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Orbital Stability Results for Soliton Solutions to Nonlinear Schrödinger Equations with External Potentials

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Orbital Stability Results for Soliton Solutions to Nonlinear Schrödinger Equations with External Potentials

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DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Joseph B. Lindgren
Lexington, Kentucky

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Lexington, Kentucky 2017

ABSTRACT OF DISSERTATION

Orbital Stability Results for Soliton Solutions to Nonlinear Schrödinger Equations with External Potentials

For certain nonlinear Schrödinger equations there exist equilibrium solutions which are called solitary waves. Addition of a potential V changes the dynamics, but for small enough $||V||_{L^\infty}$ we can still obtain stability (and approximately Newtonian motion of the solitary wave's center of mass) for soliton-like solutions up to a finite time that depends on the size and scale of the potential V. Our method is an adaptation of the well-known Lyapunov method.

For the sake of completeness, we also prove long-time stability of traveling solitons in the case $V = 0$.

KEYWORDS: traveling waves, nonlinear, schrödinger, NLS, orbital stability, Lyapunov

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Date: May 3, 2017

Orbital Stability Results for Soliton Solutions to Nonlinear Schrödinger Equations with External Potentials

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Ad Majorem Dei Gloriam
J.M.J.

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Chapter 1 Introduction

1.1 Some Background, Brief History, and the Main Theorem

In physics, solitary traveling waves are an important object of study. Previously mentioned applications include "propagation of electromagnetic waves in nonlinear media" [19, 17], "electromagnetic (Langmuir) waves in a plasma," and "motion of a vortex filament for Euler equations of fluid mechanics" [17]. From a more theoretical perspective, it is of interest to note that there exist infinite multi-parameter families of solitons that are (in some sense) equilibrium solutions of a given nonlinear schrödinger equation. In particular, soliton solutions of these equations appear closely related to the problem of Bose-Einstein condensation (according to a presentation by Kay Kirkpatrick, citing Gross and Pitaevskii, 1961), which may have a part to play in the search for suitable materials to construct a viable quantum computer. Stability of these equilibrium solutions is of interest because such behavior is critical for the modelling of physical phenomena.

This document attempts to broaden what is known about the orbital stability of solitons in the presence of an external potential V. This is a more robust model for phenomena than the case $V = 0$, given the likely imperfections of a physical environment. We take as our equation a focusing non-linear Schrödinger equation

$$\begin{cases} i\psi_t + \Delta\psi + f(\psi) = \lambda V^h \psi, \\ \psi(\cdot, 0) = \psi_0 \in H^1(\mathbb{R}^d), \\ (x,t) \in \mathbb{R}^d \times \mathbb{R} \end{cases} \tag{1.1}$$

with power nonlinearity $f(\psi) = |\psi|^{2k}\psi$ and subcritical exponent $k < 2/d$, and study the dynamics of soliton-like solutions to this equation under $H^1(\mathbb{R}^d)$ perturbations of the initial data. Here $V^h(x) := V(hx)$, $V \in C^2(\mathbb{R}^d)$, $||V||_{L^\infty} = 1$, and $\lambda > 0$ is made sufficiently small. We view this problem as an extension of the case $\lambda = 0$, given below:

$$i\psi_t + \Delta\psi + f(\psi) = 0. \tag{1.2}$$

We use notation suggesting a general dimension d, but some existence and coercivity arguments will require $d = 1$ or $d = 3$.

Existence of solutions to Equation (1.2) was shown by Ginibre and Velo[11], using conserved quantities to bound the L^2 norm of the gradient of the solution.

Theorem 1.1 (J. Ginibre and G. Velo, 1977)**.** *Let ψ be a solution of*

$$i\psi_t = (-\Delta + m)\,\psi + g(\psi),$$

where $m \in \mathbb{R}$ and g is a continuously differentiable, complex valued function with $g(0) = 0$ and having complex derivatives polynomially bounded by the variable with order less than $\frac{d+2}{d-2}$ for $d > 2$. Then for initial data $\psi_0 \in H^1\left(\mathbb{R}^d\right)$, the (unique) solution ψ with $\psi(0) = \psi_0$ lies in $C\left([0,\infty); H^1\left(\mathbb{R}^d\right)\right)$.

There is also a version of the theorem for $d = 1$. The theorem is proved in two steps by applying fixed point methods to the integral equation to show local existence, and then extending the solution globally by means of conserved quantities. In the appendix of a later paper [10], the same authors state a result for the case $d = 1$ (under some slightly relaxed conditions) which is applicable to a power nonlinearity $|\psi|^{p-1}\psi$ with $1 \leq p < 1 + \frac{4}{d}$. In that same appendix, they claim that a proof would require only "minor modifications." In 1988, Y.-G. Oh used the same (fixed point and conservation law) strategy to achieve global existence for solutions of a cubic NLS with asymptotically quadratic external potential [16]. The equation he studies is equivalent (via rescaling of x and t variables by h) to

$$i\psi_t + \frac{1}{2}\Delta\psi + |\psi|^{p-1}\psi = V^h\psi, \tag{1.3}$$

where $1 \leq p < 1 + \frac{4}{d}$. Here V is bounded below and $|D^\alpha V|$ is assumed to be bounded for all multi-indices $|\alpha| \geq 2$. The global solutions obtained are elements of a class of functions with finite Hamiltonian energy (roughly $H_0^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)$) for $t \geq 0$.

The main theorem (Theorem 1.6) of this document is an adaptation of the work of Fröhlich, Gustafson, Jonsson, and Sigal in [7]. In that paper, they rely on smallness of the semiclassical parameter h to smooth the potential and gain the desired bounds on the error. These four authors have more recently specialized their method to the case of a confining potential in [15]. In contrast, we weaken the need for small h (even allowing $h = 1$) by introducing the small coupling constant λ, and we make no assumptions about the shape of V. Our approach allows us to approximate the dynamics of a soliton moving in the presence of a (L^∞-small) random potential. We anticipate this being a step forward on the path to describing dynamics in the presence of other random potentials, perhaps those with small expectation values. For completeness, an extension (from standing waves to traveling solitons) of an earlier result by M. I. Weinstein [20] is included.

Orbital stability is one of two commonly used notions in the analysis of equilibria of dynamical systems. Below we give an abstract definition of both *orbital* and *asymptotic* stability.

Definition 1.2 (Orbital and Asymptotic Stability)**.** *We say that a solution $\psi \in C(\mathbb{R}^d; H)$ of a dynamical system is **orbitally stable** under a metric $\rho : H \times H \to \mathbb{R}$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any other solution $\phi \in C(\mathbb{R}^d; H)$,*

$$\rho\left(\phi(0), \psi(0)\right) < \delta$$

implies

$$\rho\left(\phi(t), \psi(t)\right) < \epsilon$$

for all $t > 0$.

*We say that ψ is **asymptotically stable** if, additionally,*

$$\rho\left(\phi(t), \psi(t)\right) \to 0$$

as $t \to \infty$.

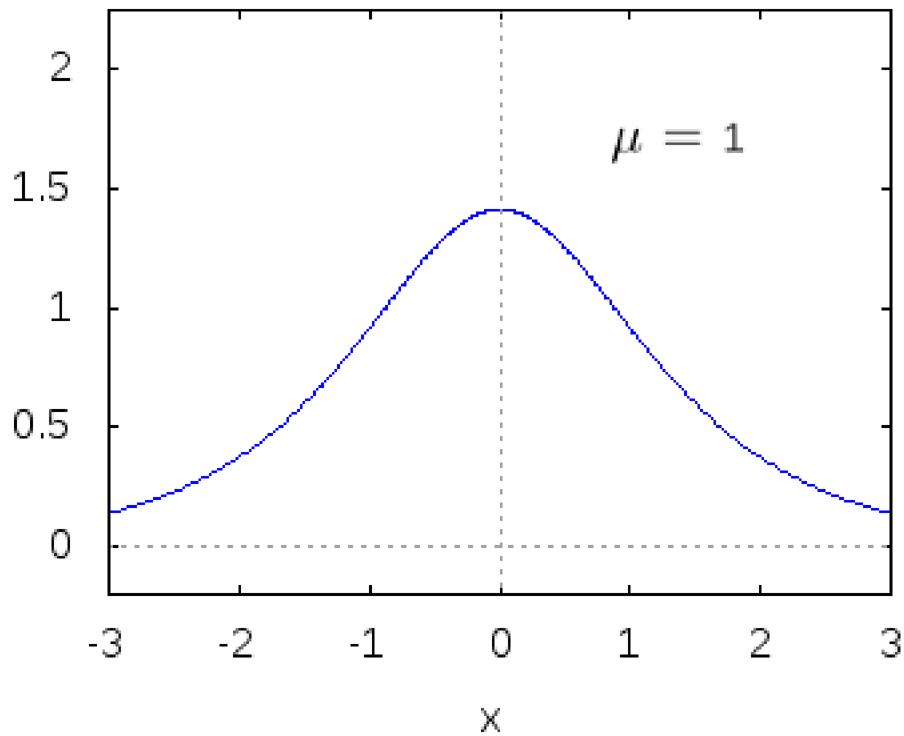

Figure 1.1: The soliton profile η_μ for $d = 1$, $k = 1$, $\mu = 1$.

In our search for stable solutions of (1.2), we look first for what are sometimes called "standing waves"; solutions that take the form

$$\psi(x,t) = e^{i\mu t}\eta_\mu(x).$$

Here μ governs the oscillation of the wave through the uni-modular factor $e^{i\mu t}$ as well as determine the amplitude of the wave. Supposing that $e^{i\mu t}\eta_\mu(x)$ solves (1.2), we find that η_μ solves

$$-\Delta\eta_\mu + \mu\eta_\mu - f(\eta_\mu) = 0. \tag{1.4}$$

The time-independent function η_μ is often called a "ground state" of the non-linear Schrödinger equation, or sometimes a "soliton profile." As an explicit example in the case $d = 1$ with cubic nonlinearity $f(\psi) = |\psi|^2\psi$, the function

$$\eta_\mu(x) = \sqrt{2\mu}\,\mathrm{sech}(\sqrt{\mu}x)$$

solves Equation (1.4) for any $\mu > 0$. A graph of such a profile is displayed in Figure 1.1. Due to a more general theorem from [2], we have the following result:

Theorem 1.3 (Berestycki-Lions (1983))**.** *Let $\mu > 0$ and $d = 1$ or $d \geq 3$. Then η_μ solving (1.4) is a positive, radially decreasing function and $\eta_\mu \in H^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$. When $d = 1$, η_μ is unique up to translations.*

An equivalent formulation, utilized in [19], is that the ground state η_μ is a minimizer of the functional

$$I[u] := \frac{||\nabla u||^{kd}||u||^{2-k(2-d)}}{||u||_{2k+2}^{2k+2}},$$

where $||\cdot||_p$ is the usual L^p norm and $||\cdot|| = ||\cdot||_2$. After a scaling argument, the condition $I'[u] = 0$ is seen to be equivalent to equation 1.4. More details are given in Appendix D.

Symmetries of the problem give rise to an infinite family of solutions to (1.2). In particular, the equation is easily seen to be invariant under translations and gauge (phase) transformations

$$\begin{aligned} T_a^{tr} &: u(x,t) \mapsto u(x-a,t), \\ T_\gamma^g &: u(x,t) \mapsto e^{i\gamma}u(x,t), \end{aligned}$$

as well as Galilean and scaling transformations

$$\begin{aligned} T_v^{gal} &: u(x,t) \mapsto e^{i\left(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t\right)} u\left(x - vt, t\right), \\ T_\mu^s &: u(x,t) \mapsto \mu^{\frac{1}{2k}} u\left(\sqrt{\mu}x, \mu t\right). \end{aligned}$$

Proofs of the invariance may be found in Appendix B. Adding an external potential as in Equation (1.1), however, breaks all except the gauge invariance.

Two comments are in order. First of all, for convenience of notation we take η_1 to be the solution of Equation (1.4) for $\mu = 1$ and let $\eta_\mu(x) = \mu^{\frac{1}{2k}}\eta_1(\sqrt{\mu}x)$. Secondly, we write the Galilean transform as a combination of T_γ^g, T_a^{tr}, and the boost transformation $T_v^b : u(x,t) \mapsto e^{i\frac{1}{2}v\cdot x}u(x,t)$.

$$\begin{aligned} T_v^{gal}u &= e^{i\left(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t\right)} u\left(x - vt, t\right) \\ &= T_v^b[e^{i\left(-\frac{1}{4}|v|^2 t\right)} u\left(x - vt, t\right)]. \\ &= T_v^b T^g_{\left(-\frac{1}{4}|v|^2 t\right)}[u\left(x - vt, t\right)]. \\ &= T_v^b T^g_{\left(-\frac{1}{4}|v|^2 t\right)} T^{tr}_{(vt)} u. \end{aligned}$$

This refactoring will later yield us a simpler set of tangent vectors, from which we will write down a basis for the tangent space $T_\eta \mathcal{G}$ (defined in Chapter 3). Since we study the tangent space only for each fixed time t, the presence of t in the phase change is not troublesome.

From these symmetries, it is possible to construct a traveling soliton solution $\eta_{\sigma(t)}$, determined by time-dependent parameters

$$\sigma(t) := \{a(t), v(t), \gamma(t), \mu(t)\}$$

such that it is now a solution of the original, time-dependent non-linear Schrödinger equation (still with $\lambda = 0$). The (soliton) parameters listed above may be interpreted as position, velocity, phase, and amplitude, respectively. Figure 1.2 depicts (for fixed t) the result of one such choice of parameters. The following proposition is proved in Appendix B.2:

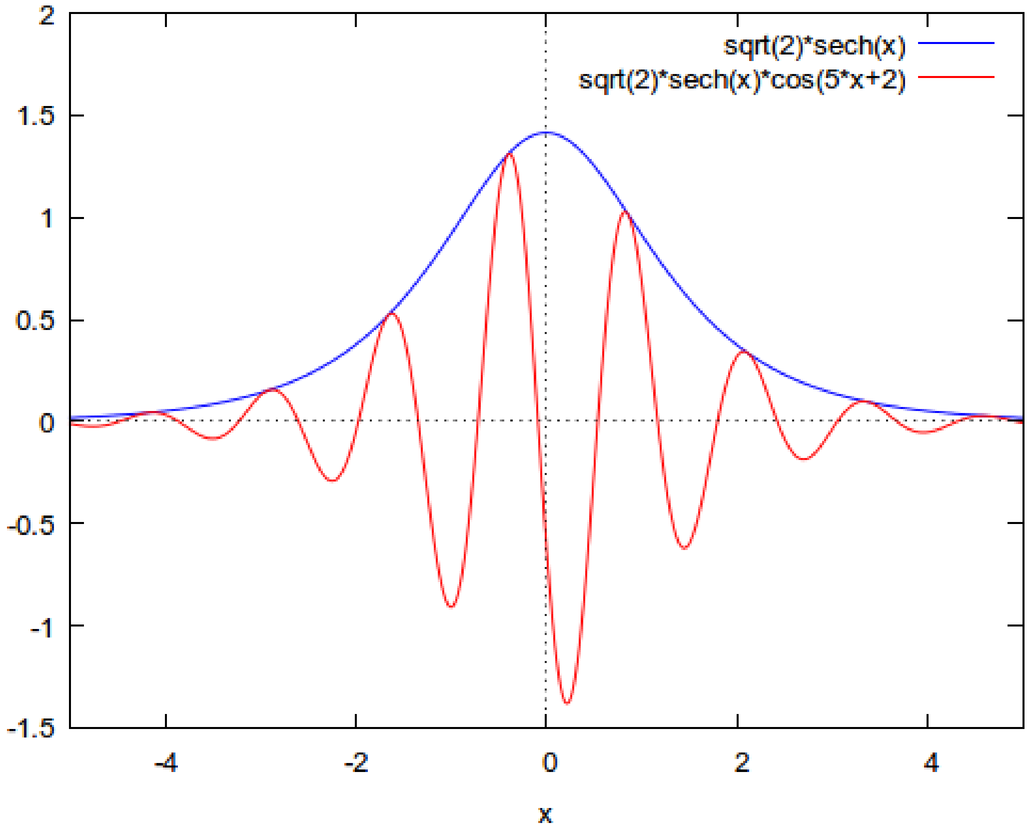

Figure 1.2: Fixed time snapshot of a traveling wave and profile for $d = 1$, $k = 1$, $\mu = 1$.

Proposition 1.4. *Define $\phi(x,t) := \frac{1}{2}v(t) \cdot x + \gamma(t)$, and let η_μ solve (1.4). Then the traveling soliton*

$$\eta_{\sigma(t)}(x,t) := e^{i\phi(x-a(t),t)}\eta_\mu(x - a(t)) \tag{1.5}$$

with initial parameters $\sigma_0 = \{a_0, v_0, \gamma_0, \mu_0\}$ solves (1.2) if and only if

$$\begin{cases} a(t) &= v_0 t + a_0, \\ v(t) &= v_0, \\ \mu &= \mu_0, \\ \gamma(t) &= \left(\mu_0 + \frac{1}{4}|v_0|^2\right) t + \gamma_0. \end{cases}$$

It is this behavior that we wish to reconstruct – up to small errors – for soliton-like solutions of (1.1) with the external potential λV^h.

Our main result (Theorem 1.6) is inspired by the result of Fröhlich, Gustafson, Jonsson, and Sigal in [7], which is summarized below in Theorem 1.5. Both theorems are fully nonlinear stability results.

Theorem 1.5 (Fröhlich-Gustafson-Jonsson-Sigal, 2004)**.** *Let ψ solve (1.1) with λ fixed, and let η_μ solve (1.4). Let $\eta_{\sigma(t)}$ be a traveling soliton with initial parameters $\sigma(0) = \sigma_0 = \{a_0, v_0, \gamma_0, \mu_0\}$, and suppose that for some $0 < \epsilon_0 \ll 1$,*

$$||\psi(\cdot,0) - \eta_{\sigma(0)}(\cdot,0)||_{H^1} \leq \epsilon_0.$$

Then for $0 < h \ll 1$, there exists some $T > 0$ and family of parameters $\sigma(t)$ with $\sigma(0) = \{a_0, v_0, \gamma_0, \mu_0\}$ such that for $t \leq T\left(h + {\epsilon_0}^2\right)^{-1}$,

$$||\psi(\cdot,t) - \eta_{\sigma(t)}(\cdot,t)||_{H^1}^2 = \mathcal{O}\left(h + \epsilon_0\right),$$

Furthermore, the parameters $\sigma(t) = \{a(t), v(t), \gamma(t), \mu(t)\}$ satisfy some (approximately Newtonian) modulation equations

$$\begin{cases} \dot{a} = v + \mathcal{O}\left(h^2 + {\epsilon_0}^2\right), \\ \frac{1}{2}\dot{v} = -\nabla\left(V_h\right)(a) + \mathcal{O}\left(h^2 + {\epsilon_0}^2\right), \\ \dot{\gamma} = \mu - \frac{1}{4}v^2 + \frac{1}{2}\dot{a}\cdot v - V_h(a) + \mathcal{O}\left(h^2 + {\epsilon_0}^2\right), \\ \dot{\mu} = \mathcal{O}\left(h^2 + {\epsilon_0}^2\right) \end{cases}$$

with initial conditions $\sigma(0) = \sigma_0$.

The condition $h \ll 1$ serves to ensure that V_h is slowly varying with respect to the scale of the support of the soliton $\eta_{\sigma(t)}$. (In fact, taking the limit $h \to 0$ sends $V_h \to V(0)$.) This 'slowly varying' condition also dictates that higher derivatives of $V_h(x)$ are controlled by higher powers of h.

To deal with potentials having less derivative decay, we introduce a small coupling constant λ to take the place of the semiclassical control. In this way, we can prove a stability result similar to Theorem 1.5 even for $h = 1$. This is stated formally in Theorem 1.6 below. There we retain the semiclassical parameter only to illustrate the possibility for interplay with λ for a stability time interval of fixed length. It should be noted that in our result, the $\sqrt{\lambda}$ error term in the $\dot{v}$ equation overwhelms the λ-linear term. This is one of the prices we pay for not requiring $h \ll 1$. What we stand to gain from our approach is the freedom to scale the size of the potential and discover if there is a transition from stable to unstable behavior as the strength of the potential increases. A possible direction of research is to seek probabilistic forms of Theorem 1.6 over a class of randomly chosen potentials $\{V_\omega\}_{\omega\in\Omega}$ with uniform L^∞ norm by studying random schrödinger operators of the form $(-\Delta + \mu) + \lambda V_\omega$.

Theorem 1.6 (Main Theorem)**.** *Suppose that $\psi \in C\left([0,\infty); H^1\left(\mathbb{R}^d\right)\right)$ solves (1.1), and let η_μ with $\mu > 0$ solve (1.4). Let $\eta_{\sigma(t)}$ be a traveling soliton with initial parameters $\sigma(0) = \sigma_0 = \{a_0, v_0, \gamma_0, \mu_0\}$, and suppose that for some $\epsilon_0 > 0$,*

$$||\psi(0) - \eta_{\sigma(0)}(0)||_{H^1} \leq \epsilon_0.$$

Then there exists some $T > 0$ and parameters $\sigma(t) := \{a(t), v(t), \gamma(t), \mu(t)\}$ such that for $t \leq T\left(\sqrt{\lambda h^3} + \epsilon_0^2\rho^{-1}\right)^{-\frac{1}{2}}$,

$$\begin{aligned} \psi(x,t) &= \eta_{\sigma(t)} + e^{i\phi(x-a(t),t)} w\left(x - a(t), t\right) \\ &= e^{i\phi(x-a(t),t)}\left[\eta_{\mu(t)}(x - a(t)) + w\left(x - a(t), t\right)\right] \end{aligned}$$

with

$$||w||_{H^1} = \mathcal{O}\left(\left(\lambda h^3\right)^{\frac{1}{4}} + \frac{\epsilon_0}{\sqrt{\rho}}\right).$$

Furthermore, the parameters $\sigma(t) = \{a(t), v(t), \gamma(t), \mu(t)\}$ satisfy some (approximately Newtonian) modulation equations

$$\begin{cases} \dot{a} & = v + \mathcal{O}\left(\sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right), \\ \frac{1}{2}\dot{v} & = -\lambda h \nabla V^h(a) + \mathcal{O}\left(\sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right), \\ \dot{\gamma} & = \mu - \frac{1}{4}|v|^2 + \frac{1}{2}\dot{a} \cdot v - \lambda V^h(a) + \mathcal{O}\left(\sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right), \\ \dot{\mu} & = \mathcal{O}\left(\sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right), \end{cases},$$

with initial conditions $\sigma(0) = \sigma_0$.

Because of the already mentioned symmetries of the problem, it is useful to introduce an equivalence class, or "orbit", of solutions generated by the symmetries. We formalize this notion as follows:

Definition 1.7 (Orbit of a solution)**.** *Let U be a group that induces the symmetries of the problem. Then the **orbit** of a function ψ (sometimes called the ψ-**orbit**) is the set*

$$\mathcal{G}_\psi := \{u\psi \text{ for some } u \in U\}. \tag{1.6}$$

Since the symmetries can be regarded as changing the frame of reference or perspective, it is typically stability of the ψ-orbit that is studied, rather than a single solution ψ. In this document we will use the notation $\mathcal{G} := \mathcal{G}_\eta$ for the orbit of η_μ when U is the group generated by the symmetries T_a^{tr}, T_v^b, T_γ^g, and T_μ^s. These symmetries are useful in obtaining the modulation equations of Theorem 1.6, as they give us parameters for minimizing the norm of $w(t) = \psi(t) - \eta_\sigma(t)$. They play a similar but less explicit role in the $\lambda = 0$ case in Chapter 9.

There is already a rich literature on the subject of soliton stability. Two papers of Grillakis, Shatah, and Strauss [12, 13] establish some general criteria under which a one-parameter family of solutions ψ of an abstract Hamiltonian system

$$\psi_t = J\mathcal{E}'(\psi) \tag{1.7}$$

is orbitally stable, for an "energy" functional $\mathcal{E}$ and a skew symmetric operator J. More specifically, they assume well-posedness of the problem on an abstract space X, as well as existence of solutions to (1.7) and also of equilibrium solutions having the form $u(t) = T(\omega t)\varphi(x)$ (named "bound states") where T is a group action inducing the one-parameter symmetry of the problem and ω is a real number. Defining the "linearized Hamiltonian" as $H_\omega = \mathcal{E}''(\varphi_\omega) - \omega\mathcal{Q}''(\varphi_\omega)$ with the aid of another conserved quantity $\mathcal{Q}$, they prove the following theorem:

Theorem 1.8 (Grillakis-Shatah-Strauss, 1987)**.** *Suppose that H_ω has exactly one negative eigenvalue and that the rest of its spectrum is positive and bounded away from 0. Then the bound state $u(t)$ defined above is orbitally stable if and only if H_ω is positive in a neighborhood of ω.*

M. I. Weinstein employs a similar strategy in [20], proving the following stability theorem for standing solitary waves solving (1.2):

Theorem 1.9 (M. I. Weinstein, 1986)**.** *Let $k < 2/d$ with $d = 1$ or $d = 3$, and let $\psi(x,t)$ be the unique solution of (1.2) with initial data $\psi_0 \in H^1$. Let η solve Equation (1.4), and let $\mathcal{G}_\eta$ be generated by the group of translation and phase changes T_a^{tr}, T_γ^g. Then for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ so that*

$$\rho(\psi_0, \mathcal{G}_\eta) < \delta(\epsilon)$$

implies

$$\rho(\psi(t), \mathcal{G}_\eta) < \epsilon$$

for all $t > 0$. (In other words, the η-orbit is orbitally stable.)

This theorem was an improvement on the work of Cazenave and Lions in [5], which was credited by M. I. Weinstein in [20] as the first to prove a fully nonlinear stability result (in the case of symmetric perturbations of the initial condition). Up to this point, he notes, stability of NLS solitons had been studied using compactness [4] and concentration-compactness [5] arguments. Weinstein's theorem applies to general H^1 perturbations of the initial condition and deals with a class of more general nonlinearities. It is this paper that our stability result for traveling waves solving (1.2) closely follows.

Y.-G. Oh's existence result (mentioned above) was closely followed by a paper [17] proving stability (instability) of ground states of (1.3). In this paper, Oh establishes a correspondence between stability (respectively, instability) and localization of the initial soliton profile u_0 at a local minimum (respectively, maximum) of the potential.

Theorem 1.10 (Y.-G. Oh, 1989)**.** *There exists some $h^* > 0$ such that the solutions $u_h = u_0 + w$ of the semiclassical problem (1.3) with perturbed initial value $u(0) = u_0$ are orbitally, or Lyapunov, stable (unstable) if $0 < h < h^*$ and the solutions u_h are localized at a local minimum (maximum) of V.*

1.2 Strategy of Proof

We follow the now well-known "Lyapunov" argument in the papers of Weinstein and Oh we have already mentioned. Treatments of this approach to stability can be found in chapter V of the Springer-Verlag textbook [3] (by Bahtia and Szegö) and elsewhere. The essence of the approach is laid out in the following definition and theorem.

Definition 1.11 (Lyapunov Functional)**.** *Let $\dot{x} = f(x)$ be a dynamical system. A functional $\mathcal{E} : V \to \mathbb{R}$ on a Banach space V is a* ***Lyapunov functional*** *for an equilibrium x^* of the dynamical system if there exists a neighborhood $U \subset V$ of x^* such that*

- *$\mathcal{E}(x) > \mathcal{E}(x^*)$, for all $x \in U \setminus x^*$ and*
- *$\frac{d}{dt}[\mathcal{E}(x(t))] \leq 0$, for all $t > 0$.*

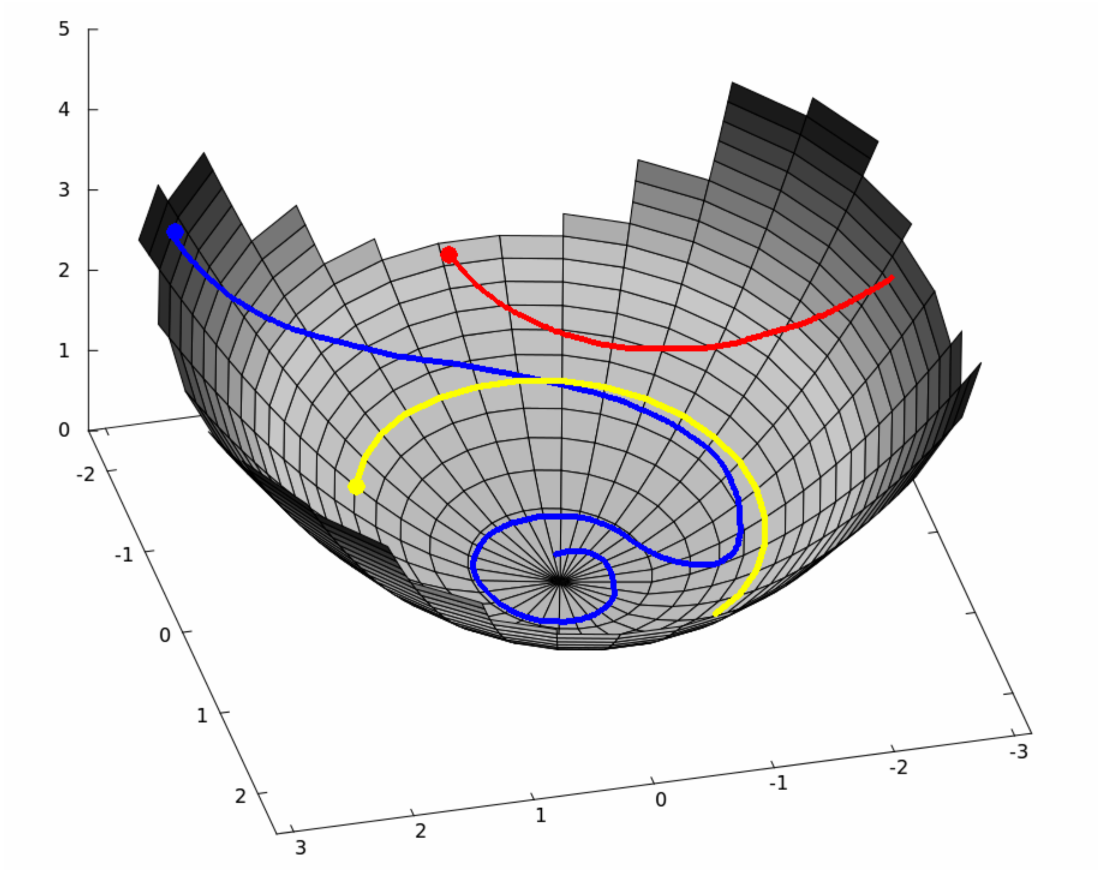

Figure 1.3: A candidate Lyapunov functional with decreasing (blue), non-increasing (yellow), and uncontrolled (red) orbits.

Theorem 1.12 (Lyapunov Stability). *If a Lyapunov functional can be constructed for an equilibrium x^* of the evolution equation $\dot{x} = f(x)$, then x^* is orbitally stable. If the inequality in the last condition of Definition 1.11 is strict for all $t > 0$, then x^* is asymptotically stable.*

The intuition motivating the definition and theorem is that as long as the solution x originates sufficiently close to x^*, its energy $\mathcal{E}(x)$ can never increase, and so it can never flow over the lip of the $\mathcal{E}$-bowl or wander far from the local minimum of $\mathcal{E}$ at x^*. The desired behavior is illustrated by the blue and yellow solution trajectories in Figure 1.3. The red trajectory in the same figure illustrates the increasing energy behavior that a Lyapunov functional cannot allow. Because of the last condition, it is natural to construct such a functional from conserved quantities of the problem.

In each of the (Lyapunov) stability results of Weinstein and Oh, the key to establishing the second condition for the Lyapunov functional is spectral analysis of the Hessian $\mathcal{L} := \mathcal{E}''$ at the ground state η. Theorem 1.8 from [12] does the same in a more general setting for $\mathcal{L} := \mathcal{E}(\cdot) - \omega\mathcal{Q}(\cdot)$, showing that this quantity is conserved in time and is locally convex at the bound state (and critical point) $u(t)$. In this manner, it is shown that $\mathcal{L}$ has only one negative eigenvalue, which can then be avoided due to some orthogonality conditions arising from the symmetries discussed above. The Lyapunov functional is thus shown to be convex enough to guarantee the second condition. The FGJS approach in [7] is of the same Lyapunov flavor, but more attention is given to the parameters σ, and only "approximate" conservation of an energy functional is used. This approximation is enough to guarantee stability for finite time, but it is an artifact of the argument that letting $V = 0$ does not recover stability of solitons for all time.

There are two conserved quantities in the problem (1.1), the Hamiltonian energy of the system,

$$\mathcal{H}_\lambda(\psi) := \frac{1}{2}\int \left(|\nabla\psi|^2 + \lambda V^h|\psi|^2\right)\, dx - F(\psi),$$

and the square integral (usually interpreted as particle mass) of the solution,

$$\mathcal{N}(\psi) = ||\psi||_2^2.$$

Here $F(u)$ is a functional with Fréchet derivative $F'(u) = f(u)$. Proofs of the conservation laws may be found in Appendix B. For the case $\lambda = 0$, the Lyapunov functional we construct is a weighted sum of these two quantities:

$$\mathcal{E}_\mu(\psi) := \mathcal{H}_0(\psi) + \mu\mathcal{N}(\psi) = \frac{1}{2}\int \left(|\nabla\psi|^2 + \mu|\psi|^2\right)\, dx - F(\psi).$$

By use of Equation (1.4) and integration by parts, we find that $\mathcal{E}_\mu(\eta_\mu) = 0$. For the case $\lambda \neq 0$, $\mathcal{H}_\lambda(\psi)$ is easily seen to be conserved in time, but it is no longer translation invariant. For this reason, following [7], we retain $\mathcal{E}_\mu$ as an "approximately conserved" energy functional. While this choice of functional allows us to retain the basic shape of the Lyapunov argument, it breaks the last two conditions in Definition 1.11. This is why we do not obtain global-time orbital stability in Theorem 1.6.

1.3 Related Research in Asymptotic Stability

All of the above results have concerned orbital stability, but some researchers are studying asymptotic stability as well. In 2013, Scipio Cuccagna and Dmitry E. Pelinovsky proved asymptotic stability (in L^∞ norm) of soliton solutions for the following cubic NLS:

$$iu_t + u_{xx} + 2|u|^2u = 0,\ u(0) = u_0.$$

In their paper [6], they employ a steepest descent method attributed by them to Deift and Zhou, as well as scattering and inverse scattering transforms. They obtain both positive and negative time asymptotics, with fixed symmetry parameters.

Theorem 1.13 (Cuccagna-Pelinovsky, 2013)**.** *Fix $s \in (1/2, 1]$, $d = 1$, let $\epsilon > 0$ be given, and consider an initial value $u(0) = u_0$ perturbed by at most ϵ in $L^{2,s}$ norm from a traveling soliton φ_{σ_0} with some initial parameters σ_0. Then there exist constant asymptotic parameters $\sigma_{\pm\infty}$ close to those of the original soliton φ such that for $\pm t \geq T > 0$,*

$$||u(t) - \varphi_{\sigma_{\pm\infty}}||_{L^\infty(\mathbb{R}^d)} \leq C\epsilon|t|^{-\frac{1}{2}}.$$

Here the $L^{2,s}$ norm is just

$$||f||_{L^{2,s}} := || \left(\sqrt{1+x^2}\right)^s f||_{L^2}.$$

Z. Gang and I. M. Sigal studied 'trapped' solitons (those localized near a minimum of the potential) of Equation 1.1 in [8, 9] with various conditions on the potential. By working in weighted L^2 spaces they obtain asymptotic stability of solitons, showing that excess energy from perturbations is radiated to infinity. Whether soliton solutions of (1.1) (for $\lambda = 0$ or otherwise) or related NLS are asymptotically stable in an unweighted L^2 norm does not appear to be known at this point.

1.4 Document Structure

The structure of this document is as follows. Chapter 2 sets the stage for spectral theory of the Hessian form $\mathcal{E}''_\mu(\eta)$ in both the $\lambda > 0$ and $\lambda = 0$ problems, beginning with a quick review of Fréchet derivatives and critical points of $\mathcal{E}_\mu$. Chapter 5 contains the specialized coercivity result for $\lambda > 0$, after Chapters 3 and 4 have introduced the necessary orthogonality conditions and local existence of the time-dependent parameters $\sigma(t)$ governing the traveling soliton $\eta_{\sigma(t)}$. Chapter 6 decomposes the solution ψ into a soliton $\eta_{\sigma(t)}$ and an error term w according to these orthogonality conditions and gives estimates on the deviation of the parameters $\sigma(t)$ from those of a free soliton. It is the concern of Chapter 7 to show that the time dependence of $\mathcal{E}_\mu(\psi(t))$ is negligible on the time scale we consider, while Chapter 8 draws all preceding chapters together and closes the proof of the main theorem.

Chapters 9 and 10 are devoted to proving the extension of Weinstein's stability result, with some rehashing of spectral analysis that was not compatible with Chapter 5. We have already asserted in the introduction (it is proved in Appendix B) that the energy $\mathcal{E}_\mu$ of the system is conserved for $\lambda = 0$. According to the definition and theorem of the Lyapunov method, it remains to show that η_σ is a local minimum of $\mathcal{E}_\mu$.

The appendices contain a proof of Ehrenfest's Law, as well as some elementary calculations and proofs deemed too unwieldy to include in the body of this document.

1.5 General Notation

Regarding notation, we follow the standard $||\cdot||_p$ and $||\cdot||_{H^1}$ for L^p and Sobolev norms respectively. As a shorthand we set

$$||\cdot|| = ||\cdot||_2,$$

and we recall that

$$||u||_{H^1}^2 = ||\nabla u||^2 + ||u||^2.$$

We use both the standard L^2 complex inner product

$$(u, v) := \int u\bar{v}\; d^dx,$$

as well as a real-valued version

$$\langle u, v \rangle := Re \int u\bar{v} \; d^d x$$

that is distinguished by its angle brackets. From the latter, we construct a symplectic form

$$\omega\left(u, v\right) := Im \int u\overline{v} \; d^d x = \left\langle u, J^{-1}v \right\rangle,$$

which we employ frequently throughout the paper. Now and hereafter, the operator J indicates multiplication by i^{-1}.

Chapter 2 Preliminary Spectral Analysis of $\mathcal{L}_\eta$

As already mentioned, our project is to prove a fully nonlinear stability result. This notion differs significantly from linear stability, so we present the reader with a brief comparison of the main ideas. In linear stability analysis, negativity of the real parts of all eigenvalues of the linearized operator indicates linear stability of an equilibrium. If A is a linearized operator with eigenvalues λ_j, then the linearized problem

$$\dot{\psi} = A\psi$$

is formally solved by combinations of exponentials of the form $e^{\lambda_j t}$. If the real parts of the eigenvalues are negative, the exponentials will decay in time. The perturbed solution thus relaxes to the equilibrium, indicating linear stability. In contrast, each negative eigenvalue of $\mathcal{E}''_\mu(\eta_\sigma)$ in the orbital (nonlinear) stability setting represents a collapse of the Lyapunov functional $\mathcal{E}_\mu$ in some direction. This prevents the functional from having a local minimum at the soliton profile η_σ, and so there is nothing preventing nearby solutions from escaping to infinity. For an illustration of this, see Figure 2.1. In this figure, one can observe that a solution with energy trajectory on the red line is not trapped by the functional, while one with trajectory remaining on the green line is trapped. In order to salvage the argument, we must obtain constraints on the perturbation of ψ from η_σ that forbid evolution of the solution ψ into the negative eigenspace of $\mathcal{L}_\eta$. Here we are saved by the manifold of infinitely many equivalent ground states, which allows us to choose parameters for a traveling soliton that closely approximates the perturbed solution ψ and stays out of the negative eigenspace of $\mathcal{L}_\eta$.

Our approach to the $\lambda = 0$ problem (1.2) (seen in Chapters 9 - 10) follows [20], choosing traveling soliton parameters a, v, γ so that the time evolving perturbation is minimized (in a modified H^1 norm) for each $t > 0$. Because the minimization problem constrains certain inner products to vanish, these constraints are often known as *orthogonality conditions*. For the $\lambda > 0$ argument, our approach follows the similar methods of the Fröhlich-Gustafson-Jonsson-Sigal paper [7], but there the symplectic form ω is introduced first and the minimization is obtained by imposing a decomposition into a soliton and an error function $w(x, t)$ that is skew orthogonal to the the tangent space of the soliton manifold $\mathcal{G}$ (Chapter 3). In each approach, these orthogonality conditions allow us to avoid the single negative eigenvalue of $\mathcal{L}_\eta$.

In the current chapter, we establish basic facts about (Fréchet) derivatives of $\mathcal{E}_\mu$. We determine the negative eigenvalues and nullspace of the Fréchet Hessian operator, which we write as $\mathcal{L}_\eta := \mathcal{E}''_\mu$. On the way, we observe that η_σ is a critical point of $\mathcal{E}_\mu$. Since $\mathcal{E}_\mu(\psi)$ is polynomially bounded in ψ in H^1-norm, it can be expanded in a Fréchet power series around some function $\eta \in H^1$ as below:

$$\mathcal{E}_\mu(\eta + \xi) = \mathcal{E}_\mu(\eta) + \mathcal{E}'_\mu(\eta)(\xi) + \mathcal{E}''_\mu(\eta)(\xi, \xi) + \ldots.$$

Thus, to show that η is a local minimum of $\mathcal{E}_\mu$, it suffices to prove that $\mathcal{E}'_\mu(\eta) = 0$ and $\mathcal{E}''_\mu(\eta)(\xi, \xi) > 0$ in a neighborhood of η.

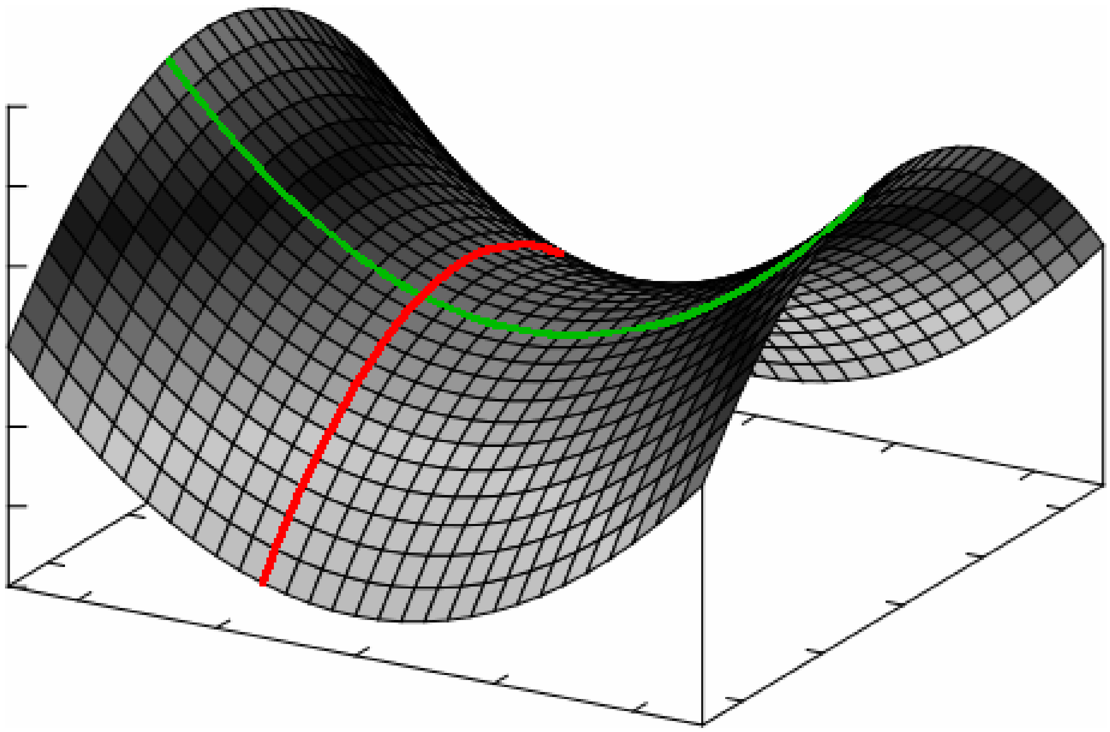

Figure 2.1: "Bad" direction(s) of the energy functional $\mathcal{E}_\mu$ near η_μ correspond to a negative eigenvalue of $\mathcal{E}_\mu''$. Shown are directions with upward (green) and downward (red) concavity of $\mathcal{E}_\mu$ on $H^1(\mathbb{R}^d)$.

We do not forget that since $\mathcal{H}_0$ is not a conserved quantity of (1.1), the case $\lambda \neq 0$ requires careful, time-dependent analysis. Instead, facts about $\mathcal{E}_\mu$ in the zero potential case provide a starting point for the careful estimates made later in this document.

One last comment should be made. In the calculations and proofs of this chapter, we concentrate on the soliton profile (or ground state) η_μ instead of the traveling soliton η_σ. This is due to a direct calculation showing that

$$\mathcal{E}_\mu(\eta_\sigma) = \mathcal{E}_{\mu+|\nabla\phi|^2}(\eta_\mu) = \mathcal{E}_{\mu+\frac{1}{4}|v(t)|^2}(\eta_\mu). \tag{2.1}$$

All that is involved in showing the above formula is translation invariance of the Lebesgue integral on $\mathbb{R}^d$, the observation that $\langle i\eta_\mu, \nabla\eta_\mu\rangle = 0$, and the definitions of $\mathcal{E}_\mu$ and the phase function $\phi(x,t)$. Because of this formula, calculations done for η_μ carry information also about η_σ.

2.1 Fréchet derivatives of $\mathcal{E}_\mu$

We turn now to straightforward calculations of the Fréchet derivatives of $\mathcal{E}_\mu$. Recall that we have defined

$$\mathcal{E}_\mu(u) = \frac{1}{2}\int \left(|\nabla u|^2 + \mu|u|^2\right)\, dx - F(u).$$

Then $\mathcal{E}_\mu'(u)$ is the (linear) operator K that satisfies

$$\lim_{||w||_{H^1}\to 0} \frac{||\mathcal{E}_\mu(u+w)-\mathcal{E}_\mu(u)-Kw||}{||w||_{H^1}} = 0.$$

To discover K, we expand the difference $\mathcal{E}_\mu(u+w)-\mathcal{E}_\mu(u)$ as follows:

$$\begin{aligned}
&\mathcal{E}_\mu(u+w)-\mathcal{E}_\mu(u)\\
=&\frac{1}{2}\int \left[|\nabla u+\nabla w|^2+\mu|u+w|^2\right]\, dx - F(u+w)\\
&-\left[\frac{1}{2}\left(\int |\nabla u|^2+\mu|u|^2\right)\, dx - F(u)\right]\\
=&\frac{1}{2}\int \left[|\nabla w|^2+2Re(\nabla u\overline{\nabla w})+\mu\left(|w|^2+2Re(u\overline{w})\right)\right]\\
&-\left[F(u+w)-F(u)\right]\\
=&\frac{1}{2}\int \left[|\nabla w|^2-2Re(\Delta u\overline{w})+\mu\left(2Re(u\overline{w})\right)+\mathcal{O}(w^2)\right]\, dx - \left[F(u+w)-Fu\right]\\
=&\int \left[-Re(\Delta u\overline{w})+\mu\left(Re(u\overline{w})\right)\right]+\mathcal{O}\left(||w||_{H^1}^2\right)\, dx - \left[F(u+w)-Fu\right].
\end{aligned}$$

Thus, $Kw = Re\left[\int\left(-\Delta+\mu\right)\overline{w}-F'(u)\overline{w}\right] = \langle\left(-\Delta+\mu\right)u - f(u), w\rangle$, or alternately,

$$\mathcal{E}_\mu'(u) = \left(-\Delta+\mu\right)u - f(u).$$

Note that for $u=\eta_\mu$ being a soliton profile with energy μ, we have

$$\mathcal{E}_\mu'(\eta_\mu) = 0$$

by Equation (1.4).

Now we compute the second Fréchet derivative, $\mathcal{E}_\mu''$. This is the linear operator L satisfying

$$\lim_{||w||_{H^1}\to 0} \frac{||\mathcal{E}_\mu'(u+w)-\mathcal{E}_\mu'(u)-Lw||}{||w||_{H^1}} = 0.$$

After some easy cancellations, we obtain

$$\begin{aligned}
0 &= \lim_{||w||_{H^1}\to 0} \frac{||\mathcal{E}_\mu'(u+w)-\mathcal{E}_\mu'(u)-Lw||}{||w||_{H^1}}\\
&= \lim_{||w||_{H^1}\to 0} \frac{||\left(-\Delta+\mu\right)(u+w)-f(u+w)-\left[\left(-\Delta+\mu\right)(u)-f(u)\right]-Lw||}{||w||_{H^1}}\\
&= \lim_{||w||_{H^1}\to 0} \frac{||\left(-\Delta+\mu\right)w-\left(f(u+w)-f(u)\right)-Lw||}{||w||_{H^1}}
\end{aligned}$$

and so

$$\mathcal{E}_\mu''(u)w = Lw = \left(-\Delta+\mu-f'(u)\right)w.$$

We will refer often to this second derivative evaluated at the function $u = \eta_\mu$ by the name

$$\mathcal{L}_\eta := \mathcal{E}''_\mu(\eta_\mu) = (-\Delta + \mu - f'(u))\,\eta_\mu.$$

Here the explicit form of $f'(u)$ is given by the following calculation:

$$\begin{aligned} f'(\eta)\xi &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} f(\eta + \epsilon\xi) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(\left|\eta + \epsilon\xi\right|^{2k} \cdot (\eta + \epsilon\xi)\right) \\ &= \left(k\left|\eta + \epsilon\xi\right|^{2(k-1)}(\eta + \epsilon\xi) 2Re(\eta\bar{\xi}) + \left|\eta + \epsilon\xi\right|^{2k}\xi\right)\Big|_{\epsilon=0} \\ &= \left|\eta\right|^{2k}(2kRe(\xi) + \xi) \end{aligned} \tag{2.2}$$

It can be determined by direct calculation that $\mathcal{L}_\eta$ is the linearized operator for the stationary equilibrium soliton $e^{i\mu t}\eta_\mu$ of the problem (1.2). We remark also that since $\mathcal{L}_\eta$ is visibly self-adjoint, its eigenvalues will be real numbers.

The spectral analysis of $\mathcal{L}_\eta$ is a well-known but lengthy argument which we take mostly from [19], [20], and [7]. Since the analysis is completely independent of the external potential, we present it here in Chapters 2 - 5.

2.2 Negative Eigenvalues of $\mathcal{L}_\eta$

To show that there exists only one such eigenvalue, we repeat here the statement and proof of an abstract proposition given in [7], but then immediately specify the result to our setting.

Let X be a Banach space, let $K \in C^3(X, \mathbb{R})$, and define

$$M := \ker K = \{u \in X : K(u) = 0\}.$$

Further, let $\mathcal{E}$ be a C^2 functional on X. Now assume there exists a Hilbert space H such that X is a dense subset, and that the Hessian quadratic form $Hess\,\mathcal{E}(u)\,(\alpha, \beta)$ defines a self-adjoint operator $\mathcal{E}''(u)$ on H such that

$$\langle \alpha, \mathcal{E}''(u)\beta\rangle = Hess\,\mathcal{E}(u)\,(\alpha, \beta)$$

for all $\alpha \in X$ and $\beta \in D(\mathcal{E}'') \subset X$.

Here the first Fréchet derivative $\mathcal{E}'(\eta) : T_\eta M \to \mathbb{R}$ is a functional acting on the tangent space of the manifold M, while the Hessian can be viewed as a bilinear form $\mathcal{E}''(\eta) : T_\eta M \times T_\eta M \to \mathbb{R}$ or as a map into the dual of the tangent space of M as below:

$$\mathcal{E}''(\eta) : T_\eta M \to (T_\eta M)^*.$$

In any case, the tangent space $T_\eta M$ is given by local coordinates as

$$T_\eta M = \left\{\partial_s \eta_s|_{s=0} : \eta_s \in C^1([0, \epsilon], M), \eta_{s=0} = \eta\right\}.$$

Then we have the following result:

Proposition 2.1. *If η is a minimizer of $\mathcal{E}$ on the set $M \subset X$, and $K'(\eta) \neq 0$, then the Hessian operator $\mathcal{E}''(\eta)$ has at most one negative eigenvalue.*

Proof. We first show that $T_\eta M = \{\xi \in X : \langle \mathcal{K}'(\eta), \xi \rangle = 0\} =: K'(\eta)^\perp$ by a double containment argument. Note that by hypothesis, $\mathcal{E}'(\eta) = 0$ and $\mathcal{E}''(\eta) \geq 0$.

[$\subseteq$]: Let $\xi \in T_\eta M$. Then by the previous characterization of the tangent space there exists η_s such that $\xi = \partial_s|_{s=0}\eta_s$, and further, η_s takes values in M. Thus, by the chain rule for Fréchet derivatives

$$0 = \partial_s (0)|_{s=0} = \partial_s K(\eta_s)\Big|_{s=0} = \langle K'(\eta_s)|_{s=0},\ \partial_s \eta_s|_{s=0} \rangle = \langle K'(\eta),\ \xi \rangle .$$

[$\supseteq$]: Let $\xi \in X$ satisfy $\langle \mathcal{K}'(\eta), \xi \rangle = 0$, and define

$$f(a,s) := \frac{1}{s^2} K\left(\eta + s\xi + s^2 a K'(\eta)\right).$$

Expanding $s^2 f(a,s)$ in a power series, we obtain

$$\begin{aligned} s^2 f(a,s) =& K\left(\eta + s\xi + s^2 a K'(\eta)\right)|_{s=0} + s\frac{d}{ds}\left[K\left(\eta + s\xi + s^2 a K'(\eta)\right)\right]|_{s=0} \\ &+ s^2 \frac{d^2}{d^2 s}\left[K\left(\eta + s\xi + s^2 a K'(\eta)\right)\right]|_{s=0} + \mathcal{O}(s^3) \\ =& K(\eta) + s\langle K'(\eta), \xi\rangle \\ &+ s^2\left[\left\langle K''(\eta), [\xi]^2\right\rangle + \langle K'(\eta), aK'(\eta)\rangle\right] + \mathcal{O}(s^3). \end{aligned} \tag{2.3}$$

Here we have used a consequence of the chain rule on second derivatives

$$\frac{d^2}{d^2 s}\left[K(q(s))\right]|_{s=0} = \left[K''(q(0)) \cdot [q'(0)]^2 + K'(q(0)) \cdot q''(0)\right]$$

to obtain the last equality above. Thus, (2.3) becomes

$$\begin{aligned} s^2 f(a,s) =& K(\eta) + s\langle K'(\eta), \xi\rangle|_{s=0} \\ &+ s^2\left[\left\langle K''(\eta), [\xi]^2\right\rangle + \langle K'(\eta), aK'(\eta)\rangle\right] + \mathcal{O}(s^3) \end{aligned}$$

and by our assumptions on K and ξ, we find that

$$s^2 f(a,s) = 0 + s\cdot 0 + s^2\left[\left\langle K''(\eta), [\xi]^2\right\rangle + \langle K'(\eta), aK'(\eta)\rangle\right] + \mathcal{O}(s^3)$$

or

$$f(a,s) = \left\langle K''(\eta), [\xi]^2\right\rangle + \langle K'(\eta), aK'(\eta)\rangle + \mathcal{O}(s).$$

Choosing

$$b = -\frac{\langle \xi, K''(\eta)\xi\rangle}{||K'(\eta)||^2}$$

now yields

$$f(b,0) = 0$$

and

$$
\begin{aligned}
\partial_a f(b,0) &= \lim_{s\to 0} \frac{1}{s^2} \left\langle K'\left(\eta + s\xi + s^2 b K'(\eta)\right),\ s^2 K'(\eta)\right\rangle \\
&= \lim_{s\to 0} \left\langle K'\left(\eta + s\xi + s^2 b K'(\eta)\right),\ K'(\eta)\right\rangle \\
&= ||K'\left(\eta\right)||^2 \\
&\neq 0.
\end{aligned}
$$

Finally, invoking the implicit function theorem yields a map $s \mapsto a(s)$ so that $f\left(a,s\right) = 0$. Thus $\eta_s := \eta + s\xi + s^2 a(s) K'(\eta)$ satisfies

$$
\eta_{s=0} = \eta,
$$

$$
\partial_s \eta_s|_{s=0} = \xi,
$$

and

$$
K(\eta_s) = s^2 f\left(a,s\right) = 0,
$$

as desired.

Having established

$$
T_\eta M = K'(\eta)^\perp,
$$

we can rewrite $\mathcal{E}''(\eta) \geq 0$ for $\eta \in M$ as

$$
\inf_{\xi \in K'(\eta)^\perp} \langle \xi, \mathcal{E}''(\eta)\xi\rangle \geq 0.
$$

Applying the max-min principle, we find that the number of non-positive eigenvalues of $\mathcal{E}''(\eta)$ is no greater than the co-dimension of $K'(\eta)^\perp$, which is 1.

□

With the proof complete, we now specify $X = H^1(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d)$, and $\mathcal{E} = \mathcal{E}_\mu$, recalling that we have previously defined

$$
\mathcal{E}_\mu(\psi) := \frac{1}{2}\int \left(|\nabla\psi|^2 + \mu|\psi|^2\right)\, dx - F(\psi).
$$

Using the scaling transformation T^s_μ from Chapter 1 to write $\eta_\mu(x) = \mu^{1/2k}\eta_1(\mu^{1/2}x)$, we quickly obtain via chain rule (and integrate by parts in the second term of the penultimate line)

$$
\begin{aligned}
\partial_\mu \int [\eta_\mu(x)]^2 =& \partial_\mu \int \mu^{1/k}\left[\eta_1(\mu^{1/2}x)\right]^2 \\
&= \left(\partial_\mu \mu^{1/k}\right)\int \left[\eta_1(\mu^{1/2}x)\right]^2 + \mu^{1/k}\int \partial_\mu \left[\eta_1(\mu^{1/2}x)\right]^2 \\
&= \left(1/k \cdot \mu^{1/k-1}\right)\int \left[\eta_1(\mu^{1/2}x)\right]^2 + \mu^{1/k-1/2}\int \eta_1(\mu^{1/2}x)\nabla\eta_1(\mu^{1/2}x) \\
&= \left(1/k \cdot \mu^{1/k-1}\right)\int \left[\eta_1(\mu^{1/2}x)\right]^2 + 0
\end{aligned}
$$

$>0.$

Thus we find, by using Theorem 3 of [12], that η minimizes $\mathcal{E}_\mu$ with respect to the constraint $K(u) := \frac{1}{2}\int |u|^2 \ d^dx \ - \ m = 0$. Applying Proposition 2.1, we see that $\mathcal{L}_\eta = \mathcal{E}''_\mu(\eta_\mu)$ has at most one negative eigenvalue.

Moreover, since $\mathcal{L}_\eta \partial_\mu \eta = -\eta$, we have

$$\langle \partial_\mu \eta, \mathcal{L}_\eta \partial_\mu \eta \rangle = \int \partial_\mu \eta \cdot (-\eta) = -\frac{1}{2}\partial_\mu \int \eta^2 < 0,$$

and so we know that $\mathcal{L}_\eta$ must also have at least one negative eigenvalue. Hence $\mathcal{L}_\eta$ has exactly one negative eigenvalue.

2.3 The Null Space of $\mathcal{L}_\eta$

A non-trivial zero eigenvalue of $\mathcal{L}_\eta$ is no safer than a negative one, since this admits degenerate local minima in the form of level troughs leading to infinity. We must therefore identify the nullspace of $\mathcal{L}_\eta$ and introduce more orthogonality conditions to keep our solution of (1.1) from falling into it. To this end, we pick a natural basis for $\mathbb{C} = \mathbb{R}^2$ and decompose complex-valued functions $u \in L^2\left(\mathbb{R}^d; \mathbb{C}\right)$ as $u_1 + iu_2$ for $u_1, u_2 \in L^2\left(\mathbb{R}^d; \mathbb{R}\right)$. This allows us to split our operator into actions on real and complex parts as follows:

$$\mathcal{L}_\eta u = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} L_1 u_1 \\ L_2 u_2 \end{bmatrix},$$

where

$$L_1 = -\Delta + \mu - (2k+1)\eta^{2k}$$

and

$$L_2 = -\Delta + \mu - \eta^{2k}.$$

This is done by using the explicit form $f'(u)\xi = \left|u\right|^{2k}(2kRe(\xi) + \xi)$ from (2.2). Then by calculations done in Chapter 3, we have

$$\mathcal{L}_\eta i\eta = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ L_2 \eta \end{bmatrix} = \vec{0}$$

and

$$\mathcal{L}_\eta \partial_{x_i} \eta = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \partial_{x_i}\eta \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 \partial_{x_i}\eta \\ 0 \end{bmatrix} = \vec{0}$$

which imply that the nullspace $Null(\mathcal{L}_\eta)$ of $\mathcal{L}_\eta$ contains $\begin{bmatrix} 0 \\ \eta \end{bmatrix}$ and $\begin{bmatrix} \partial_{x_i}\eta \\ 0 \end{bmatrix}$. In fact, we can show that this spans $Null(\mathcal{L}_\eta)$.

Proposition 2.2. *$Null\left(\mathcal{L}_\eta\right) = span_{\mathbb{R}}\left\{(0, \eta), (\partial_{x_i}\eta, 0)\right\}$.*

Proof. Since $\eta \in L^2(\mathbb{R}^d)$, $\partial_{x_i}\eta$ has at least one zero, and so (by oscillation theory) L_1 has at least one negative eigenvalue. Hence, L_2 is a non-negative operator.

Proving $Null\,(L_1) = \mathbb{C}\eta'$ is more complicated, so we give a proof for the case $d = 1$ and for other dimensions refer to [20] and [7]. Note that for $d = 1$, $L_1 w = 0$ is a homogeneous ODE, so we may use a special case of Abel's identity,

$$W\left(y_1, y_2\right)(x) = W\left(y_1, y_2\right)(x_0) \cdot \exp\left(\int_{x_0}^{x} -p\left(\xi\right)\, d\xi\right),$$

to compute the Wronskian of two solutions of the DE, where for us $p\left(\xi\right) = 0$. Hence the Wronskian is independent of x. Supposing then, that ν were another element of $Null(L_1)$, we obtain

$$W\left(\eta', \nu\right) = \det\begin{bmatrix} \eta' & \nu \\ \eta'' & \nu' \end{bmatrix} = \eta'\nu' - \nu\eta'' = \lim_{x\to\infty}\left(\eta'\nu' - \nu\eta''\right) = 0.$$

Separating variables in this new equation, we find that

$$\frac{\nu'}{\nu} = \frac{\eta''}{\eta'},$$

or

$$\ln\nu = \ln\left(\eta'\right) + C,$$

for some $C \in \mathbb{C}$. This last equation forces $\nu = C\eta' \in \mathbb{C}\eta'$.

□

Now that we know $\mathcal{L}_\eta$ has a negative eigenvalue and nullspace $Null\,(\mathcal{L}_\eta) = \{(0, \eta), (\partial_{x_i}\eta, 0)\}$, we can take steps to avoid these bad directions. It turns out that skew orthogonality to the tangent space of $\mathcal{G}$ at η (via the symplectic form ω) is a sufficient condition. In the next chapter, we introduce this tangent space, and in Chapter 5 present proof of a coercivity estimate on $\mathcal{L}_\eta$.

Chapter 3 The Tangent Space $T_\eta \mathcal{G}$

As mentioned in the introduction of this document, we wish to choose the representative of $\mathcal{G}$ closest to the solution ψ of (1.1). Recall that $\mathcal{G} := \mathcal{G}_\eta$ is the notation we use for the manifold of equivalent solitons generated by the four symmetry groups discussed (just after Theorem 1.3) in the introduction. Figure 3.1 The current chapter is devoted to constructing a basis for the space $T_\eta \mathcal{G}$ of functions tangent to the manifold of solitons $\mathcal{G}$ at the representative η and to identifying the actions of $\mathcal{L}_\eta$ on this basis. We will use this basis in Chapter 4 to obtain parameters σ that describe a decomposition of the solution ψ of Equation (1.1) into tangent and normal components.

In Chapter 6, this information and the actions of $\mathcal{L}_\eta$ on the basis of $T_\eta \mathcal{G}$ will be useful in achieving a factorization of (1.1) that reveals the potential-modified equations for the parameter family $\sigma(t)$.

3.1 The Tangent Space to the Manifold of Solitons

Because $\mathcal{G}$ is completely determined by the symmetries it describes, its tangent vectors at the representative $\eta_\mu \in \mathcal{G}$ can be easily computed. These tangent vectors define local coordinates for $\mathcal{G}$ via the cotangent bundle $T^*\mathcal{G}$, allowing us to simplify the task of minimizing over a set of parameters to management of a collection of orthogonality conditions (see Chapter 4). We define the relevant directional derivatives and include the direct calculations below:

$$\begin{aligned} z_t :=& \nabla_a \left(T_a^{tr} \eta_\mu\right) |_{a=0} = \nabla_a \left[\eta_\mu(x-a)\right] |_{a=0} \\ =& -\nabla \eta_\mu, \end{aligned}$$

$$\begin{aligned} z_g :=& \frac{\partial}{\partial \gamma} \left(T_\gamma^g \eta_\mu\right) |_{\gamma=0} = \frac{\partial}{\partial \gamma} \left[e^{i\gamma} \eta_\mu(x)\right] |_{\gamma=0} \\ =& i\eta_\mu, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} z_b :=& \nabla_v \left(T_v^{gal} \eta_\mu\right) |_{v=0,t=0} = \nabla_v \left[e^{i\left(\frac{1}{2} v\cdot x - \frac{1}{4}|v|^2 t\right)} \eta_\mu\left(x - vt\right) |_{t=0}\right] |_{v=0} \\ =& \nabla_v \left[e^{i\left(\frac{1}{2} v\cdot x\right)} \eta_\mu\left(x\right)\right] |_{v=0} \\ =& \frac{i}{2} x \eta_\mu, \end{aligned}$$

$$\begin{aligned} z_s :=& \frac{\partial}{\partial \tilde{\mu}} \left(T_{\tilde{\mu}}^s \eta_1\right) |_{\tilde{\mu}=\mu} \\ =& \partial_\mu \eta_\mu. \end{aligned}$$

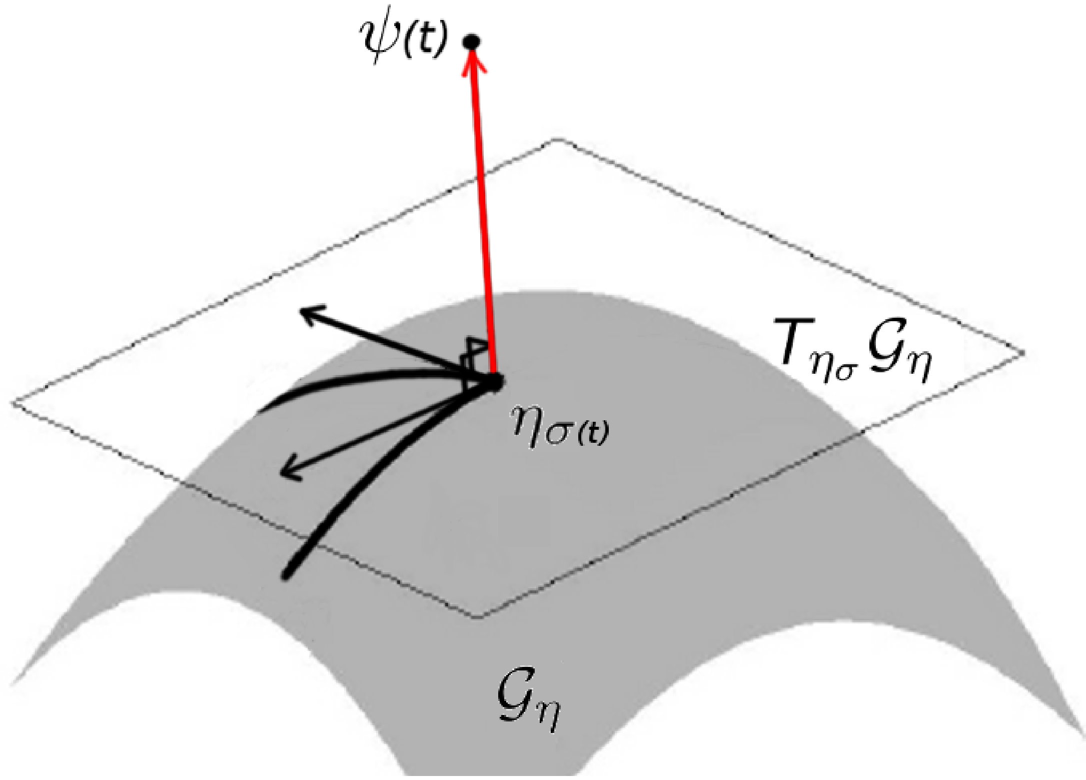

Figure 3.1: Parameters $\sigma(t)$ chosen locally on $T_{\eta_\sigma}\mathcal{G}_\eta$ to minimize error $||\psi(t) - \eta_{\sigma(t)}||_2$ (shown in red).
Credit for the original image is due to `www.researchgate.net/figure/228371900_fig1_Figure-1-Optimization-on-a-manifold-The-tangent-space-T-M-x`

Note that when we write ∇_y for $(y_1, y_2, \ldots, y_d) = y \in \mathbb{R}^d$, we mean

$$\nabla_y := (\partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_d}).$$

We are now ready to claim and prove the following proposition.

Proposition 3.1. *The set $\{z_t, z_g, z_b, z_s\}$ forms a basis of $T_\eta\mathcal{G}$.*

Proof. Due to the limited number of parameters, $\{z_t, z_g, z_b, z_s\}$ must be a spanning set of $T_\eta\mathcal{G}$. By observing that $\nabla\eta$ and $ix\eta$ are odd functions, while $\partial_\mu\eta_\mu$ and $i\eta$ are even, and that in each of these pairs one function is real while the other is pure imaginary, we conclude that $\{z_t, z_g, z_b, z_s\}$ is also a linearly independent set, hence a basis of $T_\eta\mathcal{G}$.

□

Remark: In an abuse of notation, we consider z_t and z_b as sub-collections of linearly independent functions, so we really have a basis of $2d + 2$ functions.

3.2 Actions of $\mathcal{L}_\eta$ on $T_\eta \mathcal{G}$

The current stability approach depends heavily on the nature of $\mathcal{L}_\eta = \mathcal{E}''_\mu$ and its coercivity. For technical reasons, it will be useful to know how $\mathcal{L}_\eta$ acts on the tangent space of the manifold of solitons. Recall that in the last chapter, we defined

$$\mathcal{L}_\eta u := \mathcal{E}''_\mu(\eta_\mu)u = \left(-\Delta + \mu - f'(\eta_\mu)\right) u$$

We record the following identities:

$$\begin{aligned}
\mathcal{L}_\eta z_t =& 0, \\
\mathcal{L}_\eta z_g =& 0, \\
\mathcal{L}_\eta z_b =& 2iz_t, \\
\mathcal{L}_\eta z_s =& iz_g.
\end{aligned}$$

It is also worth noting that

$$\mathcal{L}^2_\eta z_b = \mathcal{L}_\eta(-2iz_t) = 0, \quad \mathcal{L}^2_\eta z_s = \mathcal{L}_\eta iz_g = 0.$$

Below are included the explicit calculations. In the second and third calculations, we use Equation (2.2) derived from the explicit form of the nonlinearity.

$$\begin{aligned}
\mathcal{L}_\eta z_t =& \left(-\Delta + \mu\right)\left(-\nabla\eta\right) - f'(\eta)\left(-\nabla\eta\right) \\
=& -\nabla\left(-\Delta + \mu\right)(\eta) + \nabla(f(\eta)) \\
=& -\nabla\left(\left(-\Delta + \mu\right)\eta - f(\eta)\right) \\
=& 0.
\end{aligned}$$

Because η is real-valued, the explicit form of $f'(\eta)i\eta$ yields

$$\begin{aligned}
\mathcal{L}_\eta z_g =& \left(-\Delta + \mu\right)(i\eta) - f'(\eta)i\eta \\
=& i\left(-\Delta + \mu\right)\eta - i\left|\eta\right|^{2k} \cdot (\eta) \\
=& i\left(-\Delta + \mu\right)\eta - if(\eta) \\
=& 0.
\end{aligned}$$

Using $-\Delta(ix\eta) = -2i\nabla\eta - ix\Delta\eta$ and applying Equation (2.2) once more, we obtain

$$\begin{aligned}
\mathcal{L}_\eta z_b =& \left(-\Delta + \mu\right)(ix\eta) - f'(\eta)\left(ix\eta\right) \\
=& \left(-\Delta + \mu\right)(ix\eta) - ix|\eta|^2\eta \\
=& ix\left(-\Delta + \mu\right)\eta - ixf(\eta) - 2i\nabla\eta \\
=& ix\left(\left(-\Delta + \mu\right)\eta - f(\eta)\right) - 2i\nabla\eta \\
=& -2i\nabla\eta \\
=& 2iz_t.
\end{aligned}$$

Using $\partial_\mu (\mu\eta) = \eta + \mu\partial_\mu\eta$, we obtain

$$
\begin{aligned}
\mathcal{L}_\eta z_s &= (-\Delta + \mu)(\partial_\mu \eta) - f'(\eta)\partial_\mu \eta \\
&= (-\Delta + \mu)(\partial_\mu \eta) - \partial_\mu f(\eta) \\
&= \partial_\mu [(-\Delta + \mu)\eta - f(\eta)] - \eta \\
&= \partial_\mu [0] - \eta \\
&= i z_g.
\end{aligned}
$$

These formulas will be used in Chapter 6.

Chapter 4 Orthogonal Decomposition

In this chapter, we utilize the variational characterization of an L^2-norm minimizer. For $\eta \in \mathcal{G}$ to minimize the L^2 distance to ψ, the first (Fréchet) derivatives of $||\psi - \eta||_{L^2(\mathbb{R}^d)}$ with respect to ψ must vanish. A quick calculation shows that this is equivalent to the condition

$$\langle z, \eta \rangle = 0 \tag{4.1}$$

for all $z \in T_\eta \mathcal{G}$. For our purposes, however, we prefer to equip the manifold $\mathcal{G}$ with a symplectic form via its cotangent bundle $T^*\mathcal{G}$. To define the cotangent bundle, we refer to [1, p. 202]. For any $\eta \in \mathcal{G}$, define the *cotangent space* $T^*\mathcal{G}_\eta$ *of* $\mathcal{G}$ *at* η to be the collection of all 1-forms acting on $T\mathcal{G}_\eta$ (the dual of $T\mathcal{G}_\eta$). Then the *cotangent bundle* $T^*\mathcal{G}$ is the union over all $\eta \in \mathcal{G}$ of the coordinate pairs $(\eta, T^*\mathcal{G}_\eta)$. Note that $T^*\mathcal{G}$ has dimension

$$2\dim(T\mathcal{G}) = 2\dim(\mathcal{G}) = 2(2d+2).$$

As a vector space of even dimension, $T^*\mathcal{G}$ can be equipped with a symplectic form to define its geometry. Using local coordinates, an identification can be made between η and $T^*\mathcal{G}_\eta$. The following definition is taken directly from [14, p. 290].

Definition 4.1 (Symplectic Form). *Let F be a vector space. A symplectic form ω is a real-valued bilinear form $\omega : F \times F \to \mathbb{R}$ that is anti-symmetric and non-degenerate. That is, for any $v, w \in F$,*

- $\omega(v, w) = -\omega(w, v)$
- *and the linear map $v \mapsto \Phi_\omega(v)(w) := \omega(v, w)$ is an isomorphism between F and its dual F^*.*

In place of the real inner product condition, we take instead the symplectic form $\omega(v, w) := \langle v, J^{-1}w \rangle$ defined in the introduction and construct a collection of *skew* orthogonality conditions

$$\omega(z, w) = 0, \forall z \in T_\eta \mathcal{G}. \tag{4.2}$$

It is important to note that ω is extended from $T_\eta\mathcal{G} \times T_\eta\mathcal{G}$ to $H^1 \times H^1$ via projection of H^1 onto $T_\eta\mathcal{G}$ in each input. The orthogonality conditions then ensure that the components of w in $T_\eta\mathcal{G}$ are parallel to each of the tangent vectors z. But this is only possible if the projection of w onto $T_\eta\mathcal{G}$ is zero! Hence condition (4.2) yields a formulation of orthogonality equivalent to condition (4.1). We shall see later that condition (4.2) is convenient to work with.

In Section 4.1 we obtain a matrix representation Ω_η of the symplectic form ω in the basis $\{z_t, z_g, z_b, z_s\}$. In Section 4.2 we use the invertibility of Ω_η in a neighborhood of η to establish local existence of the parameter family σ that achieves the desired orthogonal decomposition of ψ.

4.1 Matrix representation of the symplectic form Ω_η

Our symplectic form, as defined in the introduction, is

$$\omega(u, v) = \langle u, \Omega_\eta v \rangle .$$

In this section, we take the collection of tangent vectors $\{z_k\}$ as our basis and compute the matrix representation of

$$\Omega_\eta|_{\{z_k\}} = \left\langle z_j, J^{-1} z_k \right\rangle = Re\left(\int z_j \overline{i z_k}\right)$$

entry by entry. Note first that because of the operation of complex conjugation on i, the matrix will be anti-symmetric.

For convenience, we define

$$m(\mu) := \frac{1}{2}||\eta_\mu||^2,$$

but sometimes omit the μ subscript. We regard $z_t = z_{1,\ldots,d}$ and $z_b = z_{d+1,\ldots,2d}$ as d-dimensional vector quantities. Finally, let $\vec{e}_j$ be the j^{th} standard basis vector in $\mathbb{R}^d$ and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} .$$

Then for any $i, j \in \{1, \ldots, d\}$, we have

$$\begin{aligned}
\left\langle \vec{e}_j \cdot z_t, J^{-1} \vec{e}_k \cdot z_t \right\rangle =& Re\left(\int \vec{e}_j \cdot z_t \overline{i \vec{e}_j \cdot z_t}\right) = \delta_{ij} Re\left(-i \int |z_t|^2\right) \\
=& 0, \\
\left\langle \vec{e}_j \cdot z_t, J^{-1} \vec{e}_k \cdot z_b \right\rangle =& Re\left(\int \vec{e}_j \cdot z_t \overline{i \vec{e}_j \cdot z_b}\right) = Re\left(\int -\eta_{x_j} \overline{i i x_k \eta}\right) = Re\left(\int x_k \cdot \eta_{x_j} \eta\right) \\
=& - Re\left(\frac{1}{2}\int \delta_{ij} |\eta|^2\right) = -\delta_{ij} m,
\end{aligned}$$

(where we integrated by parts to obtain the penultimate equality).

$$\begin{aligned}
\left\langle \vec{e}_j \cdot z_t, J^{-1} z_g \right\rangle =& Re\left(\int \vec{e}_j \cdot z_t \overline{i z_g}\right) = Re\left(\int -\eta_{x_j} \overline{i i \eta}\right) \\
=& Re\left(\int \eta_{x_j} \eta\right) \underbrace{=}_{IBP} Re\left(-\int \eta \eta_{x_j}\right) = 0, \\
\left\langle \vec{e}_j \cdot z_t, J^{-1} z_s \right\rangle =& \vec{e}_j \cdot Re\left(\int z_t \overline{i z_s}\right) = Re\left(\int -\eta_{x_j} \overline{i \partial_\mu \eta_\mu}\right) = Re\left(i \int \eta_{x_j} \partial_\mu \eta_\mu\right) \\
=& 0,
\end{aligned}$$

In the second row, we have

$$\begin{aligned}
\left\langle \vec{e}_j \cdot z_b, J^{-1}\vec{e}_k \cdot z_t \right\rangle =& -\left\langle \vec{e}_j \cdot z_t, J^{-1}\vec{e}_k \cdot z_b \right\rangle \\
=&\delta_{ij} m, \\
\left\langle \vec{e}_j \cdot z_b, J^{-1}\vec{e}_k \cdot z_b \right\rangle =&Re\left(\int \vec{e}_j \cdot z_b \overline{i\vec{e}_k \cdot z_b}\right) = Re\left(\int ix_j\eta\overline{iix_k\eta}\right) \\
=&Re\left(-i\int x_j x_k |\eta|^2\right) \\
=&0, \\
\left\langle \vec{e}_j \cdot z_b, J^{-1} z_g \right\rangle =&Re\left(\int \vec{e}_j \cdot z_b \overline{iz_g}\right) = Re\left(\int ix_j\eta\overline{ii\eta}\right) = Re\left(-i\int x_j|\eta|^2\right) \\
=&0, \\
\left\langle \vec{e}_j \cdot z_b, J^{-1} z_s \right\rangle =&Re\left(\int \vec{e}_j \cdot z_b \overline{iz_s}\right) = Re\left(\int ix_j\eta_\mu\overline{i\partial_\mu\eta_\mu}\right) = Re\left(\int x_j\eta_\mu\partial_\mu\eta_\mu\right) \\
=&\partial_\mu\left(\frac{1}{2}\int x_j|\eta_\mu|^2\right) = \partial_\mu\left(0\right) \\
=&0,
\end{aligned}$$

where the vanishing of the integral in the penultimate line follows from spherical symmetry of η_μ. The third row consists of the following entries:

$$\begin{aligned}
\left\langle z_g, J^{-1}\vec{e}_j \cdot z_t \right\rangle =& -\left\langle \vec{e}_j \cdot z_t, J^{-1} z_g \right\rangle \\
=&0, \\
\left\langle z_g, J^{-1}\vec{e}_j \cdot z_b \right\rangle =& -\left\langle \vec{e}_j \cdot z_b, J^{-1} z_g \right\rangle \\
=&0, \\
\left\langle z_g, J^{-1} z_g \right\rangle =&Re\left(\int z_g\overline{iz_g}\right) = Re\left(\int i\eta\overline{ii\eta}\right) = Re\left(-i\int|\eta|^2\right) \\
=&0, \\
\left\langle z_g, J^{-1} z_s \right\rangle =&Re\left(\int z_g\overline{iz_s}\right) = Re\left(\int i\eta\overline{i\partial_\mu\eta_\mu}\right) \\
=&Re\left(\int \eta\partial_\mu\eta_\mu\right) = \partial_\mu\left(\frac{1}{2}\int|\eta_\mu|^2\right) \\
=&m'(\mu).
\end{aligned}$$

In the last row of the matrix, we have

$$\begin{aligned}
\left\langle z_s, J^{-1}\vec{e}_j \cdot z_t \right\rangle =& -\left\langle \vec{e}_j \cdot z_t, J^{-1} z_s \right\rangle \\
=&0, \\
\left\langle z_s, J^{-1}\vec{e}_j \cdot z_b \right\rangle =& -\left\langle \vec{e}_j \cdot z_b, J^{-1} z_s \right\rangle \\
=&0, \\
\left\langle z_s, J^{-1} z_g \right\rangle =& -\left\langle z_g, J^{-1} z_s \right\rangle \\
=& -m'(\mu),
\end{aligned}$$

$$
\begin{aligned}
\left\langle z_s, J^{-1} z_s \right\rangle =& Re\left(\int z_s \overline{i z_s}\right) = Re\left(\int \partial_\mu \eta_\mu \overline{i \partial_\mu \eta_\mu}\right) = Re\left(-i ||\partial_\mu \eta_\mu||_2^2\right) \\
=& 0.
\end{aligned}
$$

Putting this all together in matrix form, we have

$$
\Omega_\eta|_{\{z_k\}} = \left\langle z_j, J^{-1} z_k \right\rangle = Re\left(\int z_j \overline{i z_k}\right) = \begin{pmatrix} 0_{ij} & -\delta_{ij} m & \vec{0}^T & \vec{0}^T \\ \delta_{ij} m & 0_{ij} & \vec{0}^T & \vec{0}^T \\ \vec{0} & \vec{0} & 0 & m' \\ \vec{0} & \vec{0} & -m' & 0 \end{pmatrix}.
$$

We will make extensive use of this operator and associated notation in Chapter 6.

4.2 Local Existence of the Parameter Family σ

We now investigate the existence of the parameter family σ for solutions ψ not far from η in a sense that we make precise below. Let $I \subset \mathbb{R}^+$ be closed and bounded, with an interior interval $I_0 \subset I \backslash \partial I$ also closed, and let $B_K \subset \mathbb{R}^d$ be bounded. Define the space of all possible parameters

$$
\Sigma := \mathbb{R}^d \times \mathbb{R}^d \times [0, 2\pi) \times I
$$

and a subset of Σ

$$
\Sigma_0 := \mathbb{R}^d \times B_K \times [0, 2\pi) \times I_0.
$$

These sets are spaces in which the parameter family $\{a, v, \gamma, \mu\} = \sigma$ lives. Finally, we define the tube set

$$
U_\delta := \left\{ \psi \in H^1 \ : \ \inf_{\sigma \in \Sigma_0} ||\psi - \eta_\sigma||_{H^1} \leq \delta \right\}
$$

of functions within $\delta > 0$ of some representative $\eta_\sigma \in \mathcal{G}$. We now use this formulation of closeness to guarantee existence of σ for ψ that deviate by no more than δ from the orbit of equivalent solitons $\mathcal{G}$. Note that $v \in B_K$ forces limited speed of the soliton, keeping the problem physically meaningful. The limitation $\mu \in I_0$ is key to invertibility of Ω_η, which is a central element of the proof of the existence theorem below:

Proposition 4.2. *There exists $\delta \ll \inf_{\mu \in I_0} m'(\mu)$ and a unique $\sigma = \sigma(\psi) \in C^1(U_\delta, \Sigma)$ such that for each $\psi \in U_\delta$,*

$$
\omega\left(\psi - \eta_{\sigma(\psi)}, z\right) = 0,
$$

i.e.,

$$
\left\langle \psi - \eta_{\sigma(\psi)}, J^{-1} z \right\rangle = 0, \ \forall z \in T_{\eta_{\sigma(\psi)}} \mathcal{G}. \tag{4.3}
$$

Proof. Let $z_{\sigma,j} := \mathcal{S}_{av\gamma} z_{\mu,j}$, where

$$
\{\sigma_1, \ldots, \sigma_{2d+2}\} = \{a, v, \gamma, \mu\},
$$

$$\{z_{\mu,1},\dots,z_{\mu,2d+2}\} = \{z_t, z_b, z_g, z_s\} = \left\{-\nabla\eta_\mu,\ \frac{i}{2}x\eta_\mu,\ i\eta_\mu,\ \partial_\mu\eta_\mu\right\},$$

and

$$\partial_k := \partial_{\sigma_k}, \forall k \in \{d+1,\dots,2d+2\},$$
$$\partial_k := \partial_{\sigma_k} + \frac{1}{2}\sigma_{k+d}\partial_{\sigma_{2d+1}}, \forall k \in \{1,\dots,d\}.$$

Note that the $z_{\sigma,j}$ span $T_{\eta_\sigma}\mathcal{G}$ and are linearly independent (though not all mutually orthogonal). Also, set

$$\phi := \frac{1}{2}v\cdot x + \gamma$$

so that

$$\eta_\sigma = \mathcal{S}_{av\gamma}\eta_\mu = e^{i\phi(x-a)}\eta_\mu(x-a).$$

The calculations below verify that $\{z_{\mu,j}\}_j$ being a basis of $T_{\eta_\mu}\mathcal{G}$ implies $\{z_{\sigma,j}\}_j$ is a basis of $T_{\eta_\sigma}\mathcal{G}$.

<u>Calculations:</u>

For $k \in \{1,\dots,d\}$,

$$\begin{aligned}
\partial_k\eta_\sigma &= \left\{\partial_{a_k} + \frac{1}{2}v_k\partial_\gamma\right\}\eta_\sigma \\
&= \left\{\partial_{a_k} + \frac{1}{2}v_k\partial_\gamma\right\}\left[e^{i\phi(x-a)}\eta_\mu(x-a)\right] \\
&= \partial_{a_k}\left[e^{i\phi(x-a)}\eta_\mu(x-a)\right] + \frac{1}{2}v_k\partial_\gamma\left[e^{i\phi(x-a)}\eta_\mu(x-a)\right] \\
&= \left[i\frac{1}{2}v_k e^{i\phi(x-a)}\eta_\mu(x-a) - e^{i\phi(x-a)}\left(\eta_\mu\right)_k(x-a)\right] + \frac{1}{2}v_k\left[ie^{i\phi(x-a)}\eta_\mu(x-a)\right] \\
&= -e^{i\phi(x-a)}\left(\eta_\mu\right)_k(x-a) \\
&= \mathcal{S}_{av\gamma}\left[-\left(\eta_\mu\right)_k\right],
\end{aligned}$$

so for $k = \{1,\dots,d\}$ (viewed as a d-tuple),

$$\nabla_k\eta_\sigma = \mathcal{S}_{av\gamma}z_t.$$

We calculate the tangent vectors corresponding to the remaining values of k below: $k = \{d+1,\dots,2d\}$ (also viewed as a d-tuple):

$$\nabla_k\eta_\sigma = \nabla_v\left[e^{i\phi}\eta_\mu(x-a)\right] = \left(\nabla_v\phi\right)\eta_\sigma = \frac{1}{2}ix\eta_\sigma = \mathcal{S}_{av\gamma}\frac{1}{2}ix\eta_\mu = \mathcal{S}_{av\gamma}z_b,$$

$k = 2d+1:$

$$\partial_k\eta_\sigma = \partial_\gamma\left[e^{i\phi}\eta_\mu(x-a)\right] = \left(\partial_\gamma\phi\right)\eta_\sigma = i\eta_\sigma = \mathcal{S}_{av\gamma}i\eta_\mu = \mathcal{S}_{av\gamma}z_g,$$

$k = 2d+2:$

$$\partial_k\eta_\sigma = \partial_\mu\left[e^{i\phi}\eta_\mu(x-a)\right] = e^{i\phi}\partial_\mu\left[\eta_\mu(x-a)\right] = \mathcal{S}_{av\gamma}\partial_\mu\eta_\mu = \mathcal{S}_{av\gamma}z_s.$$

Thus, $\{z_{\sigma,j}\}_j$ is a basis of $T_{\eta_\sigma}\mathcal{G}$, since $T_{\eta_\sigma}\mathcal{G} = \mathcal{S}_{av\gamma}T_{\eta_\mu}\mathcal{G}$.

Now we use the implicit function theorem on $G : H^1 \times \Sigma \mapsto \mathbb{R}^{2d+2}$ given by

$$G_j(\psi, \sigma) := \left\langle \psi - \eta_\sigma, J^{-1}z_{\sigma,j} \right\rangle \quad \forall j \in \{1, \dots, 2d+2\}$$

to obtain $\sigma(\psi)$. We check that

- G is C^1 in ψ (G is linear in ψ) and σ (since η_σ and $z_{\sigma,j}$ are C^1 in σ, so is G),
- $G(\eta_{\sigma_0}, \sigma_0) = 0$ for any $\sigma_0 \in \Sigma$ (this is by definition of G),
- and $\partial_\sigma G(\eta_{\sigma_0}, \sigma)\big|_{\sigma=\sigma_0}$ is invertible for any $\sigma_0 \in \Sigma$ (we check this below).

For any $1 \le j, k \le 2d+2$,

$$\begin{aligned}
\partial_k G_j(\eta_{\sigma_0}, \sigma)\big|_{\sigma=\sigma_0} =& \partial_k \left\langle \eta_{\sigma_0} - \eta_\sigma, J^{-1}z_{\sigma,j} \right\rangle \big|_{\sigma=\sigma_0} \\
=& - \left\langle \partial_k\eta_\sigma, J^{-1}z_{\sigma,j} \right\rangle \big|_{\sigma=\sigma_0} + \left\langle \eta_{\sigma_0} - \eta_\sigma, J^{-1}\partial_k z_{\sigma,j} \right\rangle \big|_{\sigma=\sigma_0} \\
=& - \left\langle z_{\sigma,k}, J^{-1}z_{\sigma,j} \right\rangle \big|_{\sigma=\sigma_0} + \left\langle \eta_{\sigma_0} - \eta_{\sigma_0}, J^{-1}\left(\partial_k z_{\sigma,j}\right)\big|_{\sigma=\sigma_0} \right\rangle \\
=& - \left\langle z_{\sigma_0,k}, J^{-1}z_{\sigma_0,j} \right\rangle + 0 \\
=& - \left\langle z_{\sigma_0,k}, J^{-1}z_{\sigma_0,j} \right\rangle \\
=& - \left\langle z_{\mu_0,k}, J^{-1}z_{\mu_0,j} \right\rangle \\
=& - \left(\Omega_{\eta_0}\right)_{kj},
\end{aligned}$$

where Ω_{η_0} is represented by a matrix with respect to a basis $\{z_{\mu_0,\ell}\}_{\ell=1}^{2d+2}$. Given our assumptions on η_0, we find that Ω_{η_0} is invertible for any $\sigma_0 \in \Sigma$, and so also is $\partial_\sigma G(\eta_{\sigma_0}, \sigma)\big|_{\sigma=\sigma_0}$. Thus, there exists $\delta > 0$ satisfying

$$\delta \ll \inf_{\mu \in I_0}\{m'(\mu), m(\mu)\} \le \inf_{\mu \in I_0} m'(\mu)$$

(see more on invertibility of Ω_η in Section 6.4) so that there is a unique C^1 map $\sigma = \sigma(\psi)$ satisfying $G(\psi, \sigma(\psi)) = 0$ for each $\psi \in B_\delta(\eta_{\sigma_0}) \subset V_{\sigma_0}$. (Here $B_r(x)$ denotes a ball of radius r centered at the point x while V_x denotes a neighborhood of x.)

Now, set $\sigma_0 = \{0, 0, 0, \mu_0\}$ and for any $\psi \in U_\delta$, note that there exists $\{n, \mu_0\} := \{a, v, \gamma, \mu_0\}$ so that

$$||\psi - \eta_{\{n,\mu_0\}}||_{H^1} \le \delta.$$

(This is because Σ_0 is a closed set and bounded in all directions except the one corresponding to $a \in \mathbb{R}^d$. Since a is merely a translation and the profile η_μ is localized, the minimizing parameter a must sit inside a bounded subset of $\mathbb{R}^d$. Hence the subset of Σ_0 on which we locally seek the minimizing parameters $\{n, \mu_0\}$ is a compact set.) Then

$$||\mathcal{S}_n^{-1}\psi - \eta_{\mu_0}||_{H^1} = ||\mathcal{S}_n^{-1}\psi - \eta_{\{0,\mu_0\}}||_{H^1} \le \delta,$$

and we can find a unique $\sigma = \sigma\left(\mathcal{S}_n^{-1}\psi\right)$ so that for all $z \in T_{\eta_{\sigma\left(\mathcal{S}_n^{-1}\psi\right)}}\mathcal{G}$,

$$\omega\left(\mathcal{S}_n^{-1}\psi - \eta_{\sigma\left(\mathcal{S}_n^{-1}\psi\right)}, z\right) = 0.$$

To transform back, we choose $\tilde{n} = \{\tilde{a}, \tilde{v}, \tilde{\gamma}\} = \{a, v, \gamma + av\}$ so that $\mathcal{S}_{\tilde{n}}\mathcal{S}_n^{-1} = id$, yielding a unique $\tilde{\sigma} = \tilde{n} \circ \sigma\left(\mathcal{S}_n^{-1}\psi\right)$ so that for all $\tilde{z} = \mathcal{S}_{\tilde{n}}z \in T_{\eta_{\sigma\left(\mathcal{S}_{\tilde{n}}\mathcal{S}_n^{-1}\psi\right)}}\mathcal{G} = T_{\eta_{\tilde{\sigma}(\psi)}}\mathcal{G}$,

$$\omega\left(\psi - \eta_{\tilde{\sigma}(\psi)}, \mathcal{S}_{\tilde{n}}z\right) = \omega\left(\mathcal{S}_n^{-1}\psi - \eta_{\sigma\left(\mathcal{S}_n^{-1}\psi\right)}, z\right) = 0.$$

□

Since

$$\langle \mathcal{S}_{av\gamma}v, \mathcal{S}_{av\gamma}w\rangle = \langle v, w\rangle$$

for any $v, w \in H^1(\mathbb{R}^d)$, Proposition (4.2) quickly yields the following corollary:

Corollary 4.3. *Given the hypotheses of Proposition (4.2), there exists a C^1 map $\sigma(t) := \{a(t), v(t), \gamma(t), \mu(t)\}$ so that*

$$\left\langle \mathcal{S}_{av\gamma}^{-1}\psi - \eta_\mu, J^{-1}z\right\rangle = 0,\ \forall z \in T_{\eta_\mu}\mathcal{G}. \tag{4.4}$$

If we define $w(x,t) := \mathcal{S}_{av\gamma}^{-1}\psi - \eta_\mu$, we have

$$\omega(w, z) = \left\langle w, J^{-1}z\right\rangle = 0,\ \forall z \in T_{\eta_\mu}\mathcal{G}. \tag{4.5}$$

Here we have indicated the dependence of η_μ and z on $\mu(t)$. The study of $\sigma \in C^1$ continues in earnest in Chapter 6. Meanwhile in Chapter 5 we establish useful lower bounds on the bilinear form $\langle\cdot, \mathcal{L}_\eta\cdot\rangle$ for functions orthogonal to the tangent space $T_{\eta_\mu}\mathcal{G}$ in the sense of Proposition 4.3.

Chapter 5 Coercivity of $\mathcal{L}_\eta$

In this chapter, we give a preview of the coercivity results used later to close stability arguments Chapters 8 and 9. Here we use information about the spectrum of $\mathcal{L}_\eta$ and tangent space $T_\eta\mathcal{G}$ from Chapters 2 and 3, respectively. These coercivity results are key in obtaining lower bounds on the energy functional $\mathcal{E}$, which in turn are critical in trapping the deviation $\psi - \eta_\sigma$. Recall that the orthogonality conditions $\omega\left(w, z\right) = 0$ were developed in Chapter 4 to avoid the negative eigenvalue of $\mathcal{L}_\eta$. The following proposition is the fruit of careful calculations which follow the methods of Weinstein [20, 19] and FGJS [7]:

Proposition 5.1. *There exists $\rho' > 0$ such that if $\omega\left(w, z\right) = 0$ for all $z \in T_\eta\mathcal{G}$, then $\langle w, \mathcal{L}_\eta w\rangle \geq \rho' ||w||_{H^1}^2$.*

To avoid exhausting author and reader alike, we have divided the proof of Proposition 5.1 into three steps. Each step builds on the one before until the proof is complete. Since the first step figures prominently in the stability analysis for the $\lambda = 0$ as well, we dignify it with a separate proposition, stated below. The proof of Proposition 5.2 is given in Appendix D, where an alternate formulation of the ground state η is brought to bear along with explicit but tedious variational calculations.

Proposition 5.2. *Let $X_1 := \{w \in H^1 : ||w|| = 1, \langle w, (\eta, 0)\rangle = 0\}$ be the set of all complex-valued H^1 functions with unit L^2 norm that are real-orthogonal to η. Then*

$$\inf_{w\in X_1} \langle w, \mathcal{L}_\eta w\rangle = 0. \tag{5.1}$$

We now proceed to prove Proposition 5.1.

Proof of Proposition 5.1.

Step 1 See Proposition 5.2.

Step 2

We now modify the space X_1 by adding the remaining orthogonality conditions from Equation (4.5) to construct

$$X := \left\{w \in H^1\left(\mathbb{R}^d, \mathbb{C}\right)\ :\ ||w|| = 1,\ \omega\left(w, z\right) = 0,\ \forall z \in T_\eta\mathcal{G}\right\}.$$

We now wish to show that

$$\alpha := \inf_{w\in X} \langle w, \mathcal{L}_\eta w\rangle > 0. \tag{5.2}$$

To this end, recall that $\omega\left(w, z\right) = \langle w, J^{-1}z\rangle$, while $(0, \eta) \in T_\eta\mathcal{G}$, so that $\omega\left(w, (\eta, 0)\right) = \int w\left(0, \eta\right) = 0$. Now consider the Euler-Lagrange equations corresponding to (5.2), which are

$$\mathcal{L}_\eta w = \alpha w + \sum_k \gamma_k J z_k.$$

By construction we have $X \subset X_1$. Thus $\alpha \geq 0$. There are three possible cases, the first two of which we show are impossible:

(i) $\alpha = 0$ and $\gamma_j \neq 0$ for at least one j

(ii) $\alpha = 0$ and $\gamma_j = 0$ for all j

(iii) $\alpha > 0$.

If (i), then for at least one $z_k \in T_\eta \mathcal{G}$,

$$\begin{aligned}
\langle z_k, \mathcal{L}_\eta w \rangle &= 0 + \sum_j \gamma_j \langle z_j, J z_k \rangle \\
&= \sum_j \gamma_j Re \left(\int z_j \overline{J z_k} \right) \\
&= \sum_j \gamma_j Re \left(- \int z_j \overline{J^{-1} z_k} \right) \\
&= - \sum_j \gamma_j \left\langle z_j, J^{-1} z_k \right\rangle \\
&= \left\langle - \sum_j \gamma_j z_j, \Omega_\eta z_k \right\rangle \\
&= \langle -\gamma_j z_j, \Omega_\eta z_k \rangle \\
&\neq 0
\end{aligned}$$

by direct observation of the matrix form of Ω_η. This contradicts (by orthogonality of w to z_k and/or by $\mathcal{L}_\eta z_k = 0$)

$$\langle z_k, \mathcal{L}_\eta w \rangle = \langle \mathcal{L}_\eta z_k, w \rangle = 0, \forall k.$$

If (ii), then $w \in Null\,(\mathcal{L}_\eta)$. Recalling Proposition 2.2, we observe that

$$Null\,(\mathcal{L}_\eta) = \{(0, \eta), (\partial_j \eta, 0)_{j=1,\ldots,d}\} \subset T_\eta \mathcal{G}.$$

Thus we must have

$$0 = \omega\,(w, z_k) = \omega \left(\sum_j \beta_j z_j, z_k \right) = \beta_k \left\langle z_k, J^{-1} z_k \right\rangle = \beta_k ||z_k||^2$$

for all k, which contradicts $||w|| = 1$. Thus, (iii) must hold, and so $\alpha > 0$.

Step 3

Step 2 implies that there exists $\rho'' > 0$ depending on μ such that

$$\langle w, \mathcal{L}_\eta w \rangle \geq \rho'' ||w||^2. \tag{5.3}$$

We improve this L^2 bound to H^1 as follows:

Let $0 < \delta < 1$, and rewrite (5.3) as

$$\langle w, \mathcal{L}_\eta w \rangle \geq (1-\delta)\,\rho''||w||^2 + \delta \,\langle w, \mathcal{L}_\eta w \rangle\,.$$

By using the explicit form of $\mathcal{L}_\eta$, we obtain

$$\langle w, \mathcal{L}_\eta w \rangle \geq ||\nabla w||^2 - C_\mu ||w||^2,$$

where $C_\mu = \sup_x \left(\mu + |f'\left(\eta\right)|\right)$. Then

$$\begin{aligned}\langle w, \mathcal{L}_\eta w \rangle &\geq (1-\delta)\,\rho''||w||^2 + \delta\left(||\nabla w||^2 - C_\mu||w||^2\right)\\ &= \left(\rho'' - \delta\rho'' - \delta C_\mu\right)||w||^2 + \delta||\nabla w||^2\\ &= \delta||w||^2_{H^1},\end{aligned}$$

where we now specify $\delta \in (0,1)$ solving $\delta = \rho'' - \delta\rho'' - \delta C_\mu$ to obtain the last equality. In other words, we set

$$\delta := \frac{\rho''}{(1+\rho''+C_\mu)}.$$

$\square$

Chapter 6 Frame Transformed NLS and the Equations of Motion

In this chapter we derive the approximate modulation equations (the first two being Newton's equations) for the perturbed solution. We have already laid the groundwork in Chapter 4 for a parametrization $\{\sigma(t), w(t,x)\}$ for the solution $\psi(t,x)$ of (1.1) satisfying

$$(\psi - \eta_\sigma) \perp J^{-1} T_{\eta_\sigma} M_s.$$

The family of parameters $\sigma(t) = \{a(t), v(t), \gamma(t), \mu(t)\}$ describes the evolution of the perturbed soliton initial condition ψ_0, while the error function $w(t)$ tracks the skew-orthogonal deviation of $\psi(t,x)$ from the traveling soliton $\eta_{\sigma(t)}(t,x)$. This parametrization is well-defined for $\psi \in U_\delta$, that is, for ψ near the soliton orbit $\mathcal{G}$, as detailed in Section 4.2. Having established existence and uniqueness of these parameters in Chapter 4, we now seek to express σ and w as solutions of coupled ODEs. These will be important in establishing an upper bound on the second variation of $\mathcal{E}_\mu$ in Chapter 7, which makes possible the conclusion of the argument in Chapter 8.

To obtain σ and w, we make a transformation $\mathcal{S}_{a,v,\gamma} : H^1 \to H^1$ to a frame of reference in which we view the profile η_μ of the soliton η_σ. That is, we define

$$\mathcal{S}_{av\gamma} u := e^{i\phi(\cdot - a(t),t)} u(\cdot - a),$$

so that

$$\eta_\mu = \mathcal{S}_{av\gamma}^{-1} \eta_\sigma,$$

and we set

$$w := \mathcal{S}_{av\gamma}^{-1} \left(\psi - \eta_\sigma\right). \tag{6.1}$$

Lemma 6.1 below is simply Equation (1.1) rewritten in this frame, with transformed solution $u := \mathcal{S}_{av\gamma}^{-1} \psi$.

Remainders of the Nonlinearity

For the sake of brevity, we also introduce notation for the remainders of various order approximations of the nonlinearity $F(u) = \int \frac{|u|^{2k+2}}{2k+2}$, an antiderivative of f. In what follows, the superscript n in $R^{(n)}$ indicates the order (in w) of the remainder:

$$\begin{aligned} R^{(2)}\left(w\right) :=& F\left(\eta + w\right) - F\left(\eta\right) - \left\langle F'\left(\eta\right), w\right\rangle, \\ R^{(3)}\left(w\right) :=& F\left(\eta + w\right) - F\left(\eta\right) - \left\langle F'\left(\eta\right), w\right\rangle - \frac{1}{2}\left\langle F''\left(\eta\right) w, w\right\rangle, \end{aligned}$$

and

$$N_\eta\left(w\right) := F'\left(\eta + w\right) - F'\left(\eta\right) - F''\left(\eta\right) w.$$

We note that

$$N_\eta\left(w\right) = R^{(3)\prime}\left(w\right). \tag{6.2}$$

Estimates

For any $\eta \in H^2(\mathbb{R}^d)$ and $w \in H^1(\mathbb{R}^d)$ with $||\eta||_{H^1} + ||w||_{H^1} \leq M$, the above nonlinearities satisfy the following bounds:

$$|R^{(2)}(w)| \leq c(M)||w||^2_{H^1},$$
$$|R^{(3)}(w)| \leq c(M)||w||^3_{H^1},$$

and

$$||N_\eta(w)||_{H^{-1}} \leq C(M)||w||^2_{H^1}.$$

The first two inequalities follow directly from the fact that $R^{(2)}$ and $R^{(3)}$ are (Fréchet derivative) Taylor sums of first and second order in w, respectively. The last bound follows directly from identity (6.2).

6.1 Frame Transforming the NLS

Recall that we have taken the power nonlinearity $f(u) = \left|u\right|^{2k} \cdot (u)$, and that the modified potential λV^h satisfies $V \in C^2$ and $||V||_{L^\infty} = 1$, with $\lambda \in \mathbb{R}$ chosen sufficiently small. Recalling also the definition of the set U_δ from Section 4.2, we now state and prove the following lemma:

Lemma 6.1. *If $\psi \in U_\delta$ solves (1.1), then $u := \mathcal{S}^{-1}_{av\gamma}\psi$ satisfies*

$$\dot{u} = J\left(\left(-\Delta + \mu\right)u - f(u)\right) + \underline{\alpha} \cdot \underline{\mathcal{K}}u + J\mathcal{R}_{V^h}u, \tag{6.3}$$

where

$$\mathcal{R}_{V^h} := \lambda\left[V^h\left(x+a\right) - V^h(a) - h\nabla V^h(a)\cdot x\right] = \mathcal{O}\left(\lambda h^2 x^2\right)$$

and $\mathcal{K}$ is a tuple of carefully chosen linear operators to be specified later.

Here we are using the shorthand notations

$$\alpha \cdot \mathcal{K} := \sum_{j=1}^{2d+2} \alpha_j \mathcal{K}_j, \tag{6.4}$$

$$\underline{\alpha} \cdot \underline{\mathcal{K}} := \sum_{j=1}^{2d+1} \alpha_j \mathcal{K}_j \tag{6.5}$$

and indicate spatial translation of a function ψ by

$$\psi_a(x) := \psi(x+a).$$

Because the choice of the linear operators $\mathcal{K}_j$ is one of convenience, we will introduce their definitions in the body of the proof. We will also use the notation

$$|\alpha| := \max\{\alpha_j\}_{j=1}^{2d+2} \tag{6.6}$$

in many future estimates. In the proof, as in some of the following chapters, we suppress the time dependence of the parameters σ to avoid unnecessary notational clutter.

Proof of Lemma 6.1. First, consider the solution ψ of (1.1) when viewed in the moving frame. Then from

$$\psi = \mathcal{S}_{av\gamma}u = e^{i\left(\frac{1}{2}v\cdot(\cdot-a)+\gamma\right)}u(\cdot - a),$$

we obtain

$$u(x) = e^{-i\left(\frac{1}{2}v\cdot x+\gamma\right)}\psi(x+a),$$

or

$$\mathcal{S}_{av\gamma}^{-1}\psi = u = e^{-i\phi}\psi_a, \tag{6.7}$$

where, as before, we take

$$\phi(x) = \frac{1}{2}v\cdot x + \gamma$$

and use the notation $f_a := f(\cdot + a)$ for translation of any function f by a in the spatial variable.

We use spatial translation to obtain

$$i\dot{\psi}(x+a) = -\left[\Delta\psi(x+a) - \lambda V_a^h\psi_a + f(\psi_a)\right] \tag{6.8}$$

from Equation (1.1), which yields the third equality in the calculation immediately following.

Differentiation of (6.7) results in

$$\begin{aligned}
i\dot{u} &= i\frac{d}{dt}\left(e^{-i\phi}\right)\psi_a + ie^{-i\phi}\frac{d}{dt}\left(\psi_a\right)\\
&= i\left[-i\dot{\phi}e^{-i\phi}\psi_a + e^{-i\phi}\nabla\psi_a\cdot\dot{a} + e^{-i\phi}\dot{\psi}(\cdot+a)\right]\\
&= \dot{\phi}u + ie^{-i\phi}\nabla\psi_a\cdot\dot{a} - e^{-i\phi}\left(\Delta\psi_a - \lambda V_a^h\psi_a + f(\psi_a)\right) + \underbrace{[\mu u - \mu u]}_{=0}\\
&= \dot{\phi}u + ie^{-i\phi}\nabla\psi_a\cdot\dot{a} + e^{-i\phi}\left(-\Delta\psi_a + \lambda V_a^h\psi_a - f(\psi_a)\right) + \left[\mu e^{-i\phi}\psi_a - \mu u\right]\\
&= e^{-i\phi}\left(-\Delta + \mu + \lambda V_a^h\right)\psi_a - \underbrace{e^{-i\phi}f(\psi_a)}_{=f(u)} + ie^{-i\phi}\nabla\psi_a\cdot\dot{a} + \left(\dot{\phi} - \mu\right)u.\\
&= e^{-i\phi}\left(-\Delta + \mu + \lambda V_a^h\right)\psi_a - f(u) + ie^{-i\phi}\nabla\psi_a\cdot\dot{a} + \left(\dot{\phi} - \mu\right)u,
\end{aligned} \tag{6.9}$$

with a quick nod to the explicit form of the nonlinearity in simplifying

$$e^{-i\phi}f(\psi_a) = e^{-i\phi}\left|\psi_a\right|^{2k}\cdot(\psi_a) = \left|e^{-i\phi}\psi_a\right|^{2k}\cdot\left(e^{-i\phi}\psi_a\right) = \left|u\right|^{2k}\cdot(u) = f(u).$$

Now, for clarity of presentation, we note that by the product and chain rules

$$e^{-i\phi}\nabla\psi_a = \nabla\left(e^{-i\phi}\psi_a\right) + i\nabla\phi e^{-i\phi}\psi_a. \tag{6.10}$$

Further, since $\nabla\phi = v/2$ is independent of x,

$$\nabla\left(-i\nabla\phi\cdot e^{-i\phi}\psi_a\right) = -i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right),$$

and so

$$\begin{aligned}\Delta\left(e^{-i\phi}\psi_a\right) &= \nabla^2\left(e^{-i\phi}\psi_a\right)\\ &= \nabla\left(-i\nabla\phi e^{-i\phi}\psi_a + e^{-i\phi}\nabla\psi_a\right)\\ &= -i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) + \nabla\left(e^{-i\phi}\nabla\psi_a\right)\\ &= -i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) - i\nabla\phi\cdot e^{-i\phi}\nabla\psi_a + e^{-i\phi}\Delta\psi_a\end{aligned}$$

(using the product rule on the middle term)

$$\begin{aligned}&= -i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) - i\nabla\phi\cdot\left(\nabla\left(e^{-i\phi}\psi_a\right) + i\nabla\phi\cdot e^{-i\phi}\psi_a\right) + e^{-i\phi}\Delta\psi_a\\ &= -2i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) + |\nabla\phi|^2\cdot e^{-i\phi}\psi_a + e^{-i\phi}\Delta\psi_a \qquad (6.11)\end{aligned}$$

We can rearrange (6.11) so that it takes the form

$$e^{-i\phi}\Delta\psi_a = \Delta\left(e^{-i\phi}\psi_a\right) + 2i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) - |\nabla\phi|^2 e^{-i\phi}\psi_a. \qquad (6.12)$$

Now we can rewrite Equation (6.9) as

$$\begin{aligned}i\dot{u} &= \left(-e^{-i\phi}\Delta\psi_a\right) + \left(\mu + \lambda V_a^h\right)u - f(u) + i\dot{a}e^{-i\phi}\nabla\psi_a + \left(\dot{\phi} - \mu\right)u\\ &= -\underbrace{\left[\Delta\left(e^{-i\phi}\psi_a\right) + 2i\nabla\phi\cdot\nabla\left(e^{-i\phi}\psi_a\right) - |\nabla\phi|^2 e^{-i\phi}\psi_a\right]}_{\text{using (6.12)}} + \left[\mu + \lambda V_a^h\right]u - f(u)\\ &\quad + i\dot{a}\underbrace{\left[\nabla\left(e^{-i\phi}\psi_a\right) + i\nabla\phi e^{-i\phi}\psi_a\right]}_{\text{using (6.10)}} + \left(\dot{\phi} - \mu\right)u\\ &= -\left[\Delta u + 2i\nabla\phi\cdot\nabla u - |\nabla\phi|^2 u\right] + \left[\mu + \lambda V_a^h\right]u - f(u)\\ &\quad + i\dot{a}\left[\nabla u + i\nabla\phi u\right] + \left(\dot{\phi} - \mu\right)u.\end{aligned}$$

Dividing both sides by i and observing that $\dot{\phi} = \frac{1}{2}\dot{v}\cdot x + \dot{\gamma}$ and $\nabla\phi = \frac{1}{2}v$, we find that

$$\begin{aligned}\dot{u} =& J\left(-\Delta u + \mu u - f(u)\right) - \left(2\nabla\phi - \dot{a}\right)\cdot\nabla u\\ &+ J\left(|\nabla\phi|^2 + \lambda V_a^h - \nabla\phi\dot{a} + \dot{\phi} - \mu\right)u\\ =& J\left(-\Delta u + \mu u - f(u)\right) - \left(v - \dot{a}\right)\cdot\nabla u\\ &+ J\left(\frac{|v|^2}{4} + \lambda V_a^h - \frac{v}{2}\dot{a} + \frac{1}{2}\dot{v}\cdot x + \dot{\gamma} - \mu\right)u.\end{aligned} \qquad (6.13)$$

Now, expanding V^h in a Taylor series about a as

$$\begin{aligned}V^h(x+a) &= V_a^h(x)\\ &= \lambda V^h(a) + \lambda h\nabla V^h(a)\cdot x + \lambda h^2 D^2 V^h(a)\cdot x^2 + \lambda h^3 D^3 V^h(a)\cdot x^3 + \dots\\ &= V^h(a) + h\nabla V^h(a)\cdot x + \mathcal{R}_{V^h}(x)\end{aligned}$$

and collecting like terms in (6.13) leads us to

$$\dot{u} = J\left(-\Delta u + \mu u - f(u)\right)$$

$$
\begin{aligned}
&- \nabla u \cdot (v - \dot{a}) \\
&- Jxu \cdot \left(-\frac{1}{2}\dot{v} - \lambda h \nabla V^h(a) \right) \\
&- Ju \left(\mu - \frac{1}{4}|v|^2 + \frac{\dot{a}v}{2} - \lambda V^h(a) - \dot{\gamma} \right) \\
&+ J\mathcal{R}_{V^h}(x)u.
\end{aligned}
$$

Finally, we define the (anti-self adjoint) operators

$$
\mathcal{K}_j = -\partial_{x_j}, \quad \mathcal{K}_{d+j} = -Jx_j, \quad \mathcal{K}_{2d+1} = -J, \quad \mathcal{K}_{2d+2} = \partial_\mu, \quad \forall j \in \{1, \ldots, d\},
$$

and the corresponding coefficients

$$
\begin{aligned}
\alpha_j =& \quad v_j - \dot{a}_j, \\
\alpha_{d+j} =& - \frac{1}{2}\dot{v}_j - \partial_{x_j} V(a), \\
\alpha_{2d+1} =& \quad \mu - \frac{1}{4}v^2 + \frac{1}{2}\dot{a} \cdot v - V(a) - \dot{\gamma}, \\
\alpha_{2d+2} =& - \dot{\mu},
\end{aligned}
$$

also for all $j \in \{1, \ldots, d\}$. In this way we obtain

$$
\dot{u} = J\left(-\Delta u + \mu u - f(u)\right) + \underline{\alpha} \cdot \underline{\mathcal{K}} u + J\mathcal{R}_{V^h} u,
$$

which is Equation (6.3). The size of $\mathcal{R}_{V^h}(x)$ of follows from the Taylor series expansion of V^h.

□

It may be of interest to note that the $\mathcal{K}_j$ act on η_μ to generate the basis vectors $\{z_j\}$ of the tangent space $T_\eta \mathcal{G}$ as below:

$$
\mathcal{K}_j \eta = z_j, \forall j \in \{1, \ldots, 2d+2\}. \tag{6.14}
$$

The α_j are coefficients of the error terms which are not due to the potential λV^h.

6.2 Solving for w

Now that we have achieved the change of frame, it is time to solve for the skew-orthogonal error w and the (so far) implicitly defined parameters σ.

This project is founded upon the following three facts: (i) $u = \eta + w$ by (6.1), (ii) $\mathcal{E}'_\mu(\eta) = 0$ (by η being an energy minimizer of $\mathcal{E}_\mu$), and (iii)

$$
\begin{aligned}
-\Delta u + \mu u - f(u) =& \mathcal{E}'_\mu(u) \\
=& \mathcal{E}'_\mu(\eta + w) \\
=& \mathcal{E}'_\mu(\eta) + \mathcal{E}''_\mu(\eta)w + w\mathcal{E}'''_\mu(\eta)w + \ldots \\
=& 0 + \mathcal{L}_\eta w + wF'''(\eta)w + \ldots,
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{L}_\eta w + \left(F'(\eta + w) - F'(\eta) - F''w\right), \\
&= \mathcal{L}_\eta w + N_\eta(w).
\end{aligned}
$$

(To get this last equation, we recall the definitions $\mathcal{L}_\eta = \mathcal{E}''_\mu(\eta)$ and N_η, the latter being given earlier in this chapter). By the chain rule, we also have

$$
\dot{u} = \dot{\eta} + \dot{w} = \dot{\mu}\frac{\partial \eta}{\partial \mu} + \dot{w},
$$

and so we can rewrite (6.3) as

$$
\dot{\mu}\frac{\partial \eta}{\partial \mu} + \dot{w} = J\mathcal{L}_\eta w + JN_\eta(w) + \underline{\alpha} \cdot \underline{\mathcal{K}}\left(\eta + w\right) + J\mathcal{R}_{V^h} \cdot \left(\eta + w\right), \tag{6.15}
$$

or, alternately, as

$$
\dot{w} = J\mathcal{L}_{\eta,\sigma} w + JN_\eta(w) + q(\sigma), \tag{6.16}
$$

where

$$
\begin{aligned}
\mathcal{L}_{\eta,\sigma} &:= \mathcal{L}_\eta w + \mathcal{R}_{V^h} w + J^{-1}\underline{\alpha} \cdot \underline{\mathcal{K}}, \\
N_\eta(w) &= F'\left(\eta + w\right) - F'\left(\eta\right) - F''\left(\eta\right) w,
\end{aligned}
$$

and

$$
q(\sigma) := \alpha \cdot \mathcal{K}\eta + J\mathcal{R}_{V^h}\eta.
$$

Thus we have implicitly defined w by the (nonlinear) ODE (6.16).

6.3 Solving for σ

To derive the ODE for the collection of parameters σ, we need to use the skew orthogonality condition (4.3) and Equation (6.15) above. We will also make use of the following short lemma, whose proof is postponed until Section 6.5:

Lemma 6.2. *For $z \in T_\eta \mathcal{G}$, with $\mathcal{L}_\eta$ and w as already defined,*

$$
\langle Jz, J\mathcal{L}_\eta w \rangle = 0.
$$

Moreover, differentiating the skew orthogonality condition and using product and chain rules, we obtain

$$
0 = \frac{d}{dt}\left\langle Jz, w \right\rangle = \left\langle Jz, \dot{w} \right\rangle + \dot{\mu}\left\langle J\partial_\mu z, w \right\rangle. \tag{6.17}
$$

Now, integrating (6.15) against Jz yields

$$
\begin{aligned}
\langle Jz, \dot{\mu}\partial_\mu \eta \rangle + \langle Jz, \dot{w} \rangle = &\underbrace{\langle Jz, J\mathcal{L}_\eta w \rangle}_{=0} + \langle Jz, JN_\eta(w) \rangle + \langle Jz, \underline{\alpha} \cdot \underline{\mathcal{K}}\left(\eta + w\right) \rangle \\
&+ \langle Jz, J\mathcal{R}_{V^h} \cdot \left(\eta + w\right) \rangle
\end{aligned}
$$

or, after applying (6.17) to the left hand side and $\langle Ja, Jb\rangle = \langle a, b\rangle$ to some terms on the right hand side, we have

$$\dot{\mu}\left\langle Jz, \partial_\mu \eta\right\rangle - \dot{\mu}\left\langle J\partial_\mu z, w\right\rangle = \left\langle z, N_\eta(w)\right\rangle + \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}\eta\right\rangle + \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}w\right\rangle + \left\langle z, \mathcal{R}_{V^h}\cdot(\eta+w)\right\rangle .$$

Rearranging terms, we obtain

$$\dot{\mu}\left\langle Jz, \partial_\mu \eta\right\rangle - \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}\eta\right\rangle = \dot{\mu}\left\langle J\partial_\mu z, w\right\rangle + \left\langle z, N_\eta(w)\right\rangle + \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}w\right\rangle + \left\langle z, \mathcal{R}_{V^h}\cdot(\eta+w)\right\rangle . \tag{6.18}$$

Now we use that $\mathcal{K}_{2d+2} = \partial_\mu$ and $\alpha_{2d+2} = -\dot{\mu}$, and recall from Equation (6.14) that $z_j = \mathcal{K}_j\eta$ for each j to rewrite the left hand side of (6.18) as

$$\begin{aligned} & -\alpha_{2d+2}\left\langle Jz, \mathcal{K}_{2d+2}\eta\right\rangle - \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}\eta\right\rangle \\ = & -\alpha\cdot\left\langle Jz, \mathcal{K}\eta\right\rangle \\ = & -\sum_j \left\langle Jz, z_j\right\rangle \alpha_j . \end{aligned}$$

We observe that $\underline{\mathcal{K}}$ is visibly anti-self-adjoint and commutes with J. We also recall the notation used for the basis vectors of the tangent space (introduced in Chapter 3)

$$\begin{aligned} z_{1,\ldots,d} = z_t = & -\nabla\eta_\mu, \\ z_{d+1,\ldots,2d} = z_b = & ix\eta_\mu, \\ z_{2d+1} = z_g = & i\eta_\mu, \\ z_{2d+2} = z_s = & \partial_\mu\eta_\mu . \end{aligned}$$

Then making the substitution

$$\dot{\mu}\left\langle J\partial_\mu z, w\right\rangle + \underline{\alpha}\cdot\left\langle Jz, \underline{\mathcal{K}}w\right\rangle = \alpha\cdot\left\langle \mathcal{K}z, Jw\right\rangle$$

into the left hand side of Equation (6.18) and choosing in particular $z = z_k$ for any k, we obtain

$$\begin{aligned} -\sum_j \left\langle z_k, J^{-1}z_j\right\rangle \alpha_j = & -\sum_j \left\langle Jz_k, z_j\right\rangle \alpha_j \\ = & \left\langle z_k, N_\eta(w) + \mathcal{R}_{V^h}\cdot(\eta+w)\right\rangle - \alpha\cdot\left\langle \mathcal{K}z_k, Jw\right\rangle . \end{aligned} \tag{6.19}$$

Invoking the matrix representation Ω_η for $\omega(z_k, z_j) = \left\langle z_k, J^{-1}z_j\right\rangle$, we find that

$$\Omega_\eta|_{\{z_k\}}\alpha_j = \begin{pmatrix} 0 & -m1 & 0 & 0 \\ m1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m' \\ 0 & 0 & -m' & 0 \end{pmatrix} \begin{pmatrix} \alpha_{1,\ldots,d} \\ \alpha_{d+1,\ldots,2d} \\ \alpha_{2d+1} \\ \alpha_{2d+2} \end{pmatrix} = \begin{pmatrix} -m\cdot\alpha_{d+1,\ldots,2d} \\ m\cdot\alpha_{1,\ldots,d} \\ m'\cdot\alpha_{2d+2} \\ -m'\cdot\alpha_{2d+1} \end{pmatrix} . \tag{6.20}$$

Finally, we observe that because η and $\mathcal{R}_{V^h}$ are real, the terms $\left\langle J\eta, \mathcal{R}_{V^h}\eta\right\rangle$ and $\left\langle x_k\eta, J\mathcal{R}_{V^h}\eta\right\rangle$ are both zero.

With all these tools in hand, and cancelling factors of J and i wherever possible, we rearrange (6.19) and view the right hand side as an error term commensurate with $||w||_{H^1}$. For the tuple $k=(1,\dots,d)$, the left hand side of (6.19) becomes

$$\begin{aligned}-m\cdot\alpha_{d+1,\dots,2d} =& \langle z_{1,\dots,d}, N_\eta(w)+\mathcal{R}_{V^h}\cdot(\eta+w)\rangle - \alpha\cdot\langle \mathcal{K}z_{1,\dots,d}, Jw\rangle\\ =& \langle -\nabla\eta, N_\eta(w)+\mathcal{R}_{V^h}w\rangle + \langle -\nabla\eta, \mathcal{R}_{V^h}\eta\rangle + \alpha\cdot\langle \mathcal{K}\nabla\eta, Jw\rangle\end{aligned}$$

or (since $\alpha_{d+1,\dots,2d} = -\frac{1}{2}\dot{v} - \nabla V(a)$),

$$\begin{aligned}\frac{1}{2}\dot{v} =& -\nabla V(a)\\ &- \frac{1}{m(\mu)}\left[\langle -\nabla\eta, N_\eta(w)+\mathcal{R}_{V^h}w\rangle + \langle -\nabla\eta, \mathcal{R}_{V^h}\eta\rangle + \alpha\cdot\langle \mathcal{K}\nabla\eta, Jw\rangle\right].\end{aligned} \tag{6.21}$$

For the tuple $k=(d+1,\dots,2d)$, we have

$$\begin{aligned}m\cdot\alpha_{1,\dots,d} =& \langle z_{d+1,\dots,2d}, N_\eta(w)+\mathcal{R}_{V^h}\cdot(\eta+w)\rangle - \alpha\cdot\langle \mathcal{K}z_{d+1,\dots,2d}, Jw\rangle\\ =& \langle ix\eta, N_\eta(w)+\mathcal{R}_{V^h}w\rangle + \langle ix\eta, \mathcal{R}_{V^h}\eta\rangle - \alpha\cdot\langle \mathcal{K}ix\eta, Jw\rangle\end{aligned}$$

or (since $\alpha_{1,\dots,d} = v - \dot{a}$),

$$\begin{aligned}\dot{a} =& v\\ &- (m(\mu))^{-1}\left[\langle x\eta, JN_\eta(w)+J\mathcal{R}_{V^h}w\rangle + \langle x\eta, J\mathcal{R}_{V^h}\eta\rangle + \alpha\cdot\langle \mathcal{K}x\eta, w\rangle\right].\end{aligned} \tag{6.22}$$

For $k=2d+1$, the left hand side of (6.19) becomes

$$\begin{aligned}m'\cdot\alpha_{2d+2} =& \langle z_{2d+1}, N_\eta(w)+\mathcal{R}_{V^h}\cdot(\eta+w)\rangle - \alpha\cdot\langle \mathcal{K}z_{2d+1}, Jw\rangle\\ =& \langle i\eta, N_\eta(w)+\mathcal{R}_{V^h}w\rangle - \alpha\cdot\langle \mathcal{K}i\eta, Jw\rangle\end{aligned}$$

or (since $\alpha_{2d+2} = -\dot{\mu}$),

$$-\dot{\mu} = 0 + (m'(\mu))^{-1}\left[\langle \eta, JN_\eta(w)+J\mathcal{R}_{V^h}w\rangle + \alpha\cdot\langle \mathcal{K}\eta, w\rangle\right]. \tag{6.23}$$

And finally, for $k=2d+2$, the left hand side of (6.19) becomes

$$-m'\cdot\alpha_{2d+1} = \langle z_{2d+2}, N_\eta(w)+\mathcal{R}_{V^h}\cdot(\eta+w)\rangle - \alpha\cdot\langle \mathcal{K}z_{2d+2}, Jw\rangle$$

or (since $\alpha_{2d+1} = \mu - \frac{v^2}{4} + \frac{\dot{a}}{2}v - V(a) - \dot{\gamma}$),

$$\begin{aligned}\dot{\gamma} =& \mu - \frac{v^2}{4} + \frac{\dot{a}}{2}v - V(a)\\ &+ (m'(\mu))^{-1}\left[\langle \partial_\mu\eta, N_\eta(w)+\mathcal{R}_{V^h}w\rangle + \langle \partial_\mu\eta, \mathcal{R}_{V^h}\eta\rangle - \alpha\cdot\langle \mathcal{K}\partial_\mu\eta, Jw\rangle\right].\end{aligned} \tag{6.24}$$

Collecting (6.21), (6.22), (6.23), and (6.24) together as a $(2d+2)$-vector equation and abbreviating this as

$$\dot{\sigma} = Y(\sigma) - \delta Y(\sigma, w),$$

we arrive at the following equation:

$$\delta Y_j(\sigma, w) = \sum_{k=1}^{2d+2} \left(\Omega_\eta^{-1}\right)_{jk} \beta_j \left[\langle z_k, N_\eta(w)\rangle + \langle z_k, \mathcal{R}_{V^h}\cdot(\eta+w)\rangle - \alpha\langle \mathcal{K}z_k, Jw\rangle\right],$$

where $\beta_j = -1$ if $j \in \{d+1, \dots, 2d\}$ and $\beta_j = 1$ otherwise. β_j is introduced purely to solve the notational issue of mismatched signs amongst the α_j used above.

We note that $||\mathcal{R}_{V^h} z_k|| = \mathcal{O}(\lambda h^2)$. Thus, by (i) $||N_\eta(w)|| \le c||w||^2_{H^1}$ for $||w||_{H^1} \le 1$ and (ii) using the notation $|\alpha| := \max_j |\alpha_j|$ from (6.6), we obtain

$$\begin{aligned}\delta Y &\le ||\Omega_\eta^{-1}|| \left(||w||^2_{H^1} + \lambda h^2 + |\alpha| \cdot ||w||\right)\\ &\le B\left(||w||^2_{H^1} + \lambda h^2 + |\alpha| \cdot ||w||\right),\end{aligned}$$

where $||\Omega_\eta^{-1}|| \le B$ is a bound on the operator norm of Ω_η^{-1}.

6.4 Estimating $||\Omega_\eta^{-1}||$

We choose $0 < R < \mu_0$ so that $I_0' := [\mu_0 - R, \mu_0 + R] \subset I_0$. To enforce $\mu(t) \in I_0'$, we choose t so that

$$|\mu(t) - \mu_0| \le \sup_{s\in(0,t)} |\dot{\mu}| \cdot t \le R. \tag{6.25}$$

Then using the matrix representation of the operator, the operator norm is bounded by the maximal eigenvalue of Ω_η^{-1}:

$$\begin{aligned}||\Omega_\eta^{-1}|| \le& \max\left\{\frac{1}{m(\mu)}, \frac{1}{m'(\mu)}\right\}\\ \le& C \cdot \max\left\{\frac{1}{\sqrt{\mu}}, \sqrt{\mu}\right\}\\ \le& C \cdot \max\left\{\frac{1}{\sqrt{\mu_0 - R}}, \sqrt{\mu_0 + R}\right\}\end{aligned}$$

We note that this last expression is finite, and abbreviate this bound as

$$||\Omega_\eta^{-1}|| \le B < \infty.$$

We will see in Chapter 8 that (6.25) is of the same order (in the relevant parameters) as the stability time bound given in Theorem 1.6.

6.5 Proof of Lemma 6.2

Proof. We recall first that $\mathcal{L}_\eta u = (-\Delta + \mu)\, u - \left|\eta\right|^{2k} (2kRe(u) + u)$. By the following argument, $\mathcal{L}_\eta$ is a self-adjoint operator on our inner product space:

Let φ, ζ be any two elements of $H^1(\mathbb{R}^d)$. Then, integrating by parts twice in the first term and shuffling some ζ and φ factors between the nested $Re(\cdot)$ operations in the third term, we obtain

$$\begin{aligned}\langle \varphi, \mathcal{L}_\eta \zeta\rangle =& Re\left(\int \varphi\left[(-\Delta + \mu)\,\overline{\zeta} - \left|\eta\right|^{2k}\left(2kRe(\overline{\zeta}) + \overline{\zeta}\right)\right]\, dx\right)\\ =& \int Re\left[\varphi\,(-\Delta + \mu)\,\overline{\zeta}\right] - Re\left[\varphi\left|\eta\right|^{2k}\left(2kRe(\overline{\zeta}) + \overline{\zeta}\right)\right]\, dx\end{aligned}$$

$$
\begin{aligned}
&= \int Re\left[\overline{\zeta}\left(-\Delta+\mu\right)\varphi\right] - Re\left[\overline{\zeta}\left|\eta\right|^{2k}\left(2kRe(\varphi)+\varphi\right)\right]\,dx \\
&=Re\left(\int\left[\left(-\Delta+\mu\right)\varphi-\left|\eta\right|^{2k}\left(2kRe(\varphi)+\varphi\right)\right]\overline{\zeta}\,dx\right) \\
&=\left\langle \mathcal{L}_\eta\varphi,\zeta\right\rangle.
\end{aligned}
$$

Further, $J\overline{J}=|J|^2=1$, and so

$$
\begin{aligned}
\left\langle Jz, J\mathcal{L}_\eta w\right\rangle &= \left\langle z,\mathcal{L}_\eta w\right\rangle \\
&= \left\langle \mathcal{L}_\eta z, w\right\rangle \\
&= \sum_{i\in\{t,g,b,s\}} c_i\left\langle \mathcal{L}_\eta z_i, w\right\rangle
\end{aligned}
$$

for any $z=\sum c_i z_i \in T_\eta\mathcal{G}$ with coefficient $c\in\mathbb{R}^{2d+2}$. Now we invoke the known zero eigenvectors and "zero modes" from Section 3.2

$$
\begin{aligned}
\mathcal{L}_\eta z_t &=0, \\
\mathcal{L}_\eta z_g &=0, \\
\mathcal{L}_\eta z_b &=2iz_t, \\
\mathcal{L}_\eta z_s &=iz_g,
\end{aligned}
$$

to get

$$
\begin{aligned}
\sum_{i\in\{t,g,b,s\}} c_i\left\langle \mathcal{L}_\eta z_i, w\right\rangle &=0+0+c_b\left\langle 2iz_t,w\right\rangle + c_s\left\langle 2iz_g,w\right\rangle \\
&=2c_b\left\langle J^{-1}z_t,w\right\rangle + 2c_s\left\langle J^{-1}z_g,w\right\rangle \\
&=0.
\end{aligned}
$$

The last step follows from the skew orthogonality condition $\left\langle J^{-1}z,w\right\rangle=0$ for any $z\in T_\eta\mathcal{G}$.

□

Chapter 7 Approximate Conservation of Energy

The main result of this chapter is Proposition 7.1, given below. This proposition shows that while the energy functional is not conserved, it is slowly varying in time relative to the current deviation from the traveling soliton $\eta_{\sigma(t)}$. This will be an important ingredient (see Lemma 8.4) in guaranteeing that the evolving solution ψ remains close to the traveling soliton. Key to this "approximate conservation" is the smallness of the coupling constant λ and the assumption $V \in C^\infty \cap L^\infty$. The semiclassical parameter is unnecessary. Further, the constant C in Proposition 7.1 depends on the size of integrals against ∇V, not norms of ∇V. Thus, large discrepancies in the scale of variations between factors in these integrals may cause the average value of the potential to be a bigger determiner of stability time scale than an L^∞ norm of V.

Recall that when a solution $\psi(t) \in H^1$ (t fixed) of Equation (1.1) lies inside the tubular neighborhood U_δ (defined at the outset of Section 4.2) we have local existence of the parameters $\sigma(t) = \{a(t), v(t), \gamma, (t)\mu(t)\}$. Recall further that we have given the name $w(t)$ to the (skew orthogonal) deviation of the frame transformed solution

$$u(t) = \mathcal{S}_{av\gamma}^{-1}\psi(t)$$

from the (time independent) soliton profile

$$\eta_\mu = \mathcal{S}_{av\gamma}^{-1}\eta_\sigma(t),$$

and that $|\alpha|(t)$ defined in Section 6.1 bounds the eccentricity of the parameters $\sigma(t)$ away from their free, traveling soliton trajectories. With this notation in force, we now can precisely formulate our "approximate conservation" result.

7.1 Main Estimate

Proposition 7.1. *Let $\psi \in U_\delta$ solve Equation (1.1). Then*

$$\frac{d}{dt}\left[\mathcal{E}_\mu(u) - \mathcal{E}_\mu(\eta_\mu)\right] \leq C\left(|\alpha| \cdot ||w||_{H^1}^2 + \lambda h ||w||_{H^1}^2 + \lambda h^2 ||w||_{H^1}\right),$$

where λ and h are the coupling constant and semi-classical parameter in (1.1), respectively.

In the proof of this result we will make use of the following lemma, which is independent of the specific choice of external potential λV^h. In much of this chapter, therefore, we use the more general notation V in place of λV^h for the external potential. We will continue to use the notation $f_a(x) := f(x+a)$ to indicate spatial translation of a function.

Lemma 7.2. *Let V be the external potential in Equation (1.1). Then*

$$\frac{d}{dt}\left(\mathcal{E}_\mu(u)\right) = \frac{1}{2}\dot{\mu}||u||^2 - \left\langle \left(\frac{1}{2}\dot{v} + \nabla V_a\right) iu, \nabla u \right\rangle.$$

Here some tools from Chapters 4 and 6 are brought to bear, and we exploit the similarities between two Hamiltonians $\mathcal{H}_\lambda$ and $\mathcal{H}_0$ defined in the introduction. We rely on the following minor lemma.

Lemma 7.3. *Let ψ solve Equation (1.1). Then*

$$\frac{d}{dt}\left\{\int V|\psi|^2\,dx\right\} = 2\left\langle (\nabla V)\, i\psi, \nabla\psi\right\rangle$$

We prove the main result modulo these two lemmas, giving their proofs in Section 7.2.

Proof of Prop. 7.1. Since η is stationary, the definition

$$\mathcal{E}_\mu(\psi) := \frac{1}{2}\int \left(|\nabla\psi|^2 + \mu|\psi|^2\right)\,dx - F(\psi)$$

yields

$$\frac{d}{dt}\mathcal{E}_\mu(\eta) = \frac{1}{2}\int \dot{\mu}|\eta|^2\,dx = \frac{1}{2}\dot{\mu}||\eta||^2.$$

We now rely on Lemma 7.2 as follows (and write $V = \lambda V^h$) to obtain

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{E}_\mu(u) - \mathcal{E}_\mu(\eta_\mu)\right) &= \frac{1}{2}\dot{\mu}||u||^2 - \left\langle \left(\frac{1}{2}\dot{v} + \nabla V_a\right) iu, \nabla u\right\rangle - \frac{1}{2}\dot{\mu}||\eta||^2 \\ &= \underbrace{\frac{1}{2}\dot{\mu}||u||^2 - \frac{1}{2}\dot{\mu}||\eta||^2}_{=:\mathcal{A}} - \underbrace{\left\langle \left(\frac{1}{2}\dot{v} + \nabla V_a\right) iu, \nabla u\right\rangle}_{=:\mathcal{B}}\end{aligned}$$

We now deal individually with the expressions marked as $\mathcal{A}$ and $\mathcal{B}$. Noting that $u = \eta + w$, while

$$\langle w, \eta\rangle = -\langle w, iz_g\rangle = -\left\langle w, J^{-1}z_g\right\rangle = 0$$

due to Proposition 4.3, and $|\dot{\mu}| = |\alpha_{2d+2}| \le |\alpha|$, we obtain

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\dot{\mu}||u||^2 - \frac{1}{2}\dot{\mu}||\eta||^2 \\ &= \frac{1}{2}\dot{\mu}\left(||\eta||^2 + ||w||^2\right) - \frac{1}{2}\dot{\mu}||\eta||^2 \\ &= \frac{1}{2}\dot{\mu}||w||^2 \\ &\le C\cdot|\alpha|\cdot||w||^2.\end{aligned}$$

Applying another orthogonality condition from Proposition 4.3, we have

$$\langle i\eta, \nabla w\rangle = \langle i\nabla\eta, w\rangle = \langle iz_t, w\rangle = 0, \tag{7.1}$$

and because of the real inner product $\langle\cdot,\cdot\rangle$, also

$$\langle iq\eta, \nabla\eta\rangle = 0$$

for any $q \in L^\infty$. So, once again decomposing $u = \eta + w$, we get

$$\begin{aligned}\mathcal{B} &= \left\langle \left(\frac{1}{2}\dot{v} + \nabla V_a\right) i(\eta + w), (\nabla\eta + \nabla w) \right\rangle \\ &= \left\langle \left(\frac{1}{2}\dot{v} + \nabla V_a\right) iw, \nabla w \right\rangle + \langle (\nabla V_a)\, i\eta, \nabla w\rangle + \langle (\nabla V_a)\, iw, \nabla\eta\rangle .\end{aligned}$$

Employing a "put-and-take" trick on the first term and using equation (7.1) once more to get

$$(\nabla V(a)) \cdot \langle i\eta, \nabla w\rangle = 0 = (\nabla V(a)) \cdot \langle iw, \nabla\eta\rangle ,$$

for use in the second and third terms, we obtain

$$\begin{aligned}\mathcal{B} &= \left\langle \left(\frac{1}{2}\dot{v} + \nabla V(a) + \nabla V_a - \nabla V(a)\right) iw, \nabla w \right\rangle \\ &\quad + \langle (\nabla V_a - \nabla V(a))\, i\eta, \nabla w\rangle + \langle (\nabla V_a - \nabla V(a))\, iw, \nabla\eta\rangle \\ &= \underbrace{\left\langle \left(\frac{1}{2}\dot{v} + \nabla V(a)\right) iw, \nabla w \right\rangle}_{\mathcal{O}\left(|\alpha|\cdot||w||^2_{H^1}\right)} + \underbrace{\langle (\nabla V_a - \nabla V(a))\, iw, \nabla w\rangle}_{\mathcal{O}\left(\lambda h||w||^2_{H^1}\right)} \\ &\quad + \underbrace{\langle (\nabla V_a - \nabla V(a))\, i\eta, \nabla w\rangle + \langle (\nabla V_a - \nabla V(a))\, iw, \nabla\eta\rangle}_{\mathcal{O}\left(\lambda h^2||w||_{H^1}\right)} .\end{aligned}$$

Thus,

$$\mathcal{B} \leq C \cdot \left(|\alpha| \cdot ||w||^2_{H^1} + \lambda h||w||^2_{H^1} + \lambda h^2||w||_{H^1}\right) .$$

The underbraces indicate the upper bounds term-by-term that establish the estimate given in Proposition 7.1. They are obtained by applying

(i) the Taylor series expansion of $V = \lambda V^h$ at a along with the hypothesis $V \in C^\infty$ and bounds

$$||D^\beta V||_\infty \leq C\lambda h^{|\beta|},\ \forall \beta \in \mathbb{N}^d,$$

(ii) compactness of the support of η, and

(iii) the estimate

$$|\frac{1}{2}\dot{v} + \nabla V(a)| \leq |\alpha|$$

obtained from $-\alpha_{d+j} = \frac{1}{2}\dot{v}_j + \partial_j V(a)$.

By putting together the estimates on $\mathcal{A}$ and $\mathcal{B}$ we finally obtain

$$\mathcal{A} + \mathcal{B} \leq C \cdot \left(|\alpha| \cdot ||w||^2_{H^1} + \lambda h||w||^2_{H^1} + \lambda h^2||w||_{H^1}\right) ,$$

as desired.

□

What now remains is to return and prove supporting Lemmas 7.3 and 7.2.

7.2 Proofs of Supporting Lemmas

The techniques used in these proofs are elementary, but the calculations are somewhat tedious. We use the notation V^n as shorthand for $(V(x))^n$.

Proof of Lemma 7.3. We integrate Equation (1.1) against $Vi\psi$ as follows to obtain:

$$0 = \langle i\psi_t, Vi\psi\rangle + \langle \Delta\psi, Vi\psi\rangle + \langle -V\psi + f(\psi), Vi\psi\rangle$$

or

$$\langle \psi_t, V\psi\rangle = -\langle \Delta\psi, Vi\psi\rangle - \langle -V\psi + f(\psi), Vi\psi\rangle .$$

Noting that $\frac{d}{dt}\langle \psi, V\psi\rangle = 2\langle \psi_t, V\psi\rangle$ and deleting imaginary terms below leads to the desired result:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int V|\psi|^2\right) =& \frac{1}{2}\frac{d}{dt}\langle \psi, V\psi\rangle \\
=& -\langle \Delta\psi, Vi\psi\rangle + Re\int i\underbrace{\left(V^2|\psi|^2 + Vf(\psi)\overline{\psi}\right)}_{\in\mathbb{R}} \\
=& -\langle \Delta\psi, Vi\psi\rangle + Re\int i\underbrace{\left(V^2|\psi|^2 + V\left|\psi\right|^{2k}\cdot(\psi)\overline{\psi}\right)}_{\in\mathbb{R}} \\
=& -\langle \Delta\psi, Vi\psi\rangle \\
=& \langle \nabla\psi, i\nabla(V\psi)\rangle \\
=& \langle \nabla\psi, i\left((\nabla V)\psi + V\nabla\psi\right)\rangle \\
=& \langle \nabla\psi, i(\nabla V)\psi\rangle + \langle \nabla\psi, iV\nabla\psi\rangle \\
=& \langle \nabla\psi, (\nabla V)i\psi\rangle - Re\int i\underbrace{\left(V|\nabla\psi|^2\right)}_{\in\mathbb{R}} \\
=& \langle \nabla\psi, (\nabla V)i\psi\rangle .
\end{aligned}$$

□

In the proof of Lemma 7.2 below, we use Lemma 7.3, as well as the definition of $\mathcal{H}_V$ (from the introduction), the phase function $\phi(x,t)$ (first introduced in Proposition 1.4) of η_σ, and Ehrenfest's theorem (stated and proved in Appendix A). This last ingredient, which could be colloquially referred to as "a Newton's law for quantum mechanical expectation values," allows us to convert a specific integral of $\nabla\psi$ into an integral of ∇V.

Proof of Lemma 7.2. Note that

$$||\mathcal{S}_{av\gamma}^{-1}\psi||^2 = ||\psi||^2 \tag{7.2}$$

and

$$\int V|\mathcal{S}_{av\gamma}^{-1}\psi|^2 = \int V_{-a}|\psi|^2. \tag{7.3}$$

From the definition of $\mathcal{H}_\lambda$, we obtain

$$\begin{aligned}
2\mathcal{H}_\lambda\left(\mathcal{S}_{av\gamma}^{-1}\psi\right) =& 2\mathcal{H}_\lambda\left(e^{-i\phi}\psi_a\right)\\
=& \int |\nabla\left(e^{-i\phi}\psi_a\right)|^2 + V|e^{-i\phi}\psi_a|^2\,dx - 2F(e^{-i\phi}\psi_a)\\
=& \int |-i\nabla\phi e^{-i\phi}\psi_a + e^{-i\phi}\nabla\psi_a|^2 + V|\psi_a|^2\,dx - 2\int\frac{\left|e^{-i\phi}\psi_a\right|^{2k+2}}{2k+2}\\
=& \int |\nabla\phi\psi_a|^2 + 2Re\left(-i\nabla\phi\psi_a\cdot\nabla\psi_a\right) + |\nabla\psi_a|^2 + V|\psi_a|^2\,dx\\
& - 2\int\frac{\left|\psi_a\right|^{2k+2}}{2k+2}
\end{aligned}$$

Since $\nabla\phi = v/2$, and the integral is invariant under the mapping of functions $f_a \mapsto f$, we determine that

$$\begin{aligned}
2\mathcal{H}_\lambda\left(\mathcal{S}_{av\gamma}^{-1}\psi\right) &= \frac{v^2}{4}||\psi_a||^2 - iv\left\langle\psi_a,\nabla\psi_a\right\rangle + \int|\nabla\psi_a|^2 + V|\psi_a|^2\,dx - 2\int\frac{\left|\psi_a\right|^{2k+2}}{2k+2}\\
&= \frac{v^2}{4}||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle + \int|\nabla\psi|^2 + V_{-a}|\psi|^2\,dx - 2\int\frac{|\psi|^{2k+2}}{2k+2}\\
&= \frac{v^2}{4}||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle + \int|\nabla\psi|^2 + V_{-a}|\psi|^2\,dx - 2F(\psi)\\
&= \frac{v^2}{4}||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle + 2\mathcal{H}_\lambda(\psi) + \int\left(V_{-a}-V\right)|\psi|^2\,dx.
\end{aligned}$$

Thus

$$2\mathcal{H}_\lambda\left(\mathcal{S}_{av\gamma}^{-1}\psi\right) = 2\mathcal{H}_\lambda(\psi) + \frac{v^2}{4}||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle + \int\left(V_{-a}-V\right)|\psi|^2\,dx. \tag{7.4}$$

Then using the definitions of $\mathcal{E}_\mu$ and $\mathcal{H}_V$ to get

$$2\mathcal{E}_\mu(u) = 2\mathcal{E}_\mu\left(\mathcal{S}_{av\gamma}^{-1}\psi\right) = 2\mathcal{H}_\lambda\left(\mathcal{S}_{av\gamma}^{-1}\psi\right) + \mu||\mathcal{S}_{av\gamma}^{-1}\psi||^2 - \int V|\mathcal{S}_{av\gamma}^{-1}\psi|^2\,dx,$$

and substituting Equations (7.2), (7.3), and (7.4) into the right hand side of this equation we obtain

$$\begin{aligned}
2\mathcal{E}_\mu(u) =& \left[2\mathcal{H}_\lambda(\psi) + \frac{v^2}{4}||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle + \int\left(V_{-a}-V\right)|\psi|^2\,dx\right]\\
& + \mu||\psi||^2 - \int V_{-a}|\psi|^2\,dx\\
=& 2\mathcal{H}_\lambda(\psi) + \left(\frac{v^2}{4}+\mu\right)||\psi||^2 - iv\left\langle\psi,\nabla\psi\right\rangle - \int V|\psi|^2\,dx.
\end{aligned}$$

Now *Ehrenfest's theorem* (see Appendix A) decrees that

$$\frac{d}{dt}\left\langle\psi,-i\nabla\psi\right\rangle = -\left\langle\psi,\left(\nabla V\right)\psi\right\rangle. \tag{7.5}$$

Recalling from Chapter 1 above that $\mathcal{H}_\lambda(\psi)$ and $||\psi||^2$ are conserved quantities, we see that

$$
\begin{aligned}
2\frac{d}{dt}\mathcal{E}_\mu(u) &= \left(\frac{v\dot{v}}{2}+\dot{\mu}\right)||\psi||^2 - i\dot{v}\left\langle \psi, \nabla\psi\right\rangle - v\cdot\frac{d}{dt}\left\langle \psi, -i\nabla\psi\right\rangle - \frac{d}{dt}\left\{\int V|\psi|^2\,dx\right\} \\
&= \left(\frac{v\dot{v}}{2}+\dot{\mu}\right)||\psi||^2 - i\dot{v}\left\langle \psi, \nabla\psi\right\rangle + \underbrace{v\left\langle \psi, (\nabla V)\,\psi\right\rangle}_{*} - \underbrace{2\left\langle (\nabla V)\,i\psi, \nabla\psi\right\rangle}_{**} \\
&= \left(\frac{v\dot{v}}{2}+\dot{\mu}\right)||\psi||^2 - i\dot{v}\left\langle \psi, \nabla\psi\right\rangle + v\left\langle (\nabla V)\,\psi, \psi\right\rangle - 2\left\langle (\nabla V)\,i\psi, \nabla\psi\right\rangle
\end{aligned}
$$

where the $(*)$ term comes from applying Ehrenfest's theorem (7.5) and $(**)$ is from applying Remark 7.3.

Now we collect $\left(\frac{\dot{v}}{2}+\nabla V\right)$ terms as follows:

$$
\begin{aligned}
2\frac{d}{dt}\mathcal{E}_\mu(u) &= \left(\frac{v\dot{v}}{2}+\dot{\mu}\right)||\psi||^2 - i\dot{v}\left\langle \psi, \nabla\psi\right\rangle + v\left\langle (\nabla V)\,\psi, \psi\right\rangle - 2\left\langle (\nabla V)\,i\psi, \nabla\psi\right\rangle \\
&= \dot{\mu}||\psi||^2 + v\left\langle \frac{\dot{v}}{2}\psi, \psi\right\rangle - 2\left\langle i\frac{\dot{v}}{2}\psi, \nabla\psi\right\rangle + v\left\langle (\nabla V)\,\psi, \psi\right\rangle - 2\left\langle i\,(\nabla V)\,\psi, \nabla\psi\right\rangle \\
&= \dot{\mu}||\psi||^2 + v\left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)\psi, \psi\right\rangle - 2\left\langle i\left(\frac{\dot{v}}{2}+\nabla V\right)\psi, \nabla\psi\right\rangle \\
&= \dot{\mu}||\psi||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)v\psi, \psi\right\rangle + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)2i\nabla\psi, \psi\right\rangle \\
&= \dot{\mu}||\psi||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)\cdot(v+2i\nabla)(\psi), \psi\right\rangle,
\end{aligned}
$$

where the penultimate equality follows from complex conjugation of $\nabla\psi\overline{\psi}$ in the last inner product. Now making the replacement $\psi = \mathcal{S}_{av\gamma}u = e^{i\phi}u_{-a}$ and cancelling the uni-modular factor $|e^{i\phi}|^2$, we see that

$$
\begin{aligned}
2\frac{d}{dt}\mathcal{E}_\mu(u) &= \dot{\mu}||u||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)\cdot(v+2i\nabla)\left(e^{i\phi}u_{-a}\right), e^{i\phi}u_{-a}\right\rangle \\
&= \dot{\mu}||u||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)\cdot\left(ve^{i\phi}u_{-a} + 2i\left[i\nabla\phi e^{i\phi}u_{-a} + e^{i\phi}\nabla u_{-a}\right]\right), e^{i\phi}u_{-a}\right\rangle \\
&= \dot{\mu}||u||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V\right)\cdot\left(vu_{-a} + 2i\left[i\nabla\phi u_{-a} + \nabla u_{-a}\right]\right), u_{-a}\right\rangle \\
&= \dot{\mu}||u||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V_a\right)\cdot\left(vu + \left[-2\nabla\phi u + 2i\nabla u\right]\right), u\right\rangle \\
&= \dot{\mu}||u||^2 + \left\langle \left(\frac{\dot{v}}{2}+\nabla V_a\right)\cdot\left(vu + \left[-vu + 2i\nabla u\right]\right), u\right\rangle \\
&= \dot{\mu}||u||^2 + \left\langle i\left(\dot{v}+2\nabla V_a\right)\nabla u, u\right\rangle.
\end{aligned}
$$

Here the last few steps involve invariance of the inner product under the mapping of the integrand by $x \mapsto (x+a)$ and the substitution $2\nabla\phi = v$.

Now, since $(\dot{v} + 2\nabla V_a) \in L^\infty(\mathbb{R}^d)$ is real-valued for each $t \in \mathbb{R}$, we can move complex conjugates once more to obtain

$$2\frac{d}{dt}\mathcal{E}_\mu(u) = \dot{\mu}||u||^2 - \langle(\dot{v} + 2\nabla V_a)\, iu, \nabla u\rangle\,,$$

which completes the proof. $\square$

Chapter 8 Short Time Orbital Stability For $0 \leq \lambda \ll 1$

In this chapter, we conclude the stability argument for soliton solutions of Equation (1.1). Of critical importance are the following two propositions, the first of which relies heavily on spectral analysis from Chapters 2 - 5:

Proposition 8.1. *Let ψ solving (1.1) be decomposed according to Proposition 4.3 as $\psi = \eta_\mu + w$, with Equation (4.4) holding. Then there exist uniform, positive constants ρ and c such that*

$$\left| (\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)) \right| \geq \frac{\rho}{2} ||w||_{H^1}^2 - c||w||_{H^1}^3. \tag{8.1}$$

Proof. By Proposition 5.1, there exists $\rho' > 0$ such that if $w \perp T_\eta \mathcal{G}$, then

$$\langle w, \mathcal{L}_\eta w \rangle \geq \rho' ||w||_{H^1}^2.$$

Computing, then, we find that

$$\begin{aligned} &2\left(\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)\right) \\ =& \int \left[|\nabla(\eta + w)|^2 + \mu|\eta + w|^2 \right] \, dx - 2F(\eta + w) - \left[\int \left[|\nabla \eta|^2 + \mu |\eta|^2 \right] \, dx - 2F(\eta) \right] \\ =& \left[2 \langle \nabla \eta, \nabla w \rangle + \langle \nabla w, \nabla w \rangle + 2 \langle \eta, w \rangle + \mu \langle w, w \rangle \right] - 2 \left[F(\eta + w) - F(\eta) \right]. \end{aligned}$$

we obtain (via integration by parts, the Taylor expansion of $F(u)$ and the equations for η and $\mathcal{L}_\eta$)

$$\begin{aligned} 2\left(\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)\right) =& \left[2 \langle \nabla \eta, \nabla w \rangle + 2 \langle \eta, w \rangle + 2 \langle F'(\eta), w \rangle \right] \\ &+ \left[\langle \nabla w, \nabla w \rangle + \mu \langle w, w \rangle - \langle F''(\eta) w, w \rangle \right] \\ &+ \mathcal{O}(||w||_{H^1}^3) \\ =& 0 + \langle w, (-\Delta + \mu - f'(\eta)) \, w \rangle + \mathcal{O}(||w||_{H^1}^3) \\ =& \langle w, \mathcal{L}_\eta w \rangle + \mathcal{O}(||w||_{H^1}^3) \end{aligned}$$

Hence we obtain

$$\left| (\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)) \right| \geq \rho ||w||_{H^1}^2 - c||w||_{H^1}^3,$$

for some $c = c(\eta) > 0$, as desired. □

As noted above, to avoid confusion we will use δY instead of δX for the difference $\dot{\sigma} - Y(\sigma)$. We define

$$X := |||w|||_{H^1} := \sup_{s \in (0,t)} ||w(s)||_{H^1}. \tag{8.2}$$

Then we have the following proposition, which will be used to impose a uniform bound on X for t less than some $T(\epsilon, \lambda, h)$:

Proposition 8.2 (9.2)**.** *Let ρ be the coercivity constant given in Proposition 8.1.There exists a constant c independent of λ and ϵ_0, such that for $||w||_{H^1} \leq 1$,*

$$\rho|||w|||_{H^1}^2 \leq c\epsilon_0^2 + ct\left(\lambda h^2|||w|||_{H^1} + \lambda h|||w|||_{H^1}^2 + |\alpha|_\infty \cdot |||w|||_{H^1}^2\right) + c|||w|||_{H^1}^3 \quad (8.3)$$

We establish two ancillary lemmas and then proceed to the proof of 8.2.

8.1 Supporting Lemmas

Define

$$\Delta\mathcal{E} := \mathcal{E}_{\mu_0}(u_0) - \mathcal{E}_{\mu_0}(\eta_{\mu_0}),$$

where $\mu_0 := \mu(0)$, $u_0 := \mathcal{S}_{av\gamma}^{-1}\psi\big|_{t=0}$, and

$$w_0 := u_0 - \eta_{\mu(0)} = \mathcal{S}_{av\gamma}^{-1}\left(\psi(0) - \eta_{\sigma(0)}\right).$$

Lemma 8.3 (9.1)**.** $|\Delta\mathcal{E}| \leq c\epsilon_0^2$ *for some $c > 0$ independent of ϵ_0 and λ.*

Proof. Using

$$\mathcal{E}'_{\mu_0}(\eta_{\mu_0}) = 0,$$

$$R_\eta^{(2)}(w) := F(\eta + w) - F(\eta) - \langle F'(\eta), w\rangle,$$

and

$$|R_\eta^{(2)}(w)| \leq c(M)||w||_{H^1}^2,$$

we obtain (using the stationary Equation (1.4) to remove cross terms in the fourth step)

$$\begin{aligned}
\Delta\mathcal{E} =& \mathcal{E}_{\mu_0}(u_0) - \mathcal{E}_{\mu_0}(\eta_{\mu_0}) \\
=& \mathcal{E}_{\mu_0}(w_0 + \eta_{\mu_0}) - \mathcal{E}_{\mu_0}(\eta_{\mu_0}) - \left\langle \mathcal{E}'_{\mu_0}(\eta_{\mu_0}), w_0\right\rangle \\
=& \left[||\nabla(w_0 + \eta_{\mu_0})||^2 + \mu_0||w_0 + \eta_{\mu_0}||^2 - F(w_0 + \eta_{\mu_0})\right] \\
&- \left[||\nabla\eta_{\mu_0}||^2 + \mu_0||\eta_{\mu_0}||^2 - F(\eta_{\mu_0})\right] - \left\langle \mathcal{L}_{\eta_{\mu_0}}(\eta_{\mu_0}), w_0\right\rangle \\
=& \left[||\nabla w_0||^2 + \mu_0||w_0||^2\right] - \left[F(w_0 + \eta_{\mu_0}) - F(\eta_{\mu_0}) - \langle F'(\eta_{\mu_0}), w_0\rangle\right] \\
=& ||\nabla w_0||^2 + \mu_0||w_0||^2 - R_{\eta_{\mu_0}}^{(2)}(w_0).
\end{aligned}$$

Hence,

$$\begin{aligned}
|\Delta\mathcal{E}| \leq& ||\nabla w_0||^2 + \mu_0||w_0||^2 + C||w_0||_{H^1}^2 \\
\leq& \tilde{c}||w_0||_{H^1}^2
\end{aligned}$$

To finish the proof, recall that $|v_0| \leq B_K$, for some $0 < K < \infty$, while $w_0 = u_0 - \eta_\sigma(0)$. Hence, invoking spatial translation invariance of the integral and later dropping the unimodular phase factor, we obtain

$$\begin{aligned}
||w_0||_{H^1}^2 =& ||\mathcal{S}_{av\gamma}^{-1}\left(\psi_0 - \eta_{\sigma(0)}\right)||_{H^1}^2 \\
=& ||e^{-i\frac{1}{2}v(0)x}(\psi_0 - \eta_{\sigma(0)})||_{H^1}^2
\end{aligned}$$

$$
\begin{aligned}
=&||e^{-i\frac{1}{2}v(0)x}(\psi_0-\eta_{\sigma(0)})||_2^2+||\nabla\left[e^{-i\frac{1}{2}v(0)x}(\psi_0-\eta_{\sigma(0)})\right]||_2^2\\
=&||(\psi_0-\eta_{\sigma(0)})||_2^2\\
&+||-i\frac{1}{2}v(0)e^{-i\frac{1}{2}v(0)x}(\psi_0-\eta_{\sigma(0)})+e^{-i\frac{1}{2}v(0)x}\nabla(\psi_0-\eta_{\sigma(0)})||_2^2\\
=&||(\psi_0-\eta_{\sigma(0)})||_2^2\\
&+||-i\frac{1}{2}v(0)(\psi_0-\eta_{\sigma(0)})+\nabla(\psi_0-\eta_{\sigma(0)})||_2^2.
\end{aligned}
$$

Now, applying the triangle inequality, we see that

$$
\begin{aligned}
||w_0||_{H^1}^2\leq&||(\psi_0-\eta_{\sigma(0)})||_2^2\\
&+\left(\frac{v(0)}{2}||(\psi_0-\eta_{\sigma(0)})||_2+||\nabla(\psi_0-\eta_{\sigma(0)})||_2\right)^2\\
\leq&\left(1+\frac{1}{2}v(0)\right)^2||(\psi_0-\eta_{\sigma(0)})||_{H^1}^2
\end{aligned}
$$

Thus we obtain

$$||w_0||_{H^1}\leq(1+v(0))||\psi_0-\eta_{\sigma(0)}||_{H^1}.$$

Moreover, by continuity of the map $\sigma:U_\delta\to\Sigma$ (See Proposition 4.2) there exists $C>0$ such that

$$||(\eta_{\sigma(0)}-\eta_{\sigma(0)})||_{H^1}^2\leq C|\sigma(0)-\sigma_0|.$$

Finally, recalling the notation $\sigma(0)=\sigma(\psi_0)$ and $\sigma_0=\sigma(\eta_{\sigma_0})$, and noting that derivatives of the map σ are uniformly bounded in the closed set U_δ, we obtain

$$|\sigma(0)-\sigma_0|\leq C||\psi_0-\eta_{\sigma_0}||_2\leq C||\psi_0-\eta_{\sigma_0}||_{H^1}.$$

Using the triangle inequality to put these last three estimates together, we find that

$$
\begin{aligned}
||w_0||_{H^1}\leq&(1+v(0))||\psi_0-\eta_{\sigma(0)}||_{H^1}\\
\leq&(1+v(0))\left[||\psi_0-\eta_{\sigma_0}||_{H^1}+||\eta_{\sigma_0}-\eta_{\sigma(0)}||_{H^1}\right]\\
\leq&C\cdot(1+v(0))||\psi_0-\eta_{\sigma_0}||_{H^1}\\
\leq&C\cdot(1+v(0))\epsilon_0.
\end{aligned}
$$

Thus, at last we have

$$|\Delta\mathcal{E}|\leq c\epsilon_0^2,$$

for $c:=(C\cdot(1+v(0)))^2$. $\square$

Lemma 8.4. *For η_μ, w, λ, h, and α defined above,*

$$\left|(\mathcal{E}_\mu(\eta_\mu+w)-\mathcal{E}_\mu(\eta_\mu))\right|\leq c\epsilon_0^2+ct\left(\lambda h^2|||w|||_{H^1}+\lambda h|||w|||_{H^1}^2+|\alpha|_\infty\cdot|||w|||_{H^1}^2\right)$$

Proof. From the modified Proposition 7.1 given above, we obtain

$$\left|(\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu))\right| \leq |\Delta\mathcal{E}| + ct\left(\lambda h^2 |||w|||_{H^1} + \lambda h |||w|||_{H^1}^2 + |\alpha|_\infty \cdot |||w|||_{H^1}^2\right)$$

by integrating on the time interval $s \in (0,t)$ and taking a supremum on the right hand side over the same interval. By Lemma 8.3, we have $|\Delta\mathcal{E}| \leq c\epsilon_0^2$ for some c independent of ϵ_0 and λ. Thus, taking suprema over $s \in (0,t)$ on the left hand side as well, we obtain

$$\sup_{s\in[0,t]}\left|(\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu))\right| \leq c\epsilon_0^2 + ct\left(\lambda h^2 |||w|||_{H^1} + \lambda h |||w|||_{H^1}^2 + |\alpha|_\infty \cdot |||w|||_{H^1}^2\right)$$

□

Proof of Proposition 8.2. Taking suprema on both sides of the coercivity estimate (8.1) given above bounds the left hand side from below. The lower (coercive) and upper bounds together immediately yield inequality 8.3. □

8.2 Proof of Main Theorem

Recall that we have η_μ solving Equation(1.4), a traveling soliton $\eta_\sigma(t)$ with initial conditions $\sigma_0 = \{a_0, v_0, \gamma_0, \mu_0\}$, and the orthogonal decomposition $\psi = \eta_\mu + w$ of the solution of Equation (1.1). Moreover, the parameters $\sigma(t)$ are governed by the approximate Newton's equations (see the main theorem 1.6), with error terms of order $|\alpha|$. It now remains to show that $||w(t)||_{H^1}$ and $|\alpha|$ remain small for $t \leq T(\epsilon_0, \lambda, h)$. Put another way, $X(t)$ and $|\alpha|_\infty$ must be small for the same time bound, where $|\alpha|_\infty$ is the notation we will use for $|\alpha|_\infty := \sup_{s\in(0,t)} |\alpha|$.

The critical inequality

Suppose that for the modified potential λV^h (that is, satisfying $V \in C^2$, $||V||_{L^\infty} = 1$, and $\lambda \in \mathbb{R}$ sufficiently small), we have

$$\rho X^2 \leq c\epsilon_0^2 + ct\left[\lambda h^2 X + (\lambda h + |\alpha|_\infty) X^2\right] + cX^3 \tag{8.4}$$

and

$$\delta Y = \mathcal{O}\left(||w||_{H^1}^2 + \lambda h^2 + |\alpha|||w||\right). \tag{8.5}$$

Now, suppose that for large enough K,

$$ct\left(\lambda h + |\alpha|_\infty\right)^{\frac{1}{2}} \leq \frac{\rho}{K} \tag{8.6}$$

and

$$\left(\lambda h + |\alpha|_\infty\right)^{\frac{1}{2}} \leq \frac{K}{2}. \tag{8.7}$$

Then from (8.4) and (8.6) we obtain

$$0 \leq c\epsilon_0^2 + ct\lambda h^2 X + ct\left(\lambda h + |\alpha|_\infty\right) X^2 - \rho X^2 + cX^3$$

$$\leq c\epsilon_0^2 + \frac{\rho}{K}\lambda^{\frac{1}{2}}h^{\frac{3}{2}}X + \frac{\rho}{K}\left(\lambda h + |\alpha|_\infty\right)^{\frac{1}{2}} X^2 - \rho X^2 + cX^3$$

(and by assuming $X \leq 1$ and the implicit time bound (8.7))

$$\leq \left(c\epsilon_0^2 + \frac{\rho}{K}\sqrt{\lambda h^3}\right) - \frac{\rho}{2}X^2 + cX^3.$$

Rescaling c yields

$$0 \leq \left(c\epsilon_0^2 + 2\rho K^{-1}\sqrt{\lambda h^3}\right) - \rho X^2 + cX^3, \tag{8.8}$$

and we define the right hand side of the last inequality (8.8) as a polynomial $P(X)$.

The first positive root of $P(X)$

Nearly all that remains to be shown is that the quantity X is bounded by the first root of $P(X)$, so long as time constraints (8.6) and (8.7) hold. The following lemma establishes the approximate location of the first root, subject to smallness assumptions on the parameters ϵ_0 and λ.

Lemma 8.5. *For any $\delta > 0$, there exist constants $q, q' > 0$ so that for ϵ_0 and λ both sufficiently small, $P(X)$ has a root in the open interval*

$$\left(q'\left(\sqrt{\frac{2\sqrt{\lambda h^3}}{K}} + \frac{\epsilon_0}{\sqrt{\rho}}\right), q\left(\sqrt{\frac{2\sqrt{\lambda h^3}}{K}} + \frac{\epsilon_0}{\sqrt{\rho}}\right)\right) \subset (0, \delta). \tag{8.9}$$

Proof. Set $a^2 = \epsilon_0^2\rho^{-1}$ and $b^2 = 2K^{-1}\sqrt{\lambda h^3}$. Then $P(X) = \rho\left(ca^2 + b^2\right) - \rho X^2 + cX^3$

Define $X_q = q \cdot (b + a)$, with $q > 0$ to be determined. Then

$$\begin{aligned}
P(X_q) - \rho\left(ca^2 + b^2\right) &= -\rho X_q^2 + cX_q^3 \\
&= \rho X_q^2\left(-1 + \frac{c}{\rho}X_q\right) \\
&= \rho q^2 (a+b)^2 \underbrace{\left(-1 + \frac{cq}{\rho}(b+a)\right)}_{<-1/2}
\end{aligned}$$

and choosing a and b small enough (done by choosing ϵ_0 and λ small enough), we can force the last factor to be less than $-\frac{1}{2}$, so that

$$\begin{aligned}
-2\rho q^2\left(b^2 + a^2\right) &\leq -\rho q^2\left(b^2 + a^2 + 2ab\right) \\
&= -\rho q^2 (a+b)^2 \\
&\leq P(X_q) - \rho\left(ca^2 + b^2\right) \\
&\leq -\frac{\rho q^2}{2}(a+b)^2 \\
&\leq -\frac{\rho q^2}{2}\left(a^2 + b^2\right),
\end{aligned}$$

or

$$\rho\left(ca^2+b^2\right)-2\rho q^2\left(b^2+a^2\right)\leq P(X_q)\leq\rho\left(ca^2+b^2\right)-\frac{\rho q^2}{2}\left(a^2+b^2\right)$$

Now choose q' so that

$$\rho\left(ca^2+b^2\right)-2\rho\left(q'\right)^2\left(b^2+a^2\right)>0$$

and q so that

$$\rho\left(ca^2+b^2\right)-2^{-1}\rho q^2\left(a^2+b^2\right)<0.$$

Then, by the Intermediate Value Theorem, there exists a root r of $P(X)$ in the desired interval $(X_{q'}, X_q)$. All that is required for this interval to be contained in $(0, \delta)$ is to choose the parameters λ and ϵ_0 smaller still. □

Conclusion of the argument

To conclude the argument, we look to Figure 8.2. There, we name the first positive

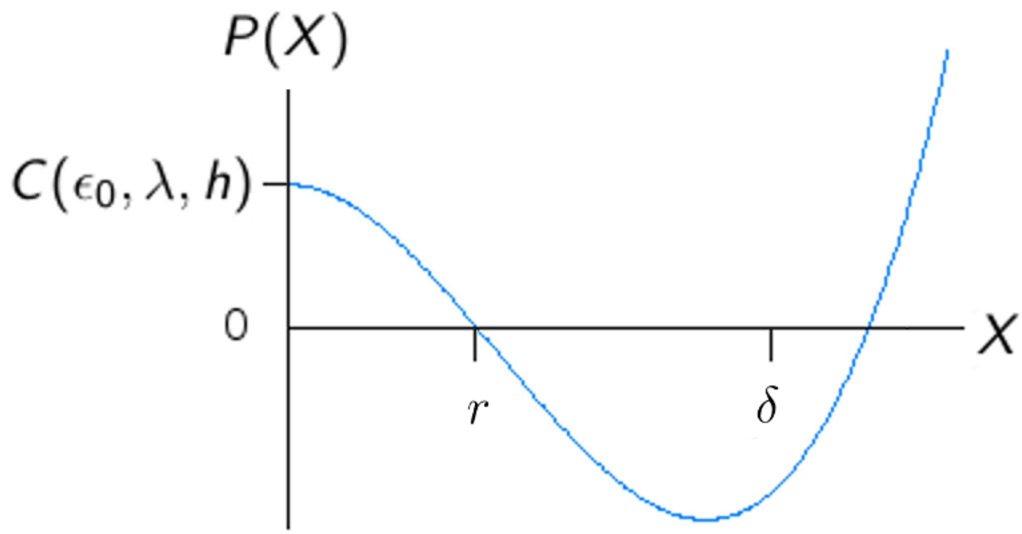

Figure 8.1: The first positive root r of $P(X)$.

root of $P(X)$ to be r and observe that the first interval on which the inequality (8.8) holds is $[0, r)$. If $X(0)$ lies in this interval, there is no way for $X(t)$ to escape past r so long as inequalities (8.6) and (8.7) hold. This is due to continuity of X in time, which comes from the local existence of σ as a C^1 function of t. More formally, what this means is that if we choose

$$||w(0)||_{H^1}\leq X_{q'}\leq r,$$

then we obtain via (8.9) that

$$||w(t)||_{H^1} \leq X(t) \leq r \leq X_q \leq q\left(\sqrt{\frac{2\sqrt{\lambda h^3}}{K}} + \frac{\epsilon_0}{\sqrt{\rho}}\right) \tag{8.10}$$

for all t satisfying (8.6) and (8.7).

We require $r < \delta$ in order to utilize the results from preceding chapters, beginning with the local existence of $\sigma(t)$ from Proposition 4.2.

Now, by recalling that $\alpha = \dot{\sigma} - Y(\sigma) = -\delta Y$ and using (8.5), we see that

$$|\alpha| \leq |\delta Y| = Bc\left(||w||_{H^1}^2 + \lambda h^2 + |\alpha|||w||\right) \tag{8.11}$$

Solving for $|\alpha|$, moreover, we have

$$|\alpha|\left(1 - Bc||w||\right) \leq Bc\left(||w||_{H^1}^2 + \lambda h^2\right) \tag{8.12}$$

and letting $||w|| \leq (2Bc)^{-1}$, we can force $(1 - Bc||w||) > \frac{1}{2}$ and then take a supremum on each side to obtain

$$|\alpha|_\infty \leq 2Bc\left(X^2 + \lambda h^2\right) \leq B\tilde{C}\left(\frac{2\sqrt{\lambda h^3}}{K} + \frac{\epsilon_0^2}{\rho} + \lambda h^2\right), \tag{8.13}$$

for t still satisfying (8.6) and additionally (6.25). (Notice that we have used $r < \delta$ for small $\delta(\lambda, h, \epsilon_0)$ again to obtain positivity of the coefficient of $|\alpha|$ in Estimate (8.12).) Since λ is assumed to be small and $h < 1$, we have $\lambda h^2 < \sqrt{\lambda h^3}$, and so

$$|\alpha|_\infty \leq \tilde{C}\left(\frac{2\sqrt{\lambda h^3}}{K} + \frac{\epsilon_0^2}{\rho}\right) = C\left(\sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right) =: A. \tag{8.14}$$

Finally, we can simplify our time scale bound by noting that $t \leq C\left(\lambda h + |\alpha|_\infty\right)^{-\frac{1}{2}}$ is implied by $t \leq C\left(\lambda h + A\right)^{-\frac{1}{2}}$. Thus (provided $\lambda h \leq \sqrt{\lambda h^3}$, that is, $\lambda \leq h$), we end up shortening the time interval to

$$t \leq C\left(\lambda h + A\right)^{-\frac{1}{2}} = \mathcal{O}\left(\lambda h + \sqrt{\lambda h^3} + \frac{\epsilon_0^2}{\rho}\right)^{-\frac{1}{2}} = \mathcal{O}\left(\sqrt{\lambda h^3} + \epsilon_0^2\rho^{-1}\right)^{-\frac{1}{2}}, \tag{8.15}$$

under which, of course, the result (8.10) will still hold. We note here that (i) up a constant and (ii) for the small $|\alpha|_\infty$ we consider, this is less generous than the bound (6.25), which now turns out to be

$$t \leq R \cdot \left(\sup_{s\in(0,t)} |\dot{\mu}|\right)^{-1} \leq \frac{\bar{C}}{|\alpha|_\infty}$$

Hence $\mu(t) \in I_0$, the argument is closed, and we have "soliton stability" for times satisfying (8.15).

8.3 Norm-changing Perturbations of ψ_0

If $||\psi_0|| \neq ||\eta||$ (the difference being at most $||w_0|| \leq \epsilon$), we use the rescaling

$$\varphi_\lambda(x,t) = \lambda e^{i\left(\lambda^2 t\right)} \eta\left(\lambda x\right)$$

to obtain

$$||\psi|| = ||\varphi_\lambda|| = \lambda ||\eta||$$

with

$$||\varphi_\lambda - \eta||_{H^1} < \mathcal{O}(\epsilon).$$

Note that $\eta_\sigma(x,t) = e^{i\phi(x-a(t),t)}\eta(x-a(t))$ solving (1.2) (for fixed μ) is equivalent to

$$-\mu\eta + \Delta\eta + f(\eta) = 0,$$

while φ_λ solving (1.2) is equivalent to

$$-(\mu+\lambda^2)\varphi_\lambda + \Delta\varphi_\lambda + f(\varphi_\lambda) = 0.$$

Now, choosing $\lambda > 0$ so that $||\psi|| = ||\varphi_\lambda|| = \lambda||\eta||$, we reapply the "equal norm"-limited proof of Theorem 9.1 to show that $\rho_{(\mu+\lambda^2)}\left(\psi\left(\cdot,t\right), \mathcal{G}_{\varphi_\lambda}\right) < \epsilon$. Finally, we use the triangle inequality and inequality (9.5) (in the penultimate inequality) to get

$$\begin{aligned}
\rho_\mu\left(\psi(t), \mathcal{G}_\eta\right) \leq & ||e^{i\phi(x,t)}\psi_{a(t)}\left(x,t\right) - \eta||_{H^1} \\
\leq & ||e^{i\phi(x,t)}\psi_{a(t)}\left(x,t\right) - \varphi_\lambda|| + ||\varphi_\lambda - \eta||_{H^1} \\
< & ||e^{i\phi(x,t)}\psi_{a(t)}\left(x,t\right) - \varphi_\lambda|| + \epsilon \\
< & C(\mu+\lambda^2)\rho_{(\mu+\lambda^2)}\left(\psi\left(\cdot,t\right), \mathcal{G}_{\varphi_\lambda}\right) + \epsilon \\
< & \Big(C(\mu+\lambda^2) + 1\Big)\,\epsilon.
\end{aligned}$$

Chapter 9 Global Time Orbital Stability For $\lambda = 0$

For the sake of completeness, we note that in the absence of an external potential we get global time orbital stability for traveling solitons. This is not at all obvious from the above result, as setting $\lambda = 0$ in Theorem 1.6 fails to recover global stability. A shorter argument that follows papers [19, 20] of M. I. Weinstein (and which does not obtain explicit modulation equations) is used instead.

9.1 The $\lambda = 0$ Setting

As mentioned in Chapter 1, setting $\lambda = 0$ reduces the problem (1.1) to (1.2), a focusing NLS with nonlinearity f. This equation is recalled below:

$$i\psi_t + \Delta\psi + f(\psi) = 0,$$

with initial data

$$\psi(0) = \psi_0 \in H^1\left(\mathbb{R}^d\right).$$

Many of the calculations can be done for arbitrary dimension, but in some cases we must restrict to the sub-critical case $k < 2/d$ in order to make use of a scaling argument. We also inherit the limitation $d \in \{1, 3\}$ from Proposition 5.2.

Well-posedness of this problem for H^1 initial data was obtained by Ginibre and Velo (see Theorem 1.1). As was mentioned in the introduction, M. I. Weinstein employed a Lyapunov argument to show stability of standing solitary waves solving Equation (1.2), that is $e^{i\mu t}\eta_\mu(x)$, where η_μ solves Equation (1.4) as follows:

$$-\left(\Delta + \mu\right)\eta_\mu + f(\eta_\mu) = 0.$$

The Lyapunov functional that (following Weinstein) we employ is identical to the one in the $\lambda \neq 0$ case. To condense some later notation we introduce

$$\Delta\mathcal{E}_\mu[\varphi, \eta] := 2\left(\mathcal{E}_\mu(\varphi) - \mathcal{E}_\mu(\eta)\right).$$

where (as before)

$$\begin{aligned}\mathcal{E}_\mu(\varphi) :=& \mathcal{H}_0[\varphi] + \mu\mathcal{N}[\varphi] \\ =& \frac{1}{2}\int\left(|\nabla\varphi|^2 + \mu|\varphi|^2\right)\,dx - F(\varphi),\end{aligned}$$

where $\mu > 0$. Since $\mathcal{H}_0[\varphi]$ and $\mathcal{N}[\varphi]$ are conserved in time for $\lambda = 0$ (see Chapter 1 or the proofs in Appendix B), the second Lyapunov condition $\frac{d}{dt}\mathcal{E}_\mu[\varphi(t)] = 0 \leq 0$ holds.It remains to verify that the first condition in Definition 1.11 is satisfied: $\mathcal{E}_\mu(\varphi) > \mathcal{E}_\mu(\eta_\mu)$ for any φ in some punctured neighborhood of η_σ. Recall the calculation (2.1), which says

$$\mathcal{E}_\mu(\eta_\sigma) = \mathcal{E}_{\mu+|\nabla\phi|^2}(\eta_\mu) = \mathcal{E}_{\mu+|v|^2/4}(\eta_\mu), \tag{9.1}$$

linking energies of soliton and ground state. We will use this in the proof of Theorem 9.1 (stated in Section 9.3) to make our estimates in the moving frame of reference.

9.2 Definitions and Notation

We recall from an earlier chapter that, given a certain conditions on $a(t), v(t) \in \mathbb{R}^d$ and $\gamma(t), \mu(t) \in \mathbb{R}$ for each $t \in \mathbb{R}$,

$$\psi_{\{a,v,\gamma\}(t)}(x,t) = e^{i\left(\frac{1}{2}v(t)(x-a(t))+\gamma(t)\right)}\eta_\mu\left(x - a(t)\right)$$

is a ground state of Equation (1.2). Recall also that we employ the notation

$$\phi(x) := \frac{1}{2}v \cdot x + \gamma$$

and

$$\psi_a(x) := \psi(x+a).$$

For convenience, we define a modified H^1 norm

$$||f||^2_{H^1_\mu(\mathbb{R}^d)} := \mu||f||_2^2 + ||\nabla f||_2^2,$$

and sometimes use the shorthand

$$||f|| := ||f||_2.$$

Finally, we specify Definition 1.7 so that

$$\mathcal{G}_\psi := \left\{e^{i\gamma)}\psi\left(\cdot + a\right) \ : \ a \in \mathbb{R}^d, \gamma \in \mathbb{R}\right\}, \tag{9.2}$$

and we measure stability using the metric ρ_μ, defined as follows in terms of the H^1_μ norm introduced earlier:

$$\left[\rho_\mu\left(\psi(t), \mathcal{G}_\eta\right)\right]^2 := \inf_{a\in\mathbb{R}^d, \gamma\in\mathbb{R}} \left\{||e^{i\phi}\psi_a(t) - \eta||^2_{H^1_\mu}\right\}. \tag{9.3}$$

Later, we will choose $a = a(t)$ and $\gamma = \gamma(t)$ so that this infimum is attained for each t. In the absence of an external potential we hold parameters v and μ constant. We note in passing that by translation and phase invariance of Lebesgue integrals, $\rho_\mu\left(\psi, \mathcal{G}_\eta\right) = \rho_\mu\left(\eta, \mathcal{G}_\psi\right)$.

9.3 Sketch of the Proof

For clarity of presentation, we give a sketch of the basic Lyapunov argument first, devoting later sections to filling in the details surrounding the assumption (9.4). In doing so, we closely follow the structure of presentation given by Weinstein in [20].

Theorem 9.1. *Let $k < 2/d$ with $d = 1$ or $d = 3$, let η_μ solve Equation (1.4), and let $\mathcal{G}_{\eta_\mu}$ be generated by translation and phase symmetries T_a^{tr}, T_γ^g.*

Then for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ so that for any $\psi(x,t)$ solving (1.2) with initial data $\psi_0 \in H^1(\mathbb{R}^d)$,

$$\rho_\mu(\psi_0, T_v^{gal}\mathcal{G}_{\eta_\mu}) < \delta(\epsilon)$$

implies

$$\rho_\mu\left(\psi(t), T_v^{gal}\mathcal{G}_{\eta_\mu}\right) < \epsilon$$

for all $t > 0$. (In other words, the η_μ-orbit is orbitally stable.)

Proof. (Sketch) We choose $a = a(t)$ and $\gamma = \gamma(t)$ to minimize $\rho_\mu(\psi(t), \mathcal{G}_\eta)$ for each t, and write

$$\mathcal{S}_{av\gamma}^{-1}\psi = e^{-i\phi(x,t)}\psi_a(x,t) = \eta(x) + w(x,t).$$

Here w is the perturbation of ψ from $\mathcal{G}_\eta$ (see the definition of ρ_μ), which we decompose as follows:

$$w =: r + is.$$

Now, if we have

$$\Delta\mathcal{E}_{\mu^*}[\psi(t), \eta_{\sigma(t)}] \geq D||w||_{H^1}^2 - D'\mathcal{O}\left(||w||_{H^1}^{2+\theta}\right), \tag{9.4}$$

for $\mu^* = \mu - \frac{|v|^2}{4}$, $\theta > 0$, and for perturbations w of the ground state η that do not change the L^2 norm of ψ, we can close the Lyapunov argument.

Noting that the deviation $\rho_\mu\left(\psi(t), T_v^{gal}\mathcal{G}_\eta\right)$ is by definition controlled by $||w(t)||_{H^1}$ as follows:

$$\sqrt{\min\{\mu, 1\}}||w(t)||_{H^1} \leq \rho_\mu\left(\psi(t), T_v^{gal}\mathcal{G}_\eta\right) \leq \sqrt{\max\{\mu, 1\}}||w(t)||_{H^1}, \tag{9.5}$$

we see that

$$\Delta\mathcal{E}_{\mu^*}[\psi(t), \eta_{\sigma(t)}] \geq g\left(\rho_\mu\left(\psi(t), T_v^{gal}\mathcal{G}_\eta\right)\right),$$

where $g(\delta) = k\delta^2 - \mathcal{O}\left(\delta^{2+\theta}\right)$ with all constants positive. Now, invoking continuity of $\mathcal{E} : H^1(\mathbb{R}^d) \to \mathbb{R}$ and the control given by (9.5), we conclude that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\rho_\mu\left(\psi_0, T_v^{gal}\mathcal{G}_\eta\right) < \delta$$

implies

$$g\left(\rho_\mu\left(\psi(t), T_v^{gal}\mathcal{G}_\eta\right)\right) \leq \Delta\mathcal{E}_{\mu^*}\left[\psi(t), \eta_\sigma(t)\right] = \Delta\mathcal{E}_{\mu^*}\left[\psi_0, \eta_{\sigma_0}\right] < g(\epsilon). \tag{9.6}$$

Since $g(0) = 0$ and $g(\delta) > 0$ (even $g'(\delta) > 0$) for $\delta > 0$ sufficiently small, we see that g must be increasing somewhere near 0. (See Figure 9.1 for a sketch of these

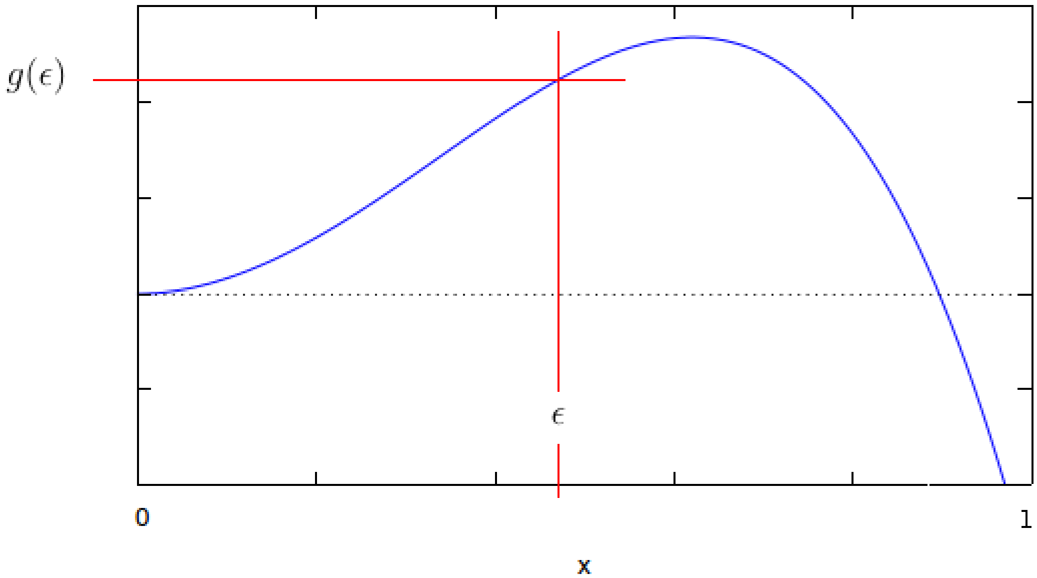

Figure 9.1: The "trapping" region of $g(x)$ near 0 corresponding to stable perturbations of initial data ψ_0.

features.) Hence g is monotone increasing on an interval $[0, \alpha)$ for some $\alpha > 0$, and so for small enough ϵ,

$$\rho_\mu \left(\psi(t), T_v^{gal} \mathcal{G}_\eta\right) < \epsilon,$$

for all $t > 0$.

$\square$

We now provide more details concerning the lower bound (9.4) on the difference $\Delta\mathcal{E}_{\mu^*}$. In the calculations that follow, we make the now familiar decomposition

$$\mathcal{S}_{a,v,\gamma}^{-1}\psi(x,t) = e^{-i\phi(x,t)}\psi_a(x,t) = \eta(x) + w(x,t),$$

where $w = r + is$ is the perturbation of ψ from the ground state η. We observe, for convenience of calculation, that for any $p, q \in H^1(\mathbb{R}^d)$,

$$\begin{aligned} ||p+q||_{H^1_\mu}^2 =& ||\nabla p + \nabla q||^2 + \mu||p+q||^2 \\ =& ||\nabla p||^2 + 2\langle \nabla p, \nabla q\rangle + ||\nabla q||^2 + \mu||p||^2 + 2\mu\langle p, q\rangle + \mu||q||^2 \\ =& ||p||_{H^1_\mu}^2 + ||q||_{H^1_\mu}^2 + 2\langle \nabla p, \nabla q\rangle + 2\mu\langle p, q\rangle\,. \end{aligned} \tag{9.7}$$

Using phase and translation invariance and time independence of $\mathcal{E}$, we obtain first

$$\begin{aligned} \Delta\mathcal{E}_{\mu^*}[\phi_0, \eta_{\sigma(0)}] =& 2\left(\mathcal{E}_{\mu^*}(\psi(t)) - \mathcal{E}_{\mu^*}(\eta_{\sigma(t)})\right) \\ =& 2\left(\mathcal{E}_\mu[e^{-i\phi(\cdot,t)}\psi_a(\cdot,t)] - \mathcal{E}_\mu[\eta_\mu(\cdot)]\right) \\ =& 2\left(\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)\right), \end{aligned}$$

and then, applying (9.7) and expanding $F(\eta_\mu + w)$ in a power series,

$$
\begin{aligned}
\Delta\mathcal{E}[\phi_0,\eta_{\sigma(0)}] = &\left[\int |\nabla\eta_\mu+\nabla w|^2+\mu|\eta_\mu+w|^2\ dx\right]-2F(\eta_\mu+w)\\
&-\int\left[|\nabla\eta_\mu|^2+\mu|\eta_\mu|^2\ dx\right]-2F(\eta_\mu)\\
=&||\eta_\mu+w||^2_{H^1_\mu}-||\eta_\mu||^2_{H^1_\mu}\\
&-2\left[F(\eta_\mu+w)-F(\eta_\mu)\right]\\
=&||w||^2_{H^1_\mu}+2\left\langle\nabla\eta_\mu,\nabla w\right\rangle+2\mu\left\langle\eta_\mu,w\right\rangle\\
&-2\left\langle F'(\eta_\mu),w\right\rangle-\left\langle F''(\eta_\mu)w,w\right\rangle+\mathcal{O}\left(\int|w|^3\right)\\
=&||w||^2_{H^1_\mu}+2\left\langle\nabla\eta_\mu,\nabla w\right\rangle+2\mu\left\langle\eta_\mu,w\right\rangle\\
&-2\left\langle f(\eta_\mu),w\right\rangle-\left\langle f'(\eta_\mu)w,w\right\rangle+\mathcal{O}\left(\int|w|^3\right).
\end{aligned}
$$

After some rearrangement, along with use of Equation (1.4) and the definition of $\mathcal{L}_\eta$, this becomes

$$
\begin{aligned}
\Delta\mathcal{E}_{\mu^*}[\phi_0,\eta_{\sigma(0)}] =&\left[2\left\langle\nabla\eta_\mu,\nabla w\right\rangle+2\mu\left\langle\eta_\mu,w\right\rangle-2\left\langle f(\eta_\mu),w\right\rangle\right]\\
&+\left[||w||^2_{H^1_\mu}-\left\langle f'(\eta_\mu)w,w\right\rangle\right]+\mathcal{O}\left(\int|w|^3\right)\\
=&0+\left\langle\mathcal{L}_\eta w,w\right\rangle+\mathcal{O}\left(\int|w|^3\right).
\end{aligned}
$$

Finally, decomposing $w=r+is$ and letting

$$L_1=-\Delta+\mu-(2k+1)\eta_\mu^{2k},$$

$$L_2=-\Delta+\mu-\eta_\mu^{2k},$$

just as in Chapter 2, we arrive at

$$
\begin{aligned}
\Delta\mathcal{E}_{\mu^*}[\phi_0,\eta_{\sigma(0)}] =&0+\left\langle\mathcal{L}_\eta w,w\right\rangle+\mathcal{O}\left(\int|w|^3\right)\\
=&(L_1r,r)+(L_2s,s)-\int\mathcal{O}\left(w^3\right)\ dx.
\end{aligned}
$$

Spectral analysis of the operators $L_\pm$ is now required to establish estimate (9.4), but under slightly different orthogonality conditions than those in our $\lambda\neq 0$ approach, as is dictated by the minimization problem in the first section of the next chapter.

Chapter 10 Variational Arguments for Coercivity of $L_{\pm}$

This chapter is devoted to establishing estimate (9.4) via careful analysis of the operators L_1, L_2 defined above. Since L_2 has no negative eigenvalues, we quickly obtain a lower bound on $(L_2 s, s)$. Due to its negative eigenvalue, the corresponding estimate for the operator L_1 requires much more work, including Lagrange multiplier and compactness arguments to construct a minimizer of $(L_1 r, r)$.

10.1 Minimization Problem

Critical to these estimates are orthogonality conditions (10.2) and (10.3) obtained by choosing a, γ so that the infimum in the definition of $\rho_\mu\left(\psi(t), \mathcal{G}_\eta\right)$ is attained. We use a calculus of variations approach.

Recall the definitions

$$\phi(x) := \frac{1}{2} v \cdot x + \gamma,$$
$$\psi_a(x) := \psi(x+a),$$

and

$$e^{-i\phi}\psi_a =: \eta(x) + w(x,t) =: \eta + r + is,$$

and vary the parameters a, γ so as to find a minimizer of

$$\inf_{(a,\gamma)\in\mathbb{R}^{2d+1}} ||e^{-i\phi}\psi_a - \eta||^2_{H^1_\mu} \tag{10.1}$$

where as before

$$||u||^2_{H^1_\mu} := \int |\nabla u|^2 + \mu |u|^2.$$

By calculations carried out in Appendix C, we arrive at the following minimizing constraints, or orthogonality conditions:

From varying γ, we obtain

$$(f(\eta), s) = 0. \tag{10.2}$$

From varying a, we get

$$2\left(\nabla\left(f(\eta)\right), r\right) - v\left(f(\eta), s\right) = 0,$$

or (after applying Equation (1.4))

$$\left(\nabla\left(f(\eta)\right), r\right) = \left(f'(\eta)\nabla\eta, r\right) = \vec{0}. \tag{10.3}$$

These two constraints are critical to the following analysis of $L_{\pm}$.

10.2 Analysis of L_2

We have already shown in Section 2.2 that L_2 is a non-negative operator, i.e.,

$$\inf_{||s||_{H^1}=1} (L_2 s, s) \geq 0.$$

If this minimum were attained by $s = \eta$ subject to the constraint $(s, f(\eta)) = 0$, then we would have

$$\begin{aligned}
0 =& (L_2\eta, \eta) \\
=& \int |\nabla\eta|^2 + \mu\eta^2 - (\eta)^{2k}\,\eta^2 \\
\geq& C(\mu)||\eta||_{H^1} - (\eta, f(\eta)) \\
=& C(\mu)||\eta||_{H^1},
\end{aligned}$$

a contradiction. So by the non-degeneracy of η, we must have

$$\inf\left\{(L_2 s, s) : (s, f(\eta)) = 0, ||s||_{H^1} = 1\right\} > 0.$$

Hence $(L_2 s, s)/(s, s) \geq C_1 > 0$, or

$$(L_2 s, s) \geq C_1(s, s).$$

Now, noting that

$$(L_2 s, s) = \int |\nabla s|^2 + s^2 - (\eta)^{2k}\, s^2 = \int |\nabla s|^2 + \left(1 - (\eta)^{2k}\right) s^2,$$

we see that

$$\begin{aligned}
||s||_{H^1}^2 =& \int |\nabla s|^2 + |s|^2 \\
=& (L_2 s, s) + \int |\eta^k s|^2 \\
\leq& (L_2 s, s) + \sup_x (\eta)^{2k} \cdot (s, s) \\
\leq& C_2\,(L_2 s, s)
\end{aligned}$$

for some $C_2 > 0$, which is the uniform H^1 bound we wanted.

10.3 Analysis of L_1

L_1 is more difficult to handle than L_2 was, since L_1 has exactly one negative eigenvalue (see again Section 2.2). First, we invoke Proposition 5.2 and reformulate it as follows by rewriting the orthogonality condition $\langle w, \eta\rangle = 0$ in terms of the L^2 inner product $(\cdot, \cdot)$:

Proposition 10.1. *Let $k < 2/d$. Then for any $\xi \in H^1(\mathbb{R}^d;\mathbb{R})$ with $(\xi,\eta) = 0$, we have the lower bound*

$$(L_1\xi,\xi) \geq 0.$$

For now, we assume that

$$||\phi|| = ||\phi_0|| = ||\eta||, \tag{10.4}$$

and for convenience $||\eta|| = 1$. This quickly yields the formula (via $||\eta+w||^2 = ||\psi||^2 = ||\eta||^2$),

$$(r,\eta) = -\frac{1}{2}||w||^2.$$

We also that recall from Section 2.3 that the nullspace of L_1 is

$$Null\,(L_1) = \text{span}\left\{\partial_{x_j}\eta, \forall 1 \leq j \leq d\right\}.$$

Now, to preserve the flow of the argument we present a proposition and defer the proof until the next subsection.

Proposition 10.2 (L_1 constrained coercivity)**.** *Let $k < 2/d$. Then subject to the constraints*

$$\begin{aligned} (r,\eta) =& \ \ 0, \\ (r, \nabla(f(\eta))) =& \ \ \vec{0}, \end{aligned}$$

we have

$$\inf_{r\in H^1} \frac{(L_1 r, r)}{(r,r)} = \lambda$$

for some $\lambda > 0$.

With this result, we are now ready to finish the coercivity argument for L_1.

Proposition 10.3 (Prop. 3.3)**.** *For r satisfying the condition $(r,\ \nabla f(\eta)) = \vec{0}$ and $||\psi|| = ||\eta||$,*

$$(L_1 r, r) \geq D||r||_{H^1}^2 - D'||w||_{H^1}^3 - D''||w||_{H^1}^4.$$

Here D, D', and D'' are positive constants.

Proof. Decomposing $r = r_\parallel \oplus r_\perp := (r,\eta)\,\eta\ \oplus\ r - (r,\eta)\,\eta$, we see that

$$(L_1 r, r) = (L_1 r_\perp, r_\perp) + 2\left(L_1 r_\parallel, r_\perp\right) + \left(L_1 r_\parallel, r_\parallel\right).$$

We estimate each term on the right hand side individually. For the first term, we refer to Proposition 10.2. Thus we find that (since we assumed $||\eta|| = 1$)

$$\begin{aligned} (L_1 r_\perp, r_\perp) \geq & D||r_\perp||^2 \\ =& D\left[||r||^2 - ||r_\parallel||^2\right] \\ =& D\left[||r||^2 - (r,\eta)^2\,||\eta||^2\right] \end{aligned}$$

$$=D\left[||r||^2 - \frac{1}{4}||w||^4\right].$$

As for the other terms, we have

$$\begin{aligned}\left(L_1 r_{\|}, r_{\|}\right) =& (r,\eta)^2 (L_1\eta,\eta)\\ =& \frac{1}{4}(L_1\eta,\eta)\,||w||^4\\ \geq& - D'||w||^4,\end{aligned}$$

since $(L_1\eta,\eta)$ is bounded below, and

$$\begin{aligned}\left(L_1 r_{\|}, r_{\perp}\right) =& (r,\eta)(L_1\eta, r_{\perp})\\ \geq& - D''||w||^3,\end{aligned}$$

where the last line, seen below, follows from Cauchy-Schwarz and the orthogonal decomposition $r = r_{\|} \oplus r_{\perp}$:

$$\begin{aligned}(L_1\eta, r_{\perp}) =& (-\Delta\eta + \eta - (2k+1)f(\eta), r_{\perp})\\ =& (-\Delta\eta - (2k+1)f(\eta), r_{\perp})\\ =& - (\Delta\eta + (2k+1)f(\eta), r_{\perp})\\ \leq& ||\Delta\eta + (2k+1)f(\eta)|| \cdot ||r_{\perp}||,\end{aligned} \tag{10.5}$$

and

$$||r_{\perp}|| \leq ||r|| \leq ||w||.$$

Note that in the first line of estimate (10.5), we used $(r_{\perp},\eta) = 0$. Collecting the estimates for each of the three terms, we find that

$$(L_1 r, r) \geq D||r||^2 - D'||w||^3 - D''||w||^4.$$

By a similar argument used for promoting the bound on $(L_2 v, v)$ from L^2 to H^1, we can replace $||r||^2$ by $||r||^2_{H^1}$ in this last inequality, which completes the proof.

□

Proof of Proposition 10.2

In this subsection we prove Proposition 10.2. The proof is adapted from Proposition 2.9 in [19]. First we prove that $\lambda = 0$ implies a non-trivial admissible minimizer exists (Step 1). Then we rely on a Lagrange multiplier argument to show that this minimizer cannot satisfy the given constraints (Step 2).

Proof of Proposition 10.2. We suppose that $\lambda = 0$ and seek a contradiction.

Step 1:

Choose a sequence of real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ such that $||f_n|| = 1$, the orthogonality constraints

$$\begin{aligned}(f_n,\eta) =&\ 0,\\ (f_n, \nabla(f(\eta))) =&\ \vec{0}\end{aligned}$$

hold, and $(L_1 f_n, f_n) \downarrow 0$ as $n \to \infty$. Then using the definition of L_1 to get

$$(L_1 f_n, f_n) = ||f_n||_{H^1}^2 - \int f'(\eta)|f_n|^2,$$

we find that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n > N$,

$$||f_n||_{H^1}^2 - \int f'(\eta)|f_n|^2 \le \epsilon,$$

and by $\eta \in L^\infty$,

$$||f_n||_{H^1}^2 \le \epsilon + \int f'(\eta)|f_n|^2 \le \epsilon + C(\eta)||f_n||^2. \tag{10.6}$$

Since $||f_n|| = 1$, we have the uniform bound

$$0 < 1 \le ||f_n||_{H^1}^2 \le \epsilon + C_\eta. \tag{10.7}$$

This implies that we can choose a subsequence $\{f_k\}$ that converges weakly H_0^1 to some $f_* \in H^1$. Weak H_0^1 convergence suffices to establish that the orthogonality constraints above hold also for f_*.

It remains to show that $\langle \mathcal{L}_\eta f_k, f_k \rangle \to \langle \mathcal{L}_\eta f_*, f_* \rangle$ as $k \to \infty$, (with $||f_*|| = 1$).

We begin by showing that the nonlinear terms converge. That is, we show that as $k \to \infty$,

$$\int f'(\eta)\left(f_k^2 - f_*^2\right) \to 0. \tag{10.8}$$

First, let $\epsilon > 0$ be given and choose $R > 0$ sufficiently large enough to guarantee that $\int_{U^c} f'(\eta) = \int_{U^c}(2k+1)\eta^{2k} \le \frac{\epsilon}{2}$ for $U = B_R(0)$, $U^c = \mathbb{R}^d \backslash U$. Thus we have

$$\begin{aligned}
\left| \int f'(\eta)\left(f_k^2 - f_*^2\right) \right| &= \left| \int_U f'(\eta)\left(f_k^2 - f_*^2\right) \right| + \left| \int_{U^c} f'(\eta)\left(f_k^2 - f_*^2\right) \right| \\
&\le \int_U f'(\eta)\left|f_k^2 - f_*^2\right| + \frac{\epsilon}{2} \cdot \left(||f_k||^2 + ||f_*||^2\right) \\
&\le \int_U f'(\eta)\left|f_k^2 - f_*^2\right| + \epsilon
\end{aligned}$$

Thus, applying Rellich-Kondrachov compactness ($L^2(U) \subset\subset W^{1,2}(U) = H^1(U)$) to the bounded sequence $\{f_k\}$, we obtain strong convergence in $L^2(U)$ of a subsequence $\{f_{k_m}\}$. In other words, $||f_{k_m} - f_*||_{L^2(U)} \to 0$ as $m \to \infty$, and hence

$$\begin{aligned}
\lim_{m\to\infty} \left| \int f'(\eta)\left(f_{k_m}^2 - f_*^2\right) \right| &\le \lim_{m\to\infty} \int_U f'(\eta)\left|f_{k_m}^2 - f_*^2\right| + \epsilon \\
&\le \lim_{m\to\infty} ||\eta||_{L^\infty(U)}^{2k} ||f_{k_m} + f_*||_{L^2(U)} ||f_{k_m} - f_*||_{L^2(U)} + \epsilon \\
&\le 0 + \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we can take the limit $\epsilon \to 0$ obtain

$$\lim_{m\to\infty} \Big| \int f'(\eta) \left(f_{k_m}^2 - f_*^2 \right) \Big| = 0. \tag{10.9}$$

We use this to estimate the nonlinear term of $(L_1 f_*, f_*)$. To show $(L_1 f_*, f_*) = 0$, it remains to bound the other two terms. Take a function $\zeta \in L^2$ with unit norm, and observe that by the weak convergence $f_n \rightharpoonup f_*$, Fatou's Lemma, and also the use of Cauchy-Schwarz,

$$(\zeta, \nabla f_*) = \liminf_{n\to\infty} (\zeta, \nabla f_n) \leq ||\zeta|| \cdot \liminf_{n\to\infty} ||\nabla f_n|| = \liminf_{n\to\infty} ||\nabla f_n||.$$

Setting $\zeta = \nabla f_* / ||\nabla f_*||$ yields

$$||\nabla f_*|| \leq \liminf_{n\to\infty} ||\nabla f_n||. \tag{10.10}$$

Putting together estimates (10.9) and (10.10) and assuming $||f_*|| = 1$, we obtain

$$(L_1 f_*, f_*) \leq \liminf_{n\to\infty} (L_1 f_n, f_n) = 0.$$

and Proposition 10.1 now forces $(L_1 f_*, f_*) = 0$, as desired.

Note: Although we do not know if $||f_*|| = 1$, we can apply Fatou's lemma once more to obtain

$$||f_*||^2 \leq \liminf_{n\to\infty} ||f_n||^2 = 1. \tag{10.11}$$

If the inequality is strict, we simply rescale so that f_* has unit norm. Then f_* is admissible (it satisfies the specified constraints) and it is a minimizer of $(L_1 f, f)$.

Step 2:

Since the infimum is attained at an admissible function f_*, the statement of Proposition 10.2 is equivalent to only having a non-trivial solution $(f_*, \lambda, \beta, \gamma)$ of the Lagrange multiplier problem

$$\begin{cases} (r, \eta) = & 0, \\ (r, \nabla (f(\eta))) = & \vec{0}, \\ ||r|| = & 1, \\ (L_1 - \lambda) r = & \beta\eta + \gamma\nabla (f(\eta)), \forall \beta \in \mathbb{R}, \gamma \in \mathbb{R}^d. \end{cases} \tag{10.12}$$

in the case $\lambda > 0$. This is because (due to our constraints)

$$((L_1 - \lambda) r, r) = \beta (\eta, r) + \gamma (\nabla (f(\eta)) \eta, r) = 0,$$

and so

$$(L_1 r, r) = \lambda (r, r) = \lambda.$$

To show coercivity of L_1, we continue to seek a contradiction to the supposition $\lambda = 0$. That is, we assume a non-trivial solution of

$$\begin{cases} (r, \eta) = & 0, \\ (r, \nabla (f(\eta))) = & \vec{0}, \\ ||r|| = & 1, \\ L_1 r = & \beta\eta + \gamma\nabla (f(\eta)), \end{cases}$$

in the variables r, β, γ. Integrating the last equation against $\nabla\eta$, we obtain

$$\begin{aligned}(L_1 r, \nabla\eta) =& \left(\beta\eta + \gamma\nabla\left(f(\eta)\right), \nabla\eta\right)\\ =& 0 + \gamma\left(\nabla\left(f(\eta)\right), \nabla\eta\right).\end{aligned}$$

Noting that L_1 is self-adjoint and $Null L_1 = span_{j\in\{1,\ldots,d\}}\left\{\eta_{x_j}\right\}$, we see that

$$(L_1 r, \nabla\eta) = (r, L_1\nabla\eta) = \left(r, \vec{0}\right) = \vec{0},$$

and we briefly calculate

$$\left(\nabla\left(f(\eta)\right), \nabla\eta\right) = \left(f'(\eta)\nabla\eta, \nabla\eta\right) = \left((2k+1)\eta^{2k}\nabla\eta, \nabla\eta\right) = (2k+1)||\eta^k\nabla\eta||^2 \neq 0.$$

Thus, $\gamma = \vec{0}$ is forced, and we move on to determine the values of r and β from the simplified equation

$$L_1 r = \beta\eta.$$

Letting $\varphi_\theta := -\frac{\beta}{2k\mu}\left(\eta + kx\cdot\nabla\eta\right) + \theta\cdot\nabla\eta$ for any $\theta \in \mathbb{R}^d$, we show by direct calculation that

$$L_1\varphi_\theta = \beta\eta.$$

Moreover, these φ_θ are the only functions to do so, since we have a complete description of the kernel of L_1 above. In the calculation we use the facts $\theta\cdot\nabla\eta \in Null(L_1)$ and the Equation (1.4) solved by η:

$$\begin{aligned}L_1\varphi_\theta =& \left\{-\Delta + \mu - (2k+1)\eta^{2k}\right\}\varphi_\theta\\ =& -\frac{\beta}{2k\mu}\left\{-\Delta + \mu - (2k+1)\eta^{2k}\right\}\left(\eta + kx\cdot\nabla\eta\right)\\ =& -\frac{\beta}{2k\mu}\left\{-\Delta\eta + \mu\eta - (2k+1)\eta^{2k+1}\right\} - \frac{\beta}{2\mu}\left\{-\Delta + \mu - (2k+1)\eta^{2k}\right\}\left(x\cdot\nabla\eta\right)\\ =& -\frac{\beta}{2}\left\{-2k\eta^{2k+1}\right\} - \frac{\beta}{2\mu}\left\{-\Delta\left(x\cdot\nabla\eta\right) + \mu\left(x\cdot\nabla\eta\right) - (2k+1)\eta^{2k}\left(x\cdot\nabla\eta\right)\right\}\\ =& \frac{\beta}{\mu}\eta^{2k+1} - \frac{\beta}{2\mu}\left\{-\left(2\Delta\eta + x\cdot\nabla\Delta\eta\right) + x\cdot\nabla\left(\mu\eta\right) - x\cdot\nabla\left(\eta^{2k+1}\right)\right\}\\ =& \frac{\beta}{\mu}\left(\eta^{2k+1} + \Delta\eta\right) - \frac{\beta}{2\mu}x\cdot\nabla\left(-\Delta\eta + \mu\eta - \eta^{2k+1}\right)\\ =& \frac{\beta}{\mu}\left(\mu\eta\right) - \frac{k\beta}{2}x\cdot\nabla\left(0\right)\\ =& \beta\eta\end{aligned}$$

It remains to discover the parameters β and θ for which the two orthogonality conditions hold. To this end, we observe that

$$
\begin{aligned}
(x \cdot \nabla \eta, \eta) &= \sum_i \int x_i \left(\partial_{x_i} \eta\right) \eta \, dx \\
&= -\sum_i \int \partial_{x_i} \left(x_i \eta\right) \eta \, dx \\
&= -\sum_i \int \eta^2 \, dx - \sum_i \int x_i \partial_{x_i} \left(\eta\right) \eta \, dx \\
&= -\sum_i ||\eta||^2 - \left(x \cdot \nabla \eta, \eta\right),
\end{aligned}
$$

or

$$
(x \cdot \nabla \eta, \eta) = -\frac{d}{2} ||\eta||^2.
$$

Thus,

$$
\begin{aligned}
(\varphi_\theta, \eta) &= \left(-\frac{\beta}{2} \left(\eta + x \cdot \nabla \eta\right) + \theta \cdot \nabla \eta, \eta\right) \\
&= -\frac{\beta}{2} \left[(\eta, \eta) + (x \cdot \nabla \eta, \eta)\right] + 0 \\
&= -\frac{\beta}{2} \left[||\eta||^2 - \frac{d}{2} ||\eta||^2\right] \\
&= 0
\end{aligned}
$$

only if $\beta = 0$ (assuming $d \neq 2$). Here we have used $(\theta \cdot \nabla \eta, \eta) = 0$, which is shown below via integration by parts:

$$
(\theta \cdot \nabla \eta, \eta) = \sum_i \int \eta \theta_i \partial_{x_i} \eta = -\sum_i \int \left(\partial_{x_i} \eta\right) \theta_i \eta = -\left(\theta \cdot \nabla \eta, \eta\right).
$$

Finally, we check $(\varphi_\theta, \nabla \left(f(\eta)\right)) = 0$. To this end, we calculate

$$
\begin{aligned}
(\varphi_\theta, \nabla \left(f(\eta)\right)) &= \left(-\frac{\beta}{2} \left(\eta + x \cdot \nabla \eta\right) + \theta \cdot \nabla \eta, \nabla \left(f(\eta)\right)\right) \\
&= -\frac{\beta}{2} \left(\eta, \nabla \left(f(\eta)\right)\right) - \frac{\beta}{2} \left(x \cdot \nabla \eta, \nabla \left(f(\eta)\right)\right) + \left(\theta \cdot \nabla \eta, \nabla \left(f(\eta)\right)\right).
\end{aligned}
$$

But

$$
\begin{aligned}
(\eta, \nabla \left(f(\eta)\right)) &= \int \eta \nabla \left(\left|\eta\right|^{2k} \cdot (\eta)\right) \\
&= -\int \left|\eta\right|^{2k} \cdot (\eta) \nabla \eta \\
&= -\frac{1}{2k+1} \int \eta \nabla \left(\left|\eta\right|^{2k} \cdot (\eta)\right) \\
&= -\frac{1}{2k+1} \left(\eta, \nabla \left(f(\eta)\right)\right)
\end{aligned}
$$

implies

$$(\eta, \nabla(f(\eta))) = \vec{0}, \tag{10.13}$$

and by spherical symmetry of η (hence also of $\left(\nabla\left((2k+1)\eta^{2k}\right)\right)^2$),

$$\begin{aligned}(x \cdot \nabla\eta, \nabla(f(\eta))) =& \sum_i \int x_i (\nabla\eta) f'(\eta) \nabla\eta \, dx \\ =& \sum_i \int x_i (2k+1)\eta^{2k} (\nabla\eta)^2 \, dx \\ =& \sum_i \int \frac{2k+1}{2k^2} x_i \left(\nabla\left(\eta^{k+1}\right)\right)^2 \, dx \\ =& \vec{0}.\end{aligned} \tag{10.14}$$

Calculating

$$(\theta \cdot \nabla\eta, \nabla(f(\eta))) = \theta \int \nabla\eta \nabla(f(\eta)) = \theta \int (2k+1)\eta^{2k} (\nabla\eta)^2 = (2k+1)\theta ||\eta\nabla\eta||^2, \tag{10.15}$$

and putting this together with (10.13) and (10.14) we find that

$$\begin{aligned}(\varphi_\theta, \nabla(f(\eta))) =& -\frac{\beta}{2}(\eta, \nabla(f(\eta))) - \frac{\beta}{2}(x \cdot \nabla\eta, \nabla(f(\eta))) + (\theta \cdot \nabla\eta, \nabla(f(\eta))) \\ =& 0 + 0 + (\theta \cdot \nabla\eta, \nabla(f(\eta))) \\ =& (2k+1)\theta ||\eta^k \nabla\eta||^2\end{aligned} \tag{10.16}$$

Thus $\theta = 0$ is also forced. Hence we must have $r = \varphi_\theta = 0$, and thus the only solution of the Lagrange multiplier problem for $\lambda = 0$ is the trivial one ($r = 0, \beta = 0, \gamma = \vec{0}$). But this contradicts our assumption that $||r||_2 = 1$, and hence we must have $\lambda > 0$.

□

Chapter A Ehrenfest's Theorem

What follows is a proof sketch of Ehrenfest's theorem. The theorem is stated for $\psi \in H^1$, but we assume two weak derivatives for the sake of the sketch. The authors in [7] assert that standard approximation arguments can remove this extra assumption.

Theorem A.1. *For $\psi(x,t)$ in $H^1(\mathbb{R}^d)$ solving (1.1),*

$$\partial_t \langle \psi, -i\nabla\psi\rangle = -\langle \psi, (\nabla V)\psi\rangle .$$

Proof. Let $I_\psi(t) := \left\langle \psi, -i\partial_{x_j}\psi \right\rangle$. Then

$$\partial_t I_\psi(t) = \left\langle \partial_t\psi, -i\partial_{x_j}\psi\right\rangle + \left\langle \psi, -i\partial_{x_j}\partial_t\psi\right\rangle = \left\langle (i\partial_t\psi), \partial_{x_j}\psi\right\rangle + \left\langle \psi, -\partial_{x_j}(i\partial_t\psi)\right\rangle$$

and using $i\partial_t\psi = (-\Delta + V)\psi - f(\psi)$ we obtain

$$\begin{aligned}
\partial_t I_\psi(t) &= \left\langle (-\Delta + V)\psi - f(\psi), \partial_{x_j}\psi\right\rangle + \left\langle \psi, -\partial_{x_j}\left((-\Delta+V)\psi - f(\psi)\right)\right\rangle \\
&= Re\int -\Delta\psi\overline{\partial_{x_j}\psi} + V\psi\overline{\partial_{x_j}\psi} - f(\psi)\overline{\partial_{x_j}\psi} + \psi\overline{\left[-\partial_{x_j}\left(-\Delta\psi + V\psi - f(\psi)\right)\right]} \\
&= Re\int \underbrace{-\Delta\psi\overline{\partial_{x_j}\psi}} + V\psi\overline{\partial_{x_j}\psi} - \underbrace{f(\psi)\overline{\partial_{x_j}\psi}} + \underbrace{\psi\overline{\partial_{x_j}\Delta\psi}} - \psi\overline{\partial_{x_j}V\psi} + \underbrace{\psi\overline{\partial_{x_j}f(\psi)}} \\
&= Re\int V\psi\overline{\partial_{x_j}\psi} - \psi\overline{\partial_{x_j}V\psi} \\
&= Re\int V\psi\overline{\partial_{x_j}\psi} - \psi V\overline{\partial_{x_j}\psi} - \psi\left(\partial_{x_j}V\right)\overline{\psi} \\
&= Re\int -\psi\left(\partial_{x_j}V\right)\overline{\psi} \\
&= \left\langle -\psi, \left(\partial_{x_j}V\right)\psi\right\rangle,
\end{aligned}$$

which is what we wanted to show. □

Chapter B Miscellaneous Calculations

This appendix is contains several calculations that are straightforward but too tedious to include in the body of this document. Proofs in order of appearance are for (i) invariant transformations of solutions to Equations (1.1) and (1.2), (ii) Proposition 1.4, and (iii) conservation of quantities $\mathcal{H}_\lambda$ and $\mathcal{N}$ for Equation (1.1).

Here, as elsewhere, we indicate translation of a function f by $f_a(x) = f(x+a)$. However, a t subscript always means a time derivative.

B.1 Invariance Proofs

Invariance for all $\lambda \in \mathbb{R}$

Proposition B.1. *Equation (1.1) is invariant under the gauge transformation*

$$T_\gamma^g \ : \ u(x,t) \mapsto e^{i\gamma} u(x,t).$$

Proof. It suffices to show that substitution of the transformed solution

$$\varphi(x,t) = T_\gamma^g \psi(x,t) = e^{i\gamma}\psi(x,t)$$

into Equation (1.1) yields Equation (1.1) once more, as follows. If

$$i\left(e^{i\gamma}\psi\right)_t + \Delta\left(e^{i\gamma}\psi\right) + |\left(e^{i\gamma}\psi\right)|^{2k}\left(e^{i\gamma}\psi\right) = \lambda V^h\left(e^{i\gamma}\psi\right),$$

then (since γ is a constant) we obtain

$$\left(i\psi_t\right)e^{i\gamma} + \left(\Delta\psi\right)e^{i\gamma} + \left(|\psi|^{2k}\psi\right)e^{i\gamma} = \left(\lambda V^h\psi\right)e^{i\gamma},$$

and cancelling the factors of $e^{i\gamma}$ yields Equation (1.1).

□

Invariance for $\lambda = 0$

Proposition B.2. *Equation (1.2) is invariant under translation, scaling, and Galilean transformations recalled below:*

$$\begin{aligned} T_a^{tr} \ &: \ u(x,t) \mapsto u(x-a,t), \\ T_\mu^s \ &: \ u(x,t) \mapsto \sqrt{\mu}u\left(\sqrt{\mu}x, \mu t\right), \\ T_v^{gal} \ &: \ u(x,t) \mapsto e^{i\left(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t\right)}u\left(x - vt, t\right). \end{aligned}$$

Proof. As before, we simply substitute into the relevant equation. In this case, we recover Equation (1.2).

1) *Translation invariance*: We substitute the transformed solution

$$\varphi(x,t) = \psi_a(x,t) = \psi(x+a,t)$$

into Equation (1.2) to get

$$i\left(\psi_a\right)_t + \Delta\left(\psi_a\right) + \left|\left(\psi_a\right)\right|^{2k}\left(\psi_a\right) = 0.$$

Passing derivatives leads directly to

$$\left(i\psi_t\right)_a + \left(\Delta\psi\right)_a + \left(|\psi|^{2k}\psi\right)_a = 0,$$

and translating the equation by $x \mapsto x - a$ we obtain Equation (1.2) once again.

2) *Scaling invariance*: We substitute the transformed solution

$$\varphi(x) = \mu^{\frac{1}{2k}} u\left(\sqrt{\mu}x, \mu t\right)$$

into Equation (1.2) to get

$$i\mu\mu^{\frac{1}{2k}}\psi_t(\sqrt{\mu}x, \mu t) + \mu\mu^{\frac{1}{2k}}\Delta\psi(\sqrt{\mu}x, \mu t) + |\mu^{\frac{1}{2k}}\psi(\sqrt{\mu}x, \mu t)|^{2k}\mu^{\frac{1}{2k}}\psi(\sqrt{\mu}x, \mu t) = 0,$$

since

$$\varphi_t(x) = \mu\mu^{\frac{1}{2k}}\psi_t(\sqrt{\mu}x, \mu t)$$

and

$$\Delta\varphi(x) = \mu\mu^{\frac{1}{2k}}\Delta\psi(\sqrt{\mu}x, \mu t).$$

Then cancelling the common factor $\mu\mu^{\frac{1}{2k}}$, we get

$$\left(i\psi_t + \Delta\psi + |\psi|^2\psi\right)\left(\sqrt{\mu}x, \mu t\right) = 0,$$

and and scaling $\sqrt{\mu}x \mapsto x$ and $\mu t \mapsto t$ yields Equation (1.2).

3) *Galilean invariance*: We take

$$\varphi(x,t) = e^{i\xi(x,t)}\psi(x - vt, t),$$

with $\xi = \frac{1}{2}v \cdot x + \frac{|v|^2}{4}t$. Noting that $\nabla\xi = v/2$ and $\xi_t = -|v|^2/4$ we get

$$\begin{aligned}\varphi_t(x,t) = &\left[i\xi_t(x,t)e^{i\xi(x,t)}\psi(x - vt, t) + e^{i\xi(x,t)}\psi_t(x - vt, t) - \nabla\xi e^{i\xi(x,t)}\nabla\psi(x - vt, t)\right]\\ =&e^{i\xi(x,t)}\left[-i\frac{|v|^2}{4}\psi(x - vt, t) + \psi_t(x - vt, t) - \frac{v}{2}\nabla\psi(x - vt, t)\right]\end{aligned}$$

and

$$\begin{aligned}\Delta\varphi(x,t) = &\left[\Delta\left(e^{i\xi(x,t)}\right)\psi(x - vt, t) + \nabla\left(e^{i\xi(x,t)}\right)\nabla\left(\psi(x - vt, t)\right)\right.\\ &\left.+\ e^{i\xi(x,t)}\Delta\left(\psi(x - vt, t)\right)\right]\\ =&e^{i\xi(x,t)}\left[-\left(\frac{|v|}{2}\right)^2\psi(x - vt, t) + i\frac{v}{2}\nabla\psi(x - vt, t) + \Delta\psi(x - vt, t)\right]\\ =&e^{i\xi(x,t)}\left[-\frac{|v|^2}{4}\psi(x - vt, t) + i\frac{v}{2}\nabla\psi(x - vt, t) + \Delta\psi(x - vt, t)\right].\end{aligned}$$

Thus, when substituting $\varphi(x,t)$ into Equation (1.2), two pairs of terms cancel so that we are left with

$$e^{i\xi(x,t)} \cdot \left(i\psi_t + \Delta\psi + |\psi|^{2k}\psi\right)(x - vt, t) = 0.$$

Then cancelling $e^{i\xi(x,t)}$ and translating $x - vt \mapsto x$, we have Equation (1.2)

$$i\psi_t + \Delta\psi + |\psi|^{2k}\psi = 0,$$

as desired.

$\square$

B.2 Proof of Proposition 1.4

Proof. We use the notation $\phi(x) := \frac{1}{2}v \cdot x + \gamma$, take $u(x,t) := e^{i\phi(x-a)}\eta_\mu(x-a)$ to be our traveling soliton, and calculate the relevant quantities from Equation (1.2) (NLS with $\lambda = 0$) below. Observe that

$$\frac{d}{dt}(\phi(x-a)) = v_t/2 \cdot (x) + v/2 \cdot (-a_t) + \gamma_t,$$

$$\nabla(\phi(x-a)) = v/2,$$

and

$$\Delta(\phi(x-a)) = 0.$$

Then

$$\begin{aligned}
iu_t =& i\frac{d}{dt}\left[e^{i\phi(x-a)}\eta_\mu(x-a)\right] \\
=& i\left[i\frac{d}{dt}(\phi(x-a))e^{i\phi(x-a)}\eta_\mu(x-a) + e^{i\phi(x-a)}\partial_\mu\left[\eta_\mu(x-a)\right]\cdot\mu_t\right. \\
&\left. + e^{i\phi(x-a)}(\nabla\eta_\mu)(x-a)\cdot(-a_t)\right] \\
=& -e^{i\phi(x-a)}\left[\frac{1}{2}v_t\cdot(x-a) + \frac{1}{2}v\cdot(-a_t) + \gamma_t\right]\eta_\mu(x-a) \\
&+ ie^{i\phi(x-a)}\left[\partial_\mu\left[\eta_\mu(x-a)\right]\cdot\mu_t + (\nabla\eta_\mu)(x-a)\cdot(-a_t)\right],
\end{aligned}$$

$$\begin{aligned}
\Delta u =& \Delta\left(e^{i\phi(x-a)}\right)\eta_\mu(x-a) + 2\nabla\left(e^{i\phi(x-a)}\right)\nabla\left(\eta_\mu(x-a)\right) + \Delta\left(\eta_\mu(x-a)\right) \\
=& -\left|\nabla\phi\right|^2 e^{i\phi(x-a)}\eta_\mu(x-a) + 2i\nabla\phi e^{i\phi(x-a)}\nabla(\eta_\mu)(x-a) + e^{i\phi(x-a)}\Delta(\eta_\mu)(x-a) \\
=& e^{i\phi(x-a)}\left[-\frac{|v|^2}{4}\eta_\mu(x-a) + 2i\left(\frac{v}{2}\right)\nabla(\eta_\mu)(x-a) + \Delta(\eta_\mu)(x-a)\right],
\end{aligned}$$

and

$$|u|^{2k}u = e^{i\phi(x-a)}|\eta_\mu(x-a)|^{2k}\eta_\mu(x-a).$$

Now, since the traveling soliton u is supposed to solve Equation (1.2), we obtain

$$0 = iu_t + u_{xx} + |u|^{2k}u$$

$$
\begin{aligned}
=&- e^{i\phi(x-a)}\left[\frac{1}{2}v_t\cdot(x-a)+\frac{1}{2}v\cdot(-a_t)+\gamma_t\right]\eta_\mu(x-a)\\
&+ie^{i\phi(x-a)}\left[\partial_\mu\left[\eta_\mu(x-a)\right]\cdot\mu_t+(\nabla\eta_\mu)(x-a)\cdot(-a_t)\right]\\
&+e^{i\phi(x-a)}\left[-\frac{|v|^2}{4}\eta_\mu(x-a)+2i\left(\frac{v}{2}\right)\nabla(\eta_\mu)(x-a)+\Delta(\eta_\mu)(x-a)\right]\\
&+e^{i\phi(x-a)}|\eta_\mu(x-a)|^{2k}\eta_\mu(x-a),
\end{aligned}
$$

or, after cancelling factors of $e^{i\phi(x-a)}$ and regrouping terms in a natural way,

$$
\begin{aligned}
0=&-\left[\frac{|v|^2}{4}+\frac{1}{2}v\cdot(-a_t)+\gamma_t\right]\eta_\mu(x-a)+\Delta(\eta_\mu)(x-a)+|\eta_\mu(x-a)|^{2k}\eta_\mu(x-a)\\
&-\frac{1}{2}v_t\cdot(x-a)\eta_\mu(x-a)+i\mu_t\partial_\mu\left[\eta_\mu(x-a)\right]+i\left(v-a_t\right)\nabla(\eta_\mu)(x-a).
\end{aligned}
$$

Observing that the right hand side is zero if and only if the coefficient of $\eta_\mu(x-a)$ is $-\mu$ (see equation 1.4) and each of the coefficients of the η_μ terms on the second line are zero (by linear independence of the $T_\eta\mathcal{G}_\eta$ basis functions established in Proposition 3.1), we conclude that the parameters of a traveling soliton constructed above must satisfy the following system of ODEs:

$$
\begin{cases}
\frac{|v|^2}{4}-\frac{1}{2}v\cdot a_t+\gamma_t &=\mu,\\
-\frac{1}{2}v_t\cdot(x-a) &=0, \forall x\in\mathbb{R}^d,\\
i\mu_t &=0,\\
i(v-a_t) &=0.
\end{cases}
$$

This is equivalent to

$$
\begin{cases}
a_t &=v,\\
v_t &=0,\\
\mu_t &=0,\\
\gamma_t &=\mu+\frac{|v|^2}{4},
\end{cases}
$$

and solving the initial value problem for a, v, γ, μ, we obtain

$$
\begin{cases}
a(t) &=v_0t+a_0,\\
v(t) &=v_0,\\
\mu(t) &=\mu_0,\\
\gamma(t) &=\left(\mu_0+\frac{|v_0|^2}{4}\right)t+\gamma_0.
\end{cases}
$$

□

B.3 Conservation Laws for Equation (1.1)

Proposition B.3. *Let ψ solve (1.1). Then the quantities*

$$
\mathcal{H}_\lambda(\psi):=\frac{1}{2}\int|\nabla\psi|^2+\lambda V^h|\psi|^2\,dx-F(\psi)
$$

and

$$\mathcal{N}(\psi) := ||\psi||_2^2$$

are conserved in time

Proof. $\mathcal{H}_\lambda(\psi)$ is seen to be conserved in time by direct calculation and use of Equation (1.1) and the definition $F'(\psi) := f(\psi)$:

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}_\lambda(\psi) &= \frac{d}{dt}\left[\frac{1}{2}\int |\nabla\psi|^2 + \lambda V^h|\psi|^2 \, dx - F(\psi)\right] \\
&= Re\int \nabla\psi_t\overline{\nabla\psi} + \lambda V^h\psi_t\overline{\psi} - \overline{f(\psi)}\psi_t \, dx \\
&= Re\int \psi_t\left(-\overline{\Delta\psi} + \lambda V^h\overline{\psi} - \overline{f(\psi)}\right) \, dx \\
&= Re\int \psi_t\left(\overline{i\psi_t}\right) \, dx \\
&= -Im||\psi_t||^2 \\
&= 0.
\end{aligned}$$

To see conservation of $\mathcal{N}(\psi)$, integrate Equation (1.1) against the solution ψ as below

$$\left(i\psi_t + \Delta\psi + f(\psi) - \lambda V^h\psi, \psi\right) = 0$$

and rearrange to get

$$(i\psi_t, \psi) = \left(-\Delta\psi - |\psi|^{2k}\psi + \lambda V^h\psi, \psi\right).$$

(Here we used the explicit form of the nonlinearity $f(\psi) = |\psi|^{2k}\psi$.) After integration by parts, the right hand side of this equation is the real-valued expression

$$\int |\nabla\psi|^2 + \lambda V^h|\psi|^2 - |\psi|^{2k+2} \, dx.$$

Thus the left hand side, $(i\psi_t, \psi)$, is also real-valued, and so we see that

$$\begin{aligned}
\frac{d}{dt}\mathcal{N}(\psi) =& \frac{d}{dt}||\psi||^2 \\
=& \int \psi_t\overline{\psi} + \psi\overline{\psi_t} \\
=& 2Re\int \psi_t\overline{\psi} \\
=& 2Im\,(i\psi_t, \psi) \\
=& 0.
\end{aligned}$$

□

Chapter C Variational Calculations for $\lambda = 0$

In this appendix, we derive the two constraints (10.3) and (10.2) on the error $w = r+is$ for the case $\lambda = 0$. The strategy, as already mentioned in Chapter 10, is to choose parameters a, γ in such a way as to minimize the H^1_μ norm of $w(t)$ for each time t. After sorting through real and imaginary parts and using Equation (1.4), we arrive at the claimed equations.

Recall the decomposition

$$e^{-i\phi}\psi_a = \eta + w,$$

and that for $p, q \in H^1(\mathbb{R}^d)$,

$$\begin{aligned} ||p+q||^2_{H^1_\mu} =&||\nabla p + \nabla q||^2 + \mu||p+q||^2 \\ =&||\nabla p||^2 + 2\langle \nabla p, \nabla q\rangle + ||\nabla q||^2 + \mu||p||^2 + 2\mu\langle p, q\rangle + \mu||q||^2 \\ =&||p||^2_{H^1_\mu} + ||q||^2_{H^1_\mu} + 2\langle \nabla p, \nabla q\rangle + 2\mu\langle p, q\rangle . \end{aligned}$$

Varying γ by $\delta\gamma$

Let $\phi(\delta\gamma) = \phi - \delta\gamma$, and consider the difference

$$\Delta_\gamma := ||e^{-i\phi(\delta\gamma)}\psi_a - \eta||^2_{H^1_\mu} - ||e^{i\phi}\psi_a - \eta||^2_{H^1_\mu}.$$

Letting $p = e^{-i\phi(\delta\gamma)}\psi_a - e^{-i\phi}\psi_a$ and $q = e^{-i\phi}\psi_a - \eta = w$, we find that

$$\begin{aligned} \Delta_\gamma =&||p+q||^2_{H^1_\mu} - ||q||^2_{H^1_\mu} \\ =&||p||^2_{H^1_\mu} + 2\langle \nabla p, \nabla q\rangle + 2\mu\langle p, q\rangle . \end{aligned}$$

Expanding $e^{i\delta\gamma}$ in a power series, as below

$$e^{i\delta\gamma} = 1 + i\delta\gamma - \frac{1}{2}(\delta\gamma)^2 + \dots$$

we see that

$$\begin{aligned} p =&e^{-i\phi(\delta\gamma)}\psi_a - e^{-i\phi}\psi_a \\ =&e^{-i\phi}\psi_a\left[e^{i\delta\gamma} - 1\right] \\ =&e^{-i\phi}\psi_a\left[i\delta\gamma + \mathcal{O}(\delta\gamma)^2\right] \\ =&(\eta + w)\left[i\delta\gamma + \mathcal{O}(\delta\gamma)^2\right], \end{aligned}$$

and thus,

$$\begin{aligned} \frac{\Delta_\gamma}{\delta\gamma} = \frac{1}{\delta\gamma}\Big[&||i\delta\gamma(\eta+w)||^2_{H^1_\mu} + 2\langle \nabla(i\delta\gamma(\eta+w)), \nabla w\rangle \\ &+2\mu\langle (i\delta\gamma(\eta+w)), w\rangle + \mathcal{O}(\delta\gamma)^2\Big]. \end{aligned}$$

Now setting $\lim_{\delta\gamma\to 0}\frac{\Delta_\gamma}{\delta\gamma}=0$ to reflect minimization (of (10.1)) over γ, we obtain the constraint

$$\begin{aligned}0=&\left[2\left\langle i\nabla\left(\eta+w\right),\nabla w\right\rangle+2\mu\left\langle i\left(\eta+w\right),w\right\rangle\right]\\=&2\left[Im\left(\nabla\eta+\nabla w,\nabla w\right)+\mu Im\left(\eta+w,w\right)\right]\\=&2Im\left[\left(\nabla\eta,\nabla w\right)+\mu\left(\eta,w\right)\right]\\=&2Im\left(-\Delta\eta+\mu\eta,w\right)\\=&2Im\left(-f(\eta),w\right)\\=&-2\left(f(\eta),s\right).\end{aligned}$$

In the last calculation we used

$$Im\left(w,w\right)=Im||w||^2=0=Im||\nabla w||^2=Im\left(\nabla w,\nabla w\right)$$

and Equation (1.4).

Varying a by δa

Consider the difference

$$\Delta_a:=||e^{-i\phi}\psi_{a+\delta a}-\eta||^2_{H^1_\mu}-||e^{-i\phi}\psi_a-\eta||^2_{H^1_\mu}.$$

Letting $p=e^{-i\phi}\psi_{a+\delta a}-e^{-i\phi}\psi_a$ and $q=e^{-i\phi}\psi_a-\eta=w$, we find that

$$\begin{aligned}\Delta_a=&||p+q||^2_{H^1_\mu}-||q||^2_{H^1_\mu}\\=&||p||^2_{H^1_\mu}+2\left\langle\nabla p,\nabla q\right\rangle+2\mu\left\langle p,q\right\rangle.\end{aligned}$$

This time, expanding $\psi_{a+\delta a}$ in a power series as below

$$\psi_{a+\delta a}=\psi_a+\delta a\nabla\psi_a+\mathcal{O}\left(\delta a^2\right),$$

we see that

$$\begin{aligned}p=&e^{-i\phi(\delta a)}\psi_{a+\delta a}-e^{-i\phi}\psi_a\\=&\delta a\nabla\psi_a e^{-i\phi}+\mathcal{O}(\delta a^2)\\=&\delta a\left[\nabla\left(\psi_a e^{-i\phi}\right)-\psi_a\nabla\left(e^{-i\phi}\right)\right]+\mathcal{O}(\delta a^2)\\=&\delta a\left[\nabla\left(\psi_a e^{-i\phi}\right)-\psi_a\frac{iv}{2}e^{-i\phi}\right]+\mathcal{O}(\delta a^2)\\=&\delta a\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right]+\mathcal{O}(\delta a^2)\end{aligned}$$

and thus

$$\frac{\Delta_a}{\delta a}=\frac{1}{\delta a}||\delta a\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right]||^2_{H^1_\mu}+\frac{1}{\delta a}\mathcal{O}(\delta a^2)$$

$$
\begin{aligned}
&+\frac{1}{\delta a}2\left\langle \delta a\nabla\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right]+\mathcal{O}(\delta a^2),\nabla w\right\rangle\\
&+\frac{1}{\delta a}2\mu\left\langle \delta a\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right]+\mathcal{O}(\delta a^2),w\right\rangle\\
=&2\left\langle\left[\Delta\left(\eta+w\right)-\frac{iv}{2}\nabla\left(\eta+w\right)\right],\nabla w\right\rangle\\
&+2\mu\left\langle\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right],w\right\rangle+\mathcal{O}\left(\delta a\right)
\end{aligned}
$$

Now, noting that

$$(\nabla w, w)=0=(\Delta w,\nabla w)\,,$$

tracking down imaginary components, integrating by parts, and recalling Equation (1.4) below

$$(-\Delta+\mu)\,\eta-f(\eta)=0,$$

we obtain (in the case of minimizing a)

$$
\begin{aligned}
0=&\lim_{\delta a\to 0}\frac{\Delta_a}{\delta a}\\
=&2\left\langle\left[\Delta\left(\eta+w\right)-\frac{iv}{2}\nabla\left(\eta+w\right)\right],\nabla w\right\rangle+2\mu\left\langle\left[\nabla\left(\eta+w\right)-\frac{iv}{2}\left(\eta+w\right)\right],w\right\rangle\\
=&2\left\langle\left[\Delta\eta-\frac{iv}{2}\nabla\eta\right],\nabla w\right\rangle+2\mu\left\langle\left[\nabla\eta-\frac{iv}{2}\eta\right],w\right\rangle\\
=&2\left\langle-\nabla\left[\Delta\eta-\frac{iv}{2}\nabla\eta\right]+\mu\left[\nabla\eta-\frac{iv}{2}\eta\right],w\right\rangle\\
=&2\left\langle\left(\nabla-\frac{iv}{2}\right)(-\Delta\eta)+\left(\nabla-\frac{iv}{2}\right)(\mu\eta)\,,w\right\rangle\\
=&2\left\langle\left(\nabla-\frac{iv}{2}\right)f(\eta),w\right\rangle\\
=&2\left[\left\langle\nabla\left(f(\eta)\right),w\right\rangle-\frac{v}{2}Im\left(f(\eta),w\right)\right]\\
=&2\left(\nabla\left(f(\eta)\right),r\right)-v\left(f(\eta),s\right),
\end{aligned}
$$

or

$$\left(\nabla\left(f(\eta)\right),r\right)=\frac{v}{2}\left(f(\eta),s\right).$$

Chapter D Proof of Proposition 5.2

In this appendix we give a proof of Proposition 5.2 derived from the paper [19] of M. I. Weinstein. The strategy of proof is to use the following alternate characterization of the ground state profile η (quoted as Proposition 2.6 in [19] and proved as Theorem B in [18]):

Theorem D.1 (Weinstein, 1983)**.** *The "ground state" η_μ is a minimizer of the functional*

$$I[u] := (2k+2)\frac{||\nabla u||^{kd}||u||^{2-k(2-d)}}{||u||_{2k+2}^{2k+2}},$$

where $||\cdot||_p$ is the usual L^p norm and $||\cdot|| = ||\cdot||_2$. [1]

It can be shown that the (first variation) condition that

$$\frac{d}{d\epsilon}I\left[\eta_{\alpha,\beta}+\epsilon\xi\right]\big|_{\epsilon=0} = 0$$

is equivalent to the statement that $\eta_{\alpha,\beta}$ satisfies

$$-\Delta\eta_{\alpha,\beta} + \mu\eta_{\alpha,\beta} - f(\eta_{\alpha,\beta}) = 0.$$

The coercivity of L_1 comes from the concavity (second variation) condition

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} I(\eta_{\alpha,\beta}+\epsilon\xi) \geq 0,$$

the orthogonality assumption $\langle w,(\eta,0)\rangle = 0$, and the subcritical exponent $k < 2/d$. Here, $(\eta,\alpha,\beta) \mapsto \eta_{\alpha,\beta}(x) := \alpha\eta(\beta x)$ is a scaling carefully chosen so that the coefficients here match those in the stationary Equation (1.4). We claim that $I[\cdot]$ is invariant under this scaling; this invariance will be shown in the course of the proof.

Proof of Proposition 5.2. As mentioned in the proof of Proposition 2.2, L_2 is a non-negative operator. By the diagonalization

$$\mathcal{L}_\eta = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$$

also introduced in Chapter 2 it thus suffices to show

$$\inf_{r\in X_1}\langle r, L_1 r\rangle = 0,$$

where $X_1 := \left\{w \in H^1(\mathbb{R}^d;\mathbb{R}) : ||w|| = 1, \langle w,(\eta,0)\rangle = 0\right\}$.

In the next two subsections, we compute the first and second variations of I at η. We use the short-hand notation $f'(u)\xi := \frac{d}{d\epsilon}|_{\epsilon=0}f(u+\epsilon\xi)$ when computing Fréchet derivatives.

[1] The constant factor $2k+2$ is only included in this definition to simplify later notation.

The first variation of the functional I

We rewrite

$$I[u] := \frac{H(u)G(u)}{F(u)} = (2k+2)\frac{||\nabla u|^{kd}||u||^{2+k(2-d)}}{||u||_{2k+2}^{2k+2}}$$

where

$$H(u) := ||\nabla u||^{kd}, \quad G(u) := ||u||^{2+k(2-d)},$$

and

$$F(u) = \frac{1}{2k+2}||u||_{2k+2}^{2k+2} = \int \frac{|u|^{2k+2}}{2k+2}$$

is simply the previously introduced antiderivative of the nonlinearity $f(u)$.

For convenience, we write down

$$H'(u)\xi = kd||\nabla u||^{kd-2}\langle \nabla u, \nabla \xi\rangle, \quad G'(u)\xi = (2+k(2-d))\,||u||^{k(2-d)}\langle u, \xi\rangle$$

before computing

$$\begin{aligned}(F(u))^2 \cdot (I'[u], \xi) =& \left[H'(u)\xi G(u)F(u) + H(u)G'(u)\xi F(u) - H(u)G(u)F'(u)\xi\right]\\ =& kd||\nabla u||^{kd-2}\langle \nabla u, \nabla \xi\rangle ||u||^{2+k(2-d)}F(u)\\ &+ ||\nabla u||^{kd}(2+k(2-d))\,||u||^{k(2-d)}\langle u, \xi\rangle F(u)\\ &- ||\nabla u||^{kd}||u||^{2+k(2-d)}\langle F'(u), \xi\rangle .\end{aligned}$$

Rewritten, this becomes

$$\begin{aligned}\frac{(F(u))^2}{||\nabla u||^{kd-2}||u||^{k(2-d)}} \cdot (I'[u], \xi) =& \underbrace{kd||u||^2F(u)}_{A_{k,d,u}}\langle \nabla u, \nabla \xi\rangle\\ &+ \underbrace{(2+k(2-d))\,||\nabla u||^2F(u)}_{B_{k,d,u}}\langle u, \xi\rangle\\ &- \underbrace{||\nabla u||^2||u||^2}_{C_{k,d,u}}\langle F'(u), \xi\rangle\\ =:& \left\langle \left(-A_{k,d,u}\Delta u + B_{k,d,u}u - C_{k,d,u}F'(u)\right), \xi\right\rangle\end{aligned}$$

Thus we see that the equation η must solve is

$$-A_{k,d,u}\Delta u + B_{k,d,u}u - C_{k,d,u}F'(u) = 0. \tag{D.1}$$

Rescaling ($\eta \mapsto \eta_{\alpha,\beta}$) and letting I be minimized at η gives us (since ξ is arbitrary) the equation for the ground state

$$-\Delta\eta + \mu\eta - f(\eta) = 0,$$

as expected.

The second variation of the functional I

To simplify future notation, we call

$$D[u] := \frac{(F(u))^2}{||\nabla u||^{kd-2}||u||^{k(2-d)}},$$

$$a_{k,d} := kd,$$

$$b_{k,d} := 2 + k\left(2-d\right),$$

and

$$c_{k,d} := 2k+2.$$

Since the first variation is zero at $u = \eta$, we can use product rule to obtain

$$\begin{aligned}
\frac{(I''[\eta]\xi,\phi)}{D[\eta]} =& D[\eta]\cdot(I''[\eta]\xi,\phi) + (D'[\eta],\phi)\cdot\underbrace{(I'[\eta],\xi)}_{=0} \\
=& \left(D[\eta]\cdot I'[\eta]\xi\right)'\phi \\
=& \big[a_{k,d}\left\langle\nabla\eta,\nabla\xi\right\rangle ||\eta||^2F(\eta) + b_{k,d}||\nabla\eta||^2\left\langle\eta,\xi\right\rangle F(\eta) \\
&\quad -||\nabla\eta||^2||\eta||^2F'(\eta)\xi\big]'\phi \\
=& a_{k,d}\left[\left\langle\nabla\xi,\nabla\phi\right\rangle ||\eta||^2F(\eta) + \left\langle\nabla\eta,\nabla\xi\right\rangle 2\left\langle\eta,\phi\right\rangle F(\eta)\right. \\
&\qquad\left. + \left\langle\nabla\eta,\nabla\xi\right\rangle ||\eta||^2\left\langle F'(\eta),\phi\right\rangle\right] \\
&+ b_{k,d}\left[2\left\langle\nabla\eta,\nabla\phi\right\rangle\left\langle\eta,\xi\right\rangle F(\eta) + ||\nabla\eta||^2\left\langle\xi,\phi\right\rangle F(\eta)\right. \\
&\qquad\left. +||\nabla\eta||^2\left\langle\eta,\xi\right\rangle\left\langle F'(\eta),\phi\right\rangle\right] \\
&- \left[2\left\langle\nabla\eta,\nabla\phi\right\rangle ||\eta||^2\left\langle F'(\eta),\xi\right\rangle + ||\nabla\eta||^2\left\langle\eta,\phi\right\rangle\left\langle F'(\eta),\xi\right\rangle\right. \\
&\qquad\left. +||\nabla\eta||^2||\eta||^2\left\langle F''(\eta)\phi,\xi\right\rangle\right].
\end{aligned}$$

By the constraint $(r,\eta) = 0$,

$$\begin{aligned}
\frac{(I''[\eta]r,r)}{D[\eta]} =& a_{k,d}\left[\left\langle\nabla r,\nabla r\right\rangle ||\eta||^2F(\eta) + 0 + \left\langle\nabla\eta,\nabla r\right\rangle ||\eta||^2\left\langle F'(\eta),r\right\rangle\right] \\
&+ b_{k,d}\left[0 + ||\nabla\eta||^2\left\langle r,r\right\rangle F(\eta) + 0\right] \\
&- \left[2\left\langle\nabla\eta,\nabla r\right\rangle ||\eta||^2\left\langle F'(\eta),r\right\rangle + 0 + ||\nabla\eta||^2||\eta||^2\left\langle F''(\eta)r,r\right\rangle\right] \\
&.
\end{aligned}$$

Noting that by (D.1) and the constraint $(r,\eta) = 0$ we have

$$\begin{aligned}
C_{k,d,\eta}\left\langle F'(\eta),r\right\rangle =& ||\nabla\eta||^2||\eta||^2\left\langle F'(\eta),r\right\rangle \\
=& \left\langle\left(-A_{k,d,\eta}\Delta\eta + B_{k,d,\eta}\eta\right),r\right\rangle \\
=& A_{k,d,\eta}\left\langle\nabla\eta,\nabla r\right\rangle \\
=& a_{k,d}||\eta||^2F(\eta)\left\langle\nabla\eta,\nabla r\right\rangle.
\end{aligned}$$

Thus defining $\mathcal{L}_{k,d,\eta} := -A_{k,d,\eta}\Delta + B_{k,d,\eta} - C_{k,d,\eta}F''(\eta)$, we obtain

$$\frac{(I''[\eta]r,r)}{D[\eta]} = a_{k,d}\left[\left\langle\nabla r,\nabla r\right\rangle ||\eta||^2F(\eta) + \left\langle\nabla\eta,\nabla r\right\rangle\left(a_{k,d}||\eta||^2F(\eta)\left\langle\nabla\eta,\nabla r\right\rangle ||\nabla\eta||^{-2}\right)\right]$$

$$
\begin{aligned}
&+ b_{k,d}\left[||\nabla\eta||^2 \langle r,r\rangle F(\eta)\right] \\
&- \left[2\langle\nabla\eta,\nabla r\rangle\left(a_{k,d}||\eta||^2 F(\eta)\langle\nabla\eta,\nabla r\rangle||\nabla\eta||^{-2}\right)\right. \\
&\qquad \left.+||\nabla\eta||^2||\eta||^2\langle F''(\eta)r,r\rangle\right]. \\
&= \langle\mathcal{L}_{k,d,\eta}r,r\rangle + (a_{k,d}-2)\left[\langle\nabla\eta,\nabla r\rangle^2\left(a_{k,d}||\eta||^2F(\eta)||\nabla\eta||^{-2}\right)\right]
\end{aligned}
$$

Finally, forcing $(I''[\eta]r,r) \geq 0$ yields

$$
0 \leq \langle\mathcal{L}_{k,d,\eta}r,r\rangle + (a_{k,d}-2)\left[\langle\nabla\eta,\nabla r\rangle^2\left(a_{k,d}||\eta||^2F(\eta)||\nabla\eta||^{-2}\right)\right]
$$

or

$$
\langle\mathcal{L}_{k,d,\eta}r,r\rangle \geq (2-kd)\langle\nabla\eta,\nabla r\rangle^2 \underbrace{a_{k,d}||\eta||^2F(\eta)}_{A_{k,d,\eta}}||\nabla\eta||^{-2}.
$$

Thus, the stipulation that $k < 2/d$ yields ‘specialized’ (i.e., constrained) coercivity.

As we have seen above, each of the operators $\mathcal{L}_{k,d,\eta}$ are coercive under orthogonality and dimensional constraints, but for various ground states η. To obtain the coercivity we desire for $L_1 = -\Delta + \mu + f'(\eta)$, we must ensure that the ground state η solves the correct equation. It is necessary, therefore, to rescale coordinates in the Equation (D.1) (hence in η itself). To justify this, we must show that the functional I is invariant under such scalings. The calculations are given below.

Making the replacement $\eta(x) \mapsto \alpha\eta(\beta x) =: \alpha\eta_\beta =: \eta_{\alpha,\beta}$ for $\alpha,\beta \in \mathbb{R}$, we obtain (by a change of variables, $y = \beta x,\ dy = \beta^d dx$)

$$
\begin{aligned}
||\eta_{\alpha,\beta}||^2 =& \int |\alpha\eta(\beta x)|^2\,dx = \alpha^2\int|\eta(\beta x)|^2\,dx = \frac{\alpha^2}{\beta^d}\int|\eta(y)|^2\,dy \\
=&\alpha^2\beta^{-d}||\eta||^2
\end{aligned}
$$

$$
\begin{aligned}
||\nabla\eta_{\alpha,\beta}||^2 =& \int|\nabla\left\{\alpha\eta(\beta x)\right\}|^2\,dx = \alpha^2\int|\beta(\nabla\eta)(\beta x)|^2\,dx = \frac{\alpha^2\beta^2}{\beta^d}\int|\nabla\eta(y)|^2\,dy \\
=&\alpha^2\beta^2\beta^{-d}||\nabla\eta||^2
\end{aligned}
$$

$$
\begin{aligned}
||\eta_{\alpha,\beta}||_{2k+2}^{2k+2} =& \int|\alpha\eta(\beta x)|^{2k+2}\,dx = \alpha^{2k+2}\int|\eta(\beta x)|^{2k+2}\,dx = \frac{\alpha^{2k+2}}{\beta^d}\int|\eta(y)|^{2k+2}\,dy \\
=&\alpha^{2k+2}\beta^{-d}||\eta||_{2k+2}^{2k+2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{I[\eta_{\alpha,\beta}]}{2k+2} =& \frac{||\nabla\eta_{\alpha,\beta}||^{kd}||\eta_{\alpha,\beta}||^{2+k(2-d)}}{||\eta_{\alpha,\beta}||_{2k+2}^{2k+2}} \\
=& \frac{\left(\alpha^2\beta^2\beta^{-d}\right)^{\frac{kd}{2}}||\nabla\eta||^{kd}\left(\alpha^2\beta^{-d}\right)^{\frac{2k+2-kd}{2}}||\eta||^{2+k(2-d)}}{\left(\alpha^{2k+2}\beta^{-d}\right)||\eta||_{2k+2}^{2k+2}} \\
=& \frac{\left(\alpha^{kd}\beta^{kd}\beta^{\frac{-kd^2}{2}}\right)||\nabla\eta||^{kd}\left(\alpha^{2k+2-kd}\beta^{-kd}\beta^{-d}\beta^{\frac{kd^2}{2}}\right)||\eta||^{2+k(2-d)}}{\left(\alpha^{2k+2}\beta^{-d}\right)||\eta||_{2k+2}^{2k+2}}
\end{aligned}
$$

$$=\frac{I[\eta]}{2k+2}.$$

We can now choose the two independent parameters α and β so that the rescaled ground state $\eta := \eta_{\alpha,\beta}$ solves (D.1) with coefficients

$$\begin{aligned} A_{k,d,\eta} &= 1, \\ B_{k,d,\eta} &= \mu, \\ C_{k,d,\eta} &= 1. \end{aligned}$$

Thus, for $k < 2/d$ (and $(r, \eta) = 0$), we have

$$\langle L_1 r, r\rangle \geq (2 - kd) \langle \nabla\eta, \nabla r\rangle^2 \, ||\nabla\eta||^{-2} > 0. \tag{D.2}$$

Since this was all that remained to be shown, the proof is now complete. □

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