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In Search of a Class of Representatives for *SU*-Cobordism Using the Witten Genus

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In Search of a Class of Representatives for SU-Cobordism Using the Witten Genus

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
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Lexington, Kentucky

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Lexington, Kentucky 2016

ABSTRACT OF DISSERTATION

In Search of a Class of Representatives for SU-Cobordism Using the Witten Genus

In algebraic topology, we work to classify objects. My research aims to build a better understanding of one important notion of classification of differentiable manifolds called cobordism. Cobordism is an equivalence relation, and the equivalence classes in cobordism form a graded ring, with operations disjoint union and Cartesian product. My dissertation studies this graded ring in two ways:

1. by attempting to find preferred class representatives for each class in the ring.
2. by computing the image of the ring under an interesting ring homomorphism called the Witten Genus.

KEYWORDS: Algebraic Topology, Cobordism, Multiplicative Genera, the Witten Genus, Minimal Surfaces

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Date: July 14, 2016

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For Lora and Brandon.

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1 Introduction

1.1 Overview and History

In the beginning, (that is, as far as topology is concerned, so around the turn of the 20th century) Henri Poincaré and some others hoped to classify all topological spaces up to homeomorphism. This problem was quickly found to be intractable, so they loosened the notion of equivalence, and some really beautiful mathematics was introduced. Around a quarter of a century after Poincaré's *Analysis Situs* founded the study of algebraic topology, René Thom introduced a method of classifying manifolds called (co)bordism. This particular notion of equivalence has been important to topologists since its introduction, and is the central object of my research.

Cobordism is a notion of equivalence that says two closed n-dimensional manifolds are the same if together they form the boundary of an $(n+1)$-dimensional manifold. That is, the two manifolds are part of the same whole, and are, therefore, the same object. All the manifolds that are cobordant to one another form a cobordism class. If we take all of the cobordism classes in a particular dimension, say n, we have a cobordism group, which we will denote Ω_n^O, with the addition operation of disjoint union and $[0]$ the class of objects that are themselves boundaries. Taking the direct sum over non-negative dimensions, and with the multiplication operation of cross product, we have a graded ring, denoted Ω_*^O.

Now, as we've described it so far, the ring Ω_*^O, which we can call unoriented cobordism, is a fairly simple object. It's a graded ring such that in each dimension the sum of any two objects in the same class is $[0]$. Thom showed in [38] that Ω_*^O is in fact a polynomial ring over $\mathbb{Z}/2$ with i-dimensional generators x_i $(i \neq 2^k - 1)$. Two years later Albrecht Dold [10] finished describing the manifolds we can take for the x_i's, and this is a very nice result. Just like when you think of fractions, you'd prefer to think of $\frac{1}{2}$ as opposed to, say, $\frac{37}{74}$, when we're thinking about unoriented cobordism classes, we have a preferred class representative, Dold manifolds. The situation becomes a little less straightforward when we require that our manifolds and the cobordisms between them have more structure.

One particularly nice choice is to require that each manifold have a complex structure on its stable tangent bundle. The resultant ring, Ω_*^U, the ring of (stably almost) complex cobordism, is beautiful and well-studied. The ring of complex cobordism is a polynomial ring over $\mathbb{Z}$ with one variable in each even dimension, and Daniel Quillen showed that Ω_*^U is the coefficient ring for Michel Lazard's universal formal group law [21].

Friedrich Hirzebruch asked whether, as with Ω_*^O, Ω_*^U should have a preferred class representative, and in particular, if every class in Ω_*^U should contain a connected non-singular algebraic variety. While the issue of connectivity is still unanswered, John Milnor showed that every class does contain a non-singular algebraic variety. This is a beautiful theorem because it gives us a class of preferred representatives for classes in Ω_*^U.

This dissertation focuses mainly on the the ring of special unitary cobordism, denoted Ω_*^{SU}, which is closely related to the ring of complex cobordism. We will observe the following:

Theorem. *Every element in Ω_2^{SU} can be represented by a non-singular algebraic variety.*

On the other, hand we will see that:

Theorem. *There exist classes in Ω_4^{SU} that are not representable by any non-singular algebraic variety.*

In fact, we will even show that

Theorem. *There exist classes in Ω_4^{SU} that are not representable by any complex, compact surface.*

We may conclude then, that despite what is, as we will see, the very close relationship between Ω_*^U and Ω_*^{SU}, the direct analog of Milnor's theorem for Ω_*^U is not true for Ω_*^{SU}.

Since we may not always choose a non-singular algebraic variety as a class representative in Ω_*^{SU}, a natural line of inquiry is to determine what types of manifolds can be chosen as a class of preferred representatives for classes in Ω_*^{SU}, and what classes in Ω_*^{SU} can be represented by a given class of manifolds. In [12], the image of an interesting ring homomorphism, the Witten genus, is computed for Calabi-Yau manifolds, a certain class of manifolds in Ω_*^{SU} that are well-studied in physics. Comparing the images of Ω_*^{SU} and the subring of Calabi-Yau manifolds may provide evidence towards what classes in Ω_*^{SU} can be represented by Calabi-Yau manifolds.

Pursuing this line of investigation, for manifolds whose cobordism class we label x_{4i}, and defining W_i to be the image of x_{4i} under the Witten genus, we have the following:

Theorem. *The image of SU-cobordism under the Witten genus is given by the span of $\mathbb{Z}[2W_i]$ and $\mathbb{Z}[W_i^2]$.*

1.2 Preliminaries in Cobordism Theory

We begin with this definition. More details may be found in [6] or [37].

Definition Two smooth, compact n-dimensional manifolds M_1 and M_2 are <u>cobordant</u> if their disjoint union forms the boundary of a smooth, compact $(n+1)$-dimensional manifold, N. That is, $\partial N = M_1 \amalg M_2$.

The classical example of cobordism is the so-called "pair of pants" due to John Milnor:

Suppose $M_1 = S^1$ and $M_2 = S^1 \amalg S^1$.

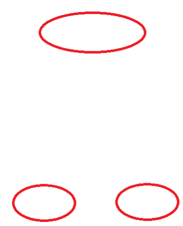

Figure 1.1: M_1 and M_2

A surface that has $M_1 \amalg M_2$ as its boundary is a cobordism between the two manifolds:

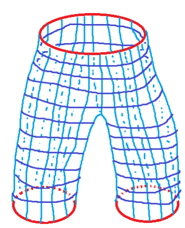

Figure 1.2: A cobordism between M_1 and M_2

Throughout this thesis we will study several structures on manifolds, and the cobordism rings associated to these manifolds with additional structure. Cobordisms of manifolds-with-structure were introduced in [20], and chapter II of [37] contains an excellent exposition on this topic. Generally speaking a cobordism class of a manifold-with-structure will be the manifold M with a choice of lift, ν, over the classifying space BO to the classifying space of the structure group, B:

$$\begin{tikzcd} & B \arrow[d, "\lambda"] \\ M \arrow[ur, "\nu"] \arrow[r, "\chi"'] & BO \end{tikzcd}$$

where λ is a fibration of B over BO, and χ is the map classifying the stable normal bundle of M.

The notion of cobordism considered will, in all cases, respect the additional structure on the manifolds. For instance in the case of oriented cobordism, Ω_*^{SO}, we have:

$$\begin{tikzcd} & BSO \arrow[d, "\lambda"] \\ M \arrow[ur, "\nu"] \arrow[r, "\chi"'] & BO \end{tikzcd}$$

where λ is a fibration of BSO over BO induced by the inclusion $SO(n) \hookrightarrow O(n)$, and cobordism will be defined as follows:

Definition Two smooth, compact, oriented n-dimensional manifolds $[M_1, \nu_1]$ and $[M_2, \nu_2]$ are <u>oriented cobordant</u> if their disjoint union forms the boundary of a smooth, compact, oriented $(n+1)$-dimensional manifold, $[N, \zeta]$, when the orientation of M_2 is reversed. That is, $\partial[N, \zeta] = [M_1 \amalg M_2, \nu_1 \amalg \bar{\nu}_2]$.

Here, $\bar{\nu}_2$ is the orientation opposite of ν_2. Typically, we will write $[M]$ for $[M, \nu]$ and $-[M]$ for $[M, \bar{\nu}]$.

At this point, there may not seem any apparent way of determining when two n-manifolds are cobordant apart from finding some $(n+1)$-manifold which their disjoint union bounds. In fact, we can accomplish this by comparing certain characteristic numbers of the manifolds. We will now recall three related characteristic classes of manifolds and their respective characteristic numbers. Further elaboration on characteristic classes and numbers can be found in [26], [15].

The Stiefel-Whitney class,

$$w(\xi) = w_0(\xi) + w_1(\xi) + \cdots + w_n(\xi),$$

of a real vector bundle $\xi = (E, \pi, B)$ of dimension n is a cohomology class satisfying the following properties:

1. $w_0(\xi) = 1$ and $w_k(\xi) \in H^k(B; \mathbb{Z}/2)$ for all k.

2. (Naturality) If $f : \xi \to \zeta$ is a bundle map, $w(\xi) = f^* w(\zeta)$, with f^* the induced homomorphism in cohomology.

3. (Whitney Sum) $w(\xi \oplus \zeta) = w(\xi) \cdot w(\zeta)$, where $\oplus$ is the Whitney sum.

4. (Normalization) If ζ is the tautological line bundle over $\mathbb{R}P^1$, then $w(\zeta) = 1 + w_1(\zeta)$ and $w_1(\zeta) \in H^1(\mathbb{R}P^1; \mathbb{Z}/2)$ is non-zero.

Similarly, we have the Chern class,

$$c(\xi) = c_0(\xi) + c_1(\xi) + \cdots + c_n(\xi),$$

of a complex vector bundle $\xi = (E, \pi, B)$ of complex dimension n is an integral cohomology class satisfying the following properties:

1. $c_0(\xi) = 1$ and $c_k(\xi) \in H^{2k}(B; \mathbb{Z})$ for all k.
2. (Naturality) If $f : \xi \to \zeta$ is a bundle map, $c(\xi) = f^* c(\zeta)$.
3. (Whitney Sum) $c(\xi \oplus \zeta) = c(\xi) \cdot c(\zeta)$.
4. (Normalization) If ζ is the tautological line bundle over $\mathbb{C}P^1$, then $w(\zeta) = 1 + c_1(\zeta)$ where $c_1(\zeta)$ is the canonical generator of $H^2(\mathbb{C}P^1; \mathbb{Z})$.

Finally, the Pontryagin class of a vector bundle,

$$p(\xi) = p_0(\xi) + p_1(\xi) + \cdots + p_n(\xi),$$

is an integral cohomology class defined in terms of the Chern classes by

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(B; \mathbb{Z}).$$

Here, $\xi \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of ξ. In the case that ξ is a complex bundle, we have that

$$p_k(\xi) = (-1)^k c_{2k}(\xi \oplus \bar{\xi})$$

where $\bar{\xi}$ is the conjugate bundle. In this case, we can then compute the Pontryagin class in terms of the Chern class by

$$p_0(\xi) - p_1(\xi) + \cdots \pm p_n(\xi) = (c_0(\xi) + c_1(\xi) + \cdots + c_n(\xi))(c_0(\xi) - c_1(\xi) + \cdots + (-1)^n c_n(\xi)).$$

If M is a manifold with stable tangent bundle τM, we denote by $w(M)$ (respectively $c(M), p(M)$) the Stiefel-Whitney class (Chern class, Pontryagin class) of the tangent bundle, $w(\tau M)$ $(c(\tau M), p(\tau M))$.

Suppose that $I = \{i_1, \ldots, i_k\}$ is a partition of a non-negative integer $m \leq n$. We can associate an integer modulo 2 to the Stiefel-Whitney class

$$w_{i_1} \ldots w_{i_k}(M) \in H^m(M, \mathbb{Z}/2)$$

by evaluating the Stiefel-Whitney class on the fundamental class μ_M of M. Similarly, we can associate integers to the Chern and Pontryagin classes

$$c_{i_1} \ldots c_{i_k}(M) \in H^{2m}(M, \mathbb{Z})$$

and

$$p_{i_1} \ldots p_{i_k}(M) \in H^{4m}(M, \mathbb{Z}).$$

We call these integers Stiefel-Whitney, Chern, and Pontryagin numbers, respectively, and denote them as

$$< w_{i_1} \dots w_{i_k}(M), \mu_M >= w_{i_1} \dots w_{i_k}[M] \text{ or } w_I[M],$$

$$< c_{i_1} \dots c_{i_k}(M), \mu_M >= c_{i_1} \dots c_{i_k}[M] \text{ or } c_I[M],$$

and

$$< p_{i_1} \dots p_{i_k}(M), \mu_M >= p_{i_1} \dots p_{i_k}[M] \text{ or } p_I[M].$$

For each of these characteristic numbers, unless I partitions n, the characteristic number evaluates to 0. Note that n represents a different value in each case. For the Stiefel-Whitney numbers, n is the real dimension of M; for Chern numbers, n is the complex dimension of M; and for Pontryagin numbers, $4n$ is the real dimension of M.

We may also recall now that for an almost complex manifold, M, the top Chern class is the Euler class of M, i.e.

$$c_n(M) = \chi(M)$$

and that the evaluation of this class

$$c_n[M] = \chi[M]$$

gives the Euler characteristic.

These characteristic numbers can help us determine when two manifolds are cobordant. For instance,

Theorem 1.2.1. (Thom, [38]) *Suppose Suppose M_1 and M_2 are manifolds. M_1 and M_2 are cobordant if and only if all of their Stiefel-Whitney numbers agree, i.e., $w_I[M_1] = w_I[M_2]$ for all I.*

Theorem 1.2.2. (Wall, [40]) *Suppose M_1 and M_2 are oriented manifolds. M_1 and M_2 are (oriented) cobordant if and only if all of their Stiefel-Whitney and Pontryagin numbers agree.*

A stably complex manifold, i.e. an object that can be studied with (stably) complex cobordism, is a smooth, compact manifold M, a complex vector bundle ξ, and a choice of real vector bundle isomorphism

$$c_M : \tau M \oplus \mathbb{R}^k \to \xi,$$

where $\mathbb{R}^k$ is the k-dimensional trivial bundle. We have a similar theorem for determining complex cobordism classes.

Theorem 1.2.3. (Milnor and Novikov (Separately), [24] and [28]) *Suppose M_1 and M_2 are stably complex manifolds. M_1 and M_2 are complex cobordant if and only if all of their Chern numbers agree.*

It will also be useful to mention another characteristic class that can be derived from either the Stiefel-Whitney, Chern, or Pontryagin classes. Consider again the partition $I = \{i_1, \ldots, i_k\}$ of $m \leq n$. Let $\sigma_1, \ldots, \sigma_m$ be the first m elementary symmetric polynomials in formal variables $t_1, \ldots, t_n$. Call $s_I = s_{i_1,\ldots,i_k}$ the unique polynomial satisfying

$$s_I(\sigma_1, \ldots, \sigma_m) = \sum t_1^{i_1} \cdots t_k^{i_k}.$$

We then have, e.g.,

$$\begin{array}{rllll}
s_1(\sigma_1) & = \sigma_1 & & & \\
& & & & \\
s_2(\sigma_1, \sigma_2) & = \sigma_1^2 & - & \sigma_2 & \\
s_{1,1}(\sigma_1, \sigma_2) & = & & \sigma_2 & \\
& & & & \\
s_3(\sigma_1, \sigma_2, \sigma_3) & = \sigma_1^3 & -3\sigma_1\sigma_2 & & +3\sigma_3 \\
s_{1,2}(\sigma_1, \sigma_2, \sigma_3) & = & & \sigma_1\sigma_2 & -3\sigma_3 \\
s_{1,1,1}(\sigma_1, \sigma_2, \sigma_3) & = & & & \sigma_3.
\end{array}$$

Table 1.1: Table of s-classes for small n.

By identifying with σ_i the Chern class c_i (similarly, Stiefel-Whitney or Pontryagin class, w_i or p_i) we obtain a characteristic class we will call the s-class[1]. Evaluating this class on the fundamental class of M, we obtain the s-number $s_n[M]$. We also have the following theorem, regarding the s-number:

Theorem 1.2.4. ([26, pg. 192]) *For any product $K^m \times L^n$ of complex manifolds of dimension $m, n \neq 0$,*

$$s_{n+m}[K^m \times L^n] = 0.$$

So, the s number of M is non-zero only if M is indecomposable, i.e. M cannot be written as a (non-trivial) product of complex manifolds.

Finally, we will make use of some characteristic numbers in the ring of real K-theory of a space X, $KO(X)$. For our purposes, it is enough to observe that the characteristic numbers we use are obtained by evaluating certain rational combinations of Pontryagin classes. A thorough treatment of K-theory can be found in [3] or [7]. The KO-theory Pontryagin numbers, π_I, which we use in §4.3, are explicitly described in [1].

[1]This characteristic class is discussed in several places in the literature, but does not have an accepted name (see Appendix A for further digression on this characteristic class). Since we define this class in terms of elementary symmetric polynomials – and the fact that it is denoted with s – the symmetric class or s-class seems to be a reasonable choice.

2 Representation of Cobordism Classes

2.1 Preferred Cobordism Class Representatives

In this chapter, we will expand on the history of preferred representation of cobordism classes discussed in §1.1. Interest in the question of preferred class representatives has existed since Thom properly introduced and began the study of cobordism. Indeed he showed that,

Theorem 2.1.1. (Thom, [38]) *Ω_*^O is isomorphic to a polynomial ring over $\mathbb{Z}/2$ with generators in dimensions not equal to $2^k - 1$, i.e. $\Omega_*^O \cong (\mathbb{Z}/2)[x_i]$, $i \neq 2^k - 1$.*

These generators x_i, are required to be indecomposable, i.e. x_i can be chosen as a generator if and only if $s_i[x_i]$ is non-zero. Immediately following this, we are presented with choices for generators x_2, x_4, x_5, x_6, and x_8, and for all even dimensions[1]. In particular,

Theorem 2.1.2. (Thom, [38]) *For n even, one may take $x_n := [\mathbb{R}P^n]$.*

For x_5 Thom suggests the so-called Wu space, which is obtained by identifying the points $(x_0, x_1, x_2) \times 0$ and $(\bar{x}_0, \bar{x}_1, \bar{x}_2) \times 1$ in the product of $\mathbb{C}P^2 \times [0, 1]$, but laments that he knows no constructions for generators in other odd dimensions.

Two years later, Dold provided such a construction. Define

$$P(m, n) := S^m \times \mathbb{C}P^n / \tau$$

with $m > 0$ and identifying points with respect to the involution

$$\tau : S^m \times \mathbb{C}P^n \to S^m \times \mathbb{C}P^n,$$

$$(x, [y]) \mapsto (-x, [\bar{y}]).$$

Dold showed that the cohomology of these manifolds is generated by two classes,

$$c \in H^1(P(m, n); \mathbb{Z}/2)$$

and

$$d \in H^2(P(m, n); \mathbb{Z}/2),$$

under the relations

$$c^{m+1} = 0,$$

$$d^{n+1} = 0,$$

and no others. We then have that

[1] There are, of course, no generators in dimension $1 = 2^1 - 1$, $3 = 2^2 - 1$, or $7 = 2^3 - 1$.

Theorem 2.1.3. (Dold, [10]) *The Stiefel-Whitney class of $P(m, n)$ are given by*

$$w(P(m,n)) = (1+c)^m(1+c+d)^{n+1}.$$

Using this, Dold was able to show

Theorem 2.1.4. (Dold, [10]) *For $i \in \mathbb{N}$ not of the form $i = 2^l - 1$, and for integers r and s such that $i + 1 = 2^r(2s+1)$ we have that*

$$x_i = \begin{cases} P(i,0) = \mathbb{R}P^i & i \text{ even} \\ P(2^r-1, s2^r) & i \text{ odd.} \end{cases}$$

With the complete and explicit description of generators given by Dold, we are equipped with a preferred class representative for any generator in unoriented cobordism. We are given a similarly complete description of generators of the oriented cobordism ring by C.T.C. Wall in [40], however it is not as explicit. Despite Wall's claims of its simplicity[2], the ring Ω_*^{SO} is somewhat more complicated than unoriented cobordism.

One may choose to work around the complication of torsion, and still arrive at nice results. Indeed, we have

Theorem 2.1.5. (Thom, [38]) *$\Omega_*^{SO} \otimes \mathbb{Q}$ is a polynomial ring with generators in dimensions divisible by 4, i.e. $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[y_{4i}]$, $i \geq 0$.*

Thom went on to show

Theorem 2.1.6. (Thom, [38]) *One may take $y_{4i} := [\mathbb{C}P^{2i}]$.*

Independent of this work, we have

Theorem 2.1.7. (S. P. Novikov, [28]) *The ring Ω_*^{SO} modulo torsion is an integral polynomial ring with generators in dimensions divisible by 4, i.e. $\Omega_*^{SO}/Tor \cong \mathbb{Z}[z_{4i}]$.*

If, on the other hand, one chooses not to ignore the torsion elements of Ω_*^{SO}, we have, for example in [26]:

Theorem 2.1.8. (Wall, [40])
$\Omega_0^{SO} \cong \mathbb{Z}$ and is generated by a signed point.
$\Omega_1^{SO} \cong \Omega_2^{SO} \cong \Omega_3^{SO} \cong 0$.
$\Omega_4^{SO} \cong \mathbb{Z}$ and is generated by $\mathbb{C}P^2$.
$\Omega_5^{SO} \cong \mathbb{Z}/2$ and is generated by the Wu space.
$\Omega_6^{SO} \cong \Omega_7^{SO} \cong 0$.
$\Omega_8^{SO} \cong \mathbb{Z} \oplus \mathbb{Z}$ and is generated by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$.
$\Omega_9^{SO} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and is generated by the Milnor hypersurface $H_{2,8}$ and $H_{2,4} \times \mathbb{C}P^2$.

[2]Wall claims in [40, pg. 294] that if anyone thinks what he's done is hard, they should try to give a description of Ω_*^{Spin}.

The Milnor hypersurface $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ is the smooth hypersurface given by

$$x_0y_0 + x_1y_1 + \cdots + x_ky_k = 0,$$

where $k = \min(i, j)$, and

$$(x_0 : x_1 : \cdots : x_i)$$

and

$$(y_0 : y_1 : \cdots : y_j)$$

are the homogeneous coordinates on $\mathbb{C}P^i$ and $\mathbb{C}P^j$ respectively.

2.2 Class of Preferred Representatives for Complex Cobordism

We will now consider a famous result of Milnor that is the inspiration for the remaining work in this thesis. Milnor answers the question of class representation for (stably almost) complex cobordism in a less explicit way than it has been answered for unoriented or oriented cobordism. Instead of providing explicit manifolds that can be taken as generators in each dimension, Milnor gives a class of manifolds that can be chosen as a representative for each class. That is, instead of providing a preferred class representative for each class, Milnor's theorem provides a class of preferred representatives.

The proof of this theorem requires the following:

Theorem 2.2.1. (Milnor, Novikov, Stong, [24, 28, 37]) *Ω^U_* is an integral polynomial ring with generators in each even dimension. The cobordism class of a stably complex manifold M^{2i} may be taken as the $2i-$dimensional generator if and only if*

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if } i+1 \neq p^s \text{ for any prime } p \\ \pm p & \text{if } i+1 = p^s \text{ for some prime } p. \end{cases}$$

Theorem 2.2.2. (Milnor, [14, 37, 39]) *Every class, $x \in \Omega^U_n$ contains a non-singular algebraic variety (not necessarily connected) if $n > 0$.*

Proof. We follow the proof given in [37, pg. 130].

Call $U_* \subset \Omega^U_*$ the set of cobordism classes that contain a non-singular algebraic variety. It is easy to see that U_* is closed under addition and multiplication (disjoint union, and Cartesian product).

We will show that there are classes $x_i, x'_i \in U_{2i}$ which satisfy the requirements of theorem 2.2.1. That is, call

$$m_i = \begin{cases} 1 & \text{if } i+1 \neq p^s \text{ for any prime } p \\ p & \text{if } i+1 = p^s \text{ for some prime p,} \end{cases}$$

we will have

$$s_i[x_i] = m_i$$

and

$$s_i[x_i'] = -m_i.$$

We will then be able to make the argument by induction. Suppose $U_{2j} = \Omega_{2j}^U$ if $j < k$. If $x \in \Omega_{2k}^U$, then

$$s_k[x] = tm_k, \ t \in \mathbb{Z}.$$

If $t > 0$, we have that $[x] = [tx_k + v]$, and if $t < 0$, we have that $[x] = [|t|x_i' + v]$ where v is a sum of decomposable elements of dimension $2k$. So, $v \in U_{2k}$ by induction, and therefore $x \in U_{2k}$.

Call $A_{2k} := \{l \in \mathbb{Z} | s_k[x] = l \text{ for some } x \in U_{2k}\}$. Note that A_{2k} is closed under sums. Also, since

$$s_k[\mathbb{C}P^k] = k + 1 > 0$$

and

$$s_k[H_{r,k-r+1}] = -\binom{k+1}{r} < 0, \text{ for } 1 < r < k-1$$

we have that A_{2k} contains both positive and negative elements.

Call p the smallest positive element in A_{2k} and n the largest negative element in A_{2k}. We then have that $p + n = 0$, since if it is positive, we have $p > p + n > 0$, and if it is negative we have $n < p+n < 0$, which are both contradictions. So, $n = -p$.

For any element $q \in A_{2k}$, we can write $q = tp + s$ with $0 \geq s < p$. But then $s = q + tn \in A_{2k}$. Since $s < p$, we have that $s = 0$. So, A_{2k} is the set of multiples of p. Since

$$\gcd_{0 \leq i \leq k+1} \binom{k+1}{i} = m_k,$$

the greatest common divisor of elements of A_{2k} is m_k, and $p = m_k$ and $n = -m_k$. Therefore, there are manifolds in U_{2k} which satisfy theorem 2.2.1 for every k. $\square$

The question of how to explicitly represent generators in complex cobordism is still being considered. In [14], Hirzebruch asked which classes in Ω_*^U can be represented by connected non-singular algebraic varieties.

Andrew Wilfong gave a partial answer to this question by showing that many generators can be represented by toric varieties. In particular,

Theorem 2.2.3. (Wilfong, [42]) *If $n < 100001$, odd, or one less than a power of a prime, the cobordism class of a smooth projective toric variety can be taken as the generator of dimension $2n$ of Ω_*^U.*

3 Special Unitary Cobordism

The remaining work of this thesis is in pursuit of an analog for special unitary cobordism to Milnor's theorem for complex cobordism. That is, we would like to determine a class of manifolds that may be taken as a representative for each class in Ω_*^{SU}. In this chapter we will describe the additive structure of Ω_*^{SU}, which is determined by the close relationship between Ω_*^{SU} and Ω_*^{U}. Finally we will show that the direct analog of Milnor's theorem is not true. That is,

Theorem. *There exist classes in Ω_4^{SU} that are not representable by non-singular algebraic varieties.*

3.1 The Ring of Special Unitary Cobordism and its Connection to Complex Cobordism

Some partial results on the structure of Ω_*^{SU} were given by Milnor [24] and Novikov [28], but the complete description was given by Conner and Floyd in [8], [9]. A very succinct treatment of their work is given in [37, Ch. X]. The work in this section mainly follows their presentations.

Definition A special unitary (SU-)manifold is a smooth, compact manifold, M, together with a choice of lift ν:

$$\begin{tikzcd} & BSU \arrow[d, "\lambda"] \\ M \arrow[ur, "\nu"] \arrow[r, "\chi"'] & BO \end{tikzcd}$$

We may also observe that SU-manifolds have the property that $c_1 c_\omega[M] = 0$ for all ω. Similar to the characterizations of cobordism classes given in §1.2, we may observe the following:

Theorem 3.1.1. ([1]) *Two SU-manifolds are cobordant if and only if they have the same Chern numbers and KO-characteristic numbers.*

Since we have that

$$\begin{tikzcd} & BSU \arrow[d, "\iota"] \\ & BU \arrow[d, "\lambda"] \\ M \arrow[uur, "\nu"] \arrow[r, "\chi"'] & BO \end{tikzcd}$$

where ι is the map induced by the inclusion $SU(n) \hookrightarrow U(n)$, we can observe that an SU-manifold is necessarily a stably almost complex manifold. For these reasons, we may think of SU-cobordism as complex cobordism with "orientation", and the work done by Conner and Floyd relating SU-cobordism to complex cobordism can be compared to the work done in [40] and by Atiyah [2] relating oriented cobordism to unoriented cobordism.

We may now define a useful homomorphism

$$\alpha : \Omega_*^{\mathrm{SU}} \to \Omega_*^{\mathrm{U}}$$

called the forgetful homomorphism, which replaces the given SU-structure with the underlying stably complex structure.

A homomorphism from Ω_*^U to Ω_*^{SU} may also be defined. Suppose M^{2n} is a stably complex manifold with orientation σ_{2n}. Define

$$\delta[M^{2n}] := V^{2n-2}$$

to be a stably complex submanifold of M^{2n} with orientation σ_{2n-2} so that if

$$j : V^{2n-2} \hookrightarrow M^{2n}$$

is the natural inclusion,

$$j_*(\sigma_{2n-2}) = c_1(M^{2n}) \frown \sigma_{2n}.$$

That is, V^{2n-2} the submanifold representing the homology class Poincaré dual to $c_1(M^{2n})$. Observe that since $c_1(V^{2n-2})$ is trivial, V^{2n-2} admits an SU-structure. So, δ is a homomorphism of degree -2:

$$\delta : \Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SU}}.$$

We may similarly define a homomorphism of degree -4. Define

$$\Delta[M^{2n}] := V^{2n-4}$$

to be a stably complex submanifold of M^{2n} with orientation σ_{2n-4} so that

$$j_*(V^{2n-4}) = -c_1^2(M^{2n}) \frown \sigma_{2n}.$$

Then,

$$\Delta : \Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{U}}.$$

Defining $\mathcal{W}_{2n}$ to be the kernel of the homomorphism

$$\Delta : \Omega_{2n}^U \to \Omega_{2n-4}^U,$$

we have the following:

Theorem 3.1.2. ([8, pg. 31]) *$\mathcal{W}_{2n}$ consists of cobordism classes $[M^{2n}] \in \Omega^U_{2n}$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes. The homomorphism $\alpha\delta : \Omega^U_{2n} \to \Omega^U_{2n-2}$ has $\alpha\delta\Omega^U_{2n} \subset \mathcal{W}_{2n-2}$. If $[M^{2n}] \in \mathcal{W}_{2n}$, then $\delta[M^{2n}] = 0$ if and only if every Chern number of which c_1 is a factor vanishes.*

We may also observe that the image α is contained in $\mathcal{W}_*$. That is,

$$\alpha : \Omega^{\mathrm{SU}}_* \to \mathcal{W}_* \subset \Omega^{\mathrm{U}}_*.$$

With these observations in hand, we may state the following:

Theorem 3.1.3. (Conner and Floyd, [37, pg. 242]) *We have the following exact sequences:*

$$\begin{tikzcd} \Omega^{SU}_* \arrow[rr, "t"] & & \Omega^{SU}_* \arrow[dl, "\alpha"] \\ & \mathcal{W}_* \arrow[ul, "\delta"] & \end{tikzcd}$$

and

$$0 \longrightarrow \mathcal{W}_* \xrightarrow{F_*} \Omega^U_* \xrightarrow{\Delta} \Omega^U_* \longrightarrow 0$$

Here $F_* : \mathcal{W}_* \to \Omega^U_*$ is the inclusion, and t is given by multiplication by the class of S^1 with a non-trivial SU-structure, which we label $[\hat{S}]$.

Recall from theorem 2.2.1 that Ω^U_{2n+1} is trivial. So, since $\mathcal{W}_{2n+1} \subset \Omega^U_{2n+1}$, the long exact sequence reduces to the exact sequence

$$0 \longrightarrow \Omega^{SU}_{2n-1} \xrightarrow{t} \Omega^{SU}_{2n} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\delta} \Omega^{SU}_{2n-2} \xrightarrow{t} \Omega^{SU}_{2n-1} \longrightarrow 0$$

Using these exact sequences, one may show the following:

Theorem 3.1.4. (Conner and Floyd, [37, pg. 243]) *$\Omega^{SU}_0 \cong \mathbb{Z}, \Omega^{SU}_1 \cong \mathbb{Z}/2$, and $\Omega^{SU}_2 \cong \mathbb{Z}/2$.*

We also have from [20]:

Theorem 3.1.5. (Lashoff and Rothenburg) *$t^3 = 0$, and so $\Omega^{SU}_3 \cong 0$.*

We may consider the long exact sequence in theorem 3.1.3 as an exact couple. In doing so, we have the derived couple:

$$\begin{tikzcd} \mathrm{Im}(t) \arrow[rr, "t"] & & \mathrm{Im}(t) \arrow[dl, "\alpha'"] \\ & H(\mathcal{W}) \arrow[ul, "\delta'"] & \end{tikzcd}$$

with $H(\mathcal{W})$ the homology group of $\{\mathcal{W}_*, \partial := \alpha\delta\}$. Note that ∂ is a boundary operator. For $M^{2n} \in \mathcal{W}_{2n}$ we have that $c_1(\partial M^{2n}) = c_1(\alpha\delta M^{2n})$ is trivial. So, $\partial^2 M^{2n} = 0$. The homology of $\{\mathcal{W}_*, \partial\}$ is studied in [8, Ch. II], and is given by the following theorem:

Theorem 3.1.6. ([8, pg. 47]) *$H_*(\mathcal{W})$ is a polynomial algebra over $\mathbb{Z}/2$ with generators in dimension 4 and $8k$, for $k \geq 2$. Moreover, multiplication by the dimension 4 generator gives an isomorphism $\mathcal{W}_{8k} \to \mathcal{W}_{8k+4}$.*

Using this derived couple and theorem 3.1.5, we have the exact sequence

$$0 \longrightarrow t(\Omega^{SU}_{2n}) \cong \Omega^{SU}_{2n+1} \longrightarrow H_{2n}(\mathcal{W}) \longrightarrow t(\Omega^{SU}_{2n-3}) \cong \Omega^{SU}_{2n-3} \longrightarrow 0.$$

Since all elements of $H(\mathcal{W})$ are of order 2, the sequence splits, and we have the following:

Theorem 3.1.7. ([8, pg. 68]) *For each n we have that*

$$H_{2n}(\mathcal{W}) \cong \Omega^{SU}_{2n+1} \oplus \Omega^{SU}_{2n-3}.$$

Applying this theorem tells us a great deal about the cobordism groups in Ω^{SU}_*. In particular, since

$$H_{8n+2}(\mathcal{W}) \cong H_{8n+6}(\mathcal{W}) \cong 0,$$

we have immediately that

$$\Omega^{SU}_{2n+3} \cong 0$$

and

$$\Omega^{SU}_{2n+7} \cong 0.$$

The isomorphism in theorem 3.1.6 gives us that

$$H_{8n}(\mathcal{W}) \cong \Omega^{SU}_{8n+1} \oplus \Omega^{SU}_{8n-3} \cong \Omega^{SU}_{8n+5} \oplus \Omega^{SU}_{8n+1} \cong H_{8n+4}(\mathcal{W}),$$

or

$$\Omega^{SU}_{8n-3} \cong \Omega^{SU}_{8n+5}.$$

Since $\Omega^{SU}_{-3} \cong 0$, induction on n gives that $\Omega^{SU}_{8n+5} \cong 0$.

We may now state a special case of theorem 3.1.7:

Theorem 3.1.8.

$$H_{8n}(\mathcal{W}) \cong \Omega^{SU}_{8n+1}.$$

This theorem, together with the structure of the homology of $\mathcal{W}_*$ given in theorem 3.1.6 gives the following theorem, which completely describes the structure of torsion in Ω^{SU}_*:

Theorem 3.1.9. [8, pg. 68])) *The torsion of Ω^{SU}_* is given as follows:*
$\mathrm{Tor}(\Omega^{SU}_n) = 0$ *unless $n = 8k+1$ or $8k+2$, in which case* $\mathrm{Tor}(\Omega^{SU}_n)$ *is a vector space over $\mathbb{Z}/2$ whose dimension is the number of partitions of k.*

This completes the description of the additive structure of Ω^{SU}_*. Since this structure depends completely on the relationship between Ω^{SU}_* and Ω^{U}_*, one may hope for an analog of 2.2.2 for SU-cobordism.

3.2 Representing SU-Cobordism Classes by Non-Singular Algebraic Varieties in Small Dimensions

In this section we will study the question of class representability by non-singular algebraic varieties for Ω_2^{SU} and Ω_4^{SU}. Recall from theorem 3.1.4 that

$$\Omega_2^{SU} \cong \mathbb{Z}/2.$$

We therefore need only find an example of an algebraic manifold of dimension 2 with a non-trivial SU-structure. Such an example is given, for instance, in [33, §2.1]. In particular, such a manifold is given by a non-singular cubic polynomial in $\mathbb{C}P^2$. We have as an immediate corollary:

Corollary 3.2.1. *Every element in Ω_2^{SU} can be represented by a non-singular algebraic variety.*

Consider now the cobordism group $\Omega_4^{SU} \cong \mathbb{Z}$. Observe that the manifolds in Ω_4^{SU} are smooth, compact manifolds of 2 complex dimensions. Since every Chern number with a factor of c_1 vanishes for SU-manifolds, we may determine the cobordism class of a manifold $M \in \Omega_4^{SU}$ by its signature:

$$\sigma[M] := \frac{c_1^2[M] - 2c_2[M]}{3} = -\frac{2}{3}c_2[M].$$

Ochanine showed in [29] that the signature of a $(8k+4)$-dimensional SU-manifold is divisible by 16. Therefore, we may take a class $[M] \in \Omega_4^{SU}$ to be an additive generator if $\sigma[M] = \pm 16$. We will observe, in particular, that while we may take an algebraic generator with $\sigma[M] = -16$, the opposite is not true. That is, the class $-[M] \in \Omega_4^{SU}$ with signature $-\sigma[M] = 16$ does not contain an algebraic variety.

We now recall the classical result of Federigo Enriques and Kunihiko Kodaira, which classifies up to bi-rational equivalence all complex compact surfaces. To understand this notion of equivalence, we define the operation of a blow-up. Suppose M is a complex manifold with complex coordinates $(z_1, \dots z_n)$, and consider $\mathbb{C}P^{n-1}$ with homogeneous coordinates $[X_0 : \cdots : X_{n-1}]$. Let D be a unit disc about the origin.

Definition The <u>blow-up of D at 0</u> is given by

$$Bl_0 D = \{(z, X) \in D \times \mathbb{C}P^{n-1} | z_i X_j = z_j X_i \forall i, j\}.$$

Applying this process to a neighborhood of a point $x \in M$ gives the blow-up of M at the point x, $Bl_x M$. The projection

$$\pi : Bl_x M \to M$$

is an isomorphism away from x, and we have that

$$\pi^{-1}(x) \cong \mathbb{C}P^{n-1}.$$

It follows from the factorization lemma [4, pg. 98] that every bi-rational map may be written as a sequence of blow-ups and (the inverse of this operation) blow-downs. So, we will define bi-rational equivalence as follows:

Definition Two complex surfaces are bi-rationally equivalent if one may be obtained from the other by a sequence of blow-ups and blow-downs.

It should be noted that, in general, bi-rational equivalence and cobordism do not produce the same equivalence classes. However, the operations of blow-up and blow-down have very a specific effect on the Chern numbers of surfaces. We have from [17, 2.5.8] that Bl_xM is diffeomorphic to $M\#\mathbb{C}P^n_*$, where $\mathbb{C}P^n_*$ is $\mathbb{C}P^n$ with reversed orientation. We also have from [35] that $[M\#N] = [M] + [N]$. So, for a surface S, we have that

$$c_1^2[Bl_xS] = c_1^2[S\#\mathbb{C}P^2_*] = c_1^2[S] + c_1^2[\mathbb{C}P^2_*] = c_1^2[S] + 1,$$

and

$$c_2[Bl_xS] = c_2[S\#\mathbb{C}P^2_*] = c_2[S] + c_2[\mathbb{C}P^2_*] = c_2[S] - 1.$$

Therefore, by studying its Chern numbers, we can determine if a class in the following classification is bi-rationally equivalent to a surface that meets the necessary conditions of an additive generator of Ω_4^{SU}, namely:

1. $c_1^2[M] = 0$ (since $c_1(M) = 0$).
2. The signature of M is ± 16.

Recall the classification of complex compact surfaces:

Theorem 3.2.2. (Enriques and Kodaira, [4, pg. 243], [11, pg. 590]) *Every complex compact surface has a minimal model in exactly one class in table* 3.1.

Class	c_1^2	$c_2 = \chi$	Signature	Algebraicity
Rational Surfaces	8 or 9	4 or 3		Always Algebraic
Surfaces of Class VII_0	≤ 0	≥ 0		Not Algebraic
Ruled Surfaces ($g \geq 1$)	$8(1-g)$	$4(1-g)$		Always Algebraic
Enriques Surfaces	0	12	-8	Always Algebraic
Bi-Elliptic Surfaces	0	0	0	Not Algebraic
Kodaira Surfaces	0	0	0	Not Algebraic
K3 Surfaces	0	24	-16	Sometimes Algebraic
2-tori	0	0	0	
Properly Elliptic Surfaces	0	≥ 0		
Surfaces of General Type	> 0	> 0		

Table 3.1: The Enriques-Kodaira Classification of Complex, Compact Surfaces

A surface M is a minimal model, if it admits no line bundle ξ with $c_1^2(\xi)[M] = -1$.

We can observe from the table that a $K3$-surface can be taken as one of our additive generators. A $K3$-surface is, in particular, a Calabi-Yau manifold, i.e. a complex manifold with an SU-structure, and therefore an SU-manifold. Since a $K3$-surface has signature -16, we now need to find an SU-surface with signature 16. In particular, we need to find a surface S bi-rationally equivalent to one given in the classification with $c_1^2[S] = 0$ and $c_2[S] = -24$. Since

$$c_1^2[Bl_x S] = c_1^2[S] + 1$$

and

$$c_2[Bl_x S] = c_2[S] - 1,$$

we need to check to see if any of the classes in the table are algebraic and have Chern numbers satisfying

$$c_2[S] + 24 = -(c_1^2[S] - 0).$$

That is,

$$c_1^2[S] + c_2[S] = -24.$$

By inspection, the only classes that don't fail this criteria are ruled surfaces.

The ruled surfaces, in fact, may give a solution to this problem. In particular,

$$8(1-g) + 4(1-g) = -24$$

precisely when $g = 3$. In this case, we have a ruled surfaces of genus 3, which we denote S_3^R. Applying the blow-up operation 16 times to S_3^R, we have a surface $Bl_{16}S_3^R$ with

$$c_1^2[Bl_{16}S_3^R] = 0$$

and $\sigma[Bl_{16}S_3^R] = 16$. These are the necessary (but not sufficient) conditions to be the desired additive generator. If, in fact, $Bl_{16}S_3^R$ admits an SU-structure, we may take $Bl_{16}S_3^R$ as the other additive generator for Ω_4^{SU}. If not, we have that if such a generator exists, it is bi-rationally equivalent to S_3^R. So, we can state the following:

Theorem 3.2.3. *If every class in Ω_4^{SU} can be represented by a non-singular algebraic variety, we may take as additive generators a $K3$-surface, and an SU-surface bi-rationally equivalent to a ruled surface of genus 3.*

To determine whether or not $Bl_{16}S_3^R$ is an SU-manifold it is sufficient to study its first Chern class. Recall from [11, Ch 4.6 §2] that the first Class of a blow-up is given by

$$c_1(Bl_x M) = \pi^* c_1(M) - E$$

where

$$\pi : Bl_x M \to M$$

is the projection and $E = \pi^{-1}(x) \cong \mathbb{C}P^1$ is the exceptional divisor. Since

$$H^*(Bl_x M; \mathbb{Z}) = \pi^* H^*(M; \mathbb{Z}) \oplus H^*(E; \mathbb{Z}) / \pi^* H^*(x),$$

we have that

$$c_1(Bl_{16}S_3^R) = \pi^* c_1(M) - 16E \neq 0.$$

So, $Bl_{16}S_3^R$ is not an SU-manifold.

Therefore, we have the following:

Theorem 3.2.4. *There exist classes in Ω_4^{SU} that are not representable by any non-singular algebraic variety.*

So, we see that the direct analog of theorem 2.2.2 is not true for Ω_*^{SU}. Since theorem 3.2.2 is a classification of all complex surfaces, we may easily relax our criteria from algebraic manifolds to complex manifolds, and ask the following question:

Question 3.2.1. *Is there a complex SU-surface with signature 16?*

We will see that this question can also be answered in the negative.

Relaxing our criteria to include any complex manifold we see that the only additional class we need now consider is that of Surfaces of Class VII_0. However, it is easy to see that any SU-surface of class VII_0 is trivial. Let S_{VII} be an SU-surface of class VII_0. We have from [4, I.7.2] that the geometric genus, $p_g(S_{VII}) = 0$. We also have from [4, pg. 245] that the difference in the geometric genus and the arithmetic genus, i.e. the irregularity

$$q(S_{VII}) = p_g(S_{VII}) - p_a(S_{VII}) = 1.$$

We have then, by Noether's Formula, for the Todd genus of S_{VII}:

$$Td[S_{VII}] = \frac{c_1^2 + c_2}{12}[S_{VII}] = p_a(S_{VII}) + 1 = 0.$$

In particular,

$$c_1^2[S_{VII}] = -c_2[S_{VII}].$$

Therefore, since S_{VII} is an SU-surface, $c_2[S_{VII}] = 0$ and S_{VII} is trivial in Ω_4^{SU}. We may then conclude:

Theorem 3.2.5. *There exist classes in Ω_4^{SU} that are not representable by any complex, compact surface.*

We may conclude, then, that the search for the class of preferred representatives for Ω_*^{SU} may be restricted to classes of stably complex manifolds.

We may also consider the following questions:

Question 3.2.2. *What classes of Ω_*^{SU} can be represented by a non-singular algebraic variety?*

and

Question 3.2.3. *What classes of Ω_*^{SU} can be represented by a complex manifold?*

In the next chapter we will refocus our attention to questions of this type.

4 Multiplicative Genera

4.1 Introduction

The image of a certain class of SU-manifolds, Calabi-Yau manifolds, under a ring homomorphism called the Witten genus is studied in [12]. Calabi-Yau manifolds were introduced by Shing-Tung Yau in [45], and can be defined as complex manifolds for which every Chern number with a factor of c_1 vanishes. That is, manifolds that are both complex and have an SU-structure are Calabi-Yau manifolds. By studying the image of Ω_*^{SU} under the Witten genus, and comparing it to the results in [12], we may be able to determine what classes are not, or perhaps are representable by Calabi-Yau manifolds.

In this chapter we will give some background in the theory of multiplicative genera. We will then study the results of D. & G. Chudnovsky, Peter Landweber, Ochanine, and Robert Stong regarding the image of Ω_*^{SU} under Ochanine's elliptic genus, and apply their methods to obtain the image of Ω_*^{SU} under the Witten genus. We will then give some partial results regarding the explicit values in this image.

4.2 Background for Multiplicative Genera

In this section we will properly introduce multiplicative genera, and give some useful results regarding the image of a given genus. Background on genera may be found in [32], [15], [16].

Definition A *(multiplicative) genus*, φ, is a ring homomorphism

$$\varphi : \Omega_*^{SO} \to \Lambda,$$

where Λ is a $\mathbb{Q}-$algebra.

As a ring homomorphism, a genus satisfies the obvious properties for manifolds $[M], [N] \in \Omega_*^{SO}$, i.e.:

1. $\varphi(\partial M) = 0$.
2. $\varphi(M \amalg N) = \varphi(M) + \varphi(N)$.
3. $\varphi(M \times N) = \varphi(M)\varphi(N)$.

If we consider the commutative diagram

$$\begin{tikzcd}
\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \arrow[dr] & \\
\Omega_*^{\mathrm{SO}} \arrow[u] \arrow[r, "\varphi"'] & \Lambda
\end{tikzcd}$$

along with theorems 2.1.5 and 2.1.6, we see that the value of any genus is completely determined by its values on the even-dimensional complex projective spaces. Therefore, a genus is completely determined by the following formal power series:

Definition The logarithm of φ is the odd formal power series in u given by

$$g(u) = \sum_{n=0}^{\infty} \varphi(\mathbb{C}P^{2n}) \frac{u^{2n+1}}{2n+1}.$$

Genera can also be defined by a related power series called the characteristic series by Hirzebruch in [15]. This series is defined by

$$Q(x) = \frac{x}{g^{-1}(x)}$$

where $g^{-1}(x)$ is the formal inverse of the power series $g(x)$.

It is sometimes convenient to associate to this characteristic series a sequence of polynomials

$$K_{\varphi} = 1 + K_1 + K_2 + \dots$$

with

$$K_{\varphi}(\sigma_1, \sigma_2, \dots) = Q(t_1)Q(t_2)\cdots$$

where σ_i is the i^{th} elementary symmetric polynomial in the formal variables t_j. $\{K_1, K_2, \dots\}$ is called the multiplicative sequence of φ.

Replacing σ_i with the Pontryagin class p_i (as in the definition of the s-numbers), we have

$$\varphi(M) = K_{\varphi}(p_1, p_2, \dots)[M].$$

Since for a $4n$-dimensional manifold M, the Pontryagin classes $p_j(M) = 0$ for $j > n$, in practice, we need only the polynomial $K_n(p_1, \dots, p_n)$ to compute the image of M under a given genus.

Examples

Suppose we want to define a genus L such that $L(\mathbb{C}P^{2n}) = 1$ for each n. In this case, we get a logarithm of

$$g(u) = \sum_{n=0}^{\infty} \frac{u^{2n+1}}{2n+1} = \operatorname{arctanh}(u).$$

In this case, the inverse of the logarithm is easy to identify, and we have a characteristic series

$$Q(x) = \frac{x}{\tanh(x)}.$$

Hirzebruch's Signature Theorem, [15, pg. 86], shows that the signature of a smooth, compact $4k$-dimensional manifold is given by evaluation of this genus, and so we may call this genus the signature, though it sometimes appears in the literature (as Hirzebruch described it) as the L-genus. Note that this is the same signature mentioned in §3.2.

Setting

$$L(p_1, p_2, \cdots) := K_L(p_1, p_2, \cdots) = \prod_{i=1}^{\infty} \frac{t_i}{\tanh(t_i)},$$

we have for the first few polynomials:

$$L_1(p_1) = \frac{p_1}{3}$$

$$L_2(p_1, p_2) = \frac{7p_2 - p_1^2}{45}$$

$$L_3(p_1, p_2, p_3) = -\frac{62p_3 - 13p_2p_1 + 2p_1^3}{945}$$

Suppose instead we want to define a genus such that

$$Q(x) = \frac{x/2}{\sinh(x/2)}.$$

We call this genus the $\hat{A}$-genus, and setting

$$\hat{A}(p_1, p_2, \cdots) := K_{\hat{A}}(p_1, p_2, \cdots) = \prod_{i=1}^{\infty} \frac{t_i/2}{\sinh(t_i/2)}$$

we have for the first few polynomials:

$$\hat{A}_1(p_1) = -\frac{p_1}{24}$$

$$\hat{A}_2(p_1, p_2) = \frac{7p_1^2 - 4p_2}{5760}$$

$$\hat{A}_3(p_1, p_2, p_3) = -\frac{31p_1^3 - 44p_1p_2 + 16p_3}{967680}$$

The $\hat{A}$-genus has the property that it takes integer values for manifolds in Ω_*^{Spin} [15, pg. 197].

The Elliptic Genus

We will now focus on the results of D. and G. Chudnovsky, Landweber, Ochanine, and Stong from their unpublished manuscript [5]. In the manuscript they compute the image of Ω_*^{SU} under Ochanine's elliptic genus. Moreover, they give a useful method by which the image of this cobordism ring may be computed for a given genus. Though [5] was unpublished, many of the ingredients necessary for their work may be found in [19] and elsewhere.

An elliptic genus is a genus, $\varphi_{\delta,\epsilon}$, with logarithm

$$g(u) = \int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}}.$$

The genus $\varphi_{\delta,\epsilon}$takes values in $\Lambda = \mathbb{Q}[\delta, \epsilon]$, and δ and ϵ are algebraically independent. This class of genera was introduced by Ochanine in [30], and have since been widely studied. We may observe that the elliptic genus generalizes our previous examples. In particular,

$$\varphi_{1,1} = \sigma,$$

and

$$\varphi_{-\frac{1}{8},0} = \hat{A}.$$

Computing the coefficient of u in $g(u)$ we see that

$$\varphi_{\delta,\epsilon}(\mathbb{C}P^2) = \delta,$$

and from [16, pg. 17] we have that

$$\varphi_{\delta,\epsilon}(\mathbb{H}P^n) = \begin{cases} 0, & n \text{ odd}, \\ \epsilon^{\frac{n}{2}}, & n \text{ even}. \end{cases}$$

Computing the Image of Special Unitary Cobordism

Consider again the commutative diagram

$$\begin{tikzcd} \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \arrow[dr] & \\ \Omega_*^{\mathrm{SO}} \arrow[u] \arrow[r, "\varphi"'] & \Lambda \end{tikzcd}$$

and observe that

Theorem 4.2.1. ([37, pg. 336])

$$\Omega_*^{Spin} \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \Omega_*^{SO} \otimes \mathbb{Z}\left[\frac{1}{2}\right].$$

In particular,

$$\Omega_*^{Spin} \otimes \mathbb{Q} \cong \Omega_*^{SO} \otimes \mathbb{Q}.$$

So, we may extend the commutative diagram to

$$\begin{tikzcd}
\Omega_*^{Spin} \otimes \mathbb{Q} \arrow[r, leftrightarrow, "\cong"] & \Omega_*^{SO} \otimes \mathbb{Q} \arrow[dr] & \\
& \Omega_*^{SO} \arrow[u] \arrow[r, "\varphi"'] & \Lambda
\end{tikzcd}$$

Following [37], define the the ring $B_* \subset \Omega_*^{Spin} \otimes \mathbb{Q}$ to be the ring of elements in $\Omega_*^{Spin} \otimes \mathbb{Q}$ having all integral KO-characteristic numbers. We then have

Theorem 4.2.2. (Stong, [36], [37, pg. 280])

i)

$$B_{8k} \cong \Omega_{8k}^{Spin}/\mathrm{Tor}$$

and

$$2B_{8k+4} \cong \Omega_{8k+4}^{Spin}/\mathrm{Tor}.$$

ii)

$$B_* \cong \mathbb{Z}[x_4, x_8, x_{12}, \dots].$$

To characterize the classes x_{4i} it is sufficient to consider their characteristic numbers modulo 2 by theorem 4.2.1. In this case, we have

Theorem 4.2.3. (Stong, [5], [37, pg. 280]) *The generators of B_* at the prime 2 may be characterized by:*

1. *$\hat{A}(x_4)$ is odd.*
2. *$s_n(p_1, p_2, \dots, p_n)[x_{4n}]$ is odd for $n \neq 2^s$.*
3. *$s_{\frac{n}{2},\frac{n}{2}}(p_1, p_2, \dots, p_n)[x_{4n}]$ is odd for n a power of 2.*

We can now define a homomorphism

$$\varphi : B_* \to \Lambda$$

via the commutative diagram

$$\begin{tikzcd}
B_* \subset \Omega_*^{Spin} \otimes \mathbb{Q} \arrow[r, leftrightarrow, "\cong"] & \Omega_*^{SO} \otimes \mathbb{Q} \arrow[dr] & \\
& \Omega_*^{SO} \arrow[u] \arrow[r, "\varphi"'] & \Lambda.
\end{tikzcd}$$

Since we are interested in computing the image of Ω_*^{SU}, we now must consider the image of the forgetful homomorphism $\Omega_*^{SU} \to \Omega_*^{Spin}/\mathrm{Tor}$, which is given in the following theorem:

Theorem 4.2.4. (Stong, [37, pg. 282]) *The image of the* Ω_*^{SU} *in* $\Omega_*^{Spin}/\mathrm{Tor}$ *is given by the integral span of* $\mathbb{Z}[2x_{4n}]$ *and* $\mathbb{Z}[x_{4n}^2]$.

Finally, we may now define a homomorphism

$$\varphi : \Omega_*^{SU} \to \Lambda$$

via the commutative diagram

$$\begin{tikzcd} \Omega_*^{SU} \arrow[r] & \Omega_*^{Spin}/\mathrm{Tor} \subset B_* \subset \Omega_*^{Spin} \otimes \mathbb{Q} \arrow[r, leftrightarrow, "\cong"] & \Omega_*^{SO} \otimes \mathbb{Q} \arrow[dr] & \\ & & \Omega_*^{SO} \arrow[u] \arrow[r, "\varphi"'] & \Lambda. \end{tikzcd}$$

In particular,

Theorem 4.2.5. (D.& G. Chudnovsky, Landweber, Ochanine, and Stong, [5])

$$\varphi\left(\Omega_*^{SU}\right) = 2\mathbb{Z}[\varphi(x_{4n})] + \mathbb{Z}[\varphi(x_{4n})^2].$$

So, the problem of computing the image of Ω_*^{SU} under a given genus, is reduced to computing the image of B_*. We may wish now to determine specific manifolds whose class we may take for each of the x_{4n}. In [5] Stong suggests manifolds which satisfy the conditions in theorem 4.2.3.

First, note that

$$\hat{A}[\mathbb{C}P^2] = -\frac{1}{8}.$$

So,

$$\hat{A}(8[\mathbb{C}P^2]) = -\frac{8}{8} = -1.$$

Therefore, $8[\mathbb{C}P^2]$ is an appropriate choice for x_4.

Secondly, we can easily check that

$$s_{1,1}[\mathbb{H}P^2] = p_2[\mathbb{H}P^2] = 7.$$

So, we may choose $x_8 = [\mathbb{H}P^2]$.

If $n = 2^j$, we may take

$$x_{4\cdot n} = [\mathbb{H}P^n - \left(\mathbb{H}P^2\right)^{\frac{n}{2}}],$$

since modulo 2,

$$s_{\frac{n}{2},\frac{n}{2}}(p_1, p_2, \ldots, p_n)[\mathbb{H}P^n] \equiv \binom{2n+2}{2} \equiv 1,$$

and

$$s_{\frac{n}{2},\frac{n}{2}}(p_1,p_2,\ldots,p_n)[\left(\mathbb{H}P^2\right)^{\frac{n}{2}}] \equiv 0.$$

If $n \neq 2^j$, we can write $2n = 1 + 2t + k$ and consider the complex $(4n+2)$-dimensional manifold N given by the total space of the fibration

$$\mathbb{C}P(\xi_1^{-2} \otimes \xi_2^{-(2t+1)} \oplus (k+1))$$

over

$$\mathbb{C}P^1 \times \mathbb{C}P^{2t}$$

where $(k+1)$ indicates $k+1$ copies of the trivial line bundle. Recall the map

$$\delta : \Omega_*^U \to \Omega_*^{SU}$$

from chapter 3, and consider the manifold

$$M^{4n} = \delta N = \mathbb{C}P(\xi_1^{-2} \otimes \xi_2^{-(2t+1)} \oplus (k)).$$

We have from [37, pg. 281] that

$$\mathrm{Im}(\delta) \subset 2B_*,$$

so

$$\frac{1}{2}[M^{4n}] \in B_*.$$

Following [5], if we choose t so that

$$\binom{2n-2}{2t} \equiv 0 \mod 2,$$

we have that

$$s_n(p_1,p_2,\ldots,p_n)[\frac{1}{2}M^{4n}] \equiv 1 \mod 2.$$

So, we may take

$$x_{4n} = [\frac{1}{2}M^{4n}].$$

Returning to the example of the elliptic genus, $\varphi_{\delta,\epsilon}$we have the following:

$$\varphi_{\delta,\epsilon}(x_4) = \varphi_{\delta,\epsilon}\left(8\mathbb{C}P^2\right) = 8\varphi_{\delta,\epsilon}\left(\mathbb{C}P^2\right) = 8\delta,$$

and

$$\varphi_{\delta,\epsilon}(x_8) = \varphi_{\delta,\epsilon}\left(\mathbb{H}P^2\right) = \epsilon.$$

For $n = 2^j$ we have

$$\varphi_{\delta,\epsilon}(x_{4n}) = \varphi_{\delta,\epsilon}\left(\mathbb{H}P^n - \left(\mathbb{H}P^2\right)^{\frac{n}{2}}\right) = \varphi_{\delta,\epsilon}\left(\mathbb{H}P^n\right) - \varphi_{\delta,\epsilon}\left(\mathbb{H}P^2\right)^{\frac{n}{2}} = \epsilon^{\frac{n}{2}} - (\epsilon)^{\frac{n}{2}} = 0.$$

For $n \neq 2^j$, we have that x_{4n} is the projectivization of an even-dimensional complex vector bundle. That such projectivizations vanish under the elliptic genus is the main result of [30].

So, we have that $\varphi_{\delta,\epsilon}\left(B_*\right) = \mathbb{Z}[8\delta,\epsilon]$. Applying theorem 4.2.5 we have

Theorem 4.2.6. (D.& G. Chudnovsky, Landweber, Ochanine, and Stong, [5])

$$\varphi_{\delta,\epsilon}\left(\Omega_*^{SU}\right) = \mathbb{Z}[16\delta,(8\delta)^2,2\epsilon,16\delta\epsilon,\epsilon^2].$$

4.3 The Image of Special Unitary Cobordism Under the Witten Genus

In this section, we will discuss the image of Ω_*^{SU} under the Witten genus.

The Witten Genus

The Witten genus, φ_{w} was introduced by Edward Witten in [43], [44], and is defined by the characteristic series

$$Q(x) = \frac{x}{\sigma(x)} = \exp\left(\sum_{k\geq 1} \frac{2}{(2k)!} G_{2k} x^{2k}\right).$$

Here, $\sigma(x)$ is the Weierstrass σ function [41, pg. 447], and

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1}\right) q^n,$$

where B_k is the k^{th} Bernoulli number. Note that G_k is an Eisenstein series, as described in [13].

For manifolds equipped with a string structure, i.e. a lift

$$\begin{tikzcd} & BO\langle 8\rangle \arrow[d, "\lambda"] \\ M \arrow[ur, "\nu"] \arrow[r, "\chi"'] & BO, \end{tikzcd}$$

where $BO\langle 8\rangle$ is the 7-connected cover of BO, we have that

$$\varphi_{\mathrm{w}}[M] \in \mathbb{Z}[G_4, G_6],$$

where $\mathbb{Z}[G_4, G_6]$ is the ring of modular forms with integral Fourier expansion [46]. There is a famous conjecture of Stolz regarding the vanishing of certain manifolds of this type. In particular:

Conjecture (Stolz, [34]). *Suppose $[X] \in \Omega_*^{String}$ admits a Riemannian metric with positive Ricci curvature. Then*

$$\varphi_w(X) = 0.$$

In general, we have

$$\varphi_{\mathrm{w}} : \Omega_*^{SO} \to \Lambda$$

where, as we see in [46],

$$\Lambda = \mathbb{Q}[G_2, G_4, G_6],$$

the ring of quasimodular forms, which is described in [18].

Definition A modular form of weight k is a holomorphic function, f, on the upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$$

such that for for any $\tau \in \mathcal{H}$ and

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

we have that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

We have that G_2 instead satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f(\tau) - \frac{c(c\tau + d)}{4\pi i}.$$

Since G_2 satisfies this "modular-like" equation is called almost modular, or quasimodular in the literature. Elements in the ring $\mathbb{Q}[G_2, G_4, G_6]$ are then called quasimodular forms.

Defining

$$W_i := \varphi_{\mathrm{w}}(x_{4i}),$$

and applying the results of the previous sections, we have

Theorem 4.3.1.

$$\varphi_w\left(\Omega_*^{SU}\right) = \mathbb{Z}[2W_i] + \mathbb{Z}[W_i^2].$$

Computations

We will now give some partial results regarding the explicit values of the W_i's.

Proposition 4.3.2. *$W_1 = \varphi_w(x_4) = 24G_2$.*

Proof. Since the Witten genus is the power series associated to the characteristic series

$$Q(x) = \exp\left(\sum_{k=1}^{\infty} \frac{2}{(2k)!} G_{2k} \cdot x^{2k}\right)$$

we can compute

$$g(x) = \left(\frac{x}{Q(x)}\right)^{-1} = x + \sum_{n=1}^{\infty} \frac{\varphi_{\mathrm{w}}(\mathbb{C}P^n)}{n+1} x^{n+1} = x + G_2 x^2 + \left(\frac{5}{2}G_2^2 + \frac{1}{12}G_4\right)x^5 + ...$$

We then observe that

$$W_1 = \varphi_{\mathrm{w}}(x_4) = 8\varphi_{\mathrm{w}}[\mathbb{C}P^2] = 24G_2.$$

□

Proposition 4.3.3. $W_2 = \varphi_w(x_8) = 2G_2^2 - \frac{5}{6}G_4$.

Proof. To determine $\varphi_{\mathrm{w}}(x_8)$, we decompose (as a manifold in Ω_*^{SO})

$$[\mathbb{H}P^2] = 3[\mathbb{C}P^2]^2 - 2[\mathbb{C}P^4].$$

Standard computations then give us that

$$W_2 = \varphi_{\mathrm{w}}(x_8) = 2G_2^2 - \frac{5}{6}G_4.$$

□

Proposition 4.3.4. *We may choose* $x_{12} = \frac{1}{8}[\mathbb{H}P^3]$*, and* $W_3 = \varphi_w(x_{12}) = -\frac{4}{3}G_2^3 + \frac{1}{3}G_4G_2 + \frac{7}{360}G_6$.

Proof. First, we need to show that $\frac{1}{8}[\mathbb{H}P^3] \in B_*$. Since $\mathbb{H}P^3$ is spin, we just need to show that $\frac{1}{8}[\mathbb{H}P^3]$ has all integral KO-characteristic numbers. For this we recall from [1, 31] that we may compute the KO-theory Pontryagin numbers, π_I, of $[\mathbb{H}P^3]$. One can compute that:

$$\pi_1[\mathbb{H}P^3] = 0 \qquad \pi_1^3[\mathbb{H}P^3] = p_1^3[\mathbb{H}P^3] = 64$$

$$\pi_1^2[\mathbb{H}P^3] = -8 \qquad \pi_1\pi_2[\mathbb{H}P^3] = p_1p_2[\mathbb{H}P^3] = 48$$

$$\pi_2[\mathbb{H}P^3] = 0 \qquad \pi_3[\mathbb{H}P^3] = p_3[\mathbb{H}P^3] = 8$$

Since each of these is divisible by 8, the KO-Pontryagin numbers for $\frac{1}{8}[\mathbb{H}P^3]$ are all integral. Therefore, its KO-characteristic numbers are all integral.

Next, we need to show that $s_3[\frac{1}{8}\mathbb{H}P^3]$ is odd. Now,

$$s_3(M) = p_1^3 - 3p_1p_2 + 3p_3.$$

Since [16],

$$p(\mathbb{H}P^3) = \frac{(1+u)^8}{1+4u} = 1 + 4u + 12u^2 + 8u^3,$$

we have that

$$s_3[\mathbb{H}P^3] = -56.$$

So,

$$s_3[\frac{1}{8}\mathbb{H}P^3] = -7.$$

So, $\frac{1}{8}[\mathbb{H}P^3]$ is an appropriate choice for x_{12}.

Finally, we just need to determine the image of $\frac{1}{8}\mathbb{H}P^3$ under the Witten genus. To do this, we decompose

$$\frac{1}{8}[\mathbb{H}P^3] = [\mathbb{C}P^6] - 3[\mathbb{C}P^4] \times [\mathbb{C}P^2] + 2[\mathbb{C}P^2]^3$$

and compute

$$W_3 = \varphi_{\mathrm{w}}(x_{12}) = \varphi_{\mathrm{w}}(\frac{1}{8}\mathbb{H}P^3) = -\frac{4}{3}G_2^2 + \frac{1}{3}G_4G_2 + \frac{7}{360}G_6.$$

□

We have that

$$W_1 = 24G_2,$$

$$W_2 = 2G_2^2 - \frac{5}{6}G_4,$$

and

$$W_3 = -\frac{4}{3}G_2^3 + \frac{1}{3}G_4G_2 + \frac{7}{360}G_6.$$

For x_{4i} with $i \geq 4$, we refer to [5] and use manifolds similar to those suggested by Landweber. If $i = 2^j$ for some j, we may take

$$x_{4\cdot 2^s} = [\mathbb{H}P^{2^s}].$$

If $i \neq 2^j$, we may take

$$x_{4i} = \frac{1}{2}[M^{4i}]$$

where M^{4i}. It is clear from the previous section that these are valid choices for the classes x_{4i}.

We have from [18, pg. 3] that every quasimodular form can be expressed as a polynomial in G_2 with coefficients that are modular forms, i.e. rational polynomials in G_4 and G_6. So one may hope that W_i for $i > 3$ is expressible as a polynomial in W_1, W_2, and W_3. The question of whether or not this is true is, as yet open. In the final chapter, we will discuss this, and other open questions.

5 The Open Road Ahead

The results presented in this thesis lead to several questions. In fact, it could rightly be stated that the author, at the end of this work, is left with more questions than answers. In this final chapter, we will discuss and state several such questions.

In Chapter 3, we showed that neither non-singular algebraic varieties nor complex manifolds can be taken as representatives for every class in Ω_*^{SU}. So, we may conclude that whatever class of manifold is the class of preferred representatives for SU-manifolds, it must be a class of stably almost complex SU-manifolds. So, the first question is this:

Question 5.0.1. *What class of stably almost complex SU-manifolds is the class of preferred representatives for SU-cobordism?*

Immediately following question 5.0.1, and in light of the results of Chapter 4, we may ask the following:

Question 5.0.2. *Can one choose a more convenient set of generators for B_*?*

Aside from its application to the work of this thesis, the image of the Witten genus is of general interest. In particular, refining the results of Chapter 4 may lead to new classes of manifolds which vanish under the Witten genus.

With respect to our interest in the Witten genus, this "more convenient" set of generators may help to make answering the following question more straightforward:

Question 5.0.3. *is the image of Ω_*^{SU} under the Witten genus finitely generated?*

The ring of quasimodular forms is finitely generated by G_2, G_4, and G_6, and one may express G_2, G_4, and G_6 rationally in terms of W_1, W_2, and W_3. However, since the image of Ω_*^{SU} is in the integral span of the W_i's it is unclear whether or not we can express W_i with $i \geq 4$ in terms of W_1, W_2, and W_3. If we could choose manifolds in the classes x_{4i} that vanish under the Witten genus, as in theorem 4.2.5, then the question would be answered. However, recalling the conjecture of Stolz mentioned in §4.3, finding manifolds whose image vanishes on the Witten genus is not an entirely trivial matter.

Returning to questions like question 5.0.1, there are classes in Ω_*^{SU} that deserve some special attention. In [8], Conner and Floyd completely determined the torsion of Ω_*^{SU}. They also describe, quite explicitly, the torsion elements themselves:

Theorem 5.0.4. (Conner and Floyd, [8, pg. 69]) *Consider the forgetful homomorphism*

$$\alpha : \Omega_*^{SU} \to \Omega_*^U,$$

and also

$$\partial := \alpha\delta : \Omega_*^U \to \Omega_*^U.$$

The image of α *contains the image of* ∂. *There exist closed SU-manifolds,* W^{8k} *with* $k \geq 1$*, such that*

$$Im(\alpha)/Im(\delta) \cong \mathbb{Z}/2[W^{8k}].$$

Every torsion element of Ω_*^{SU} *is of the form*

$$[V^{8k} \times \hat{S}]$$

or

$$[V^{8k} \times \hat{S} \times \hat{S}]$$

where V^{8k} *is a polynomial in the* W^{8k} *with coefficients* 0 *or* 1.

Furthermore, they provide a characterization of the W^{8k}:

Theorem 5.0.5. *The* W^{8k} *in theorem 5.0.4 are characterized by:*

1. W^8 *has odd Todd genus.*
2. *If* $k \neq 2^j$*, then* $s_{2k,2k}(c_1, ..., c_{4k})[M^{8k}]$ *is odd.*
3. *If* $k = 2^j$ *for some* j*, then* $s_{k,k,k,k}(c_1, ..., c_{4k})[M^{8k}]$ *is odd.*

As we noted in §3.2, the non-trivial element in Ω_2^{SU} can be represented by a non-singular algebraic variety. So, if we could find algebraic representatives for the W^{8k}, we would be able to represent every torsion element in Ω_*^{SU} by a non-singular algebraic variety. Therefore, we can consider the following question:

Question 5.0.6. *Can we represent torsion elements in* Ω_{8k+2}^{SU} *with non-singular algebraic varieties?*

Finally, it would be nice to see an answer to the following:

Question 5.0.7. *What obstruction to algebraicity is obtained by putting an SU-structure on a stably complex manifold?*

Since, as we observed in §3.1, every SU-manifold is also a stably complex manifold, we have from theorem [24] that every SU-manifold is complex cobordant to a non-singular algebraic variety. The author would very much like to understand what aspect of the SU-structure prevents extending this property to SU-cobordism.

Appendix

A The s-class

A.1 What's in a name?

One of the most frequently used characteristic classes in the study of cobordism theory is what we've called the s-class. This class, which detects when a manifold is indecomposable, appears in virtually every reference given in this thesis, and in many others. However, perplexingly, it does not seem to have an accepted name in the literature.

Wilfong, in [42], calls these numbers 'Milnor numbers', due to the frequency of their use in his work. Milnor has attributed this number to Thom [25]. Thom, on the other hand, attributes the numbers to Pontryagin [39]. Wall in [40] also attributes these numbers to Pontryagin. Stong, chooses different naming conventions, and calls them both 'Chern numbers' [37, pg. 259] and, as we do here, s-numbers [37, pg. 263].

In [26, pg. 189], the s-number is not named explicitly, but Milnor says more information is available in Percy Macmahon's *Combinatory Analysis*. In that volume, Macmahon credits the polynomials that define the s-numbers to Albert Girard, and gives reference to his 1629 paper *Invention Nouvelle en L'Algebre.*

A.2 Results on the s-number of Certain Special Unitary Manifolds

While working to directly emulate the proof of theorem 2.2.2 using SU-manifolds we discovered an interesting number-theoretic problem. The minimal value of the s-number for SU-manifolds is given in [37, pg. 262]. We wondered if the greatest common divisor of the s-numbers of manifolds

$$V^{2n-2} = \delta[\mathbb{C}P^{i_1} \times \mathbb{C}P^{i_2} \times \cdots \times \mathbb{C}P^{i_k}],$$

where δ is the homomorphism given in §3.1 and $I = \{i_1, \ldots, i_k\}$ is a partition of n other than $\{1, (n-1)\}$, would match this value. It is easy to see that s-numbers of the V^{2n-2}'s is a multiple of a multinomial coefficient. So, we were left with this number-theoretic problem regarding the greatest common divisors of certain multinomial coefficients, which, although eventually unnecessary for the work of this thesis, turned out to provide some interesting results in and of itself[1].

We will denote the set of all partitions of n by $P(n)$, and the set of all partitions with parts of size at most $n-2$ as $\hat{P}(n)$. A generic partition contained in $\hat{P}(n)$ will be denoted $\sigma \in \hat{P}(n)$.

[1]This section of the appendix appears in [27].

Definition The multinomial coefficient $\binom{n}{\sigma_1,\sigma_2,\ldots,\sigma_t}$ is defined by

$$(x_1+x_2+\ldots+x_t)^n = \sum_{(\sigma_1,\sigma_2,\ldots,\sigma_t)\in P(n)} \binom{n}{\sigma_1,\sigma_2,\ldots,\sigma_t} x_1^{\sigma_1}x_2^{\sigma_2}\ldots x_t^{\sigma_t}.$$

It will occasionally be convenient to denote the multinomial coefficient $\binom{n}{\sigma_1,\sigma_2,\ldots,\sigma_t}$ associated to the partition $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_t)$ of n by $\binom{n}{\sigma}$.

Definition The p-adic expansion of n is the unique expansion $n = \sum_{i=0}^{\infty} a_i p^i$ with $0 \le a_i \le p-1$.

Definition The p-adic order of n, denoted $\nu_p(n)$, is the largest power k of p such that $p^k|n$.

Main Result of the Appendix

The goal of this section of the appendix is to prove the following:

Proposition A.2.1.

$$\gcd_{\sigma\in\hat{P}(n)} \binom{n}{\sigma} = \begin{cases} p & \text{if } n=p^s \\ q & \text{if } n=q^t+1 \\ p\cdot q & \text{if } n=p^s \text{ and } n=q^t+1 \\ 1 & \text{else} \end{cases}$$

Now, recall that

$$\binom{n}{\sigma} = \binom{n}{\sigma_1,\sigma_2,\ldots,\sigma_t} = \frac{n!}{\sigma_1!\sigma_2!\ldots\sigma_t!}.$$

So, our goal will be to determine when we can find, for each fixed prime $p<n$, a partition $\sigma\in\hat{P}(n)$ with

$$\nu_p(\sigma) := \nu_p(\sigma_1!)+\nu_p(\sigma_2!)+\ldots+\nu_p(\sigma_t!) = \nu_p(n!).$$

First, let's recall from [22], the value of $\nu_p(n!)$.

Theorem (Legendre, 1808). $\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$.

We will also use the following two corollaries of this theorem:

Corollary A.2.1. *Let $n=a_0+a_1p+a_2p^2+\ldots+a_sp^s$ be the p-adic expansion of n. Then,*

$$\nu_p(n!) = a_1\cdot\nu_p(p!)+a_2\cdot\nu_p(p^2!)+\ldots+a_s\cdot\nu_p(p^s!).$$

Proof. We begin with the formula in Legendre's theorem, and replace n by its p-adic expansion.

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{a_0 + a_1 p + a_2 p^2 + ... + a_s p^s}{p^i} \right\rfloor .$$

Expanding the sum, we get

$$\begin{aligned}
(a_1 + a_2 p + ... + a_s p^{s-1}) + (a_2 + a_3 p + ... + a_s p^{s-2}) + ... + (a_{s-1} + a_s p) + (a_s) \\
= a_1 + a_2(1+p) + a_3(1+p+p^2) + ... + a_s(1+p+...+p^{s-1}) \\
= \sum_{i=1}^{s} a_i \sum_{j=0}^{i-1} p^j \\
= \sum_{i=1}^{s} a_i \sum_{j=1}^{i} \left\lfloor \frac{p^i}{p^j} \right\rfloor \\
= a_1 \cdot \nu_p(p!) + a_2 \cdot \nu_p(p^2!) + ... + a_s \cdot \nu_p(p^s!).
\end{aligned}$$

□

Corollary A.2.2. *$\nu_p(p^m!) = 1 + p \cdot \nu_p(p^{m-1}!)$ for all $m \geq 1$.*

Proof. Again, we begin with the formula in Legendre's theorem, apply a few elementary algebraic operations, and mathematical induction.

$$\begin{aligned}
\nu_p(p^m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{p^m}{p^i} \right\rfloor = \sum_{i=1}^{m} \left\lfloor \frac{p^m}{p^i} \right\rfloor \\
= 1 + \sum_{i=1}^{m-1} \left\lfloor \frac{p^m}{p^i} \right\rfloor \\
= 1 + p \cdot \sum_{i=1}^{m-1} \left\lfloor \frac{p^{m-1}}{p^i} \right\rfloor \\
= 1 + p \cdot \nu_p(p^{m-1}!).
\end{aligned}$$

□

We will also make use of the following proposition:

Proposition A.2.2. *Suppose $n = p^s$ or $n = q^t + 1$, then p (respectively, q) divides $\binom{n}{\sigma}$ for every $\sigma \in \hat{P}(n)$.*

Proof. For brevity, we prove only the case that $n = p^s$. The case that $n = q^t + 1$ is proved similarly.

Suppose that $n = p^s$, and let $\sigma \in \hat{P}(n)$. For each part σ_i of σ we can consider its p-adic expansion:

$$\sigma_i = a_{i,0} + a_{i,1}p + ... + a_{i,(s-1)}p^{s-1}.$$

Note that since $n = p^s$, no p-adic expansion of any part of $\sigma \in \hat{P}(n)$ has non-zero coefficient on the p^s term. Also, since

$$n = p^s = \sigma_1 + \sigma_2 + ... + \sigma_f$$

$$= (a_{1,0}+a_{1,1}p+...+a_{1,(s-1)}p^{s-1})+(a_{2,0}+a_{2,1}p+...+a_{2,(s-1)}p^{s-1})+...+(a_{f,0}+a_{f,1}p+...+a_{f,(s-1)}p^{s-1})$$

$$= (a_{1,0} + ... + a_{f,0}) + (a_{1,1} + ... + a_{f,1})p + ... + (a_{1,(s-1)} + ... + a_{f,(s-1)})p^{s-1}$$

we can observe that $a_{1,(s-1)} + ... + a_{f,(s-1)}$ is at most p.

Now we observe the following two cases. If $a_{1,(s-1)} + ... + a_{f,(s-1)} = p$, then we are in the case presented in Corollary $A.2.2$, and we have that

$$p \cdot \nu_p(p^{s-1}!) < \nu_p(p^s!).$$

So, $p | \binom{n}{\sigma}$.

On the other hand, if $a_{1,(s-1)} + ... + a_{f,(s-1)} < p$, then there is some $0 < j < s-1$ for which the sum $a_{1,j} + ... + a_{f,j} > 1$, and it follows from Corollary 2 that

$$\nu_p(\sigma) \leq p \cdot \nu_p(p^{s-2}!) + (p-1) \cdot \nu_p(p^{s-1}!) < p \cdot \nu_p(p^{s-1}!) < \nu_p(p^s!).$$

So, $p^k | \binom{n}{\sigma}$ for some $k \geq 2$.

Therefore, p divides $\binom{n}{\sigma}$ for every $\sigma \in \hat{P}(n)$. $\square$

Finally, define for each p the p-adic partition of n to be

$$\sigma_p(n) := \left(\underbrace{p^s, ..., p^s}_{a_s \text{ entries}}, \underbrace{p^{s-1}, ..., p^{s-1}}_{a_{s-1} \text{ entries}}, ..., \underbrace{p, ..., p}_{a_1 \text{ entries}}, \underbrace{1, ..., 1}_{a_0 \text{ entries}} \right).$$

Proof of Proposition A.2.1. We are now ready to prove the main result of the appendix. The goal, again, is to determine when we can find, for each prime $p < n$, a partition of n whose associated multinomial coefficient is not divisible by p.

First, suppose that n is neither a prime power nor one more than a prime power. It follows from Corollary $A.2.1$ that for each prime $p < n$, $p \nmid \binom{n}{\sigma_p(n)}$. Since, for each prime $p < n$, we have a multinomial coefficient not divisible by p, the greatest common divisor of multinomial coefficients over all partitions in $\hat{P}(n)$ is 1.

On the other hand, suppose that $n = p^s$ or $n = q^t + 1$. Then, we have that $\sigma_p = (p^s)$ or $\sigma_q = (q^t, 1)$. Note that these partitions are not in $\hat{P}(n)$. Let's define instead

$$\hat{\sigma}_p(n) := \left(\underbrace{p^{s-1}, ..., p^{s-1}}_{p \text{ entries}} \right),$$

and

$$\hat{\sigma}_q(n) := \left(\underbrace{q^{t-1}, ..., q^{t-1}}_{q \text{ entries}}, 1 \right).$$

It follows from Corollary $A.2.2$ that $p | \binom{n}{\hat{\sigma}_p(n)}$, but $p^2 \nmid \binom{n}{\hat{\sigma}_p(n)}$, and from Proposition $A.2.2$ that $p | \binom{n}{\sigma}$ for every $\sigma \in \hat{P}(n)$. Similarly, $q | \binom{n}{\hat{\sigma}_q(n)}$, but $q^2 \nmid \binom{n}{\hat{\sigma}_q(n)}$, and $q | \binom{n}{\sigma}$ for every $\sigma \in \hat{P}(n)$. So, if $n = p^s$ or $n = q^t + 1$ we can consider the multinomial coefficient associated to the r-adic partition for any prime, r, less than $n - 1$, and $\hat{\sigma}_p(n)$ (respectively, $\hat{\sigma}_q(n)$). Then, the greatest common divisor of multinomial coefficients over all partitions in $\hat{P}(n)$ is p (respectively, q).

Finally, if $n = p^s$ and $n = q^t + 1$, we can consider the multinomial coefficient associated to the r-adic partition for any prime, r, less than $n - 1$, $\hat{\sigma}_p$, and $\hat{\sigma}_q$. This gives that the greatest common divisor of multinomial coefficients over all partitions in $\hat{P}(n)$ is $p \cdot q$.

$\square$

Further Refinement and Interesting Connections

The penultimate case stated in Proposition A.2.1, when $n = p^s = q^t + 1$, is of particular interest. Of course, if $p^s = q^t + 1$, then $p^s - q^t = 1$. Solutions to this equation are the subject of Eugène Catalan's famous conjecture from 1844 that was proved by Preda Mihăilescu in 2004 [23]:

Theorem (Mihăilescu, 2004, conj. Catalan, 1844). *For p, q prime, and $s, t > 1$, the Diophantine equation $p^s - q^t = 1$ admits only one solution. In particular, $3^2 - 2^3 = 9 - 8 = 1$.*

We note, however, that there are other solutions when either s or t is 1. In particular, if $s = t = 1$, we have that

$$3^1 - 2^1 = 1.$$

If $s = 1$ with $t > 1$, we have that $p^1 = q^t + 1$ must be odd, since if $p = 2$, $q^t \leq 1$. So, p must be an odd prime, and $q = 2$. Primes of this form, $p = 2^t + 1$, are called *Fermat primes*. Some examples of Fermat primes are

$$5 = 2^2 + 1,$$

$$17 = 2^4 + 1,$$
$$257 = 2^8 + 1,$$

and

$$65537 = 2^{16} + 1.$$

In fact, this list, together with the case $3 = 2^1 + 1$ is the complete list of known Fermat primes.

Conversely, if $t = 1$ with $s > 1$, we have that $p^s = q^1 + 1$ must be even, since if $q = 2$, $p^s = 3$, so $p = 3$ and $s = 1$. So, q must be an odd prime, and $p = 2$. Primes of this form, $q = 2^s - 1$, are called *Mersenne primes*. Some examples of Mersenne primes are

$$7 = 2^3 - 1,$$
$$31 = 2^5 - 1,$$

and

$$127 = 2^7 - 1.$$

There are currently 48 known Mersenne primes, the largest of which is $2^{57885161} - 1$.

Now we can observe the following refinement of Proposition A.2.1. It is interesting to note that it is currently unknown if there are infinitely many examples satisfying cases 3 and 4 of this corollary.

Corollary A.2.3.

$$\gcd_{\sigma \in \hat{P}(n)} \binom{n}{\sigma} = \begin{cases} p & \textit{if } n = p^s \\ q & \textit{if } n = q^t + 1 \\ 2 \cdot q & \textit{if } n = p^s = q^t + 1 \textit{ and } n \textit{ even} \\ 2 \cdot p & \textit{if } n = p^s = q^t + 1 \textit{ and } n \textit{ odd} \\ 1 & \textit{else} \end{cases}$$

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