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Matching and Independence Complexes Related to Small Grids

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Abstract

The topology of the matching complex for the $2 \times n$ grid graph is mysterious. We describe a discrete Morse matching for a family of independence complexes $\mathrm{Ind}(\Delta_n^m)$ that include these matching complexes. Using this matching, we determine the dimensions of the chain spaces for the resulting Morse complexes and derive bounds on the location of non-trivial homology groups for certain $\mathrm{Ind}(\Delta_n^m)$. Furthermore, we determine the Euler characteristic of $\mathrm{Ind}(\Delta_n^m)$ and prove that several homology groups of $\mathrm{Ind}(\Delta_n^m)$ are non-zero.

Keywords: Grid Graphs; Independence Complexes; Recursions; Homology

1 Introduction

Consider a simple graph $G = (V(G), E(G))$. A *matching* on G is a subgraph $H = (V(G), S)$ with $S \subseteq E(G)$ and maximum vertex degree 1. We make no distinction between a matching and its edge set S. The *matching complex of* G, denoted $M(G)$, is the simplicial complex with vertex set $E(G)$ and faces given by the matchings on G.

We find it useful to reframe matchings in the language of independent sets as follows. An *independent set* in a graph G is a set $T \subseteq V(G)$ such that no two elements of T are adjacent in G. The *independence complex of* G, denoted $\mathrm{Ind}(G)$, is the abstract simplicial complex with vertex set $V(G)$ and faces given by the independent sets in G. Recall that the *line graph* of G, denoted $L(G)$, has vertex set $E(G)$ with two vertices of $L(G)$ adjacent if they are adjacent edges in G. A key observation is that $M(G) = \mathrm{Ind}(L(G))$ for a finite simple graph G.

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Recall that the *simplicial join* of two abstract simplicial complexes Δ and Γ is the abstract simplicial complex $\Delta * \Gamma = \{\sigma \cup \tau \mid \sigma \in \Delta, \tau \in \Gamma\}$. It is straightforward from this definition to verify that $\mathrm{Ind}(A \uplus B) \cong \mathrm{Ind}(A) * \mathrm{Ind}(B)$ and $M(A \uplus B) \cong M(A) * M(B)$ for graphs A and B, where $\uplus$ denotes the disjoint union.

For the path on n vertices (denoted Pa_n) and the cycle on n vertices (denoted C_n), the homotopy type of the matching and independence complexes are known [19]; see also [16, Section 11.4]. However, matching and independence complexes quickly become quite complicated, e.g. [3, 4, 5, 7, 10, 11, 17, 21, 22, 23]. Jonsson [16] provides a thorough survey regarding these and other simplicial complexes arising from graphs, including special emphasis on the matching complex for complete graphs and complete bipartite graphs.

We focus our attention in this paper on $G(2, n)$, the $2 \times n$ grid graph with $V = \{1, 2\} \times [n]$ and where two vertices (x_0, y_0) and (x_1, y_1) are adjacent when their Euclidean distance is exactly 1.

Definition 1. We define $\Gamma_n := G(2, n+2)$ and $D_n := L(\Gamma_n)$. The indexing shift is chosen so that n is the number of interior rungs on the ladder of Γ_n as well as the number of interior vertices of degree 4 in D_n. For example, Γ_3 and D_3 are isomorphic to the graphs below, respectively.

Figure 1: Γ_3 and D_3

In an unpublished manuscript [15], Jonsson establishes basic results regarding the matching complexes for Γ_n and more general grid graphs. For example, Jonsson shows that the homotopical depth of $M(\Gamma_n)$ is $\lceil 2n/3 \rceil$, which implies that this skeleton of the complex is a wedge of spheres. However, Jonsson states [15, page 3] that "it is probably very hard to determine the homotopy type of" matching complexes of grid graphs.

In [6], Bousquet-Mélou, Linusson, and Nevo introduced *matching trees* as a way to apply discrete Morse theory to study independence complexes of simple graphs. In this paper, we will use matching trees to produce a Morse matching on the face poset of $M(\Gamma_n) = \mathrm{Ind}(D_n)$. Our matching algorithm has a recursive structure that allows us to enumerate the number and dimension of cells in a cellular complex homotopy equivalent to $\mathrm{Ind}(D_n)$. We then use this recursion to determine topological properties of $\mathrm{Ind}(D_n)$.

Our techniques actually apply to independence complexes of a larger class of graphs that include the D_n graphs. Before introducing these more general graphs, we need to define a family of useful related graphs.

Definition 2. For $m \geqslant 1$ and $n \geqslant 1$, let $\widehat{Y}_n^m$ be two vertices connected by m disjoint paths each having $n+1$ edges. We ignore the degenerate cases $m = 0$ and $n = 0$. For example, $\widehat{Y}_4^3$ is isomorphic to the graph below. Also, observe that $\widehat{Y}_n^1 \cong Pa_{n+2}$ and $\widehat{Y}_n^2 \cong C_{2n+2}$.

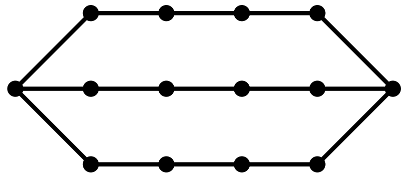

Figure 2: $\widehat{Y}_4^3$

We will impose a specific labeling on these $\widehat{Y}_n^m$ graphs for use throughout this paper: the leftmost vertex is a, the rightmost vertex is b, and the k-th vertex away from a on the j-th path is (j,k).

Definition 3. Let Δ_n^m denote the (labeled) graph $\widehat{Y}_{n+1}^m$ with n additional vertices labeled $\{1,\dots,n\}$ and edges $\{k,(j,k)\}$ and $\{k,(j,k+1)\}$ for each $j \in [m]$ and each $k \in [n]$. As an example, Δ_3^4 is depicted in the figure below. The indexing convention is chosen so that n is the number of interior vertices of degree $2m$. Therefore, we further define $\Delta_0^m := \widehat{Y}_1^m$ and $\Delta_{-1}^m := K_1$ where K_1 denotes an isolated vertex with no loops.

It is straightforward to verify that $\Delta_n^2 = D_n$, and hence Δ_n^m is a family generalizing D_n.

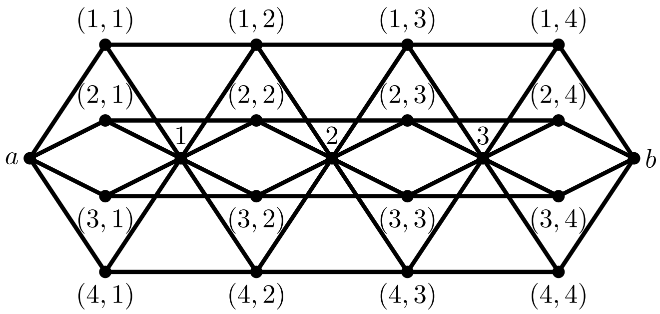

Figure 3: Labeled Δ_3^4

The article is structured as follows. In Section 2, we review discrete Morse theory and matching trees for independence complexes. In Section 3, we describe a matching tree procedure for $\mathrm{Ind}(\Delta_n^m)$ which we call the Comb Algorithm. This matching tree produces a cellular complex X_n^m that is homotopy equivalent to $\mathrm{Ind}(\Delta_n^m)$. In Section 4, we use the Comb Algorithm to establish enumerative properties regarding dimensions of the chain spaces of X_n^m. Finally, in Section 5, we apply these enumerative results to derive some homological properties of $\mathrm{Ind}(\Delta_n^m)$. We conclude with two questions for further research.

2 Discrete Morse Theory

In this section, we introduce tools from discrete Morse theory. Discrete Morse theory was introduced by R. Forman in [14] and has since become a standard tool in topological combinatorics. Similar ideas were developed around the same time by Brown [8] and Chari [9].

The main idea of simplicial discrete Morse theory is to pair cells in a simplicial complex in a manner that allows them to be cancelled via elementary collapses, which reduces the complex under consideration to a homotopy-equivalent cellular complex, typically having fewer cells. Further details regarding the following definitions and theorems can be found in [16] and [18].

Definition 4. A *partial matching* on a poset P is a subset $\mu \subseteq P \times P$ such that

- $(a, b) \in \mu$ implies b covers a, and
- each $a \in P$ belongs to at most one element in μ.

When $(a, b) \in \mu$, we write $a = d(b)$ and $b = u(a)$. A partial matching on P is *acyclic* if there does not exist a cycle of the form

$$b_1 > d(b_1) < b_2 > d(b_2) < \cdots < b_n > d(b_n) < b_1$$

with $n \geqslant 2$ and all $b_i \in P$ being distinct. Given an acyclic partial matching μ on P, we say that the unmatched elements of P are *critical*.

The following theorem asserts that an acyclic partial matching on the face poset of a polyhedral cell complex is exactly the pairing needed to produce the homotopy equivalence promised by discrete Morse theory.

Theorem 5. (Main Theorem of Discrete Morse Theory, [8, Proposition 1], [9, Proposition 3.3]). *Let Δ be a polyhedral cell complex, and let μ be an acyclic partial matching on the face poset of Δ. If c_i denotes the number of critical i-dimensional cells of Δ, then the space Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geqslant 0$, plus a single 0-dimensional cell in the case where the empty set is paired in the matching.*

We now define how to create a matching tree on a simple graph $G = (V, E)$. For $A, B \subseteq V$ such that $A \cap B = \emptyset$, let

$$\Sigma(A, B) := \{I \in \mathrm{Ind}(G) : A \subseteq I \text{ and } B \cap I = \emptyset\}.$$

For a vertex $p \in V(G)$, let $N(p)$ denote the neighbors of p in G.

A *matching tree* $\tau(G)$ for G is a directed tree constructed according to the following algorithm.

Algorithm 6 (Matching Tree Algorithm, MTA)**.** Begin by letting $\tau(G)$ be a single node labeled $\Sigma(\emptyset, \emptyset)$.

WHILE $\tau(G)$ has a leaf node $\Sigma(A, B)$ with out-degree 0 and $|\Sigma(A, B)| \geqslant 2$,
DO ONE OF THE FOLLOWING:

1. If there exists a vertex $p \in V \setminus (A \cup B)$ such that $N(p) \subseteq (A \cup B)$, create a directed edge from $\Sigma(A, B)$ to a new node labeled $\emptyset_p$. Refer to p as a *free vertex* of $\tau(G)$.

2. If there exist vertices $p \in V \setminus (A \cup B)$ and $v \in N(p)$ such that $N(p) \setminus (A \cup B) = \{v\}$, create a directed edge from $\Sigma(A, B)$ to a new node labeled $\Sigma(A \cup \{v\}, B \cup N(v))$. Refer to v as a *matching vertex* of $\tau(G)$ with respect to p.

3. Choose a vertex $v \in V \setminus (A \cup B)$ and created two directed edges from $\Sigma(A, B)$ to new nodes labeled $\Sigma(A, B \cup \{v\})$ and $\Sigma(A \cup \{v\}, B \cup N(v))$. Refer to v as a *splitting vertex* of $\tau(G)$.

The node $\Sigma(\emptyset, \emptyset)$ is called the *root* of the matching tree, while any non-root node of out-degree 1 in $\tau(G)$ is called a *matching site* of $\tau(G)$ and any non-root node of out-degree 2 is called a *splitting site* of $\tau(G)$.

Steps 2 and 3 of the above algorithm ensure that for any $\Sigma(A, B)$ in $\tau(G)$, $a \in A$ implies $N(a) \in B$. Hence, if $p \notin A \cup B$, neither p nor any of its neighbors are in A. For a vertex p satisfying the hypotheses of Step 1, we know that all neighbors of p are in B since $N(p) \subseteq (A \cup B)$. Consequently, given $\sigma \in \Sigma(A, B)$, we also have $\sigma \cup \{p\} \in \Sigma(A, B)$, which means we may pair σ and $\sigma \cup \{p\}$ in the face poset of $\mathrm{Ind}(G)$.

Similarly, for vertices p and v satisfying the hypotheses of Step 2, all of p's neighbors (except for v) are in B. Note that performing Step 3 with this choice of v implies that the branch with $\Sigma(A, B \cup \{v\})$ has p as a free vertex, so we can then perform Step 1 on that branch. This two-step sequence is an equivalent operation to performing Step 2 itself. Also, note that the empty set is always matched at the last node of the form $\Sigma(\emptyset, B)$.

A key observation from [6] is that a matching tree on G yields an acyclic partial matching on the face poset of $\mathrm{Ind}(G)$ as follows.

Theorem 7 ([6], Section 2)**.** *A matching tree $\tau(G)$ for G yields an acyclic partial matching on the face poset of* $\mathrm{Ind}(G)$ *whose critical cells are given by the non-empty sets $\Sigma(A, B)$ labeling non-root leaves of $\tau(G)$. In particular, for such a set $\Sigma(A, B)$, the set A is a critical cell in* $\mathrm{Ind}(G)$.

3 The Comb Matching Algorithm

In this section, we define a specific matching tree for the $\mathrm{Ind}(\Delta_n^m)$ complexes. First, it is appropriate to determine the homotopy type of $\mathrm{Ind}(Y_n^m)$ and $\mathrm{Ind}(\widehat{Y}_n^m)$, where Y_n^m (defined below) is a graph related to $\widehat{Y}_n^m$.

Definition 8. For $m \geqslant 1$ and $n \geqslant 1$, let Y_n^m denote the extended star graph with a central vertex of degree m and paths of n edges emanating outward. We refer to each of these paths as a *tendril*. We ignore the degenerate cases $m = 0$ and $n = 0$ as we did with $\widehat{Y}_n^m$.

As an example, Y_4^3 is isomorphic to the graph below. Observe that removing one of the vertices of degree m and all of its edges from $\widehat{Y}_n^m$ produces Y_n^m. Also, observe that $Y_n^1 \cong Pa_{n+1}$ and $Y_n^2 \cong Pa_{2n+1}$.

Since Y_n^m is a tree for $m \geqslant 1$ and $n \geqslant 0$, we know by work of Ehrenborg and Hetyei [12] that $\mathrm{Ind}(Y_n^m)$ is either contractible or homotopy equivalent to a single sphere.

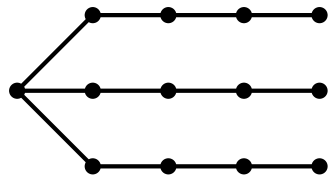

Figure 4: Y_4^3

Lemma 9. *For $m \geqslant 1$ and $n \geqslant 0$,*

$$\mathrm{Ind}(Y_n^m) \simeq \begin{cases} * & \text{if } n = 3k \\ S^{mk} & \text{if } n = 3k+1 \\ S^{m(k+1)-1} & \text{if } n = 3k+2 \end{cases}.$$

Proof. **Case 1:** $n = 3k$. We use induction on m. If $m = 1$, then $Y_n^1 \cong Pa_{3k+1}$; hence, $\mathrm{Ind}(Y_n^1)$ is contractible by [18, Prop 11.16]. Suppose the induction hypothesis holds for $\ell < m$. Select a tendril of Y_n^m and label the vertices 1 through n starting at the leaf. We consider a matching tree on $\mathrm{Ind}(Y_n^m)$. Perform Step 2 of the MTA with $p = 1$ and $v = 2$. Repeat with $p = 4$ and $v = 5$ and so on, modulo 3. Since $n = 3k$, we will eventually perform Step 2 with $p = n-2$ and $v = n-1$. The remaining subgraph of Y_n^m from which we may select vertices is isomorphic to Y_n^{m-1}. Since $\mathrm{Ind}(Y_n^{m-1})$ is contractible by assumption, $\mathrm{Ind}(Y_n^m)$ is contractible as well.

Case 2: $n = 3k+1$ or $n = 3k+2$. Let a be the vertex of degree m in Y_n^m. We again consider a matching tree on $\mathrm{Ind}(Y_n^m)$. We apply Step 3 of the MTA with $v = a$. At the $\Sigma(\{a\}, N(a))$ and $\Sigma(\emptyset, \{a\})$ nodes, the remaining subgraphs of Y_n^m from which we may select vertices are isomorphic to an m-fold disjoint union of Pa_{n-1}'s and an m-fold disjoint union of Pa_n's respectively. When $n = 3k+1$, the union of Pa_n's is contractible by [18, Prop 11.16], and each subcomplex $\mathrm{Ind}(Pa_{n-1})$ contributes $\left\lfloor \frac{n-2}{3} \right\rfloor + 1 = k$ vertices toward a single critical cell. In total, the vertex a and the vertices from each $\mathrm{Ind}(Pa_{n-1})$ factor combine to form a single critical cell of dimension mk. When $n = 3k+2$, the union of the Pa_{n-1}'s is contractible by [18, Prop 11.16], and each subcomplex $\mathrm{Ind}(Pa_n)$ contributes $\left\lfloor \frac{n-1}{3} \right\rfloor + 1 = k+1$ vertices toward a single critical cell. In total, the vertices from each $\mathrm{Ind}(Pa_n)$ factor combine to form a single critical cell of dimension $m(k+1)-1$. This gives the result. $\square$

Remark 10. An alternative method of proof relies on the fact that, when $N(v) \subseteq N(w)$, $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G \backslash w)$ per [13, Lemma 3.2] and [2, Prop 3.1]. On each tendril of an arbitrary Y_n^m, set v equal to the leaf, set w equal to the vertex two vertices in from the leaf, and then use this result to obtain that $\mathrm{Ind}(Y_n^m)$ is homotopy-equivalent to the independence complex of the disjoint union of Y_{n-3}^m and m copies of K_2. Hence, $\mathrm{Ind}(Y_n^m) \simeq \mathrm{susp}^m(\mathrm{Ind}(Y_{n-3}^m))$. It is straightforward to determine $\mathrm{Ind}(Y_0^m)$, $\mathrm{Ind}(Y_1^m)$, and $\mathrm{Ind}(Y_2^m)$ by hand, and then we obtain the result by induction.

Lemma 11. *For $m \geqslant 2$ and $n \geqslant 1$,*

$$\mathrm{Ind}(\widehat{Y}_n^m) \simeq \begin{cases} S^{mk} & \text{if } n = 3k \\ S^{mk} & \text{if } n = 3k+1 \\ S^{mk+1} \vee S^{m(k+1)-1} & \text{if } n = 3k+2 \end{cases} .$$

Proof. In $\widehat{Y}_n^m$, label the two vertices of degree m as a and b respectively. We consider a matching tree on $\mathrm{Ind}(\widehat{Y}_n^m)$. First, we apply Step 3 of the MTA with $v = b$. At the $\Sigma(\{b\}, N(b))$ and $\Sigma(\emptyset, \{b\})$ nodes, the remaining subgraphs of $\widehat{Y}_n^m$ from which we may select vertices are isomorphic to Y_{n-1}^m and Y_n^m respectively. For $n = 3k$ and $n = 3k+1$, the result is immediate from applying Lemma 9 as one of the branches will produce contractible information.

For the $n = 3k+2$ case with $m \geqslant 3$, Lemma 9 only shows that two cells of the appropriate dimension exist, but they may not necessarily form a wedge. This is sufficient for the remainder of the article, but we prove that the two cells do, in fact, form a wedge for sake of completeness. Given the matching tree defined above for $\mathrm{Ind}(\widehat{Y}_n^m)$, let τ denote the cell of dimension $mk+1$, and let σ denote the cell of dimension $m(k+1)-1$. In the style of [20, Theorem 2.2], we argue that the feasibility domain of σ (see [20, Def 2.1]) is such that τ and σ must form a wedge. Suppose there exists a generalized alternating path from σ to τ as per [20, Def 2.1]. Our choice of matching tree implies $b \in \tau$ while $b \notin \sigma$. Let x_i be the last element in the alternating path with $b \notin x_i$, so $b \in x_{i+1}$. If x_i is covered by x_{i+1}, then x_i and x_{i+1} are matched in the matching tree and so b was designated as a free vertex during some application of Step 1 of the MTA. This is not possible as b is included in $A \cup B$ in all tree nodes except for the root. If $x_i > x_{i+1}$, then $x_{i+1} \subseteq x_i$ as sets. This contradicts that $b \notin x_i$ and $b \in x_{i+1}$. Consequently, no such generalized alternating path can exist between σ and τ. The feasibility region of σ does not contain τ, and so σ and τ form a wedge per [20, Theorem 2.2]. □

We now develop a matching tree for $\mathrm{Ind}(\Delta_n^m)$.

Algorithm 12 (Comb Algorithm, CA)**.** Fix $m \geqslant 2$, $n \geqslant 1$ and use the labeling of the vertices of Δ_n^m from Section 1.

Step 1: Perform Step 3 of the MTA for $v = 1$, which produces two leaves $\Sigma(\{1\}, N(1))$ and $\Sigma(\emptyset, \{1\})$ respectively.

Step 2: For each $k \in \{2, \ldots, n\}$, inductively perform Step 3 of the MTA for $v = k$ on the leaf $\Sigma(\emptyset, \{1, 2, \ldots, k-1\})$, successively producing leaves $\Sigma(\{k\}, N(k) \cup \{1, 2, \ldots, k-1\})$ and $\Sigma(\emptyset, \{1, 2, \ldots, k\})$.

Step 3: At the $\Sigma(\{1\}, N(1))$ leaf, we may perform Step 1 of the MTA with $p = a$.

Step 4: For each $k \in \{2, \ldots, n-1\}$, consider the leaf

$$\Sigma(\{k\}, N(k) \cup \{1, 2, \ldots, k-1\}) .$$

Now, the remaining subgraph of Δ_n^m from which we may select vertices is isomorphic to the graph $Y_{k-1}^m \biguplus \Delta_{n-(k+1)}^m$. Since $\mathrm{Ind}(Y_{k-1}^m)$ is known, we can determine the number and dimension of critical cells below this node by inductively applying this algorithm to $\Delta_{n-(k+1)}^m$.

Step 5: At the $\Sigma(\{n\}, N(n) \cup \{1, 2, \ldots, n-1\})$ leaf, we may perform Step 1 of the MTA with $p = b$.

Step 6: At the $\Sigma(\emptyset, \{1, 2, \ldots, n\})$ leaf, the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to $\widehat{Y}_{n+1}^m$. Since $\mathrm{Ind}(\widehat{Y}_{n+1}^m)$ is known, we can determine the number and dimension of critical cells arising below this node.

We call this process for generating a matching tree for $\mathrm{Ind}(\Delta_n^m)$ the "Comb Algorithm" because of the visual shape of the resulting matching tree. Steps 1 and 2 produce the backbone of the "comb," while Steps 3 through 6 produce the teeth. For example, applying Steps 1 and 2 of the comb algorithm to $\mathrm{Ind}(\Delta_4^m)$ leads to the (partial) matching tree in Figure 3.

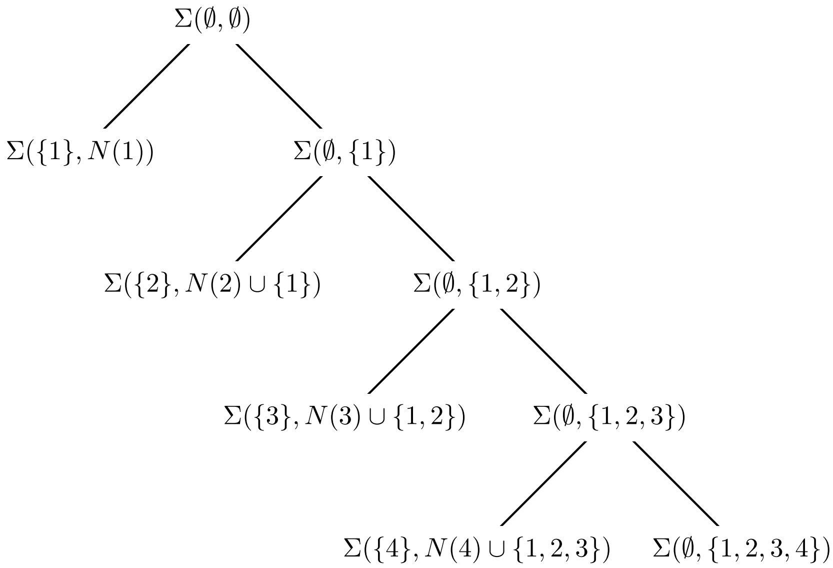

Figure 5: Example of the Comb Algorithm

4 Chain Spaces of X_n^m

Definition 13. Denote by X_n^m the cellular complex arising from the Comb Algorithm applied to $\mathrm{Ind}(\Delta_n^m)$ for $m \geqslant 2$ and $n \geqslant 1$. Since we cannot apply the Comb Algorithm to $\mathrm{Ind}(\Delta_0^m)$, we define $X_0^m := S^0$ in agreement with the fact that $\Delta_0^m \cong \widehat{Y}_1^m$. Now, for fixed $m \geqslant 2$ and arbitrary $d \geqslant 1$, let C_n^d be the number of d-dimensional cells in X_n^m.

Since the Comb Algorithm will always pair the empty set with a 0-cell, we insist that $C_n^{-1} = 0$. Also, we set C_n^0 to be one less than the number of 0-dimensional cells in X_n^m to avoiding including the extra 0-cell generated by the empty set pairing. Furthermore, the overall context implies that $C_n^d = 0$ if $d < 0$ or $n < 0$.

Proposition 14. *Suppose* $0 \leqslant n \leqslant 3$. *Then,* $C_n^d = 0$ *for all* $d \geqslant 0$ *except the following:*

When $m = 2$

	$n = 0$	1	2	3
$d = 0$	1	-	-	-
1	-	2	-	-
2	-	-	1	2

When $m \geqslant 3$

	$n = 0$	1	2	3
$d = 0$	1	-	-	-
1	-	1	-	-
2	-	-	-	1
$m-1$	-	1	-	-
m	-	-	1	1

Table 1: Initial conditions of the Comb Algorithm recursion

Proof. Fix $m \geqslant 2$. We separately consider $\mathrm{Ind}(\Delta_n^m)$ for $n \in \{0, 1, 2, 3\}$.

__Case 1__**:** Suppose $n = 0$. Then $\Delta_0^m \cong \widehat{Y}_1^m$, which implies $\mathrm{Ind}(\Delta_0^m) \simeq S^0$ by Lemma 11. Consequently, $C_0^0 = 1$ while $C_0^d = 0$ for all other d.

__Case 2__**:** Suppose $n = 1$. We apply Step 1 followed by Step 3 of the CA to $\mathrm{Ind}(\Delta_1^m)$. At the $\Sigma(\emptyset, \{1\})$ node, the remaining graph from which we may select vertices is isomorphic to $\widehat{Y}_2^m$. Thus, $\mathrm{Ind}(\Delta_1^m) \simeq \mathrm{Ind}(\widehat{Y}_2^m)$, from which we can apply Lemma 11. So, $C_1^1 = C_1^{m-1} = 1$ if $m \geqslant 3$, and $C_1^1 = 2$ if $m = 2$. In either case, $C_1^d = 0$ for all other d.

__Case 3__**:** Suppose $n = 2$. First, apply the Comb Algorithm to $\mathrm{Ind}(\Delta_2^m)$. We note that Step 5 subsumes Step 4 in this particular instance. Now, Steps 3 and 5 imply that no critical cells are picked out below the nodes $\Sigma(\{1\}, N(1))$ and $\Sigma(\{2\}, N(2) \cup \{1\})$. Consequently, Step 6 implies that $\mathrm{Ind}(\Delta_2^m) \simeq \mathrm{Ind}(\widehat{Y}_3^m) \simeq S^m$ via Lemma 11. Thus, $C_2^m = 1$ while $C_2^d = 0$ for all other d.

__Case 4__**:** Suppose $n = 3$. First, apply the Comb Algorithm to $\mathrm{Ind}(\Delta_3^m)$. Now, Steps 3 and 5 imply that no critical cells are generated below $\Sigma(\{1\}, N(1))$ and $\Sigma(\{3\}, N(3) \cup \{1, 2\})$. Per Step 4 at the $\Sigma(\{2\}, N(2) \cup \{1\})$ leaf, the remaining subgraph of Δ_3^m from which we may select vertices is isomorphic to $Y_1^m \uplus \Delta_0^m$. We already know that $\mathrm{Ind}(Y_1^m)$ and $\mathrm{Ind}(\Delta_0^m)$ are both homotopy equivalent to S^0, thus each has one critical 0-cell with a single vertex. Consequently, $\mathrm{Ind}(Y_1^m \uplus \Delta_0^m)$ must have a single critical cell consisting of two vertices, so it is homotopy equivalent to S^1. Accounting for the vertex 2 as well, we see that the Comb Algorithm generates a 2-cell below this node. At the node $\Sigma(\emptyset, \{1, 2, 3\})$ generated in Step 6, the remaining subgraph of Δ_3^m from which we may select vertices is isomorphic to $\widehat{Y}_4^m$. Since $\mathrm{Ind}(\widehat{Y}_4^m) \simeq S^m$, the Comb Algorithm generates an m-cell below this node. In total, we have $C_3^2 = C_3^m = 1$ if $m \geqslant 3$, otherwise $C_3^2 = 2$. In either case, $C_3^d = 0$ for all other d. $\square$

Theorem 15. *Using Proposition 14 as initial conditions, we have*

$$C_n^d = C_{n-3}^{d-2} + C_{n-4}^{d-(m+1)} + C_{n-3}^{d-m}\,, \tag{1}$$

when $n \geqslant 4$ for fixed $m \geqslant 2$. In this formula, a summand is zero if the subscript or superscript is negative.

Proof. Assume $n \geqslant 4$ and $d \geqslant 0$. Applying the Comb Algorithm to $\mathrm{Ind}(\Delta_n^m)$ generates factors of the form $\mathrm{Ind}(Y_{k-1}^m \biguplus \Delta_{n-(k+1)}^m)$ for $1 \leqslant k \leqslant n$, each of which are identically $\mathrm{Ind}(Y_{k-1}^m) * \mathrm{Ind}(\Delta_{n-(k+1)}^m)$. We let $C_n^d(k)$ be the number of d-dimensional cells in X_n^m produced by the Comb Algorithm below the node $\Sigma(\{k\}, N(k) \cup \{1, 2, \ldots, k-1\})$, that is, the cells referenced in Step 4 of the Comb Algorithm. We use $C_n^d(\emptyset)$ to denote the number of d-dimensional cells arising from Step 6 of the Comb Algorithm. It is clear that $C_n^d = \sum_{k=1}^n C_n^d(k) + C_n^d(\emptyset)$.

First, whenever $k-1 \equiv 0 \bmod 3$, $\mathrm{Ind}(Y_{k-1}^m)$ is contractible and, consequently, so is $\mathrm{Ind}(Y_{k-1}^m) * \mathrm{Ind}(\Delta_{n-(k+1)}^m)$. Thus, $C_n^d(k) = 0$ when $k-1 \equiv 0 \bmod 3$, and so we may assume that $k = 3\ell$ or $k = 3\ell + 2$ for some non-negative integer ℓ. Also, note that $\mathrm{Ind}(Y_{k-1}^m) * \mathrm{Ind}(\Delta_{n-(k+1)}^m)$ is contractible for $k = n$ since $\mathrm{Ind}(\Delta_{-1}^m)$ is contractible, i.e. $C_n^d(n) = 0$. These observations agree respectively with Steps 3 and 5 of the Comb Algorithm.

Next, we consider $C_n^d(2)$. Such a d-cell must correspond to the set of $d+1$ vertices consisting of the vertex 2, a single vertex contributed from $\mathrm{Ind}(Y_1^m)$, and $d-1$ vertices contributed from $\mathrm{Ind}(\Delta_{n-3}^m)$. Therefore, the d-cells coming from $\mathrm{Ind}(Y_1^m) * \mathrm{Ind}(\Delta_{n-3}^m)$ are in bijective correspondence with the $(d-2)$-cells of $\mathrm{Ind}(\Delta_{n-3}^m)$. Hence, $C_n^d(2)$ equals C_{n-3}^{d-2}. Note that if $d < 2$, then $C_n^d(2) = 0$.

Similarly, we consider $C_n^d(3)$. The $d+1$ vertices corresponding to such a d-cell consist of the vertex 3, m vertices contributed from $\mathrm{Ind}(Y_2^m)$, and $d-m$ vertices contributed from $\mathrm{Ind}(\Delta_{n-4}^m)$, provided $d-m > 0$. Therefore, the d-cells coming from $\mathrm{Ind}(Y_2^m) * \mathrm{Ind}(\Delta_{n-4}^m)$ are in bijective correspondence with the $(d-(m+1))$-cells of $\mathrm{Ind}(\Delta_{n-4}^m)$. Hence, $C_n^d(3) = C_{n-4}^{d-(m+1)}$. Note that if $d < m+1$, then $C_n^d(3) = 0$.

Lastly, we simultaneously consider $C_n^d(k)$ for $k \in \{4, 5, \ldots, n, \emptyset\}$. As before, we can disregard $k \equiv 1 \bmod 3$ and $k = n$. We first consider the case when $k = 3\ell$ for some positive integer ℓ, which implies that $\mathrm{Ind}(Y_{k-1}^m) \simeq S^{m\ell-1}$. A d-cell contributed from the factor $\mathrm{Ind}(Y_{k-1}^m) * \mathrm{Ind}(\Delta_{n-(k+1)}^m)$ consists of (i) the vertex k, (ii) $m\ell$ vertices from $\mathrm{Ind}(Y_{k-1}^m)$, and (iii) $d - m\ell$ vertices from $\mathrm{Ind}(\Delta_{n-(k+1)}^m)$, provided that $d - m\ell > 0$. We observe that a similar factor of $\mathrm{Ind}(Y_{(k-1)-3}^m \biguplus \Delta_{n-(k+1)}^m)$ is generated when the Comb Algorithm is applied to $\mathrm{Ind}(\Delta_{n-3}^m)$. It is straightforward to show that the difference in dimension of the critical cell in $\mathrm{Ind}(Y_{k-1}^m)$ from that of the critical cell in $\mathrm{Ind}(Y_{k-4}^m)$ is m. This implies that the $d - m\ell$ vertices from $\mathrm{Ind}(\Delta_{n-(k+1)}^m)$ that generate a given d-cell in the factor $\mathrm{Ind}(Y_{k-1}^m) * \mathrm{Ind}(\Delta_{n-(k+1)}^m)$ for $\mathrm{Ind}(\Delta_n^m)$ also generate a cell of dimension $d - m$ in the factor $\mathrm{Ind}(Y_{k-4}^m \biguplus \Delta_{n-(k+1)}^m)$ for $\mathrm{Ind}(\Delta_{n-3}^m)$ and vice versa. Consequently, $C_n^d(k) = C_{n-3}^{d-m}(k-3)$, provided $d \geqslant m$. A similar argument holds for the case when $k \equiv 2 \bmod 3$.

Next, we see that $C_n^d(\emptyset) = C_{n-3}^{d-m}(\emptyset)$ if $d \geqslant m$. This observation follows because the difference in dimensions of the critical cells in $\mathrm{Ind}(\widehat{Y}^m_{n+1})$ from those of the critical cells in $\mathrm{Ind}(\widehat{Y}^m_{n-2})$ is m while the number of critical cells is constant modulo 3.

Hence, we must have

$$\sum_{k=4}^{n} C^d_n(k) + C^d_n(\emptyset) = \sum_{k=1}^{n-3} C^{d-m}_{n-3}(k) + C^{d-m}_{n-3}(\emptyset) = C^{d-m}_{n-3},$$

which gives

$$C^d_n = \sum_{k=1}^{n} C^d_n(k) + C^d_n(\emptyset) = C^{d-2}_{n-3} + C^{d-(m+1)}_{n-4} + C^{d-m}_{n-3}. \qquad \square$$

Now that we have a recursive formula that gives the number of critical cells generated by the Comb Algorithm, we can manipulate this formula to get a recursive formula for the reduced Euler characteristic of both $\mathrm{Ind}(\Delta^m_n)$ and X^m_n. We denote the reduced Euler characteristic by χ^m_n. Note that since C^0_n is one less than the number of 0-dimensional cells in X^m_n, we have $\chi^m_n = \sum_{d\geqslant 0}(-1)^d C^d_n$.

Corollary 16. *Given the initial conditions from Proposition 14, when $m \geqslant 2$ and $n \geqslant 4$, we have*

$$\chi^m_n = (1 + (-1)^m)\chi^m_{n-3} + (-1)^{m+1}\chi^m_{n-4}.$$

Proof. Fix m and n as above. Using formula (1) for C^d_n, we obtain

$$\begin{aligned}
\chi^m_n &= \sum_{d\geqslant 0}(-1)^d \left(C^{d-2}_{n-3} + C^{d-(m+1)}_{n-4} + C^{d-m}_{n-3}\right) \\
&= \left(\sum_{d\geqslant 0}(-1)^d C^{d-2}_{n-3}\right) + \left(\sum_{d\geqslant 0}(-1)^d C^{d-(m+1)}_{n-4}\right) + \left(\sum_{d\geqslant 0}(-1)^d C^{d-m}_{n-3}\right) \\
&= \left(\sum_{d\geqslant 0}(-1)^{d-2} C^{d-2}_{n-3}\right) + \left((-1)^{m+1}\sum_{d\geqslant 0}(-1)^{d-(m+1)} C^{d-(m+1)}_{n-4}\right) \\
&\quad + \left((-1)^m \sum_{d\geqslant 0}(-1)^{d-m} C^{d-m}_{n-3}\right) \\
&= \left(\sum_{d\geqslant 0}(-1)^d C^d_{n-3}\right) + \left((-1)^{m+1}\sum_{d\geqslant 0}(-1)^d C^d_{n-4}\right) + \left((-1)^m \sum_{d\geqslant 0}(-1)^d C^d_{n-3}\right) \\
&= \chi^m_{n-3} + (-1)^{m+1}\chi^m_{n-4} + (-1)^m \chi^m_{n-3} \\
&= (1 + (-1)^m)\chi^m_{n-3} + (-1)^{m+1}\chi^m_{n-4}
\end{aligned}$$

The fourth equality above is obtained by reindexing and noting that $C^{d-(m+1)}_{n-4} = 0$ for $d < m$ and $C^{d-m}_{n-3} = 0$ for $d < m - 1$. $\square$

Corollary 17. *When m is even, χ_n^m satisfies the recursion $a_n = a_{n-3} - a_{n-2} - a_{n-1}$ with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = 1$, and hence has generating function $\frac{1-x}{1+x+x^2-x^3}$. (This sequence is the A078046 entry in the OEIS [1].)*

Proof. Assume that $m \geqslant 2$ is even. First, observe that $\chi_0^m = 1$, $\chi_1^m = -2$, and $\chi_2^m = 1$ by Proposition 14, so both relations have the same initial conditions. We can easily verify that $\chi_3^m = 2 = 1 - (-2) - 1 = a_0 - a_1 - a_2 = a_3$. Now, for fixed n, assume that χ_ℓ^m satisfies both relations for $\ell < n$. Since m is even, we have that $\chi_n^m = 2 \cdot \chi_{n-3}^m - \chi_{n-4}^m = \chi_{n-3}^m + (\chi_{n-3}^m - \chi_{n-4}^m)$. By assumption, $\chi_{n-1}^m = \chi_{n-4}^m - \chi_{n-3}^m - \chi_{n-2}^m$, which implies that $\chi_{n-3}^m - \chi_{n-4}^m = -\chi_{n-2}^m - \chi_{n-1}^m$. Therefore, we obtain by substitution that $\chi_n^m = \chi_{n-3}^m + (\chi_{n-3}^m - \chi_{n-4}^m) = \chi_{n-3}^m - \chi_{n-2}^m - \chi_{n-1}^m$. Consequently, χ_n^m satisfies both relations by induction. □

Remark 18. When m is odd, $\chi_n^m = \chi_{n-4}^m$. It is easy to verify that $\chi_0^m = 1$, $\chi_1^m = 0$, $\chi_2^m = -1$, and $\chi_3^m = 1$ from Proposition 14. Therefore, $\chi_n^m \in \{-1, 0, 1\}$ depending on the value of n modulo 4.

For the special case $m = 2$, the dimensions of C_n^d have an interesting enumerative interpretation. In particular, the sequence A201780 in OEIS [1] is the Riordan array of

$$\left(\frac{(1-x)^2}{1-2x}, \frac{x}{1-2x}\right)$$

which can be alternatively defined by

$$T(j,k) = 2 \cdot T(j-1,k) + T(j-1,k-1) \tag{2}$$

with initial conditions $T(0,0) = 1$, $T(1,0) = 0$, $T(2,0) = 1$, and $T(j,k) = 0$ if $k < 0$ or $j < k$.

Proposition 19. *When $m = 2$, formula* (1) *reduces to $C_n^d = 2C_{n-3}^{d-2} + C_{n-4}^{d-3}$. We can convert between our C_n^d array and the above Riordan array by the relations*

$$C_n^d = T(n-d+2, 3d-2n) \qquad \textit{and} \qquad T(j,k) = C_{2(j-2)+k}^{3(j-2)+k}.$$

Proof. The initial conditions of C_d^n are realized as entries in this Riordan array as follows. First, it is clear that we have $C_0^0 = 1 = T(2,0)$. It is straightforward to obtain the following:

$$\begin{aligned} C_1^1 &= 2 \\ &= 2(2 \cdot 0 + 1) + 0 \\ &= 2(2 \cdot T(0,1) + T(0,0)) + T(1,0) \\ &= 2 \cdot T(1,1) + T(1,0) \\ &= T(2,1) \end{aligned}$$

$$
\begin{aligned}
C_2^2 &= 1 \\
&= 2 \cdot 0 + 1 \\
&= 2 \cdot T(1,2) + T(1,1) \\
&= T(2,2)
\end{aligned}
\qquad\qquad
\begin{aligned}
C_2^3 &= 2 \\
&= 2 \cdot 1 + 0 \\
&= 2 \cdot T(2,0) + T(2,-1) \\
&= T(3,0)
\end{aligned}
$$

Now, define expressions $J_n^d := n - d + 2$ and $K_n^d := 3d - 2n$, which means that $T(J_n^d, K_n^d) = T(n - d + 2, 3d - 2n)$. It is straightforward to verify that applying the relation (2) to this entry gives

$$
\begin{aligned}
T(J_n^d, K_n^d) &= T(n - d + 2, 3d - 2n) \\
&= 2 \cdot T(n - d + 1, 3d - 2n) + T(n - d + 1, 3d - 2n - 1) \\
&= 2 \cdot T(J_{n-3}^{d-2}, K_{n-3}^{d-2}) + T(J_{n-4}^{d-3}, K_{n-4}^{d-3})
\end{aligned}
$$

Thus, the recursion applied to $T(n - d + 2, 3d - 2n)$ matches that of C_d^n. The proof of the second half of the claim is similar and omitted. $\square$

5 Dimension Range of Critical Cells and Homology

In this section, we calculate the dimension range of the critical cells generated by the Comb Algorithm as well as consider a specific homological implication for $m \geqslant 4$.

Theorem 20. *Fix $m \geqslant 2$ and $n \geqslant 0$. Define*

$$
d_n^{min} := \begin{cases} \left\lfloor \dfrac{2n+2}{3} \right\rfloor & \text{if } n = 3k \text{ or } n = 3k+1 \\ 2\left\lfloor \dfrac{n-1}{3} \right\rfloor + m & \text{if } n = 3k+2 \end{cases} .
$$

Then, $C_n^d = 0$ if $0 \leqslant d < d_n^{min}$ (excluding the base 0-cell) while $C_n^{d_n^{min}}$ is nonzero. When $m = 2$, these two formulas coincide.

Proof. By Proposition 14, the claim holds for the base cases of $n \in \{0, 1, 2, 3\}$. We proceed by strong induction. For $n \geqslant 4$, suppose that the claim is true for all $0 \leqslant i < n$. For fixed j, consider the leaf $\Sigma(\{j\}, N(j) \cup \{1, 2, \ldots, j-1\})$ from the Comb Algorithm applied to $\mathrm{Ind}(\Delta_n^m)$. Steps 3 and 4 of the Comb Algorithm allow us to assume that $j \in \{2, \ldots, n\}$. If $j < n$, then the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to $Y_{j-1}^m \uplus \Delta_{n-(j+1)}^m$, which corresponds to a subcomplex of $\mathrm{Ind}(\Delta_n^m)$ of the form $\mathrm{Ind}(Y_{j-1}^m) * \mathrm{Ind}(\Delta_{n-(j+1)}^m)$. Moreover, by Lemma 9, $\mathrm{Ind}(Y_{j-1}^m)$ is contractible when $j \equiv 1 \bmod 3$. Since joins respect homotopy equivalences, $\mathrm{Ind}(Y_{j-1}^m) * \mathrm{Ind}(\Delta_{n-(j+1)}^m)$ is contractible when $j \equiv 1 \bmod 3$, thus we may further assume that j is of the form 3ℓ or $3\ell + 2$ for some non-negative integer ℓ. Observe that when $j = 3\ell$ or $j = 3\ell + 2$, $\mathrm{Ind}(Y_{j-1}^m)$ is homotopy equivalent to $S^{m\ell-1}$ or $S^{m\ell}$ respectively. We let δ_j denote the dimension of this sphere.

Still considering $j \in \{2, \ldots, n-1\}$, we have $n-(j+1) < n$, and so the induction hypothesis holds for $\mathrm{Ind}(\Delta^m_{n-(j+1)})$. We now count the minimum number of vertices in a critical cell in the matching tree below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \ldots, j-1\})$. We have the vertex j itself, $\delta_j + 1$ vertices from $\mathrm{Ind}(Y^m_{j-1})$, and $d^{min}_{n-(j+1)} + 1$ vertices from $X^m_{n-(j+1)}$. This total number of vertices corresponds to a cell of dimension $\delta_j + d^{min}_{n-(j+1)} + 2$ below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \ldots, j-1\})$. In the special case $j = n$, the remaining subgraph of Δ^m_n from which we may query vertices is isomorphic to $\widehat{Y}^m_{n+1}$, so we can also expect the subcomplex $\mathrm{Ind}(\widehat{Y}^m_{n+1})$ to contribute one or two cells of the appropriate dimension per Lemma 11.

Next, we explicitly calculate d^{min}_n for each value of n mod 3.

Case 1: Suppose that $n = 3k$. The proposed d^{min}_n is $\left\lfloor \frac{2n+2}{3} \right\rfloor = \left\lfloor \frac{6k+2}{3} \right\rfloor = 2k$.

Subcase 1a: If $j = 3\ell$, then we have $n-(j+1) = 3(k-\ell-1)+2$, which implies $d^{min}_{n-(j+1)} = 2(k-\ell-1)+m$. Thus,

$$\begin{aligned}\delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell - 1) + 2(k-\ell-1) + m + 2 \\ &= 2k + (m-2)\ell + (m-1).\end{aligned}$$

Subcase 1b: If $j = 3\ell+2$, then we have $n-(j+1) = 3(k-\ell-1)$, which implies $d^{min}_{n-(j+1)} = 2(k-\ell-1)$. Thus,

$$\begin{aligned}\delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell) + 2(k-\ell-1) + 2 \\ &= 2k + (m-2)\ell.\end{aligned}$$

By Lemma 11, the cell contributed by the subcomplex $\mathrm{Ind}(\widehat{Y}^m_{n+1})$ is of dimension mk. Observe that each of these cellular dimensions is no less than $2k$ since $m \geqslant 2$. Hence, none of the cells in X^m_n are of dimension smaller than $\left\lfloor \frac{2n+2}{3} \right\rfloor$. Furthermore, when $j = 2$, we have that the factor $\mathrm{Ind}(Y^m_1) * \mathrm{Ind}(\Delta^m_{n-3})$ produces at least one cell of dimension exactly $2k$, which implies that $C^{d^{min}_n}_n$ is non-zero.

Case 2: Suppose that $n = 3k+1$. The proposed d^{min}_n is $\left\lfloor \frac{2n+2}{3} \right\rfloor = \left\lfloor \frac{6k+4}{3} \right\rfloor = 2k+1$.

Subcase 2a: If $j = 3\ell$, then we have $n-(j+1) = 3(k-\ell)$, which implies that $d^{min}_{n-(j+1)} = 2(k-\ell)$. Thus,

$$\begin{aligned}\delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell - 1) + 2(k-\ell) + 2 \\ &= 2k + (m-2)\ell + 1.\end{aligned}$$

Subcase 2b: If $j = 3\ell+2$, then we have $n-(j+1) = 3(k-\ell-1)+1$, which implies that $d^{min}_{n-(j+1)} = 2(k-\ell)-1$. Thus,

$$\begin{aligned}\delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell) + 2(k-\ell) - 1 + 2 \\ &= 2k + (m-2)\ell + 1.\end{aligned}$$

By Lemma 11, the cells contributed by the subcomplex $\mathrm{Ind}(\widehat{Y}^m_{n+1})$ are of dimensions $mk+1$ and $m(k+1)-1$. Observe that each of these cellular dimensions is no less than $2k+1$ since $m \geqslant 2$. Therefore, none of the cells in X^m_n are of dimension smaller than $\left\lfloor \frac{2n+2}{3} \right\rfloor$. Furthermore, when $j=2$, we have that the factor $\mathrm{Ind}(Y^m_1) * \mathrm{Ind}(\Delta^m_{n-3})$ produces at least one cell of dimension exactly $2k+1$, which implies that $C_n^{d_n^{min}}$ is non-zero.

<u>Case 3</u>: Suppose that $n = 3k+2$. The proposed d_n^{min} is $2\left\lfloor \frac{n-1}{3} \right\rfloor + m = 2k+m$.

Subcase 3a: If $j = 3\ell$, then we have $n-(j+1) = 3(k-\ell)+1$, which implies $d^{min}_{n-(j+1)} = 2(k-\ell)+1$. Thus,

$$\begin{aligned} \delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell - 1) + 2(k-\ell) + 1 + 2 \\ &= 2k + (m-2)\ell + 2. \end{aligned}$$

Because $j = 3\ell$ and $j \geqslant 2$, we have $\ell \geqslant 1$, which implies that

$$2k + (m-2)\ell + 2 \geqslant 2k + (m-2) + 2 = 2k + m.$$

Subcase 3b: If $j = 3\ell+2$, then we have $n-(j+1) = 3(k-\ell-1)+2$, which implies $d^{min}_{n-(j+1)} = 2(k-\ell-1)+m$. Thus,

$$\begin{aligned} \delta_j + d^{min}_{n-(j+1)} + 2 &= (m\ell) + 2(k-\ell-1) + m + 2 \\ &= 2k + (m-2)\ell + m. \end{aligned}$$

By Lemma 11, the subcomplex $\mathrm{Ind}(\widehat{Y}^m_{n+1})$ produces a cell of dimension $m(k+1)$. Observe that each of these cellular dimensions is at least $2k+m$ since $m \geqslant 2$. Therefore, none of the cells in X^m_n are of dimension smaller than $2\left\lfloor \frac{n-1}{3} \right\rfloor + m$. Furthermore, when $j=2$, we have that the factor $\mathrm{Ind}(Y^m_1) * \mathrm{Ind}(\Delta^m_{n-3})$ produces at least one cell of dimension exactly $2k+m$, which implies that $C_n^{d_n^{min}}$ is non-zero.

For all three cases, d_n^{min} is a sharp lower bound on the dimension of cells generated by the Comb Algorithm applied to $\mathrm{Ind}(D^m_n)$. As a final observation, when $m=2$ and $n = 3k+2$, we have $\lfloor \frac{2n+2}{3} \rfloor = 2k+2 = 2\lfloor \frac{n-1}{3} \rfloor + m$. □

Remark 21. Theorem 20 shows that X^m_n is at least d_n^{min}-connected. After a suitable adjustment of notation, this agrees with results of Jonsson [15, Proposition 2.7] regarding the connectivity of $\mathrm{Ind}(\Delta^2_n)$.

Theorem 22. *Fix $m \geqslant 2$ and $n \geqslant 0$. Define*

$$d_n^{max} := \begin{cases} \left\lfloor \dfrac{3n+2}{4} \right\rfloor & \text{if } m = 2 \\ n + 1 + (m-3)\left\lfloor \dfrac{n+2}{3} \right\rfloor & \text{otherwise} \end{cases}.$$

Then, $C^d_n = 0$ if $d > d_n^{max}$, and $C_n^{d_n^{max}}$ is nonzero.

Proof. By Proposition 14, the claim holds for the bases cases of $n \in \{0,1,2,3\}$. We proceed by strong induction. For $n \geqslant 4$, suppose the claim is true for all $0 \leqslant i < n$. For fixed j, we will be considering the maximum dimension of a cell produced below the node $\Sigma(\{j\}, N(j) \cup \{1,2,\ldots,j-1\})$ from the Comb Algorithm applied to $\mathrm{Ind}(\Delta_n^m)$. As before, we may assume $j \in \{2,\ldots,n\}$. If $j = n$, the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to $\widehat{Y}_{n+1}^m$. If $j < n$, then the remaining subgraph is $Y_{j-1}^m \uplus \Delta_{n-(j+1)}^m$, which corresponds to a subcomplex of $\mathrm{Ind}(\Delta_n^m)$ of the form $\mathrm{Ind}(Y_{j-1}^m) * \mathrm{Ind}(\Delta_{n-(j+1)}^m)$. We will again use the notation δ_j from the proof of Theorem 20.

Again considering $j \in \{2,\ldots,n-1\}$, we have $n-(j+1) < n$, and so the induction hypothesis holds for $\mathrm{Ind}(\Delta_{n-(j+1)}^m)$. We now count the maximum number of vertices in a critical cell in the matching tree below the node $\Sigma(\{j\}, N(j) \cup \{1,2,\ldots,j-1\})$. We have the vertex j itself, $\delta_j + 1$ vertices from $\mathrm{Ind}(Y_{j-1}^m)$, and $d_{n-(j+1)}^{max} + 1$ vertices from $X_{n-(j+1)}^m$. This total number of vertices corresponds to a cell of dimension $\delta_j + d_{n-(j+1)}^{max} + 2$ below the node $\Sigma(\{j\}, N(j) \cup \{1,2,\ldots,j-1\})$. As before, in the special case $j = n$, we expect the subcomplex corresponding to $\mathrm{Ind}(\widehat{Y}_{n+1}^m)$ to contribute one or two cells of the appropriate dimension per Lemma 11.

Next, we explicitly calculate d_n^{max} for the two cases of m.

<u>Case 1</u>: Suppose that $m = 2$. The proposed d_n^{max} is $\left\lfloor \frac{3n+2}{4} \right\rfloor$.

Subcase 1a: If $j = 3\ell$, then we have

$$\begin{aligned}
\delta_j + d_{n-(j+1)}^{max} + 2 &= (2\ell - 1) + \left\lfloor \frac{3(n-(3\ell+1))+2}{4} \right\rfloor + 2 \\
&= \left\lfloor \frac{3n - \ell + 3}{4} \right\rfloor \quad \leqslant \quad d_n^{max}
\end{aligned}$$

since $\ell \geqslant 1$ as a consequence of $j \geqslant 2$.

Subcase 1b: If $j = 3\ell + 2$, then we have

$$\begin{aligned}
\delta_j + d_{n-(j+1)}^{max} + 2 &= 2\ell + \left\lfloor \frac{3(n-(3\ell+3)+2}{4} \right\rfloor + 2 \\
&= \left\lfloor \frac{3n - \ell + 1}{4} \right\rfloor \quad \leqslant \quad d_n^{max}.
\end{aligned}$$

We now consider the contribution of the subcomplex corresponding to $\mathrm{Ind}(\widehat{Y}_{n+1}^m)$. When $n = 3k$, $d_n^{max} = \left\lfloor \frac{9k+2}{4} \right\rfloor \geqslant 2k$ while the $\widehat{Y}_{n+1}^m$ contribution has dimension $2k$. When $n = 3k+1$, $d_n^{max} = \left\lfloor \frac{9k+5}{4} \right\rfloor \geqslant 2k+1$ while the $\widehat{Y}_{n+1}^m$ contributions have dimension $2k+1$. When $n = 3k+2$, $d_n^{max} = \left\lfloor \frac{9k+8}{4} \right\rfloor \geqslant 2k+2$ while the $\widehat{Y}_{n+1}^m$ contribution has dimension $2k+2$. So, all things considered, no cells of X_n^m exceed the proposed maximum dimension. Furthermore, when $j = 3$, the $\mathrm{Ind}(Y_2^m) * \mathrm{Ind}(\Delta_{n-4}^m)$ factor produces at least one cell of dimension exactly d_n^{max}, which implies that $C_n^{d_n^{max}}$ is non-zero.

<u>Case 2</u>: Suppose that $m \geqslant 3$. The proposed d_n^{max} is $n + 1 + (m-3)\left\lfloor \frac{n+2}{3} \right\rfloor$.

Subcase 2a: If $j = 3\ell$, then we have $\delta_j + d_{n-(j+1)}^{max} + 2$ is equal to the following

$$
\begin{aligned}
&(m\ell - 1) + \left(n - (3\ell+1) + 1 + (m-3)\left\lfloor \frac{n-(3\ell+1)+2}{3} \right\rfloor\right) + 2 \\
&= n + 1 + (m-3)\left(\left\lfloor \frac{n-3\ell+1}{3} \right\rfloor + \ell\right) \\
&= n + 1 + (m-3)\left\lfloor \frac{n+1}{3} \right\rfloor \quad \leqslant \quad d_n^{max}.
\end{aligned}
$$

Subcase 2b: If $j = 3\ell + 2$, then we have $\delta_j + d_{n-(j+1)}^{max} + 2$ is equal to the following

$$
\begin{aligned}
&(m\ell) + \left(n - (3\ell+3) + 1 + (m-3)\left\lfloor \frac{n-(3\ell+3)+2}{3} \right\rfloor\right) + 2 \\
&= n + (m-3)\left(\left\lfloor \frac{n-3\ell-1}{3} \right\rfloor + \ell\right) \\
&= n + (m-3)\left\lfloor \frac{n-1}{3} \right\rfloor \quad \leqslant \quad d_n^{max}.
\end{aligned}
$$

We now consider the contribution of the subcomplex corresponding to $\mathrm{Ind}(\widehat{Y}_{n+1}^m)$. When $n = 3k$, $d_n^{max} = 3k + 1 + (m-3)\left\lfloor \frac{3k+2}{3} \right\rfloor = mk + 1$ while the $\widehat{Y}_{n+1}^m$ contribution has dimension mk. When $n = 3k+1$, $d_n^{max} = (3k+1) + 1 + (m-3)\left\lfloor \frac{(3k+1)+2}{3} \right\rfloor = m(k+1) - 1$ while the $\widehat{Y}_{n+1}^m$ contributions have dimension $mk+1$ and $m(k+1)-1$ respectively. When $n = 3k+2$, $d_n^{max} = (3k+2)+1+(m-3)\left\lfloor \frac{(3k+2)+2}{3} \right\rfloor = mk+m$ while the $\widehat{Y}_{n+1}^m$ contribution has dimension $mk + m$. So, all things considered, no cells of X_n^m exceed the proposed maximum dimension. Moreover, when $n = 3k+1$ or $n = 3k+2$, the contributions from the $\widehat{Y}_{n+1}^m$ factor imply that $C_n^{d_n^{max}}$ is non-zero. When $n = 3k$ and $j = 3$, we have that the factor $\mathrm{Ind}(Y_2^m) * \mathrm{Ind}(\Delta_{n-4}^m)$ produces at least one cell of dimension exactly d_n^{max}, which again implies that $C_n^{d_n^{max}}$ is non-zero. □

Using Theorems 14 and 15, we can create data tables containing dimensions of the integral cellular chain spaces of X_n^m for reasonable values of n and m. For $m \geqslant 4$, it is interesting that gaps appear in the dimensions of the chain spaces for low values of d relative to n. For example, the Comb Algorithm eliminates all cells of dimension $\left\lfloor \frac{2n+2}{3} \right\rfloor + 1$ through $\left\lfloor \frac{2n+2}{3} \right\rfloor + (m-3)$ when $n = 3k$ or $n = 3k+1$. Furthermore, we can explicitly determine the lowest non-vanishing homology for $n = 3k$ and $n = 3k+1$ when $m \geqslant 4$; see Jonsson [15, Lemma 2.3 and Proposition 2.7] for analogous results when $m = 2$.

Theorem 23. *Suppose that $m \geqslant 4$, and let $\nu_n = \left\lfloor \frac{2n+2}{3} \right\rfloor$. If $n = 3k$ or $n = 3k+1$, then $\nu_n = d_n^{min}$ from Theorem 20, and $C_n^{\nu_n} = 1$ while $C_n^{\nu_n+1} = 0$. This implies that $H_{\nu_n}(X_n^m; \mathbb{Z}) \cong \mathbb{Z}$. If $n = 3k+2$, then $C_n^{\nu_n} = 0$, which implies that $H_{\nu_n}(X_n^m; \mathbb{Z})$ is trivial.*

Proof. We consider three cases, one for each value of n mod 3.

Case 1: Suppose $n = 3k$. We know that $C_n^\ell = 0$ for $\ell < \nu_n$ from our cellular dimension range. We argue by induction on k that $C_n^{\nu_n} = 1$ while $C_n^{\nu_n+1} = 0$, which proves the claim for $n = 3k$. Begin by recalling that $C_0^0 = 1$ and $C_0^1 = 0$, which provides a base case.

Now, assume that $C_{3\ell}^{\nu_{3\ell}} = 1$ while $C_{3\ell}^{\nu_{3\ell}+1} = 0$ for $0 \leqslant \ell < k$. We know that

$$C_n^{\nu_n} = C_{3k-3}^{\nu_{3k}-2} + C_{3k-4}^{\nu_{3k}-m-1} + C_{3k-3}^{\nu_{3k}-m}$$

by our cellular recursion. Observe that $\nu_{3k} - 2 = 2k - 2 = \nu_{3k-3}$, so $C_{3k-3}^{\nu_{3k}-2} = 1$ by the induction hypothesis. Since $\nu_{3k} - m - 1 < \nu_{3k} - 2 = \nu_{3k-4}$, it follows that $C_{3k-4}^{\nu_{3k}-m-1} = 0$. Similarly, $\nu_{3k} - m < \nu_{3k} - 2 = \nu_{3k-3}$, so $C_{3k-3}^{\nu_{3k}-m} = 0$. Hence, $C_n^{\nu_n} = 1$.

Our cellular recursion also gives

$$C_n^{\nu_n+1} = C_{3k-3}^{\nu_{3k}-1} + C_{3k-4}^{\nu_{3k}-m} + C_{3k-3}^{\nu_{3k}-m+1}.$$

Observe that $\nu_{3k} - 1 = 2k - 1 = \nu_{3k-3} + 1$, so $C_{3k-3}^{\nu_{3k}-1} = 0$ by the induction hypothesis. Now, we note that $\nu_{3k} - m < \nu_{3k} - 2 = \nu_{3k-4}$ still, which implies $C_{3k-4}^{\nu_{3k}-m} = 0$. Similarly, $\nu_{3k} - m + 1 < \nu_{3k} - 2 = \nu_{3k-3}$, so $C_{3k-3}^{\nu_{3k}-m+1} = 0$. Hence, $C_n^{\nu_n+1} = 0$. By induction, we conclude that $C_{3k}^{\nu_{3k}} = 1$ while $C_{3k}^{\nu_{3k}+1} = 0$ for all k, from which the result follows.

Case 2: Suppose $n = 3k + 1$; this argument is similar to that of the previous case. We argue by induction on k that $C_n^{\nu_n} = 1$ while $C_n^{\nu_n+1} = 0$. We obtain our base case by recalling that $C_1^1 = 1$ and $C_1^2 = 0$ for $m \geqslant 4$. Next, we know that

$$C_n^{\nu_n} = C_{3k-2}^{\nu_{3k+1}-2} + C_{3k-3}^{\nu_{3k+1}-m-1} + C_{3k-3}^{\nu_{3k}-m}$$

by our cellular recursion. Observe that $\nu_{3k+1} - 2 = 2k - 1 = \nu_{3k-2} = \nu_{3(k-1)+1}$, so $C_{3k-2}^{\nu_{3k+1}-2} = 1$ by the induction hypothesis. Now, $\nu_{3k+1} - m - 1 = 2k - m < 2k - 2$, which is precisely ν_{3k-3}, implying that $C_{3k-3}^{\nu_{3k+1}-m-1} = 0$. Similarly, we see that $\nu_{3k+1} - m$ equal $2k - m + 1 < 2k - 1 = \nu_{3k-2}$, so $C_{3k-2}^{\nu_{3k+1}-m} = 0$. Hence, $C_n^{\nu_n} = 1$.

We also know that

$$C_n^{\nu_n+1} = C_{3k-2}^{\nu_{3k+1}-1} + C_{3k-3}^{\nu_{3k+1}-m} + C_{3k-2}^{\nu_{3k+1}-m+1}$$

by our cellular recursion. Observe that $\nu_{3k+1}-1 = \nu_{3k+1}-2+1 = \nu_{3k-2}+1$, so $C_{3k-2}^{\nu_{3k+1}-1} = 0$ by the induction hypothesis. Now, $\nu_{3k+1} - m = 2k - m + 1 < 2k - 2$, which is again ν_{3k-3}. Therefore, $C_{3k-3}^{\nu_{3k+1}-m} = 0$. Similarly, $\nu_{3k+1} - m + 1 < 2k - 1 = \nu_{3k-2}$, so $C_{3k-2}^{\nu_{3k+1}-m+1} = 0$. Hence, $C_n^{\nu_n+1} = 0$. By induction, we conclude that $C_{3k+1}^{\nu_{3k+1}} = 1$ while $C_{3k+1}^{\nu_{3k+1}+1} = 0$ for all k, from which the result follows.

Case 3: Suppose $n = 3k+2$. Recall from Theorem 20 that for $n = 3k+2$ and $m \geqslant 3$, the minimum dimension of critical cells produced by the Comb Algorithm is $2\left\lfloor \frac{n-1}{3} \right\rfloor + m$. It is easy to check that $\left\lfloor \frac{2n+2}{3} \right\rfloor = 2k + 2 < 2k + m = 2\left\lfloor \frac{n-1}{3} \right\rfloor + m$. Therefore, $C_n^{\nu_n} = 0$ when $n = 3k + 2$, i.e. $H_{\nu_n}(X_n^m; \mathbb{Z})$ is trivial. □

For other homology groups, the Comb Algorithm provides less comprehensive results. For example, when $m = 2$, that is, when X_n^m is homotopy equivalent to the matching complex on the $2 \times (n+2)$ grid graph, a direct analysis of the chain space dimensions on a data table yields the following.

Observation 24. *X_n^2 has non-trivial free integral homology in dimension $\lfloor \frac{9n+9}{13} \rfloor$ for $0 \leqslant n \leqslant 99$, except possibly for $n \in \{48, 61, 74, 84, 87, 90, 94, 97\}$. This arises because the rank of the chain space of X_n^2 in dimension $\lfloor \frac{9n+9}{13} \rfloor$ exceeds the sum of the ranks of the chain spaces in dimensions $\lfloor \frac{9n+9}{13} \rfloor - 1$ and $\lfloor \frac{9n+9}{13} \rfloor + 1$ for these values of n. Consequently, even if we were to try to further match away the critical cells in dimension $\lfloor \frac{9n+9}{13} \rfloor$, there are not enough cells in the adjacent dimensions to completely pair them all away.*

As an interesting side note, when $m = 2$, the values of d_n^{min} and d_n^{max} imply that X_n^2 is a wedge of spheres for $n \in \{0, 1, 2, 3, 4, 5, 7, 8, 11\}$.

As n grows larger, the data suggest that the rank of the $\lfloor \frac{9n+9}{13} \rfloor$-dimensional chain space ceases to "typically" exceed the sum of the ranks of the neighboring chain spaces. This suggests that the behavior of $\mathrm{Ind}(\Delta_n^m)$ for "small" values of n, including many values of n for which by-hand computations appear prohibitive, is not indicative of the general behavior of these complexes.

In conclusion, the topology of $\mathrm{Ind}(\Delta_n^m)$ remains generally mysterious. It would be of interest to investigate the following two questions.

1. Does torsion occur in the homology of $\mathrm{Ind}(\Delta_n^m)$? If so, for which p does $\mathbb{Z}/p\mathbb{Z}$ appear as a summand?

2. There is a natural action of the symmetric group $\mathfrak{S}_m$ on $\mathrm{Ind}(\Delta_n^m)$. What is the $\mathfrak{S}_m$-module structure of $H_*(\mathrm{Ind}(\Delta_n^m); \mathbb{C})$?

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