a stream during periods of stormwater runoff.

This model is not without certain limitations to the range of its application and it is imperative to note these limitations. The governing equation was derived based on the assumption that the pollutant concentration is well mixed across the cross-section of flow. Thus, for the model to be a reasonable approximation of the effects of lateral inflow, its application is limited to those streams that exhibit rapid lateral mixing characteristics so that transverse concentration gradients are quickly reduced. Such streams are those with a small width to depth ratio, natural and man-made roughness that induces added turbulent mixing, and numerous meanders and bends. Streams which have these characteristics are normally the smaller streams where the primary inputs are lateral inflows and not point sources. For those cases when simulations are required for larger streams that do not mix well laterally, either a two-dimensional model is required or else the assumption is made that the significant waste inputs are point sources, such as the discharges from municipal treatment plants and the confluence with lower order streams, and lateral inflow is neglected. In this case, the convective-longitudinal dispersion model may also apply.

CHAPTER 3

Unsteady Streamflow Model

Flood Routing

Flood routing is that label applied to a set of models used to predict the temporal and spatial variations of a flood wave (runoff hydrograph) as it travels through a river reach. Routing techniques are classified into two categories, hydraulic or hydrologic. Hydraulic routing utilizes system models formulated in terms of the physics of the system, i.e., both the equations of continuity and momentum (equation of motion) are used. These equations are fully descriptive of one-dimensional, gradually varied, unsteady flow.

Parallel to the simplification of Eq. 1 to the form of Eq. 11, through time averaging of turbulence fluctuations and spatial averaging of transverse and vertical velocity profiles, it can be shown that the popularly used hydraulic routing equations represent simplifications to the Navier-Stokes equations for fluid flow. It should be noted that one-dimensional models are consistent with the quality of streamflow data normally measured.

Hydrologic routing models employ only the continuity relationship coupled usually with an empirically based relationship between storage and discharge that requires historical data for statistical parameter determination. Though not as exact as the hydraulic models, the hydrologic models are widely used due to their small data requirements and ease of solution. Further, research recently has begun to provide hydraulic based interpretations to hydrologic parameters. (Cunge, 1969 and Dooge, 1973)

The primary interests in flood routing are (1) the extent of subsidence undergone by a flood wave as it moves downstream, and (2) the speed with which the wave crest moves downstream. These two considerations relate directly to the convective and dispersive transport properties of a flood wave (unsteady flow).

Governing Hydraulic Equations

The principal laws governing the movement of water (flood waves) in a stream are the laws for the conservation of mass and momentum. These laws are expressed mathematically as partial differential equations which, in one-dimensional form,* are known as the gradually-varied unsteady open channel flow equations. The major assumptions made in their derivation are:

1. The water surface profile varies gradually, which is equivalent to stating the pressure distribution is approximately hydrostatic, or that vertical accelerations are small;

2. Resistance to flow can be closely approximated by steady flow formulas;

3. Velocity distribution across the wetted area does not affect substantially flood wave propagation;

4. Momentum carried to the streamflow from lateral inflow is negligible; and

5. The slope of the channel is small.

Conservation of Mass (Continuity): The principle of conservation of mass states that the difference in mass flux into a control volume is equal to the rate of change of mass stored within the control volume.

---

*These equations describe the change in streamflow in two dimensions: vertical and longitudinal. They are classified as one-dimensional since only one spatial variable occurs as an independent variable. That is, the vertical variation is expressed as a function of longitudinal position.

Consider the elemental fluid volume shown in Fig. 5, where Q is the streamflow rate in cubic feet per second, q is the lateral inflow rate in cfs per unit length of channel, y and A are depth and cross-sectional area of flow in feet and square feet, respectively, and x and t are the space and time coordinates in feet and seconds. Applying the continuity principle yields:

$$Q + q\,dx - (Q + \frac{\partial Q}{\partial x}\,dx) = \frac{\partial A}{\partial t}\,dx \qquad \text{(Eq. 16)}$$

which simplifies to

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \qquad \text{(Eq. 14)}$$

which is the equation of continuity for one dimensional streamflow. The first term accounts for nonuniform flow, the second for unsteady flow, and the third for lateral inflow (stormwater runoff).

<u>Conservation of Momentum</u>: This second equation is given by Newton's second law of motion which states that the rate of change of momentum is equal to the applied forces. The applied forces, as seen in Fig. 6, are (1) pressure, (2) gravity, and (3) resistive frictional forces. Considering forces in the downstream direction as positive, upon equating the upstream and downstream pressure forces, there results a pressure gradient given by:

$$-\rho g A(\partial y/\partial x)\,\Delta x$$

where ρ is the mass density of water and g is the universal gravity constant. Similarly, it can be shown that the gravity or weight force is given by:

$$\rho g A\,\Delta x \tan\alpha$$

FIGURE 5

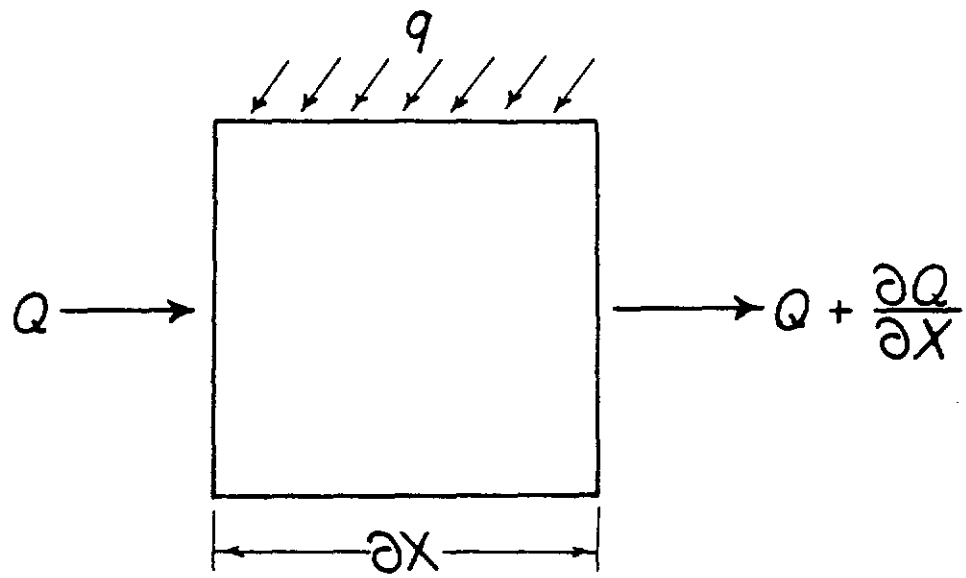


Figure 5: Elemental Stream Control Volume

FIGURE 6

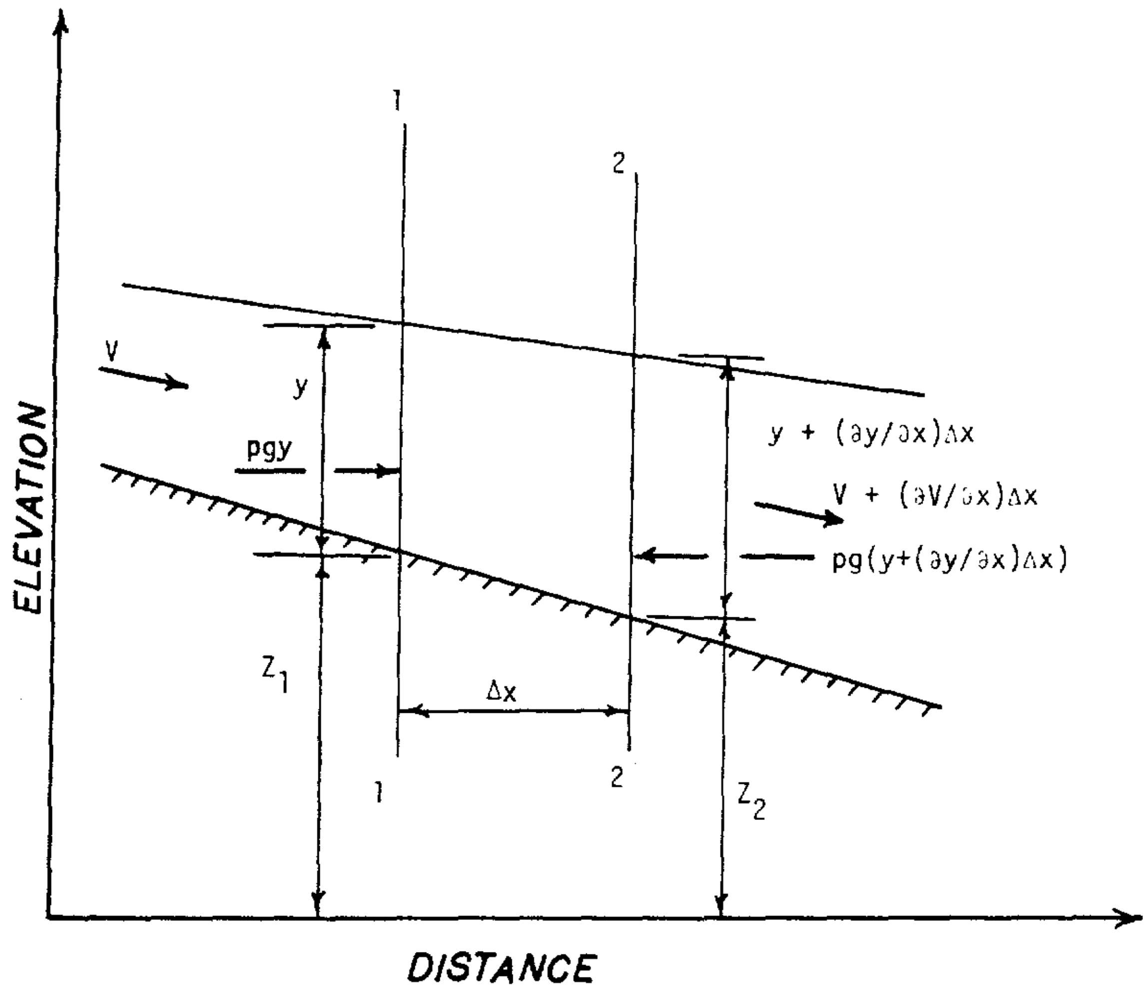


Figure 6. Definition Diagram for Open Channel Flow

where α is the angle of inclination of the bed relative to some horizontal datum. For gradually varied flow, tan α closely corresponds to the channel slope, S_0, and may be expressed as such. Finally, the friction force acting to retard the flow is expressed in terms of an average shear stress

$$\tau P\ \Delta x$$

where τ is shear stress and P is wetted perimeter. Expressing head loss in terms of friction slope, S_f, and equating head (energy) loss to the work done by the shear force yields the expression for friction force

$$\tau P\ \Delta x \quad = \quad \gamma S_f A \Delta x \qquad \text{(Eq. 17)}$$

The resultant force on the fluid volume in the direction of flow is the summation of the three applied external forces.

$$\rho g A \Delta x\ (\ -\ (\partial y/\partial x) + S_0 + S_f)$$

The rate of change of momentum for the fluid volume is equal to the summation of the applied external forces. The rate of change of momentum is described by the local (temporal) and spatial (convective) rates of momentum change.

The momentum of the fluid inside the control volume is $(\rho A \Delta x)V$. The temporal momentum change is just the time derivative.

$$\frac{\partial}{\partial t}\,(\rho A \Delta x)V = \rho \Delta x\left(\frac{A\partial V}{\partial t} + \frac{V\partial A}{\partial t}\right) \qquad \text{(Eq. 18)}$$

The spatial change in momentum is the rate of momentum change across the control surface. The momentum flux through the control surface is $\rho V^2 A$.

The spatial change is the x-derivative

$$\frac{\partial}{\partial x}(\rho V^2 A) = 2\rho AV \frac{\partial V}{\partial x} + \rho V^2 \frac{\partial V}{\partial x} \qquad \text{(Eq. 19)}$$

The total rate of momentum change is the sum of the temporal and spatial momentum changes.

$$\rho\Delta x \left(A \frac{\partial V}{\partial t} + V \frac{\partial A}{\partial t}\right) + \rho V\Delta x\left(V \frac{\partial A}{\partial x} + 2A \frac{\partial V}{\partial x}\right)$$

Substituting the following equivalence from continuity

$$A \frac{\partial V}{\partial x} + V \frac{\partial A}{\partial x} = -q - \frac{\partial A}{\partial t} \qquad \text{(Eq. 20)}$$

allows the rate of momentum change to be written as

$$\rho A\Delta x \left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x}\right) - \rho Vq\Delta x$$

Finally, in accordance with Newton's second law, the external forces are equated to the momentum change yielding the mathematical expression for the conservation of momentum.

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x} = g(S_o - S_f) - \frac{qV}{A} \qquad \text{(Eq. 21)}$$

As with Eq. 14, this equation also accounts for the effects of nonuniform and unsteady flow and lateral inflow.

Simplified Hydraulic Models

Equation 14 and 21 are accepted as fully descriptive of one-dimensional flood routing. These equations describe both the forward or downstream wave propagation characteristics as well as backward or upstream characteristics. As such, these equations are known generally as the dynamic wave equations, where dynamic waves are characterized as having both forward and backward propagation characteristics. As a flood wave passes through a channel reach, the combined effects of channel irregularity, pool and riffle patterns, natural and manmade resistance characteristics and gravity forces act to reduce the flood peak* while lengthening the time base of the hydrograph. That is, the peak of the flood hydrograph is attenuated and the hydrograph shape is dispersed in time (also in space). The dynamic wave equations account well for hydrograph attenuation. However, two drawbacks to the wholesale general use of this modeling approach are the large data requirements and the necessity for numerical integration of the model equations. Very often, based on channel geometry and alignment and flood wave characteristics, it is possible to make very valid simplifying assumptions that allow one to utilize approximations to the dynamic wave equations. When this is possible, advantages in terms of ease of solution and data requirements oftentimes are realized.

Two approximations that have found wide application in engineering practice are the diffusion and kinematic wave models. The diffusion wave model assumes that the inertia terms in the equation of motion, Eq. 21, are negligible compared with the pressure, friction, and gravity terms. Thus, the diffusion model equations are continuity, Eq. 14, and the following simplified form of the conservation of momentum.

$$\frac{\partial y}{\partial x} = S_o - S_f \qquad \text{(Eq. 22)}$$

---

*This general statement assumes minor lateral inflow.

The kinematic model assumes that the inertia and pressure terms are negligible compared with the friction and gravity terms, further reducing Eq. 22 to

$$S_o \approx S_f \tag{Eq. 23a}$$

which states simply that the equation of motion can be approximated by a uniform flow formula of the general form

$$Q = ay^b \tag{Eq.23b}$$

where a, b are parameters.

Although approximate, both the diffusion and kinematic models have been shown to be fairly good descriptions of the physical phenomenon in a variety of flood routing cases. The kinematic model has been successfully applied to overland flow (Overton and Meadows, 1976), to small streams draining upland watersheds (Brakensiek, 1967), and to the description of the travel of slow-rising flood waves. This later case occurs both in major streams, such as the Mississippi River, when long duration flood hydrographs resulting from, as an example, spring snowmelt, occurs and in small streams where the streamflow hydrograph results principally from lateral stormwater inflow along the stream reach. The subsidence (attentuation and dispersion) of the flood wave, however, is better described by the diffusion model since the kinematic model, by definition (Henderson, 1966 and Ponce, et.al., 1978), does not allow for subsidence.

Kinematic Waves

In order to better understand the physical significance of kinematic and non-kinematic waves and the subsequent physical interpretation of hydrologic routing models, it is imperative that the wave speed and crest subsidence (hydrograph dispersion) characteristics be investigated.

Wave Speed: The kinematic wave speed is determined by comparing the continuity equation with no lateral inflow.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \qquad \text{(Eq. 24)}$$

to the definition of the total derivative

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Q}{\partial t} \qquad \text{(Eq. 25)}$$

An observer moving with wave speed, c,

$$c = \frac{dx}{dt} = \frac{dQ}{dA} \qquad \text{(Eq. 26)}$$

would observe that the flow rate is constant, i.e.

$$\frac{dQ}{dt} = 0 \qquad \text{(Eq. 27)}$$

For prismatic or nearly prismatic channels, since Q is a unique function of y alone

$$\frac{dx}{dt} = \frac{1}{T} \quad \frac{dQ}{dy} \qquad \text{(Eq. 28)}$$

where T is the channel top width in feet. This relationship is analogous to that of Seddon (1900) who observed that the main body of flood waves on the Mississippi River moved at a rate given by Eq. 28.

Eq. 28 implies that equal depths on both the leading and recession limbs of a hydrograph travel at the same speed. Since greater depths move at faster rates, Eq. 28, it follows that the leading limb of the hydrograph will steepen and the recession limb will develop an elongated "tail."

Crest Subsidence: From Eqs. 23 and 27, it is found, that to an observer moving with wave speed c,

$$\frac{dy}{dt} = \frac{\partial y}{\partial x} \quad \frac{dx}{dt} + \frac{\partial y}{\partial t} = 0 \qquad \text{(Eq. 29)}$$

Manipulating this equation yields:

$$\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \quad \frac{dt}{dx} = 0 \qquad \text{(Eq. 30)}$$

which establishes that theoretically, the kinematic wave crest does not subside as the wave moves downstream. That is, to an observer, the depth at the crest does not change as the crest moves through a uniform reach with wave speed c.

These results show that a kinematic wave can alter in shape but without crest subsidence. Further, the maximum discharge rate occurs with the maximum depth of flow.

Hydraulic Geometry and Rating Curves: One important aspect of the kinematic wave model is the replacement of the momentum equation with a uniform flow formula, which is nothing more than a single valued rating between discharge and depth (area) at a point in the stream. As previously discussed, the fact that natural channels are not prismatic leads to subsidence and dispersion of a hydrograph, suggesting that the relationship between discharge and depth is not unique but varies over the hydrograph. If the dispersive characteristics are small such that a variable rating relationship does not differ significantly from the single valued rating, the conclusion can be drawn that the main body of a hydrograph does move kinematically. In which case, the kinematic model or the diffusive wave model should be sufficient for most simulation purposes. This represents an economy of data and computational requirements over the dynamic wave model.

That the flow in many streams behaves essentially kinematically is evident from the relationships between hydraulic geometry and discharge first set forth by Leopold and Maddock (1953). The fact that the channel characteristics of natural rivers seemed to constitute an interdependent system which could be described by a series of graphs having a simple geometric form suggested the term "hydraulic geometry" to Leopold and Maddock. Subsequent studies have verified and expanded upon this initial work with the result that hydraulic geometry equations may be used to provide a general estimate of channel characteristics at any location within the drainage system.

Leopold and Maddock (1953), and later researchers, based their studies on the premise that at a given time, all points on a river are experiencing the same frequency of discharge; and, that at a given channel cross section, different discharges have different frequencies. Frequency of discharge is defined as the number of times, on the average, that a given flow will be equaled or exceeded in any given year. Thus a flow with a 10% frequency will, on the average, have a 10% chance of being equaled or exceeded in any given year. Based on this premise, the researchers were able to compare different discharges occurring at the same time at different points along a river system.

As a result of their analysis of the variation of hydraulic characteristics at a particular cross section in a river, Leopold and Maddock (1953) proposed that discharge be related to other hydraulic factors in the following manner:

$$w = aQ^b \qquad \text{(Eq. 31a)}$$

$$d = cQ^f \qquad \text{(Eq. 31b)}$$

$$V = kQ^m \qquad \text{(Eq. 31c)}$$

where w is width, d is depth, V is cross-sectional mean velocity, Q is discharge and a, b, c, f, k, and m are best fit constants. It follows that since width, depth, and mean velocity are each functions of discharge, then $b + f + m = 1.0$; and $ack = 1.0$. Betson (1979) noted that a fourth relationship is sometimes also presented

$$A = nQ^P \qquad \text{(Eq. 31d)}$$

and that $f = p - b$ and $m = 1 - p$. The relationships shown in Eqs. 31 are for individual stations in that they relate channel measures to concurrent discharge.

The results from several studies are shown in Table 1. It is notable that the values do not vary widely, particularly for the depth-discharge relationship. These results enforce the use of single valued rating curves and simplified routing models.

TABLE 1. TYPICAL STATION EXPONENT TERMS FOR GEOMORPHIC EQUATIONS

LOCATION OF BASIN(S)	Exponents width b	depth f	velocity m	area p	Reference
Midwest	0.26	0.40	0.34	0.66	Langbein, et al. (1954)
Brandywine, Pa.	0.04	0.41	0.55	0.45	ditto
158 Stations in U.S.	0.12	0.45	0.43	0.57	ditto
Big Sandy River, Ky.	0.23	0.41	0.36	0.64	Stall and Yang (1976)
Cumberland Plateau, Ky.	0.245	0.487	0.268	0.732	Betson (1979)
Johnson City, Tn.	0.08	0.43	0.49	0.51	Weeter and Meadows (1978)
Theoretical	0.23	0.42	0.35	0.67	Leopold and Langbein (1962)

Non-Kinematic Waves

The result of Eq. 30 frequently does not agree with nature. Rather, due to previously mentioned factors, flow peaks are seen to subside. This fact suggests that the applicability of the kinematic model is limited, and that either the diffusive or dynamic wave model is preferable.

Differences between the two non-kinematic models can be investigated by examining the significance of each of the dynamic terms in the momentum equation, Eq. 21. First using the definition for discharge at a point in a stream

$$Q = VA \qquad \text{(Eq. 32)}$$

Eq. 21 can be rewritten as follows:

$$\frac{Q}{A^2}\frac{\partial Q}{\partial x} - \frac{Q^2}{A^3}\frac{\partial A}{\partial x} + \frac{1}{A}\frac{\partial Q}{\partial t} - \frac{Q}{A^2}\frac{\partial A}{\partial t} + g\frac{\partial y}{\partial x} = g(S_o - S_f) - \frac{Vq}{A} \qquad \text{(Eq. 33)}$$

The partial of A with respect to time is removed in terms of the spatial derivative of Q using the continuity expression, Eq. 14. After this substitution and rearranging, Eq. 33 becomes

$$\frac{2Q}{gA^2}\frac{\partial Q}{\partial x} - \frac{Q^2}{gA^3}\frac{\partial A}{\partial x} + \frac{1}{gA}\frac{\partial Q}{\partial t} + \frac{\partial y}{\partial x} = S_o - S_f \qquad \text{(Eq. 34)}$$

At any cross-section

$$\frac{dA}{dy} = T \qquad \text{(Eq. 35)}$$

and for most natural channels, the wave speed (celerity) is approximated by the kinematic wave speed. If Chezy's resistance is assumed

$$c = \frac{3}{2}\frac{Q}{A} \qquad \text{(Eq. 36)}$$

Drawing on these two relationships and the definition for Froude number

$$F^2 = \frac{V^2}{gy} \simeq \frac{Q^2 T}{gA^3} \quad \text{(Eq. 37)}$$

the various terms in Eq. 39 can be rewritten as

$$\frac{2Q}{gA^2} \frac{\partial Q}{\partial x} = 3F^2 \frac{\partial y}{\partial x} \quad \text{(Eq. 38a)}$$

$$- \frac{Q^2}{gA^3} \frac{\partial A}{\partial x} = - F^2 \frac{\partial y}{\partial x} \quad \text{(Eq. 38b)}$$

and

$$\frac{1}{gA} \frac{\partial Q}{\partial t} = - \frac{9}{4} F^2 \frac{\partial y}{\partial x} \quad \text{(Eq. 38c)}$$

Tracing back, the contributions of each term in Eq. 21 are found.

$$v \frac{\partial v}{\partial x} = (0.5\ F^2) \frac{\partial y}{\partial x} \quad \text{(Eq. 39a)}$$

and

$$\frac{\partial v}{\partial t} = (-0.75\ F^2) \frac{\partial y}{\partial x} \quad \text{(Eq. 39b)}$$

which allows Eq. 21 to be rewritten as

$$(1 - 0.25\ F^2) \frac{\partial y}{\partial x} = S_o - S_f \quad \text{(Eq. 40)}$$

An equivalent expression was found by Dooge (1973) in determining the cumulants of a linearized version of Eq. 21.

Examination of Eqs. 39 and 40 reveals that the convective and temporal acceleration terms are of nearly equal magnitude but are of opposite sign, and hence, act to nearly cancel each other. These two terms are significant for Fronde numbers greater than 0.60, where significance is taken as 10 percent of the coefficient value in Eq. 40. Evidence of Froude numbers less than 0.60 for unsteady events in small streams is documented in the literature, e.g. (Gburek and Overton, 1973). Further, using the theoretical values for hydraulic elements of Leopold and Langbein (1962) it can be shown that

$$F \propto Q^{0.14} \qquad \text{(Eq. 41)}$$

demonstrating that Froude number is largely insensitive to increasing discharge in most natural streams for flow in bank. These results suggest the diffusive wave model can be confidently applied to most flood routing events.

Wave Speed: The wave speed of a flood routing model composed of Eqs. 18 and 40 can be found by combining the two equations into a single expression that removes the term for friction slope, and the partial derivatives of A and y. Assuming Chezy's resistance, Eq. 40 is rewritten as

$$(1 - 0.25\, F^2)\, \frac{\partial y}{\partial x} = S_o - \frac{Q^2}{C^2 A^2 R} \qquad \text{(Eq. 42)}$$

where R is hydraulic radius defined as area divided by wetted perimeter and C is Chezy's resistance coefficient. Taking the partial with respect to time of this equation, the partial with respect to space of the continuity equation, and combining the two, using Eq. 35 to remove y and A, and then rearranging, the following expression is obtained.

$$\frac{\partial Q}{\partial t} + c(Q)\, \frac{\partial Q}{\partial x} = \frac{Q}{2TS_o}\, [1 - 0.25F^2]\, \frac{\partial^2 Q}{\partial x^2} + c(Q)q \qquad \text{(Eq. 43)}$$

where $c(Q)$ is wave speed or a function of discharge. This equation is of the form of the classical convective - diffusion (dispersion) relationship. See Eq. 11. Examination of this equation shows that the wave speed is given by

$$c = c(Q) = \frac{1}{T}\ \frac{dQ}{dy} \tag{Eq. 28}$$

which, of course, is the kinematic wave speed. Thus, the main body of a diffusive wave moves with kinematic speed, a result consistent with the observations of Seddon (1900) and subsequent researchers.

An interesting result is obtained if the rating between area and discharge, Eq. 31d, is utilized in a parallel derivation of Eq. 43. The following equation is obtained.

$$\frac{\partial Q}{\partial t} + c(Q)\,\frac{\partial Q}{\partial x} = \frac{Q}{2TS_o}\left[1-\left(\frac{1}{p}-1\right)^2 F^2\right]\frac{\partial^2 Q}{\partial x^2} + c(Q)q \tag{Eq. 44}$$

Comparison with Eq. 43 yields the value for $p=0.67$, which is the theoretical value determined by Leopold and Langbein (1962).

<u>Crest Subsidence</u>: From Eq. 43, it is found that to an observer moving with wave speed, c, that

$$\frac{dQ}{dt} = \frac{Q}{2TS_o}\,(1-0.25F^2)\,\frac{\partial^2 Q}{\partial x^2} + c(Q)q \tag{Eq. 45}$$

utilizing the relationship

$$\frac{dQ}{dx} \approx \frac{dQ}{dt}\cdot\frac{dt}{dx} = \frac{dQ}{dt}\ \frac{1}{c} \tag{Eq. 46}$$

Eq. 45 can be rewritten as

$$\frac{dQ}{dx} = \frac{Q}{2TS_o c} [1-0.25F^2] \frac{\partial^2 Q}{\partial x^2} + q \qquad \text{(Eq. 47)}$$

which shows that the diffusive discharge hydrograph subsides as it moves through a river reach.

Further properties of the crest region of a diffusive wave can be derived by rewriting Eq. 42 in terms of Q as

$$Q \simeq CyT \sqrt{y(S_o - (1-0.25F^2) \frac{\partial y}{\partial x}} \qquad \text{(Eq. 48)}$$

where hydraulic radius has been approximated as y, the condition for a wide channel. Taking the derivative with respect to x and equating to zero yields

$$3 \frac{\partial y}{\partial x} [S_o - (1-0.25F^2) \frac{\partial y}{\partial x}] = y (1-0.25F^2) \frac{\partial^2 y}{\partial x^2} \qquad \text{(Eq. 49)}$$

In the region of the crest, the shape of the flood wave is concave, and $\frac{\partial^2 y}{\partial x^2} < 0$, and therefore, $\frac{\partial y}{\partial x} < 0$, also. That is, the peak flow rate does not occur where depth is a maximum, but at a point in advance of the maximum depth.

Determination of an explicit statement for the rate of subsidence of the wave crest requires explicit statement of y as a function of x. This, obviously, will vary with each hydrograph. Henderson (1963) developed a first approximation to the rate of crest subsidence based on the assumption that the region of the crest is parabolic in shape. In the region of maximum depth, he found that

$$\frac{dy}{dx} = \frac{Q}{2TS_o c} \quad \frac{\partial^2 y}{\partial x^2} \qquad \text{(Eq. 50)}$$

and in the region of maximum discharge

$$\frac{dy}{dx} \simeq \left(1 - \frac{1}{9r^2}\right) \frac{Q}{2TS_o c} \quad \frac{\partial^2 y}{\partial x^2} \qquad \text{(Eq. 51)}$$

where r is defined as the ratio of channel bed slope to the slope of the leading limb of the flood wave. He reported that from the examination of a number of flood records, r was found to never be less than 10 and often was greater than 100.

The region of maximum discharge is known as the local crest and the region of maximum depth is known as the wave crest. At any moment in time, the position of the local crest represents the maximum discharge that will pass that point, and the river crest is the maximum depth along the river and occurs upstream of the local crest.

<u>Looped Rating Curves</u>: Eq. 49 clearly demonstrates that a single valued rating between discharge and depth (area) does not hold for non-kinematic waves. An approximate expression for the discharge variable (looped) rating curve is given by

$$\frac{Q}{Qn} = \sqrt{1 - \frac{(1-0.25F^2)}{S_o} \frac{\partial y}{\partial x}} \qquad \text{(Eq. 52)}$$

where Q_n is the uniform flow at a given depth. This expression is rendered more useful if the spatial derivative is replaced by some alternate quantity, deducible from data at-a-station, i.e. at one point in the stream. Making use of the kinematic relationship

$$\frac{\partial y}{\partial x} = \frac{1}{c}\frac{\partial y}{\partial t} \qquad \text{(Eq. 29)}$$

Eq. 51 can be written as

$$\frac{Q}{Q_n} = \sqrt{1 - \frac{(1-0.25F^2)}{S_o c}\frac{\partial y}{\partial t}} \qquad \text{(Eq. 53)}$$

It must be noted that Eq. 53 is not entirely correct since the kinematic relationship, Eq. 29, was included.

A typical looped rating curve is shown in Figure 7. Comparison with the associated discharge hydrograph illustrates that as a flood hydrograph passes a point, the maximum discharge is first observed, then the maximum depth, and finally a point where the flow is uniform. The uniform flow occurs when the flood wave is essentially horizontal and therefore has a slope, $\frac{\partial y}{\partial x}$, that is very small relative to the bed slope. This, obviously, will occur close to the region of maximum depth. The occurence of uniform flow is illustrated graphically as the point of intersection of the looped rating curve with the single valued uniform flow rating curve.

It should be noted that the scale of Figure 7 is exaggerated for clarity. The occurence of the three points in question likely occurs much closer together than implied by the figure.

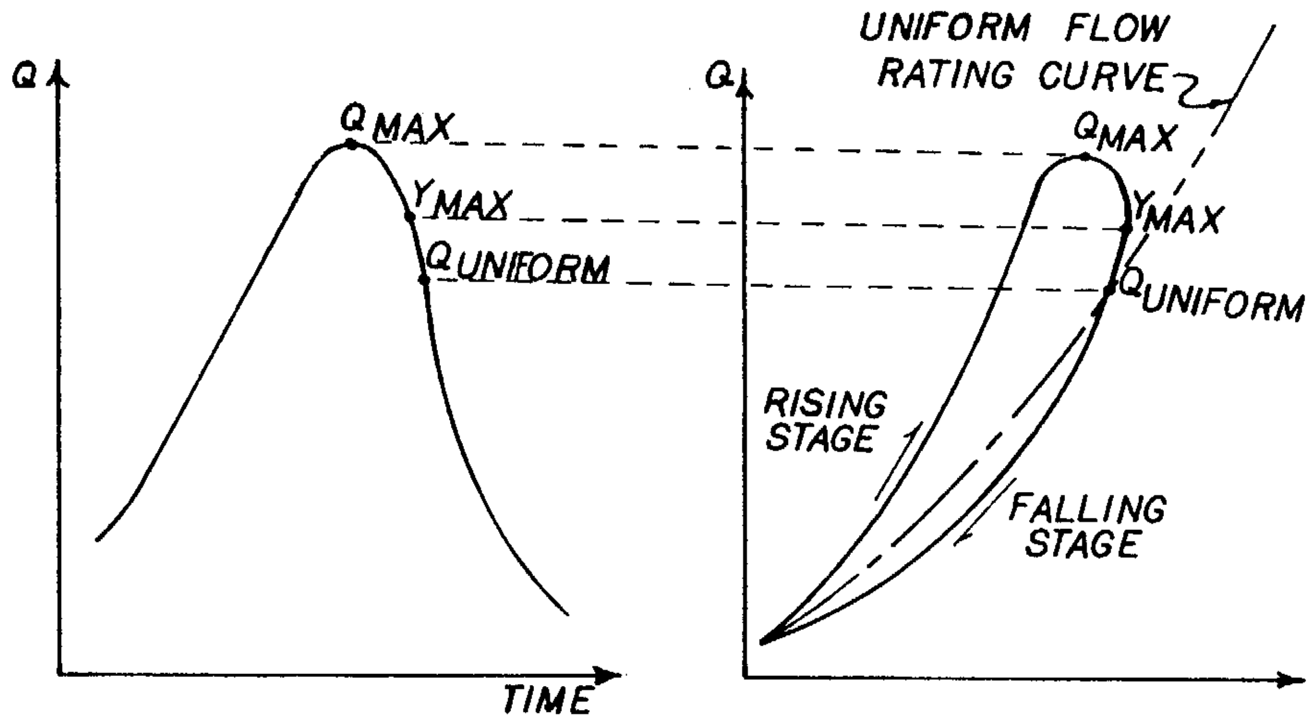


Figure 7. Loop Stage-Discharge Rating Curve and Associated Discharge Hydrograph for Attenuating Wave.

The utility of a looped rating curve instead of a single valued rating curve is determined by how wide the loop is, relative to the single valued curve. It must be noted though that most published streamflow data and associated rating curves determined from field discharge measurements generally are better approximated by a single valued relationship. Looped curves can be approximated using Eq. 52 and time series records of river stage at a station.

Hydrologic Routing Equations

The family of hydrologic routing models are formulated in terms of a continuity relationship and an assumed relationship between streamflow and reach storage. As such, hydrologic models are not strictly founded on the physics of the system and therefore have parameters that must be determined by fitting to historical data.

Muskingum River Routing: Perhaps the best known and most widely used of the hydrologic models is the Muskingum routing model. This model was originally developed for application to flood routing on the Muskingum River in Central Ohio, hence the origin of the name. The model utilizes continuity

$$I - O = \frac{dS}{dt} \qquad \text{(Eq. 54)}$$

where I is inflow to a river reach, O is outflow and S in the storage within the reach. Continuity holds that the net flow into a reach must equal the rate of change of water storage within the reach. Coupled with continuity is the storage relationship.

$$S = K\,[\theta I + (1-\theta)O] \qquad \text{(Eq. 55)}$$

where K is a characteristic storage time approximated as the travel time through a reach, and θ is a weighting coefficient. For attenuating waves, $0 \leq \theta \leq 0.5$.

Eqs. 54 and 55 are solved using a finite differencing technique. Defining $I_1 = I(t)$ and $I_2 = (t + \Delta t)$, and similarly, O_1, O_2, S_1 and S_2, the following approximation to Eq. 54 is written

$$\frac{I_1 + I_2}{2} + \frac{O_1 + O_2}{2} = \frac{S_2 - S_1}{\Delta t} \qquad \text{(Eq. 56)}$$

The inflow hydrograph provides I_1 and I_2, and O_2 is the desired quantity. O_1 is either known from initial conditions or a previous calculation. S_1 and S_2 are expressed in terms of I and O as follows

$$S_2 - S_1 = K\,[\theta(I_2 - I_1) + (1-\theta)\,(O_2 - O_1)] \qquad \text{(Eq. 57)}$$

Substituting Eq. 57 into Eq. 56 and simplifying gives

$$O_2 = C_o I_2 + C_1 I_1 + C_2 O_1 \qquad \text{(Eq. 58)}$$

where

$$C_o = \frac{-K\theta + 0.5\Delta t}{K - K\theta + 0.5\Delta t} \qquad \text{(Eq. 59a)}$$

$$C_1 = \frac{K\theta + 0.5\Delta t}{K - K\theta + 0.5\Delta t} \qquad \text{(Eq. 59b)}$$

and

$$C_2 = \frac{K - K\theta - 0.5\Delta t}{K - K\theta + 0.5\Delta t} \qquad \text{(Eq. 59c)}$$

Note that K and Δt must have the same time unit and that the coefficients sum to 1.0.

Estimation of Model Parameters: The success of using the Muskingum model is quite sensitive to the selection of model parameters. Historically, K and θ have been estimated by matching model output with actual inflow-outflow records. The obvious shortcoming to this is that the model is limited to gaged streams. Oftentimes, it is desired to route flood hydrographs along ungaged streams. To do so requires the capacity for estimating model parameters from available channel and hydrograph characteristics.

Cunge (1969), while investigating the numerical properties of Eq. 58, used Taylor series expansions to each of the terms in Eq. 58 and found that it could be represented by an equivalent equation of the convective-diffusive form

$$\frac{\partial Q}{\partial t} + \frac{\Delta x}{K}\frac{\partial Q}{\partial x} = \left[\Delta x(1-\theta)c(Q) - \frac{1}{2}\frac{\Delta x^2}{K}\right]\frac{\partial^2 Q}{\partial x^2} \qquad \text{(Eq. 60)}$$

where Δx is the reach length. Comparison of this equation with Eq. 43 shows that

$$K = \frac{\Delta x}{c(Q)} \qquad \text{(Eq. 61a)}$$

and

$$\theta = \frac{1}{2}\left[1 - \frac{Q(1-0.25F^2)}{2TS_o \Delta x c(Q)}\right] \qquad \text{(Eq. 61b)}$$

Cunge (1969) and later researchers developed similar expressions to Eqs. 61, but only Ponce and Yevjevich (1978) considered the variation of K and θ with Q. Only Dooge (1973) included the correction for dynamic effects, $(1-0.25F^2)$, in the equation for θ, but he considered c(Q) to be constant and not a function of Q. Therefore, in the opinion of this writer, Eqs. 61 are the most general expressions for K and θ to date.

Another very important feature of Eq. 60 is that it demonstrates the Muskingum routing model is diffusive, and offers the same advantages of the diffusive wave model.

Lateral Inflow: Examination of Eq. 58 reveals no term involving lateral inflow. By retracing the steps in deriving Eq. 60 and recognizing that Eqs. 43 and 60 should be equivalent, it is established that Eq. 58 should be written

$$O_2 = C_0 I_2 + C_1 I_1 + C_2 O_1 + \bar{q}\Delta x \qquad \text{(Eq. 62)}$$

where $\bar{q}$ is the average lateral inflow during a computational time interval. In summary, the steps involved in deriving Eq. 60 are as follows: (1) rewriting Eqs. 59 and 55 as

$$\frac{dS}{dt} = \frac{Q(x,t) - Q(x + \Delta x, t)}{\Delta t} \qquad \text{(Eq. 63)}$$

and

$$S = K\,[\theta Q(x,t) + (1-\theta)\,Q(x+\Delta x,t)] \qquad \text{(Eq. 69)}$$

(2) expanding the function $Q(x+\Delta x, t)$ in a Taylor series in x, truncated to second order;

(3) taking the time derivative of the modified Eq. 64, considering K and θ constants (first approximation), dropping terms with derivatives higher than order two and equating the result with the modified Eq. 63; and

(4) utilizing the continuity equation, Eq. 14, and the equation for kinematic celerity.

These steps were outlined by Kousis (1978).

Only Ponce (1979) included a term for lateral inflow. However, his term was $2q\Delta x/3$ and not $q\Delta x$ as given in Eq. 62, and therefore, did not consider the full contribution of lateral inflow.

CHAPTER 4

Evaluation of Longitudinal Dispersion

Understanding the ability of streams to disperse pollutants is essential to the effective abatement of pollution in streams. As seen in the theoretical development of the longitudinal dispersion model, the mixing effects due to turbulence and nonuniform velocity are lumped into the dispersion coefficient, E. As noted by Liu (1977) many formulas, empirical as well as theoretical, for E have been proposed in the literature, but they are, for the most part, inapplicable to natural streams. To illustrate this point, he used field data to evaluate several equations and found that they occasionally were in error by a factor of 720. The best prediction was still in error by a factor of 18. This wide range only highlights the fact that the state-of-the-art in predicting longitudinal dispersion is not that well perfected.

Perhaps one of the shortcomings to accurately predicting E is a lack of understanding of the process and the various factors involved. Typical equations for E include such terms as discharge, cross-sectional velocity, depth of flow, hydraulic radius, shear velocity and channel slope. Liu (1977) and later Beltaos (1980) concluded that the effects of channel geometry and unsteady flow on E are not well understood.

The purpose of Liu's work was to propose an improved equation for predicting longitudinal dispersion. His equation can be expressed as

$$E = \frac{0.5\ T^3\sqrt{gRS_o}}{A} \qquad \text{(Eq. 65)}$$

where the terms are as previously defined. He tested his equation on a large group of data and found that in most cases predicted E was within a factor of six of observed E.

Based on the results of Eq. 43, and results from Chapter 2 that dispersion is essentially a convective process, it was felt that the expression for dispersion of a hydrograph might also be a good predictor of the dispersion of a buoyant tracer. From Eq. 43, the expression for E is

$$E = \frac{Q}{2TS_o}\,[1-0.25F^2] \qquad \text{(Eq. 66)}$$

This equation was tested using the data from Liu (1977).

The necessary field data and the observed values of E for eleven test cases are shown in Table 2. The prediction results of Eqs 65 and 66 are shown in Table 3. It is obvious from Table 3 that Liu's equation outperformed Eq. 66 in all cases but one. However, in four of the cases, Eqs. 65 and 66 predicted within a factor of two of each other. Both equations did poorly on test number 2, a very wide channel. In wide channels, the opportunity for secondary currents, braided channels, and large transverse mixing characteristics basically violate the assumptions behind a one-dimensional model.

TABLE 2

Dispersion Field Data, From Liu (1977)

Test Number	Q,cfs	A,ft^2	S_o,ft/ft	T,ft	E,Ft^2/sec
1	54	71	.0013	52	162
2	323	391	.00036	221	98
3	35	57	.00292	56	98
4	140	261	.000324	113	289
5	240	405	.000329	114	213
6	300	149	.0013	60	451
7	3000	1091	.0004	197	120
8	900	403	.000121	81	191
9	950	407	.000121	81	191
10	1800	836	.0004	194	120
11	48	61	.0013	53	166

TABLE 3

Comparison of Observed and Predicted Dispersion

Longitudinal Dispersion, Ft^2/sec

Test Number	Observed	Liu	Eq. 66
1	162	251	405
2	98	2,629	2,021
3	98	524	107
4	289	470	1,912
5	213	274	325
6	451	244	1,908
7	120	974	18,853
8	191	93	45,556
9	191	92	48,056
10	120	1,089	11,501
11	166	301	348

In an effort to determine a range of applicability for Eq. 66, the ratio of E predicted to E observed were plotted versus channel slope and top width, as shown in Figure 8. Close examination of the data indicates that Eq. 66 predicts within a range of error less than a factor of ten for channels simultaneously satisfying $S_o \geq 0.0003$ and $T \leq 115$ feet. This result might suggest physical limits to the applicability of the diffusive model and the one-dimensional approximation, respectively.

A review of the slope of the natural channels studied by Betson (1979) and Stall and Yang (1970) revealed that only four of 46 listed streams have a slope less than 0.0003. This evidence plus the results demonstrated in Figure 8 are encouraging regarding the application of the diffusive wave model to smaller natural streams and the estimation of longitudinal dispersion with Eq. 66. It must be noted that Liu's data were for steady flow conditions. The performance of Eq. 66 and other formulas remains to be tested under unsteady flow conditions.

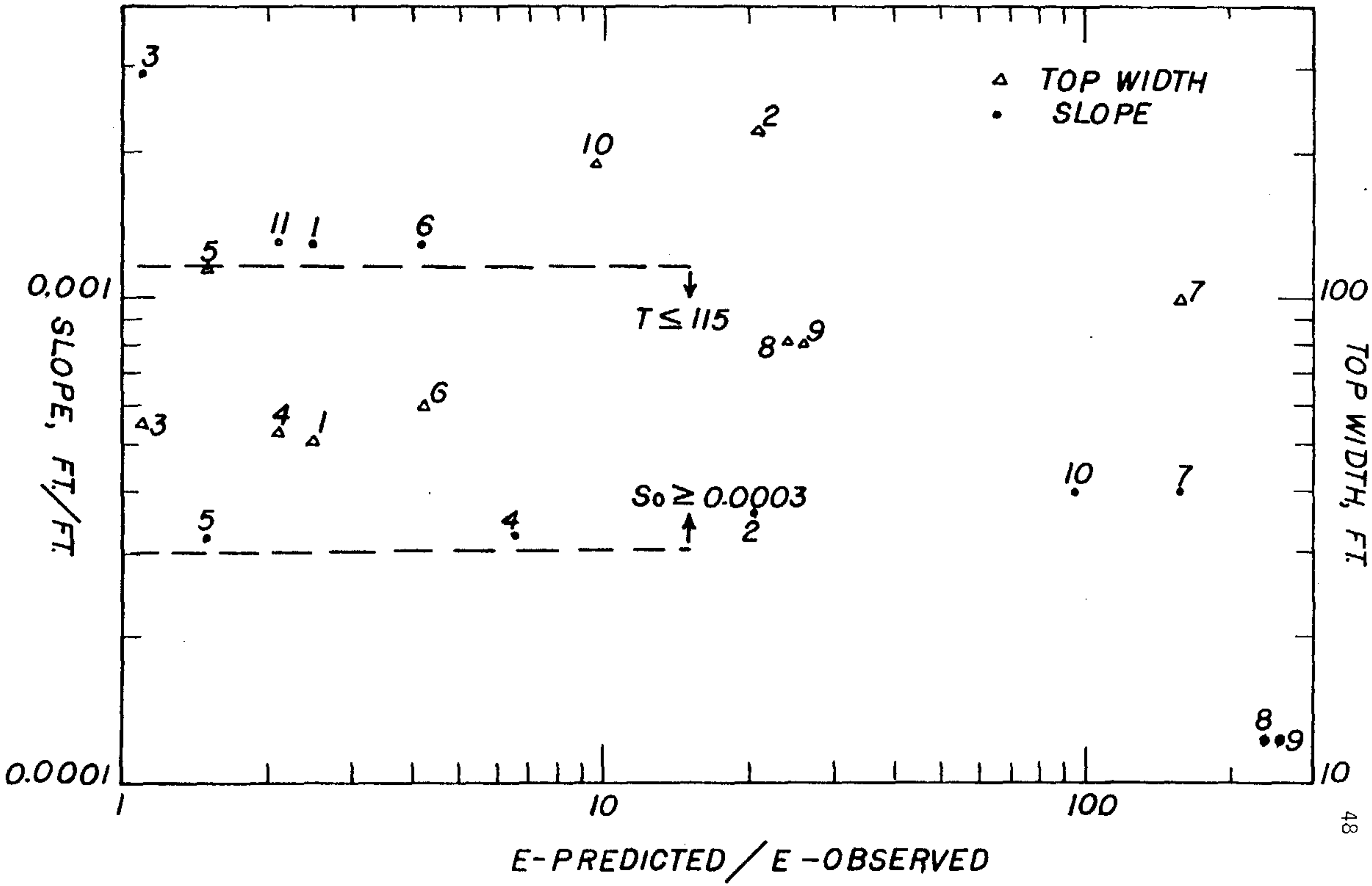


FIGURE 8

Figure 8. The Effects of Slope and Top width on Prediction of Longitudinal Dispersion.

CHAPTER 5

Conclusions and Recommendations

This research was conducted to provide a rigorous theoretical investigation of the various processes involved in pollutant transport and hydrograph dispersion during periods of stormwater runoff, and to propose a suitable model for simulating the impact of stormwater on streams. Based on the results of this research, the following conclusions are drawn:

1. The diffusive wave model is the appropriate flood routing model for small streams.
2. The diffusive wave is accurately simulated with the improved Muskingum routing model.
3. A suitable stream water quality model for simulating the impact of stormwater should couple the convective - diffusive equation, Eq. 15, with the improved version of the Muskingum routing model. The resultant model would be relatively simple and inexpensive to use, and would still be consistent with the principal transport and dispersion processes involved.
4. An expression for longitudinal dispersion during unsteady flow possibly has been developed.
5. Preliminary criteria for the applicability of the recommended model, and perhaps one dimensional models in general, have been established. It is suggested

that these models be applied only to streams with top width less than 115 feet and channel bed slopes greater than 0.0003 feet/feet (1.6 feet per mile).

The following recommendations are made:

1. Investigate the complications of coupling the water quality and streamflow equations.
2. Establish the best solution strategy for the coupled equations during unsteady streamflow.
3. Obtain data to test Eq. 66 and other equations for longitudinal dispersion during unsteady flow.

LIST OF REFERENCES

Beltaos, Spyridon, "Longitudinal Dispersion in Rivers," J. Hydr. Div., ASCE, Vol. 106, No. HY1, pp. 151-172, Jan., 1980.

Betson, Roger P., "A Geomorphic Model for Use in Streamflow Routing," Water Resources Research, Vol. 15, No. 1, pp. 95-101, Feb., 1979.

Brakensiek, D.L., "Kinematic Flood Routing," Trans: ASAE, Vol. 10, No. 3, pp. 390-343, 1967.

Cunge, J.A., "On the Subject of a F-ood Propagation Computation Method (Muskingum Method)," J. Hydr. Res., Vol. 7, No. 2, pp. 205-230, 1969.

Dooge, J.C.I., "Linear Theory of Hydrologic Systems," Agri. Res. Ser. Tech Bull. No. 1968, Oct., 1973.

Gburek, W.J., and D.G. Overton, "Subcritical Kinematic Flow in a Stable Stream," J. Hydr. Div., ASCE, Vol. 99, No. HY9, pp. 1433-1447, Sept., 1973.

Henderson, F.M., "Flood Waves in Prismatic Channels," J. Hydr. Div., ASCE, Vol. 89 No. HY4, pp. 39-67, July, 1963.

Henderson, F.M., Open Channel Flow, MacMillan, New York, NY, 1966.

Holley, E.R., "Unified View of Diffusion and Dispersion," H. Hydr. Div., ASCE, Vol. 95, No. HY2, pp. 621-HY4, March, 1969.

Kousis, A.D., "Theoretical Estimations of Flood Routing Parameters," J. Hydr. Div., ASCE, Vol. 104, No. H41, pp. 109-115, Jan. 1978.

Leopold, L.B., and T. Maddock, Jr., "The Hydraulic Geometry of Stream Channels and Some Physiographic Implications," U.S.G.S. Prof. Pap. 252, 1953.

Leopold, L.B. et al., "Flurial Processes in Geomorphology," N.H. Freeman, San Francisco, Cal., 1954.

Leopold, L.B. and W.B. Langbein, "The Concept of Endropy in Landscape Evolution," U.S. Geol. Surv. Prof. Pap. 500-A, 20 pp., 1962.

Liu, Henry, "Predicting Dispersion Coefficient of Streams," J. EED, ASCE, Vol. 103, No. EE1, pp. 59-69, Feb., 1977.

Overton, D.E., and M.E. Meadows, Stormwater Modeling, Academic Press, New York, 1976.

Ponce, V.M., Ruh-Ming Li, and D.B. Simons, "Applicability of Kinematic and Diffusion Wave Models," J. Hydr. Div., ASCE, Vol. 104, No. HY3, pp. 353-360, March, 1978.

Ponce, V.M., and Vujica Yerjerich, "Muskingum - Cunge Method with Variable Parameters," J. Hydr. Div., ASCE, Vol. 109, No. HY12, pp. 1663-1667, Dec., 1978.

Ponce, V.M., "Simplified Muskingum Routing Equation," J. Hydr. Div., ASCE, Vol. 105, No. HY1, pp. 85-91, Jan., 1979.

Roesner, Larry A., et al., "Computer Documentation for the Stream Quality Model QUAL-II," US EPA Preprint, Environmental Research Laboratory, Athens, Georgia, July, 1977.

Seddon, J.A., "River Hydraulics," Trans., ASCE, Vol. 43, p. 179, 1900.

Soonen, Michael B., "Urban Runoff Quality: Information Needs," J. Tech. Councils, ASCE, Vol. 106, No. TCI, pp. 29-40, August, 1980.

Stall, J.B., and C.T. Yang, "Hydraulic Geometry of Illinois Streams," Res. Rep. 15, Univ. of Ill. Water Resour. Res. Center, Urbana, 1970.

Weeter, D.W., and M.E. Meadows, "Water Quality Modeling for Small Streams," First Tennessee-Virginia Development District, 208 Water Quality Management Program, Johnson City, TN, February, 1978.