12-9-2015

Finite Groups in Which Pronomality and ?-Pronormality Coincide

Adolfo Ballester-Bolinches
Universitat de València, Spain

James C. Beidleman
University of Kentucky, clark@ms.uky.edu

Arnold D. Feldman
Franklin and Marshall College

Matthew F. Ragland
Auburn University at Montgomery

J. Group Theory **19** (2016), 323–329
DOI 10.1515/jgth-2015-0035

Finite groups in which pronormality and $\mathfrak{F}$-pronormality coincide

Adolfo Ballester-Bolinches, James C. Beidleman,
Arnold D. Feldman and Matthew F. Ragland

Communicated by Evgenii I. Khukhro

Abstract. For a formation $\mathfrak{F}$, a subgroup U of a finite group G is said to be $\mathfrak{F}$-pronormal in G if for each $g \in G$, there exists $x \in \langle U, U^g \rangle^{\mathfrak{F}}$ such that $U^x = U^g$. If $\mathfrak{F}$ contains $\mathfrak{N}$, the formation of nilpotent groups, then every $\mathfrak{F}$-pronormal subgroup is pronormal and, in fact, $\mathfrak{N}$-pronormality is just classical pronormality. The main aim of this paper is to study classes of finite soluble groups in which pronormality and $\mathfrak{F}$-pronormality coincide.

1 Introduction and statements of results

Note that all groups considered in this paper are finite and we use the notation of [1, 5, 9].

The third author [6] and Müller [8] independently generalised pronormality of a subgroup of a finite soluble group to $\mathfrak{F}$-pronormality, where $\mathfrak{F}$ is a subgroup-closed saturated formation containing $\mathfrak{N}$, the class of all nilpotent groups. Both used the concept of an $\mathfrak{F}$-base, a generalisation of a Hall system of a soluble group. But $\mathfrak{F}$-bases cannot be defined for non-soluble groups, so we use the following definition due to Müller:

Definition 1.1. Let G be a group and let U be a subgroup of G. Then U is said to be $\mathfrak{F}$-pronormal in G if, for each $g \in G$, there exists $x \in \langle U, U^g \rangle^{\mathfrak{F}}$ such that $U^x = U^g$.

See [3, remark after Definition 2] for a brief proof that $\mathfrak{N}$-pronormality is simply pronormality, so that this is a legitimate generalisation.

It is easy to see that when $\mathfrak{F}$ contains $\mathfrak{N}$, every $\mathfrak{F}$-pronormal subgroup is pronormal. This leads immediately to the question of how to characterise the groups G such that a subgroup is $\mathfrak{F}$-pronormal in G if and only if it is pronormal in G. In the soluble case, Müller showed that for $\mathfrak{F}$ saturated and containing $\mathfrak{N}$, the class

The first author has been supported by the grant MTM2010-19938-C03-01 from MICINN (Spain) and Project of NSFC (11271085).

of groups whose maximal subgroups are all $\mathfrak{F}$-pronormal is the same as the class of groups whose $\mathfrak{F}$-normalisers are nilpotent if and only if the latter class is a saturated formation [8, Satz 7.3].

In [2, 3] we dealt with generalisations of pronormality and subnormality. For recent work by others on generalisations of subnormality, see [7, 10, 11]. This paper is a natural continuation of [4], which dealt principally with generalisations of pronormality. Our aim here is to investigate conditions for soluble and supersoluble groups to belong to the class $\mathcal{P}_{\mathfrak{F}}$ of all groups in which pronormality and $\mathfrak{F}$-pronormality coincide, where $\mathfrak{F}$ is a formation containing $\mathfrak{N}$ satisfying a certain closure property. A precise description of the largest subgroup-closed classes of soluble and supersoluble groups contained in $\mathcal{P}_{\mathfrak{F}}$ appears as a consequence of our study.

We begin by introducing some definitions and notation that are needed to state our main results.

Definition 1.2. The class $\mathfrak{E}_{\mathfrak{F}}$ consists of all soluble groups G such that if $H \leq L \leq G$, then H is $\mathfrak{F}$-pronormal in L if and only if H is pronormal in L.

Definition 1.3. Given distinct prime numbers p and q, let $\mathfrak{M}_{(p,q)}$ be the class of groups G such that $G = PQ$, where P is an elementary abelian p-group, Q is cyclic of order q, P is normal in G, $C_Q(P) = 1$ and P, regarded as a Q-module, is homogeneous. Furthermore, let $\mathfrak{X}_{(p,q)}$ be the subclass of $\mathfrak{M}_{(p,q)}$ in which Q acts irreducibly on P.

It is clear that a group $G \in \mathfrak{M}_{(p,q)}$ always has an epimorphic image in $\mathfrak{X}_{(p,q)}$.

Let $\mathfrak{M}$ be the union of the classes $\mathfrak{M}_{(p,q)}$ for all pairs (p,q), and let $\mathfrak{X}$ be the union of the classes $\mathfrak{X}_{(p,q)}$. Write $\mathfrak{M}_{\mathfrak{F}} = \mathfrak{M} \cap \mathfrak{F}$ and $\mathfrak{X}_{\mathfrak{F}} = \mathfrak{X} \cap \mathfrak{F}$.

For each prime p, let $\pi_{\mathfrak{F}}(p)$ be the set of primes q such that $\mathfrak{X}_{(p,q)}$ is contained in $\mathfrak{F}$. Then set $g_{\mathfrak{F}}(p)$ to be the class of soluble groups whose orders are not divisible by any prime in $\pi_{\mathfrak{F}}(p)$, i.e., the soluble $\pi_{\mathfrak{F}}(p)'$-groups.

Definition 1.4. We say a formation $\mathfrak{F}$ possesses *property* (δ) if $\mathfrak{F}$ is closed under taking normal $\mathfrak{M}$-subgroups of soluble groups.

Definition 1.5. If $\mathfrak{F}$ is a formation, a soluble group G is said to be a $\mathfrak{P}_{\mathfrak{F}}$-group if no section of G is isomorphic to a member of $\mathfrak{X}_{\mathfrak{F}}$.

Theorem 1.6. *Let $\mathfrak{F}$ be a formation that contains the class $\mathfrak{N}$. Then $\mathfrak{P}_{\mathfrak{F}}$ is a saturated formation which is locally defined by the formation function f given by $f(p) = g_{\mathfrak{F}}(p)$ for all primes p. Furthermore, $\mathfrak{E}_{\mathfrak{F}}$ is contained in $\mathfrak{P}_{\mathfrak{F}}$. If $\mathfrak{F}$ has property (δ), then $\mathfrak{E}_{\mathfrak{F}} = \mathfrak{P}_{\mathfrak{F}}$.*

Property (δ) is necessary in Theorem 1.6, as the following example constructed in [2] shows.

Example 1.7. Let h be a function that takes each prime to a class of groups such that $h(5)$ is the class of groups that are trivial or cyclic of order 4 and $h(r)$ is trivial for all primes other than 5. As in [5, Chapter IV, Proposition (1.3)], let $\mathfrak{H}$ be the formation of soluble groups G such that for each p-chief factor S of G, $\mathrm{Aut}_G(S) \in h(p)$. Then $\mathfrak{X}_{(5,2)}$ is not contained in $\mathfrak{H}$, and, in fact, neither is any other member of $\mathfrak{X}$. Thus $\mathfrak{X}_{\mathfrak{H}}$ is empty, and $\mathfrak{P}_{\mathfrak{H}}$ is the class of all soluble groups. Now consider the semidirect product G of a cyclic group of order 5 and its full automorphism group, which is cyclic of order 4, and let T be a Sylow 2-subgroup of G. Then T is maximal and pronormal in G, and $G \in \mathfrak{H}$. So if $G \in \mathfrak{E}_{\mathfrak{H}}$, T is $\mathfrak{H}$-pronormal in G, and therefore normal in G by Lemma 2.1 below. But T is not normal in G, so G is in $\mathfrak{P}_{\mathfrak{H}}$ and not $\mathfrak{E}_{\mathfrak{H}}$. Note that G has a subgroup of index 2 that is in $\mathfrak{X}_{(5,2)}$ but not in $\mathfrak{H}$, violating property (δ).

For the class $\mathfrak{U}$ of supersoluble groups, the solution of the problem is much nicer.

Theorem 1.8. *Let $\mathfrak{F}$ be a formation that contains $\mathfrak{N}$ and has property (δ). Then $\mathfrak{E}_{\mathfrak{F}} \cap \mathfrak{U} = \mathcal{P}_{\mathfrak{F}} \cap \mathfrak{U}$, that is, if G is a supersoluble group, then the following statements are pairwise equivalent:*

(i) $G \in \mathfrak{E}_{\mathfrak{F}}$.

(ii) *For all primes p dividing $|G|$, if S is a p-chief factor of G, then we have* $\mathrm{Aut}_G(S) \in g_{\mathfrak{F}}(p)$.

(iii) *If U is pronormal in G, then U is $\mathfrak{F}$-pronormal in G.*

(iv) $G \in \mathfrak{P}_{\mathfrak{F}}$.

2 Preliminaries

We assume here that $\mathfrak{F}$ contains $\mathfrak{N}$. Note that in this case, all $\mathfrak{F}$-pronormal subgroups are pronormal. The following lemmas will be used in the proofs of our main results.

Lemma 2.1 ([4, Lemma 3]). *Let U be a subgroup of a group G and let $\mathfrak{F}$ be a formation.*

(i) *If $U \leq H$ and U is $\mathfrak{F}$-pronormal in G, then U is $\mathfrak{F}$-pronormal in H.*

(ii) *If N is a normal subgroup of G and U is $\mathfrak{F}$-pronormal in G, then UN/N is $\mathfrak{F}$-pronormal in G/N.*

(iii) *If N is a normal subgroup of G and U/N is $\mathfrak{F}$-pronormal in G/N, then U is $\mathfrak{F}$-pronormal in G.*

(iv) *If U is maximal, $\mathfrak{F}$-pronormal, and $G \in \mathfrak{F}$, then U is normal in G.*

Lemma 2.2 ([3, Proposition 1]). *Let U be a subgroup of a group G and let N be a normal subgroup of G such that $U \leq N \leq G$. Then if $\mathfrak{F}$ is a formation, the following conditions are equivalent:*

(i) *U is $\mathfrak{F}$-pronormal in G.*

(ii) *U is $\mathfrak{F}$-pronormal in N and $G = N_G(U)N$.*

Lemma 2.3. *The class $\mathfrak{E}_{\mathfrak{F}}$ is closed under the taking of sections.*

Proof. It is clear from the definition that $\mathfrak{E}_{\mathfrak{F}}$ is subgroup-closed, so suppose N is normal in $G \in \mathfrak{E}_{\mathfrak{F}}$ and H/N is pronormal in $L/N \leq G/N$. Then H is pronormal in $L \leq G$, so by hypothesis, H is $\mathfrak{F}$-pronormal in L, and by Lemma 2.1, H/N is $\mathfrak{F}$-pronormal in L/N. Hence $G/N \in \mathfrak{E}_{\mathfrak{F}}$. Thus if A/B is a section of $G \in \mathfrak{E}_{\mathfrak{F}}$, then $A \leq G$ implies $A \in \mathfrak{E}_{\mathfrak{F}}$ as remarked above, so $A/B \in \mathfrak{E}_{\mathfrak{F}}$ as just proved. □

3 Proofs of the main results

Proof of Theorem 1.6. Let $\mathfrak{Y} = LF(f)$ be the saturated formation locally defined by the formation function f given by $f(p) = g_{\mathfrak{F}}(p)$ for all primes p. Next, we see that $\mathfrak{Y} = \mathfrak{E}_{\mathfrak{F}}$. Assume, arguing by contradiction, that $\mathfrak{E}_{\mathfrak{F}}$ is not contained in $\mathfrak{Y}$ and let $G \in \mathfrak{E}_{\mathfrak{F}} \setminus \mathfrak{Y}$ of minimal order. By Lemma 2.3, $G/N \in \mathfrak{Y}$ for every non-trivial normal subgroup N of G. Therefore, G is a primitive group. Applying [5, Chapter A, Theorem 15.2], $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G which is complemented by a maximal subgroup M in G, $C_G(N) = N$ and $G/N \in \mathfrak{Y}$. Let p be the prime dividing the order of N. Then $G/N \notin f(p)$ and so there exists a prime $q \neq p$ dividing the order of M such that $\mathfrak{X}_{(p,q)}$ is contained in $\mathfrak{F}$. Let A be a subgroup of order q of M. According to [5, Chapter A, Proposition 12.5], $N = C_N(A) \times [A, N]$. Since $C_G(N) = N$, it follows that $[A, N] \neq 1$. Let N_0 be a minimal normal subgroup of AN contained in $[A, N]$. Then $C_{N_0}(A) = 1$ and so $B = AN_0 \in \mathfrak{X}_{(p,q)}$. In particular, $B \in \mathfrak{F}$. Since A is maximal in B and $B \in \mathfrak{E}_{\mathfrak{F}}$, it follows that A is normal in B by Lemma 2.1, contrary to the choice of N_0. Consequently, $\mathfrak{E}_{\mathfrak{F}}$ is contained in $\mathfrak{Y}$.

Applying [5, Chapter IV, Proposition 3.14], $\mathfrak{Y}$ is subgroup-closed. Assume that $\mathfrak{Y}$ is not contained in $\mathfrak{P}_{\mathfrak{F}}$ and let $G \in \mathfrak{Y} \setminus \mathfrak{P}_{\mathfrak{F}}$ be a group of minimal order. Then every proper section of G belongs to $\mathfrak{P}_{\mathfrak{F}}$. Since $G \notin \mathfrak{P}_{\mathfrak{F}}$, we conclude that G has a section which belongs to some $\mathfrak{X}_{(p,q)} \cap \mathfrak{F}$. The minimal choice of G implies that $G \in \mathfrak{X}_{(p,q)} \cap \mathfrak{F}$. Since $G \in \mathfrak{Y}$, it follows that $G/\mathrm{Soc}(G) \in f(p) = g_{\mathfrak{F}}(p)$, against supposition. Therefore $\mathfrak{Y}$ is contained in $\mathfrak{P}_{\mathfrak{F}}$.

Suppose that $\mathfrak{P}_{\mathfrak{F}}$ is not contained in $\mathfrak{Y}$, and let $G \in \mathfrak{P}_{\mathfrak{F}} \setminus \mathfrak{Y}$ be of minimal order. Then G is an $\mathfrak{Y}$-critical group, that is, every proper subgroup of G belongs to $\mathfrak{Y}$. Since $\mathfrak{Y}$ is saturated, it follows that G is primitive. Let $N = \mathrm{Soc}(G)$ and p the prime dividing $|N|$. Let M be a core-free maximal subgroup of G complementing N in G. Then $NM_1 \in \mathfrak{Y}$ for every maximal subgroup M_1 of M. Since $O_{p'p}(NM_1) = O_p(NM_1)$, we have that the Hall p'-subgroups of NM_1 belong to $g_{\mathfrak{F}}(p)$. Hence, if M is not of prime order, it follows that $G/N \in f(p)$. Applying [5, Chapter IV, Remark 3.5 c)], $G \in \mathfrak{Y}$, against the choice of G. Hence $G \in \mathfrak{X}_{(p,q)}$ for some $q \neq p$, and $q \in \pi_{\mathfrak{F}}(p)$, i.e. $G \in \mathfrak{F}$. This contradiction yields $\mathfrak{P}_{\mathfrak{F}} \subseteq \mathfrak{Y}$. Therefore we have $\mathfrak{E}_{\mathfrak{F}} \subseteq \mathfrak{P}_{\mathfrak{F}} = \mathfrak{Y}$.

Assume now that $\mathfrak{F}$ has property (δ). We prove that $\mathfrak{E}_{\mathfrak{F}} = \mathfrak{P}_{\mathfrak{F}}$. Suppose, arguing by contradiction, that G is a soluble group of minimal order with respect to being a member of $\mathfrak{P}_{\mathfrak{F}}$ but not $\mathfrak{E}_{\mathfrak{F}}$. Then there exists $U \leq L \leq G$ such that U is pronormal in L but not $\mathfrak{F}$-pronormal in L. If $L < G$, by minimality of G, U is $\mathfrak{F}$-pronormal in L. Hence $L = G$, and U is pronormal in G but not $\mathfrak{F}$-pronormal in G. Let $C = \mathrm{Core}_G(U)$. Then if $C > 1$, by minimality of G, $G/C \in \mathfrak{P}_{\mathfrak{F}}$ implies $G/C \in \mathfrak{E}_{\mathfrak{F}}$, so that, since U/C is pronormal in G/C, U/C is $\mathfrak{F}$-pronormal in G/C, implying U $\mathfrak{F}$-pronormal in G by Lemma 2.1. Hence $\mathrm{Core}_G(U) = 1$.

Now consider the normal closure $V = U^G$ of U in G. If $V < G$, then by minimality of G, since U is pronormal in V, U is $\mathfrak{F}$-pronormal in V. Also, the pronormality of U in G implies $G = N_G(U)V$, so by Lemma 2.2, U is $\mathfrak{F}$-pronormal in G. Hence $V = G$, i.e., U is contained in no proper normal subgroup of G.

Since U is not $\mathfrak{F}$-pronormal in G, there exists $g \in G$ such that U and U^g are not conjugate via an element of $J^{\mathfrak{F}}$, where $J = \langle U, U^g \rangle$. But U is pronormal in G, so there exists $x \in J$ such that $U^g = U^x$. Thus if $J < G$, by minimality of G, U pronormal in J implies U $\mathfrak{F}$-pronormal in J, so there does exist $y \in J^{\mathfrak{F}}$ conjugating U to $U^y = U^g$. Hence $J = G$, $J^{\mathfrak{F}} = G^{\mathfrak{F}}$, and we are assuming that U and U^g are not conjugate via any element of $G^{\mathfrak{F}}$.

Assume now that G is not in $\mathfrak{F}$, so that $G^{\mathfrak{F}} > 1$, and let N be any minimal normal subgroup of G contained in $G^{\mathfrak{F}}$. Note that UN/N is pronormal in G/N, so by minimality of G, UN/N is $\mathfrak{F}$-pronormal in G/N, so that UN is $\mathfrak{F}$-pronormal in G by Lemma 2.1. Hence UN and $(UN)^g$ are conjugate by an element of $\langle UN, (UN)^g \rangle^{\mathfrak{F}} = G^{\mathfrak{F}}$. Thus there exists $z \in G^{\mathfrak{F}}$ such that $(UN)^z = (UN)^g$. Therefore, $gz^{-1} \in N_G(UN)$. But because U is pronormal in G, $N_G(UN)$ is equal to $N_G(U)N$, so $gz^{-1} = hw$, where $h \in N_G(U)$ and $w \in N$. Therefore, $gz^{-1}w^{-1} \in N_G(U)$, so $U^g = U^{wz}$. But $N \leq G^{\mathfrak{F}}$, so $wz \in G^{\mathfrak{F}}$, a contradiction.

Thus we may assume $G \in \mathfrak{F}$. Let N be any minimal normal subgroup of G, so that $G/N \in \mathfrak{F}$, and by minimality of G, any maximal subgroup L/N of G/N, being pronormal in G/N, is $\mathfrak{F}$-pronormal in G/N. Then by Lemma 2.1 (iv), L/N is normal in G/N. Thus G/N is nilpotent. If N is not unique, then G itself is nilpotent, so that any pronormal subgroup of G, being subnormal as well, is normal in G and therefore $\mathfrak{F}$-pronormal in G. Hence N is the unique minimal normal subgroup of G and UN/N, being pronormal and subnormal in G/N, is normal in G/N. Thus UN is normal in G. But $U^G = G$, so $UN = G$, U is a core-free maximal subgroup of G and $C_G(N) = N$. Then U belongs to $f(p) = g_{\mathfrak{F}}(p)$. Let q be a prime dividing $|U|$ and let A be a subgroup of U of order q in $\mathrm{Z}(U)$. Then N, regarded as A-module, is homogeneous by [5, Chapter B, Corollary 9.4]. Therefore $C_A(N) = 1$ and $B = NA$ is a normal subgroup of G in $\mathfrak{M}_{(p,q)}$. Since $\mathfrak{F}$ has property (δ), it follows that $B \in \mathfrak{F}$, contrary to $G \in \mathfrak{P}_{\mathfrak{F}}$.

The proof of the theorem is complete. □

Proof of Theorem 1.8. Conditions (i), (ii) and (iv) are equivalent by Theorem 1.6, and clearly (i) implies (iii), so we need only prove that (iii) implies (iv). Hence suppose G is a supersoluble group of minimal order such that every pronormal subgroup of G is $\mathfrak{F}$-pronormal in G, but G is not a member of $\mathfrak{P}_{\mathfrak{F}}$. Applying Lemma 2.1, the condition on G is inherited in every epimorphic image of G. Therefore, $G/N \in \mathfrak{P}_{\mathfrak{F}}$ for all non-trivial normal subgroups of G. Applying Theorem 1.6, $\mathfrak{P}_{\mathfrak{F}}$ is a saturated formation. Hence G is a supersoluble primitive group. Applying [5, Chapter A, Theorem 15.2], $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G of prime order, p say, which is complemented in G by a core-free maximal subgroup M of G. Moreover, $C_G(N) = N$ and so M is a cyclic group of order dividing $p-1$. Since $G/N \notin g_{\mathfrak{F}}(p)$, there exists a prime q dividing $|M|$ such that $\mathfrak{X}_{(p,q)} \cap \mathfrak{F}$ is not empty. Note that if A is a subgroup of order q of M, we have that $NA \in \mathfrak{X}_{(p,q)}$. Therefore $NA \in \mathfrak{F}$. Note that NA is normal in G and so A is pronormal in G since A is a Sylow q-subgroup of NA. By hypothesis, A is $\mathfrak{F}$-pronormal in G and so is $\mathfrak{F}$-pronormal in NA.

By Lemma 2.1, A is normal in AN, which is not possible. Consequently, we have $\mathfrak{P}_{\mathfrak{F}} \cap \mathfrak{U} = \mathcal{P}_{\mathfrak{F}} \cap \mathfrak{U}$. □

Acknowledgments. The third author wishes to thank Franklin & Marshall College for its generous sabbatical policy, and National University of Ireland, Galway and Universitat de València and Universitat Politècnica de València for their support.

Bibliography

[1] A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite Groups*, Math. Appl. 584, Springer, New York, 2006.

[2] A. Ballester-Bolinches, A. D. Feldman, M. C. Pedraza-Aguilera and M. F. Ragland, A class of generalised finite T-groups, *J. Algebra* **333** (2011), no. 1, 128–138.

[3] A. Ballester-Bolinches, J. C. Beidleman, A. D. Feldman and M. F. Ragland, On generalised subnormal subgroups of finite groups, *Math. Nachr.* **286** (2013), no. 11–12, 1066–1071.

[4] A. Ballester-Bolinches, J. C. Beidleman, A. D. Feldman and M. F. Ragland, On generalised pronormal subgroups of finite groups, *Glasg. Math. J.* **56** (2014), 691–703.

[5] K. Doerk and T. Hawkes, *Finite Soluble Groups*, De Gruyter, Berlin, 1992.

[6] A. Feldman, $\mathfrak{F}$-bases and subgroup embedding in finite solvable groups, *Arch. Math.* **47** (1986), 481–492.

[7] V. A. Kovaleva and A. N. Skiba, Finite soluble groups with all n-maximal subgroups $\mathfrak{F}$-subnormal, *J. Group Theory* **17** (2014), no. 2, 273–290.

[8] N. Müller, *$\mathfrak{F}$-Pronormale Untergruppen endlich auflösbarer Gruppen*, Diplomarbeit, Johannes Gutenberg-Universität, Mainz, 1985.

[9] L. A. Shemetkov and A. N. Skiba, *Formations of Algebraic Systems* (in Russian), Nauka, Moscow, 1989.

[10] A. F. Vasil'ev, T. I. Vasil'eva and V. N. Tyutyanov, On K-$\mathbb{P}$-subnormal subgroups of finite groups (in Russian), *Mat. Zametki* **95** (2014), no. 4, 517–528; translation in *Math. Notes* **95** (2014), no. 3–4, 471–480.

[11] A. F. Vasil'ev, T. I. Vasil'eva and V. N. Tyutyanov, On the products of $\mathbb{P}$-subnormal subgroups of finite groups (in Russian), *Sibirsk. Mat. Zh.* **53** (2012), no. 1, 59–67; translation in *Sib. Math. J.* **53** (2012), no. 1, 47–54.

Received May 15, 2015.

Author information

Adolfo Ballester-Bolinches, Departament d'Àlgebra, Universitat de València,
Dr. Moliner, 50, 46100 Burjassot, València, Spain.
E-mail: adolfo.ballester@uv.es

James C. Beidleman, University of Kentucky, Lexington, KY 40506-0027, USA.
E-mail: james.beidleman@uky.edu

Arnold D. Feldman, Department of Mathematics, Franklin and Marshall College,
Lancaster, PA 17604-3003, USA.
E-mail: afeldman@fandm.edu

Matthew F. Ragland, Department of Mathematics and Computer Science, Auburn
University at Montgomery, P.O. Box 244023, Montgomery, AL 36124-4023, USA.
E-mail: mragland@aum.edu