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Bounded Point Derivations on Certain Function Spaces

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Bounded Point Derivations on Certain Function Spaces

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Stephen Deterding
Lexington, Kentucky

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2018

ABSTRACT OF DISSERTATION

Bounded Point Derivations on Certain Function Spaces

Let X be a compact subset of the complex plane and denote by $R^p(X)$ the closure of rational functions with poles off X in the $L^p(X)$ norm. We show that if a point x_0 admits a bounded point derivation on $R^p(X)$ for $p > 2$, then there is an approximate derivative at x_0. We also prove a similar result for higher order bounded point derivations. This extends a result of Wang, which was proven for $R(X)$, the uniform closure of rational functions with poles off X. In addition, we show that if a point x_0 admits a bounded point derivation on $R(X)$ and if X contains an interior cone, then the bounded point derivation can be represented by the difference quotient if the limit is taken over a non-tangential ray to x_0. We also extend this result to the case of higher order bounded point derivations. These results were first shown by O'Farrell; however, we prove them constructively by explicitly using the Cauchy integral formula.

KEYWORDS: uniform and L^p rational approximation, bounded point derivations, approximate derivatives, non-tangential limits

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Date: April 30, 2018

To Dr. Seddon, for showing me the joy of mathematics.

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1.1 Capacities

A powerful tool in the study of approximation problems on the complex plane is the notion of capacities. Much like measures, capacities provide information about the structure of a set; however, in general the two are not comparable. Unlike measures, which quantify the size of a set, capacities were originally motivated as a way to measure the ability of a set to hold an electric charge. However, this physical interpretation does not hold true for every kind of capacity. We now specify the two capacities that will be used in what follows: analytic capacity and Sobolev q-capacity.

The analytic capacity of a set is a measure of how large functions that are analytic off the set can become. Notably, the compact sets with analytic capacity zero are the compact sets which are removable singularities for bounded analytic functions [12, Theorem 1.10]. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and let X be a compact subset of $\mathbb{C}$. A function f is said to be admissible for X if

1. f is analytic on $\hat{\mathbb{C}} \setminus X$.
2. $|f(z)| \leq 1$ on $\hat{\mathbb{C}} \setminus X$.
3. $f(\infty) = 0$.

The analytic capacity of the compact set X is defined by

$$\gamma(X) = \sup |f'(\infty)|.$$

where the supremum is taken over all admissible functions f. Note that $f'(\infty)$ is the derivative at the point at infinity and not the limit as the derivative tends to infinity. In fact, $f'(\infty) = \lim_{z\to\infty} zf(z)$. If X is not compact, then we define the analytic capacity of X as $\gamma(X) = \sup \gamma(K)$, where the supremum is taken over all compact subsets of X. We review some of the properties of analytic capacity that will be used in our proofs. More information on analytic capacity can be found in Gamelin's book [12].

1. Let B_r be a disk with radius r. Then $\gamma(B_r) = r$.
2. Analytic capacity is semi-additive [30]. This means that there exists a constant C such that for any countable collection of Borel sets E_n,

$$\gamma\left(\bigcup_{n=1}^{\infty} E_n\right) \leq C \sum_{n=1}^{\infty} \gamma(E_n).$$

A powerful theorem of Mel'nikov [20, Theorem 4] (See also [19] for the special case of an annulus) shows how the Cauchy integral of a nice-enough function can be bounded by the analytic capacity of the set where the function is not analytic.

Theorem 1.1.1. *Let Γ be a closed curve that encloses a region U. Let f be any function continuous and bounded by M_0 on U and analytic on $U \setminus K$, where K is a compact subset of U. Then there is a constant C which only depends on the curve Γ such that*

$$\left| \int_\Gamma f(z)dz \right| \leq CM_0\gamma(K).$$

If f is analytic on U, then Theorem 1.1.1 reduces to Cauchy's theorem. Thus Theorem 1.1.1 gives an upper bound for the failure of Cauchy's theorem for non-analytic continuous functions.

We will also make use of Sobolev q-capacity. For $1 < q < 2$, the q-capacity of a compact set X, denoted by $\Gamma_q(X)$ is defined by

$$\Gamma_q(X) = \inf \int |\nabla u|^q dA,$$

where dA is 2 dimensional Lebesgue (Area) measure and the infimum is taken over all infinitely differentiable functions u with compact support such that $u \equiv 1$ on X. If X is not compact then we define the q-capacity of X as $\Gamma_q(X) = \sup \Gamma_q(K)$, where the supremum is taken over all compacts subsets of X. We review some of the properties of q-capacity that will be used in our proofs. Proofs of these results and additional information on q-capacity can be found in the book of Adams and Hedberg [1].

1. For $1 < q < 2$, the q-capacity of a ball of radius r is equal to r^{2-q}.
2. q-capacity is monotonic; that is, if $E \subseteq F$ are sets then $\Gamma_q(E) \leq \Gamma_q(F)$.
3. q-capacity is sub-additive. [17] This means that for any countable collection of Borel sets E_n,

$$\Gamma_q\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \Gamma_q(E_n).$$

1.2 Bounded point derivations on R(X)

A key result in the study of rational approximation is Runge's Theorem.

Theorem 1.2.1. *Let X be a compact subset of the complex plane. If A is a set that contains at least one point from all of the bounded connected components of the compliment $\mathbb{C} \setminus X$ and f is a function that is analytic on an open neighborhood of X, then there exists a sequence of rational functions that converges uniformly to f such that the poles of the rational functions lie in A.*

Runge's theorem shows that a function that is analytic in a neighborhood of a set can always be uniformly approximated by rational functions with poles outside the set. Motivated by this result, mathematicians considered the problem of approximating other types of functions by rational functions. One notable example is the question of when continuous functions can be approximated by rational functions. It is useful to rephrase this question in the language of functional analysis. Let $R_0(X)$ denote the set of all rational functions whose poles lie off X. For instance if X is the closed unit disk, then $\frac{1}{z-2}$ belongs to $R_0(X)$, but $\frac{1}{z}$ does not. Now let $C(X)$ denote the set of all continuous functions on X and let $R(X)$ denote the subset of $C(X)$ that consists of all functions in $C(X)$ which on X are uniformly approximable by functions in $R_0(X)$. The question of approximating continuous functions by rational functions is the same as asking for which sets X does $R(X) = C(X)$. It is easy to see that a necessary condition for $R(X) = C(X)$ is that X must not contain any interior points, so from now on we make this assumption. The problem of determining both necessary and sufficient conditions for $R(X) = C(X)$ was first solved by Vitushkin [31], who found conditions in terms of analytic capacity.

Theorem 1.2.2. *Let X be a compact subset of the complex plane. Then the following are equivalent.*

1. $R(X) = C(X)$.
2. *For all open sets U,* $\gamma(U \setminus X) = \gamma(U)$.
3. *For almost all $z \in X$,* $\limsup_{r \to 0^+} \dfrac{\gamma(B_r(z) \setminus X)}{r^2} > 0$.

Vitushkin's Theorem answers the original question of determining the sets on which continuous functions can be approximated by rational functions; however, there are other aspects of rational approximation that are of interest. One example is the question of how well differentiability is preserved under convergence in the uniform norm. All functions in $R(X)$ are continuous, but they may not be differentiable. In fact it is a result of Dolzhenko [10] that there is a nowhere differentiable function in $R(X)$ whenever X is a nowhere dense set. For this reason, we will consider weaker notions of analyticity, such as monogenicity to answer this question. A function f defined on a set E is said to be monogenic at a point $x_0 \in E$ if the following limit

$$\lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}$$

holds throughout the points of E. If E is an open set then a function f is monogenic at each point of E if and only if f is analytic on E. If E is not open, then in general a monogenic function will not be analytic; however, in many cases it will possess many of the properties of analytic functions. As an example, Borel [3] constructed a compact set X with no interior that contains a large dense subset E such that every function f monogenic on X is infinitely differentiable and uniquely determined by its value and the values of all its derivatives at any fixed point of E. Thus Borel's construction gives an example of a quasianalytic class of functions. There have also been constructions that extend the uniqueness property of analytic functions to $R(X)$. Sinanjan [28] (See also [6, pg. 223]) credits Keldysh with the construction of a nowhere dense set X such that the functions in $R(X)$ are monogenic and such that if any two of them coincide on an arbitrary portion of X, then they are identical on all of X; however, he does not provide a specific reference. Another construction is due to Gonchar; although, it only appears in a survery article of Mel'nikov and Sinanjan [21]. Gonchar constructed a compact nowhere dense set X such that if any two functions coincide on a set of positive one-dimensional Hausdorff measure, then they coincide everywhere on X.

For our present purposes of studying the differentiability of functions in $R(X)$ we introduce a weaker notion of the derivative known as a bounded point derivation. A bounded point derivation on $R(X)$ at a point x_0 is a bounded linear functional, D, such that for all pairs f, g in $R(X)$, $D(fg) = D(f)g(x_0) + f(x_0)D(g)$. Thus even if functions in $R(X)$ fail to be differentiable, it is still possible for them to possess some type of analytic structure. How close the functions in $R(X)$ can come to being differentiable can be determined by the existence of bounded point derivations. The next theorem gives several equivalent notions of a bounded point derivation.

Theorem 1.2.3. *The following are equivalent.*

1. *There is a bounded point derivation on $R(X)$ at x_0.*
2. *The map $f \to f'(x_0)$ extends as a bounded linear functional from the rational functions with poles off X to $R(X)$.*
3. *There exists a constant $C > 0$ such that for every $f \in R_0(X)$, $|f'(x_0)| \leq C||f||$. Here $||\cdot||$ denotes the sup norm on X.*

Proof. (3) $\implies$ (2): If f is a function in $R(X)$ then there is a sequence $\{f_j\}$ of rational functions with poles off X that converges to f uniformly. Thus $|f_j'(x_0) - f_k'(x_0)| \leq C||f_j - f_k||$, which tends to 0 as $j, k \to \infty$. So $\{f_j'(x_0)\}$ is a Cauchy sequence and hence converges. Hence the map $f \to f'(x_0)$ can be extended from the space of rational functions with poles off X to a bounded linear functional on $R(X)$, which we denote as D. Moreover, it follows that $Df = \lim_{j\to\infty} f_j'(x_0)$, where $\{f_j\}$ is a sequence of rational functions which converges to f uniformly, and that the value of Df does not depend on the choice of this sequence.

(2) $\implies$ (1): Suppose f and g belong to $R(X)$. Then there exists sequences $\{f_j\}$ and $\{g_j\}$ of rational functions with poles off X which converge uniformly to f and g respectively. Let D be the extension of the map $f \to f'(x_0)$. Then $D(fg) = \lim_{j\to\infty}(f_jg_j)'(x_0) = \lim_{j\to\infty} f_j'(x_0)g_j(x_0) + f_j(x_0)g_j'(x_0) = D(f)g(x_0) + f(x_0)D(g)$, so there is a bounded point derivation at x_0.

(1) $\implies$ (3): We will show that a bounded point derivation must send f to a fixed constant multiple of $f'(x)$ for all rational functions f. First note that by definition $D(1) = D(1\cdot 1) = 1D(1) + D(1)1 = 2D(1)$. Hence $D(1) = 0$. Now suppose $D(x) = c$. Then $D(x^2) = D(x)x + xD(x) = 2xD(x) = 2cx$. Similarly $D(x^3) = D(x^2)x + x^2D(x) = 2cx^2 + cx^2 = 3cx^2$. Likewise it follows that if f is a polynomial then $D(f) = cf'(x)$. Now let $f = \dfrac{p}{q}$ be a rational function. Then $D(p) = D\left(q\cdot\dfrac{p}{q}\right) = D(q)\cdot\dfrac{p}{q} + q\cdot D\left(\dfrac{p}{q}\right)$. Hence $D\left(\dfrac{p}{q}\right) = \dfrac{qD(p) - pD(q)}{q^2}$ and thus a bounded point derivation must send a rational function f to a fixed constant multiple of $f'(x)$ from which (3) follows.

$\square$

These equivalent notions of a bounded point derivation can be used to extend the definition to higher order derivations. For a non-negative integer t, we say that $R(X)$ has a bounded point derivation of order t at x_0 if there exists a constant $C > 0$ such that $|f^{(t)}(x_0)| \le C||f||$ for all rational functions f with poles off X. An equivalent definition is to say that $R(X)$ has a bounded point derivation of order t at x_0 if the map $f \to f^{(t)}(x_0)$ extends as a bounded linear functional from $R_0(X)$ to $R(X)$. If $t = 0$, we take the 0-th order derivative to be the evaluation of the function at x_0. For this reason, a 0-th order bounded point derivation is usually called a bounded point *evaluation*. Bounded point evaluations have been widely studied in both rational approximation theory and operator theory. (See for instance [4], [11], [15], and [9, §2.7].) From now on we will use the term *bounded point derivation* to refer to a first order bounded point derivation and will specify the order if we mean to refer to a higher order bounded point derivation.

Not every point admits a bounded point derivation, so it is of great importance to characterize those points for which bounded point derivations exist. Some examples show that the amount of bounded point derivations in a set can vary widely. Wermer [33] gave an example of a nowhere dense set X with the property that $R(X)$ admits bounded point derivations of order 1 at almost every point of $R(X)$. This example was generalized by Hallstrom [13] who constructed a nowhere dense set X with the property that almost every point of X admits bounded point derivation of all orders. At the other extreme, Wermer also constructed a nowhere dense set with no bounded point derivations in [33] and O'Farrell [24] constructed a nowhere dense set which contains a single bounded point derivation. The following theorem summarizes some conditions for the existence of bounded point derivations.

Theorem 1.2.4. *Let X be a compact subset of the complex plane.*

1. *Every interior point of X admits bounded point derivations of all orders for $R(X)$.*
2. *There is a bounded point derivation of order t on $R(X)$ at x_0 if and only if there exists a constant $C > 0$ such that $|f^{(t)}(x_0)| \leq C||f||$ for every $f \in R_0(X)$. Here $|| \cdot ||$ denotes the sup norm on X.*

Proof. (1) follows since the uniform limit of a holomorphic function is holomorphic. Thus at an interior point every function in $R(X)$ has a derivative. (2) can be proven in the same way as part (3) of Theorem 1.2.3.

□

The following example illustrates the use of Theorem 1.2.4.

Example 1.2.5. Let $a_n = \dfrac{3}{2^{n+2}}$, let $B_n = \left\{ z : |z - a_n| < \dfrac{1}{2^{n+2}} \right\}$, and let $X = \overline{\Delta} \setminus \bigcup B_n$. Let $f_n(z) = \dfrac{1}{z - a_n}$. Then $|f_n'(0)| = \dfrac{1}{a_n^2} = \frac{16}{9} \cdot 4^n$ and $||f_n|| = 2^{n+2}$. Hence for every $C > 0$, there exists n such that $|f_n'(0)| > C||f_n||_X$. Thus there is no bounded point derivation on $R(X)$ at 0.

We say that a boundary point of a set is part of the outer boundary if it is a point on a the boundary of a single connected component of the compliment of the set and is otherwise part of the inner boundary. Notice that in the above example, 0, which admitted a bounded point derivation on $R(X)$, is part of the inner boundary. In fact, in order for a boundary point to admit a bounded point derivation it must belong to the inner boundary.

Theorem 1.2.6. *Let X be a compact subset of the plane and let x_0 belong to the outer boundary of X. Then x_0 does not admit a bounded point derivation for $R(X)$.*

Proof. Since x_0 is on the boundary of a connected component of the compliment of X, there exists a sequence of points $x_n \notin X$ which converges to x_0 and a constant $k > 0$ such that for all $z \in X$, $|x_n - x_0| \leq k|x_n - z|$. Now let

$$f_n(z) = \frac{x_0 - x_n}{z - x_n}.$$

Then for each n, $f_n(z) \in R_0(X)$ and $|f_n(z)| \leq k$. However $f_n'(x_0) = \dfrac{-1}{x_0 - x_n}$ which tends to ∞ as $n \to \infty$. Thus x_0 does not admit a bounded point derivation for $R(X)$. $\square$

1.3 Necessary and sufficient conditions for bounded point derivations on R(X)

It is often quite difficult to determine whether $R(X)$ admits bounded point derivations using the definition alone. Fortunately there are necessary and sufficient conditions for the existence of bounded point derivations, which are given in terms of analytic capacity. As these conditions only depend on analytic capacity, they are geometric rather than analytic.

Theorem 1.3.1. *Let $A_n(x_0)$ be the annulus $\left\{x : \frac{1}{2^{n+1}} < |x - x_0| < \frac{1}{2^n}\right\}$ and let t be a non-negative integer. Then there is a bounded point derivation of order t on $R(X)$ at x_0 if and only if*

$$\sum_{n=1}^{\infty} 2^{(t+1)n} \gamma(A_n(x_0) \setminus X) < \infty.$$

Theorem 1.3.1 was first proved for the case of $t = 0$ (bounded point evaluations) by Mel'nikov [19]. Hallstrom [13] extended his result to the general case. In Example 1.2.5, We considered a set X such that $R(X)$ did not have a bounded point derivation at 0. By utilizing Theorem 1.3.1 we can attain the same conclusion.

Example 1.3.2. Let $a_n = \dfrac{3}{2^{n+2}}$, let $B_n = \left\{z : |z - a_n| < \dfrac{1}{2^{n+2}}\right\}$, and let $X = \overline{\Delta} \setminus \bigcup B_n$. Since each B_n is entirely contained in the corresponding A_n it follows that $\gamma(A_n \setminus X) = \gamma(B_n) = \dfrac{1}{2^{n+2}}$ and hence

$$\sum_{n=1}^{\infty} 4^n \gamma(A_n(x_0) \setminus X) = \sum_{n=1}^{\infty} 2^{n-2} = \infty.$$

Thus there is no bounded point derivation on $R(X)$ at 0.

Hallstrom [13] also proved the following necessary condition for the existence of bounded point derivations.

Theorem 1.3.3. *Let $B_r(x_0)$ be the ball $\{x : |x - x_0| < r\}$. If there is a bounded point derivation of order t on $R(X)$ at x_0 then*

$$\lim_{r \to 0^+} \frac{\gamma(B_r(x_0) \setminus X)}{r^{t+1}} = 0.$$

Theorem 1.3.3 provides a necessary condition for the existence of bounded point derivations. We now show that this condition is not sufficient.

Theorem 1.3.4. *There exists a set $X \subseteq \mathbb{C}$ containing 0 such that*

$$\lim_{r\to 0^+} \frac{\gamma(B_r(0)\setminus X)}{r^2} = 0,$$

but 0 does not admit a bounded point derivation on $R(X)$.

Proof. Let D denote the closed unit disk and let $A_n = \{\frac{1}{2^{n+1}} < |z| < \frac{1}{2^n}\}$. For each n, let D_n be a disk entirely contained in A_n with radius $\dfrac{1}{n+1}\cdot\dfrac{1}{4^{n+1}}$. Let $X = D \setminus \bigcup D_n$.

We will first show that 0 does not admit a bounded point derivation on $R(X)$. Note that $\gamma(A_n \setminus X) = \gamma(D_n) = \dfrac{1}{n+1}\cdot\dfrac{1}{4^{n+1}}$. Hence

$$\sum_{n=1}^{\infty} 4^n\gamma(A_n\setminus X) = \sum_{n=1}^{\infty} 4^n \frac{1}{n+1}\cdot\frac{1}{4^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{4}\cdot\frac{1}{n+1} = \infty.$$

Hence by Theorem 1.3.1, 0 does not admit a bounded point derivation on $R(X)$.

Next we will show that

$$\lim_{r\to 0^+} \frac{\gamma(B_r(0)\setminus X)}{r^2} = 0.$$

Choose $r > 0$. Then there exists N such that $\frac{1}{2^N} < r < \frac{1}{2^{N-1}}$. Thus $B_r(0)\setminus X$ contains all of the disks D_n for $n \geq N$, possibly contains part or all of the disk D_{N-1}, and contains none of the the disks D_n for $n < N-1$. So $B_r(0)\setminus X = \bigcup_{n=N}^{\infty} D_n \bigcup D_r$ where D_r is the part of D_{N-1} that is contained in $B_r(0)$. Hence

$$\gamma(B_r(0)\setminus X) \leq \gamma(\bigcup_{n=N-1}^{\infty} D_n),$$

and by the semi-additivity of analytic capacity we obtain

$$\lim_{r\to 0^+}\frac{\gamma(B_r(0)\setminus X)}{r^2} \leq \lim_{N\to\infty}\frac{\gamma(\bigcup_{n=N-1}^{\infty} D_n)}{\left(\frac{1}{2^N}\right)^2} \leq \lim_{N\to\infty} C4^N\sum_{n=N}^{\infty}\frac{1}{n}\cdot\frac{1}{4^n}.$$

If we expand the sum we have

$$C4^N \sum_{n=N}^{\infty} \frac{1}{n} \cdot \frac{1}{4^n} = C4^N \left(\frac{1}{N} \cdot \frac{1}{4^N} + \frac{1}{N+1} \cdot \frac{1}{4^{N+1}} + ... \right) = C \left(\frac{1}{N} + \frac{1}{N+1} \cdot \frac{1}{4} + ... \right).$$

Let $\vec{u} = \left(\frac{1}{N}, \frac{1}{N+1}, ... \right)$ and let $\vec{v} = \left(1, \frac{1}{4}, \frac{1}{16} ... \right)$. Then

$$\left(\frac{1}{N} + \frac{1}{N+1} \cdot \frac{1}{4} + ... \right) = \vec{u} \cdot \vec{v}.$$

Let $||\vec{x}|| = \vec{x} \cdot \vec{x}$. By the Cauchy-Schwarz inequality, we know that $\vec{u} \cdot \vec{v} \leq ||\vec{u}|| \cdot ||\vec{v}||$. Hence we have that

$$\begin{aligned} \left(\frac{1}{N} + \frac{1}{N+1} \cdot \frac{1}{4} + ... \right) &\leq \left(\frac{1}{N^2} + \frac{1}{(N+1)^2} + ... \right) \left(1 + \frac{1}{16} + \frac{1}{16^2} + ... \right) \\ &= \left(\sum_{n=N}^{\infty} \frac{1}{n^2} \right) \left(\sum_{n=0}^{\infty} \frac{1}{16^n} \right). \end{aligned}$$

The second sum is bounded and the first sum tends to 0 as $N \to \infty$. Hence we conclude that

$$\lim_{r \to 0^+} \frac{\gamma(B_r(0) \setminus X)}{r^2} = 0.$$

□

1.4 Rational approximation in the areal mean and bounded point derivations

We now consider rational approximation in the L^p norm where the underlying measure is 2-dimensional Lebesgue (area) measure. We first review some important results concerning the L^p norm and L^p spaces. Recall that the L^p norm of a function is $||f||_p = \left(\int |f|^p dx \right)^{\frac{1}{p}}$. Given a compact set X, we define $L^p(X)$ to be the space of functions on X with finite L^p norm. The L^p spaces have the following containment property: For a compact set X, if $p < p'$ then $L^{p'}(X) \subseteq L^p(X)$. This follows directly from Hölder's inequality.

Theorem 1.4.1 (Hölder's Inequality). *Let $1 < p < \infty$ and let $q = \frac{p}{p-1}$. If $f \in L^p(X)$ and $g \in L^q(X)$ then*

$$\int_X |fg| dx \leq \left(\int_X |f|^p dx \right)^{\frac{1}{p}} \cdot \left(\int_X |g|^q dx \right)^{\frac{1}{q}}.$$

We now turn our focus to some of the algebraic properties of the L^p spaces. Recall that a linear functional on a Banach space B is a map $\phi : B \to \mathbb{C}$ such that $\phi(\alpha f + g) = \alpha\phi(f) + \phi(g)$. A linear functional ϕ is said to be bounded if there exists a positive number C such that

$$\sup_f \frac{|\phi(f)|}{||f||} < C,$$

where the supremum is taken over all $f \in B$ and the norm $|| \cdot ||$ is the norm on B. The dual space of a Banach space B is the space of bounded linear functionals that act on B. It is well known that the dual space of $L^p(X)$ for $1 < p < \infty$ is $L^q(X)$ where $q = \frac{p}{p-1}$. Thus we have the following theorem.

Theorem 1.4.2. *Let $1 < p < \infty$ and let $q = \frac{p}{p-1}$. If ϕ is a bounded linear functional on $L^p(X)$ then there exists $k \in L^q(X)$ such that*

$$\phi(f) = \int f k dA$$

for all $f \in L^p(X)$.

The measure kdA is called a representing measure for the linear functional.

By analogy with the uniform case, we define the space $R^p(X)$ to be the closure of rational functions with poles off X in the L^p norm. It follows from Hölder's inequality that $R(X)$ is contained in $R^p(X)$, thus proving results for $R^p(X)$ also includes results for $R(X)$. An interesting problem in rational approximation is to determine under what conditions on X can every function in $L^p(X)$ be approximated in the L^p norm by rational functions with poles off X. It is straightforward to show that $R^p(X) \neq L^p(X)$ unless X has empty interior, so from now on, we will make this assumption. The full problem was solved in the case of $p > 2$ by Hedberg [16] who gave necessary and sufficient conditions for $R^p(X) = L^p(X)$ in terms of Sobolev q-capacity.

Theorem 1.4.3. *Let X be a compact set and let $2 < p < \infty$. Then the following are equivalent.*

1. *$R^p(X) = L^p(X)$.*
2. *For almost every $x \in X$,*

$$\limsup_{r \to 0^+} \frac{\Gamma_q(B_r(x) \setminus X)}{r^{2-q}} > 0.$$

When $1 \leq p < 2$, the situation is quite different. Sinanjan has shown that if X has no interior and $1 \leq p < 2$, then $R^p(X) = L^p(X)$ [28]. This leaves the case of $p = 2$. As there is a noticeable difference in approximation in the case of $1 \leq p < 2$ and the case of $p > 2$, the transition case of $p = 2$ tends to exhibit properties that differ from both of the other cases. For instance Hedberg [16] has shown that the necessary and sufficient conditions for approximation in the L^2 norm, while similar to the ones for $p > 2$, are not what is expected based on Theorem 1.4.3.

Theorem 1.4.4. *Let X be a compact set. Let Γ_2 denote Sobolev 2-capacity, which is defined in Section 1.2.3. Then the following are equivalent.*

1. $R^2(X) = L^2(X)$.
2. *For almost every $x \in X$,*

$$\limsup_{r \to 0^+} \frac{\Gamma_2(B_r(x) \setminus X)}{r^2} > 0.$$

Our chief interest concerns how much differentiability is preserved under convergence of rational functions in the L^p norm. In general functions in $R^p(X)$ will not even be continuous much less differentiable. Moreover, since $R(X)$ is contained in $R^p(X)$, we know from Dolzhenko's result for $R(X)$ that $R^p(X)$ contains a nowhere differentiable function whenever X is a nowhere dense set. Hence, as in the case of uniform rational approximation, we will consider weaker notions of analyticity. One problem we mention in particular is the problem of constructing a set X so that the functions in $R^p(X)$ possess the uniqueness property that whenever two functions in $R^p(X)$ agree on some subset of X, then they are identical on all of X. In 1965, Sinanjan [28] constructed the first example of a set with this kind of uniqueness property. He showed the existence of a compact nowhere dense set X with positive area measure such that whenever two functions in $R^p(X)$ agree on a relatively open subset of X, then they agree almost everywhere in X. A set A is said to be relatively open in X if there is an open set U such that $X \cap U = A$. In 1973, Brennan [5] strengthened this result by showing the existence of a compact nowhere dense set X with positive area measure such that whenever two functions in $R^p(X)$ agree on a set of positive measure in X then they agree almost everywhere in X. Notably Brennan's construction did not make any use of capacity and instead made use of bounded point derivations defined on $R^p(X)$.

We now define bounded point derivations on $R^p(X)$. Note that we cannot define then in exactly the same way that we did for $R(X)$, by the formula $D(fg) = D(f)g + fD(g)$, because the product fg may not be in $R^p(X)$. Instead we make use of Theorem 1.2.3 to extend the definition from $R(X)$ to $R^p(X)$. We say that there is a bounded point derivation on $R^p(X)$ if there exists a constant $C > 0$ such that for every rational function f with poles off X, $|f'(x_0)| \leq C||f||_p$. Likewise, for a non-negative integer

t, we say that $R^p(X)$ has a bounded point derivation of order t at x_0 if there exists a constant $C > 0$ such that $|f^{(t)}(x_0)| \leq C||f||_p$ for all rational functions f with poles off X.

If f is a function in $R^p(X)$ then there is a sequence $\{f_j\}$ of rational functions with poles off X that converges to f in the L^p norm. If there is a bounded point derivation at x_0 then $|f_j^{(t)}(x_0) - f_k^{(t)}(x_0)| \leq C||f_j - f_k||_p$, which tends to 0 as j and k tend to infinity. Thus $\{f_j^{(t)}(x_0)\}$ is a Cauchy sequence and hence converges. Hence the map $f \to f^{(t)}(x_0)$ can be extended from the space of rational functions with poles off X to a bounded linear functional on $R^p(X)$, which we denote as $D^t_{x_0}$. It follows that $D^t_{x_0} f = \lim_{j\to\infty} f_j^{(t)}(x_0)$, where $\{f_j\}$ is a sequence of rational functions which converges to f in the L^p norm, and the value of $D^t_{x_0} f$ does not depend on the choice of this sequence. Thus bounded point derivations can be used to define a derivative for functions in $R^p(X)$ which are not differentiable in the usual sense. In addition, the existence of different orders of bounded point derivations shows to what extent differentiability is preserved under convergence in the L^p norm. For these reasons, it is important to understand the relationship between bounded point derivations and the usual notion of the derivative. We consider these questions in Chapters 2 and 3.

1.5 Necessary and sufficient conditions for bounded point derivations on spaces of rational functions in the areal mean

In analogy to the theorems which give necessary and sufficient conditions in terms of analytic capacity for the existence of bounded point derivations on $R(X)$, there are necessary and sufficient conditions in terms of q-capacity for the existence of bounded point derivations on $R^p(X)$. The two theorems in this section were both discovered by Hedberg [15].

Theorem 1.5.1. *Let $2 < p < \infty$ and let $q = \frac{p}{p-1}$. Let $A_n(x_0)$ be the annulus $\left\{x : \frac{1}{2^{n+1}} < |x - x_0| < \frac{1}{2^n}\right\}$. Then there is a bounded point derivation of order t on $R^p(X)$ at x_0 if and only if*

$$\sum_{n=0}^{\infty} 2^{(t+1)nq} \Gamma_q(A_n(x_0) \setminus X) < \infty.$$

Theorem 1.5.2. *Let $2 < p < \infty$ and let $q = \frac{p}{p-1}$. Let $B_r(x_0)$ be the ball $\{x : |x - x_0| < r\}$. If there is a bounded point derivation of order t on $R^p(X)$ at x_0, then*

$$\lim_{r\to 0^+} \frac{\Gamma_q(B_r(x_0) \setminus X)}{r^{q(t+1)}} = 0.$$

This next example demonstrates the utility of Theorem 1.5.1.

Example 1.5.3. Let $1 < p < \infty$ and let $q = \frac{p}{p-1}$. Let $a_n = \dfrac{3}{2^{n+2}}$, let $B_n = \left\{z : |z - a_n| < (4^{-nq} \cdot n^{-2})^{\frac{1}{2-q}}\right\}$, and let $X = \overline{\Delta} \setminus \bigcup B_n$. Since each B_n is entirely contained in the corresponding $A_n(x_0)$ it follows that $\Gamma_q(A_n(x_0) \setminus X) = \Gamma_q(B_n) = 4^{-nq} \cdot n^{-2}$ and hence

$$\sum_{n=1}^{\infty} 4^{nq} \Gamma_q(A_n(x_0) \setminus X) = \sum_{n=1}^{\infty} n^{-2} < \infty$$

Thus there is a (first order) bounded point derivation on $R^p(X)$ at 0.

Theorem 1.5.2 provides a necessary condition for the existence of bounded point derivations. We now show that this condition is not sufficient.

Theorem 1.5.4. *There exists a plane set X containing 0 such that*

$$\lim_{r \to 0^+} \frac{\Gamma_q(B_r(0) \setminus X)}{r^{2q}} = 0$$

but 0 does not admit a bounded point derivation on $R^p(X)$.

Proof. Let D denote the closed unit disk and let $A_n = \{\frac{1}{2^{n+1}} < |z| < \frac{1}{2^n}\}$. For each n, let D_n be a disk entirely contained in A_n with radius $\left(\dfrac{1}{n+1} \cdot \dfrac{1}{4^{(n+1)q}}\right)^{\frac{1}{2-q}}$. Let $X = D \setminus \bigcup D_n$.

We will first show that 0 does not admit a bounded point derivation on $R(X)$. Note that $\Gamma_q(A_n \setminus X) = \Gamma_q(D_n) = \dfrac{1}{n+1} \cdot \dfrac{1}{4^{(n+1)q}}$. Hence

$$\sum_{n=1}^{\infty} 4^{nq} \Gamma_q(A_n \setminus X) = \sum_{n=1}^{\infty} 4^{nq} \cdot \frac{1}{n+1} \cdot \frac{1}{4^{(n+1)q}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \cdot \frac{1}{4^q} = \infty.$$

Hence by Theorem 1.5.1, 0 does not admit a bounded point derivation on $R^p(X)$.

Next we will show that

$$\lim_{r \to 0^+} \frac{\Gamma_q(B_r(0) \setminus X)}{r^{2q}} = 0.$$

Choose $r > 0$. Then there exists N such that $\frac{1}{2^N} < r < \frac{1}{2^{N-1}}$. Thus $B_r(0) \setminus X$ contains all of the disks D_n for $n \geq N$, possibly contains part or all of the disk D_{N-1}, and contains none of the the disks D_n for $n < N - 1$. So $B_r(0) \setminus X = \bigcup_{n=N}^{\infty} D_n \bigcup D_r$ where D_r is the part of D_{N-1} that is contained in $B_r(0)$. Hence

$$\Gamma_q(B_r(0)\setminus X) \le \Gamma_q(\bigcup_{n=N-1}^{\infty} D_n),$$

and by the sub-additivity of q-capacity we obtain

$$\lim_{r\to 0^+} \frac{\Gamma_q(B_r(0)\setminus X)}{r^{2q}} \le \lim_{N\to\infty} \frac{\Gamma_q(\bigcup_{n=N-1}^{\infty} D_n)}{\left(\frac{1}{2^N}\right)^{2q}} \le \lim_{N\to\infty} C4^{Nq} \sum_{n=N}^{\infty} \frac{1}{n}\cdot\frac{1}{4^{nq}}.$$

If we expand the sum we have

$$\begin{aligned} C4^{Nq} \sum_{n=N}^{\infty} \frac{1}{n}\cdot\frac{1}{4^{nq}} &= C4^{Nq}\left(\frac{1}{N}\cdot\frac{1}{4^{Nq}} + \frac{1}{N+1}\cdot\frac{1}{4^{(N+1)q}} + \dots\right) \\ &= C\left(\frac{1}{N} + \frac{1}{N+1}\cdot\frac{1}{4^q} + \dots\right). \end{aligned}$$

Let $\vec{u} = \left(\frac{1}{N}, \frac{1}{N+1}, \dots\right)$ and let $\vec{v} = \left(1, \frac{1}{4^q}, \frac{1}{16^q} \dots\right)$. Then

$$\left(\frac{1}{N} + \frac{1}{N+1}\cdot\frac{1}{4^q} + \dots\right) = \vec{u}\cdot\vec{v}.$$

By the Cauchy-Schwarz inequality, we know that $\vec{u}\cdot\vec{v} \le ||\vec{u}||\cdot||\vec{v}||$. Hence we have that

$$\begin{aligned} \left(\frac{1}{N} + \frac{1}{N+1}\cdot\frac{1}{4^q} + \dots\right) &\le \left(\frac{1}{N^2} + \frac{1}{(N+1)^2} + \dots\right)\left(1 + \frac{1}{16^q} + \frac{1}{16^{2q}} + \dots\right) \\ &= \left(\sum_{n=N}^{\infty} \frac{1}{n^2}\right)\left(\sum_{n=0}^{\infty}\left(\frac{1}{16^q}\right)^n\right). \end{aligned}$$

The second sum is bounded and the first sum tends to 0 as $N\to\infty$. Hence we conclude that

$$\lim_{r\to 0^+} \frac{\Gamma_q(B_r(0)\setminus X)}{r^{2q}} = 0.$$

□

Chapter 2 Difference quotient formulas and approximate derivatives

2.1 Uniform rational approximation

We now consider the relationship between bounded point derivations and the usual notion of the derivative. In particular, if there is a bounded point derivation at x_0 then we would like to know how close the functions in $R(X)$ or $R^p(X)$ come to being differentiable. One way to do this would be to find a difference quotient formula that represents the bounded point derivation. That is, given a bounded point derivation at $x_0 \in X$, can we find a set E such that

$$Df = \lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.1.1}$$

for all f in $R(X)$? The size of the set E tells us how close functions in $R(X)$ come to being differentiable. Since Dolzhenko showed that $R(X)$ always contains a nowhere differentiable function when X has no interior, we know that that E cannot be all of X. The next possibility is that E is a set with full area density at x_0. This means that if we let $\Delta_n(x_0)$ denote the ball centered at x_0 with radius $\frac{1}{n}$ and let m denote 2 dimensional Lebesgue measure, then $\lim\limits_{n\to\infty} \dfrac{m(\Delta_n(x_0) \setminus E)}{m(\Delta_n(x_0))} = 0$. If the set E in (2.1.1) has full area density at x_0 then we say that f has an approximate derivative at x_0. The concept of an approximate derivative stretches back to the work of Men'shov, although he used the term *asymptotic derivative*. In [22], Men'shov proved that a function that is continuous on a domain and has an approximate derivative at almost every point of the domain must be analytic on the domain. Wang [32] has proven the following theorem which shows that if $R(X)$ has a bounded point derivation at x_0 then every function in $R(X)$ has an approximate derivative at x_0.

Theorem 2.1.1. *Suppose that there is a bounded point derivation on $R(X)$ at x_0 denoted by $D^1_{x_0}$. Then given a function f in $R(X)$, there exists a set E of full area density at x_0 such that*

$$D^1_{x_0} f = \lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}.$$

We will also consider the extension of Theorem 2.1.1 to higher order bounded point derivations. To do so we first consider how to define higher order approximate derivatives and higher order difference quotients. Intuitively, a higher order approximate derivative at x_0 should be defined in the same way as a higher order derivative except that the limit of the difference quotient should be taken over a set with full area density at x_0. However, a function in $R^p(X)$ may not have derivatives of any orders and thus we cannot define an approximate higher order derivative in terms of any of the lower order derivatives. Hence we will use the following definition for higher order difference quotients.

Definition 2.1.2. Let t be a positive integer, let f be a function in $R(X)$ or $R^p(X)$, let x_0 be a point in X, and choose $h \in \mathbb{C}$ so that f is defined at x_0+sh for $s = 0, 1, ..., t$. The **t-th order difference quotient of f at x_0 and h** is denoted by $\Delta_h^t f(x_0)$ and defined by

$$\Delta_h^t f(x_0) = h^{-t} \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} f(x_0 + sh).$$

For this definition to be reasonable, it should agree with the usual definition for higher order derivatives when f has derivatives of all orders.

Theorem 2.1.3. *Suppose that f has derivatives of all orders on a neighborhood of x_0. Then for all positive integers t, $f^{(t)}(x_0) = \lim_{h\to 0} \Delta_h^t f(x_0)$.*

Proof. The proof is by induction. Since $\Delta_h^1 f(x_0) = \dfrac{f(x_0 + h) - f(x_0)}{h}$ the theorem is true for $t = 1$. Now assume that $f^{(t-1)}(x_0) = \lim_{h\to 0} \Delta_h^{t-1} f(x_0)$. Then

$$f^{(t)}(x_0) = \lim_{h\to 0} \frac{\Delta_h^{t-1} f(x_0 + h) - \Delta_h^{t-1} f(x_0)}{h} = \lim_{h\to 0} \Delta_h^1 \circ \Delta_h^{t-1} f(x_0).$$

Thus to show that $f^{(t)}(x_0) = \lim_{h\to 0} \Delta_h^t f(x_0)$ it is enough to prove that $\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) = \Delta_h^t f(x_0)$. It follows from Definition 2.1.2 that

$$\begin{aligned}\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) h^{-t} = \Bigg\{ &\sum_{s=0}^{t-1} (-1)^{t-1-s} \binom{t-1}{s} f(x_0 + (s+1)h) \\ &- \sum_{s=0}^{t-1} (-1)^{t-1-s} \binom{t-1}{s} f(x_0 + sh) \Bigg\}.\end{aligned}$$

A change of variable of $s = s - 1$ in the first sum yields

$$\begin{aligned}\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) = h^{-t} \Bigg\{ &\sum_{s=1}^{t} (-1)^{t-s} \binom{t-1}{s-1} f(x_0 + sh) \\ &- \sum_{s=0}^{t-1} (-1)^{t-1-s} \binom{t-1}{s} f(x_0 + sh) \Bigg\}.\end{aligned}$$

Multiplying the second sum by (-1) changes the subtraction to addition. Then moving the t-th term of the first sum outside the sum and doing the same to the 0-th term of the second sum yields

$$\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) = h^{-t} \left\{ f(x_0+th) + \sum_{s=1}^{t-1} (-1)^{t-s} \binom{t-1}{s-1} f(x_0+sh) \right.$$
$$\left. + \sum_{s=1}^{t-1} (-1)^{t-s} \binom{t-1}{s} f(x_0+sh) + (-1)^t f(x_0) \right\}.$$

The two sums can be combined using the binomial identity $\binom{t-1}{s-1} + \binom{t-1}{s} = \binom{t}{s}$. Hence

$$\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) = h^{-t} \left\{ f(x_0+th) + \sum_{s=1}^{t-1} (-1)^{t-s} \binom{t}{s} f(x_0+sh) + (-1)^t f(x_0) \right\}.$$

In addition since $\binom{t}{0} = \binom{t}{t} = 1$ the two terms outside the sum can be put back into the sum and thus

$$\Delta_h^1 \circ \Delta_h^{t-1} f(x_0) = h^{-t} \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} f(x_0+sh) = \Delta_h^t f(x_0).$$

□

We now define higher order approximate derivatives using Definition 2.1.2.

Definition 2.1.4. Let t be a positive integer. A function f in $R(X)$ or $R^p(X)$ has an **approximate derivative of order** t at x_0 if there exists a set E' with full area density at 0, and a number L such that

$$\lim_{h \to 0, h \in E'} \Delta_h^t f(x_0) = L$$

We say that L is the approximate derivative of order t at x_0.

Thus a t-th order approximate derivative at x_0, is a t-th order difference quotient in which the limit as $h \to 0$ is taken over a set with full area density at 0. The reason that the set E' has full area density at 0 instead of at x_0 is that the limits in the definitions of usual higher order derivatives are taken as $h \to 0$ and therefore, the higher order approximate derivatives must be defined similarly.

The following higher order extension of Theorem 2.1.1 was also proven by Wang [32].

Theorem 2.1.5. *Let t be a positive integer and suppose that there exists a bounded point derivation of order t on $R(X)$ at x_0 denoted by $D_{x_0}^t$. Then given a function f in $R(X)$ there exists a set E' with full area density at 0, such that*

$$D_{x_0}^t f = \lim_{h \to 0, h \in E'} \Delta_h^t f(x_0).$$

2.2 Rational approximation in the areal mean

We will show that Theorems 2.1.1 and 2.1.5 can be extended from $R(X)$ to $R^p(X)$. Our first result is the following theorem.

Theorem 2.2.1. *For $2 < p < \infty$, suppose that there is a bounded point derivation on $R^p(X)$ at x_0 denoted by $D_{x_0}^1$. Then given a function f in $R^p(X)$, there exists a set E of full area density at x_0 such that*

$$D_{x_0}^1 f = \lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}.$$

We remark that this theorem is only valid for $2 < p < \infty$. Recall that when $1 \leq p < 2$, $R^p(X) = L^p(X)$ and thus there are no bounded point derivations on $R^p(X)$. In fact there are not even bounded point *evaluations* [4, Lemma 3.5]. This still leaves open the case of $p = 2$. It is possible for bounded point derivations on $R^2(X)$ to exist; however, we do not know whether Theorem 2.2.1 still holds for $R^2(X)$.

We will also prove the following higher order extension of Theorem 2.2.1.

Theorem 2.2.2. *Let t be a positive integer. For $2 < p < \infty$ suppose that there exists a bounded point derivation of order t on $R^p(X)$ at x_0 denoted by $D_{x_0}^t$. Then given a function f in $R^p(X)$ there exists a set E' with full area density at 0, such that*

$$D_{x_0}^t f = \lim_{h \to 0, h \in E'} \Delta_h^t f(x_0).$$

2.3 Results from measure theory

In this section, we briefly review some results from measure theory to be used in our proofs. From now on q denotes the conjugate exponent to p; that is, $q = \frac{p}{p-1}$, and dA denotes 2 dimensional Lebesgue (area) measure. Since a bounded point derivation is a bounded linear functional, it follows from Theorem 1.4.2 that there exists a function k in $L^q(X)$ such that the measure kdA represents the bounded point derivation. If the representing measure for a t-th order bounded point derivation on $R^p(X)$ is known, then it would be useful to have a method for finding the representing measures for bounded point derivations of lesser orders. The next lemma, which describes such a method, is based on a theorem of Wilken [34].

Lemma 2.3.1. *Let $1 \leq p < \infty$. Let t be a positive integer and suppose that there is a t-th order bounded point derivation on $R^p(X)$ at x_0 with representing measure $k_t dA$. For each m with $0 \leq m \leq t$, let $k_m = \frac{m!}{t!}(z - x_0)^{t-m} k_t$. Then k_m belongs to $L^q(X)$ and $k_m dA$ represents an m-th order bounded point derivation on $R^p(X)$ at x_0.*

Proof. Since k_t belongs to $L^q(X)$, k_m also belongs to $L^q(X)$. To prove that k_m represents an m-th order bounded point derivation on $R^p(X)$ at x_0, we first suppose that f is a rational function with poles off X. Hence $f(z)(z - x_0)^{t-m}$ is a rational function and integrating $f(z)(z - x_0)^{t-m}$ against the measure $k_t dA$ is the same as evaluating the t-th derivative of $f(z)(z - x_0)^{t-m}$ at $z = x_0$, which can be done using the general Leibniz rule. The only term that will not vanish is the term which puts exactly $t - m$ derivatives on $(z - x_0)^{t-m}$ and m derivatives on $f(z)$. Hence

$$\int f(z)(z - x_0)^{t-m} k_t(z) dA_z = \binom{t}{m}(t - m)! f^{(m)}(x_0) = \frac{t!}{m!} f^{(m)}(x_0),$$

and

$$\int f(z) k_m(z) dA_z = f^{(m)}(x_0).$$

Hence by Hölder's inequality, $|f^{(m)}(x_0)| \leq ||k_m||_q ||f||_p$. So there is a bounded point derivation of order m at x_0 and the measure $k_m dA$ represents the bounded point derivation.

$\square$

Lastly, we review the definitions of the Cauchy transform and Newtonian potential of a measure.

Definition 2.3.2. Let $k \in L^q(X)$.

1. The **Cauchy transform** of the measure kdA, which is denoted by $\hat{k}(x)$ is defined by

$$\hat{k}(x) = \int \frac{k(z)}{z - x} dA_z.$$

2. The **Newtonian potential** of the measure kdA, which is denoted by $\tilde{k}(x)$ is defined by

$$\tilde{k}(x) = \int \frac{|k(z)|}{|z - x|} dA_z.$$

2.4 A set with full area density at a certain point

In this section a method is given to construct a set with full area density at x_0 which also possesses the properties needed for the proofs of Theorems 2.2.1 and 2.2.2. Constructing this set can be accomplished by first listing the desired properties and then showing that the set with these desired properties has full area density at x_0.

Theorem 2.4.1. *Suppose* $1 < q < 2$. *Let* $k \in L^q(X)$, *and let* $0 < \delta_0 < 1$. *Let* E *be the set of* x *in* X *that satisfy the following properties.*

1. $\displaystyle\int_X \frac{|(x-x_0)k(z)|^q}{|z-x|^q} dA < \delta_0.$
2. $|x - x_0|\tilde{k}(x) < \delta_0.$

Then E *has full area density at* x_0.

To prove Theorem 2.4.1, we will need a few lemmas. The first lemma is an extension of a result of Browder [7, Lemma 1].

Lemma 2.4.2. *Suppose* $1 < q < 2$. *Let* $\chi_{\{x_0\}}$ *be the characteristic function of the point* x_0 *and let* m *denote 2 dimensional Lebesgue measure. For* n *positive, let* $\Delta_n = \{x : |x - x_0| < \frac{1}{n}\}$ *and let* $w_n(z) = \dfrac{1}{m(\Delta_n)} \displaystyle\int_{\Delta_n} \frac{|x-x_0|^q}{|z-x|^q} dm_x$. *Then* $w_n(z) \leq \dfrac{2}{2-q}$ *for all* z *and all* n, *and* $w_n(z) \to \chi_{\{x_0\}}$ *pointwise as* $n \to \infty$.

Proof. We first show that $w_n(z) \to \chi_{\{x_0\}}$ pointwise as $n \to \infty$. If $z = x_0$, then the integrand is identically 1 and $w_n(z) = 1$ for all n. Now suppose that $z \neq x_0$. If n is sufficiently large, then $|z - x_0| > \frac{1}{n}$ and thus z need not be in Δ_n for large n. Since the measure of Δ_n is $\frac{\pi}{n^2}$, we can rewrite $w_n(z)$ as $\dfrac{n^2}{\pi} \displaystyle\int_{\Delta_n} \frac{|x-x_0|^q}{|z-x|^q} dm_x$. In addition since x belongs to Δ_n, $|x - x_0| \leq \frac{1}{n}$. Therefore $w_n(z) \leq \dfrac{n^{2-q}}{\pi} \displaystyle\int_{\Delta_n} \frac{1}{|z-x|^q} dm_x$. If n is sufficiently large, it follows from the reverse triangle inequality that

$$|z - x| \geq \Big| |z - x_0| - |x_0 - x| \Big| \geq |z - x_0| - \frac{1}{n} > 0.$$

Thus $|z - x|^q > (|z - x_0| - \frac{1}{n})^q > 0$ and

$$w_n(z) \leq \frac{n^{2-q}}{\pi(|z-x_0| - \frac{1}{n})^q} \int_{\Delta_n} dm_x \leq \frac{n^{-q}}{(|z-x_0| - \frac{1}{n})^q},$$

which tends to 0 as $n \to \infty$. Thus if $z \neq x_0$ then $w_n(z) \to 0$ pointwise as $n \to \infty$ and hence $w_n(z) \to \chi_{\{x_0\}}$ pointwise as $n \to \infty$.

To show that $w_n(z) \leq \dfrac{2}{2-q}$ for all z and all n, we first recall the inequality

$$w_n(z) \leq \frac{n^{2-q}}{\pi} \int_{\Delta_n} \frac{1}{|z-x|^q} dm_x$$

which was proved above. Now, the value of the integral would be larger if the integration was performed over $B(z, \frac{1}{n})$, the disk with radius $\frac{1}{n}$ centered at z instead of integrating over Δ_n. Hence,

$$w_n(z) \leq \frac{n^{2-q}}{\pi} \int_{B(z,\frac{1}{n})} \frac{1}{|z-x|^q} dm_x.$$

It follows from a calculation that $\displaystyle\int_{B(z,\frac{1}{n})} \frac{1}{|z-x|^q} dm_x = \frac{2\pi n^{-(2-q)}}{2-q}$. Hence $w_n(z) \leq \dfrac{2}{2-q}$.

□

We note that it is in the above lemma, that our proof breaks down for the case of $p = 2$. If $p = 2$, then $q = 2$, but $w_n(z)$ is no longer bounded in this case since $\frac{1}{z^2}$ is not locally integrable.

Lemma 2.4.3. *Suppose* $1 < q < 2$. *Let* $\Delta_n = \left\{x \in X : |x - x_0| < \frac{1}{n}\right\}$, *let* $k \in L^q(X)$ *and let* m *denote* 2 *dimensional Lebesgue measure. Then*

$$\frac{1}{m(\Delta_n)} \int_{\Delta_n} \left\{ \int_X \frac{|x-x_0|^q |k(z)|^q}{|z-x|^q} dm_z \right\} dm_x \to 0$$

as $n \to \infty$.

Proof. Let $w_n(z)$ be as in the previous lemma. Since $w_n(z)$ is uniformly bounded for all n, $\displaystyle\int_X w_n(z)|k(z)|^q \leq C \int_X |k(z)|^q$ and because $k(z) \in L^q(X)$, it follows that this integral is bounded. Since $w_n(z) \to 0$ almost everywhere as $n \to \infty$, it follows from the dominated convergence theorem that $\displaystyle\int_X w_n(z)|k(z)|^q \to 0$ as $n \to \infty$. Recall that $w_n(z) = \dfrac{1}{m(\Delta_n)} \displaystyle\int_{\Delta_n} \frac{|x-x_0|^q}{|z-x|^q} dm_x$. Hence interchanging the order of integration yields

$$\frac{1}{m(\Delta_n)}\int_{\Delta_n}\left\{\int_X \frac{|x-x_0|^q|k(z)|^q}{|z-x|^q}dm_z\right\}dm_x \to 0$$

as $n \to \infty$. $\square$

Lemma 2.4.4. *Suppose* $1 < q < 2$. *Choose* $\delta > 0$, *let* $k \in L^q(X)$ *and let* m *denote 2 dimensional Lebesgue measure. Let*

$$E_\delta = \left\{x \in X : \int_X \frac{|x-x_0|^q|k(z)|^q}{|z-x|^q}dm_z < \delta\right\}.$$

Then E_δ *has full area density at* x_0.

Proof. It follows immediately from the definition of E_δ that

$$\frac{1}{m(\Delta_n)}\int_{\Delta_n\setminus E_\delta}\left\{\int_X \frac{|x-x_0|^q|k(z)|^q}{|z-x|^q}dm_z\right\}dm_x \geq \frac{\delta m(\Delta_n \setminus E_\delta)}{m(\Delta_n)}.$$

By Lemma 2.4.3 the left hand side tends to 0 as $n \to \infty$. Thus $\lim_{n\to\infty}\frac{m(\Delta_n \setminus E_\delta)}{m(\Delta_n)} = 0$ and E_δ has full area density at x_0.

$\square$

The proof of Theorem 2.4.1 now follows from Lemma 2.4.4.

Proof. **(Theorem 2.4.1)**

Lemma 2.4.4 immediately implies that the set of x in X where property 1 holds has full area density at x_0. To show that the set where property 2 holds also has full area density at x_0 note that by Hölder's inequality

$$\int_X \frac{|x-x_0||k(z)|}{|z-x|}dm_z \leq \left\{\int_X \frac{|x-x_0|^q|k(z)|^q}{|z-x|^q}dm_z\right\}^{\frac{1}{q}} \cdot m(X)^{\frac{1}{p}}.$$

It follows from Lemma 2.4.4 that the integral on the right is bounded. If $m(X) = 0$, then property 2 holds for any choice of $\delta_0 > 0$ and we are done. Thus we can assume that $m(X) \neq 0$. If the integral on the right hand side is less that $\frac{\delta_0}{m(X)^{\frac{1}{p}}}$ then the left hand side will be less than δ_0. This can be done by choosing $\delta = \frac{\delta_0}{m(X)^{\frac{1}{p}}}$ in Lemma 2.4.4. Thus property 2 also holds on a set with full area density at x_0 and thus the set E has full area density at x_0. $\square$

2.5 The existence of approximate derivatives

The goal of this section is to prove Theorem 2.2.1 by showing that, for $2 < p < \infty$, the existence of a bounded point derivation on $R^p(X)$ at x_0 implies that every function in $R^p(X)$ has an approximate derivative at x_0. Choose f in $R^p(X)$ and let $g(z) = f(z) - D^0_{x_0}f - D^1_{x_0}f \cdot (x - x_0)$. Then to show that $f(z)$ has an approximate derivative at x_0, it suffices to show that $g(z)$ has an approximate derivative at x_0 since $g(z)$ differs from $f(z)$ by a polynomial. The reason that it is more advantageous to work with $g(z)$ rather than $f(z)$ is that $D^0_{x_0}(g) = D^1_{x_0}(g) = 0$.

Consider the following family of linear functionals defined for every $x \in X$: $L_x(F) = \dfrac{F(x)}{x - x_0} - D^1_{x_0}F$. To prove Theorem 2.2.1, it suffices to show that there is a set E with full area density at x_0 such that $L_x(g) \to 0$ as $x \to 0$ through the points of E. Once this is shown, it follows that $\lim_{x \to x_0} \dfrac{g(x)}{x - x_0} - D^1_{x_0}g = 0$ and since $g(x_0) = 0$, this shows that g has an approximate derivative at x_0.

Since $R^p(X)$ has a bounded point derivation at x_0, there exists a function k_1 in $L^q(X)$ such that the measure $k_1 dA$ represents the bounded point derivation. Hence by Lemma 2.3.1, the function $k = (z - x_0)k_1$ belongs to $L^q(X)$ and kdA is a representing measure for x_0. Fix $0 < \delta_0 < 1$ and let E be the set of x in X that satisfies the following properties.

1. $\displaystyle\int_X \frac{|(x - x_0)k_1|^q}{|z - x|^q} dA < \delta_0.$
2. $\displaystyle\int_X \frac{|(x - x_0)k|^q}{|z - x|^q} dA < \delta_0.$
3. $|x - x_0|\tilde{k}(x) < \delta_0.$

It follows from Theorem 2.4.1 that E has full area density at x_0.

To show that $L_x(g) \to 0$ through E it is useful to consider how $g(z)$ can be approximated by rational functions with poles off X. Since f is in $R^p(X)$, there is a sequence $\{f_j\}$ of rational functions with poles off X which converges to $f(z)$ in the L^p norm. Let $g_j(z) = f_j(z) - D^0_{x_0}f_j - D^1_{x_0}f_j \cdot (x - x_0)$. Then $\{g_j\}$ is a sequence of rational functions with poles off X that possesses the following properties:

1. $\{g_j\} \to g(z)$ in the L^p norm.
2. For each j, $D^0_{x_0}g_j = D^1_{x_0}g_j = 0$.
3. $L_x(g_j) \to 0$ as $x \to x_0$.

The first two properties are easy to verify. The third property follows since $g_j(z)$ is a rational function with poles off X and thus $D^1_{x_0} g_j = g_j'(x_0)$.

It now follows from the linearity of L_x and the triangle inequality that $|L_x(g)| \leq |L_x(g - g_j)| + |L_x(g_j)|$. Hence to show that $L_x(g) \to 0$ as $x \to x_0$, it follows from property 3 that it is enough to show that $L_x(g - g_j) \to 0$ as $j \to \infty$. By property 1, it suffices to prove that there is a constant C which does not depend on x such that for all x in E, $|L_x(g - g_j)| \leq C||g - g_j||_p$. Moreover, since a bounded point derivation is already a bounded linear functional, it is enough to show that there is a constant C which does not depend on j or x such that $\left|\dfrac{g(x) - g_j(x)}{x - x_0}\right| \leq C||g - g_j||_p$. This is done in Lemma 2.5.2.

We will first need to construct a representing measure for x in E, which allows $\dfrac{g(x) - g_j(x)}{x - x_0}$ to be expressed by an integral, from which the desired bound can be obtained. To do this, we borrow a technique of Bishop [2]. Bishop showed that if μ is an annihilating measure on $R(X)$ (i.e $\int f d\mu = 0$ for all f in $R(X)$) and if the Cauchy transform $\hat{\mu}(x)$ is defined and nonzero, then the measure defined by $\dfrac{1}{\hat{\mu}(x)} \dfrac{\mu(z)}{z - x}$ is a representing measure for x. If kdA is a representing measure for x_0 on $R^p(X)$ then $(z - x_0)kdA$ is an annihilating measure on $R^p(X)$ and thus Bishop's technique can be used to construct a representing measure for x on $R^p(X)$.

Lemma 2.5.1. *Let k be a function in $L^q(X)$ such that kdA is a representing measure for x_0. Choose x in X and suppose that $|x - x_0|\tilde{k} < \delta < 1$, and that $\dfrac{(x - x_0)k}{z - x}$ belongs to $L^q(X)$. Let $c = \widehat{(z - x_0)k}(x)$ and let $k_x(z) = \dfrac{1}{c}\dfrac{(z - x_0)k(z)}{z - x}$. Then there exists a bounded point evaluation on $R^p(X)$ at x and $k_x dA$ is a representing measure for x.*

Proof. Before we begin the proof, we note a few things. First

$$c = \widehat{(z - x_0)k}(x) = \int \frac{(z - x_0)k}{z - x} dA_z = 1 + \int \frac{(x - x_0)k}{z - x} dA_z = 1 + (x - x_0)\hat{k}(x).$$

Thus $1 - |x - x_0|\tilde{k}(x) \leq |c| \leq 1 + |x - x_0|\tilde{k}(x)$ and hence, $1 - \delta \leq |c| \leq 1 + \delta$. Since $\delta < 1$, k_x is well defined. Second, k_x can also be written as follows:

$$k_x(z) = \frac{(z - x_0)k(z)}{(z - x)(1 + (x - x_0)\hat{k}(x))}.$$

Finally, $\dfrac{(z-x_0)k(z)}{z-x} = 1 + \dfrac{(x-x_0)k(z)}{z-x}$ and hence k_x belongs to $L^q(X)$.

If F is a rational function with poles off X, $\dfrac{[F(z)-F(x)](z-x_0)}{z-x}$ is also a rational function with poles off X. Since kdA is a representing measure for x_0, $\displaystyle\int \frac{[F(z)-F(x)](z-x_0)}{z-x} k(z)dA_z = 0$ and hence

$$\int \frac{F(z)(z-x_0)}{z-x} k(z)dA_z - \int \frac{F(x)(z-x_0)}{z-x} k(z)dA_z = 0.$$

Since $z - x_0 = z - x + x - x_0$, it follows that

$$\begin{aligned}\int \frac{F(x)(z-x_0)}{z-x} k(z)dA_z &= \int F(x)k(z)dA_z + \int \frac{F(x)(x-x_0)k(z)}{z-x} dA_z \\ &= F(x)(1 + (x-x_0)\hat{k}(x)).\end{aligned}$$

Hence $F(x) = \displaystyle\int \frac{F(z)(z-x_0)k(z)}{(z-x)(1+(x-x_0)\hat{k}(x))} dA_z$. So $F(x) = \int F(z)k_x(z)dA$ whenever F is a rational function with poles off X. Thus by Hölder's inequality $|F(x)| \leq ||k_x||_q ||F||_p$ and since k_x is an L^q function, it follows that x admits a bounded point evaluation on $R^p(X)$ and that $k_x dA$ is a representing measure for x.

$\square$

Lemma 2.5.2. *Suppose that x belongs to E and let j be a positive integer. Then there exists a constant C which does not depend on x or j such that* $\dfrac{|g(x)-g_j(x)|}{|x-x_0|} \leq C||g-g_j||_p$.

Proof. If x belongs to E, then the hypotheses of Lemma 2.5.1 are satisfied and $k_x dA$ is a representing measure for x. Thus

$$|g(x) - g_j(x)| = \frac{1}{|c|} \left| \int [g(z) - g_j(z)] \left(\frac{z-x_0}{z-x} \right) k(z)dA_z \right|.$$

Since $D^0_{x_0}[g(z) - g_j(z)] = 0$, it follows that $\int [g(z) - g_j(z)]k(z)dA_z = 0$. Now since

$$\int [g(z) - g_j(z)] \left(\frac{z - x_0}{z - x}\right) k(z) dA_z = \int [g(z) - g_j(z)] k(z) dA_z + \int [g(z) - g_j(z)] \left(\frac{x - x_0}{z - x}\right) k(z) dA_z,$$

we obtain that

$$|g(x) - g_j(x)| = \frac{|x - x_0|}{|c|} \left| \int [g(z) - g_j(z)] \frac{k(z)}{z - x} dA_z \right|. \tag{2.5.1}$$

Next, observe that $\frac{1}{z-x} = \frac{1}{z - x_0} + \frac{x - x_0}{(z-x)(z-x_0)}$. Applying this observation to (2.5.1) yields

$$|g(x) - g_j(x)| = \frac{|x - x_0|}{|c|} \left| \int [g(z) - g_j(z)] \frac{k(z)}{z - x_0} dA_z + \int [g(z) - g_j(z)] \frac{(x - x_0) k(z)}{(z - x)(z - x_0)} dA_z \right|. \tag{2.5.2}$$

The first integral in (2.5.2) is the same as the bounded point derivation at x_0 applied to $g(z) - g_j(z)$ which is 0, and hence

$$|g(x) - g_j(x)| = \frac{|x - x_0|}{|c|} \left| \int [g(z) - g_j(z)] \frac{(x - x_0) k_1(z)}{(z - x)} dA_z \right|.$$

Finally by Hölder's inequality,

$$\frac{|x - x_0|}{|c|} \left| \int [g(z) - g_j(z)] \frac{(x - x_0) k_1(z)}{(z - x)} dA_z \right| \leq \frac{|x - x_0|}{|c|} \|g - g_j\|_p \left\| \frac{(x - x_0) k_1}{(z - x)} \right\|_q$$

and since it follows from property 1 of E that $\left\| \frac{(x - x_0) k_1}{(z - x)} \right\|_q \leq \delta_0$, there is a constant C that does not depend on x or j such that

$$|g(x) - g_j(x)| \leq C |x - x_0| \cdot ||g - g_j||_p.$$

□

2.6 Higher order bounded point derivations

The goal of this section is to prove Theorem 2.2.2 by modifying the proof of Theorem 2.2.1. Choose f in $R^p(X)$ and let $g(z) = f(z) - D^0_{x_0} f - D^1_{x_0} f \cdot (z - x_0) - ... - \frac{1}{t!} D^t_{x_0} f \cdot (z - x_0)^t$. Then $D^m_{x_0} g = 0$ for $0 \leq m \leq t$. As before, to show that $f(z)$ has a t-th order approximate derivative at x_0 it suffices to show that $g(z)$ has a t-th order approximate derivative at x_0.

Consider the following family of linear functionals defined for every h in $\mathbb{C}$: $L_h(F) = \Delta^t_h F(x_0) - D^t_{x_0} F$. To prove Theorem 2.2.2, it suffices to show that there is a set E' with full area density at 0 such that $L_h(g) \to 0$ as $h \to 0$ through the points of E'. Once this is shown, it follows that $\lim_{h \to 0, h \in E'} \left| \Delta^t_h g(x_0) - D^t_{x_0} g \right| = 0$ and thus g has a t-th order approximate derivative at x_0.

Since there is a t-th order bounded point derivation on $R^p(X)$ at x_0, there exists a function k_t in $L^q(X)$ such that the measure $k_t dA$ represents this t-th order bounded point derivation. Hence by Lemma 2.3.1, the function $k = \frac{(z - x_0)k_t}{t!}$ belongs to $L^q(X)$ and kdA is a representing measure for x_0. Fix $0 < \delta_0 < 1$ and let E be the set of x in X that satisfies the following properties.

1. $\displaystyle\int_X \frac{|(x - x_0)k_t|^q}{|z - x|^q} dA < \delta_0.$

2. $\displaystyle\int_X \frac{|(x - x_0)k|^q}{|z - x|^q} dA < \delta_0.$

3. $|x - x_0|\tilde{k}(x) < \delta_0.$

It follows from Theorem 2.4.1 that E has full area density at x_0. Now, for $1 \leq s \leq t$, let $E_s = \{h \in \mathbb{C} : x_0 + sh \in E\}$ and let $E' = \bigcap_{s=1}^{t} E_s$. Then for each s, E_s has full area density at 0 and hence E' also has full area density at 0.

As in the previous section, to show that $L_h(g)$ tends to 0 through E' it is useful to consider how $g(z)$ can be approximated by rational functions with poles off X. Since f belongs to $R^p(X)$, there is a sequence $\{f_j\}$ of rational functions with poles off X which converges to $f(z)$ in the L^p norm. Let $g_j(z) = f_j(z) - D^0_{x_0} f_j - D^1_{x_0} f_j \cdot (x - x_0) - ... - \frac{1}{t!} D^t_{x_0} f_j \cdot (x - x_0)^t$. Then $\{g_j\}$ is a sequence of rational functions with poles off X that possesses the following properties.

1. $\{g_j\} \to g(z)$ in the L^p norm.

2. For each j, $D^m_{x_0} g_j = 0$ for $0 \leq m \leq t$.
3. $L_h(g_j) \to 0$ as $h \to 0$.

The first two properties are easy to verify. The third property follows since $g_j(z)$ is a rational function with poles off X and thus $D^t_{x_0} g_j = g_j^{(t)}(x_0)$.

It now follows from the linearity of L_h and the triangle inequality that $|L_h(g)| \leq |L_h(g - g_j)| + |L_h(g_j)|$. Hence to show that $L_h(g) \to 0$ as $h \to 0$, it follows from property 3 that it is enough to show that $L_h(g - g_j) \to 0$ as $j \to \infty$. By property 1 it suffices to prove that there is a constant C which does not depend on h such that for all h in E', $|L_h(g - g_j)| \leq C||g - g_j||_p$. Moreover, since a bounded point derivation is already a bounded linear functional, it is enough to show that there is a constant C which does not depend on j such that $|\Delta^t_h(g(x_0) - g_j(x_0))| \leq C||g - g_j||_p$. Furthermore, since the difference quotient is a finite linear combination of terms of the form $g(x_0 + sh) - g_j(x_0 + sh)$, it is enough to show that for each s between 0 and t, $|g(x_0 + sh) - g_j(x_0 + sh)| \leq C||g - g_j||_p$. This is done in Lemma 2.6.2. First, however, we prove the following factorization lemma.

Lemma 2.6.1. *Let t be a positive integer. Then*

$$\frac{1}{z-x} = \sum_{m=1}^{t} \frac{(x-x_0)^{m-1}}{(z-x_0)^m} + \frac{(x-x_0)^t}{(z-x)(z-x_0)^t}.$$

Proof. The proof is by induction. For the base case, note that

$$\frac{1}{z-x} = \frac{1}{z-x_0} + \frac{x-x_0}{(z-x)(z-x_0)}. \tag{2.6.1}$$

Now assume that we have shown that

$$\frac{1}{z-x} = \sum_{m=1}^{t-1} \frac{(x-x_0)^{m-1}}{(z-x_0)^m} + \frac{(x-x_0)^{t-1}}{(z-x)(z-x_0)^{t-1}}.$$

Then

$$\frac{1}{z-x} = \sum_{m=1}^{t-1} \frac{(x-x_0)^{m-1}}{(z-x_0)^m} + \frac{1}{z-x} \cdot \frac{(x-x_0)^{t-1}}{(z-x_0)^{t-1}},$$

and applying (2.6.1) to the $\dfrac{1}{z-x}$ term in the sum proves the lemma.

□

Lemma 2.6.2. *Suppose that h belongs to E' and let j be a positive integer. Let $0 \leq s \leq t$. Then there exists a constant C which does not depend on h or j such that* $\dfrac{|g(x_0 + sh) - g_j(x_0 + sh)|}{|h|^t} \leq C||g - g_J||_p$.

Proof. Let $x = x_0 + sh$. Then x belongs to E and the hypotheses of Lemma 2.5.1 are satisfied, so $k_x dA$ is a representing measure for x. Thus

$$|g(x) - g_j(x)| = \frac{1}{|c|} \left| \int [g(z) - g_j(z)] \left(\frac{z - x_0}{z - x} \right) k(z) dA_z \right|.$$

Since $D^0_{x_0}[g(z) - g_j(z)] = 0$, it follows that $\int [g(z) - g_j(z)] k(z) dA_z = 0$. Now since

$$\begin{aligned}\int [g(z) - g_j(z)] \left(\frac{z - x_0}{z - x} \right) k(z) dA_z &= \int [g(z) - g_j(z)] k(z) dA_z \\ &+ \int [g(z) - g_j(z)] \left(\frac{x - x_0}{z - x} \right) k(z) dA_z,\end{aligned}$$

we obtain that

$$|g(x) - g_j(x)| = \frac{|x - x_0|}{|c|} \left| \int [g(z) - g_j(z)] \frac{k(z)}{z - x} dA_z \right|. \tag{2.6.2}$$

Applying Lemma 2.6.1 to the $\dfrac{k(z)}{z - x}$ term in the rightmost integral in (2.6.2) shows that

$$\begin{aligned}|g(x) - g_j(x)| - = \frac{|x - x_0|}{|c|} \Bigg| \sum_{m=1}^{t} \int [g(z) - g_j(z)] \frac{(x - x_0)^{m-1} k(z)}{(z - x_0)^m} dA_z \\ + \int [g(z) - g_j(z)] \frac{(x - x_0)^t k(z)}{(z - x)(z - x_0)^t} dA_z \Bigg|.\end{aligned}$$

We can factor out the powers of $x - x_0$ from each integral since integration is with respect to z. Thus each integral in the sum is of the form $\int [g(z) - g_j(z)] \dfrac{k(z)}{(z - x_0)^m} dA$ where $1 \leq m \leq t$. This integral simplifies to $\int [g(z) - g_j(z)] \dfrac{(z - x_0)^{t-m} k_t(z)}{t!} dA_z$ and by Lemma 2.3.1, the integral reduces to a constant times the m-th order bounded point derivation of $g(z) - g_j(z)$, which is 0 for $1 \leq m \leq t$. Hence

$$|g(x) - g_j(x)| = \frac{|x - x_0|}{|c|t!} \left| \int [g(z) - g_j(z)] \frac{(x - x_0)^t k_t(z)}{(z - x)} dA_z \right|,$$

which simplifies to

$$\frac{|x - x_0|^t}{|c|t!} \left| \int [g(z) - g_j(z)] \frac{(x - x_0) k_t(z)}{(z - x)} dA_z \right|.$$

Finally by Hölder's inequality,

$$\frac{|x - x_0|^t}{|c|t!} \left| \int [g(z) - g_j(z)] \frac{(x - x_0) k_t(z)}{(z - x)} dA_z \right| \leq \frac{|x - x_0|^t}{|c|t!} \|g - g_j\|_p \left\| \frac{(x - x_0) k_t}{(z - x)} \right\|_q$$

and since it follows from property 1 of E that $\left\| \frac{(x - x_0)k}{(z - x)(z - x_0)^t} \right\|_q \leq \delta_0$, there is a constant C that does not depend on h or j such that

$$|g(x) - g_j(x)| \leq C|x - x_0|^t ||g - g_j||_p.$$

Since $x = x_0 + sh$, it follows that $|g(x_0 + sh) - g_j(x_0 + sh)| \leq C|s|^t |h|^t ||g - g_j||_p$ and thus

$$\frac{|g(x_0 + sh) - g_j(x_0 + sh)|}{|h|^t} \leq C||g - g_j||_p.$$

□

2.7 The set of full area density

We saw in Section 2.1.2 that if there is a bounded point derivation on $R(X)$ at a point x_0, then there exists a set E with full area density such that

$$D_{x_0} f = \lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}, \tag{2.7.1}$$

for all f in $R(X)$, and that a similar result holds for $R^p(X)$.

Not much can be said about the structure of E and it would be of great importance to be able to replace this set with one that is better understood. Since Lemma 2.5.1

shows that there is a bounded point derivation for $R(X)$ at every $x \in E$, a reasonable candidate to replace E is the set of $x \in X$ that admit bounded point evaluations for $R(X)$ or $R^p(X)$. However, we will show that this cannot occur in general by giving an example of a set X, a point $x_0 \in X$ which admits a bounded point derivation for $R(X)$, and sequence of interior points of X which converges to x_0 such that (2.7.1) does not hold. Since interior points admit bounded point evaluations (in fact they admit bounded point derivations of all orders) this shows that the set E cannot be replaced with the set of bounded point evaluations on X.

We first construct the set X. For each positive integer n, let $x_n = \dfrac{3}{2^{n+2}} \in A_n(0)$ and let $r_n = 4^{-n} \cdot n^{-2}$. Let B_n denote the ball centered at x_n with radius r_n. Now let D denote the open unit disk, and let $X = D \setminus \bigcup_n B_n$. It follows from Theorem 1.3.1 that $R(X)$ admits a bounded point derivation at 0

Theorem 2.7.1. *Let X be the set constructed in the above paragraph. For each positive integer m, let $y_m = \dfrac{3}{2^{m+2}} + 4^{-m}$. Then there exists $f \in R(X)$ such that*

$$D_0 f \neq \lim_{m\to\infty} \frac{f(y_m) - f(0)}{y_m}.$$

Proof. Let $f_N(z) = \displaystyle\sum_{n=1}^{N} \frac{-1}{z - x_n} \cdot 4^{-n} \cdot n^{-4}$. If $z \in X$, then $|z - x_n| > 4^{-n} \cdot n^{-2}$. Hence $|f_N(z)| \leq \displaystyle\sum_{n=1}^{N} n^{-2}$ and it follows from the Weierstrass M-test that $f_N(z)$ converges uniformly to a limit function $f(z) \in R(X)$.

It follows from the definition of a bounded point derivation that $D_0 f = \lim_{N\to\infty} f_N'(0)$. Hence

$$\begin{aligned} D_0 f &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{x_n^2} \cdot 4^{-n} \cdot n^{-4} \\ &= \sum_{n=1}^{\infty} \frac{16}{9} \cdot n^{-4} \\ &= \frac{16}{9} \cdot \frac{\pi^4}{90}. \end{aligned}$$

We will prove the theorem by showing that $\dfrac{f(y_m) - f(0)}{y_m}$ grows without bound as m gets large. A computation shows that

$$\frac{f(y_m)-f(0)}{y_m} = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{-1}{y_m - x_n} \cdot 4^{-n} \cdot n^{-4} - \sum_{n=1}^{\infty} \frac{1}{x_n} \cdot 4^{-n} \cdot n^{-4}}{y_m}$$

$$= \frac{\displaystyle\sum_{n=1}^{\infty} \frac{-y_m}{(y_m - x_n)x_n} \cdot 4^{-n} \cdot n^{-4}}{y_m}$$

$$= \sum_{n=1}^{\infty} \frac{-1}{(y_m - x_n)x_n} \cdot 4^{-n} \cdot n^{-4}.$$

This sum is real, but has positive and negative terms. The terms are positive if $n < m$ and negative when $n \geq m$. Note that the m-th term of this sum is

$$\frac{-1}{(y_m - x_m)x_m} \cdot 4^{-m} \cdot m^{-4} = \frac{-2^{m+2}}{3m^4},$$

and this term grows without bound as m gets large. We will show that the sum of the positive terms is bounded independent of m and hence the difference quotient grows without bound as m gets large.

A computation shows that the $m-1$ term is

$$\frac{-1}{(4^{-m} + \frac{3}{2^{m+2}} - \frac{3}{2^{m+1}}) \cdot \frac{3}{2^{m+1}}} \cdot 4^{-(m-1)} \cdot (m-1)^{-4} = \frac{-2^{m+1} \cdot 4^{-(m-1)} \cdot (m-1)^{-4}}{3 \cdot (4^{-m} - \frac{3}{2^{m+2}})}$$

$$= \frac{-2^{-m-1} \cdot (m-1)^{-4}}{3 \cdot 2^{-m-2} \cdot (2^{-m+2} - 3)} = \frac{-2 \cdot (m-1)^{-4}}{3 \cdot (2^{-m+2} - 3)}$$

Hence the $m-1$ term is bounded by $\frac{1}{3}$. We now suppose that $n < m-1$. The n-th term of the sum is

$$\frac{-4^{-n} n^{-4}}{(4^{-m} + \frac{3}{2^{m+2}} - \frac{3}{2^{n+2}}) \cdot \frac{3}{2^{n+2}}} = \frac{\frac{4}{3} \cdot 2^{-n} n^{-4}}{\frac{3}{2^{n+2}} - 4^{-m} - \frac{3}{2^{m+2}}}.$$

Since $4^{-m} \leq \frac{3}{2^{m+2}}$, it follows that

$$\frac{\frac{4}{3}\cdot 2^{-n}n^{-4}}{\frac{3}{2^{n+2}} - 4^{-m} - \frac{3}{2^{m+2}}} \leq \frac{\frac{4}{3}\cdot 2^{-n}n^{-4}}{\frac{3}{2^{n+2}} - \frac{6}{2^{m+2}}} \leq \frac{\frac{4}{3}\cdot 2^{-n}n^{-4}}{\frac{3}{4}\left(\frac{2^{m-1}-2^{n}}{2^{n+m-1}}\right)} = \frac{\frac{16}{9}\cdot 2^{m-1}n^{-4}}{2^{m-1}-2^{n}}.$$

Since $2^{m-1} - 2^n \geq 2^{m-2}$ when $n < m-1$, it follows that

$$\frac{\frac{16}{9}\cdot 2^{m-1}n^{-4}}{2^{m-1}-2^{n}} \leq \frac{32}{9}n^{-4}$$

Thus the sum of the positive terms is bounded by

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{32}{9}n^{-4} = \frac{1}{3} + \frac{32}{9}\cdot\frac{\pi^4}{90}.$$

while the m-th term grows unboundedly as m gets large. Hence

$$D_0 f \neq \lim_{m\to\infty} \frac{f(y_m) - f(0)}{y_m}.$$

$\square$

Chapter 3 Difference quotient formulas and interior cones

3.1 Uniform rational approximation

In the previous chapter we saw that we could evaluate a bounded point derivation on $R(X)$ at x_0 using the following difference quotient formula

$$Df = \lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0},$$

where E is a set with full area density at x_0. One drawback of this result is that it doesn't provide any information about the structure of the set E; it only shows that such a set exists. For this reason, it would be preferable to have a formula which provided more explicit information concerning the set that the limit is taken over. Such a formula was discovered by O'Farrell [26, Corollary 3], who found a different representation for a bounded point derivation on $R(X)$ provided that X satisfies an additional geometric condition. We say that X has an interior cone at x_0 if there is a segment J ending at x_0 and a constant $k > 0$ such that $\text{dist}(x, \partial X) \geq k|x - x_0|$ for all x in J. The segment J is called a non-tangential ray to x_0. O'Farrell proved the following theorem.

Theorem 3.1.1. *Suppose that there is a bounded point derivation on $R(X)$ at x_0, which we denote by $D^1_{x_0}$, and that X has an interior cone at x_0 and J is a non-tangential ray to x_0. Then*

$$D^1_{x_0} f = \lim_{x \to x_0, x \in J} \frac{f(x) - f(x_0)}{x - x_0}.$$

Although O'Farrell's result requires an additional hypothesis compared to Theorem 2.1.1, it has the advantage of being more concrete, as the set where the limit is taken over is clearly described. However, the set that the limit is taken over does not have full area density. Nevertheless, it is a subset of a set of full area density over which a derivative can be computed.

If we let d_{x_0} denote the operation of taking a limit of the difference quotient over a non-tangential ray and suppose that $\{f_n\}$ is a sequence of rational functions with poles off X that converges to f uniformly, then we can interpret the result of O'Farrell as saying that the following diagram commutes.

$$\begin{array}{ccc} f_j & \xrightarrow[j\to\infty]{\lim} & f \\ \Big\downarrow{\scriptstyle d_{x_0}} & & \Big\downarrow{\scriptstyle d_{x_0}} \\ f_j'(x_0) & \xrightarrow[j\to\infty]{\lim} & D^1_{x_0} f \end{array}$$

3.2 A constructive proof

O'Farrell's proof of Theorem 3.1.1 uses duality arguments and abstract measures, as well as results from functional analysis such as the Riesz representation theorem. We will provide a different proof of this result which is constructive in nature, making direct use of the Cauchy integral formula.

Because of the length of the proof, it is broken into a series of smaller results. The strategy of the proof is as follows. First, we define a family of bounded linear functionals by $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - D^1_{x_0} f$ where x is a fixed point in J. To prove the theorem, it is enough to show that the linear functionals $L_x(f)$ tend to the 0 functional as $x \to x_0$ through the points of J. Now given a function f in $R(X)$, there exists a sequence $\{f_j\}$ of rational functions which converges to f uniformly. Thus by linearity and the triangle inequality, $|L_x(f)| \leq |L_x(f - f_j)| + |L_x(f_j)|$. We claim that for x in J, $|L_x(f - f_j)| \leq C||f - f_j||_X$ where $|| \cdot ||_X$ denotes the sup norm on X and the constant C does not depend on x. Assuming the claim for a moment, we see that since $f_j \to f$ uniformly, $L_x(f - f_j) \to 0$ as $j \to \infty$ independent of x. Now since each f_j is a rational function with poles off X, $D^1_{x_0} f_j = f_j'(x_0)$ and thus $L_x(f_j) \to 0$ as $x \to x_0$. It thus follows that $L_x(f)$ tends to the 0 functional as $x \to x_0$.

To prove the claim, note that since a bounded point derivation is a bounded linear functional, we only need to prove the bound for the difference quotient term of $L_x(f)$. Hence it is enough to show that $\dfrac{|f(x) - f(x_0)|}{|x - x_0|} \leq C||f||_X$ for all f in $R(X)$, where the constant C does not depend on x or f. We will first prove this bound for rational functions with poles off X and then extend the result to arbitrary functions in $R(X)$.

Lemma 3.2.1. *Suppose that X has an interior cone at x_0 and let J be a non-tangential ray to x_0. Suppose that $R(X)$ has a bounded point derivation at x_0, and let f_j be a rational function with poles off X. Then for all x in X,*

$$\frac{|f_j(x) - f_j(x_0)|}{|x - x_0|} \leq C||f_j||_X, \tag{3.2.1}$$

where the constant C does not depend on x or f_j.

The proof of Lemma 3.2.1 almost follows directly from the cone condition and the definition of a bounded point derivation. For, if there is a bounded point derivation on $R(X)$ at x_0, then there exists a constant k such that $|f_j'(x_0)| \leq k||f_j||_X$ for all rational functions f_j with poles off X. Let U be an open neighborhood of x_0 on which f is analytic. Then by the Cauchy integral formula

$$\left|\frac{1}{2\pi i}\int_{\partial U}\frac{f_j(z)}{(z-x_0)^2}dz\right| \leq k||f_j||_X. \tag{3.2.2}$$

Now it also follows from the Cauchy integral formula that

$$\left|\frac{f_j(x)-f_j(x_0)}{(x-x_0)}\right| \leq \frac{1}{2\pi}\int_{\partial U}\frac{|f_j(z)|}{|z-x_0|\cdot|z-x|}dz.$$

If there is an interior cone at x_0 and if x lies on a non-tangential ray to x_0, then there exists a constant $k > 0$ such that $\dfrac{|x-x_0|}{|z-x|} < k^{-1}$ for all x in J, which implies that $\dfrac{|z-x_0|}{|z-x|} < 1+k^{-1}$. Thus $\dfrac{1}{|z-x|\cdot|z-x_0|} \leq \dfrac{1+k^{-1}}{|z-x_0|^2}$ and hence,

$$\left|\frac{f_j(x)-f_j(x_0)}{(x-x_0)}\right| \leq \frac{1+k^{-1}}{2\pi}\int_{\partial U}\frac{|f_j(z)|}{|z-x_0|^2}dz.$$

So the right hand side is almost, but not quite, the same as the left hand side of (3.2.2). If it was the same, then Lemma 3.2.1 would follow immediately, but as it is, a different method is required.

Proof (Lemma 3.2.1). We first make a couple of preliminary observations. First, since f_j is a rational function with poles off X, there exists a neighborhood U of X such that f_j is analytic on U. Let B_n denoted the ball centered at x_0 with radius 2^{-n}. Then there exists an integer $N > 0$ such that U contains B_N and hence f_j is analytic inside the ball B_N. In addition, there also exists an integer $M < 0$ such that U is itself contained inside the ball B_M. Now, we can modify f_j so that it is continuous on B_M but still analytic on U and by multiplication with a cutoff function, we can make it so that the modified function is 0 on the boundary of B_M. Thus there exists a function $\tilde{f}_j$ such that

1. $\tilde{f}_j$ is continuous on B_M.
2. $\tilde{f}_j = f_j$ on U.
3. $\tilde{f}_j = 0$ on the circle $|z-x_0| = 2^{-M}$.
4. $||\tilde{f}_j||_X \leq 2||f_j||_X$

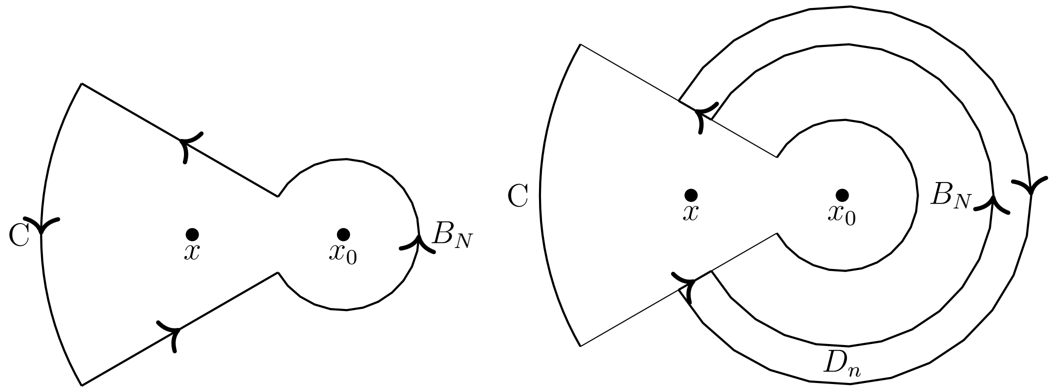

Figure 3.1: The contour of integration for Theorem 3.1.1

Recall that J is a non-tangential ray to x_0. Since X has an interior cone, it follows that there is a sector in $\mathring{X}$ with vertex at x_0 that contains J and a constant $k > 0$ such that $\text{dist}(x, \partial X) \geq k|x - x_0|$ for all $x \in J$. Let C denote this sector. It follows from the Cauchy integral formula and the construction of $\tilde{f}_j$ that

$$\frac{f_j(x) - f_j(x_0)}{x - x_0} = \frac{1}{2\pi i} \int_{\partial(C \bigcup B_N)} \frac{f_j(z)}{(z - x)(z - x_0)} dz = \frac{1}{2\pi i} \int_{\partial(C \bigcup B_N)} \frac{\tilde{f}_j(z)}{(z - x)(z - x_0)} dz$$

where the boundary is oriented so that the interior of $C \bigcup B_N$ is always to the left of the path of integration. (See Figure 3.1.) Let $D_n = A_n \setminus C$. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial(C \bigcup B_N)} \frac{\tilde{f}_j(z)}{(z - x)(z - x_0)} dz &= \frac{1}{2\pi i} \sum_{n=M}^{N} \int_{\partial D_n} \frac{\tilde{f}_j(z)}{(z - x)(z - x_0)} dz \\ &+ \frac{1}{2\pi i} \int_{|z - x_0| = 2^{-M}} \frac{\tilde{f}_j(z)}{(z - x)(z - x_0)} dz. \end{aligned}$$

Since $\tilde{f}_j = 0$ on $|z - x_0| < 2^{-M}$, the last integral vanishes and hence

$$\frac{f_j(x) - f_j(x_0)}{x - x_0} = \frac{1}{2\pi i} \sum_{n=M}^{N} \int_{\partial D_n} \frac{\tilde{f}_j(z)}{(z - x)(z - x_0)} dz. \tag{3.2.3}$$

Applying Mel'nikov's integral estimate (Theorem 1.1.1) to the (pairwise similar) regions D_n yields that there exists a constant $C > 0$ such that for $n = 1, 2, \cdots$,

$$\left| \int_{\partial D_n} g(z) dz \right| \leq C ||g||_{D_n} \gamma(D_n \cap K), \tag{3.2.4}$$

where g is continuous on D_n and analytic on $D_n \setminus K$. (See [25] for another use of this estimate.) Since $\dfrac{\tilde{f}_j}{(z-x)(z-x_0)}$ is continuous on D_n, and analytic on X, the above estimate implies that

$$\left| \int_{\partial D_n} \frac{\tilde{f}_j(z)}{(z-x)(z-x_0)} dz \right| \leq C \sup_{z \in D_n} \left\{ \frac{\tilde{f}_j(z)}{(z-x)(z-x_0)} \right\} \gamma(D_n \setminus X). \tag{3.2.5}$$

Since J is a non-tangential ray to x_0, there exists a constant k, such that $\dfrac{|x-x_0|}{|z-x|} \leq k^{-1}$ for all x in J and $z \notin C$. Thus on D_n, $\dfrac{|z-x_0|}{|z-x|} \leq 1 + \dfrac{|x-x_0|}{|z-x|} \leq 1 + k^{-1}$ and $\dfrac{1}{|z-x| \cdot |z-x_0|} \leq \dfrac{1+k^{-1}}{|z-x_0|^2}$. Hence on D_n

$$\left| \frac{\tilde{f}_j(z)}{(z-x)(z-x_0)} \right| \leq \frac{(1+k^{-1})|\tilde{f}_j(z)|}{|z-x_0|^2} \leq 4^n ||\tilde{f}_j||_X. \tag{3.2.6}$$

It follows from (3.2.5) and (3.2.6) that

$$\left| \int_{\partial D_n} \frac{\tilde{f}_j(z)}{(z-x)(z-x_0)} dz \right| \leq C ||\tilde{f}_j||_X 4^n \gamma(A_n \setminus X). \tag{3.2.7}$$

Applying this estimate to (3.2.3) yields

$$\left| \frac{f_j(x) - f_j(x_0)}{x - x_0} \right| \leq C ||\tilde{f}_j||_X \sum_{n=1}^{\infty} 4^n \gamma(A_n \setminus X).$$

Thus by Hallstrom's Theorem (Theorem 1.3.1) we have that

$$\left| \frac{f_j(x) - f_j(x_0)}{x - x_0} \right| \leq C ||\tilde{f}_j||_X,$$

where the constant C does not depend on x. (3.2.1) then follows from the above inequality and the fact that $||\tilde{f}_j||_X \leq 2||f_j||_X$. □

Thus we have shown that (3.2.1) holds for all rational functions with poles off X. The last step in proving Theorem 3.1.1 is to extend (3.2.1) to all functions in $R(X)$. This proves the claim at the beginning of section 2, and completes the proof of Theorem 3.1.1.

Lemma 3.2.2. *Suppose that there is a bounded point derivation on $R(X)$ at x_0 and also suppose that X has an interior cone at x_0. Let J be a non-tangential ray to x_0. Then for every function f in $R(X)$,*

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} \leq C||f||_X.$$

Proof. Let $\{f_j\}$ be a sequence of rational functions that converges uniformly to f. Then by Lemma 3.2.1, (3.2.1) holds. It follows from uniform convergence that $f_j(x_0) \to f(x_0)$ and $f_j(x) \to f(x)$ as $j \to \infty$. Hence taking the limit of both sides of (3.2.1) yields

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} \leq C||f||_X.$$

□

3.3 An example

In this section, we give a concrete example of how Theorem 3.1.1 can be used to determine the value of a bounded point derivation evaluated at a particular function. For each $n > 1$ let $r_n = \frac{1}{4^n} \cdot \frac{1}{2^n}$, and let $x_n = \frac{3}{2^{n+2}}$. Then $B_n := B_{r_n}(x_n) \subseteq A_n$. Let Δ denote the open unit disk and let $X = \overline{\Delta} \setminus \bigcup_{n=1}^{\infty} B_n$. It follows from Theorem 1.3.1 that 0 admits a bounded point derivation for $R(X)$.

Now let

$$f_N(z) = \sum_{n=1}^{N} \frac{1}{z - x_n} \cdot \frac{1}{4^n} \cdot \frac{1}{6^n}.$$

If z is in X then $|z - x_n| \geq r_n$ and hence

$$f_N(z) \leq \sum_{n=1}^{N} 4^n \cdot 2^n \cdot \frac{1}{4^n} \cdot \frac{1}{6^n} = \sum_{n=1}^{N} \frac{1}{3^n}.$$

Thus it follows from the Weierstrass M-test that $f_N(z)$ converges uniformly to a limit function $f(z)$. Thus $f(z)$ belongs to $R(X)$. We will use Theorem 3.1.1 to evaluate the value of the bounded point derivation applied to $f(z)$. First note that since the x_n all have positive real parts, X contains an interior cone whose midline, henceforth denoted by J, is the negative real axis. J is thus a non-tangential ray to x_0 and so

$$
\begin{aligned}
D_0^1 f &= \lim_{x\to 0, x\in J} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0, x\in J} \frac{\displaystyle\sum_{n=1}^{\infty} \frac{1}{x-x_n}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n} - \sum_{n=1}^{\infty}\frac{1}{0-x_n}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n}}{x} \\
&= \lim_{x\to 0, x\in J} \frac{\displaystyle\sum_{n=1}^{\infty} \frac{x}{x_n(x-x_n)}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n}}{x} \\
&= \lim_{x\to 0, x\in J} \sum_{n=1}^{\infty} \frac{1}{x_n(x-x_n)}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n}.
\end{aligned}
$$

The last sum converges uniformly when x is in J and thus we can interchange the sum and the limit. Hence

$$D_0^1 f = \sum_{n=1}^{\infty} \frac{-1}{x_n^2}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n}.$$

Since $x_n^2 = \frac{9}{16}\cdot\frac{1}{4^n}$, it follows that

$$D_0^1 f = \frac{-16}{9}\sum_{n=1}^{\infty}\frac{1}{6^n},$$

and by applying the formula for the sum of a geometric series starting at 1, we have that

$$D_0^1 f = \frac{-16}{9}\cdot\frac{1}{5} = \frac{-16}{45}.$$

We now compute the value of $D_0^1 f$ using the definition of a bounded point derivation to show that the two are the same.

$$D_0^1 f = \lim_{N\to\infty} f_N'(0) = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{-1}{x_n^2}\cdot\frac{1}{4^n}\cdot\frac{1}{6^n},$$

which we saw converges to $\dfrac{-16}{45}$, so both Theorem 3.1.1 and the definition can be used to evaluate the bounded point derivation.

3.4 Higher order derivations

In this section, we show how our proof of Theorem 3.1.1 can be modified to apply to the higher order case. We will show that when $R(X)$ has a t-th order bounded point derivation at x_0 and X has an interior cone at x_0, then the bounded point derivation can be represented by a higher order difference quotient where the limit is taken over a non-tangential ray to x_0. We use the same definitions and notations for higher order notations as in Definition 2.1.2 in this section. The higher order formulation of Theorem 3.1.1 is as follows.

Theorem 3.4.1. *Suppose that $R(X)$ has a t-th order bounded point derivation at x_0, which we denote by $D^t_{x_0}$ and that X has an interior cone at x_0. Let J be a non-tangential ray to x_0. Then*

$$D^t_{x_0} f = \lim_{h \to 0, x_0+h \in J} \Delta^t_h f(x_0).$$

To prove Theorem 3.4.1, we will make use of the following lemma which provides a Cauchy integral formula for higher order difference quotients.

Lemma 3.4.2. *Let f be an analytic function on an open set U containing x. Suppose that h is chosen so that $x+h$, $x+2h$,..., $x+th$ all belong to U. Then*

$$\Delta^t_h f(x) = \frac{t!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-x)(z-x-h)...(z-x-th)} dz.$$

Proof. The proof is by induction. When $t = 0$, then Theorem 3.4.2 is the usual Cauchy integral formula. Now we assume that it is true that

$$\Delta^t_h f(x) = \frac{t!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-x)(z-x-h)...(z-x-th)} dz,$$

and we will show that

$$\Delta^{t+1}_h f(x) = \frac{(t+1)!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-x)(z-x-h)...(z-x-th)(z-x-(t+1)h)} dz.$$

We saw in the proof of Theorem 2.1.3 that $\Delta^{t+1}_h f(x) = \dfrac{\Delta^t_h f(x+h) - \Delta^t_h f(x)}{h}$. Thus by the induction hypothesis,

$$\Delta_h^{t+1}f(x) = \frac{1}{h}\left\{\frac{t!}{2\pi i}\int_{\partial U}\frac{f(z)}{(z-x-h)(z-x-2h)...(z-x-(t+1)h)}dz\right.$$
$$\left.-\frac{t!}{2\pi i}\int_{\partial U}\frac{f(z)}{(z-x)(z-x-h)...(z-x-th)}dz\right\},$$

and hence it follows that

$$\Delta_h^{t+1}f(x) = \frac{(t+1)!}{2\pi i}\int_{\partial U}\frac{f(z)}{(z-x)(z-x-h)...(z-x-th)(z-x-(t+1)h)}dz,$$

which completes the proof.

□

The proof of Theorem 3.4.1 can then be obtained by making a few modifications to our proof of Theorem 3.1.1. This time define a family of linear functionals by $L_h(f) = \Delta_h^t f(x_0) - D_{x_0}^t f$ where $h \in \mathbb{C}$. Then to prove Theorem 3.4.1, it suffices to show that the linear functionals L_h converge to the 0 functional as $h \to 0$. Now given a function f in $R(X)$, there is a sequence $\{f_j\}$ of rational functions such that f_j converges to f uniformly. By the linearity of L_h and the triangle inequality, $|L_h(f)| \leq |L_h(f-f_j)| + |L_h(f_j)|$. Since $L_h(f_j) \to 0$ as $h \to 0$ whenever f_j is a rational function, it is enough to show that $|L_h(f-f_j)| \leq C||f-f_j||_X$, where C does not depend on h or j. Furthermore, since $D_{x_0}^t$ is a bounded linear functional, it suffices to show that $|\Delta_h^t f(x_0)| \leq C||f||_X$ for all f in $R(X)$. As in the proof of Theorem 3.1.1, this can be done by first proving the result for rational functions with poles off X and then taking limits on both sides of the equation to obtain the general result.

Proving the result for rational functions is done in the same way as Lemma 3.2.1 except that one has to use Lemma 3.4.2 to obtain an integral formula for the difference quotient. The remainder of the proof of Theorem 3.4.1 follows in the same manner as the proofs of Lemmas 3.2.1 and 3.2.2.

3.5 Convergence in the Gleason metric

We now adapt the techniques of this chapter to give a shorter and more constructive proof of a result of O'Farrell concerning convergence in the Gleason metric. Let $||\cdot||$ denote the sup norm on X. We define the t-th order Gleason metric on X as follows.

$$d^t(x,y) = \sup\left\{|f^{(t)}(x) - f^{(t)}(y)| : f \in R_0(X), ||f|| \leq 1\right\}.$$

The following theorem was proven by O'Farrell [26].

Theorem 3.5.1. *Suppose $t = 0$ and x is not a peak point for $R(X)$ or $t \geq 1$ and $R(X)$ admits a t-th order bounded point derivation at x. Suppose there is a positive constant k and a sequence of points $\{y_n\} \subseteq \mathring{X}$, which converges to x in the Euclidean norm, such that*

$$\text{dist }[y_n, \partial X] \geq k|y_n - x|,$$

for $n = 1, 2, \cdots$. Then $\{y_n\}$ converges to x in the d^t metric.

The original proof of Theorem 3.5.1 was rather non-constructive making use of abstract representing measures. Our objective is to provide a simpler constructive proof of this result. Instead of utilizing representing measures we will make direct use of the Cauchy integral formula as well as necessary geometric conditions for the existence of bounded point derivations.

Proof. The hypothesis immediately implies that for each n there exists a cone contained in X with vertex at x such that y_n lies on the midline of the cone and that for all $z \notin X$ and $n = 1, 2, \cdots$,

$$|y_n - x| \leq k^{-1}|z - y_n|,$$

and by applying the triangle inequality, we also obtain

$$|z - x| \leq (k^{-1} + 1)|z - y_n|. \tag{3.5.1}$$

Now choose $f \in R_0(X)$ and let $g(z) = f(z) - \dfrac{f^{(t)}(x)z^t}{t!}$. Then $g(z) \in R_0(X)$, $||g||$ is bounded and $g^{(t)}(y_n) = f^{(t)}(y_n) - f^{(t)}(x)$. Thus it is enough to show that $|g^{(t)}(y_n)| \to 0$ as $n \to \infty$. Since g is a rational function with poles off X, then there exists a neighborhood U of X such that g is analytic on U. Let B_j denoted the ball centered at x with radius 2^{-j}. Then there exists an integer $N > 0$ such that U contains B_N, and hence g is analytic inside the ball B_N. Now fix y_n and let C_n denote a cone in X with vertex at x whose midline passes through y_n. Let $M(n)$ be the smallest integer so that C_n is entirely contained in the ball B_M. Now, we can modify g so that it is continuous on B_M but still analytic on $U \cap B_M$ and by multiplication with a cutoff function, we can make it so that the modified function is 0 on the boundary of B_M. Thus for each n there exists a function g_n such that

1. g_n is continuous on B_M.

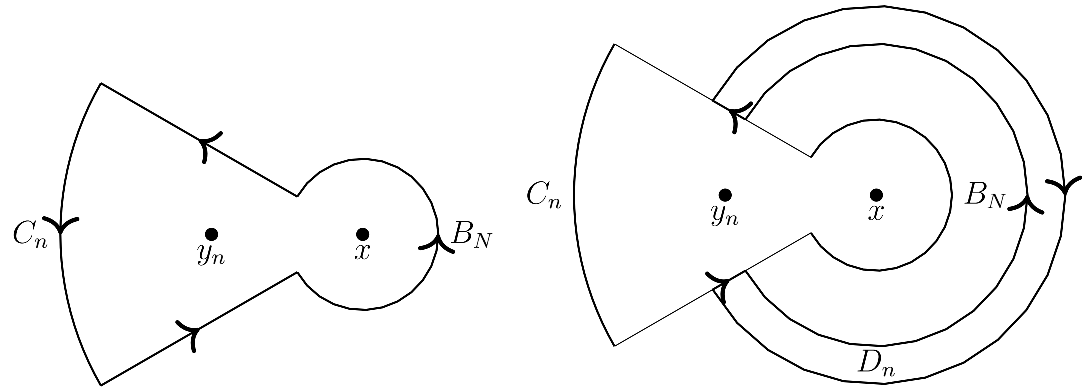

Figure 3.2: The contour of integration for Theorem 3.5.1

2. $g_n = g$ on $U \cap B_M$.
3. $g_n = 0$ on the circle $|z - x| = 2^{-M}$.
4. $\sup_X g_n \le 2 \sup_X g$.

It then follows from the Cauchy integral formula and the construction of g_n that

$$g^{(t)}(y_n) = \frac{t!}{2\pi i} \int_{\partial(C_n \bigcup B_N)} \frac{g(z)}{(z - y_n)^{t+1}} dz = \frac{t!}{2\pi i} \int_{\partial(C_n \bigcup B_N)} \frac{g_n(z)}{(z - y_n)^{t+1}} dz,$$

where the boundary is oriented so that the interior of $C_n \bigcup B_N$ is always to the left of the path of integration. (See Figure 3.2.) We denote by A_m the annulus $\{z \in \mathbb{C} : \frac{1}{2^{m+1}} < |z - x| < \frac{1}{2^m}\}$ and let $D_m = A_m \setminus C_n$. Then

$$\begin{aligned} \frac{t!}{2\pi i} \int_{\partial(C_n \bigcup B_N)} \frac{g_n(z)}{(z - y_n)^{t+1}} dz = & \frac{t!}{2\pi i} \sum_{m=M}^{N} \int_{\partial D_m} \frac{g_n(z)}{(z - y_n)^{t+1}} dz \\ & + \frac{t!}{2\pi i} \int_{|z - x_0| = 2^{-M}} \frac{g_n(z)}{(z - y_n)^{t+1}} dz. \end{aligned}$$

Since $g_n = 0$ on $|z - x_0| < 2^{-M}$, the last integral vanishes and hence

$$g^{(t)}(y_n) = \frac{t!}{2\pi i} \sum_{m=M}^{N} \int_{\partial D_m} \frac{g_n(z)}{(z - y_n)^{t+1}} dz.$$

Note that the integrand is continuous on D_m and analytic on X; thus we can apply Mel'nikov's integral estimate for D_n (Equation 3.2.4) to obtain

$$|g^{(t)}(y_n)| \leq C \sum_{m=M}^{N} \sup_{D_m} \left\{ \frac{g_n(z)}{(z-y_n)^{t+1}} \right\} \gamma(D_m \setminus X).$$

It follows from (3.5.1) that

$$\begin{aligned}
\sup_{D_m} \left\{ \frac{g_n(z)}{(z-y_n)^{t+1}} \right\} &\leq ||g_n|| \cdot \sup_{D_m} \left\{ \frac{1}{|z-y_n|^{t+1}} \right\} \\
&\leq ||g_n|| \cdot \sup_{D_m} \left\{ \frac{(k^{-1}+1)^{t+1}}{|z-x|^{t+1}} \right\} \\
&\leq ||g_n|| \cdot (k^{-1}+1)^{t+1} \cdot 2^{(t+1)n}.
\end{aligned}$$

Hence

$$\begin{aligned}
|g^{(t)}(y_n)| &\leq C||g_n|| \sum_{m=M}^{N} 2^{(t+1)n} \gamma(D_m \setminus X) \\
&\leq C||g_n|| \sum_{m=M}^{\infty} 2^{(t+1)n} \gamma(A_m \setminus X).
\end{aligned}$$

As $n \to \infty$, $y_n \to x$ and thus $M \to \infty$. It thus follows from Theorem 1.3.1 that the above sum tends to 0 as $n \to \infty$. Thus $\{y_n\} \to x$ in the d^t metric.

$\square$

Chapter 4 Ideas for future work

I plan on pursuing my research in several directions as there are still several interesting problems that remain unsolved. First, as previously mentioned, a remarkable outstanding problem is to determine whether Theorem 2.2.1 remains true in the case of $p = 2$. Determining if this theorem holds would, along with my previous work, completely characterize the values of p for which a bounded point derivation on $R^p(X)$ implies the existence of approximate derivatives. It seems to me that it would be very difficult to show that Theorem 2.2.1 fails for $p = 2$ as this would require the construction of a function which must be in $R^2(X)$ but not $R^p(X)$ for any higher values of p. For this reason, it seems to me that it would be more fruitful to attempt to prove the theorem true for $p = 2$. This could involve the creation of a brand new proof of Theorem 2.2.1 that would remain true for $p = 2$. In particular, a more constructive proof of Theorem 2.2.1 would make it easier to understand why the result is true and could also provide more information about the structure of the set of full area density. Currently all that is known is that the set of full area density is not, in general, the same as the set of bounded point evaluations. (See Section 2.7.)

A natural continuation of the topics of my research would be to consider harmonic approximation on compact subsets of the complex plane. Let $H(X)$ denote the uniform closure of functions that are harmonic on a neighborhood of X and let $C_h(X)$ denote the continuous functions on X which are harmonic on the interior of X. The problem of harmonic approximation is to find conditions on X so that $H(X) = C_h(X)$ or $H(X) = C(X)$. This problem was completely solved by Keldysh. Shortly thereafter it came to light that Keldysh's necessary and sufficient conditions are equivalent to necessary and sufficient conditions for $R^2(X) = L^2_a(X)$. Here $L^2_a(X)$ denotes the spaces of L^2 functions analytic on the interior of X. Thus $H(X) = C_h(X)$ if and only if $R^2(X) = L^2_a(X)$. These conditions can be given in terms of the fine topology, which is the weakest topology that makes all superharmonic functions continuous.

A set X is said to be 2-thin at a point x_0 if

$$\int_0 \frac{\Gamma_2(X \cap B_r(x_0))}{r} dr < \infty.$$

Alternatively, it follows from the subadditivitiy of q-capacity that a set is 2-thin at x_0 if and only if

$$\sum_{n=1}^{\infty} n\Gamma_2(X \cap A_n(x_0)) < \infty,$$

where $A_n(x_0) = \{z : 2^{-n-1} < |z - x_0| < 2^{-n}\}$. A set U is said to be a fine neighborhood for x_0 if $\mathbb{C}\backslash U$ is 2-thin at x_0. The fine neighborhoods of a point form the neighborhood basis for the fine topology. A function f is said to be finely differentiable at a point z_0 if there exists a set E that is 2-thin at x_0 such that the limit

$$f'(z_0) = \lim_{z \to z_0, z \notin E} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. As the concept of a fine derivative is closely related to the concept of an approximate derivative, it would be interesting to know if the existence of a bounded point derivation on $R(X)$ or $R^p(X)$ also implies the existence of a fine derivative at x_0.

Another area of investigation is to consider whether Theorem 3.1.1 can be extended to the case of bounded point derivations on $R^p(X)$. That is, given a bounded point derivation D on $R^p(X)$ at x_0 and supposing that X has an interior cone at x_0 with midline J, is it true that

$$Df = \lim_{x \to x_0, x \in J} \frac{f(x) - f(x_0)}{x - x_0}?$$

It would seem that a proof similar to the one we gave for Theorem 3.1.1 could be used to prove this result, but we have not yet been able to show it.

I am also interested in studying bounded point derivations defined for other function spaces. One space in particular that I would like to work with is the space of functions analytic on a fixed open set which also satisfy a Lipschitz condition on the entire plane. It would be interesting to know whether my results for bounded point derivations on $R^p(X)$ would also be valid for these new spaces of functions.

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