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Some Take-Away Games on Discrete Structures

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SOME TAKE-AWAY GAMES ON DISCRETE STRUCTURES

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Kristen M. Barnard
Lexington, Kentucky

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Lexington, Kentucky 2017

ABSTRACT OF DISSERTATION

SOME TAKE-AWAY GAMES ON DISCRETE STRUCTURES

The game of Subset Take-Away is an impartial combinatorial game posed by David Gale in 1974. The game can be played on various discrete structures, including but not limited to graphs, hypergraphs, polygonal complexes, and partially ordered sets. While a universal winning strategy has yet to be found, results have been found in certain cases. In 2003 R. Riehemann focused on Subset Take-Away on bipartite graphs and produced a complete game analysis by studying nim-values. In this work, we extend the notion of Take-Away on a bipartite graph to Take-Away on particular hypergraphs, namely oddly-uniform hypergraphs and evenly-uniform hypergraphs whose vertices satisfy a particular coloring condition. On both structures we provide a complete game analysis via nim-values. From here, we consider different discrete structures and slight variations of the rules for Take-Away to produce some interesting results. Under certain conditions, polygonal complexes exhibit a sequence of nim-values which are unbounded but have a well-behaved pattern. Under other conditions, the nim-value of a polygonal complex is bounded and predictable based on information about the complex itself. We introduce a Take-Away variant which we call Take-As-Much-As-You-Want, and we show that, again, nim-values can grow without bound, but fortunately they are very easily described for a given graph based on the total number of vertices and edges of the graph. Finally we consider Take-Away on a specific type of partially ordered set which we call a rank-complete poset. We have results, again via an analysis of nim-values, for Take-Away on a rank-complete poset for both ordinary play as well as misère play.

KEYWORDS: Take-Away, combinatorial game, misère game, hypergraph, poset, polytopal complex

Author's signature: Kristen M. Barnard

Date: May 1, 2017

SOME TAKE-AWAY GAMES ON DISCRETE STRUCTURES

By
Kristen M. Barnard

Director of Dissertation: Carl W. Lee

Director of Graduate Studies: Dr. Peter Hislop

Date: May 1, 2017

Dedicated to my mother, Debbie, who hated math but loved me beyond measure.
I hope I always make you proud.

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One thing I always loved about getting a new album, whether it be on cassette, CD, or vinyl, is reading the artist's acknowledgments where they took the time to thank all the people that helped them along the way, from close friends and family to the companies that made their equipment. This is my moment to do the same.

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Kristen Barnard uses PaperMate Flair felt-tip pens.

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Chapter 1 Introduction

With respect to the creation and completion of this dissertation, two very important things happened in 1982: I was born, and Elwyn Berlekamp, John Conway, and Richard Guy published the first volume of their series *Winning Ways For Your Mathematical Plays*. In this first chapter we will reference this text and other works to introduce the fundamentals of combinatorial game theory, and impartial games in particular.

1.1 What Is A Combinatorial Game?

In this section we will begin to outline the basics of combinatorial game theory.

There are several conditions associated with combinatorial games:

- Combinatorial games are two-player games, say player A and player B, and the players move alternately.
- Combinatorial games are games of complete information; that is, the rules of the game are clearly defined and at any given point in the game—henceforth known as a *position* of the game—both players know the current position and all of the possible moves that can be made from that position.
- Combinatorial games involve no elements of chance such as spinners, dice, drawing or dealing cards, et cetera.

- Alternating play continues between the players until one player is unable to move per the rules of the game, and the rules of the game are such that the game will end after a finite number of moves.

- The convention of *normal play* is that if a player makes the final possible move of the game then she is the winner. Alternatively, if the rules are such that the player who makes the last possible move of the game is declared the loser then we call this *misère play*. In particular, no ties are permitted [3].

The games in this dissertation will follow all of these conditions, but it is important to note that combinatorial game theory techniques can also be used to study some games where some of the conditions are relaxed. The conditions which must hold, though, are the condition of complete information and the condition of having no elements of chance involved. In order to determine a winning strategy for a combinatorial game, it is important to know what positions are options after each move. Further, of the games we discuss in this work, most are played under normal play; however we do investigate some misère games in Chapter 5.

Of course, the goal of studying combinatorial games is to determine a *winning strategy* for them. A winning strategy for player A (respectively, B) in a combinatorial game is a strategy that specifies a move from each position that guarantees a win for player A (respectively, B) [1].

One possible way to search for a winning strategy is a brute force method enumerating every possible game scenario. It is not hard to see that as the size of the game grows this task also grows, and could soon become unmanageable and likely very tedious. Also, doing this forces us to examine games in which players make moves that we recognize as mistakes—think, for example, of instances where a player makes a move that fails to block a 3-in-a-row in Tic-Tac-Toe. Thinking of the brute force method just described and that these mistakes must be taken into account, we see that there are scenarios in which player A could win as well as scenarios in which player B could win, and this does not efficiently lead to describing a winning strategy.

To study combinatorial games in search of a winning strategy more efficiently, we must assume that both players *play perfectly* [1]. Playing perfectly means that, since both players know the rules of the game and have complete information available to them, if a player knows that she can make a move that will force her opponent to lose the game then she must make that move. If no such move is available, playing perfectly is just making a legal move in the game.

With this assumption of perfect play, studying the game strategies becomes much more interesting and, in some cases, more straightforward. We will assume, of course, that both players are playing perfectly in the games studied in this dissertation. With the assumption, we have that one of the players, either player A or player B, will have a guaranteed winning strategy in a game, but not both. This is the Fundamental Theorem of Combinatorial Games.

Theorem 1.1 (Fundamental Theorem of Combinatorial Games)**.** *Let G be a combinatorial game, played between two players A and B, with player A moving first. Then either a winning strategy in G exists for player A or a winning strategy exists for player B, but not both.*

This is a standard result, and proofs can be found in most references on combinatorial games [8, 1]. It is important to note that Theorem 1.1 guarantees that there will be *a* winner, but it does not provide any details about the winning strategy or how to determine the winner. Determining who has the winning strategy and describing it is the focus of much research in combinatorial game theory, and this dissertation in particular. Some games, like Nim (which will be fully described in the next section), have been completely solved and a winning strategy is known. Other games, like the game Chomp, are such that we know which player has a winning strategy, but that strategy is not yet known.

Chomp [8] is a combinatorial game that is played on an m by n rectangle created from mn smaller rectangles—most descriptions of the game describe this as a chocolate candy bar, so we will do the same. We assume for playing Chomp that the chocolate rectangle in the lower left corner is poisoned. Play begins with player A removing (or "chomping") a rectangular piece of chocolate of any size she chooses from the upper right hand corner of the bar. The players then alternately remove rectangles from this side of the bar while following the rule that if they remove the piece in row i and column j of the bar that they must remove all chocolate rectangles above and including i and to the right of and including$j$. The winning player is the player who removes the last piece of "safe" chocolate, leaving her opponent with the poisoned rectangle.

The winning strategy for Chomp is unknown, but it is known that the winning strategy will be for player A. Suppose, to the contrary, that player B has a winning strategy for Chomp. If that were the case, player A can move first by removing the single chocolate rectangle in the upper right of the bar and then player B can employ her winning strategy. However, any move available to player B at this point was available to player A during her move. This means that player A could have used the winning strategy from the beginning, and, since Theorem 1.1 states only one player has a winning strategy if both players are playing perfectly, we conclude that the first player had the winning strategy in the game all along.

Another interesting game to discuss at this point is *Kayles* [3]. Kayles is a combinatorial game which is best described as being played with a series of bowling pins placed in a line and when it is her turn to move a player may either remove one pin or else two pins that are immediately adjacent to each other in the line. The player that removes the last pin of all is declared the winner. Like Chomp, Kayles has a winning strategy for player A, but this strategy is known and easily described. If the number of pins in the line is odd, player A should remove the single pin in the middle of the line leaving an equal number of pins in two now disjoint lines. If the number of pins in the line is even, player A should remove the two pins in the middle of the line, again leaving an equal number of pins in two now disjoint lines. From there, player A will observe player B's move, which will be in only one of the two lines, and make that same move in her turn in the *other* line. Player A then continues to copy player B's moves, and eventually player A will have the final move and hence win the game. This mirroring type of strategy is referred to as the *Tweedledum and Tweedledee Strategy* or simply Tweedledum-Tweedledee [3].

Based on this winning strategy of Kayles, we see that after player A makes the first move that the line of pins becomes broken into two separate lines of pins. Due to the rules of the game, it is impossible for a player to make a single move that will make a change in both lines. It actually seems like playing two games of Kayles simultaneously, except when it is her turn to play a player must choose to make a move in only one of the games. This is exactly the idea behind the sum of games.

We can define the *sum of games* as follows. If G_1 and G_2 are two combinatorial games then $G_1 + G_2$ is a combinatorial game such that copies of G_1 and G_2 are placed next to each other and when it is her turn to move a player can either play in G_1 or G_2, but not both. Under normal play conditions, the winner is the player that leaves a position in which her opponent has no move in either game. We can see how this easily extends from 2 games to k games and $G_1 + G_2 + \cdots + G_k$ where a player can move in exactly one summand game G_i.

On a technical note, we should point out that the sum of two finite games will not necessarily always be itself a finite game [3]. However, the games we consider in this dissertation will not have this problem.

The sum of games has been introduced here through modeling a decomposition of a single game like Kayles, but it is important to note that any combinatorial games can be summed for play in this way. For example, we could construct a game that has one component that is a Chomp rectangle, one component that is a line of pins for Kayles, and a third component that is a heap of tokens for Nim, a game we will discuss in the next section.

1.2 Nim and Impartial Games

In this section we will look at the highly important game of Nim and look at its relation to the study of impartial games.

An *impartial game* is a combinatorial game in which all of the possible moves available to player A are also available to player B, and the reverse is also true. Combinatorial games which are not impartial are called *partizan games*; in these games, we would limit the moves of player A to certain conditions and the moves of player B to other conditions. For example, if we were to consider chess as a combinatorial game then it would be a partizan game because at any given point in the game the options available to the player moving the black chess pieces are not the same as those available to the player moving the white chess pieces.

For an impartial game we can determine a winning strategy by working recursively from the terminal positions of the game. Here, a terminal position refers to a position from which no further moves can be made. Recall that under normal play the player that reaches a terminal position is the winner. We can label these terminal positions with a $\mathcal{P}$ to signify that the player that *previously* moved has won the game. Now we can look backwards in the game from these terminal positions to the positions that are a single move in the game away from a terminal position. We label these positions with an $\mathcal{N}$ to signify that the *next* player to move in the game will win. We continue this backward trek through the game, labeling positions as follows:

- if *any* move can be made from the current position to a $\mathcal{P}$-position, label the current position with an $\mathcal{N}$, and

- if *every* move that can be made from the current position is an $\mathcal{N}$-position, label the current position with a $\mathcal{P}$.

Thus, when playing the game, a player will want to make a move to a position labeled $\mathcal{P}$ because from that position her opponent can only move to a position labeled $\mathcal{N}$. Continuing this pattern will give the player the opportunity to make a move to a terminal position and win the game.

The above labeling sketches the proof of Theorem 1.1 for impartial games. We note that we can do a similar sketch for partizan games, but this labeling would require an ordered pair because if the players have different moves available to them then each game position might have a different label for each player [3]. We will only be investigating impartial games in this dissertation.

The game of *Nim* is an impartial game and is truly a cornerstone of combinatorial game theory. In 1901 Nim was the first combinatorial game to have its full strategy published [1], and this strategy has far-reaching implications in the theory of combinatorial games. While there are many variants, the traditional play of Nim begins with multiple heaps of tokens where the heaps can have varying height. When it is her turn to move, a player must remove a positive number of tokens from any one heap of tokens. The winning player is the player that removes the final token from the last remaining heap of tokens. Certainly this game is not very interesting when there is only one heap of tokens as player A could remove the entire heap and win the game. The game is not much more interesting, in fact, when played with two heaps. Analogous to the previously described Tweedledum-Tweedledee strategy for Kayles, if the two heaps of tokens are the same height then player B has the advantage by copying the moves of player A but in the other heap of tokens. If the two heaps of tokens are not the same height, player A has the advantage by removing enough tokens from the taller heap to make it equal in height to the shorter heap. Player A's subsequent moves in this optimal strategy will be to copy the moves of player B but in the other heap of tokens.

When a game of Nim begins with three or more heaps, however, we cannot always rely on the Tweedledum-Tweedledee strategy. Instead, the strategy for solving Nim (regardless of the number of heaps) lies in numbers which we will call *nim-values* that are associated with positions of the game. The nim-value of a position is the non-negative integer associated with that position by the *Sprague-Grundy Function.*

Definition 1.2. For a game position P we define the *Sprague Grundy Function*, denoted $g(P)$, recursively as follows:

- every final winning position P has $g(P) = 0$, and
- for any position P in the game, $g(P)$ is found by examining all positions reachable from it by one move, call them P_1, P_2, $\ldots$. Suppose the nim-values for these positions are n_1, n_2,$\ldots$, respectively. Then $g(P)$ is the least non-negative integer that does not appear in the list n_1, n_2,$\ldots$. That is, $g(P) = \text{mex}\{n_1, n_2, \ldots\}$ where *mex* stands for minimum excluded value.

While, by definition, we note that final winning positions will have $g(P) = 0$, there could potentially be other positions within the play of the game that have nim-value

0. The positions with nim-value 0 are critical to our winning strategy for Nim; in fact, we can refer to the set of positions in G which have nim-value 0 as the *winning positions* of G. We can summarize the winning strategy of Nim as follows: if the initial position of the game G has $g(G) = 0$ then you should be player B and make your moves in the game always to a position with nim-value 0; otherwise be player A and always move to a position with nim-value 0 when it is your turn.

Let's begin by thinking of a game of Nim with one single heap. The final winning position of this game, P_0, is when all the tokens have been removed; that is, when there are 0 tokens. Thus $g(P_0) = 0$. When the heap consists of just one token, P_1, we could make a move to P_0, so the nim-value cannot be 0. Since the move to P_0 is the only move that can be made, the smallest non-negative integer we cannot reach through a move is 1, and $g(P_1) = 1$. If P_2 is the position consisting of 2 tokens in one heap, we can make moves to both P_1 and P_0, thus we can reach nim-values of 0 and 1. We conclude $g(P_2) = 2$. From here, it is not difficult to see that a game of Nim consisting of one single heap with n tokens has a nim-value of n.

In the case of two heap Nim, it is a little less immediate to see the nim-values. From our winning Tweedledum-Tweedledee strategy we already know that the winning positions will be where the two heaps have equal size. How will we find the nim-values of the positions where the heaps have unequal size? And how does this extend to three or more heaps to help us find winning strategies? We are now trying to find nim-values of positions that can be viewed, instead of as a position in a single game of multi-heap Nim, as a position in a sum of games of single-heap Nim. This leads us to the idea of a sum of nim-values, or *nim-sum*.

Definition 1.3. The binary representation of the *nim-sum* of numbers a, b, $\ldots$, k, denoted $a \oplus b \oplus \cdots \oplus k$, is obtained by adding the binary representations of the numbers without carrying.

We could express this definition in some equivalent ways, first by saying that the binary representation of the nim-sum of a and b, $a \oplus b$, is $a \veebar b$, the logical "exclusive or" or "XOR" of the binary representations. A second reframing of the definition would be to add the binary representations of a and b by adding each place value in $\mathbb{Z}_2$.

Take, for example, $5 \oplus 7$. The binary representation of 5 is 101. The binary representation of 7 is 111. Adding these without carrying produces the binary representation 010, which in decimal is 2. Therefore $5 \oplus 7 = 2$. We can also see that this definition agrees with our earlier observation that two nim-heaps of the same size should combine to have a nim-value equal to 0 since our Tweedledum-Tweedledee strategy tells us these are winning positions. Adding without carrying shows us that $n \oplus n = 0$ and we can use this fact to conclude $a \oplus b \oplus c = 0$ if and only if $a \oplus b = c$. We can now use these two facts to find the nim-value for any game of Nim, regardless of how many nim-piles we have.

Theorem 1.4. *If G, H, and J are impartial combinatorial games then $G = H + J$ implies $g(G)$=$g(H)$$\oplus$$g(J)$.*

R. P. Sprague (1935) and P. M. Grundy (1939) independently showed that the game of Nim implicitly contains the additive theory of all impartial games.

Theorem 1.5. *Every impartial game is equivalent to a nim-heap.*

Theorem 1.5 may seem small, but it really has profound implications. Because of this theorem we know that we can analyze an impartial game to find winning positions and game play strategy by investigating the nim-values of the different positions within the game. Further, Theorem 1.4 shows us that when we are playing games—like Kayles, for example—that naturally break down into components that are smaller copies of themselves, we can use nim-sum to find the nim-values of those positions and continue to use our strategy of moving to positions with nim-value 0.

1.3 Poset Take-Away

In this section we will look at basic definitions for partially ordered sets, in particular ones that will be of use to us in later chapters.

The games we are investigating in this dissertation are all impartial and are also all played on discrete structures. Moreover, we could describe each game as a game on a *partially ordered set* or poset. A poset is a set P together with a binary relation $\leq$ satisfying the following:

- For all $x \in P$, $x \leq x$,
- If $x \leq y$ and $y \leq x$ then $x = y$, and
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

From the definition it is clear that a poset could be infinite. We will assume in this dissertation that all posets—and other discrete structures—will be finite. For more definitions and theory about partially ordered sets, see *Enumerative Combinatorics Volume I* by Richard P. Stanley [9]).

In a poset, P, it could be the case that two elements are not related to one another; that is, there could be elements x and y such that neither $x \leq y$ nor $y \leq x$ are true. In this case we say x and y are *incomparable*; otherwise we say x and y are *comparable*. We define the relation $<$ as $x < y$ if $x \leq y$ but $x \neq y$. If $x < y$ and there is no other element $z \in P$ such that $x < z < y$, then we say y *covers* x. With knowledge of these covering relations within P we can draw what is known as a *Hasse diagram* of P. We use dots to represent the elements of P and connect the dots with a line segment if one of the elements covers the other. The Hasse diagram is oriented such that if y covers x then y is above x in the diagram. For an example, please refer to Figure 5.1.

Playing combinatorial games on posets can be best seen when played on the Hasse diagram. We will take a closer look at how all of the games in this dissertation can be viewed as posets in Chapter 5, but for now we will describe a general rule for a broad class of games that is well-suited to play on a poset. We will define a *Take-Away* game as follows: Players will play on a poset, P. When it is her turn to move, a player can remove any element of the poset, x, and take with it all other elements y such that $x \leq y$. In general, the winner is defined to be the player that removes the last element from P.

We call a poset (or subposet) in which all of the elements are comparable a *chain*. An example of a chain would be the set of natural numbers with the relation "less than", $<$. Let C be a finite chain within a poset P. The *length* of C is defined to be $|C| - 1$. If every maximal chain of P has the same length, n, then we call P a *graded* poset with *rank* n. The Hasse diagram of a graded poset P will have some of its elements at the very bottom; these elements will cover no other elements and will be incomparable with each other. We will say these elements have rank 0. Next we can have a collection of elements of P which cover the rank 0 elements in the relation; we will call these the rank 1 elements. This continues until we have our final set of elements at the top of the Hasse diagram and these elements will be the rank n elements.

The concept of rank of a poset will be critical to our investigation and analysis in Chapter 5. For now, however, we will need to describe another important discrete structure that will be used repeatedly in this dissertation: the graph.

1.4 The Bipartite Graph Game

In this section we will look at basic definitions of graph theory as well as a partial solution to a game on graphs which was the inspiration for the original work that is in this dissertation.

Definition 1.6. A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and an incidence relation that associates with each edge two (not necessarily distinct) members of the vertex set. The elements of the vertex set are called *vertices* and are represented in a drawing of the graph by a dot or point. The elements of the edge set are called *edges* and are represented in a drawing of the graph by a curve connecting two vertices. If an edge has the same vertex for both of its endpoints that edge is called a *loop*. The number of edges incident to a given vertex, v, is called the *degree* of that vertex and is denoted $deg(v)$ [10]. Loops count as two edges when determining the degree of a vertex; that is, if v is a vertex with only a loop on it then

$\deg(v) = 2$.

We note that the definition allows for a graph to possibly be infinite, but we will restrict our focus to *finite graphs*. Finite graphs have a finite number of vertices as well as a finite number of edges. All of the graphs we consider will have no loops.

The Graph Game is a version of Take-Away played on a finite graph, G. When it is her turn to move, a player may take either a single edge from the graph or otherwise she may take a vertex and with it any edges that have that vertex as an endpoint. The winner will be the player that removes the final piece of the graph. When we remove pieces of the graph the new positions we reach are themselves graphs, and these positions are referred to as *subgraphs* of G.

Definition 1.7. A *subgraph* of a graph G is a graph H such that $V(H)$ is a subset of $V(G)$, $E(H)$ is a subset of $E(G)$, and the assignment of endpoint vertices to edges in H is the same as in G [10].

There are a few types of graphs that will be particularly interesting in either the discussion of the Graph Game here or in later chapters of the dissertation. One such type is a *path*. A path is a sequence $P = v_0e_1v_1e_2v_2 \ldots e_kv_k$, $k \geq 2$, whose terms are alternating vertices and edges such that for $1 \leq i \leq k$ the endpoints of e_i are v_{i-1} and v_i and no vertices nor edges are repeated in the sequence. We say the path P has length k. Another important graph for our purposes is the *cycle*. A cycle can be defined as a closed path; that is, a cycle is a sequence $C = v_0e_1v_1e_2v_2 \ldots e_kv_0$, $k \geq 2$, whose terms are alternating vertices and edges such that for $1 \leq i \leq k-1$ the endpoints of e_i are v_{i-1} and v_i, the endpoints of e_k are v_0 and v_{k-1}, and no vertices nor edges are repeated in the sequence (except for v_0) [4]. While paths and cycles can be graphs on their own, they can also be subgraphs of other graphs.

If every pair of vertices in G has a path between them then we say that G is *connected*. Otherwise, we say G is *disconnected*. A graph G which is disconnected is a collection of two or more connected graphs. We call these connected graphs the *components* of G. We denote the number of components of G by $\omega(G)$. An edge, e, is called a *cut edge* of G if $\omega(G - e) > \omega(G)$; that is, a cut edge of G is an edge that when removed from G increases the number of components of the resulting graph [4].

A connected graph that contains no cycles is called a *tree*. There are a few standard facts about trees that will be helpful to know.

Theorem 1.8. *If G is a tree with at least two vertices then G has at least two vertices with degree 1.*

Theorem 1.9. *A connected graph G is a tree if and only if every edge is a cut edge.*

In the Graph Game it is easy to see that a player could make a move on a connected component of the graph for which the resulting position becomes disconnected. Since the Graph Game is an impartial game, however, in these instances the single graph game just becomes a sum of two or more graph games and a player may only make a move in one component at a time.

The Graph Game has yet to be completely solved. Certain classes and types of graphs have been investigated specifically, and of those we are particularly interested in the results for *bipartite graphs*. A bipartite graph, G, is a graph such that $V(G)$

can be partitioned into two sets, $V_1(G)$ and $V_2(G)$, such that all edges in G must have one endpoint in $V_1(G)$ and the other endpoint in $V_2(G)$.

The following theorem is due to R. Riehemann (2003) [7]; see also T. Khandhawit and L. Ye (2011) [6].

Theorem 1.10 (Riehemann, 2003)**.** *Given a bipartite graph G, the nim-value of G in the Graph Game depends on $|V(G)|$ and $|E(G)|$ as follows:*

- *If $|V(G)|$ is even and $|E(G)|$ is even, then $g(G) = 0$,*
- *If $|V(G)|$ is odd and $|E(G)|$ is even, then $g(G) = 1$,*
- *If $|V(G)|$ is even and $|E(G)|$ is odd, then $g(G) = 2$, and*
- *If $|V(G)|$ is odd and $|E(G)|$ is odd, then $g(G) = 3$.*

This result was the primary inspiration for the games we investigate in Chapter 2 and Chapter 3. Riehemann proved through example that nim-values for graphs in general can grow arbitrarily large, so Theorem 1.10 is quite exciting in that it says a bipartite graph of any size has a nim-value determined exclusively by the parity of its edges and vertices.

1.5 Polytopal Complexes

Another discrete structure that we will play a game of Take-Away on in this dissertation is the polytopal complex. For definitions and theory about polytopal complexes, see *Lectures on Polytopes* by Günter M. Ziegler [11]. A *polytopal complex* is a finite, nonempty collection C of convex polytopes in $\mathbb{R}^d$ such that

- If $P \in C$ and Q is a (possibly empty) face of P, then $Q \in C$, and
- If P, $Q \in C$ with $P \neq Q$, then $P \cap Q$ is a common (possibly empty) face of P and Q.

The *dimension* of a polytopal complex is the largest dimension of a polytope in C [11]. Let f_i denote the number of polytopes of dimension i in a polytopal complex K; in situations where it might be unclear as to which complex we are referring, we will use the notation $f_i(K)$. If K does not have any faces of dimension i then $f_i(K) = 0$.

We will define C to be *connected* if its underlying graph is connected as in the definition in Section 1.4. The vertices of the underlying graph are the f_0 faces of C and the edges are the f_1 faces of C. As with graphs, if C is not connected we will refer to its connected components.

We will prove some results about polytopal complexes in Chapter 4, but we will focus on a specific class of polygonal complexes called *polygonal complexes* in Chapter 3 and sections of Chapter 2. A polygonal complex is a polytopal complex of dimension 2; that is, a polygonal complex is a polytopal complex which consists only of f_0 faces (vertices), f_1 faces (edges), and f_2 faces (polygons). Again, we will assume a polygonal complex to be connected if its underlying graph is connected.

1.6 A Dissertation Road Map

We close Chapter 1 with a guide to what follows in this dissertation. In Chapter 2 we investigate Take-Away on hypergraphs and ultimately extend the Bipartite Graph Game to hypergraphs with particular characteristics. We also consider examples of hypergraphs that do not have the proper characteristics and investigate nim-values of these examples. In Chapter 3 we turn our focus to a particular kind of hypergraph, the polygonal complex. In particular, we look at playing Take-Away on a structure that has an even-sided polygon with paths on vertices and edges extending from some of the vertices of the polygon. These results stand in stark contrast to what we saw in Chapter 2. In Chapter 4 we return to looking at graphs, but alter the rules of Take-Away to allow a player to take more than one element of a graph at a time. We then extend this altered version of Take-Away to polytopal complexes. Chapter 5 takes us to the world of posets, in which we look at results of Take-Away on a very specific type of poset. In this chapter we also consider misère play on the poset, and misère play on the sum of posets. Finally, in Chapter 6 we discuss some areas for further investigation related to what we have discovered in Chapters 2 through 5.

The table that follows summarizes the main results of this dissertation.

Structure	Game	Results
Oddly Uniform Hypergraph	Type 1 Take-Away	Theorem 2.4
Evenly Uniform Hypergraph with a Marked Coloring	Type 1 Take-Away	Theorem 2.8
Even Polygon with One Tail	Type 2 Take-Away	Theorem 3.5
Even Polygon with Two Tails (one of fixed length 1)	Type 2 Take-Away	Theorem 3.6
Even Polygon with Two Tails (one of fixed length 2)	Type 2 Take-Away	Theorem 3.9
Even Polygon with Two Tails (both of length $\ell \geq 3$)	Type 2 Take-Away	Theorem 3.10
Quadrilateral with Four Tails	Type 2 Take-Away	Theorem 3.11
Even Polygon with a Tail at Every Vertex	Type 2 Take-Away	Theorem 3.12
Connected Graph with No Loops	Take-As-Much-As-You-Want	Theorem 4.2
Connected Polytopal Complex with $d \geq 3$	Take-As-Much-As-You-Want	Theorem 4.4
Finite Rank-Complete Poset	Take-Away	Theorem 5.1
Finite Rank-Complete Poset	Misère Take-Away	Theorem 5.2
Direct Sum of Finite Rank-Complete Posets	Take-Away	Theorem 5.3

Chapter 2 Hypergraph Games

In this chapter we will consider the game of Take-Away played on various hypergraphs. We will consider different classes of hypergraphs, look at specific examples of these classes, and prove some results about their nim-values and overall strategy for playing Take-Away for each class.

2.1 Game Description

A *hypergraph*, $H = (V(H), E(H))$, is a generalization of a graph in which each edge (also referred to as *hyperedge*) $e \in E(H)$ is incident to a nonempty finite subset of $V(H)$. Throughout this dissertation, we consider only hypergraphs with $|V(H)|$ and $|E(H)|$ finite and each $e \in E(H)$ is incident to at least two vertices. Similar to its definition in a regular graph, we define the *degree* of a vertex in a hypergraph to be the number of hyperedges incident to that vertex.

In this dissertation we will investigate two different versions of Take-Away on a hypergraph. Recall that in Take-Away on a graph, a player has two options when it is her turn to move: she can remove an edge, or she can remove a vertex and take with it all edges which are incident to that vertex. The player who removes the last vertex of the graph is the winner. On a hypergraph, the game is played in much the same way. If a player chooses to take a vertex, all hyperedges containing that vertex will also be removed. A player can also choose to take a hyperedge. An important difference to note between Take-Away on a graph and Take-Away on a hypergraph

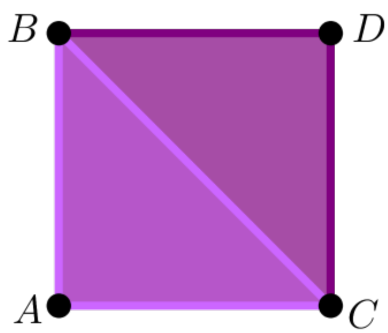

Figure 2.1: A hypergraph with hyperedges $\{A, B\}$, $\{A, C\}$, $\{B, C\}$,$\{B, D\}$, $\{C, D\}$, $\{A, B, C\}$, and $\{A, B, C, D\}$.

is that a hyperedge e_i may be completely contained within another hyperedge e_j. It is this fact that will split the game into two types.

The first type of Take-Away on a hypergraph that we will discuss we will call *Type 1 Take-Away*. In Type 1 Take-Away we will consider the hyperedges to be independent objects. In this case, if a player chooses to remove e_i, then e_j (and any other hyperedge that completely contains e_i) will remain in the hypergraph to be removed in another round of the game. As in the graph game, the player who removes the last vertex from the hypergraph is declared the winner.

The second type of Take-Away on a hypergraph that we will discuss we will call *Type 2 Take-Away*. In Type 2 Take-Away we will consider the hyperedges to be linked by inclusion as subsets of the vertex set. In this case, if a player chooses to remove e_i, then e_j (and any other hyperedge that completely contains e_i) will also be removed. As in the graph game, the player who removes the last vertex from the hypergraph is declared the winner.

For example, consider the hypergraph pictured in Figure 2.1. This hypergraph has vertex set $V(H) = \{A, B, C, D\}$ and the following hyperedges: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$,$\{B, D\}$, $\{C, D\}$, $\{A, B, C\}$, and $\{A, B, C, D\}$. Suppose a player moves in the game on this hypergraph by removing vertex A. In both Type 1 Take-Away and Type 2 Take-Away, removing vertex A will also remove any hyperedge containing vertex A, so the remaining hypergraph will have vertex set $V(H)' = \{B, C, D\}$ and the hyperedges $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$. Also, in either version of the game, choosing to take the hyperedge with the largest cardinality, $\{A, B, C, D\}$, will not remove any additional parts of the hypergraph.

If, instead, the player chose to remove the hyperedge $\{A, B\}$, then the two different types of Take-Away would produce different subsequent positions of the game. Under Type 1 play, the remaining graph would have vertex set $V'(H) = V(H) = \{A, B, C, D\}$ and the hyperedges $\{A, C\}$,$\{B, C\}$, $\{B, D\}$, $\{C, D\}$, and $\{A, B, C, D\}$. Under Type 2 play, the remaining graph would have vertex set $V'(H) = V(H) = \{A, B, C, D\}$ and the hyperedges $\{A, C\}$,$\{B, C\}$, $\{B, D\}$, and $\{C, D\}$, as $\{A, B, C, D\}$ completely contains $\{A, B\}$.

In Chapter 2 we will focus on Type 1 Take-Away games. In Chapter 3 we will focus on Type 2 Take-Away games.

2.2 Oddly Uniform Hypergraphs

A hypergraph is said to be *oddly uniform* if every hyperedge has odd cardinality [2]. We can see that the hypergraph in Figure 2.1 is **not** oddly uniform because it has hyperedges of even cardinality. The graph pictured in Figure 2.2, however, is oddly uniform. The vertex set is $V(H) = \{E, F, J, K, L\}$ and the hyperedges are

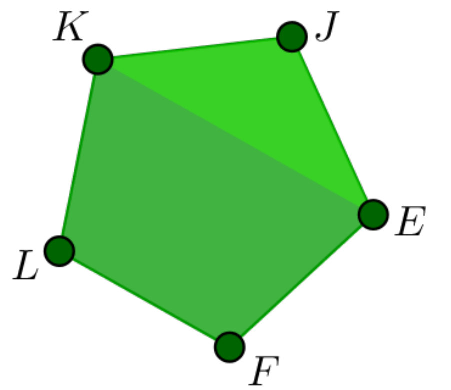

Figure 2.2: An oddly uniform hypergraph with hyperedges $\{E, F, J, K, L\}$ and $\{E, J, K\}$.

$\{E, F, J, K, L\}$ and $\{E, J, K\}$.

We now prove some useful lemmas about oddly uniform hypergraphs.

Lemma 2.1. *Let H be an oddly uniform hypergraph. If $|E(H)|$ is even and $|V(H)|$ is odd, then H must have at least one vertex with an even degree.*

Proof. Assume, to the contrary, that every vertex in H has odd degree. Let n_e be the number of vertices incident to the hyperedge e. Since H is oddly uniform, n_e is odd for every $e \in E(H)$. Now we have

$$\sum_{v \in V(H)} \deg(v) = \sum_{e \in E(H)} n_e$$

Since each n_e is odd and $|E(H)|$ is even, $\sum_{e \in E(H)} n_e$ is even. Hence $\sum_{v \in V(H)} \deg(v)$ is even. By the assumption, every vertex in H has odd degree, and $|V(H)|$ is odd, hence $\sum_{v \in V(H)} \deg(v)$ is odd. Since we have a contradiction, we find that there must be at least one vertex in H that has even degree. $\square$

Lemma 2.2. *Let H be an oddly uniform hypergraph. If $|E(H)|$ is odd and $|V(H)|$ is odd, then H must have at least one vertex with an odd degree.*

Proof. Assume, to the contrary, that every vertex in H has even degree. Let n_e be the number of vertices incident to the hyperedge e. Since H is oddly uniform, n_e is odd for every $e \in E(H)$. Now we have

$$\sum_{v \in V(H)} \deg(v) = \sum_{e \in E(H)} n_e$$

Since each n_e is odd and $|E(H)|$ is odd, $\sum_{e \in E(H)} n_e$ is odd. Hence $\sum_{v \in V(H)} \deg(v)$ is odd. By the assumption, every vertex in H has even degree, and $|V(H)|$ is odd, hence $\sum_{v \in V(H)} \deg(v)$ is even. Since we have a contradiction, we find that there must be at least one vertex in H that has odd degree. □

Lemma 2.3. *Let H be an oddly uniform hypergraph. If $|E(H)|$ is odd and $|V(H)|$ is even, then H must have at least one vertex with an odd degree.*

Proof. Assume, to the contrary, that every vertex in H has even degree. Let n_e be the number of vertices incident to the hyperedge e. Since H is oddly uniform, n_e is odd for every $e \in E(H)$. Now we have

$$\sum_{v \in V(H)} \deg(v) = \sum_{e \in E(H)} n_e$$

Since each n_e is odd and $|E(H)|$ is odd, $\sum_{e \in E(H)} n_e$ is odd. Hence $\sum_{v \in V(H)} \deg(v)$ is odd. By the assumption, every vertex in H has even degree, and $|V(H)|$ is even, hence $\sum_{v \in V(H)} \deg(v)$ is even. Since we have a contradiction, we find that there must be at least one vertex in H that has odd degree. □

With these lemmas in mind, we now turn our focus to Take-Away on oddly uniform hypergraphs.

2.3 Take-Away on Oddly Uniform Hypergraphs

We will describe a complete solution to the game of Type 1 Take-Away on an oddly uniform hypergraph. The strategy for playing can be determined by analyzing the nim-values of the positions, and those nim-values can be determined based exclusively on the number of vertices and the number of hyperedges in the hypergraph.

Theorem 2.4. *Let H be an oddly uniform hypergraph. In the game of Type 1 Take-Away on H, $g(H)$ can be determined based on the number of vertices, $V(H)$, and the number of hyperedges, $E(H)$. Specifically,*

- *if $|V(H)|$ is even and $|E(H)|$ is even, $g(H) = 0$*
- *if $|V(H)|$ is odd and $|E(H)|$ is even, $g(H) = 1$*
- *if $|V(H)|$ is odd and $|E(H)|$ is odd, $g(H) = 2$, and*
- *if $|V(H)|$ is even and $|E(H)|$ is odd, $g(H) = 3$.*

Proof. The proof is by induction on $v + e$, where $v = |V(H)|$ and $e = |E(H)|$.

Base case: Certainly it is clear that a hypergraph with no vertices (and thus no hyperedges) is the winning position of the game, and hence has $g(H) = 0$.

Assume, then, that the theorem holds for $v + e = i$ for all $0 \leq i \leq k$ and consider a hypergraph, H, where $v + e = k + 1$. There are four cases to examine.

- *Case 1:* $|V(H)|$ is even and $|E(H)|$ is even. Consider the possible positions that could be achieved after one move in the game. The resulting hypergraph, H', could have one of the following combinations: $|V(H')|$ is odd and $|E(H')|$ is even, $|V(H')|$ is odd and $|E(H')|$ is odd, or $|V(H')|$ is even and $|E(H')|$ is odd. These positions have $g(H')$ equaling 1, 3, and 2, respectively. Since these are the only positions that can be reached from H after one move, $g(H) = 0$.

- *Case 2:* $|V(H)|$ is odd and $|E(H)|$ is even. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is even and $|E(H')|$ is odd, or $|V(H')|$ is odd and $|E(H')|$ is odd. It is always possible to move to a H' with $|V(H')|$ even and $|E(H')|$ even because H must have at least one vertex with an even degree (see Lemma 2.1), and this H' has $g(H') = 0$. The remaining possibilities have $g(H')$ equaling 2 and 3, thus $g(H) = 1$.

- *Case 3:* $|V(H)|$ is odd and $|E(H)|$ is odd. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is even and $|E(H')|$ is odd, or $|V(H')|$ is odd and $|E(H')|$ is even. Since H has an odd number of hyperedges it is certainly possible to remove one hyperedge and result in a H' with $|V(H')|$ odd and $|E(H')|$ even; this H' has $g(H') = 1$. It is always possible to move to a H' with $|V(H')|$ even and $|E(H')|$ even because H must have at least one vertex with an odd degree (see Lemma 2.2), and this H' has $g(H') = 0$. The only other position that is possible has $g(H') = 3$. Therefore $g(H) = 2$.

- *Case 4:* $|V(H)|$ is even and $|E(H)|$ is odd. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is odd and $|E(H')|$ is odd, or $|V(H')|$ is odd and $|E(H')|$ is even. Since H has an odd number of hyperedges, it is certainly possible to remove one hyperedge and result in a H' with $|V(H')|$ even and $|E(H')|$ even; this H' has $g(H') = 0$. It is always possible to move to a position H' with $|V(H')|$ odd and $|E(H')|$ even because H must have a vertex with odd degree (see Lemma 2.3), and this H' has $g(H') = 1$. It is also always possible to move to a position H' with $|V(H')|$ is odd and $|E(H')|$ is odd because H must have a vertex of even degree (see Lemma 2.3, again), and for this H' $g(H') = 2$. Therefore $g(H) = 3$.

□

This result is exciting because there are special cases of oddly uniform hypergraphs that come up somewhat naturally and would be reasonable choices for structures on which to play a game of Take-Away. We describe some in the next section.

2.4 Corollaries and Related Games

In this section, we will consider some of the special cases of oddly uniform hypergraphs.

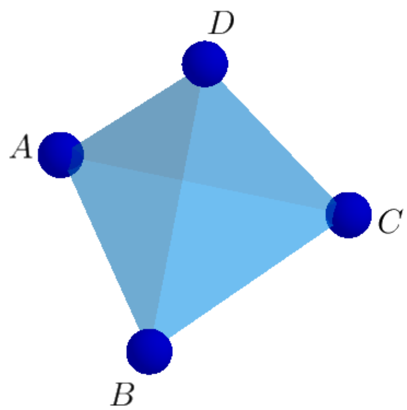

Figure 2.3: A tetrahedron.

3-D Solids

Any polyhedron composed of odd polygons can be viewed as an oddly uniform hypergraph. Take, for example, the tetrahedron. The tetrahedron has four triangular faces and the faces connect with each other at a total of 4 corners. We can place a vertex at each corner and ignore the edges of the tetrahedron, and in doing so we create an oddly uniform hypergraph. Since the tetrahedron, T, has 4 faces and 4 corners it is a hypergraph with an even number of hyperedges and an even number of vertices, $g(T) = 0$.

There are several more classical 3-D solids like the tetrahedron that we can regard as oddly uniform hypergraphs, ready for a game of Take-Away. Among them are three more Platonic solids (the octahedron, the dodecahedron, and the icosahedron), certain Archimedean solids (the icosidodecahedron, and the snub dodecahedron), antiprisims over odd polygons, duals of prisms, deltahedra, and most duals of Archimedean solids (the Catalan solids).

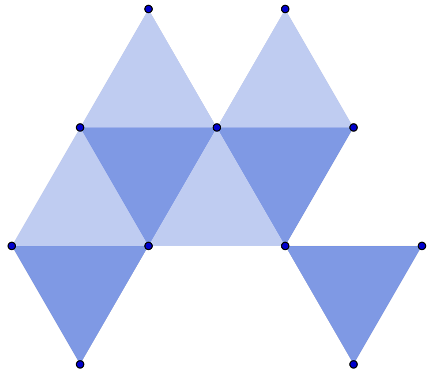

Figure 2.4: A polygonal complex consisting of triangles without edges.

Odd Polygonal Complexes

Consider a polygonal complex comprised of odd polygons. If we include all of the vertices of these polygons but exclude all of the edges of the polygons, then we again have examples of oddly uniform hypergraphs.

We now turn our attention to a different class of hypergraphs, the evenly uniform hypergraphs.

2.5 Evenly Uniform Hypergraphs

Taking a cue from the definition of oddly uniform hypergraphs, we define an *evenly uniform hypergraph* to be a hypergraph with all hyperedges having even cardinality.

As we did with the oddly uniform hypergraphs, we will now prove an important lemma about evenly uniform hypergraphs that will be necessary for studying their nim-values in the game of Type 1 Take-Away.

Lemma 2.5. *If H is an evenly uniform hypergraph with $|V(H)|$ odd and $|E(H)|$ even, then H has a vertex with even degree.*

Proof. To the contrary, assume that every vertex in H has odd degree. Let n_e be the number of vertices incident to the hyperedge e. Since H is evenly uniform, n_e is even for every $e \in E(H)$. Now we have

$$\sum_{v \in V(H)} \deg(v) = \sum_{e \in E(H)} n_e.$$

Since each n_e is even and $|E(H)|$ even, $\sum_{e \in E(H)} n_e$ is even. Hence $\sum_{v \in V(H)} \deg(v)$ is even. By the assumption, every vertex in H has odd degree, and $|V(H)|$ is odd, hence $\sum_{v \in V(H)} \deg(v)$ is odd. Since we have a contradiction, we find that there must be at least one vertex in H that has even degree. □

Some additional lemmas will be useful for our proof to describe nim-values of evenly uniform hypergraphs in the game of Type 1 Take-Away, but in order for these to hold we will need to introduce a new coloring condition on the hypergraphs.

2.6 Hypergraph Colorings

The classic definition of a hypergraph coloring is a coloring of the vertices such that no hyperedge has a monochromatic coloring. Of course, this is consistent with the standard coloring of a graph [10]. For a hypergraph where the hyperedges have 3 or more vertices, this definition means that every hyperedge must have at least two colors, but the same color can be repeated on multiple vertices within the same hyperedge. We note that this definition is for hypergraphs in general, not just evenly uniform hypergraphs.

We will call a hypergraph coloring a *marked coloring* if there is an odd number of vertices on each hyperedge which receive a particular color—call this color c_1. Let

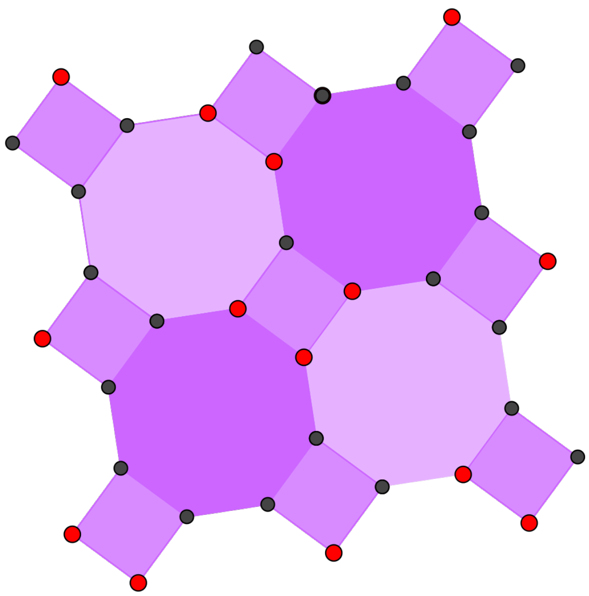

Figure 2.5: An evenly uniform hypergraph with a standard coloring.

H be a hypergraph and $C_1 \subseteq V(H)$ be the subset of vertices receiving color c_1. It is important to notice that if we sum the degrees of the vertices in C_1, since each hyperedge of H has an odd number of marked vertices, we have that

$$\sum_{v \in C_1} \deg(v) \equiv |E(H)| \pmod{2}.$$

Again, it is important to note that a marked coloring is not exclusive to an evenly uniform hypergraph. There are evenly uniform hypergraphs that do not have marked colorings as well as hypergraphs which exhibit a marked coloring that are not evenly uniform.

In Figure 2.6 we see an example of an evenly uniform hypergraph which has a marked coloring. The vertices in red are the subset C_1, and the other vertices are marked in black. This particular marked coloring allows each hyperedge to have *exactly* one red vertex, so $\sum_{v \in C_1} \deg(v) = |E(H)|$ here.

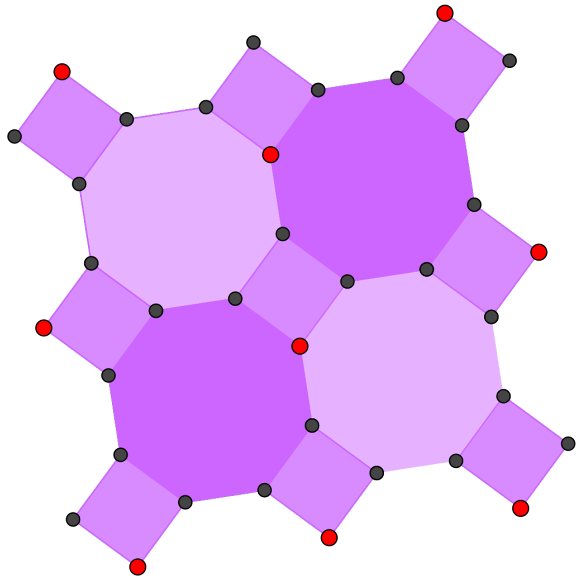

Figure 2.6: An evenly uniform hypergraph with a marked coloring.

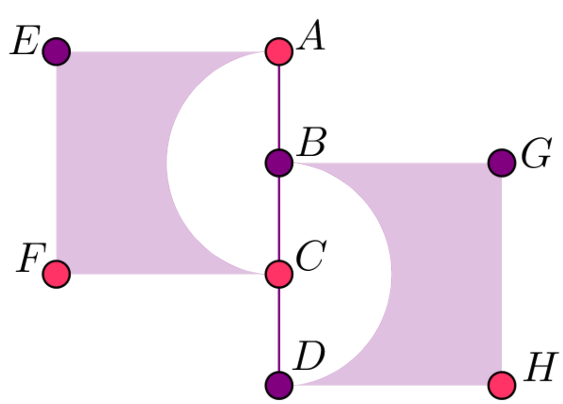

Figure 2.7: Another evenly uniform hypergraph with a marked coloring.

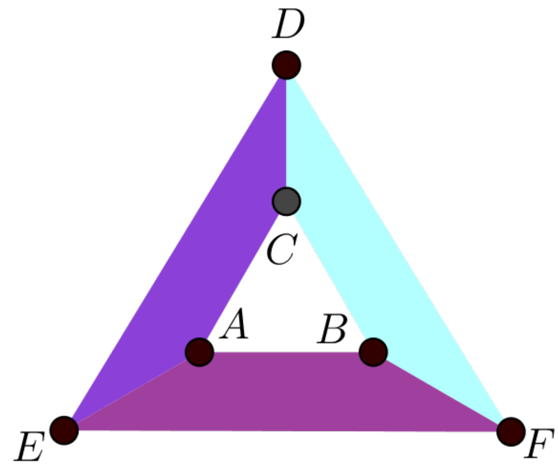

Figure 2.8: A hypergraph that does *not* have a marked coloring.

In Figure 2.7 we have an example of an evenly uniform hypergraph which has a marked coloring, but it cannot be colored in such a way that each hyperedge has only one vertex with color c_1. In this hypergraph we have the hyperedges $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$, $\{A, C, E, F\}$, and $\{B, D, G, H\}$. Suppose we could color this hypergraph so that every hyperedge has only one vertex with color c_1. The problem lies within the chain of hyperedges $\{A, B\}$, $\{B, C\}$, and $\{C, D\}$. In order to properly color this chain we must give A and C the same color and B and D the same color. Unfortunately, however, A and C are both on hyperedge $\{A, C, E, F\}$ and B and D are both on hyperedge $\{B, D, G, H\}$. Properly coloring the hyperedge path $\{A, B\}$, $\{B, C\}$, $\{C, D\}$ prevents our ability to only allow one vertex with color c_1 on every hyperedge; however, we can give each hyperedge an odd number of vertices colored with c_1 as shown.

In Figure 2.8 we have an example of an evenly uniform hypergraph that *cannot* be colored with a marked coloring. To the contrary, suppose that it could. Without loss of generality, we could choose any vertex to be the first to receive color c_1, so we will choose vertex D. So the hyperedges $\{A, C, D, E\}$ and $\{B, C, D, F\}$ have one vertex with color c_1. Now hyperedge $\{A, B, E, F\}$ needs to have either one or three vertices colored with color c_1. If we color only one vertex, say vertex A, then hyperedge $\{A, C, D, E\}$ will have two vertices with color c_1. Since a marked coloring has an odd number of vertices on each edge receiving the special color, we must choose a third vertex on hyperedge $\{A, C, D, E\}$ to color with c_1. Choosing either vertex C or vertex E will result in one of the other hyperedges to now have two vertices with color c_1. As we continue to add vertices to the set C_1 we will always force at least one of the hyperedges to have an even number of vertices with color c_1, hence this hypergraph does not have a marked coloring.

Given this definition of marked coloring, we can now prove our other necessary lemmas.

Lemma 2.6. *If H is an evenly uniform hypergraph with a marked coloring, $|V(H)|$ is even, and $|E(H)|$ is odd, then H has a vertex of odd degree.*

Proof. Assume, to the contrary, that every vertex in H has even degree. Since

$$\sum_{v \in C_1} \deg(v) \equiv |E(H)| \pmod 2,$$

$E(H)$ must be even. But $|E(H)|$ is odd. Since we have a contradiction, it must be that there is at least one vertex in H that has odd degree. □

Lemma 2.7. *If H is an evenly uniform hypergraph with a marked coloring, $|V(H)| \geq 3$ and is odd, and $|E(H)|$ is odd, then H has at least one even degree vertex and at* *least one odd degree vertex.*

Proof. Let n_e be the number of vertices in the hyperedge e. Since H is evenly uniform, n_e is even for every $e \in E(H)$. Now we have

$$\sum_{v \in V(H)} \deg(v) = \sum_{e \in E(H)} n_e.$$

Since each n_e is even, $\sum_{e \in E(H)} n_e$ is even. Hence $\sum_{v \in V(H)} \deg(v)$ is even. Since there is an odd number of vertices, this implies that there is at least one vertex with even degree in $V(H)$.

Since $\sum_{v \in C_1} \deg(v) \equiv |E(H)| \pmod 2$, and $E(H)$ is odd, $\sum_{v \in C_1} \deg(v)$ must also be odd. Therefore there must be at least one odd degree vertex in C_1. □

As the earlier lemmas were useful in analyzing the game of Type 1 Take-Away on oddly uniform hypergraphs, the lemmas in this section will be used to help prove a strategy for game play of Type 1 Take-Away on evenly uniform hypergraphs.

2.7 Take-Away on Evenly Uniform Hypergraphs

Recall that for an oddly uniform hypergraph that the game of Type 1 Take-Away is completely solved. Unfortunately, we cannot say the same thing for evenly uniform hypergraphs. In fact, if Take-Away on evenly uniform hypergraphs were completely solved then the solution to the graph game would follow as a corollary as every graph could be viewed as a 2-uniform hypergraph.

Since there are certain cases where the game of Take-Away on a graph is solved, it is logical to try to extend those ideas to the game of Take-Away on an evenly uniform hypergraph. Recall that Take-Away is completely solved on bipartite graphs (see Theorem 1.10), so a reasonable next step is to find an appropriate condition to impose on the hypergraphs that mimics the quality that bipartite graphs have that allows the game to be solvable.

As it turns out, that condition is that the hypergraph have a marked coloring. This should not be surprising, as a marked coloring is one possibility of an extension of a bipartite coloring to a hypergraph. In fact, if we again consider a graph as a 2-uniform hypergraph then the ones that have marked colorings are exactly the bipartite graphs.

Theorem 2.8. *Let H be an evenly uniform hypergraph with a marked coloring. In the game of Type 1 Take-Away on H, $g(H)$ can be determined based on the number of vertices, $|V(H)|$, and the number of hyperedges, $|E(H)|$. Specifically,*

- *if $|V(H)|$ is even and $|E(H)|$ is even, $g(H) = 0$*
- *if $|V(H)|$ is odd and $|E(H)|$ is even, $g(H) = 1$*
- *if $|V(H)|$ is even and $|E(H)|$ is odd, $g(H) = 2$, and*
- *if $|V(H)|$ is odd and $|E(H)|$ is odd, $g(H) = 3$.*

Proof. The proof is by induction on $v + e$, where $v = |V(H)|$ and $e = |E(H)|$.

Base case: Certainly it is clear that a hypergraph H with no vertices (and hence no hyperedges) is the winning position of the game, and hence $g(H) = 0$.

Assume, then, that the theorem holds for evenly uniform hypergraphs with marked colorings where $v + e = i$, $0 \le i \le k$. Consider an evenly uniform hypergraph, H, which has a marked coloring and where $v+e = k+1$. There are four cases to examine.

- *Case 1:* $|V(H)|$ is even and $|E(H)|$ is even. Consider the possible positions that could be achieved after one move in the game. The resulting hypergraph, H', could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is odd, $|V(H')|$ is odd and $|E(H')|$ is odd, or $|V(H')|$ is odd and $|E(H')|$ is even. These positions have $g(H')$ equal to 2, 3, and 1, respectively. Since these are the only positions that could be reached from H after one move, $g(H) = 0$.

- *Case 2:* $|V(H)|$ is odd and $|E(H)|$ is even. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is odd and $|E(H')|$ is odd, or $|V(H')|$ is even and $|E(H')|$ is odd. In the first instance, since H is guaranteed to have a vertex of even degree (see Lemma 2.5) it is certainly possible to remove this vertex and result in a H' with $|V(H')|$ even and $|E(H')|$ even; this H' has $g(H') = 0$. The other two possibilities, if achievable, have $g(H')$ equal to 3 and 2, respectively. Therefore $g(H) = 1$.

- *Case 3:* $|V(H)|$ is even and $|E(H)|$ is odd. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is odd and $|E(H')|$ is even, or $|V(H')|$ is odd and $|E(H')|$ is odd. It is always possible to move to a H' with $|V(H')|$ even and $|E(H')|$ even because you can remove a single hyperedge; this H' has $g(H') = 0$.

It is also always possible to move to a H' with $|V(H')|$ odd and $|E(H')|$ even because H must have a vertex with odd degree (see Lemma 2.6), and this H' has $g(H') = 1$. The third combination—$|V(H')|$ odd and $|E(H')|$ odd—is only possible when H has an even degree vertex. If this particular H' is possible, it has $g(H') = 3$. Therefore $g(H) = 2$.

- *Case 4:* $|V(H)|$ is odd and $|E(H)|$ is odd. The possible positions, H', resulting after one move could have one of the following combinations: $|V(H')|$ is even and $|E(H')|$ is even, $|V(H')|$ is odd and $|E(H')|$ is even, or $|V(H')|$ is even and $|E(H')|$ is odd. Since H has an odd number of hyperedges it is certainly possible to remove one hyperedge and result in a H' with $|V(H')|$ odd and $|E(H')|$ even; this H' has $g(H') = 1$. It is always possible to move to a H' with $|V(H')|$ even and $|E(H')|$ odd because H must have at least one vertex with an even degree (see Lemma 2.7), and this H' has $g(H') = 2$. It is also always possible to move to a H' with $|V(H')|$ even and $|E(H')|$ even because every H must have a vertex with odd degree (see Lemma 2.7, again), and this H' has $g(H') = 0$. Therefore $g(H) = 3$.

□

Figure 2.8 has been shown to not have a marked coloring therefore we cannot use the results of Theorem 2.8 in order to find its nim-value and determine whether or not it is a winning position in a game of Type 1 Take-Away. Does the nim-value for Figure 2.8 agree with what this theorem predicts despite not being able to use it? If so, that might imply that there is a stronger condition that could be found which would include both the hypergraphs which are marked colorable and structures like Figure 2.8.

It turns out that the nim-value for Figure 2.8 does *not* agree with the predicted value by these theorems. Since the hypergraph has an even number of vertices and an odd number of hyperedges, the theorem would predict it to have nim-value 2. The hypergraph is small, and we can analyze subsequent moves quickly and easily. Based on the symmetry in the hypergraph, we can see that there are only two unique moves that could be made in the game of Type 1 Take-Away: a player could remove a vertex, taking with it two hyperedges, or the player could remove one of the hyperedges and leave all vertices as well as the two remaining hyperedges. In the first case, the structure that is left behind will be a single hyperedge on 4 vertices and an isolated vertex. This position can be given a marked coloring, so by Theorem 2.8, this position has nim-value 3. In the second case, the structure that is left behind will be 6 vertices with two non-overlapping hyperedges, each on 4 vertices. Again, this hypergraph can be given a marked coloring, so by Theorem 2.8, this position has nim-value 0. So, according to Definition 1.2 we conclude that the nim-value for Figure 2.8 is 1. Thus if structures like this hypergraph have a pattern to their nim-values it is not the same pattern as the one we have shown for evenly uniform hypergraphs with marked colorings.

Marked evenly uniform hypergraphs can be created from any polygonal complex. It is important to note, however, that these evenly uniform hypergraphs are not equivalent to the polygonal complexes from which they are made. The construction is as follows: Place a new vertex in the center of every polygon, then extend a line

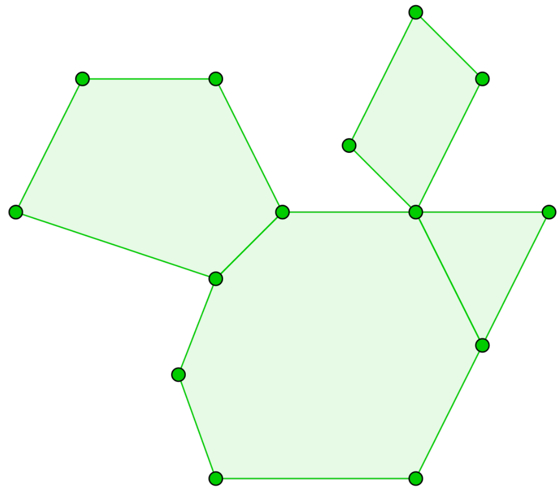

Figure 2.9: A finite polygonal complex.

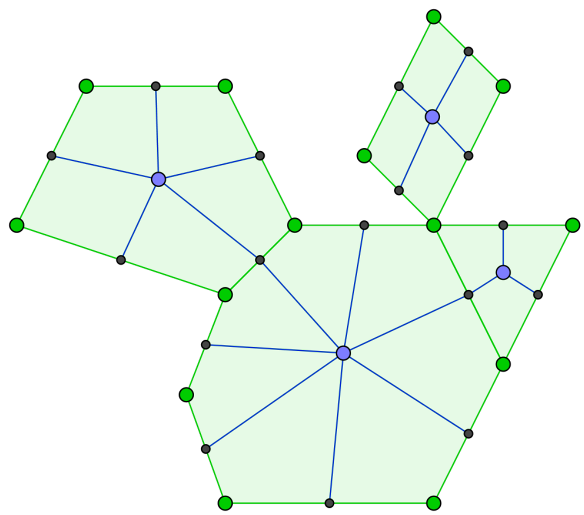

Figure 2.10: A finite polygonal complex divided into quadrilaterals and with a marked coloring.

segment from the new vertex to the midpoint of each existing edge of the polygon. This will turn every existing polygon into a collection of quadrilaterals, and the vertices receiving color c_1 are exactly the set of new vertices added in the center of each original polygon. Figure 2.10 shows the result of this construction when done on the polygonal complex in Figure 2.9.

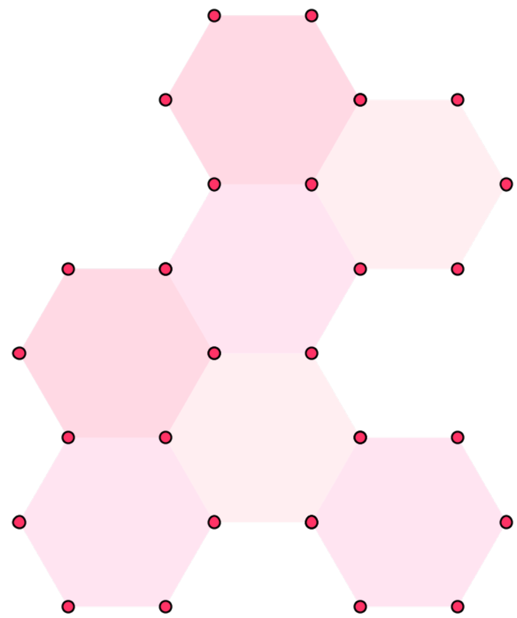

Figure 2.11: A finite polygonal complex comprised of hexagons.

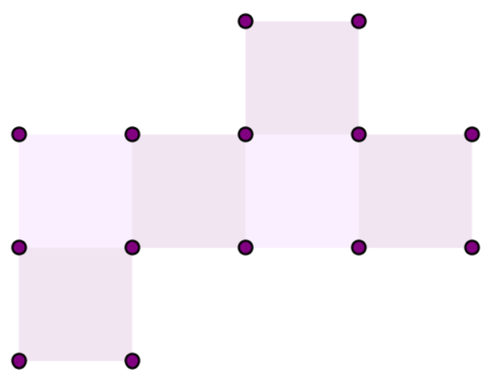

Figure 2.12: A finite polygonal complex comprised of squares.

2.8 Even Polygonal Complexes

A special case of the evenly uniform hypergraph would be a nonempty finite polygonal complex where all the polygons are regular and even-sided. Here, finite means that we have a finite number of vertices and a finite number of regular even-sided polygons.

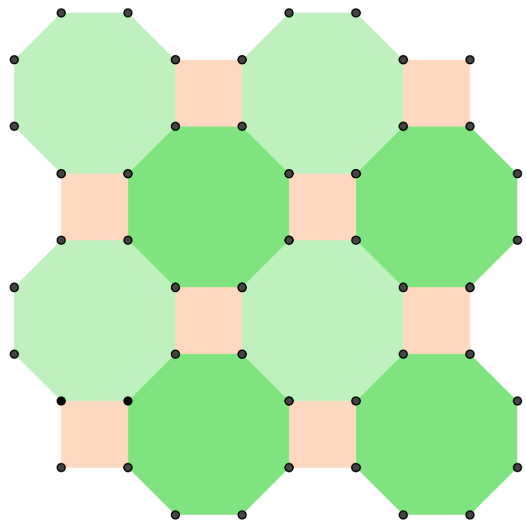

Figure 2.13: A finite polygonal complex comprised of octagons and squares.

After removing the 2-hyperedges—the elements generally referred to as edges—this structure can be colored using the marked coloring described earlier. To prove this, we rely on the fact that these structures with the edges removed always have a vertex with degree 1. We prove the lemma assuming that we do *not* include the edges of the polygon but do contain at least one polygon.

Lemma 2.9. *If G is a nonempty finite polygonal complex consisting of regular even-sided polygons with the edges removed and has at least one polygon, then G has at least one vertex which has degree* 1*.*

Proof. First, recall that for any convex polygon the sum of the exterior angles of the polygon will be 360°. Let P be a regular even-sided polygon with n vertices. Then P is convex and $n \geq 4$. Then the external angle measurement at each vertex of P will be at most 90° because $360/n \leq 90$. Hence, if α is the interior angle measure at each vertex of P then $\alpha \geq 90°$.

Now, let G be a nonempty finite polygonal complex consisting of regular even-sided polygons with the edges removed and at least one polygon. Delete all vertices of G that are not incident to any polygon to form G', and let H be the convex hull of G'. Let $x \in V(G')$ such that x is a corner of H. Since x is a corner of H, the interior angles of all polygons in G' which meet at x must sum to be less than $180°$. Then, since the measure of the interior angles of all polygons in G' must be greater than or equal to $90°$, it is not possible for more than one polygon to be incident to vertex x. Therefore we can conclude that vertex x has degree 1 in G', and hence x has degree 1 also in G. □

The regularity of the polygons is critical in the argument for Lemma 2.9. Clearly, if we do not have regularity then we cannot make the claims about the measures of the interior angles being at least $90°$. We can refer to Figure 2.8 to see an example of a nonempty finite polygonal complex which does *not* use regular polygons and does not have a vertex of degree 1.

Lemma 2.9 guarantees an odd degree vertex in any a nonempty finite polygonal complex consisting of regular even-sided polygons with the edges removed and at least one polygon. This was exactly the condition we needed the marked coloring for before, so now we can use this lemma to prove that a finite polygonal complex consisting of regular even-sided polygons with the edges removed will have a marked coloring.

Theorem 2.10. *If all edges are removed from a finite polygonal complex consisting of regular even-sided polygons and at least one polygon then the resulting G has a* *marked coloring.*

Proof. Let G be as described. The proof is by induction on $|V(G)|$.

Base Case: The smallest G that meets the requirements is a square on four vertices. Certainly this complex can receive a marked coloring by choosing one of the four vertices to receive color c_1.

Assume that for all G with $|V(G)| = i$, $4 \leq i \leq k$, that G has a marked coloring. Consider a finite polygonal complex consisting of regular even-sided polygons and at least one polygon and remove all of the edges to form H with $|V(H)| = k + 1$. We are guaranteed by Lemma 2.9 that H has a vertex, say v, with degree 1. Removing v and its incident polygon leaves H' with $|V(H')| = k$, so by induction H' has a marked coloring. Now, replace v and its incident polygon. If the polygon is properly marked under the coloring on H', we are done. If the polygon has an even number of vertices colored with c_1 under the coloring on H' then color v with c_1 so that the polygon now has an odd number of marked vertices. □

A direct corollary of Theorem 2.10 is a result that is a slight variant of Theorem 2.8.

Corollary 2.11. *For a game of Type 1 Take-Away played on a finite nonempty polygonal complex comprised of regular even-sided polygons with the edges removed, F, $g(F)$ can be determined based on the number of vertices of F, $|V(F)|$, and the number of polygons, $|P(F)|$. Specifically,*

- *if $|P(F)|$ is even and $|V(F)|$ is even, then $g(F) = 0$*

- *if* $|P(F)|$ *is even and* $|V(F)|$ *is odd, then* $g(F) = 1$

- *if* $|P(F)|$ *is odd and* $|V(F)|$ *is even, then* $g(F) = 2$*, and*

- *if* $|P(F)|$ *is odd and* $|V(F)|$ *is odd, then* $g(F) = 3$*.*

Proof. Let F be a finite nonempty polygonal complex comprised of regular even-sided polygons with the edges removed. By Theorem 2.10, F has a marked coloring. Thus we can apply Theorem 2.8.

□

2.9 Hypergraphs With Both Odd And Even Hyperedges

Thus far this chapter has focused on separating hypergraphs into those whose hyperedges all have an odd number of vertices, and those whose hyperedges all have an even number of vertices and a marked coloring. What conclusions, if any, can be drawn for hypergraphs which contain both even and odd hyperedges? We will consider this question in this section.

To attempt to answer the question, we look at a complete analysis of Type 1 Take-Away on the truncated tetrahedron with the edges removed. Figures 2.14 through 2.19 show Schlegel diagrams for all the combinatorial types of connected positions with at least one hyperedge we can obtain in a game of Type 1 Take-Away on the truncated tetrahedron. A *Schlegel diagram* is a projection of, in this case, a polyhedron into the plane through one of its faces [11]. Having this two dimensional representation of the three dimensional figure makes it easier to carry out the hyperedge and vertex removal analysis. In the figures, if the front hexagon—the hyperedge

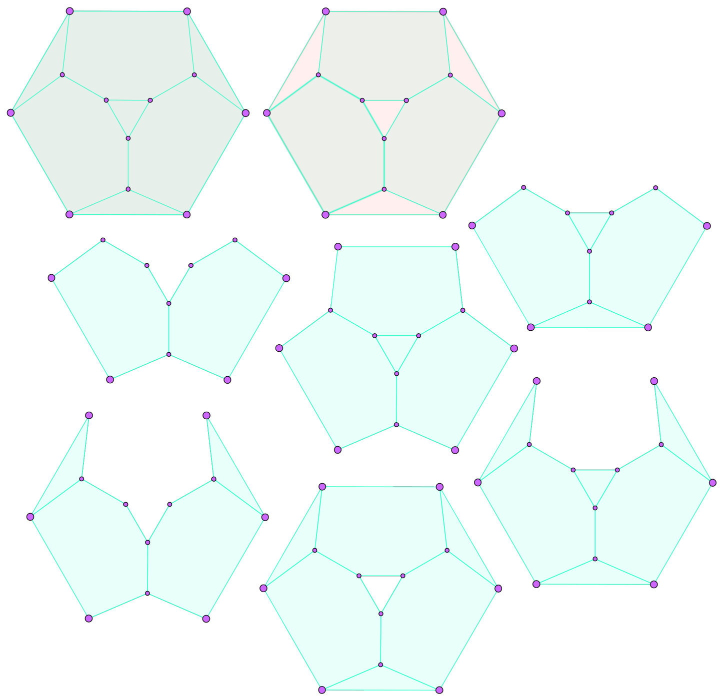

Figure 2.14: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 0.

we imagine looking through to create the Schlegel diagram—is present we see a pink shading.

We also note that the truncated tetrahedron has a marked coloring. Choose one of the triangles and give all three vertices of that triangle color c_1. Then, on the hexagon opposite that triangle, choose every other vertex to receive color c_1. This coloring, seen in Figure 2.20, puts an odd number of vertices with color c_1—shown in red—on each polygon, not just the even polygons.

Recall that for the oddly uniform hypergraphs and the evenly uniform hypergraphs with a marked coloring that the nim-value for any given position can be determined based on the cardinality of the vertex set and the cardinality of the edge set. The same cannot be said for the truncated tetrahedron, and therefore cannot be said in general

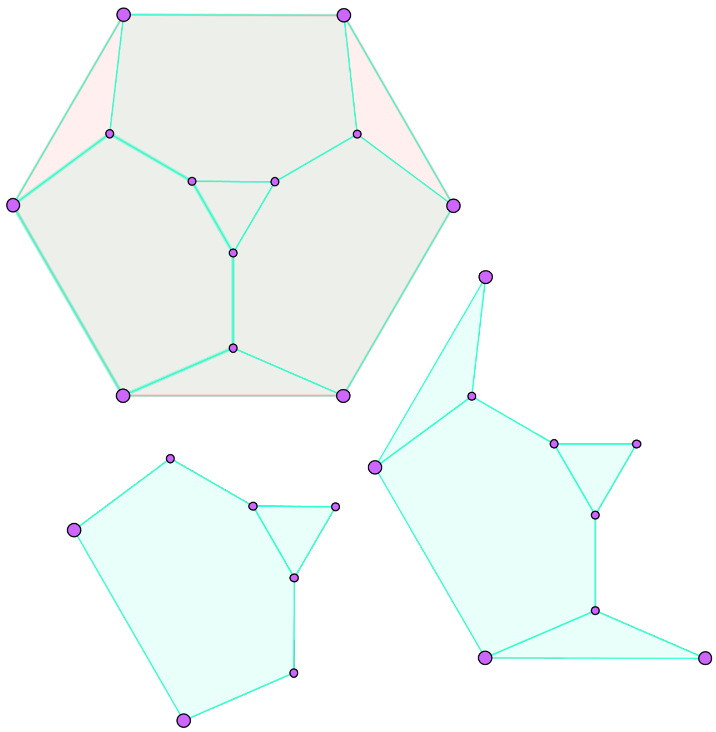

Figure 2.15: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 1.

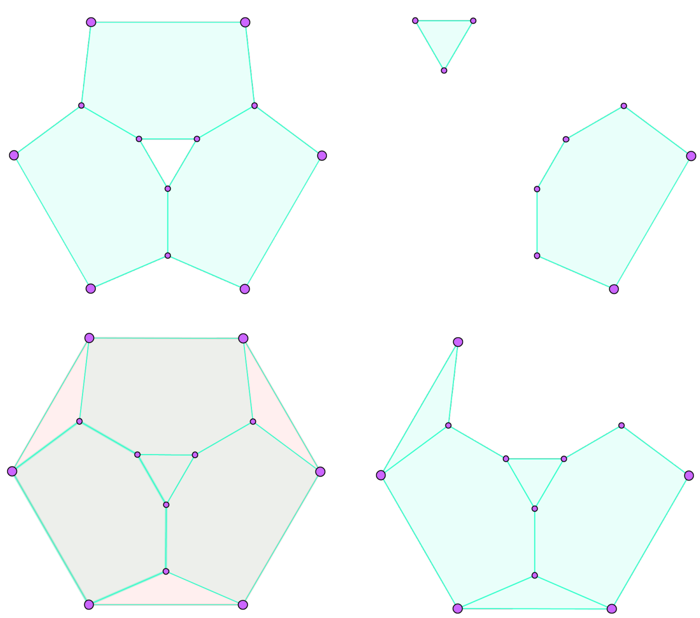

Figure 2.16: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 2.

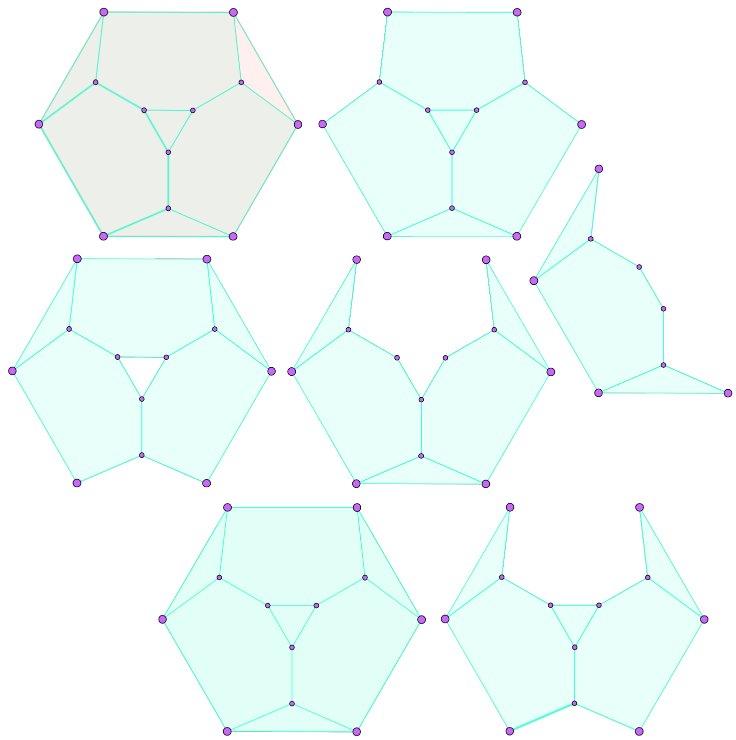

Figure 2.17: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 3.

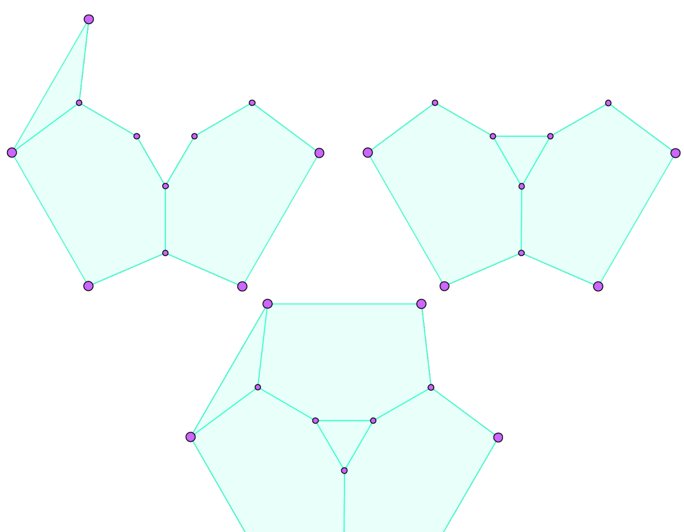

Figure 2.18: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 4.

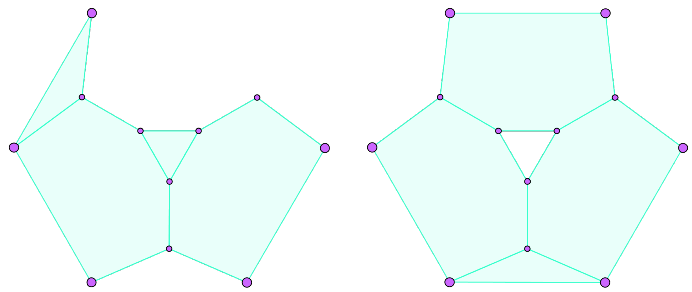

Figure 2.19: Schlegel diagrams of positions obtained from the truncated tetrahedron that have nim-value 6.

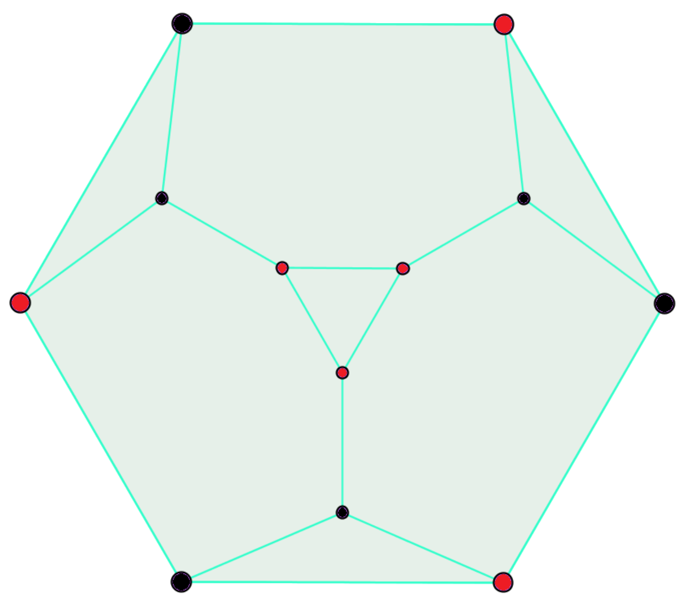

Figure 2.20: Schlegel diagram of a truncated tetrahedron showing a marked coloring.

# of hex	# of triangles	total # polygons	total # vertices	# vertices with odd degree	#vertices with even degree	#vertices with odd hex-degree	# vertices with even hex-degree	#vertices with odd t-degree	#vertices with even t-degree	# EE vertices	# EO vertices	# OE vertices	# OO vertices	NIM Value
4	4	8	12	12	0	0	12	12	0	0	12	0	0	0
4	0	4	12	0	12	0	12	0	12	12	0	0	0	0
2	0	2	10	8	2	8	2	0	10	2	0	8	0	0
3	1	4	12	9	3	6	6	3	3	3	3	6	0	0
2	2	4	10	6	4	8	2	6	4	0	2	4	4	0
2	2	4	12	6	6	8	4	6	6	2	2	4	4	0
3	3	6	12	3	9	0	12	9	3	3	3	0	6	0
2	4	6	12	6	6	8	4	12	0	0	4	0	8	0
4	2	6	12	6	6	0	12	6	6	6	6	0	0	1
1	1	2	7	5	2	6	1	3	4	0	1	4	2	1
1	3	4	9	3	6	6	3	9	0	0	3	0	6	1
3	0	3	12	6	6	6	6	0	12	6	0	6	0	2
0	1	1	3	3	0	0	3	3	0	0	3	0	0	2
1	0	1	6	6	0	6	0	0	6	0	0	6	0	2
4	1	5	12	3	9	0	12	3	9	9	3	0	0	2
2	3	5	11	4	7	8	3	9	2	0	3	2	6	2
4	3	7	12	9	3	0	12	9	3	3	9	0	0	3
3	2	5	12	8	4	6	6	6	6	2	4	4	2	3
3	2	5	12	4	8	6	6	6	6	4	2	2	4	3
2	3	5	12	5	7	8	4	9	3	1	3	2	6	3
1	2	3	8	4	4	6	2	6	2	0	2	2	4	3
3	4	7	12	6	6	0	12	12	0	0	6	0	6	3
2	3	5	12	5	7	8	4	9	3	1	3	2	6	3
2	1	3	11	7	4	8	3	3	8	2	1	6	2	4
2	1	3	10	7	3	8	2	3	7	1	1	6	2	4
3	3	6	12	7	5	6	6	9	3	1	5	2	4	4
2	2	4	11	6	5	8	3	6	5	1	2	4	4	6
3	1	4	12	5	7	6	6	3	9	5	1	4	2	6

Figure 2.21: A table with observed data about truncated tetrahedron game positions with different nim-values.

for hypergraphs with both odd cardinality and even cardinality hyperedges, even if they can be marked. This can immediately be seen given that we now have positions with nim-value 4 and positions with nim-value 6. Considering that these positions could be paired with an isolated vertex, we then have positions which, through nim-sum, have nim-value 5 and nim-value 7. We now have eight unique nim-values; could nim-values, then, be controlled by the cardinality of three specific parameters of the hypergraph?

From the parameters we observed, this does not seem to be the case. In Figure 2.21 we see for each of the positions pictured in the truncated tetrahedron figures above, we have examined fourteen parameters:

- the number of hexagons,
- the number of triangles,

- the total number of polygons,
- the total number of vertices,
- the total number of odd degree vertices,
- the total number of even degree vertices,
- the number of vertices incident to an odd number of hexagons,
- the number of vertices incident to an even number of hexagons,
- the number of vertices incident to an odd number of triangles,
- the number of vertices incident to an even number of triangles,
- the number of vertices incident to both an even number of hexagons and triangles,
- the number of vertices incident to an even number of hexagons and an odd number of triangles,
- the number of vertices incident to an odd number of hexagons and an even number of triangles, and
- the number of vertices incident to both an odd number of hexagons and an odd number of triangles.

The values with even cardinality are shaded green and the values with odd cardinality are shaded orange. Looking at the table we can see examples of positions that match parity on all fourteen parameters yet have different nim-values. For example,

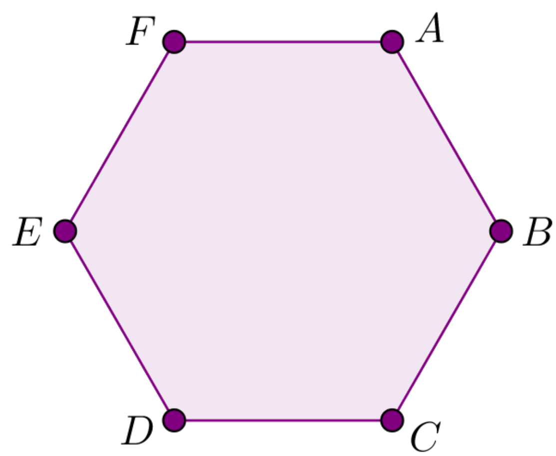

Figure 2.22: A hexagon with its edges.

the lowest position on the table with nim-value 4 and the lowest position on the table with nim-value 6 have the same parities across the table.

2.10 Type 1 Take-Away versus Type 2 Take-Away

We end this chapter by considering the results about oddly uniform hypergraphs and evenly uniform hypergraphs and how they do not hold under the rules of Type 2 Take-Away. Recall that in Type 2 Take-Away that if a hyperedge, e, is removed then any other hyperedge containing e will also be removed.

For example, consider the hypergraph, H, in Figure 2.22. This hypergraph has vertex set $\{A, B, C, D, E, F\}$ and hyperedge set $\{\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{E, F\}, \{A, F\}, \{A, B, C, D, E, F\}\}$. This hypergraph is also markable; vertices A, C, and E can be given color c_1 and then every hyperedge has an odd number of marked vertices. By Theorem 2.8 we know that $g(H) = 2$ under the rules of Type 1 Take-Away.

Consider H under the rules of Type 2 Take-Away. Due to the symmetry of H there are only three unique positions a player could move to via one move in the game. She could remove the hyperedge $\{A, B, C, D, E, F\}$—the hexagon—which leaves a position H' that is a bipartite graph with six vertices and six edges; hence, $g(H') = 0$. She could remove an edge, taking with it the hexagon, which leaves a position H' that is a bipartite graph with six vertices and five edges; hence, $g(H') = 2$. Lastly, she could remove a vertex, taking with it two edges and the hexagon, which leaves a position H' that is a bipartite graph with five vertices and four edges; hence, $g(H') = 1$. So, by Definition 1.2, we find $g(H) = 3$ under the rules of Type 2 Take-Away.

A critical observation that is made in the proofs of Theorem 2.4 and Theorem 2.8 is that making a move in an oddly uniform hypergraph or a marked evenly uniform hypergraph changes the parity of at least one of $|V(G)|$ or $|E(G)|$. In particular, if G is a hypergraph with $|V(G)|$ even and $|E(G)|$ odd then there is no single move that could be made in the game of Take-Away Type 1 that will yield a position G' which also has $|V(G)|$ even and $|E(G)|$ odd. Were this the case, then we could not determine the nim-value of a position based on these parities alone. In the hexagon in Figure 2.22 under Take-Away Type 2, however, we can remove an edge and move to a position with $|V(G)|$ even and $|E(G)|$ odd since removing the edge also removes the hexagon. Therefore the proofs used for the Type 1 Take-Away games do not carry over to the Type 2 Take-Away games.

In the next chapter we will look at some results of games played with Type 2 Take-Away.

Chapter 3 Polygon Games

In the previous chapter we saw results about the game of Type 1 Take-Away when played on very particular types of hypergraphs as well as Type 1 Take-Away on polygonal complexes with the edges removed. In this chapter we will continue to investigate the game of Take-Away played on polygonal complexes; this time, though, we will be using the rules of Type 2 Take-Away. These games have yet to be completely solved, but we do have some interesting partial results to share.

First we will discuss the basics of game play on the polygonal complexes. From there we will start with small cases and build up, proving sometimes surprising results about the cases as we go along. We also will look at a very interesting yet currently unsolved case.

3.1 Game Play

As the reader can surely presume, the evenly uniform hypergraphs and oddly uniform hypergraphs were not where we began our investigation. In fact, the first question we investigated was about playing Take-Away on a portion of a square grid such as the one pictured in Figure 3.1. The goal at that time was to solve the game played on a grid of squares, then move on to other finite portions of a tiling of the plane.

These structures are special types of polygonal complexes. The difference between these structures and the polygonal complexes we considered in the previous chapter is that the polygons we are considering here also include the 2-vertex hyperedges,

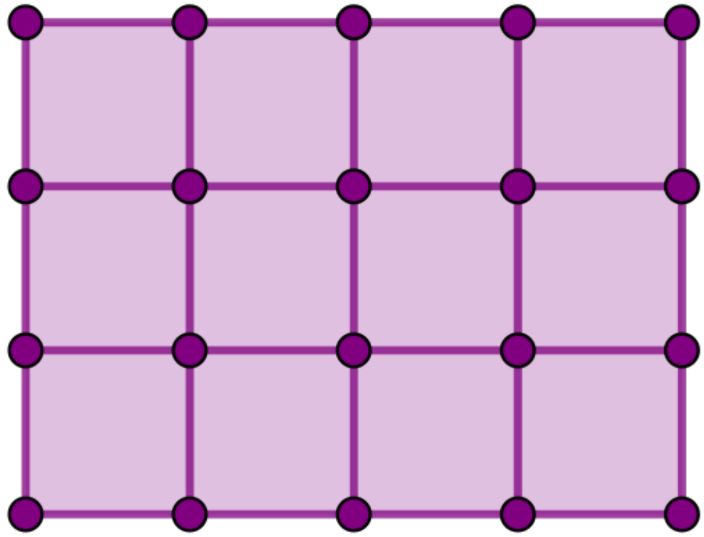

Figure 3.1: A portion of a square grid.

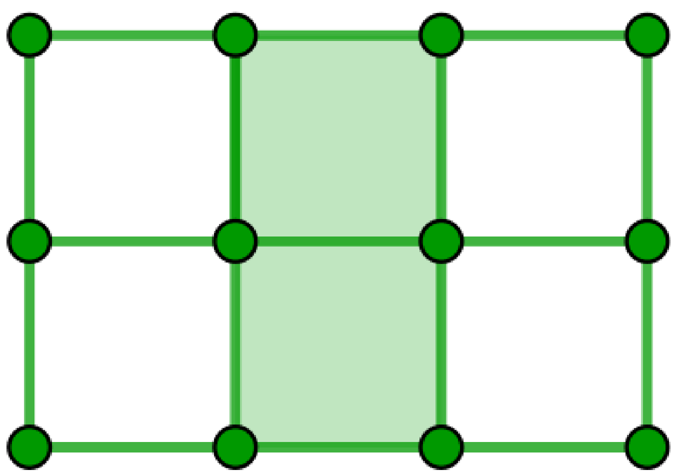

Figure 3.2: A portion of a square grid with even degree corner vertices.

the components we usually call edges. These complexes will not always meet the requirements of the theorems of the previous chapter, and even if they do we saw that the nim-values under Type 2 Take-Away can differ from the nim-values under Type 1 Take-Away.

Recall that the game of Take-Away on a graph is still unsolved, and it is known that the nim-values of a general graph could grow without bound [7, 6]. Take-Away on a bipartite graph, however, is completely solved with well-behaved nim-values as we saw in Theorem 1.10. Because of this, when investigating Take-Away on a polygonal complex, it is logical to start with a polygonal complex where if the polygonal faces were removed—leaving the edges and vertices—the underlying graph is bipartite. Of course, to try to understand the game at large, it is necessary to study small cases.

For this game of Type 2 Take-Away on a polygonal complex, we begin with an even polygon. Game play is is similar to play in the regular graph game. When it is her turn to play, a player can remove the polygonal face, an edge—taking with it all polygonal faces incident to that edge, or a vertex—taking with it all edges and polygonal faces incident to the vertex. We note here that in this chapter when we refer to an edge we are talking about a curve which connects exactly two vertices, which is the standard definition for edges in graphs. Further, when we reference $E(K)$ in a polygonal complex K, we are referring to the set of ordinary edges on two vertices, and this set does not include the polygons of the complex.

We are almost ready to discuss the results about Type 2 Take-Away on an even polygon with tails. However, before we can prove anything about an even polygon with a single tail, we must know the nim-value of the polygon *without* a tail. This result is in Theorem 3.1.

Theorem 3.1. *Let P be an even polygon with n vertices. Then, in Type 2 Take Away,* $g(P) = 3$.

Proof. Let P be as described. Then P has n vertices, n edges, and 1 polygonal face, where n is even. There are only three types of moves that could be made from P to a position P':

- The player could remove the polygonal face, leaving behind the vertices and edges. Since n is even, this P' has $g(P') = 0$ by the result for bipartite graphs.
- The player could remove an edge. This will also remove the polygonal face, leaving behind a bipartite graph with an even number of vertices and an odd number of edges; hence $g(P') = 2$.
- The player could remove a vertex. This will also remove the polygonal face and two edges, leaving behind a bipartite graph with an odd number of vertices and an even number of edges. Therefore $g(P') = 1$.

By definition, since the only positions we can reach through just one move have nim-values 0, 1, and 2, we conclude that $g(P) = 3$. □

We now discuss some partial results. It is important to note that we will frequently be referring to Theorem 1.10 in the proofs that follow in this chapter.

3.2 Even Polygons with One Tail

Our results build up from a single even polygon. In this investigation we will attach to each polygon vertex at most one path of vertices and edges of length $l \geq 1$. For ease, we will call such a path of vertices and edges a *tail*. We will regard the length of the tail to be the number of edges in the tail; note that this is equal to the number of vertices that are on the tail but not on the polygon itself.

Proposition 3.2. *Let K be an even polygon with n vertices and a single tail connected to k vertices of the polygon, where $1 \leq k \leq n-1$. Then it is possible in the game of Type 2 Take-Away on K to move to individual positions K' with $g(K') = 0$, 1, 2, and 3. Hence $g(K) \geq 4$.*

Proof. Let K be an even polygon with n vertices and a single tail connected to k vertices of the polygon, where $1 \leq k \leq n-1$. Recall that $|E(K)|$ is the number of edges in K and does not count the polygon itself. There are specific vertices, edges, and the polygonal face itself, in K that, when removed, yield the desired positions due to the results of the bipartite graph game. Namely:

- *Case 1:* Removing the polygonal face leaves a bipartite graph K' with an equal number of edges and vertices. If n is even, $g(K') = 0$. If n is odd, $g(K') = 3$.

- *Case 2:* Let v be a vertex of the polygon to which a tail is attached. Removing v will result in a bipartite graph K' with $|V(K')| = |V(K)| - 1$ and $|E(K')| = |E(K)| - 3$; both reductions are odd. Thus, if n is even, $g(K') = 3$. If n is odd, $g(K') = 0$.

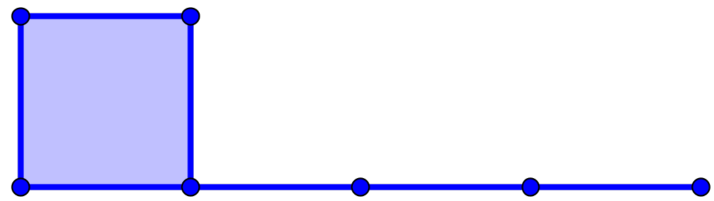

Figure 3.3: A square with a single tail.

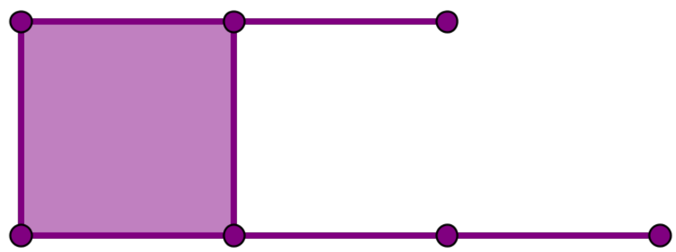

Figure 3.4: A square with two tails.

- *Case 3:* Removing any edge of the polygon also removes the polygonal face and leaves a position that is a connected graph K' with $|V(K')| = |E(K')|+1$. This is a tree, and is hence bipartite, so $g(K') = 2$ if n is even and $g(K') = 1$ if n is odd.

- *Case 4:* Removing any vertex, w, of the polygon that does *not* connect to a tail will reduce $|V(K)|$ by 1 and $|E(K)|$ by 2, thus one will be odd and the other even. Since this resulting position is a bipartite graph, $g(K') = 1$ if n is even and $g(K') = 2$ if n is odd.

□

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	3	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26
1	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
2	6	4	6	10	8	10	14	12	14	18	16	18	22	20	22	26	24	26

Figure 3.5: A table showing some nim-values for an even polygon with one tail (row 1), two tails with one having fixed length 1 (row 2), and two tails with one having fixed length 2 (row 3).

Looking at Figure 3.3 and Figure 3.4 we can see all the cases described in the 4 cases of Proposition 3.2. Therefore, in playing a game of Type 2 Take-Away on either of these figures, the first player will have the advantage because she knows how to move to a subgraph K' with $g(K') = 0$.

In the next lemmas, we investigate a pattern that will be observed in the nim-values of two different even polygon-with-tail structures, namely an even polygon with one tail, and an even polygon with two tails, one of which has fixed length 1. This pattern is similar to one described later for an even polygon with two tails, one of which has fixed length 2. We show the patterns in the table in Figure 3.5. The columns and rows of the table index the tail lengths of two tails attached to separate vertices of an even polygon. The first (color-coded) row of the table shows nim-values for an even polygon with only one tail. The second row shows the nim-values for an even polygon with two tails, one of which has a fixed length of 1. The last row shows the nim-values for an even polygon with two tails, one of which has a fixed length of 2.

Before we prove our results on these patterns, we prove some helpful lemmas.

Lemma 3.3. *For $k \geq 0$, let G_k be defined as follows:*

- *If* $k \equiv 1 \bmod 3$ *then* $G_k = \frac{4k+8}{3}$

- *If* $k \equiv 2 \bmod 3$ *then* $G_k = \frac{4k+10}{3}$

- *If* $k \equiv 0 \bmod 3$ *then* $G_k = \frac{4k}{3}$

If $k \equiv 1 \bmod 3$ *then:*

- $G_k \oplus 1 = \frac{4k+11}{3}$

- $G_k \oplus 2 = \frac{4k+14}{3}$

- $G_{k+1} \oplus 1 = \frac{4k+8}{3}$

- $G_{k+1} \oplus 2 = \frac{4k+17}{3}$

- $G_{k+2} \oplus 1 = \frac{4k+11}{3}$

- $G_{k+2} \oplus 2 = \frac{4k+14}{3}$

This forms a set of 4 consecutive integers, $S_k = \{G_k \oplus 1, G_k \oplus 2, G_{k+1} \oplus 1, G_{k+1} \oplus 2, G_{k+2} \oplus 1, G_{k+2} \oplus 2\} = \{G_k \oplus 1, G_k \oplus 2, G_{k+1} \oplus 1, G_{k+1} \oplus 2\} = \{\frac{4k+8}{3}, \frac{4k+11}{3}, \frac{4k+14}{3}, \frac{4k+17}{3}\}$*, the first of which is a multiple of four.*

Proof. Let G_k be as described and assume $k \equiv 1 \bmod 3$.

First, $k \equiv 1 \bmod 3$, then $k + 2 \equiv 0 \bmod 3$, and hence so is $4k + 8 \equiv 0 \bmod 3$. Thus $\frac{k+2}{3}$ (and $\frac{4k+8}{3}$) is an integer.

Then $G_k = \frac{4k+8}{3} = \frac{4(k+2)}{3}$, and we see that G_k is divisible by 4. Therefore G_k is even and its binary representation ends in 00. So, by the rules of nim addition, the binary representation of $G_k \oplus 1$ will end in 01 and be equivalent to ordinary integer addition: $\frac{4k+8}{3} + 1 = \frac{4k+11}{3}$. Similarly, the binary representation of $G_k \oplus 2$ will end in 10 and be equivalent to ordinary integer addition: $\frac{4k+8}{3} + 2 = \frac{4k+14}{3}$.

Now $G_{k+1} = \frac{4(k+1)+10}{3} = \frac{4k+14}{3} = \frac{2(2k+7)}{3}$. So we see that $2k + 7 \equiv 0 \bmod 3$ and so $\frac{2k+7}{3}$ is an odd integer. Thus G_{k+1} is divisible by 2 but not divisible by 4, so its binary representation ends in 10. Again, by the rules of nim addition, the binary representation of $G_{k+1} \oplus 1$ will end in 11 and be equivalent to ordinary integer addition: $\frac{4k+14}{3} + 1 = \frac{4k+17}{3}$. This will not be the case, however, for $G_{k+1} \oplus 2$. Under the rules of nim addition, the binary representation of $G_{k+1} \oplus 2$ will end in 00 since we are taking the nim-sum of two binary values which each end in 10. Therefore, here, the nim-sum would actually be regular integer *subtraction*: $G_{k+1} \oplus 2 = \frac{4k+14}{3} - 2 = \frac{4k+8}{3}$.

Last, we see that $G_{k+2} = \frac{4(k+2)}{3} = \frac{4k+8}{3}$, which is exactly the same as our first case.

So, we have the set $S_k = \{\frac{4k+8}{3}, \frac{4k+11}{3}, \frac{4k+14}{3}, \frac{4k+17}{3}\}$ which is $\{\frac{4k+8}{3}, \frac{4k+8}{3}+1, \frac{4k+8}{3}+2, \frac{4k+8}{3}+3\}$, and hence a set of four consecutive integers, the first of which is a multiple of four. $\square$

Lemma 3.4. *For $k \geq 1$ and $k \equiv 1 \bmod 3$, let S_k be a set defined as in Lemma 3.3. Then $S_1 \cup S_4 \cup \cdots \cup S_k = \{4, 5, 6, 7, ..., \frac{4k+8}{3}, \frac{4k+11}{3}, \frac{4k+14}{3}, \frac{4k+17}{3}\}$.*

Proof. By the definition of S_k, $S_1 = \{\frac{4(1)+8}{3}, \frac{4(1)+11}{3}, \frac{4(1)+14}{3}, \frac{4(1)+17}{3}\} = \{4, 5, 6, 7\}$. Similarly, $S_4 = \{\frac{4(4)+8}{3}, \frac{4(4)+11}{3}, \frac{4(4)+14}{3}, \frac{4(4)+17}{3}\} = \{8, 9, 10, 11\}$. Then $S_1 \bigcup S_4 = \{4, 5, 6, 7, 8, 9, 10, 11\}$.

Observe, in general, the highest integer in S_{k-3} is $\frac{4k+5}{3}$ and the lowest value in S_k is $\frac{4k+8}{3} = \frac{4k+5}{3} + 1$. $\square$

Note that while Lemma 3.3 involves nim-sum, neither Lemma 3.3 nor Lemma 3.4 relies on any particular game or structure.

Finally we are ready to begin proving results about Type 2 Take-Away on an even polygon with a single tail.

Theorem 3.5. *Let K be an even polygon with n vertices and a tail with length at least 1 attached to exactly one vertex of the polygon. Then, in Type 2 Take-Away, $g(K)$ is determined by the length, ℓ, of the tail. Specifically,*

- *If $\ell \equiv 1$ mod 3 then $g(K) = \frac{4\ell+8}{3}$,*
- *If $\ell \equiv 2$ mod 3 then $g(K) = \frac{4\ell+10}{3}$,*
- *If $\ell \equiv 0$ mod 3 then $g(K) = \frac{4\ell}{3}$.*

Proof. Let K be as described. Let ℓ be the length of the tail. The proof is by induction on ℓ.

Base case: Let $\ell = 1$. Proposition 3.2 tells us that we can move to positions K' with $g(K') = 0$, 1, 2, and 3, and the proof tells us what to remove to reach the position associated with each value. It is important to note that all of these moves are made in a way that removes the polygonal face; either we remove the polygonal face itself, one of its edges (hence taking the polygonal face), or one of its vertices (taking the incident edges and the polygonal face with it.) The only moves that we can make on K that are *not* on the polygon are to remove the end vertex of the tail (taking the edge of the tail with it) or to remove the edge of the tail. If we remove the former, we will be left with just the polygon, and this K' has $g(K') = 3$ (see Theorem

3.1). If we remove the latter, K' will be the polygon with an additional unattached vertex, so we find the nim-value of K' via nim-sum and $g(K') = 3 \oplus 1 = 2$. Hence, by Definition 1.2, $g(K) = 4$, as desired.

Assume, then, that the statement is true for all even polygons with tail of length j, $1 \leq j \leq \ell - 1$. Consider an even polygon with tail of length ℓ. We have three cases to consider.

- *Case 1:* Suppose $\ell \equiv 1 \ mod \ 3$. We have 4 types of moves to consider:
  - *Case 1a:* Removing the polygonal face or a vertex or edge on the polygon will result in a position K' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(K')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(K') \geq 4$.
  - *Case 1b:* Removing the vertex at the end of the tail, v, leaves a position K' with a tail of length $\ell - 1$ and $\ell - 1 \equiv 0 \ mod \ 3$. By induction, $g(K') = \frac{4(\ell-1)}{3} = \frac{4\ell-4}{3}$.
  - *Case 1c:* Removing the vertex, u, which is on the tail and adjacent to a vertex of the polygon, results in a position K' that consists of an even polygon on n vertices and a disjoint path. Similarly, if we remove the edge that is incident to the polygon vertex and to u, we will also result in a position K' that consists of an even polygon on n vertices and a disjoint path that is longer than the previous path by one edge. By Theorem 3.1, we know that the even polygon has nim-value 3. A path is a bipartite

graph and will have nim-value 1 or 2. Since we find the nim-values here using nim-sum and the path lengths differ by 1, one of these positions will have $g(K') = 3 \oplus 1 = 2$ and the other will have $g(K') = 3 \oplus 2 = 1$.

  - *Case 1d:* Removing any other vertex or edge from the tail leaves a position K' which is an even polygon with a tail of length j, $1 \leq j \leq \ell - 2$, and a disjoint path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of smaller polygon with a shorter tail of length j, $1 \leq j \leq \ell - 2$ and a disjoint path; that is, the nim-values will be found via nim-sum. So the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell+5}{3}\}$.

Since these are all the moves we can make, we conclude the nim-value is the least integer that we cannot reach; therefore $g(K) = \frac{4\ell+5}{3} + 1 = \frac{4\ell+8}{3}$.

- *Case 2:* Suppose $\ell \equiv 2 \ mod\ 3$. We have 4 types of moves to consider:

  - *Case 2a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position K' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(K')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(K') \geq 4$.

  - *Case 2b:* Removing the vertex at the end of the tail, v, is a move to a position K' with tail length $\ell - 1$, and by induction $g(K') = \frac{4(\ell-1)+8}{3} = \frac{4\ell+4}{3}$.

Removing the last edge of the tail but leaving v takes us to a position K' which has $g(K') = \frac{4\ell+4}{3} \oplus 1 = \frac{4\ell+7}{3}$.

- *Case 2c:* Removing the vertex, u, which is on the tail and adjacent to a vertex of the polygon, results in a position K' that consists of an even polygon on n vertices and a disjoint path. Similarly, if we remove the edge that is incident to the polygon vertex and to u, we will also result in a position K' that consists of an even polygon on n vertices and a disjoint path that is longer than the previous path by one edge. By Theorem 3.1, we know that the even polygon has nim-value 3. A path is a bipartite graph and will have nim-value 1 or 2. Since we find the nim-values here using nim-sum and the path lengths differ by 1, one of these positions will have $g(K') = 3 \oplus 1 = 2$ and the other will have $g(K') = 3 \oplus 2 = 1$.

- *Case 2d:* Removing any other vertex or edge from the tail leaves a position K' which is an even polygon with a tail of length j, $1 \leq j \leq \ell - 2$, and a disjoint path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of smaller polygon with a shorter tail of length j, $1 \leq j \leq \ell - 2$ and a disjoint path; that is, the nim-values will be found via nim-sum. So the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell+1}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4\ell+10}{3}$, and therefore this is $g(K)$.

- *Case 3:* Suppose $\ell \equiv 0$ *mod* 3. We have 5 types of moves to consider:

  - *Case 3a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position K' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(K')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(K') \geq 4$.

  - *Case 3b:* Removing the end vertex, v, of the tail leaves a position K' with $g(K') = \frac{4(\ell-1)+10}{3} = \frac{4\ell+6}{3}$. Removing the last edge of the tail but leaving v yields a position K' with $g(K') = \frac{4\ell+6}{3} \oplus 1 = \frac{4\ell+9}{3}$.

  - *Case 3c:* Removing the vertex, u, which is on the tail and adjacent to a vertex of the polygon, results in a position K' that consists of an even polygon on n vertices and a disjoint path. Similarly, if we remove the edge that is incident to the polygon vertex and to u, we will also result in a position K' that consists of an even polygon on n vertices and a disjoint path that is longer than the previous path by one edge. By Theorem 3.1, we know that the even polygon has nim-value 3. A path is a bipartite graph and will have nim-value 1 or 2. Since we find the nim-values here using nim-sum and the path lengths differ by 1, one of these positions will have $g(K') = 3 \oplus 1 = 2$ and the other will have $g(K') = 3 \oplus 2 = 1$.

  - *Case 3d:* Removing the vertex in the tail adjacent to v, say w, leaves a position K' with a polygon and shorter tail which has nim-value $\frac{4(\ell-2)+8}{3} =$

$\frac{4\ell}{3}$ and a single vertex. Therefore $g(K') = \frac{4\ell}{3} \oplus 1 = \frac{4\ell+3}{3}$. Removing the edge incident to w but not to v leaves the same polygon and shorter tail with nim-value $\frac{4\ell}{3}$ and a path on two vertices. Therefore $g(K') = \frac{4\ell}{3} \oplus 2 = \frac{4\ell+6}{3}$.

– *Case 3e:* Removing any other vertex or edge from the tail leaves a position K' which is an even polygon with a shorter tail and a path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of smaller polygon with a shorter tail and a path; that is, the nim-values will be found via nim-sum, and we the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell-3}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4\ell}{3}$, and therefore this is $g(K)$.

□

So, as the length of the tail, ℓ, grows, we get a sequence of these structures and the nim-values of the structures follow the pattern

$$4, 6, 4, 8, 10, 8, 12, 14, 12, 16, 18, 16, 20, 22, 20, 24, ...$$

. These values, however, are not really relevant to the actual game play of Take-Away on K. We are assured by Proposition 3.2 that there exists a move from K to a K' with $g(K') = 0$, and the proof even provides the move to make. From there, the subsequent game positions will all be bipartite graphs, so the strategy for game play is known. However, we recall that the motivation for studying this particular structure was to build up from here to eventually understand and find a winning strategy for playing Type 2 Take-Away on a grid or tiling of the plane. Further, if we borrow a cue from multi-heap nim and begin a game with multiple polygons with single tails then the nim-values become very important for determining the strategy for play.

3.3 Even Polygons with Two Tails

The next step, then, is to add another tail to the polygon. In the following theorems we will show that nim-values of these structures are also easily determined, something we can see from the table of partial values introduced earlier in Figure 3.5.

Theorem 3.6. *Let T be an even polygon with n vertices and two tails, each adjacent to a separate vertex on the polygon. Suppose one of those tails has length 1. Then, in Type 2 Take-Away, $g(T)$ is determined by the combined length, t, of the two tails. Specifically,*

- *If $t \equiv 1$ mod 3 then $g(T) = \frac{4t+8}{3}$*
- *If $t \equiv 2$ mod 3 then $g(T) = \frac{4t+10}{3}$*
- *If $t \equiv 0$ mod 3 then $g(T) = \frac{4t}{3}$*

Proof. Let T be as described. Let t be the combined length of the tails. The proof is by induction on t.

Base Cases: Let T be an even polygon with two tails. Suppose one tail has length 1 and the other tail has length 0. Then the combined tail length, t, is 1. By Theorem 3.5 we know this structure has $g(T) = 4$.

Now, let T be an even polygon with two tails attached, each to a different vertex of the polygon. Let each tail have length 1 so that $l_t = 2$. By Proposition 3.2, we know that it is possible to reach positions T' through one move that have $g(T') = 0$, 1, 2, and 3. Recall that these values are reached by removing particular components from the polygon, whether it be the polygonal face itself, or an edge or a vertex from around the polygon. Let's consider the nim-values of the other positions that it is possible to reach through one move. Removing the end vertex, v, from one of the tails will result in an even polygon with one tail which has length 1. By Theorem 3.5 we know this structure has $g(T') = 4$. Alternatively, we could remove the edge incident to both v and the polygon; this results in a T' with an even polygon with one tail of length 1 and also a disconnected vertex. We must use the nim-sum to find the nim-value, and $g(T') = 4 \oplus 1 = 5$. Hence, by Definition 1.2 we find that $g(T) = 6$.

Now, assume that the statement is true for all k, $1 \leq k \leq t-1$ and let T be an even polygon with two tails, one of fixed lengh 1, and the total tail length be t. Let us consider the positions we can reach from T through just one move.

- *Case 1:* Suppose $t \equiv 1 \ mod\ 3$. We have 5 types of moves to consider:
  - *Case 1a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this

type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.

- *Case 1b:* Removing the vertex at the end of the tail of length 1, v, leaves a position T' with a single tail of length $t-1$ and $t-1 \equiv 0$ *mod* 3. By Theorem 3.5, $g(T') = \frac{4(t-1)}{3} = \frac{4t-4}{3}$. Removing the edge incident to v leaves a position with the aforementioned T' plus a disjoint vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4t-4}{3} \oplus 1 = \frac{4t+7}{3}$.

- *Case 1c:* Removing the vertex, u, that is on the tail of length $t-1$ that is adjacent to a vertex on the polygon results in a position T' which has an even polygon with a single tail of length 1 and a disjoint path. Removing the edge incident to both u and the polygon results in a position T' which has an even polygon with a single tail of length 1 and a disjoint path that is longer than the previous path by one edge. A path is a bipartite graph and will have nim-value 1 or 2. The even polygon with a single tail of length 1 has nim-value 4 (see Theorem 3.5). So, between these two positions we will have a move so that we can get to $g(T') = 4 \oplus 1 = 5$ and a move so that we can get to $g(T') = 4 \oplus 2 = 6$.

- *Case 1d:* Removing the end vertex from the tail of length $t-1$ results in a position T' that, by induction, has $g(T') = \frac{4t-4}{3}$.

- *Case 1e:* Removing any other vertex or edge from the tail of length $t-1$ leaves a position T' which can have two components: a polygon with two tails, one with fixed length 1, but with a total tail length t', $1 \leq t' \leq t-1$

and a disjoint path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of a polygon with total tail length t' and a disjoint path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4t+5}{3}\}$.

Since these are all the moves we can make, we conclude the nim-value is the least integer that we cannot reach, therefore $g(T) = \frac{4t+8}{3}$.

- *Case 2:* Suppose $t \equiv 2 \ mod \ 3$. We have 5 types of moves to consider:
  - *Case 2a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.
  - *Case 2b:* Removing the vertex at the end of the tail of length 1, v, leaves a position T' with a single tail of length $t-1$ and $t-1 \equiv 1 \ mod \ 3$. By Theorem 3.5 $g(T') = \frac{4(t-1)+8}{3} = \frac{4t+4}{3}$. Removing the edge incident to v leaves a position with the aforementioned T' plus a disjoint vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4tl+4}{3} \oplus 1 = \frac{4t+7}{3}$.
  - *Case 2c:* Removing the vertex, u, that is on the tail of length $t-1$ that is adjacent to a vertex on the polygon results in a position T' which has an

even polygon with a single tail of length 1 and a disjoint path. Removing the edge incident to both u and the polygon results in a position T' which has an even polygon with a single tail of length 1 and a disjoint path that is longer than the previous path by one edge. A path is a bipartite graph and will have nim-value 1 or 2. The even polygon with a single tail of length 1 has nim-value 4 (see Theorem 3.5). So, between these two positions we will have a move so that we can get to $g(T') = 4 \oplus 1 = 5$ and a move so that we can get to $g(T') = 4 \oplus 2 = 6$.

– *Case 2d:* Removing the end vertex from the tail of length $t-1$ results in a position T' that, by induction, has $g(T') = \frac{4(t-1)+8}{3} = \frac{4t+4}{3}$.

– *Case 2e:* Removing any other vertex or edge from the tail of length $t-1$ leaves a position T' which can have two components:a polygon with two tails, one with fixed length 1, but with a total tail length t', $1 \leq t' \leq t-1$ and a disjoint path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of a polygon with total tail length t' and a disjoint path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4t+1}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4t+10}{3}$, and therefore this is $g(T)$.

- *Case 3:* Suppose $t \equiv 0 \ mod \ 3$. We have 4 types of moves to consider:

– *Case 3a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.

– *Case 3b:* Removing the end vertex, v, of the tail of length 1 leaves a position T' with $g(T') = \frac{4(t-1)+10}{3} = \frac{4t+6}{3}$. Removing the edge of the tail adjacent to v yields a position T' with $g(T') = \frac{4t+6}{3} \oplus 1 = \frac{4t+9}{3}$.

– *Case 3c:* Removing the vertex, u, that is on the tail of length $t-1$ that is adjacent to a vertex on the polygon results in a position T' which has an even polygon with a single tail of length 1 and a disjoint path. Removing the edge incident to both u and the polygon results in a position T' which has an even polygon with a single tail of length 1 and a disjoint path that is longer than the previous path by one edge. A path is a bipartite graph and will have nim-value 1 or 2. The even polygon with a single tail of length 1 has nim-value 4 (see Theorem 3.5). So, between these two positions we will have a move so that we can get to $g(T') = 4 \oplus 1 = 5$ and a move so that we can get to $g(T') = 4 \oplus 2 = 6$.

– *Case 3d:* Removing the end vertex, w, of the other tail results in a position T' with a polygon and total tail length $t-1$ which, by induction, has nim-value $\frac{4(t-1)+10}{3} = \frac{4t+6}{3}$. Removing the edge of the tail adjacent to w yields a position T' with $g(T') = \frac{4t+6}{3} \oplus 1 = \frac{4t+9}{3}$.

– *Case 3e:* Removing any other vertex or edge from the tail of length $t-1$ leaves a position T'which can have two components:a polygon with two tails, one with fixed length 1, but with a total tail length t', $1 \leq t' \leq t-1$ and a disjoint path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of a polygon with total tail length t' and a disjoint path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.3 and Lemma 3.4 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4t-3}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4t}{3}$, and therefore this is $g(T)$.

□

Our next result is for an even polygonal tile with two attached tails, one of which has a fixed length of 2. Just as the previous results about an even polygon with one tail and an even polygon with two tails, one of length 1, there is an easily described pattern to the nim-values of these structures when playing Type 2 Take-Away. While this pattern follows a very similar structure, it is not the same pattern, hence we prove Lemmas similar to Lemma 3.3 and Lemma 3.4 to reflect the new pattern we see here.

Lemma 3.7. *Let G_k be defined as follows:*

- *If* $k \equiv 0 \bmod 3$ *then* $G_k = \frac{4k+18}{3}$

- *If* $k \equiv 1 \bmod 3$ *then* $G_k = \frac{4k+8}{3}$

- *If* $k \equiv 2 \bmod 3$ *then* $G_k = \frac{4k+10}{3}$

If $k \equiv 0 \bmod 3$ *then:*

- $G_k \oplus 1 = \frac{4k+21}{3}$

- $G_k \oplus 2 = \frac{4k+12}{3}$

- $G_{k+1} \oplus 1 = \frac{4k+15}{3}$

- $G_{k+1} \oplus 2 = \frac{4k+18}{3}$

- $G_{k+2} \oplus 1 = \frac{4k+21}{3}$

- $G_{k+2} \oplus 2 = \frac{4k+12}{3}$

This forms a set of 4 consecutive integers, $S_k = \{\frac{4k+12}{3}, \frac{4k+15}{3}, \frac{4k+18}{3}, \frac{4k+21}{3}\}$*, the first of which is a multiple of four.*

Proof. Let G_k be as described and assume $k \equiv 0 \bmod 3$.

First, we know that $\frac{4k+18}{3}$ is an integer because if $k \equiv 0 \bmod 3$ then $4k \equiv 0 \bmod 3$ and since $18 \equiv 0 \bmod 3$, $4k + 18 \equiv 0 \bmod 3$. This means that $4k + 18$ is divisible by 3, hence $\frac{4k+18}{3}$ is an integer.

Then $G_k = \frac{4k+18}{3} = \frac{2(2k+9)}{3}$, and we see that G_k is divisible by 2 but is not divisible by 4. Therefore G_k is even and its binary representation ends in 10. So, by the rules of nim addition, the binary representation of $G_k \oplus 1$ will end in 01 and be equivalent to regular integer addition: $\frac{4k+18}{3} + 1 = \frac{4k+21}{3}$. This will not be the case, however, for

$G_k \oplus 2$. Under the rules of nim addition, the binary representation of $G_k \oplus 2$ will end in 00 since we are adding two binary values which each end in 10. Therefore, here, the nim-sum would actually be regular integer *subtraction*: $\frac{4k+18}{3} - 2 = \frac{4k+12}{3}$.

Now $G_{k+1} = \frac{4(k+1)+8}{3} = \frac{4k+12}{3} = \frac{4(k+3)}{3}$. So we see that G_{k+1} is divisible by 4, so its binary representation ends in 00. Again, by the rules of nim addition, the binary representation of $G_{k+1} \oplus 1$ will end in 01 and be equivalent to regular integer addition: $\frac{4k+12}{3} + 1 = \frac{4k+15}{3}$. Similarly, $G_{k+1} \oplus 2 = \frac{4k+12}{3} + 2 = \frac{4k+18}{3}$.

Last, we see that $G_{k+2} = \frac{4(k+2)+10}{3} = \frac{4k+18}{3}$, which is exactly the same as our first case.

So, we have the set $S_k = \{\frac{4k+12}{3}, \frac{4k+15}{3}, \frac{4k+18}{3}, \frac{4k+21}{3}\}$ which is $\{\frac{4k+12}{3}, \frac{4k+12}{3} + 1, \frac{4k+12}{3} + 2, \frac{4k+12}{3} + 3\}$, and hence a set of four consecutive integers, the first of which is a multiple of four. $\square$

Lemma 3.8. *Let S_k be a set defined as in Lemma 3.7. Then $S_1 \bigcup S_4 \bigcup ... \bigcup S_k = \{4, 5, 6, 7, ..., \frac{4k+12}{3}, \frac{4k+15}{3}, \frac{4k+18}{3}, \frac{4k+21}{3}\}$.*

Proof. By the definition of S_k, $S_1 = \{\frac{4(0)+12}{3}, \frac{4(0)+15}{3}, \frac{4(0)+18}{3}, \frac{4(0)+21}{3}\} = \{4, 5, 6, 7\}$. Similarly, $S_4 = \{\frac{4(3)+12}{3}, \frac{4(3)+15}{3}, \frac{4(3)+18}{3}, \frac{4(3)+21}{3}\} = \{8, 9, 10, 11\}$. Then $S_1 \bigcup S_4 = \{4, 5, 6, 7, 8, 9, 10, 11\}$.

Observe, in general, the highest integer in S_{k-3} is $\frac{4k+5}{3}$ and he lowest value in S_k is $\frac{4k+8}{3} = \frac{4k+5}{3} + 1$. $\square$

With these lemmas, we can now prove the result about the pattern found in the nim-values of an even polygons with two tails, each connected to a separate vertex on the polygon.

Theorem 3.9. *Let T be an even polygon with n vertices and two tails, each connected to a separate vertex on the polygon. Let one of those tails have length 2. Then, in Type 2 Take-Away, $g(T)$ is determined by the length, ℓ, of the other tail. Specifically,*

- *If $\ell \equiv 0$ mod 3 then $g(T) = \frac{4\ell+18}{3}$*
- *If $\ell \equiv 1$ mod 3 then $g(T) = \frac{4\ell+8}{3}$*
- *If $\ell \equiv 2$ mod 3 then $g(T) = \frac{4\ell+10}{3}$*

Proof. Let T be as described. Let ℓ be the length of the tail that does not have fixed length 2. The proof is by induction on ℓ.

Base Case: Let T be an even polygon with two tails attached, each to a different vertex of the polygon. Let one tail have length 2 and the other tail have length $\ell = 0$. By Theorem 3.5 we know this structure has $g(T) = 6$.

Assume, then, that the statement is true for all even polygons with two tails, one with a fixed length of 2 and the other tail of length j, $1 \leq j \leq \ell - 1$. Consider an even polygon with a tail with fixed length 2 and a tail of length ℓ. We have three cases to consider.

- *Case 1:* Suppose $\ell \equiv 0$ *mod* 3. We have 5 types of moves to consider:
  - *Case 1a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.

- *Case 1b:* Removing the vertex at the end of the tail with fixed length 2, v, leaves a position T' with a tail of length 1 and a tail of ℓ. By Theorem 3.6, $g(T') = \frac{4(\ell+1)+8}{3} = \frac{4\ell+12}{3}$. Removing the edge adjacent to v leaves a position with the same T' just described and an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+12}{3} \oplus 1 = \frac{4\ell+15}{3}$.

- *Case 1c:* Removing the vertex from the tail of length 2 that is adjacent to the polygon, w, leaves a position T' with a tail of length ℓ and an unattached vertex. By Theorem 3.5 and nim-sum, we know $g(T') = \frac{4\ell}{3} \oplus 1 = \frac{4\ell+3}{3}$. Instead of removing w, we could remove the edge which is incident to both w and the polygon. This leaves a position with a tail of length ℓ and an unattached path of length 1. Again, we use the results of Theorem 3.5, the bipartite graph game, and nim-sum to find the nim-value of this position: $\frac{4\ell}{3} \oplus 2 = \frac{4\ell+6}{3}$.

- *Case 1d:* Removing the end vertex of the tail of length ℓ, x, leaves a position T' with a tail of length 2 and a tail of length $\ell-1$, and $\ell-1 \equiv 2 \; mod \; 3$. By induction, $g(T') = \frac{4(\ell-1)+10}{3} = \frac{4\ell+6}{3}$. Removing the edge connecting x to the rest of the tail leaves a position with the aforementioned T' plus an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+6}{3} \oplus 1 = \frac{4\ell+9}{3}$.

- *Case 1e:* Removing any other vertex or edge from the tail of length ℓ leaves a position T' which is an even polygon with a tail with length 2 and a tail with length $j \leq \ell$ and an unattached path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of such a structure and a path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.7 and Lemma 3.8 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell+9}{3}\}$.

Since these are all the moves we can make, we conclude the nim-value is the least integer that we cannot reach, therefore $g(K) = \frac{4\ell+18}{3}$.

- *Case 2:* Suppose $\ell \equiv 1 \ mod\ 3$. We have 5 types of moves to consider:
  - *Case 2a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.
  - *Case 2b:* Removing the vertex at the end of the tail with fixed length 2, v, leaves a position T' with a tail of length 1 and a tail of ℓ. By Theorem 3.6, $g(T') = \frac{4(\ell+1)+10}{3} = \frac{4\ell+14}{3}$. Removing the edge adjacent to v leaves a position with the same T' just described and an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+14}{3} \oplus 1 = \frac{4\ell+17}{3}$.
  - *Case 2c:* Removing the vertex from the tail of length 2 that is adjacent to the polygon, w, leaves a posiion T' with a tail of length ℓ and an unattached

vertex. By Theorem 3.5 and nim-sum, we know $g(T') = \frac{4\ell+8}{3} \oplus 1 = \frac{4\ell+11}{3}$. Instead of removing w, we could remove the edge which is incident to both w and the polygon. This leaves a position with a tail of length ℓ and an unattached path of length 1. Again, we use the results of Theorem 3.5, the bipartite graph game, and nim-sum to find the nim-value of this position: $\frac{4\ell+8}{3} \oplus 2 = \frac{4\ell+14}{3}$.

– *Case 2d:* Removing the end vertex of the tail of length ℓ, x, leaves a position T' with a tail of length 2 and a tail of length $\ell-1$, and $\ell-1 \equiv 0 \; mod \; 3$. By induction, $g(T') = \frac{4(\ell-1)+18}{3} = \frac{4\ell+14}{3}$. Removing the edge connecting x to the rest of the tail leaves a position with the aforementioned T' plus an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+14}{3} \oplus 1 = \frac{4\ell+17}{3}$.

– *Case 2e:* Removing any other vertex or edge from the tail of length ℓ leaves a position T' which is an even polygon with a tail with length 2 and a tail with length $j \leq \ell$ and an unattached path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of such a structure and a path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.7 and Lemma 3.8 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell+5}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4\ell+8}{3}$, and therefore this is $g(T)$.

- *Case 3:* Suppose $\ell \equiv 2 \ mod \ 3$. We have 5 types of moves to consider:

  - *Case 3a:* Removing the polygonal face or a vertex or edge on the polygon will be to a position T' which is a bipartite graph. Proposition 3.2 shows that we can arrive at positions with $g(T')$ equal to 0, 1, 2, and 3 via this type of move, and by the results for Take-Away on a bipartite graph we will not arrive at a position with $g(T') \geq 4$.

  - *Case 3b:* Removing the vertex at the end of the tail with fixed length 2, v, leaves a position T' with a tail of length 1 and a tail of ℓ. By Theorem 3.6, $g(T') = \frac{4(\ell+1)}{3} = \frac{4\ell+4}{3}$. Removing the edge adjacent to v leaves a position with the same T' just described and an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+4}{3} \oplus 1 = \frac{4\ell+7}{3}$.

  - *Case 3c:* Removing the vertex from the tail of length 2 that is adjacent to the polygon, w, leaves a position T' with a tail of length ℓ and an unattached vertex. By Theorem 3.5 and nim-sum, we know $g(T') = \frac{4\ell+10}{3} \oplus 1 = \frac{4\ell+13}{3}$. Instead of removing w, we could remove the edge which is incident to both w and the polygon. This leaves a position with a tail of length ℓ and an unattached path of length 1. Again, we use the results of Theorem 3.5, the bipartite graph game, and nim-sum to find the nim-value of this position: $\frac{4\ell+10}{3} \oplus 2 = \frac{4\ell+16}{3}$.

  - *Case 3d:* Removing the end vertex of the tail of length ℓ, x, leaves a position T' with a tail of length 2 and a tail of length $\ell-1$, and $\ell-1 \equiv 1 \ mod \ 3$. By induction, $g(T') = \frac{4(\ell-1)+8}{3} = \frac{4\ell+4}{3}$. Removing the edge connecting x

to the rest of the tail leaves a position with the aforementioned T' plus an unattached vertex. Through nim-sum, we find the nim-value of this position to be $\frac{4\ell+4}{3} \oplus 1 = \frac{4\ell+7}{3}$.

– *Case 3e:* Removing any other vertex or edge from the tail of length ℓ leaves a position T' which is an even polygon with a tail with length 2 and a tail with length $j \leq \ell$ and an unattached path. A path is a bipartite graph and will have nim-value 1 or 2. By selectively removing a vertex—taking with it two edges—or a single edge, it is possible to move to every combination of such a structure and a path; that is, the nim-values will be found via nim-sum, and the results of Lemma 3.7 and Lemma 3.8 show us that we can get all nim-values from the set $\{4, 5, ..., \frac{4\ell+5}{3}\}$.

Combining these values we see that the first unreachable integer is $\frac{4\ell+10}{3}$, and therefore this is $g(K)$.

□

The nim-values we have seen here are always even and follow easily describable patterns but are unbounded. That is, if we wanted a structure that has a nim-value of $2k$ for any $k \in \mathbb{N}$, $k \geq 3$, then we could easily create it based on these theorems. It is because of this that the next result is particularly interesting; we now continue to look at an even polygon with two attached tails, but this time we consider the length of both tails to be at least 3. What we find under this condition is that the nim-values not only remain easy to determine based on tail length, but now they are bounded.

Theorem 3.10. *Let T be an even polygon with n vertices and two tails, each connected to a separate vertex on the polygon. Let both of those tails have length at least 3. Then, in Type 2 Take-Away, $g(T)$ is determined by the combined length, ℓ_t, of the two tails. Specifically,*

- *If ℓ_t is even then $g(T) = 4$*
- *If ℓ_t is odd then $g(T) = 7$.*

Proof. Let T be as described. By Proposition 3.2, we know that it is possible to reach positions through one move that have nim-values of 0, 1, 2, and 3 by removing the polygonal face or an edge or vertex incident to it. Let's consider the nim-values of the other positions that are reachable through one move.

- A player could remove a vertex or edge that completely disconnects one tail from T. The position now has two components, one is a tree and the other is a polygon with a single tail as in Theorem 3.5; we will call this component K. From this same theorem we know that $g(K)$ will be determined by the length of the remaining connected tail, and, since the tail is at least length 3, the smallest possible $g(K)$ is 8. The tree, of course, will have a nim-value of either 1 or 2. When we use nim-sum to find the nim-value of our current position, we see that there is no way to get a value smaller than $10 \oplus 2 = 8$.
- A player could remove a vertex or edge that leaves one tail from T with length 1. The position now has two components, one is a tree and the other is a polygon with two tails, one which has length 1, as in Theorem 3.6; we will call this

component L. From this same theorem, we know that $g(L)$ will be determined by the combined length, t, of the two tails, and, since t is at least 4, the smallest possible $g(L)$ is 8. The tree, of course, will have a nim-value of either 1 or 2. When we use nim-sum to find the nim-value of our current position, we see that there is no way to get a value smaller than $10 \oplus 2 = 8$.

- A player could remove a vertex or edge that leaves one tail from T with length 2. The position now has two components, one is a tree and the other is a structure as in Theorem 3.9; we will call this component M. From this same theorem, we know that the $g(M)$ will be determined by the length, l, of the longer tail, and, since l is at least 3, the smallest possible $g(M)$ is 8. The tree, of course, will have a nim-value of either 1 or 2. When we use nim-sum to find the nim-value of our current position, we see that there is no way to get a value smaller than $10 \oplus 2 = 8$.

- If the tail lengths are sufficiently large, a player could remove a vertex or edge from T which leaves a position with two components, a tree and a structure like T but with fewer vertices. The tree will have nim-value 1 or 2. Inductively, the smaller two-tailed structure will have nim-value 4 or 7. Therefore the only possibilities for a nim-value for this position are

$$4 \oplus 1 = 5$$

$$4 \oplus 2 = 6$$

$$7 \oplus 1 = 6$$

$$7 \oplus 2 = 5$$

.

- Lastly, for tails of sufficient length, a player could move an end edge or end vertex from one of the tails of T. If the sum of the tail lengths of T is even, removing the end vertex from one tail will move to a position with a similar structure and a tail length sum which is odd, so inductively this position has nim-value 7. If, instead, the player removes just the last edge, we have the same structure with nim-value 7 and an isolated vertex, and under Nim sum this is a position with nim-value 6. If the sum of the tail lengths of T is odd, removing the end vertex from one tail will move to a position with a similar structure and a tail length sum which is even, so inductively this position has nim-value 4. If, instead, the player removes just the last edge, we have the same structure with nim-value 4 and an isolated vertex, and under Nim sum this is a position with nim-value 5.

This analysis shows that if ℓ_t is even, it is impossible to move to a position with nim-value 4, so the nim-value of T is 4. Similarly, if ℓ_t is odd, it is impossible to move to a position with nim-value 7, so the nim-value of T is 7. □

With these three theorems in mind, we find that the case of one even polygon with two tails is completely solved, and in most cases is quite well-behaved.

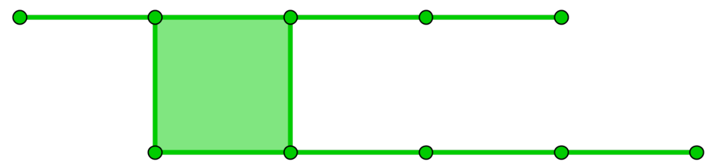

Figure 3.6: An even polygon with tails connected to each of three vertices.

3.4 Even Polygons with Three Tails

In the previous section we found that an even polygon with two connected tails was able to be completely solved with respect to finding nim-values of these structures in a game of Type 2 Take-Away. The next step in our analysis is to add another tail to the polygon. In Figure 3.6 we see an example of an even polygon with 3 tails.

It would be exciting to see results for an even polygon with three tails similar to what we saw for even polygons with 1 and 2 tails. Unfortunately, though, the figures that follow show that this is not the case. What we see in Figures 3.7 through 3.24 are tables of nim-values for even polygons with three tails. In each case, we hold the length of one tail constant, and the table shows some of the nim-values as the other tail lengths vary. Each particular number is given a unique color, and because of this when we look at the tables we can see that there *are* many interesting patterns in the three tail data, but describing them with a formula or a simple sentence like our previous lemmas does not seem obvious.

Some of the observed patterns include:

- When the fixed length tail is of length n, if another tail also has length n then as the third tail varies it follows the 4, 6, 4, 8, 10, 8, ... pattern we saw in

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	3	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26
1	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
2	6	4	6	10	8	10	14	12	14	18	16	18	22	20	22	26	24	26
3	4	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
4	8	10	8	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
5	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
6	8	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
7	12	14	12	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
8	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
9	12	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
10	16	18	16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
11	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
12	16	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
13	20	22	20	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
14	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
15	20	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
16	24	26	24	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
17	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4

Figure 3.7: A table showing the nim-values for an even polygon with three tails, one having fixed length 0.

Theorem 3.5.

- Let the fixed length tail be of length n. If $n \equiv 0 \ mod\ 3$ then the table for a tail of fixed length n is almost identical to the table for a tail with fixed length $n+2$ and the table for a tail of fixed length $n+1$ is related in a noticeable way.
- Values mostly repeat in blocks of 9 values down the main diagonal of the table for any fixed length n.
- If the tail lengths are sufficiently large and sufficiently far apart then it appears the nim-value of the position will be 4 or 7.

There could very well be additional patterns in this data, and patterns that could be easily proved, but at this time we have chosen not to focus on this task.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
1	6	4	6	10	8	10	14	12	14	18	16	18	22	20	22	26	24	26
2	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
3	8	10	8	6	4	6	16	18	16	12	14	12	24	26	24	20	22	20
4	10	8	10	4	6	4	18	16	18	14	12	14	26	24	26	22	20	22
5	8	10	8	6	4	6	16	18	16	12	14	12	24	26	24	20	22	20
6	12	14	12	16	18	16	6	4	6	8	10	8	28	30	28	32	34	32
7	14	12	14	18	16	18	4	6	4	10	8	10	30	28	30	34	32	34
8	12	14	12	16	18	16	6	4	6	8	10	8	28	30	28	32	34	32
9	16	18	16	12	14	12	8	10	8	6	4	6	32	34	32	28	30	28
10	18	16	18	14	12	14	10	8	10	4	6	4	34	32	34	30	28	30
11	16	18	16	12	14	12	8	10	8	6	4	6	32	34	32	28	30	28
12	20	22	20	24	26	24	28	30	28	32	34	32	6	4	6	8	10	8
13	22	20	22	26	24	26	30	28	30	34	32	34	4	6	4	10	8	10
14	20	22	20	24	26	24	28	30	28	32	34	32	6	4	6	8	10	8
15	24	26	24	20	22	20	32	34	32	28	30	28	8	10	8	6	4	6
16	26	24	26	22	20	22	34	32	34	30	28	30	10	8	10	4	6	4
17	24	26	24	20	22	20	32	34	32	28	30	28	8	10	8	6	4	6

Figure 3.8: A table showing the nim-values for an even polygon with three tails, one having fixed length 1.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	6	4	6	10	8	10	14	12	14	18	16	18	22	20	22	26	24	26
1	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
2	6	4	6	10	8	10	14	12	14	18	16	18	22	20	22	26	24	26
3	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
4	8	10	8	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
5	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
6	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
7	12	14	12	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
8	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
9	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
10	16	18	16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
11	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
12	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
13	20	22	20	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
14	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
15	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
16	24	26	24	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
17	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4

Figure 3.9: A table showing the nim-values for an even polygon with three tails, one having fixed length 2.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	4	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	8	10	8	6	4	6	16	18	16	12	14	12	24	26	24	20	22	20
2	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
4	7	4	7	10	8	10	7	4	7	4	7	4	7	4	7	4	7	4
5	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
6	7	16	7	12	7	12	8	4	8	4	7	4	7	4	7	4	7	4
7	4	18	4	14	4	14	4	8	4	7	4	7	4	7	4	7	4	7
8	7	16	7	12	7	12	8	4	8	4	7	4	7	4	7	4	7	4
9	4	12	4	16	4	16	4	7	4	8	4	8	4	7	4	7	4	7
10	7	14	7	18	7	18	7	4	7	4	8	4	7	4	7	4	7	4
11	4	12	4	16	4	16	4	7	4	8	4	8	4	7	4	7	4	7
12	7	24	7	20	7	20	7	4	7	4	7	4	8	4	8	4	7	4
13	4	26	4	22	4	22	4	7	4	7	4	7	4	8	4	7	4	7
14	7	24	7	20	7	20	7	4	7	4	7	4	8	4	8	4	7	4
15	4	20	4	24	4	24	4	7	4	7	4	7	4	7	4	8	4	8
16	7	22	7	26	7	26	7	4	7	4	7	4	7	4	7	4	8	4
17	4	20	4	24	4	24	4	7	4	7	4	7	4	7	4	8	4	8

Figure 3.10: A table showing the nim-values for an even polygon with three tails, one having fixed length 3.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	8	10	8	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	10	8	10	4	6	4	18	16	18	14	12	14	26	24	26	22	20	22
2	8	10	8	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	4	7	10	8	10	7	4	7	4	7	4	7	4	7	4	7	4
4	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
5	7	4	7	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7
6	4	18	4	7	12	4	10	8	10	20	12	20	12	15	12	15	12	15
7	7	16	7	4	14	7	8	10	8	4	7	4	7	4	7	4	7	4
8	4	18	4	7	12	4	10	8	10	7	4	7	4	7	4	7	4	7
9	7	14	7	4	16	7	20	4	7	10	7	10	7	4	7	4	7	4
10	4	12	4	7	18	4	12	7	4	7	10	7	4	7	4	7	4	7
11	7	14	7	4	16	7	20	4	7	10	7	10	7	4	7	4	7	4
12	4	26	4	7	20	4	12	7	4	7	4	7	10	7	10	7	4	7
13	7	24	7	4	22	7	15	4	7	4	7	4	7	10	7	4	7	4
14	4	26	4	7	20	4	12	7	4	7	4	7	10	7	10	7	4	7
15	7	22	7	4	24	7	15	4	7	4	7	4	7	4	7	10	7	10
16	4	20	4	7	26	4	12	7	4	7	4	7	4	7	4	7	10	7
17	7	22	7	4	24	7	15	4	7	4	7	4	7	4	7	10	7	10

Figure 3.11: A table showing the nim-values for an even polygon with three tails, one having fixed length 4.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	8	10	8	6	4	6	16	18	16	12	14	12	24	26	24	20	22	20
2	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
4	7	4	7	10	8	10	4	7	4	7	4	7	4	7	4	7	4	7
5	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
6	7	16	7	12	4	12	8	4	8	4	7	4	7	4	7	4	7	4
7	4	18	4	14	7	14	4	8	4	7	4	7	4	7	4	7	4	7
8	7	16	7	12	4	12	8	4	8	4	7	4	7	4	7	4	7	4
9	4	12	4	16	7	16	4	7	4	8	4	8	4	7	4	7	4	7
10	7	14	7	18	4	18	7	4	7	4	8	4	7	4	7	4	7	4
11	4	12	4	16	7	16	4	7	4	8	4	8	4	7	4	7	4	7
12	7	24	7	20	4	20	7	4	7	4	7	4	8	4	8	4	7	4
13	4	26	4	22	7	22	4	7	4	7	4	7	4	8	4	7	4	7
14	7	24	7	20	4	20	7	4	7	4	7	4	8	4	8	4	7	4
15	4	20	4	24	7	24	4	7	4	7	4	7	4	7	4	8	4	8
16	7	22	7	26	4	26	7	4	7	4	7	4	7	4	7	4	8	4
17	4	20	4	24	7	24	4	7	4	7	4	7	4	7	4	8	4	8

Figure 3.12: A table showing the nim-values for an even polygon with three tails, one having fixed length 5.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	8	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	12	14	12	16	18	16	6	4	6	8	10	8	28	30	28	32	34	32
2	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	16	7	12	7	12	8	4	8	4	7	4	7	4	7	4	7	4
4	4	18	4	7	12	4	10	8	10	20	12	20	12	15	12	15	12	15
5	7	16	7	12	4	12	8	4	8	4	7	4	7	4	7	4	7	4
6	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
7	7	4	7	4	8	4	14	12	14	4	7	4	7	4	7	4	7	4
8	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
9	7	8	7	4	20	4	16	4	16	12	7	12	7	4	7	4	7	4
10	4	10	4	7	12	7	18	7	18	7	12	7	4	7	4	7	4	7
11	7	8	7	4	20	4	16	4	16	12	7	12	7	4	7	4	7	4
12	4	28	4	7	12	7	20	7	20	7	4	7	12	7	12	7	4	7
13	7	30	7	4	15	4	22	4	22	4	7	4	7	12	7	4	7	4
14	4	28	4	7	12	7	20	7	20	7	4	7	12	7	12	7	4	7
15	7	32	7	4	15	4	24	4	24	4	7	4	7	4	7	12	7	12
16	4	34	4	7	12	7	26	7	26	7	4	7	4	7	4	7	12	7
17	7	32	7	4	15	4	24	4	24	4	7	4	7	4	7	12	7	12

Figure 3.13: A table showing the nim-values for an even polygon with three tails, one having fixed length 6.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	12	14	12	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	14	12	14	18	16	18	4	6	4	10	8	10	30	28	30	34	32	34
2	12	14	12	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	18	4	8	4	8	4	8	4	7	4	7	4	7	4	7	4	7
4	7	16	7	4	14	7	8	10	8	4	7	4	7	4	7	4	7	4
5	4	18	4	8	7	8	4	8	4	7	4	7	4	7	4	7	4	7
6	7	4	7	4	8	4	14	12	14	4	7	4	7	4	7	4	7	4
7	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
8	7	4	7	4	8	4	14	12	14	4	7	4	7	4	7	4	7	4
9	4	10	4	7	4	7	4	16	4	14	4	14	4	7	4	7	4	7
10	7	8	7	4	7	4	7	18	7	4	14	4	7	4	7	4	7	4
11	4	10	4	7	4	7	4	16	4	14	4	14	4	7	4	7	4	7
12	7	30	7	4	7	4	7	20	7	4	7	4	14	4	14	4	7	4
13	4	28	4	7	4	7	4	22	4	7	4	7	4	14	4	7	4	7
14	7	30	7	4	7	4	7	20	7	4	7	4	14	4	14	4	7	4
15	4	34	4	7	4	7	4	24	4	7	4	7	4	7	4	14	4	14
16	7	32	7	4	7	4	7	26	7	4	7	4	7	4	7	4	14	4
17	4	34	4	7	4	7	4	24	4	7	4	7	4	7	4	14	4	14

Figure 3.14: A table showing the nim-values for an even polygon with three tails, one having fixed length 7.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	12	14	12	16	18	16	6	4	6	8	10	8	28	30	28	32	34	32
2	14	12	14	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	16	7	12	7	12	8	4	8	4	7	4	7	4	7	4	7	4
4	4	18	4	7	12	4	10	7	10	7	4	7	4	7	4	7	4	7
5	7	16	7	12	4	12	8	4	8	4	7	4	7	4	7	4	7	4
6	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
7	7	4	7	4	7	4	14	12	14	4	7	4	7	4	7	4	7	4
8	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
9	7	8	7	4	7	4	16	4	16	12	7	12	7	4	7	4	7	4
10	4	10	4	7	4	7	18	7	18	7	12	7	4	7	4	7	4	7
11	7	8	7	4	7	4	16	4	16	12	7	12	7	4	7	4	7	4
12	4	28	4	7	4	7	20	7	20	7	4	7	12	7	12	7	4	7
13	7	30	7	4	7	4	22	4	22	4	7	4	7	12	7	4	7	4
14	4	28	4	7	4	7	20	7	20	7	4	7	12	7	12	7	4	7
15	7	32	7	4	7	4	24	4	24	4	7	4	7	4	7	12	7	12
16	4	34	4	7	4	7	26	7	26	7	4	7	4	7	4	7	12	7
17	7	32	7	4	7	4	24	4	24	4	7	4	7	4	7	12	7	12

Figure 3.15: A table showing the nim-values for an even polygon with three tails, one having fixed length 8.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	12	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	16	18	16	12	14	12	8	10	8	6	4	6	32	34	32	28	30	28
2	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	12	4	8	4	8	4	7	4	8	4	8	4	7	4	7	4	7
4	7	14	7	4	16	7	20	4	7	10	7	10	7	4	7	4	7	4
5	4	12	4	8	7	8	4	7	4	8	4	8	4	7	4	7	4	7
6	7	8	7	4	20	4	16	4	16	12	7	12	7	4	7	4	7	4
7	4	10	4	7	4	7	4	16	4	14	4	14	4	7	4	7	4	7
8	7	8	7	4	7	4	16	4	16	12	7	12	7	4	7	4	7	4
9	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
10	7	4	7	4	7	4	7	4	7	18	16	18	7	4	7	4	7	4
11	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
12	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
13	4	34	4	7	4	7	4	7	4	22	4	22	4	16	4	7	4	7
14	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
15	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
16	7	30	7	4	7	4	7	4	7	26	7	26	7	4	7	4	16	4
17	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16

Figure 3.16: A table showing the nim-values for an even polygon with three tails, one having fixed length 9.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	16	18	16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	18	16	18	14	12	14	10	8	10	4	6	4	34	32	34	30	28	30
2	16	18	16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	14	7	18	7	18	7	4	7	4	8	4	7	4	7	4	7	4
4	4	12	4	7	18	4	12	7	4	7	10	7	4	7	4	7	4	7
5	7	14	7	18	4	18	7	4	7	4	8	4	7	4	7	4	7	4
6	4	10	4	7	12	7	18	7	18	7	12	7	4	7	4	7	4	7
7	7	8	7	4	7	4	7	18	7	4	14	4	7	4	7	4	7	4
8	4	10	4	7	4	7	18	7	18	7	12	7	4	7	4	7	4	7
9	7	4	7	4	7	4	7	4	7	18	16	18	7	4	7	4	7	4
10	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
11	7	4	7	4	7	4	7	4	7	18	16	18	7	4	7	4	7	4
12	4	34	4	7	4	7	4	7	4	7	20	7	18	7	18	7	4	7
13	7	32	7	4	7	4	7	4	7	4	22	4	7	18	7	4	7	4
14	4	34	4	7	4	7	4	7	4	7	20	7	18	7	18	7	4	7
15	7	30	7	4	7	4	7	4	7	4	24	4	7	4	7	18	7	18
16	4	28	4	7	4	7	4	7	4	7	26	7	4	7	4	7	18	7
17	7	30	7	4	7	4	7	4	7	4	24	4	7	4	7	18	7	18

Figure 3.17: A table showing the nim-values for an even polygon with three tails, one having fixed length 10.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	16	18	16	12	14	12	8	10	8	6	4	6	32	34	32	28	30	28
2	18	16	18	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	12	4	8	4	8	4	7	4	8	4	8	4	7	4	7	4	7
4	7	14	7	4	16	7	20	4	7	10	7	10	7	4	7	4	7	4
5	4	12	4	8	7	8	4	7	4	8	4	8	4	7	4	7	4	7
6	7	8	7	4	20	4	16	4	16	12	7	12	7	4	7	4	7	4
7	4	10	4	7	4	7	4	16	4	14	4	14	4	7	4	7	4	7
8	7	8	7	4	7	4	16	4	16	12	7	12	7	4	7	4	7	4
9	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
10	7	4	7	4	7	4	7	4	7	18	16	18	7	4	7	4	7	4
11	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
12	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
13	4	34	4	7	4	7	4	7	4	22	4	22	4	16	4	7	4	7
14	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
15	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
16	7	30	7	4	7	4	7	4	7	26	7	26	7	4	7	4	16	4
17	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16

Figure 3.18: A table showing the nim-values for an even polygon with three tails, one having fixed length 11.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	16	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	20	22	20	24	26	24	28	30	28	32	34	32	6	4	6	8	10	8
2	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	24	7	20	7	20	7	4	7	4	7	4	8	4	8	4	7	4
4	4	26	4	7	20	4	12	7	4	7	4	7	10	7	10	7	4	7
5	7	24	7	20	4	20	7	4	7	4	7	4	8	4	8	4	7	4
6	4	28	4	7	12	7	20	7	20	7	4	7	12	7	12	7	4	7
7	7	30	7	4	7	4	7	20	7	4	7	4	14	4	14	4	7	4
8	4	28	4	7	4	7	20	7	20	7	4	7	12	7	12	7	4	7
9	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
10	4	34	4	7	4	7	4	7	4	7	20	7	18	7	18	7	4	7
11	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
12	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
13	7	4	7	4	7	4	7	4	7	4	7	4	22	20	22	4	7	4
14	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
15	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
16	4	10	4	7	4	7	4	7	4	7	4	7	26	7	26	7	20	7
17	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20

Figure 3.19: A table showing the nim-values for an even polygon with three tails, one having fixed length 12.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	20	22	20	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	22	20	22	26	24	26	30	28	30	34	32	34	4	6	4	10	8	10
2	20	22	20	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	26	4	22	4	22	4	7	4	7	4	7	4	8	4	7	4	7
4	7	24	7	4	22	7	15	4	7	4	7	4	7	10	7	4	7	4
5	4	26	4	22	7	22	4	7	4	7	4	7	4	8	4	7	4	7
6	7	30	7	4	15	4	22	4	22	4	7	4	7	12	7	4	7	4
7	4	28	4	7	4	7	4	22	4	7	4	7	4	14	4	7	4	7
8	7	30	7	4	7	4	22	4	22	4	7	4	7	12	7	4	7	4
9	4	34	4	7	4	7	4	7	4	22	4	22	4	16	4	7	4	7
10	7	32	7	4	7	4	7	4	7	4	22	4	7	18	7	4	7	4
11	4	34	4	7	4	7	4	7	4	22	4	22	4	16	4	7	4	7
12	7	4	7	4	7	4	7	4	7	4	7	4	22	20	22	4	7	4
13	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
14	7	4	7	4	7	4	7	4	7	4	7	4	22	20	22	4	7	4
15	4	10	4	7	4	7	4	7	4	7	4	7	4	24	4	22	4	22
16	7	8	7	4	7	4	7	4	7	4	7	4	7	26	7	4	22	4
17	4	10	4	7	4	7	4	7	4	7	4	7	4	24	4	22	4	22

Figure 3.20: A table showing the nim-values for an even polygon with three tails, one having fixed length 13.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	20	22	20	24	26	24	28	30	28	32	34	32	6	4	6	8	10	8
2	22	20	22	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	24	7	20	7	20	7	4	7	4	7	4	8	4	8	4	7	4
4	4	26	4	7	20	4	12	7	4	7	4	7	10	7	10	7	4	7
5	7	24	7	20	4	20	7	4	7	4	7	4	8	4	8	4	7	4
6	4	28	4	7	12	7	20	7	20	7	4	7	12	7	12	7	4	7
7	7	30	7	4	7	4	7	20	7	4	7	4	14	4	14	4	7	4
8	4	28	4	7	4	7	20	7	20	7	4	7	12	7	12	7	4	7
9	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
10	4	34	4	7	4	7	4	7	4	7	20	7	18	7	18	7	4	7
11	7	32	7	4	7	4	7	4	7	20	7	20	16	4	16	4	7	4
12	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
13	7	4	7	4	7	4	7	4	7	4	7	4	22	20	22	4	7	4
14	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
15	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
16	4	10	4	7	4	7	4	7	4	7	4	7	26	7	26	7	20	7
17	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20

Figure 3.21: A table showing the nim-values for an even polygon with three tails, one having fixed length 14.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	20	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	24	26	24	20	22	20	32	34	32	28	30	28	8	10	8	6	4	6
2	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	20	4	24	4	24	4	7	4	7	4	7	4	7	4	8	4	8
4	7	22	7	4	24	7	15	4	7	4	7	4	7	4	7	10	7	10
5	4	20	4	24	7	24	4	7	4	7	4	7	4	7	4	8	4	8
6	7	32	7	4	15	4	24	4	24	4	7	4	7	4	7	12	7	12
7	4	34	4	7	4	7	4	24	4	7	4	7	4	7	4	14	4	14
8	7	32	7	4	7	4	24	4	24	4	7	4	7	4	7	12	7	12
9	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
10	7	30	7	4	7	4	7	4	7	4	24	4	7	4	7	18	7	18
11	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
12	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
13	4	10	4	7	4	7	4	7	4	7	4	7	4	24	4	22	4	22
14	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
15	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7	26	24	26
17	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24

Figure 3.22: A table showing the nim-values for an even polygon with three tails, one having fixed length 15.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	24	26	24	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
1	26	24	26	22	20	22	34	32	34	30	28	30	10	8	10	4	6	4
2	24	26	24	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7
3	7	22	7	26	7	26	7	4	7	4	7	4	7	4	7	4	8	4
4	4	20	4	7	26	4	12	7	4	7	4	7	4	7	4	7	10	7
5	7	22	7	26	4	26	7	4	7	4	7	4	7	4	7	4	8	4
6	4	34	4	7	12	7	26	7	26	7	4	7	4	7	4	7	12	7
7	7	32	7	4	7	4	7	26	7	4	7	4	7	4	7	4	14	4
8	4	34	4	7	4	7	26	7	26	7	4	7	4	7	4	7	12	7
9	7	30	7	4	7	4	7	4	7	26	7	26	7	4	7	4	16	4
10	4	28	4	7	4	7	4	7	4	7	26	7	4	7	4	7	18	7
11	7	30	7	4	7	4	7	4	7	26	7	26	7	4	7	4	16	4
12	4	10	4	7	4	7	4	7	4	7	4	7	26	7	26	7	20	7
13	7	8	7	4	7	4	7	4	7	4	7	4	7	26	7	4	22	4
14	4	10	4	7	4	7	4	7	4	7	4	7	26	7	26	7	20	7
15	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7	26	24	26
16	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
17	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7	26	24	26

Figure 3.23: A table showing the nim-values for an even polygon with three tails, one having fixed length 16.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
1	24	26	24	20	22	20	32	34	32	28	30	28	8	10	8	6	4	6
2	26	24	26	4	7	4	7	4	7	4	7	4	7	4	7	4	7	4
3	4	20	4	24	4	24	4	7	4	7	4	7	4	7	4	8	4	8
4	7	22	7	4	24	7	15	4	7	4	7	4	7	4	7	10	7	10
5	4	20	4	24	7	24	4	7	4	7	4	7	4	7	4	8	4	8
6	7	32	7	4	15	4	24	4	24	4	7	4	7	4	7	12	7	12
7	4	34	4	7	4	7	4	24	4	7	4	7	4	7	4	14	4	14
8	7	32	7	4	7	4	24	4	24	4	7	4	7	4	7	12	7	12
9	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
10	7	30	7	4	7	4	7	4	7	4	24	4	7	4	7	18	7	18
11	4	28	4	7	4	7	4	7	4	24	4	24	4	7	4	16	4	16
12	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
13	4	10	4	7	4	7	4	7	4	7	4	7	4	24	4	22	4	22
14	7	8	7	4	7	4	7	4	7	4	7	4	24	4	24	20	7	20
15	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24
16	7	4	7	4	7	4	7	4	7	4	7	4	7	4	7	26	24	26
17	4	6	4	8	10	8	12	14	12	16	18	16	20	22	20	24	26	24

Figure 3.24: A table showing the nim-values for an even polygon with three tails, one having fixed length 17.

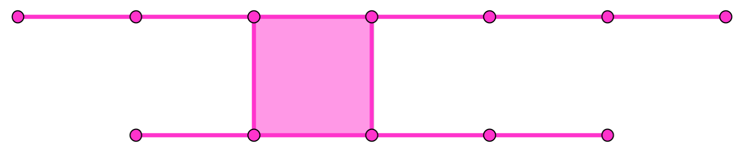

Figure 3.25: A quadrilateral with a tail connected to every vertex.

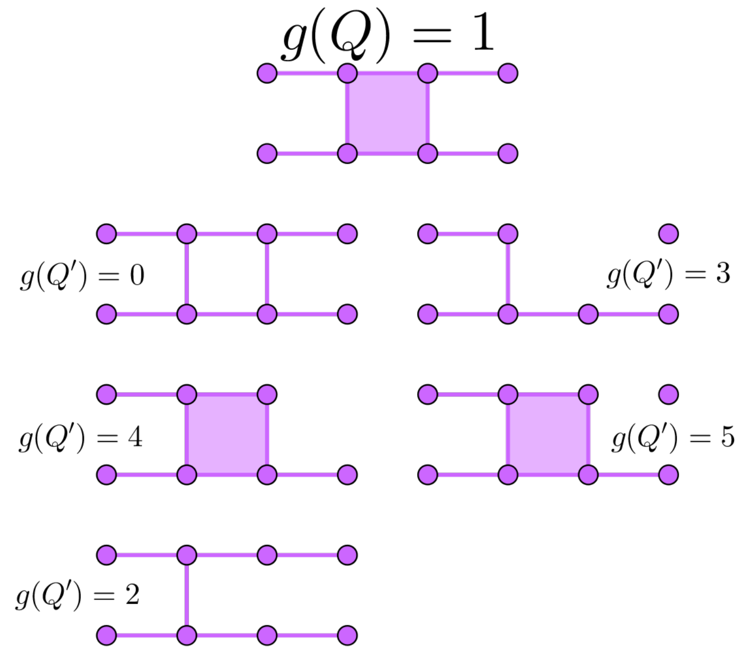

Figure 3.26: A quadrilateral with a tail connected to every vertex and all the moves that can be made from that position in a game of Type 2 Take-Away. The nim-value of each position is also included.

3.5 Quadrilaterals with Four Tails

After all of the interesting behavior we see for an even polygon with three tails, it is somewhat surprising to find that it is possible for the nim-values to settle back down and become easily describable again. When examining the case of a quadrilateral with 4 tails of any (non-zero) length, we find that the nim-values are very well-behaved.

Theorem 3.11. *For a quadrilateral with a tail of nonzero length connected to every* *vertex, Q, $g(Q)$ is determined by the total number of vertices. Specifically,*

- *If $|V(Q)|$ is even then $g(Q) = 1$*
- *If $|V(Q)|$ is odd then $g(Q) = 2$.*

Proof. Let Q be a quadrilateral with a nonempty tail connected to every vertex. The proof is by induction on n, the number of vertices of Q.

Base case: Let Q be a quadrilateral with a tail of length 1 connected to every vertex as in the top of Figure 3.26. In that figure we see every position that can be reached from Q in one move of the game of Type 2 Take-Away. Based on this analysis and the definition of nim-value, we find that $g(Q) = 1$.

Now, consider all the possible positions that could be achieved after one move in the game of Type 2 Take-Away:

- Case 1: You could remove the polygonal face and move to a position that is a bipartite graph with an equal number of vertices and edges; this position will have either nim-value 0 or 3 depending on whether the number of vertices and edges are both even or both odd, respectively.
- Case 2: You could remove a vertex adjacent to the quadrilateral, which will also remove the quadrilateral and 3 total edges. This will reduce the number of vertices by 1 and the number of edges by 3, so the total number of vertices and edges will still have the same parity, either both even or both odd. This position, then, would have nim-value 3 or 0, depending on whether Q initially

had an even number of both vertices and edges or an odd number of both vertices and edges, respectively.

- Case 3: Removing an end vertex from one of the tails will move to a position which looks like a smaller quadrilateral with 4 tails, and will have one less vertex than Q and one less edge than Q . Inductively, the nim-value of this position will be 2 if Q initially had an even number of both vertices and edges and 1 if Q initially had an odd number of both vertices and edges.

- Case 4: Removing an edge from the quadrilateral, which will also remove the quadrilateral, results in a position which is a tree. Trees are bipartite graphs, of course, so the nim-value of this position will be 2 if Q initially had an even number of vertices and an even number of edges and the nim-value of the position will be 1 if Q initially had an odd number of vertices and an odd number of edges.

- Case 5: A single vertex (and all incident edges) or a single edge can be removed from any tail.

We now see that if Q initially has an even number of vertices and an even number of edges that we can move to positions with nim-values 0, and 2 or higher. If Q initially has an odd number of vertices and an odd number of edges then we can move to positions with nim-values 0, 1, and 3 or higher. The remainder of the proof is to show that removing a vertex (and incident edges) or an edge from a tail will not result in a nim-value of 1 for a Q with an even number of vertices and will not result in a nim-value of 2 for a Q with an odd number of vertices. Removal of an edge from a tail or removal of internal vertices and their incident edges from a tail will result in a position which is no longer connected, so to find the nim-value of the position we will have to rely on nim-sum. One of the components will be a tree, which we know will have nim-value 1 or 2. The other component will either be a smaller Q, or it will be a quadrilateral with 3 tails attached. If the second component is a smaller structure like Q, then it has nim-value 1 or 2. Looking at all cases of nim-sum with these values

$$1 \oplus 1 = 0$$

$$1 \oplus 2 = 3$$

$$2 \oplus 1 = 3$$

$$2 \oplus 2 = 0$$

we can see that it is impossible to get a nim-value of 1 or 2. If the second component is a quadrilateral with 3 tails, we can see from the tables in Figure 3.7 through Figure 3.24 that the nim-values of that position are 4 or higher, and using nim-sum with these values and either 1 or 2 will not result in a position with nim-value 1 or 2; in fact, all the nim-sums will be 4 or greater. Since none of these result in a position with nim-value 1 or nim-value 2, we conclude that if Q has an even number of vertices (and hence an even number of edges), then $g(Q) = 1$ and if Q has an odd number of vertices (and hence an odd number of edges), then $g(Q) = 2$. □

Looking carefully at the proof we see that it will *not* hold for any even polygon with 4 tails; it is critical that each vertex of the polygon has an attached tail. Recall from Proposition 3.2 that if an even polygon has at least one tail and at least one vertex without an attached tail then it is possible to move to positions with nim-value 0, 1, 2, and 3. One of those values, either the 1 or the 2, depending on the number of verticies of the original structure, was attained by removing a vertex with no tail from the polygon and taking with it the two incident edges. Without these 'tail-free' vertices, we were unable to guarantee moving to a position with nim-value 1 or 2.

3.6 Even Polygons with Tails at Every Vertex

Another careful look at the proof of Theorem 3.11 invites the reader to wonder if it works for even polygons larger than the quadrilateral with a tail at each vertex of the polygon. We suspect that this may hold for any even polygon with a tail at every vertex, but what further work needs to be done to verify this? The proof of Theorem 3.11 requires we know information about an even polygon with three tails, because in this case it is possible to disconnect a tail and arrive at this position. If we are playing Take-Away on a hexagon with six tails, however, it is not possible to make one move and arrive at a position with a hexagon with three tails and some number of disconnected trees. Instead, to have a proof similar to that of Theorem 3.11, we would assume we need to know some information about the nim-value behavior of an even polygon with 5 tails. Similarly, to make statements about an octagon with 8 tails we need information about the nim-values of an even polygon with 7 tails, and so forth.

As it turns out, however, we really do not need to fully describe these structures to be able to prove that any even polygon with a tail at every vertex will have nim-value

1 or 2. All we need is Definition 1.2 and Proposition 3.2.

Theorem 3.12. *For any even polygon, P, with n vertices and a tail of nonzero length connected to each vertex, $g(P)$ is determined by the total number of vertices. Specifically,*

- *If $|V(P)|$ is even then $g(P) = 1$*
- *If $|V(P)|$ is odd then $g(P) = 2$.*

Proof. We refer to the proof of Theorem 3.11. The five cases listed in that proof are the same five options for any even polygon with a vertex at every tail, but case 5 specifically needs to be addressed as it was the one that relied on the information about a quadrilateral with 3 tails.

Case 5: A single vertex (and all incident edges) or a single edge can be removed from any tail. This will result in one of two positions.

- The resulting position could be in two components: a polygon with tails at every vertex and fewer overall vertices and a tree. Inductively, the nim-value of the polygon with tails is either a 1 or a 2, and so is the nim-value of the tree. Therefore the nim-sum will either be a 0 or a 3.
- The resulting position could now be a polygon with $n - 1$ tails and (possibly) a tree; we will call this position P'. Proposition 3.2 guarantees that we could reach in one move from P' positions with nim-values 0, 1, 2, and 3, and therefore by Definition 1.2 we know that $g(P')$ is greater than or equal to 4. A tree has nim-value 1 or 2. Therefore the nim-sum will always be at least 4.

Since neither of these result in a position with nim-value 1 or nim-value 2, and the results from cases 1 through 4 of Lemma 3.11 all hold, we conclude that if P has an even number of vertices (and hence an even number of edges), then $g(P) = 1$ and if P has an odd number of vertices (and hence an odd number of edges), then $g(P) = 2$. □

We close this chapter, having proved results about some interesting patterns of nim-values, by recognizing that the pattern

$$4, 6, 4, 8, 10, 8, 12, 14, 12, \ldots$$

that we have seen variations of throughout the chapter has been previously seen elsewhere in a different context. Riehemann [7] observed this same pattern in his partial analysis of Take-Away on graphs containing one odd cycle. In particular, this pattern and slight variations of it are the nim-values for graphs which have a 3-cycle with a single edge attached to one vertex and two tails attached to the vertex of that edge that is not in the 3-cycle. This suggests there might be connections to be made between the graphs with odd cycles in the game of Take-Away and the polygonal complexes studied in this chapter in Type 2 Take-Away.

Chapter 4 Take-As-Much-As-You-Want

4.1 Game Play

In the standard game of Take-Away on a graph, a player may choose to take a single edge or to take a vertex and all the edges adjacent to that vertex. In this chapter we will explore a new game on graphs which we will call Take-As-Much-As-You-Want. The game play is in the name; when it is her turn, a player can take as much as she wants—that is, as many edges and as many vertices with their adjacent edges as she wants—from any one component of the graph, and the player who removes the last part of the graph is the winner. If the graph is connected, the possible positions that you can move to at any stage of the game are all the subgraphs of the current position. It is not difficult to see that if your graph has only one component that this game is just as interesting as a game of single heap Nim—if you are lucky enough to go first you take all the vertices and edges and you are the winner. The game becomes more interesting, of course, when the graph has two or more components. Also interesting is that the nim-values for the individual components have a describable pattern.

4.2 Take-As-Much-As-You-Want on a Graph

Before we get to the result about Take-As-Much-As-You-Want on a graph, we first prove a lemma that will be helpful in verifying the nim-values of graphs under this game.

Lemma 4.1. *Let G be a nonempty connected graph with no loops and let $G^* = G_1 \cup \cdots \cup G_k$ be a nonempty disconnected graph with k components, $G_1, ..., G_k$, obtained from G by removing vertices and/or edges of G and such that not all G_i are isolated vertices. If $|V(G_i)| = 1$, let $h(G_i) = 1$. Otherwise, let $h(G_i) = |V(G_i)| + |E(G_i)| - 1$. Then $\sum_{i=1}^{k} h(G_i) < |V(G)| + |E(G)| - 1$.*

Proof. Let G and G^* be as described. Then each G_i will be a connected graph. Without loss of generality, we will arrange the components so that G_1 through G_j are the connected graphs on at least 2 vertices, and G_{j+1} through G_k are the isolated vertices; for convenience, we will say there are m total isolated vertices. Let a_i be the number of vertices in G_i and b_i be the number of edges in G_i.

If there are m isolated vertices, since $j \geq 1$, then at least m edges must be removed from G. Therefore $b_1 + \cdots + b_j + m \leq |E(G)|$. We also know that $a_1 + \cdots + a_j + m \leq |V(G)|$.

Now for each G_i, $1 \leq i \leq j$, $h(G_i) = |V(G_i)| + |E(G_i)| - 1 = a_i + b_i - 1$. Thus $\sum_{i=1}^{k} h(G_i) = (a_1+b_1-1)+\cdots+(a_j+b_j-1)+m = (a_1+\cdots+a_j)+(b_1+\cdots+b_j)-j+m = (a_1 + \cdots + a_j + m) + (b_1 + \cdots + b_j + m) - j - m \leq |V(G)| + |E(G)| - (j + m) \leq |V(G)| + |E(G)| - 1$. We note that we only have equality in the last statement when $j = 1$ and $m = 0$, but this case does not occur because G^* is defined to be disconnected. □

Note that the addition used in Lemma 4.1 is ordinary integer addition. We refer to the definition of nim addition, Definition 1.3, to see that $a \oplus b \leq a + b$ for all $a, b \in \mathbb{Z}_+$. This fact will be important in the proof of Theorem 4.2.

Theorem 4.2. *Let G be a connected graph with no loops. In the game Take-As-Much-As-You-Want,*

- *if* $|V(G)| = 0$, $g(G) = 0$
- *if* $|V(G)| = 1$, $g(G) = 1$
- *if* $|V(G)| \geq 2$, $g(G) = |V(G)| + |E(G)| - 1$.

Proof. The result is clear for $|V(G)| = 0$ and $|V(G)| = 1$, so we focus on the case where $|V(G)| \geq 2$.

Let G be a connected graph with $|V(G)| \geq 2$. The proof will be by induction on $n = |V(G)| + |E(G)|$.

Base case: Let G be the graph that is a single edge on 2 vertices. The possible moves are:

- You could remove the entire graph. This would be a move to the empty graph, which has nim-value 0.
- You could remove a single vertex. This would be to a position G' that is a single vertex, so $g(G') = 1$.
- You could remove the edge. This would be to a position G' which has two disconnected vertices, and using nim-sum we know that $g(G') = 0$.

The first non-negative integer that cannot be reached by a single move from G is 2; therefore $g(G) = 2$. Note that $|V(G)| = 2$, $|E(G)| = 1$, and $|V(G)| + |E(G)| - 1 = 2 + 1 - 1 = 2$.

Assume that the theorem is true for every connected graph with no loops, G, such that $|V(G)| + |E(G)| = k$, $3 \leq k \leq n-1$. Now let H be a graph with $|V(H)| + |E(H)| = n$. We have two cases to consider.

- *Case 1:* Suppose H contains at least one cycle. If H contains a cycle it is not minimally connected and therefore contains an edge, e, that is not a cut-edge (see Theorem 1.9). If we remove e the result is a position H' with $|V(H')| + |E(H')| = n-1$, hence $g(H') = |V(H')| + |E(H')| - 1 = (n-1) - 1 = n-2$. That means that for any $j \in \{0, 1, 2, \ldots, n-3\}$ we are able to move from H' to a position H'' with $g(H'') = j$. But, since a move to any of these H''s was possible directly from H, we conclude that for any $j \in \{0, 1, 2, \ldots, n-2\}$ we are able to move from H to a position H^* with $g(H^*) = j$. Hence $g(H) \geq n-1$.

- *Case 2:* Suppose H is a tree. Then H must have a vertex, v, of degree 1. Let e be the edge of H which is incident to v. Removing e results in a position H' that is a smaller tree on at least two vertices and a disconnected vertex. This smaller tree, H', has $|V(H')| + |E(H')| = n-2$, so by induction $g(H') = n-3$. Now, since H is a tree, $|V(H)| = |E(H)| + 1$. Thus $|V(H)| + |E(H)| = 2|E(H)| + 1$, and we conclude that n is odd. Then $g(H') = n-3$ is even, so $n-3 \oplus 1$ will be equal to the ordinary sum $n-2$. So, for any $j \in \{0, 1, 2, \ldots, n-2\}$ we are able to move from H to a position H^* with $g(H^*) = j$. Hence $g(H) \geq n-1$.

It remains to show, for each case, that we cannot move to a position H' with $g(H') = n-1$. It is clear that we cannot make a move to a connected subgraph H' with $g(H') = n-1$ by the inductive hypothesis and by the fact that if H' is an isolated vertex then $g(H') = 1$. Therefore we must investigate the possibilities where the move to H' disconnects H.

Let $H' = H_1 \cup \cdots \cup H_k$ where $H_1, \ldots, H_k$ are the components of H'. If each H_i is an isolated vertex, $\mathrm{g}(H_1) \oplus \cdots \oplus \mathrm{g}(H_k)$ equals 0 if k is even and 1 if k is odd.

Suppose that not all H_i are isolated vertices. By Lemma 4.1, $\mathrm{g}(H_1) \oplus \cdots \oplus \mathrm{g}(H_k) \leq \sum_{i=1}^{k} \mathrm{g}(H_i) = \sum_{i=1}^{k} h(H_i) < |V(H)| + |E(H)| - 1 = n - 1$.

Thus, we conclude that it is impossible to move from H to a position with nim-value $n-1$, and since we can move to positions with nim-values 0 through $n-2$, $g(H) = n - 1$.

□

Graphs with loops do not necessarily have the same results. Take, for example. a single vertex with a single loop attached. We could move to the empty position with nim-value 0, or to the position with just the vertex which has nim-value 1. These are the only moves that could be made, and therefore the nim-value of the single vertex with a loop is 2. The equation in Theorem 4.2 would predict the nim-value to be $1 + 1 - 1 = 1$.

4.3 Take-As-Much-As-You-Want on a Polytopal Complex

Extending the Take-As-Much-As-You-Want game to general hypergraphs presents some interesting challenges. One challenge in particular is appropriately defining a component of a hypergraph.

We can, however, extend the notion of the Take-As-Much-As-You-Want game to polytopal complexes. Let C be a polytopal complex. Recall that we will define C to be *connected* if its underlying graph is connected as in the definition in Chapter 1. The vertices of the underlying graph are the f_0 0-faces of C and the edges are the f_1 1-faces of C. As with graphs, if C is not connected we will refer to its connected components.

In this game, when it is her turn to move a player can take any nonempty collection of nonempty faces from any component of the polytopal complex under the condition that the resulting position is also a polytopal complex. It is important to note that $\emptyset \in C$ but it is not a removable polytope for the purpose of the game.

Lemma 4.3. *Let P be a nonempty connected polytopal complex of dimension d and let $P^* = P_1 \cup \cdots \cup P_k$ be a nonempty disconnected polytopal complex with k components, $P_1, ..., P_k$, obtained from P by removing polytopes from P and such that not all P_i are isolated vertices. If $f_0(P_i) = 1$, let $h(P_i) = 1$. Otherwise, let $h(P_i) = f_0(P_i) + f_1(P_i) + \cdots + f_d(P_i) - 1$. Then $\sum_{i=1}^{k} h(P_i) < f_0(P) + \cdots + f_d(P) - 1$.*

Proof. Let P and P^* be as described. Then each P_i will be a polytopal complex. Without loss of generality, we will arrange the complexes so that P_1 through P_j are the polytopal complexes with at least 2 vertices, and P_{j+1} through P_k are the isolated vertices; for convenience, we will say there are m total isolated vertices.

If there are m isolated vertices, since $j \geq 1$, then at least m faces of dimension 1 must be removed from P. Therefore $f_1(P_1) + \cdots + f_1(P_j) + m \leq f_1(P)$. We also know that $f_0(P_1) + \cdots + f_0(P_j) + m \leq f_0(P)$, and $f_i(P_1) + \cdots + f_i(P_j) \leq f_i(P)$ for all $i \geq 2$.

Now for each P_i, $1 \leq i \leq j$, $h(P_i) = f_0(P_i) + f_1(P_i) + \cdots + f_d(P_i) - 1$. Thus,

$$\begin{aligned}
\sum_{i=1}^{k} h(P_i) &= (f_0(P_1) + \cdots + f_d(P_1) - 1) + \cdots + (f_0(P_j) + \cdots + f_d(P_j) - 1) + m \\
&= (f_0(P_1) + \cdots + f_0(P_j)) + (f_1(P_1) + \cdots + f_1(P_j)) + \cdots \\
&\quad +(f_d(P_1) + \cdots + f_d(P_j)) - j + m \\
&= (f_0(P_1) + \cdots + f_0(P_j) + m) + (f_1(P_1) + \cdots + f_1(P_j) + m) + \cdots \\
&\quad +(f_d(P_1) + \cdots + f_d(P_j)) - j - m \\
&\leq f_0(P) + f_1(P) + \cdots + f_d(P) - (j + m) \\
&\leq f_0(P) + f_1(P) + \cdots + f_d(P) - 1.
\end{aligned}$$

We note that we only have equality in the last statement when $j = 1$ and $m = 0$, but this case does not occur because P^* is defined to be disconnected. □

Theorem 4.4. *Let K be a connected polytopal complex of dimension $d \geq 0$. In the game of Take-As-Much-As-You-Want,*

- *if $d = 0$ then $g(K) = f_0 \pmod 2$, and*
- *if $d > 0$ then $g(K) = f_0 + \cdots + f_d - 1$.*

Proof. Let K be a connected polytopal complex of dimension d. The proof is by induction on n, where $n = f_0 + \cdots + f_d$.

Base Case: Let K be the polytopal complex that is a single edge on 2 vertices. The possible moves are:

- You could remove the entire polytopal complex. This would be a move to the empty graph, which has nim-value 0.
- You could remove a single vertex. This would be to a position K' that is a single vertex, so $g(K') = 1$.
- You could remove the edge. This would be to a position K' which has two disconnected vertices, and using nim-sum we know that $g(K') = 0$.

The first non-negative integer that cannot be reached by a single move from K is 2; therefore $g(K) = 2$. Note that $f_0 = 2$, $f_1 = 1$, and $f_0 + f_1 - 1 = 2 + 1 - 1 = 2$.

Assume that the theorem is true for every polytopal complex K with dimension d such that $f_0 + \cdots + f_d = \ell$, $3 \leq \ell \leq n - 1$. Now let M be a complex with $f_0 + \cdots + f_d = n$. We have two cases to consider.

- *Case 1:* Suppose M is of dimension 1. Then M is a graph, and $g(M) = f_0 + f_1 - 1$, as in Theorem 4.2.

- *Case 2:* Suppose M has at least one face of dimension 2 or higher. Then M must have a face, F, of maximal dimension. Removing F results in a position M' that is a polytopal complex with $f_0 + \cdots + f_d = m - 1$. By induction, this smaller polytopal complex has $g(M') = f_0 + \cdots + f_d - 1 = (n - 1) - 1 = n - 2$. Since every reachable position by one move made from M' could have been reached in one move from M, we conclude $g(M) \geq n - 1$.

It remains to show, for Case 2, that we cannot move to a position M' with $g(M') = n - 1$. It is clear that we cannot make a move to a connected polytopal complex M' with $g(M') = n-1$ by the inductive hypothesis and by the fact that if M' is an isolated vertex then $g(M') = 1$. Therefore we must investigate the possibilities where the move to M' disconnects M.

Let $M' = M_1 \cup \cdots \cup M_k$ where M_1,. . .,M_k are the connected components of M'. If each M_i is an isolated vertex, $\mathrm{g}(M_1) \oplus \cdots \oplus \mathrm{g}(M_k)$ equals 0 if k is even and 1 if k is odd.

Suppose not all M_i are isolated vertices. By Lemma 4.3, $\mathrm{g}(M_1) \oplus \cdots \oplus \mathrm{g}(M_k) \leq \sum_{i=1}^{k} \mathrm{g}(M_i) = \sum_{i=1}^{k} h(M_i) < f_0 + \cdots + f_n - 1$.

Thus, we conclude that it is impossible to move from M to a position with nim-value $f_0 + \cdots + f_d - 1$, and since we can move to positions with nim-values 0 through $f_0 + \cdots + f_d - 2$, $g(M) = f_0 + \cdots + f_d - 1$. □

As with the graph version of Take-As-Much-As-You-Want, playing the game on a single connected component is very uninteresting as player A will win every time by taking the entire polytopal complex. This result does, however, provide the necessary insight into the strategy for playing the game on multiple connected components, which is a much more interesting game.

Chapter 5 Poset Games

The last stop on our journey through Take-Away on discrete structures brings us to posets. The basic definitions that we need for this Chapter were given in Chapter 1.

The idea of Take-Away on a poset is not a new concept. In fact, several of the well-known impartial games can be viewed as versions of Take-Away on a poset. If the poset, P, is a finite totally ordered set—the Hasse diagram of P would just be a chain—then the game of Take-Away is actually just the standard game of one-heap Nim. Taking the product of two finite total orders of lengths k and ℓ, respectively, produces a poset isomorphic to the divisor lattice of an integer $n = a^k b^\ell$ where a and b are prime. Removing the bottom element of this poset and playing Take-Away is equivalent to the game Chomp (see Chapter 1). In a sense, Take-Away on a poset without a unique bottom element can be viewed as a generalization of Chomp.

All of our previously examined Take-Away games can be viewed as poset Take-Away games.

- A hypergraph, H, from Chapter 2 can be viewed as a poset, P, where the vertices of H are the rank 0 elements of P and the hyperedges of H are the rank 1 elements of P. A rank 1 element, u, covers a rank 0 element, v, if v is a vertex of hyperedge u in the hypergraph. The rule for Type 1 Take-Away on these posets is that when it is her turn a player may take at most one element out of any single rank of P and if a rank 0 element is removed then any rank 1 elements covering it will also be removed.

- A hypergraph, H, can also be viewed as a poset, P, where the vertices of H are the rank 0 elements of P and the hyperedges are ordered by set inclusion. In this case, P might not be a ranked poset. The rule for Type 2 Take-Away on these posets is that when it is her turn a player may take any element of the poset, and in doing so she will also remove any element that is above it in the poset.

- A polygonal complex, Q, from Chapter 3 can be viewed as a poset, P, where the vertices of Q are the rank 0 elements of P, the edges of Q are the rank 1 elements of P, and the polygons in Q are the rank 2 elements of P. A rank 1 element, e, covers a rank 0 element, v, if v is a vertex of edge u in the polygonal complex, and a rank 2 element p covers a rank 1 element e if e is an edge of the polygon p. The rule for Type 2 Take-Away on these posets is that when it is her turn a player may take at most one element out of any single rank of P, and she will take with it any element above it in the associated poset.

- A graph, G, from the Take-As-Much-As-You-Want game in Chapter 4 can be viewed as a poset, P, where the vertices of G are the rank 0 elements of P and the edges of G are the rank 1 elements of P. A rank 1 element, u, covers a rank 0 element, v, if v is a vertex of edge u in the graph. The rule for Take-As-Much-As-You-Want on the poset is when it is her turn a player may take as many elements as she desires from either rank of the poset, and any rank 0 element that is removed takes with it any rank 1 elements covering it. There is a natural extension of this to Take-As-Much-As-You-Want on polytopal complexes.

In this chapter we will look at results about Take-Away on a very specific type of poset which we will call a *rank-complete* poset. We will first look at the standard play of Take-Away on such a structure and fully describe the nim-values of positions in the game. From there, we look at the game from a slightly different perspective, namely that of *misère play.* Under misère play, the player that makes the last move in the game is now the player that loses the game. We will fully describe nim-values for rank-complete posets under misère play, and wrap up the chapter with further results about misère play and rank-complete posets, namely finding nim-values for the sum of rank-complete posets under misère Take-Away.

5.1 Take-Away on a Rank-Complete Poset

A *rank-complete poset* is a finite graded poset where every element in each rank n is related to every element of rank $n-1$. An example is shown in Figure 5.1. When playing Take-Away on a rank-complete poset, a player may take any single element of her choice out of any single rank of the poset, including rank 0. When an element is removed, any elements of the poset related to it in a higher rank will also be removed. Thus if a player takes an element with rank k, *all* elements of rank $j \geq k+1$ are removed as well. Some examples of positions that can be reached in one move from the poset in Figure 5.1 are in Figure 5.2.

The nim-values of rank-complete posets are very easily described based on the

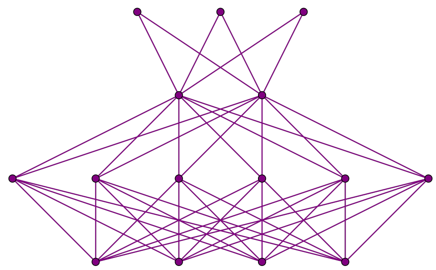

Figure 5.1: The Hasse diagram of a rank-complete poset.

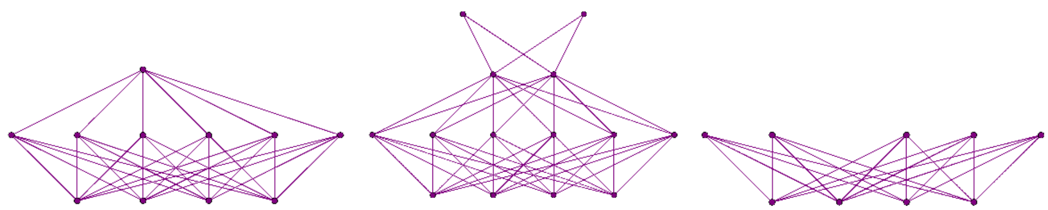

Figure 5.2: Possible positions after one move in Take-Away played on Figure 5.1

cardinality of each rank in the poset.

Theorem 5.1. *Let P be a finite rank-complete poset. In Take-Away, $g(P) = m + 1$, where m is the highest rank of odd cardinality in P. If every rank of P has even cardinality, then $g(P) = 0$.*

Proof. Let P be a finite rank-complete poset. The proof is by induction on the number of elements in P.

Base Case: Clearly a poset with no elements has a nim-value of 0. A poset with a single element in rank 0 has a nim-value of 1.

Suppose that every rank-complete poset R with k elements, $0 \leq k \leq n$, has $g(R) = m+1$, where m is the highest rank of the poset with odd cardinality. Consider a poset, P, with $n + 1$ elements.

- If every rank of P has even cardinality, any move made will be to a subposet of P, P', where the highest rank has odd cardinality and the ranks below remain unchanged and have even cardinality. Since P' has fewer elements, we have by induction that $g(P') = m' + 1$ where m' is the lone rank of odd cardinality. Hence we cannot move from P to a P' with nim-value 0, so we conclude that $g(P) = 0$.

- Assume P has at least one rank with odd cardinality. Let m be the highest rank with odd cardinality in P. This means any rank above m in P will have even cardinality. A move in the game made in one of these ranks to a position P' will change the rank from even cardinality to odd cardinality, and since P' will have fewer elements than P, by induction $g(P') > m + 1$.

  We can also move to positions P' which have nim-values 0 through m. Suppose you want to move to a P' with $g(P') = j$, $1 \leq j \leq m$. This can be done in one of two ways:

  - If rank $j-1$ has even cardinality, make a move in this rank. The resulting position will have rank $j-1$ as the highest rank with odd cardinality, and by induction $g(P') = j$.

  - If rank $j-1$ has odd cardinality in P, make a move in the next rank of odd cardinality *above* rank $j-1$. Doing so will make rank $j-1$ the highest rank with odd cardinality in P', and by induction $g(P') = j$.

  Finally, to move to a position P' with $g(P') = 0$, make a move in the *lowest* rank with odd cardinality. This will produce a position where each rank has even cardinality, and by induction $g(P') = 0$.

We are, however, unable to move to a P' with $g(P') = m + 1$. Since rank m of P is of odd cardinality, making a move in this rank will change the cardinality to even, and therefore the resulting position will have a nim-value less than $m + 1$. Finally, any move made in a lower odd cardinality rank than m will have $g(P') < m + 1$. Therefore $g(P) = m + 1$.

□

We could view this game in a way closer to the standard version of one-heap Nim. Instead of visualizing a Hasse diagram for playing this version of Take-Away, consider having a heap of tokens of different colors, with all the tokens of the same color being adjacent in the heap. For example, we could have some number of purple tokens with a set of blue tokens stacked on top of them and finally a set of green tokens on top. When it is her turn, a player could remove the top green token, the top blue token along with all of the green tokens, or the top purple token along with all of the blue and green tokens. As usual in Nim, the winner would be the player to remove the final token from the heap.

Under either visualization—posets or heaps of tokens—one could easily evaluate nim-values of the sum of these games. What is interesting about the sum of games here, particularly when viewed in the token/heap sense, is that the structures which have the same nim-value do not necessarily have to look similar at all. Since the nim-value of the poset is determined by the highest rank of odd cardinality, all that

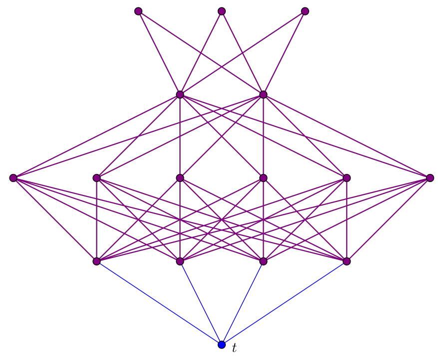

Figure 5.3: The Hasse diagram of a rank-complete poset with an extra element, t, appended beneath the rank 0 elements.

matters is that rank has an odd number of elements and any rank above it has an even number of elements. The actual number of elements in that rank could be just 1 or perhaps 101, yet the two posets would have the same nim-value.

5.2 Misère Take-Away on a Rank-Complete Poset

Now, consider playing the same game, but instead of the winner being the player to remove the last element from the poset we will say that the winner is the player that forces her opponent to remove the last element from the poset. As previously mentioned, this version of game play is referred to as misère play.

In this section we will fully describe the nim-values of rank-complete posets under misère play. Much like misère play on one-heap Nim, the nim-values of many positions do not change, and the positions that do change are the ones that have normal nim-values of 0 or 1 [5]. For posets, we can study misère play in the same way we study normal play by appending an extra element, t (referred to in most poset literature as $\hat{0}$), covered by the rank 0 elements of P—we can then, in a sense, regard t as an element with rank -1. An example is shown in Figure 5.3. We then have the added condition that t cannot be removed until all other elements of the poset have been removed; think of t as a buried treasure beneath the poset which we cannot claim until it has been completely exposed via removing all elements of P. Let $P_t = P \cup \{t\}$. Misère play on P is equivalent to normal play on P_t under the specified condition for t. We will denote the misère nim-value of a poset P as $g_m(P)$. Then $g_m(P) = g(P_t)$.

Theorem 5.2. *Let P be a rank-complete poset. When playing misère Take-Away on P,*

- *if the only rank with odd cardinality is rank 0, then $g_m(P) = 0$*
- *if every rank of the poset has even cardinality, then $g_m(P) = 1$, and*
- *otherwise $g_m(P) = g(P) = r + 1$ where r is the highest rank of the poset that has odd cardinality.*

Proof. The proof is by induction on the number of elements in the poset.

Base Cases: Under misère play, $\emptyset$, the empty poset, is a losing position for the first player; hence this position has $g_m(\emptyset) = g(\emptyset_t) = 1$. Let P be the poset that consists of only one element. Then the first player can remove the element, leaving an empty poset, which as we already discovered has nim-value 1; hence $g_m(P) = g(P_t) = 0$. Let P be the poset which consists of two elements of rank 0 and no other elements. Our only move is to the poset with just one element, which has nim-value 0; hence $g_m(P) = g(P_t) = 1$. Lastly, let P be the poset with one element in rank 0 and one element in rank 1 which covers the rank 0 element. Our only moves are to the empty poset (with nim-value 1) or to the poset with only one element (with nim-value 0); hence $g_m(P)$ =$g(P_t) = 2$.

Suppose that in misère Take-Away the theorem is true for every rank-complete poset P with k elements, $0 \leq k \leq n$, $n \geq 2$. Consider a poset, P, with $n+1$ elements and let P_t be P with t located below the rank 0 elements.

- If the only rank with odd cardinality in P is rank 0, any move made on P_t in rank j above rank 0 will be to a position P_t' with a rank of odd cardinality above rank 0; by induction, $g_m(P')$ =$g(P_t') = j+1 > 1$. A move made on P_t in rank 0 will be to a position P_t' with an even number of elements in rank 0 and no other elements in the poset; by induction, $g_m(P')$ =$g(P_t') = 1$. Since no other moves can be made, we conclude $g_m(P)$ =$g(P_t') = 0$.

- If every rank of P has even cardinality, any move made on P_t in rank j above rank 0 will be to a position P_t' with rank j of odd cardinality; by induction, $g_m(P')$ =$g(P_t') = j+1 > 1$. A move made on P_t in rank 0 will be to a position P_t' with an odd number of elements in rank 0 and no other elements in the poset; by induction, $g_m(P')$ =$g(P_t') = 0$. Since no other moves can be made, we conclude that the $g_m(P)$ =$g(P_t) = 1$.

- Assume P_t has at least one rank above rank 0 with odd cardinality. Let r be the highest rank with odd cardinality in P. This means any rank above r in P_t will have even cardinality. A move in the game made in one of these ranks will change that rank from even cardinality to odd cardinality, and since that position will have fewer elements than P_t, by induction that position will have a nim-value larger than $r+1$. Thus $g_m(P)$ =$g(P_t) > r+1$.

We can also move to positions P_t' which have nim-values 2 through r. Suppose you want to move to a P_t' with nim-value j, $2 \leq j \leq r$. This can be done in one of two ways:

– If rank $j-1$ has even cardinality, make a move in this rank. The resulting position P' will have rank $j-1$ as the highest rank with odd cardinality; by induction $g_m(P')$ =$g(P_t') = j$.

– If rank $j-1$ has odd cardinality in P_t, make a move in the next rank *above* rank $j-1$ which has odd cardinality. Doing so will make rank $j-1$ the highest rank with odd cardinality in P_t', and by induction $g_m(P')$ =$g(P_t') = j$.

To move to a position P_t' with $g_m(P')$ =$g(P_t') = 0$, we do one of two options:

– If rank 0 has even cardinality, make a move in this rank. The resulting position P' will have rank 0 as the highest rank with odd cardinality; by induction $g_m(P')$ =$g(P_t') = 0$.

– If rank 0 has odd cardinality in P_t, make a move in the next rank *above* rank 0 that has odd cardinality. Doing so will make rank 0 the highest rank with odd cardinality in P_t', and by induction $g_m(P')$ =$g(P_t') = 0$.

Finally, to move to a position P_t' with $g_m(P')$ =$g(P_t') = 1$, make a move in the *lowest* rank with odd cardinality. This will produce a position where each rank has even cardinality, and by induction $g_m(P') = g(P'_t) = 1$.

We are, however, unable to move to a P' with $g_m(P') = g(P'_t) = r+1$. Suppose that in P (and hence in P_t) there is a rank s, $0 < s < r$, with odd cardinality. Removing an element of rank s results in a position P'_t with an even cardinality at rank s and a fewer number of elements than P_t; hence, by induction, $g(P'_t) \leq s < r+1$. Since rank r of P is of odd cardinality, making a move in this rank will change the cardinality to even, and therefore the resulting position will have a nim-value less than $r+1$. Since any rank that may exist in P above rank r has even cardinality, making a move in any of these ranks will produce a position with nim-value greater than $r+1$. Therefore the $g_m(P) = g(P'_t) = r+1$.

□

While either the standard version of Take-Away on a rank-complete poset or the misère version are somewhat more interesting than regular one-heap Nim, at least to the casual player, we see from the nim-values that the optimal play on a rank-complete poset will make the game quite uninteresting, particularly in the misère case. Since the positions with nim-value 0 are very specific—all ranks having even cardinality for regular play and only rank 0 having odd cardinality for misère play—the actual game play would be very tedious or very quick, but not very interesting. In fact, with these facts about the game discovered there is no real value in knowing the actual nim-values for all positions, just knowing where the zeros are is sufficient for playing perfectly.

A way to make the game more complex—and hence more interesting—is to play on the *direct sum* of posets. The direct sum of posets P and Q, also known as the *disjoint union* of P and Q, is the poset $P+Q$ whose elements are from the set $P \cup Q$ and that has its relation defined as for x, $y \in P+Q$, $x \leq y$ in $P+Q$ if either x, $y \in P$ with $x \leq y$ in P or x, $y \in Q$ with $x \leq y$ in Q [9]. For normal game play—play such that the last player to move is the winner—we know that the nim-value of a sum of game positions is the nim-sum of the nim-values of the individual positions. (See Theorem 1.4). In the next section we will focus on the misère play of this sum.

5.3 Misère Take-Away on the Sum of Rank-Complete Posets

As we have seen and previously discussed, summing games certainly makes the game play more challenging. In this section we provide a result on misère play on a sum of games. We will consider the play of Take-Away on the sum of rank-complete posets, $P = P_1 + \cdots + P_n$, to be as follows: when it is her turn to move, a player may take an element from any rank of one single summand poset, P_i, taking with it all elements above it in that poset. The player that removes the last remaining element(s) from the collection of posets loses the game.

Returning to the game of Nim, we consider misère play where the game has multiple heaps. Let G be a game of k-heap Nim with misère play. The (misère) nim-value is determined as follows: if G contains no heaps of size $n \geq 2$ then $g_m(G) = 0$ if k is odd and $g_m(G) = 1$ if k is even—we note that this is the reverse of what we find under normal play conditions. Otherwise, that is if G contains at least one heap of size 2 or larger, then $g_m(G)$ =$g(G)$ [8].

To study misère games in a more familiar way, we can put the game of misère Take-Away on the sum of rank-complete posets in the context of regular play by appending a bottom element t to $P_1 \oplus \cdots \oplus P_n$ to get P_t, and adding the condition that t cannot be removed until all other elements from all summand posets have been removed. Then, as before, misère play on P is equivalent to normal play on P_t with this condition on t.

For our analysis, there are two particular types of rank-complete posets that we will label as *special*. One of these is a rank-complete poset in which every rank of the poset has even cardinality; we will call this Type 0. The other is a rank-complete poset which has odd cardinality in only the lowest rank of the poset, rank 0; we will call this Type 1. These posets, of course, are the ones that under regular play have nim-values of 0 and 1, respectively (see Theorem 5.1), and under misère play have nim-values of 1 and 0, respectively (see Theorem 5.2). The role of these posets is of particular importance in studying misère play on the sum of posets.

Theorem 5.3. *Let $C = P_1 + \cdots + P_k$ be a collection of k rank-complete posets, $P_1, ..., P_k$. When playing misère Take-Away on C, the nim-value of a position is based on the cardinalities of the ranks of the posets $P_1, ..., P_k$. Specifically,*

- *If each P_i is special, $g_m(C) = g(P_1) \oplus \cdots \oplus g(P_k) \oplus 1$.*
- *Otherwise, $g_m(C) = g(P_1) \oplus \cdots \oplus g(P_k)$.*

Proof. Let C be a collection of rank-complete posets, $P_1, P_2, ..., P_k$. The proof is by induction on the total number of elements in the combined posets.

Base Cases: Under misère play, $\emptyset$, the empty poset, is a losing position for the first player; hence this position has $g_m(\emptyset) = g(\emptyset_t) = 1$. Let P be the poset that consists of only one element. Then the first player can remove the element, leaving an empty poset, which as we already discovered has nim-value 1; hence $g_m(P) = g(P_t) = 0$. Let P be the poset which consists of two posets, P_1 and P_2, each with one element of rank 0 and no other elements. Our only move is to the poset with just one element, which has nim-value 0; hence $g_m(P) = g(P_t) = 1$. Lastly, let P be the poset with one element in rank 0 and one element in rank 1 which covers the rank 0 element. Our only moves are to the empty poset (with nim-value 1) or to the poset with only one element (with nim-value 0); hence $g_m(P) = g(P_t) = 2$.

Without loss of generality, we can arrange the P_i in C so that all of the special posets are together and all of the non-special posets are together. For convenience we will rename the special posets Q_1 through Q_j, and rename the non-special posets R_1 through R_ℓ.

- If the position is made up of only special posets, there are two cases:
  - *Case 1:* A move made in rank 0 of any poset in C will result in a C' which still has all special posets but has a fewer total number of elements. Let s be the number of Type 1 special posets in C'. If s is odd then $g_m(C') = (0 \oplus \cdots \oplus 0 \oplus 1 \oplus \cdots \oplus 1) \oplus 1 = 0$. For s to be odd then C must have an even number of Type 1 special posets because either a Type 1 from C became a Type 0 in C' or a Type 0 from C became a Type 1 in C'. Likewise, if s is even then $g(C') = (0 \oplus \cdots \oplus 0 \oplus 1 \oplus \cdots \oplus 1) \oplus 1 = 1$.

For s to be even then C must have an odd number of Type 1 special posets because either a Type 1 from C became a Type 0 in C' or a Type 0 from C became a Type 1 in C'. Therefore if s is odd then $g_m(C') = 0$ and if s is even then $g_m(C') = 1$.

  - *Case 2:* Remove an element in rank r, $r > 0$, in *any* poset, say Q_1, to reach a poset Q_1'. Then Q_1' has odd cardinality in rank r, so it is a non-special poset. By Theorem 5.1, $g(Q_1') = r + 1 \geq 2$. Now we have a position $C' = Q_1' + Q_2 + \cdots + Q_j$ which has fewer total elements than C, so by induction, $g_m(C') = g(Q_1') \oplus g(Q_2) \oplus \cdots \oplus g(Q_j) = g(Q_1') \oplus 0 \oplus \cdots \oplus 0 \oplus 1 \cdots \oplus 1$, where there are a total of $j - 1$ zeros and ones. Since $g(Q_1') \geq 2$, $g_m(C') \geq 2$.

Thus, if $C = Q_1 + Q_2 + \cdots + Q_j$ with $j \geq 1$, then if s is even $g_m(C) = 0$ and if s is odd $g_m(C) = 1$. Therefore, if each P_i is special, $g_m(C) = g(P_1) \oplus \cdots \oplus g(P_k) \oplus 1$.

- If C has at least one non-special poset:

  - *Case 1:* $\ell \geq 2$

    $C = Q_1 + \cdots + Q_j + R_1 + \cdots + R_\ell$. Making any move in any poset, say R_1, results in a position $C' = Q_1 + \cdots + Q_j + R_1' + \cdots + R_\ell$ that has fewer total elements than C and still has at least one non-special poset. By induction, $g_m(C') = g(C')$. It follows in this case that we obtain the same set of values through one move in the misère game as we would in the normal play game, so $g_m(C)$ is found by adding the regular nim-values of each poset; see Theorem 1.4.

- *Case 2:* $\ell = 1$

  * *Case 2a:* Remove a rank 0 element from R_1 to reach a poset R_1' that is a special poset. Then $C' = Q_1 + \cdots + Q_j + R_1'$. By induction, $g_m(C') = 0$ or 1, depending on the value of j and whether R_1' is Type 0 special or Type 1 special.

  * *Case 2b:* Remove an element from the lowest rank of odd cardinality above rank 0 in R_1 to reach a poset that is a special poset. Then $C' = Q_1 + \cdots + Q_j + R_1'$. By induction, $g_m(C') = 0$ or 1, depending on the value of j and whether R_1' is Type 0 special or Type 1 special. Further, the R_1' from Case 2a and the R_1' in this case are different types of special posets, therefore one will have nim-value 0 and the other have nim-value 1. Thus it is always possible to make move in R_1 that will yield a position with $g_m(C') = 0$ and make a move in R_1 that will yield a position with $g_m(C') = 1$.

  * *Case 2c:* Making any other move in R_1 or any move any poset other than R_1, say Q_1, results in a position C' that has fewer total elements than C and still has at least one non-special poset. By induction, $g_m(C') = g(C')$.

  It follows in this case that we obtain the same set of values through one move in the misère game as we would in the normal play game, so $g_m(C)$ is found by adding the regular nim-values of each poset; see Theorem 1.4.

Thus, if C has at least one non-special poset, $g_m(C) = g(P_1) \oplus \cdots \oplus g(P_k)$.

□

This result is a nice parallel of the relationship between $g(P)$ and $g_m(P)$ for a single rank-complete poset. In the earlier sections of this chapter we saw that the nim-values under regular play and misère play were the same for most rank-complete posets; the only places where the values actually differed were these special posets.

This result is also nice in that it generalizes misère play on multiple nim heaps. If we were to consider a rank-complete poset with only one single element in each rank this would be isomorphic to a nim heap. Playing Take-Away on a sum of these very specific types of rank-complete posets, then, would be the same as playing a game of multi-heap Nim, under both normal play conditions as well as misère play conditions.

With this game generalizing misère Nim, one might assume that the strategy for misère play could be, as Conway describes, "play as in normal play until the game is nearly over, and then make a sensible move" [5], alluding to the fact that misère Nim is played as normal until the heaps are of size 1 and the game is nearly done. However, Grundy showed that in general misère games can be much more complicated than this strategy can allow [5].

Chapter 6 Areas for Further Research

Reflecting on all of the results presented in Chapters 2 through 5, we can see that there are many opportunities for further research. The following questions are particularly interesting.

- In Chapter 2 we saw from the analysis of the nim-values of the truncated tetrahedron that the nim-values are all between 0 and 7, but we have not identified three unique characteristics about the hypergraph that could predict the nim-value in the way that the parity of the number of edges and the parity of the number of vertices did for the oddly uniform hypergraphs and the evenly uniform hypergraphs with a marked coloring. Will these hypergraphs with both odd and even sized hyperedges with a marked coloring always have nim-values between 0 and 7? Are there particular characteristics that can be identified that will quickly predict the nim-values?

- In Chapter 3 there are several places for further study. As the size of the even polygon increases and more tails are appended to the vertices, what patterns can we observe in the nim-values? We saw that for an even polygon with 3 tails the nim-values were very interesting and there seemed to be quite a few patterns all working together, making it difficult to describe them succinctly. We did notice, though, that in the places where the tail lengths seemed significantly far apart and sufficiently large that the nim-values were often 4 or 7. Can we prove that this will always be the case? Do interesting patterns arise when we append more than one tail from a single vertex? What, if anything, is the connection with the pattern found for the nim-values of an even polygon with one tail and the pattern for the graph with a single odd cycle with a particular tree attached to one vertex observed by Riehemann?

- In Chapter 4 we looked at the game of Take-As-Much-As-You-Want. What would this game look like on a general hypergraph?

- In Chapter 5 we looked at Take-Away on a very specific type of poset, followed by looking at misère play on one or more of these posets. From here, questions arise about studying Take-Away on other types of posets and looking for patterns in their nim-values. Also, the investigation into misère play suggests that it might be interesting to take a look at misère play on the other games in this dissertation.

- How would the nim-values change if the rules of any of these games were altered to include limited options for a player to pass, that is not make a move, when it is her turn? Of course we cannot allow a pass at every turn, since this could prevent the game from terminating.

- It happens that in many combinatorial games that at some positions of the game with nonzero nim-value that there are multiple positions within one move of the game that have nim-value 0. Is one of these winning positions a better choice than the others in terms of efficiency, e.g., guaranteeing to end the game sooner? For small games this could be checked using a directed graph with all the possible positions of the game as the vertices and investigating recursively from the final winning positions. This task will quickly get out of hand as the size of the game increases. Is there another way to study these games with efficiency as a factor in the winning strategy?

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