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UNIVERSAL CONSTRAINTS ON 2D CFTS AND 3D GRAVITY

Joshua D. Qualls
University of Kentucky, joshua.qualls@uky.edu

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Joshua D. Qualls, Student

Dr. Alfred Shapere, Major Professor

Dr. Timothy Gorringe, Director of Graduate Studies

UNIVERSAL CONSTRAINTS ON 2D CFTS AND 3D GRAVITY

DISSERTATION

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy in the
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By

Joshua Qualls

Lexington, Kentucky

Director: Dr. Alfred Shapere, Professor of Department of Physics and Astronomy

Lexington, Kentucky

2014

ABSTRACT OF DISSERTATION

UNIVERSAL CONSTRAINTS ON 2D CFTS AND 3D GRAVITY

We study constraints imposed on a general unitary two-dimensional conformal field theory by modular invariance. We begin with a review of previous bounds on the conformal dimension Δ_1 of the lowest primary operator assuming unitarity, a discrete spectrum, modular invariance, $c, \bar{c} > 1$, and no extended chiral algebra. We then obtain bounds on the conformal dimensions Δ_2, Δ_3 using no additional assumptions. We also show that in order to find a bound for Δ_4 or higher Δ_n, we need to assume a larger minimum value for c_{tot} that grows logarithmically with n. We next extend the previous results to remove the requirement that our two-dimensional conformal field theories have no extended chiral algebra.

We then show that modular invariance also implies an upper bound on the total number of states of positive energy less than $c_{\text{tot}}/24$ (or equivalently, states of conformal dimension Δ between $c_{\text{tot}}/24$ and $c_{\text{tot}}/12$), in terms of the number of negative energy states. Finally, we consider the case where the CFT has a gravitational dual and investigate the gravitational interpretation of our results. Using the $\text{AdS}_3/\text{CFT}_2$ correspondence, we obtain an upper bound on the lightest few massive excitations (both with and without the constraint of no chiral primary operators) in a theory of 3D matter and gravity with $\Lambda < 0$. We show our results are consistent with facts and expectations about the spectrum of BTZ black holes in 2+1 gravity. We then discuss the upper and lower bounds on number of states and primary operators in the dual gravitational theory, focusing on the case of AdS_3 pure gravity.

KEYWORDS: Conformal Field Theory, Modular Invariance, AdS/CFT Correspondence, BTZ Black Holes, Bounds

Joshua Qualls

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UNIVERSAL CONSTRAINTS ON 2D CFTS AND 3D GRAVITY

By

Joshua Qualls

Dr. Alfred Shapere
(Director of Dissertation)

Dr. Timothy Gorringe
(Director of Graduate Studies)

9 May, 2014
(Date)

Dedicated to Kylie. Your love and support over the years have made the difference. I am better for knowing you, and I will be thankful for the rest of my life for absolutely everything that you've done in helping me become a better version of myself.

And to Isaac. For too long we had only the sounds of your footsteps and laughter in our dreams. Yet even from these smallest of things, we knew we could not wait to love you forever.

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Chapter 1

Introduction

Quantum field theory is the mathematical and conceptual framework based on the physical principles of special relativity and quantum mechanics for constructing models describing the fundamental constituents and interactions of nature. In spite of its many successes, however, quantum field theory is not without shortcomings. For example, perturbation theory is not applicable to systems in which the couplings become large. So although the Lagrangian approach to quantum field theory is powerful and even captures some important semi-classical non-perturbative effects (such as solitons and instantons [1]), it can obscure important, genuinely non-perturbative effects. There are several powerful techniques for studying non-perturbative aspects of quantum field theory. A particularly fruitful approach has been to study quantum field theory in two dimensions. With examples ranging from the non-linear sigma model with non-perturbative mass generation [2, 3] to the Schwinger model with the confinement of electric charge [4] to the Thirring model and the fermion-boson equivalence to the sine-Gordon model [5, 6, 7], two-dimensonal quantum field theories have become a crucial tool for studying non-perturbative effects. Much of what we have learned about general non-perturbative phenomena in quantum field theory has its origin in such two-dimensional models.

Another important method for understanding non-perturbative phenomena in quantum field theories is conformal invariance. Conformal field theories are quantum field theories that, in addition to the usual invariance under Poincaré transformations, possess invariance under local conformal transformations (i.e. transformations that preserve angles but not lengths). It has been shown that in two-dimensional unitary interacting quantum field theories with well-defined correlation functions, scale invariance implies conformal invariance [8, 9]. Thus in the cases we are to consider conformal field theories can be thought of as quantum field theories with scale invariance. Conformal field theories (CFTs) have received a great deal of attention during the last few decades due to their importance in many different areas of theoretical physics. Their utility ranges from playing a central role in string theory to serving as useful models for interacting quantum field theories to describing two-dimensional critical phenomena and phase transitions. Conformal field theories have also had an important impact in several branches of mathematics, from the theory of vertex operator algebras and generalized Kac-Moody algebras to number theory and low-dimensional topology [10, 11, 12, 13, 14].

While relatively little is known about CFTs in general, in recent years a number of constraints on their spectra and amplitudes have been obtained by means of conformal bootstrap techniques [15, 16, 17]. The conformal bootstrap condition says that the operator algebra must be associative— when calculating the correlator of four primaries

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle,$$

for example, the answer should be the same when we reduce it to a sum of two-point functions via operator product expansions whether we are in the (12)(34) or (14)(23) channel. This results in a quadratic condition on the structure constants of the operator product expansions, and the hope is that imposing this associativity on sufficiently many primary fields will allow one to determine the CFT data and therefore solve the CFT. In [18, 19], bounds on the conformal dimensions of operators in four-dimensional unitary CFTs were derived from the condition of crossing symmetry of the four point function, using explicit expressions for the conformal blocks obtained in [20, 21]. More recently, similar methods have been applied to CFTs in diverse dimensions with great success [22, 23, 24, 25, 26, 27, 28, 29].

Local conformal symmetry is of special importance in two dimensions since the corresponding symmetry algebra in this case is infinite-dimensional. As a consequence, two-dimensional conformal field theories have an infinite number of conserved quantities and are completely solvable by symmetry considerations. The requirement of local conformal invariance might seem overly restrictiv; because the theory is scale invariant, for example, all particle-like excitations are necessarily massless. This could be seen as an argument against any possible physical applications for such theories. At energy scales far above or below all mass scales, all quantum field theories approach scale invariant theories. Thus the infrared and ultraviolet limits of the renormalization group flow of any 2D quantum field theory is a CFT. Deformations away from these CFTs obrtained by turning on masses or other parameters can be described perturbatively. [30, 31].

In two dimensions, Hellerman [32] used modular invariance of the partition function to derive a bound on Δ_1, the conformal dimension of the lowest non-vacuum primary operator, in any unitary 2D CFT with no chiral primary operators other than the identity operator and with left and right central charge $c, \tilde{c} > 1$:

$$\Delta_1 \leq \frac{c + \bar{c}}{12} + 0.4736... \tag{1.1}$$

Assuming $c > 1$ has well-known and useful implications for the structure of representations of the Virasoro algebra [33, 34, 35] (compact, unitary CFTs with $c < 1$ are also completely classified [36]), and assuming the theory has no chiral algebra beyond the Virasoro algebra simplifies the Virasoro representations. More recently, Friedan and Keller [37] investigated additional constraints from modular invariance systematically. Building on the work of [32], they applied the next several differential constraints using the linear functional method and found that for finite c_{tot} the bound (1.1) can be lowered somewhat. For large c_{tot}, however, the bounds apparently all asymptote to $\frac{c_{\text{tot}}}{12}$ as in (1.1).

The goal of this work is to study constraints imposed on a general unitary two-dimensional conformal field theory by modular invariance. We will begin by introducing the essentials of two-dimensional conformal field theories in Chapter 2, with special emphasis on the results that have direct relevance to our derivations. Topics include the two-dimensional conformal group, conformal transformations and generators, primary and descendant fields, radial quantization, the Virasoro algebra, central charge, and conditions required for unitarity. We then review modular invariance and basic consequences following from this property.

In Chapter 3, we provide a detailed review of Hellerman's work [32] on deriving a bound on the conformal dimension Δ_1 of the lowest primary operator assuming unitarity, a discrete spectrum, modular invariance, $c, \bar{c} > 1$, and no extended chiral algebra. We then extend the work of [32] to obtain bounds on the conformal dimensions Δ_2, Δ_3 using no additional assumptions. The bounds we obtain take the same form as (1.1), with the same asymptotic bound $c_{\rm tot}/12$. We also investigate the possibility of deriving bounds on primary operator conformal dimensions Δ_n for $n > 3$. We find that in order to obtain a bound for Δ_4 or higher, we need to assume a larger minimum value for $c_{\rm tot}$ that grows logarithmically with n. For asymptotically large $c_{\rm tot}$ with $c_{\rm tot} \gtrsim \frac{12}{\pi} \log n$, we show that all the Δ_n obey a bound of the same form as (1.1):

$$\Delta_n \leq \frac{c_{\rm tot}}{12} + O(1).$$

These bounds, satisfied for fixed $c_{\rm tot}$ by all Δ_n with $\log n \lesssim \pi c_{\rm tot}/12$, collectively imply that the total number of primaries of dimension $\Delta \lesssim c_{\rm tot}/12$ grows at least exponentially with $c_{\rm tot}$

$$N(c_{\rm tot}/12) \gtrsim \exp\left(\frac{\pi c_{\rm tot}}{12}\right)$$

In Chapter 4, we extend the previous results to remove the requirement that our two-dimensional conformal field theories have no extended chiral algebra. Assuming the theory has no chiral algebra beyond the Virasoro removed from consideration primary operators with left and right dimensions $h = 0, \bar{h} \neq 0$ or vice versa. Including such operators changes the partition function and thus the resulting constraints. Removing this restriction on the CFTs under consideration results in an only slightly weaker bound on the conformal dimensions Δ_1, Δ_2. Following ideas introduced in Chapter 3, we extend the results to a similar bound on Δ_n for $n \lesssim \exp(\pi c_{\rm tot}/12)$, constraining the CFT spectrum and resulting in another asymptotic lower bound on the number of primary operators with dimension less than or equal to $c_{\rm tot}/12$.

In Chapter 5, we turn our focus to different consequences of invariance of the partition function under the S-transformation. Rather than considering the ratio of two different-order polynomial constraints (suitable for bounding conformal dimensions), we turn our attention to consequences of single polynomial constraints (suitable for bounding the number of states in a given energy range). We find an upper bound on the number of states of positive energy less than $c_{\rm tot}/24$ (or, equivalently, of conformal dimension $c_{\rm tot}/24 < \Delta \leq c_{\rm tot}/12$) in terms of the number of negative energy states. In the case of pure gravity and large central charge, this gives a leading dependence on the central charge going as $\exp(\pi c_{\rm tot}/6)$. We then perform a similar analysis to obtain an upper bound on the number of primaries with positive energy less than $c_{\rm tot}/24$ before discussing possible extensions to higher-order single polynomial constraints.

Finally, we consider in Chapter 6 the case where the CFT has a gravitational dual and investigate the gravitational interpretation of our results. Using the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, Hellerman's bound on Δ_1 implies a bound on the lightest massive black hole state in the corresponding dual theory of 3D matter and gravity subject to his assumptions about the boundary CFT. We provide a brief summary of the

relevant dictionary between the two theories, before translating the results of work in the previous chapters into gravitational language. We thus find an upper bound on the lightest few massive excitations (both with and without the constraint of no chiral primary operators) in a theory of 3D matter and gravity with $\Lambda < 0$. We show our results are consistent with facts and expectations about the spectrum of BTZ black holes in 2+1 gravity. In particular, our bound on the number of primary operators corresponds to a lower bound in the flat-space limit on the number of gravitational states without boundary excitations, of mass less than or equal to $1/4G_N$. We then discuss the upper bounds on number of states and primary operators in the dual gravitational theory, focusing on the case of AdS_3 pure gravity.

Chapter 2

Unitary 2D CFTs and Modular Invariance

In this chaper, we introduce the basics of unitary, two-dimensional conformal field theories. The topics include the two-dimensional conformal group, conformal transformations and generators, primary and descendant fields, radial quantization, the Virasoro algebra, central charge, and conditions required for unitarity. We then review modular invariance, and basic consequences coming from modular invariance. The discussion in section (2.1) ff. is based heavily on [36, 38, 39, 40, 41]. We refer the reader to these references for a more complete discussion.

2.1 The conformal group in two dimensions

We begin by studying the conformal group and conformal transformations in two dimensions. A conformal transformation is defined as an invertible mapping $x \to x'$ that leaves the metric tensor invariant up to a local scale transformation

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{2.1}$$

For an infinitesimal conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the metric changes to first order in ϵ according to

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{2.2}$$

Requiring that this transformation be conformal as in eq. (2.1) implies

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \lambda(x) g_{\mu\nu}, \tag{2.3}$$

where the factor $\lambda(x)$ can be found to be

$$\lambda(x) = \frac{2}{d} \partial_\rho \epsilon^\rho. \tag{2.4}$$

To simplify the discussion here, we will assume the conformal transformation is an infinitesimal deformation of the flat Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu} \equiv \mathrm{diag}(1, 1, ..., 1)$. By taking linear combinations and permutations of eqs. (2.3) and (2.4) and their derivatives, it is possible to show

$$(2 - d)\partial_\mu \partial_\nu \lambda = \eta_{\mu\nu} \partial^2 \lambda. \tag{2.5}$$

From this expression, it is clear that the $d = 2$ case will be special; it is to this case we turn our attention.

For two-dimensional conformal field theories, we define the coordinates (z^0, z^1) on the plane and consider a change of coordinates $z^\mu \to w^\mu(x)$. For $d = 2$, eq. (2.5) implies λ is harmonic. This observation motivates the use of complex coordinates

$z \equiv z^0 + iz^1$ and $\bar{z} \equiv z^0 - iz^1$ as well as holomorphic (and antiholomorphic) derivative $\partial \equiv \frac{1}{2}(\partial_0 - i\partial_1)$ (and $\bar{\partial}$) and metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Using these variables, the holomorphic Cauchy-Riemann equations become

$$\bar{\partial} w(z, \bar{z}) = 0$$

whose solution is any holomorphic mapping $z \to w(z)$ (with a similar statement for antiholomorphic mappings). Therefore the group of conformal transformations in two dimensions is homeomorphic to the group of analytic maps with composition as the group operation. This group is clearly infinite-dimensional, as infinitely many Laurent coefficients are required to specify all functions analytic in some neighborhood. It is precisely this infinite dimensionality that allows so much to be known about two-dimensional conformal field theories.

We have not yet considered global conformal transformations, which are required to be defined everywhere in the complex plane and be invertible. The set of global conformal transformations form the special conformal group, and the complete set of these mappings as we will soon see is

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}. \tag{2.6}$$

These mappings are called projective transformations. They can be associated with matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that the composition of two maps corresponds to matrix multiplication. Clearly sending a, b, c, d to their negatives does not affect the transformation (2.6), and thus the global conformal group in two dimensions is isomorphic to $PSL(2, \mathbb{C})$.

2.2 Classical conformal generators and primaries

We now study the classical local conformal group. Although not the complete story, the classical case provides a natural introduction to important ideas and serves as a foundation for later work. In order to calculate the commutation relations of the generators of the infinite-dimensional conformal algebra, we consider infinitesimal holomorphic tranformations. Any holomorphic infinitesimal transformation can be expressed as

$$z' = z + \epsilon(z), \quad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}. \tag{2.7}$$

The effect of this mapping on a dimensionless scalar field $\phi(z, \bar{z})$, for example, is

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}) = \phi(z, \bar{z}) - \epsilon(z')\partial'\phi(z, \bar{z}) - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z, \bar{z}) \tag{2.8}$$

$$\Rightarrow \delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_n \left[c_n \ell_n \phi(z,\bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z,\bar{z})\right].$$

Here we have introduced the generators

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_m = -\bar{z}^{m+1}\bar{\partial} \tag{2.9}$$

which obey the Witt algebras

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \\ [\ell_n, \bar{\ell}_m] &= 0. \end{aligned} \tag{2.10}$$

In the quantum case, we will see the algebras receive an additional, anomalous term.

Both of these algebras contains a finite subalgebra associated with the global conformal group generated by ℓ_{-1},ℓ_0, and ℓ_1 (and antiholomorphic counterparts). By considering the associated infinitesimal transformations, we see that ℓ_{-1} ($\bar{\ell}_{-1}$) generates translations, ℓ_1 ($\bar{\ell}_1$) generates special conformal transformations, $i(\ell_0 - \bar{\ell}_0)$ generates rotations, and $\ell_0 + \bar{\ell}_0$ generates scale transformations. The special conformal transformation can be thought of as a translation, preceded and followed by an inversion $x^\mu \to x^\mu/x^2$:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu.$$

Scale transformations are defined as

$$x' = \lambda x$$

$$\Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x),$$

where λ is the dilatation factor and Δ is the scaling dimension of the field Φ. The finite form of these infinitesimal tranformations associated with this finite subalgebra is precisely eq. (2.6) (and its antiholmorphic counterpart)—the global conformal group.

For a field with scaling dimension Δ and spin s, we define the holomorphic conformal dimension h and its antiholomorphic counterpart $\bar{h}$ as

$$h = \frac{1}{2}(\Delta + s) \qquad \bar{h} = \frac{1}{2}(\Delta - s).$$

With these, we call a field quasi-primary of dimension $(h, \bar{h})$ if it transforms under a conformal map $z \to w(z)$ as

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}). \tag{2.11}$$

If the variation of a field under *any* local conformal transformation is of this form, the field is known as a primary field. Not all fields in a conformal field theory will have this property. Fields which are not primary are called secondary. For example, the derivative of a primary field of conformal dimension $h \neq 0$ is secondary, as well as the stress-energy tensor T (in most cases).

2.3 Radial quantization and conformal charge

We now turn to the quantum properties of two-dimensional conformal field theories. We begin by defining our Minkowski quantum theory on an infinite space-time cylinder, with t going from $-\infty$ to ∞ along the length of the cylinder and space being compactified with coordinate x going from 0 to L. If we contine to Euclidean space, our cylinder is described by the complex coordinate $\xi = t + ix$. Minkowski left- and right-moving fields correspond to Euclidean holomorphic and antiholomorphic fields, so these terms will be used interchangeably. The cylinder can be mapped to the complex plane (or more accurately the Riemann sphere, since with our complex plane we must include a point at infinity) via the mapping

$$z = e^{2\pi\xi/L}. \tag{2.12}$$

The remote past is situated at the origin, and the remote future lies on the point at infinity. Equal time surfaces correspond to circles of constant radius in the complex plane, thus dilatations $z \to e^{\alpha}z$ in the plane are just time translations on the cylinder. This means that the dilatation generator on the plane is proportional to the generator of time translations—the Hamiltonian of the system.

We now quantize the system with respect to this Hamiltonian and the radial "time" coordinate. In this radial quantization scheme, the time-ordering required for products of operators becomes a radial-ordering, defined by

$$\mathcal{R}\Phi_1(z)\Phi_2(w) = \begin{cases} \Phi_1(z)\Phi_2(w) & : |z| > |w| \\ \Phi_2(w)\Phi_1(z) & : |z| < |w| \end{cases}$$

If the fields were fermionic, the second expression would pick up a minus sign. The equal-time commutator $[A, B]$ of two operators, such as in eq. (2.16), can be shown to equal the contour integral of a radially ordered product:

$$[A, B] = \oint_0 dw \oint_w dz\, a(z)b(w), \text{ where } A = \oint a(z)dz,\ B = \oint b(z)dz, \tag{2.13}$$

and where the integral over z is taken around w and the integral over w is taken around the origin.

We will also assume the existence of a unique vacuum state $|0\rangle$ upon which a Hilbert space is constructed by applying creation operators. The vacuum for free-field theories can be defined as the state annihilated by the positive frequency part of the field (as is typical in the study of quantum field theories).

Now we use the standard Noether procedure to investigate the conserved conformal charge. A quantum field theory with an exact symmetry has an associated conserved current j^μ such that $\partial_\mu j^\mu = 0$. Integrating over a fixed-time slice gives the conserved charge $Q = \int j_0(x)dx$. The conserved charge generates the infinitesimal symmetry variation in a field ϕ according to $\delta_\epsilon\phi = \epsilon[Q, \phi]$. Local coordinate transformations are generated by a charge constructed from the (generally) symmetric and divergence-free stress-energy tensor $T_{\mu\nu}$. In conformally invariant theories, $T_{\mu\nu}$ is traceless—this follows from conservation of the dilatation current $T_{\mu\nu}x^\nu$. To study the conformal charge in the complex plane, we express the stress-energy tensor in terms of its holomorphic and antiholomorphic coordinates. Based on the above properties, the only nonvanishing components are

$$T(z) \equiv T_{zz}(z) \quad \text{and} \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}). \tag{2.14}$$

In radial quantization, the integral of the conserved current becomes $\int j_0(x)dx \to \int j_r(x)d\theta$. The stress-energy tensor generates local conformal transformations, so

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\, \epsilon(z)T(z) + d\bar{z}\, \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}). \tag{2.15}$$

The variation of a field due to a conformal transformation is thus given by

$$\delta_\epsilon \Phi(w,\bar{w}) = \frac{1}{2\pi i}\oint [dz\, \epsilon(z)T(z), \Phi(w,\bar{w})] + [d\bar{z}\, \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}), \Phi(w,\bar{w})]. \tag{2.16}$$

We also introduce the notion of the operator product expansion. It is typical of products of operators to have singularities when the positions of two or more fields coincide. In general, the singularities that occur when operators approach one another are encoded in the operator product expansion

$$A(x)B(y) = \sum_i C_i(x-y)O_i(y), \tag{2.17}$$

where the O_i's are a complete set of local operators and the C_i's are singular numerical coefficients. In two-dimensional conformal field theories, we can always take a basis of operators with fixed conformal weights and thus can determine by dimensional analysis $C_i \sim |x-y|^{\Delta_{O_i}-\Delta_A-\Delta_B}$. While eq. (2.17) is normally an asymptotic expression, dimensional analysis lets us argue that it converges–other types of functional dependence would require a length scale that is absent in conformal field theories.

2.4 Central charge and Virasoro algebra

Evaluating (2.16) using (2.13) and the result of the transformation (2.11) in the case of infinitesimal $f(z) = z + \epsilon(z)$, we infer the short distance singularities of T with Φ are

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \Phi(w,\bar{w}) + \cdots, \tag{2.18}$$

and similarly for antiholomorphic $\bar{T}(\bar{z})$. Thus the operator product expansion of a primary field with the stress tensor will be of this form. Unlike eq. (2.18), a secondary field will have higher than a double pole singularity in its operator product expansion with T or $\bar{T}$. The stress tensor itself is an example of a secondary field for which this is true. By performing two conformal transformations in succession, we can determine its operator product expansion with itself to be

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w). \tag{2.19}$$

This new singular term is allowed by scale invariance, analyticity, and ($z \leftrightarrow w$) symmetry. The constant c is known as the central charge, and its value will depend in general on the particular theory in question. A free scalar field contributes a central charge of 1; a free fermionic field contributes a central charge of 1/2. An identical result applies for $\bar{T}$ with $\bar{c}$.

The mode expansion of the energy-momentum tensor is given by

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \qquad L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z) \tag{2.20}$$

and similarly for $\bar{T}(\bar{z})$ in terms of $\bar{L}_n$. We can also expand the infinitesimal conformal change as

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1} \epsilon_n.$$

and similarly for $\bar{\epsilon}(\bar{z})$. Then the conformal charge has the mode expansion

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n + \bar{\epsilon}_n \bar{L}_n.$$

The mode operators L_n and $\bar{L}_n$ of the energy-momentum tensor are the generators of local conformal transformations, the quantum equivalents to ℓ_n and $\bar{\ell}_n$. Likewise, the generators of $SL(2,\mathbb{C})$ are L_{-1}, L_0, and L_1 (and their antiholomorphic counterparts). In particular, $L_0 + \bar{L}_0$ generates dilatations, which are nothing but time translations in radial quantization. Thus, $L_0 + \bar{L}_0$ is proportional to the Hamiltonian of the system.

We can derive the Virasoro algebra for these quantum generators using eqs. (2.13) and (2.19):

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12} n(n^2-1)\delta_{n+m,0} \\
[L_n, \bar{L}_m] &= 0.
\end{aligned} \tag{2.21}$$

The Virasoro algebra differs from the classical Witt algebra by the presence of a central term. Because energy and momentum density are observables (and thus real), we have the usual hermiticity condition $(L_n)^\dagger = L_{-n}$ As before, the generators L_{-1}, L_0, and L_1 (and their antiholomorphic counterparts) form a finite subalgebra corresponding to the global conformal group. The vacuum state $|0\rangle$ must be invariant under global conformal transformations, and thus the vacuum will be annihilated by these generators.

2.5 Primary and descendant states and unitarity constraints

We now consider the effects of Virasoro generators on different states. We first consider the state

$$|h, \bar{h}\rangle = \Phi(0,0)|0\rangle, \tag{2.22}$$

created by a holomorphic field $\Phi(z,\bar{z})$ with weights $h,\bar{h}$. Using equation (2.18), we find that

$$[L_0,\Phi(w,\bar{w})] = h\Phi(w,\bar{w}) + \bar{w}\bar{\partial}\Phi(w,\bar{w}) \tag{2.23}$$

and similarly for the antiholomorphic Virasoro generator. We thus conclude that

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle. \tag{2.24}$$

Therefore $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian. Using the Virasoro algebra (2.21), it is trivial to see that L_{-n} raises the energy by n units. Similarly L_n lowers the energy by n units.

We will be considering field theories where the spectrum of states has a minimum energy — the vacuum. Thus we cannot apply L_n indefinitely; there must be states that are annihilated by these operators. These highest weight states are precisely the primary states. They are therefore characterized by the relations

$$\begin{aligned} L_n|h,\bar{h}\rangle &= 0, \quad n>0, \\ L_0|h,\bar{h}\rangle &= h|h,\bar{h}\rangle. \end{aligned} \tag{2.25}$$

The hermiticity of $L_0,\bar{L}_0$ means that $h,\bar{h}\in\mathbb{R}$.

All other states can be generated by the raising operators L_{-n} and will be of the form

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle \qquad 1\le k_1\le\cdots\le k_n, \tag{2.26}$$

where for now we consider a primary state that is not the vacuum and by convention the L_{-k_i} appear in increasing order of k_i. The state (2.26) is an eigenstate of L_0 with eigenvalue

$$h' = h + k_1 + k_2 + \cdots + k_n \equiv h + N.$$

States of the form (2.26) are descendants of the non-vacuum state $|h\rangle$ and the integer N is the level of the descendant. The number of distinct, linearly independent states at level N is the number $P(N)$ of partitions of the integer N. The relevance of descendant states lies in the fact that the effect of a conformal transformation on a state is obtained by acting on it with a suitable function of the generators L_m. The subset of the full Hilbert space generated by the state $|h\rangle$ and its decendants is closed under the action of the Virasoro generators and thus forms a representation of the Virasoro algebra.

We are interested in unitary theories and therefore in unitary representations of the Virasoro algebra. These can be studied by requiring that any state at level N has a positive norm. At level zero, we have by a choice of normalization $||h\rangle|^2 = 1$. At level one, unitarity demands that the unique descendant state satisfy

$$\langle h|L_1L_{-1}|h\rangle = 2h\langle h|h\rangle = 2h \ge 0. \tag{2.27}$$

Thus the holomorphic conformal weight must satisfy $h\ge 0$. We see that $h=0$ only if $L_{-1}|h\rangle = 0$, i.e. only if $|h\rangle$ is the invariant vacuum $|0\rangle$. By considering $L_{-m}|h\rangle$, we find

$$|L_{-m}|h\rangle|^2 = \left(\frac{c_{\rm tot}}{12}(m^3-m) + 2mh\right)\langle h|h\rangle \tag{2.28}$$

Choosing very large m means that this norm will be positive definite only if $c_{\text{tot}} \geq 0$. It can also be shown that for $c = 0$ there are no interesting unitary representations; all states would have zero norm and thus can be set to zero.

We point out two facts before continuing. First, in the case of $h = 0$ the appropriate condition on n_i is that $2 \leq k_1 \leq \cdots \leq k_n$. This is because the generator L_{-1} (corresponding to translations on the complex plane) is a member of the finite subalgebra associated with the global conformal group. We must demand that this subalgebra annihilates the vacuum to preserve conformal invariance of our theory. Thus the conformal block of the vacuum (meaning the primary vacuum state and all of its descendants) has no $N = 1$ states. Second, we remark on the vacuum energy of two dimensional conformal field theories. Typically in a (nongravitational) quantum field theory one can always shift the energy of the vacuum by a constant. This is equivalent to changing the normalization of the functional integrals. In conformal field theory, this is not the case; scale and rotational invariance of the $SL(2, \mathbb{C})$ invariant vacuum fix the vacuum eigenstates of L_0 and $\bar{L}_0$ to be zero. Thus our conformal field theory on the plane is fixed with ground state energy $E_0 = 0$.

Additional unitarity constraints will arise from considering norms of excited states; the details are too involved to be reproduced here. At arbitrary level, we get the condition that the so-called Kac determinant be nonnegative:

$$\det(M_N(c, h)) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)} \geq 0, \tag{2.29}$$

where $P(x)$ is the number of partitions, the numbers $h_{p,q}$ are given by

$$h_{p,q}(m) = \frac{[(m+1)p - mq^2] - 1}{4m(m+1)}, \tag{2.30}$$

and we define

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25 - c}{1 - c}}. \tag{2.31}$$

It can be shown that for $c > 1$, there are no negative normed states. For $c < 1$, there are generically negative normed states unless two different $h_{p,q}$ have the same value. This happens precisely when

$$c = 1 - \frac{6}{m(m+1)} \tag{2.32}$$

(from inverting eq. (2.31)). Then the only possible values of h are given by eq. (2.30) where p, q are in the range

$$m - 1 \geq p \geq q \geq 1. \tag{2.33}$$

There are the so-called mimimal models. They are very constrained and well understood models (including the Ising model and the 3-state Potts model, for example). We will be considering only theories with $c > 1$ in the work to follow.

2.6 Modular invariance

We turn now to the topological properties of our two-dimensional CFTs. The complex plane with a point at infinity is topologically equivalent to a sphere, a Riemann surface of genus $g = 0$. One may study CFTs defined on Riemann surfaces of arbitrary genus g (in string theory, this is the basis for calculating multiloop scattering amplitudes). We will restrict our considerations to the simplest non-spherical case: that of a torus ($g = 1$), equivalent to a plane with periodic boundary conditions in two directions. A torus may be defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. On the complex plane, these lattice vectors may be represented by two complex numbers w_1 and w_2 which we call the periods of the lattice. The properties of a CFT defined on a torus do not depend on the overall scale of the lattice nor on the absolute orientation of the lattice vectors. Thus the relevant parameter is the ratio $\tau = w_2/w_1$, called the modular parameter.

Our strategy for defining CFTs on the torus is to make use of the local properties of operators on the plane, map them to the cylinder via (2.12), and then use a discrete identification to get the torus. This procedure will preserve the local properties of operators, but not necessarily the global properties (for example, only the generators of dilatations and rotations survive as global symmetry generators). An important particular case occurs when mapping a secondary field, such as the stress tensor $T(z)$, to the torus. According to eq. (2.16), under conformal transformation $w \to z = e^w$, $T(z)$ picks up a piece proportional to the central charge

$$T_{cyl}(w) = z^2 T(z) - \frac{c}{24}. \tag{2.34}$$

Substituting in the mode expansions (2.20) gives the translation generator $(L_0)_{cyl}$ on the cylinder in terms of the dilatation generator L_0 on the plane as

$$(L_0)_{cyl} = L_0 - \frac{c}{24}, \tag{2.35}$$

with a similar expression for $(\bar{L}_0)_{cyl}$. This means that any formulas that involved L_0 or $\bar{L}_0$ will be shifted; in particular the ground state energy for the torus will equal $E_0 = -\frac{c+\bar{c}}{24}$.

The relevant quantity we consider is the (Euclidean) partition function Z and its dependence on the modular parameter τ. We define our space and time axes respectively as the real and imaginary axes of the torus, and orient our lattice vector w_1 along the real axis. We are also free to resize all lattice vectors so that $w_1 = 2\pi$. Consider first the partition function with $\mathrm{Re}\,\tau = 0$ (meaning w_2 is strictly imaginary). Compactifying the Euclidean time with period $2\pi(\mathrm{Im}\ \tau)$ is equivalent to putting the theory at an inverse temperature $\beta = 2\pi(\mathrm{Im}\,\tau)$:

$$Z(\tau) = \mathrm{Tr}\ e^{-\beta H} = \mathrm{Tr}\ e^{-2\pi(\mathrm{Im}\tau)H},$$

where H is the Hamiltonian generating a translation in the "time" direction. For the case of a skewed torus where $\mathrm{Re}\ \tau \neq 0$, a given point is transformed along the spatial direction of the torus by Re τ. The operator that implements this translation is $e^{2\pi i(\mathrm{Re}\,\tau)P}$, where P is the momentum generating a translation in the "space" direction. On the plane we saw that $L_0 \pm \bar{L}_0$ generated dilatations and rotations, so according to the discussion of radial quantization we have

$$H = (L_0)_{cyl} + (\bar{L}_0)_{cyl}, \qquad P = (L_0)_{cyl} - (\bar{L}_0)_{cyl} \tag{2.36}$$

Thus the partition function on our torus is

$$Z(\tau) = \mathrm{Tr}\; e^{-2\pi(\mathrm{Im}\tau)(L_0+\bar{L}_0)} e^{2\pi i(\mathrm{Re}\,\tau)(L_0-\bar{L}_0)} e^{2\pi(\mathrm{Im}\tau)c_{\mathrm{tot}}/12}, \tag{2.37}$$

where $c_{\mathrm{tot}} \equiv c + \bar{c}$. Defining $q = \exp(2\pi i\tau)$ (and $\bar{q}$ similarly), we thus arrive at the partition function

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right). \tag{2.38}$$

Studying CFTs on the torus gives constraints coming from the requirement that the partition function for a given torus be independent of the choice of periods. Let $w'_{1,2}$ be periods describing the same lattice as $w_{1,2}$. Since the points $w'_{1,2}$ belong to the same lattice, they must be expressible as integer combinations of w_1 and w_2:

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}. \tag{2.39}$$

Clearly the above matrix should be invertible, since we should be able to express $w_{1,2}$ in terms of $w'_{1,2}$. And because the unit cell should have the same area whatever periods we use, the determinant of this matrix should be unity: $ad - bc = 1$. We are thus led to consider the group $SL(2, \mathbb{Z})$. Under a transformation of this form, the modular parameter transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{2.40}$$

The sign of parameters a, b, c, d can be simultaneously changed without affecting the transformation. Thus the group we are interested in is the modular group $PSL(2, \mathbb{Z})$.

The modular transformations we are interested in are

$$T : \tau \to \tau + 1 \qquad \text{or} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.41}$$

$$S : \tau \to -\frac{1}{\tau} \qquad \text{or} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

These two transformations satisfy

$$(ST)^3 = S^2 = 1$$

and it can be shown that they generate the entire modular group. In the work that follows, we will be considering conformal field theories with modular invariance—that is, CFTs with partition functions that are invariant under modular transformations expressible as successive applications of S and T.

2.7 Consequences of modular invariance and Cardy's formula

Modular invariance imposes powerful constraints on conformal field theories. One of the most basic constraints comes from demanding invariance of the partition function (2.38) under the T-transformation $T : \tau \to \tau+1$. Demanding that this transformation leave the partition function unchanged gives the condition

$$h - \bar{h} \in \mathbb{Z}. \tag{2.42}$$

The constraints on CFTs from requiring invariance of the partition function under the S-transformation are not nearly so straightforward.

In [41], Cardy studied basic consequences of invariance under the S-transformation. He showed that an S-invariant conformal field theory with a finite number of primary operators must have $c < 1$. He also proved the contrapositive to this statement: if a two-dimensional CFT is modular invariant with $c \geq 1$, then the theory has an infinite number of primary operators. His insight was that the S-transformation allows us to relate the low-temperature ($\beta \gg 1$) partition function to the high-temperature regime ($\beta \ll 1$). The partition function Z is generically given by

$$Z(\tau, \bar{\tau}) = \langle 0 | e^{2\pi i [\tau(L_0 - c/24) - \bar{\tau}(\bar{L}_0 - \bar{c}/24)]} | 0 \rangle + (\text{excited states}). \tag{2.43}$$

For $\beta = \operatorname{Im} \tau \gg 1$, this trace is well approximated by

$$Z_{low} \approx e^{2\pi i (-\tau \frac{c}{24} + \bar{\tau} \frac{\bar{c}}{24})} + O(e^{-\operatorname{Im}\tau}). \tag{2.44}$$

To consider the high-temperature regime we use the fact that for a diagonalized Hamiltonian, $\operatorname{Tr} e^{-\beta H} = \int d\epsilon \, e^{S(\epsilon) - \beta\epsilon}$, where $e^{S(\epsilon)} = \rho(\epsilon)$ is the density of states and S is the entropy. Then a saddle-point approximation gives the leading-order behavior in the high-temperature regime with $h \gg, \bar{h} \gg \bar{c}$ as

$$\log Z_{high} \approx S(h, \bar{h}) + 2\pi i \left[\tau \left(h - \frac{c}{24} \right) - \bar{\tau} \left(\bar{h} - \frac{\bar{c}}{24} \right) \right], \tag{2.45}$$

where h and $\bar{h}$ are respectively functions of τ and $\bar{\tau}$ that maximize the RHS.

Doing the inverse Legendre transform gives the entropy as

$$S(h, \bar{h}) \approx \log Z_{high} - 2\pi i \left[\tau \left(h - \frac{c}{24} \right) - \bar{\tau} \left(\bar{h} - \frac{\bar{c}}{24} \right) \right]. \tag{2.46}$$

But invariance of the partition function under the S-transformation means that

$$\log Z_{high} \approx \log Z_{low} = 2\pi i \left(-\tau \frac{c}{24} + \bar{\tau} \frac{\bar{c}}{24} \right). \tag{2.47}$$

Substituting this into (2.46) gives

$$S(h, \bar{h}) \approx 2\pi i \left(-\tau \frac{c}{24} + \bar{\tau} \frac{\bar{c}}{24} \right) - 2\pi i \left[\tau \left(h - \frac{c}{24} \right) - \bar{\tau} \left(\bar{h} - \frac{\bar{c}}{24} \right) \right]. \tag{2.48}$$

Finally, extremizing over τ and $\bar{\tau}$ gives the Cardy formula

$$S \approx 2\pi \left(\sqrt{\frac{c(h - \frac{c}{24})}{6}} + \sqrt{\frac{\bar{c}(\bar{h} - \frac{\bar{c}}{24})}{6}} \right). \tag{2.49}$$

Cardy's formula alone does not make a statement about a CFT spectrum that can be tested at finite energies or temperatures; we considered only leading terms and the formula only applies for $h \gg c, \bar{h} \gg \bar{c}$. In order to obtain and prove bounds in the general case using modular invariance, a discrete spectrum, and unitarity, what we actually study are the set of modular invariant functions with a discrete Fourier expansion having positive integer coefficients. Equation (2.49) can be used to show that the partition function and all its derivatives converge and are continuous in the upper half plane. We can use this fact to study fixed points of the partition function and derive bounds, for example, on operator dimensions.

Chapter 3

Bound on Δ_n without chiral primary operators

In this chapter, we extend the results of [32] to derive an upper bound on the conformal dimension Δ_2 of the next-to-lowest nontrival primary operator in unitary, modular-invariant two-dimensional conformal field theories without chiral primary operators, with total central charge $c_{\text{tot}} > 2$. The bound we find is of the same form as found in [32] for Δ_1: $\Delta_2 \le \frac{c_{\text{tot}}}{12} + O(1)$. We obtain a similar bound on the conformal dimension Δ_3, and present a method for deriving bounds on Δ_n for any n, under slightly modified assumptions. For asymptotically large c_{tot} and $n \lesssim \exp(\pi c_{\text{tot}}/12)$, we show that $\Delta_n \le \frac{c_{\text{tot}}}{12} + O(1)$. This implies an asymptotic lower bound of $\exp(\pi c_{\text{tot}}/12)$ on the number of primary operators of dimension $\le c_{\text{tot}}/12 + O(1)$, in the large-$c$ limit. This chapter is based heavily on the work [42].

3.1 Review of the bound on Δ_1

We begin by reviewing the methods and results of [32]. Consider a 2D CFT on the torus with modular parameter close to the fixed point $\tau \equiv (K+i\beta)/2\pi = i$, where β is the inverse temperature and K is the thermodynamic potential for spatial momentum in the compact spatial direction σ_1. We can parameterize the neighborhood of this fixed point conveniently using $\tau \equiv i\, \exp(s)$. Then invariance of the partition function $Z(\tau, \bar{\tau})$ under the modular S-transformation $\tau \to -\frac{1}{\tau}$ can be expressed as

$$Z\left(ie^{s}, -ie^{\bar{s}}\right) = Z(ie^{-s}, -ie^{-\bar{s}}) \tag{3.1}$$

By taking derivatives of this expression with respect to $s, \bar{s}$, one obtains an infinite set of equations

$$\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N_R} Z(\tau,\tilde{\tau})\bigg|_{\tau=i} = 0, \quad N_L + N_R \text{ odd} \tag{3.2}$$

For purely imaginary complex structure $\tau = i\beta/2\pi$, this condition implies

$$\left(\beta\frac{\partial}{\partial\beta}\right)^{N} Z(\beta)\bigg|_{\beta=2\pi} = 0, \quad N \text{ odd} \tag{3.3}$$

We will assume a unique vacuum and a discrete spectrum. By further assuming cluster decomposition and no chiral operators other than the stress tensor, the Virasoro structure theorem implies that the partition function $Z(\beta)$ can be expressed as a sum over conformal families:

$$Z(\beta) = Z_{id}(\beta) + \sum_{A} Z_A(\beta). \tag{3.4}$$

Here $Z_{id}(\beta)$ is the sum over states in the conformal family of the identity; $Z_A(\beta)$ is the sum over all states in the conformal family of the A^{th} primary operator, which has conformal weights $h_A, \tilde{h}_A$ and conformal dimension $\Delta_A = h_A + \tilde{h}_A$.

Hellerman considers CFTs with $c, \tilde{c} > 1$ and with no chiral operators other than the stress tensor, which implies the following explicit forms for $Z_{id}(\beta)$ and $Z_A(\beta)$:

$$Z_{id}(\tau) = q^{-\frac{c}{24}}\bar{q}^{-\frac{\tilde{c}}{24}} \prod_{m=2}^{\infty}(1-q^m)^{-1}\prod_{n=2}^{\infty}(1-\bar{q}^n)^{-1} \tag{3.5}$$

$$Z_A(\tau) = q^{h_A-\frac{c}{24}}\bar{q}^{\bar{h}_A-\frac{\tilde{c}}{24}} \prod_{m=1}^{\infty}(1-q^m)^{-1}\prod_{n=1}^{\infty}(1-\bar{q}^n)^{-1} \tag{3.6}$$

where $q = \exp(2\pi i\tau)$. The full partition function with $\tau = i\beta/2\pi$ is then given by the expression

$$Z(\beta) = M(\beta)Y(\beta) + B(\beta) \tag{3.7}$$

with

$$M(\beta) \equiv \frac{\exp(-\beta\hat{E}_0)}{\eta(i\beta/2\pi)^2} \tag{3.8}$$

and

$$B(\beta) \equiv M(\beta)\left(1-\exp(-\beta)\right)^2, \tag{3.9}$$

where $\hat{E}_0 \equiv E_0 + \frac{1}{12} = \frac{1}{12} - \frac{c+\tilde{c}}{24}$ and η is the Dedekind eta function. For real β, the partition function over primaries $Y(\beta)$ is

$$Y(\beta) = \sum_{A=1}^{\infty} e^{-\beta\Delta_A}. \tag{3.10}$$

Next, Hellerman applies the differential constraints (3.3) to the partition function (4.5). To simplify the analysis, we introduce polynomials $f_p(z)$ defined by

$$(\beta\partial_\beta)^p M(\beta)Y(\beta)\bigg|_{\beta=2\pi} = (-1)^p\eta(i)^{-2}\exp(-2\pi\hat{E}_0)\sum_{A=1}^{\infty}\exp(-2\pi\Delta_A)f_p(\Delta_A+\hat{E}_0). \tag{3.11}$$

The first few polynomials are explicitly

$$\begin{aligned} f_0(z) &= 1 \\ f_1(z) &= (2\pi z) - \frac{1}{2} \\ f_2(z) &= (2\pi z)^2 - 2(2\pi z) + \left(\frac{7}{8} + 2r_{20}\right) \\ f_3(z) &= (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r_{20}\right)(2\pi z) - \left(\frac{17}{16} + 3r_{20}\right) \end{aligned} \tag{3.12}$$

where

$$r_{20} \equiv \frac{\eta''(i)}{\eta(i)} \approx 0.0120...$$

We also define the polynomials $b_p(z)$ by

$$(\beta\partial_\beta)^p B(\beta)\Big|_{\beta=2\pi} = (-1)^p \eta(i)^{-2}\exp(-2\pi\hat{E}_0)b_p(\hat{E}_0), \tag{3.13}$$

Explicitly,

$$\begin{gathered} b_0(z) = 1 - 2e^{-2\pi} + e^{-4\pi} \\ b_1(z) = \left((2\pi z) - \frac{1}{2}\right) - 2e^{-2\pi}\left((2\pi(z+1)) - \frac{1}{2}\right) + e^{-4\pi}\left((2\pi(z+2)) - \frac{1}{2}\right) \\ b_p(z) = f_p(z) - 2e^{-2\pi} f_p(z+1) + e^{-4\pi} f_p(z+2). \end{gathered} \tag{3.14}$$

Using these polynomials, the equations (3.3) for modular invariance of $Z(\beta)$ for odd p become

$$\sum_{A=1}^{\infty} f_p(\Delta_A + \hat{E}_0)\exp(-2\pi\Delta_A) = -b_p(\hat{E}_0) \tag{3.15}$$

It is this expression that is used to derive an upper bound on the conformal dimension Δ_1. Hellerman takes the ratio of the $p=3$ and $p=1$ expressions to get

$$\frac{\sum_{A=1}^{\infty} f_3(\Delta_A + \hat{E}_0)\exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\exp(-2\pi\Delta_B)} = \frac{b_3(\hat{E}_0)}{b_1(\hat{E}_0)} \equiv F_1. \tag{3.16}$$

Or, upon rearrangement,

$$\frac{\sum_{A=1}^{\infty}\left[f_3(\Delta_A + \hat{E}_0) - F_1(\hat{E}_0)f_1(\Delta_A + \hat{E}_0)\right]\exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\exp(-2\pi\Delta_B)} = 0. \tag{3.17}$$

Next assume that $\Delta_1 > \Delta_1^+$, where Δ_1^+ is defined as the largest root of the numerator, and proceeds to obtain a contradiction. Because $\Delta_A \geq \Delta_1$, this assumption implies that every term in both the numerator and denominator is strictly positive. Then equation (3.17) says that a positive number equals zero — an impossibility. Therefore

$$\Delta_1 \leq \Delta_1^+.$$

Finally, by analyzing Δ_1^+ as a function of c_{tot} Hellerman proves that for the given assumptions, $\Delta_1^+ \leq \frac{c_{\text{tot}}}{12} + \frac{(12-\pi)+(13\pi-12)e^{-2\pi}}{6\pi(1-e^{-2\pi})}$, implying the bound

$$\Delta_1 \leq \frac{c_{\text{tot}}}{12} + 0.4736... \tag{3.18}$$

3.2 Bounds on Δ_2, Δ_3

In this section, we extend the methods described above to derive bounds on primary operators of second and third-lowest dimension. In order to bound the conformal dimension Δ_2, we move the Δ_1 term of equation (3.15) to the RHS. We then form the ratio of the $p=3$ and $p=1$ equations to get

$$\frac{\sum_{A=2}^{\infty} f_3(\Delta_A+\hat{E}_0)e^{-2\pi\Delta_A}}{\sum_{B=2}^{\infty} f_1(\Delta_B+\hat{E}_0)e^{-2\pi\Delta_B}} = \frac{f_3(\Delta_1+\hat{E}_0)e^{-2\pi\Delta_1}+b_3(\hat{E}_0)}{f_1(\Delta_1+\hat{E}_0)e^{-2\pi\Delta_1}+b_1(\hat{E}_0)} \equiv F_2(\Delta_1, c_{\text{tot}}). \quad (3.19)$$

Moving F_2 to the left side, we get

$$\frac{\sum_{A=2}^{\infty}\left[f_3(\Delta_A+\hat{E}_0)-f_1(\Delta_A+\hat{E}_0)F_2\right]\exp(-2\pi\Delta_A)}{\sum_{B=2}^{\infty} f_1(\Delta_B+\hat{E}_0)\exp(-2\pi\Delta_B)} = 0 \qquad (3.20)$$

In Appendix A, we prove that F_2 is finite and nonzero for $c, \tilde{c} > 1$ and Δ_1 in the allowed range and thus our derivations will carry through without issue.

Before proceeding, we make some definitions. Define $\Delta_{f_p}^+$ to be the largest root of $f_p(\Delta+\hat{E}_0)$ viewed as a polynomial in Δ. The bracketed expression in the numerator is a polynomial cubic in Δ_2; we denote it by $P_2(\Delta_2)$, and define the largest root of P_2 to be $\Delta_2^+(c_{\text{tot}}, \Delta_1)$, where $\hat{E}_0$ dependence has been replaced by c_{tot}.

We now assume that $\Delta_2 > \max(\Delta_{f_1}^+, \Delta_2^+)$ and attempt to obtain a contradiction. From our explicit polynomial expressions, we see that the leading coefficients of both f_1 and f_3 are positive. Thus both $P_2(\Delta_2) > 0$ and $f_1(\Delta_2+\hat{E}_0) > 0$ for $\Delta_2 > \max(\Delta_{f_1}^+, \Delta_2^+)$. Because $\Delta_n \geq \Delta_2$ for all $n > 2$, we also have $P_2(\Delta_n) > 0$ and $f_1(\Delta_n+\hat{E}_0) > 0$ for $\Delta_2 > \max(\Delta_{f_1}^+, \Delta_2^+)$. Thus every term in both the numerator and denominator of the left side of equation (3.20) is positive for $\Delta_2 > \max(\Delta_{f_1}^+, \Delta_2^+)$. The left side thus can not be equal to zero, and we have a contradiction. We have thus derived a bound on the conformal dimension Δ_2:

$$\Delta_2 \leq \max(\Delta_{f_1}^+, \Delta_2^+). \qquad (3.21)$$

From the explicit form of $f_1(\Delta+\hat{E}_0)$ in (2.12), we see that

$$\Delta_{f_1}^+ = \frac{c_{\text{tot}}}{24} + \frac{(3-\pi)}{12\pi}. \qquad (3.22)$$

We will spend the next section trying to simplify our bound by deriving a manageable expression for Δ_2^+.

Asymptotic expansion for large central charge

We begin by considering the limit of large positive total central charge c_{tot}. In the limit $c_{\text{tot}} \to \infty$, it is easy to see that Δ_2^+ is proportional to c_{tot}, plus corrections of order c_{tot}^0. We thus expand Δ_2^+ as a series at large central charge:

$$\Delta_2^+ \equiv \sum_{a=-1}^{\infty} d_{-a}(\Delta_1)\left(\frac{c_{\text{tot}}}{24}\right)^{-a}. \qquad (3.23)$$

By definition Δ_2^+ satisfies

$$P_2(\Delta_2^+) = 0$$

and is the largest real value with that property. Substituting equation (3.23) into the explicit form of $P_2(\Delta_2^+) = 0$, the equation to leading order in c_{tot} is:

$$\frac{4(d_1-1)^2\pi^2}{24^2} - \frac{\pi^2}{12^2} = 0. \tag{3.24}$$

The solution $d_1 = 2$ gives the largest root Δ_2^+,

$$\Delta_2^+ = \frac{c_{\text{tot}}}{12} + d_0(\Delta_1) + O(c_{\text{tot}}^{-1}). \tag{3.25}$$

Note how this compares to the value $\Delta_{f_1}^+$ in equation (3.22). Since we are taking the maximum of these two quantities, the true upper bound on Δ_2 will generically be given by Δ_2^+.

To determine d_0 we expand P_2 to the next order in c_{tot}. Quoting the result obtained in Appendix B, we find that the largest term possible at this order is given by $d_0 \approx 0.4736...$—the same bound to this order as for Δ_1 in [32]. Thus for large enough central charge c_{tot}, we can always bound the conformal dimension Δ_2 using the expression

$$\Delta_2 \le \frac{c_{\text{tot}}}{12} + 0.4736... + O(c_{\text{tot}}^{-1}). \tag{3.26}$$

An absolute bound on Δ_2 can be obtained numerically. We seek a linear bound of the form $\Delta_2 \le \frac{c_{\text{tot}}}{12} + D_2$, where D_2 is a numerical constant independent of Δ_1. In order for this bound to universal, we need to find D_2 so that the inequality is valid for all possible values of Δ_1 and all $c_{\text{tot}} > 2$. This can be done by explicitly solving the cubic polynomial P_2 (in terms of radicals of exponentials) and maximizing the expression $\Delta_2^+ - \frac{c_{\text{tot}}}{12}$ for $c_{\text{tot}} > 2$ and $0 < \Delta_1 \le \frac{c_{\text{tot}}}{12} + \delta_0$. This function attains a global maximum $D_2 \approx 0.5338...$ (for $c_{\text{tot}} \approx 2,\ \Delta_1 \approx 0.2717...$). Therefore

$$\Delta_2 \le \frac{c_{\text{tot}}}{12} + 0.5338... \tag{3.27}$$

Proof and numerical bound for Δ_3

Now that we have obtained a bound on Δ_2, it is natural to extend our arguments to primary operators of higher dimension. A necessary condition for our arguments to work for Δ_n is that F_n, defined as

$$F_n(\hat{E}_0, \Delta_1, \cdots, \Delta_{n-1}) \equiv \frac{\sum_{i=1}^{n-1} f_3(\Delta_i + \hat{E}_0)\exp(-2\pi\Delta_i) + b_3(\hat{E}_0)}{\sum_{i=1}^{n-1} f_1(\Delta_i + \hat{E}_0)\exp(-2\pi\Delta_i) + b_1(\hat{E}_0)}, \tag{3.28}$$

be well-defined for all relevant values of its arguments. We prove in Appendix A that F_3 is well-defined for $c_{\text{tot}} > 2$ and thus that there will be no issues. We can thus proceed with another proof by contradiction. The result is that

$$\Delta_3 \le \max(\Delta_{f_1}^+, \Delta_3^+), \tag{3.29}$$

where $\Delta_{f_1}^+$ is the expression (3.22) from above and Δ_3^+ is the largest real root of the polynomial

$$P_3(\Delta_3) \equiv f_3(\Delta_3 + \hat{E}_0) - f_1(\Delta_3 + \hat{E}_0)F_3. \tag{3.30}$$

At large central charge, one easily finds that $\Delta_3^+ \approx \frac{c_{\text{tot}}}{12}$. Maximizing the expression $\Delta_3^+ - \frac{c_{\text{tot}}}{12}$ numerically as a function of Δ_1 and Δ_2 subject to the contraints $0 < \Delta_1 \leq \frac{c_{\text{tot}}}{12} + \delta_0$, $0 < \Delta_2 \leq \frac{c_{\text{tot}}}{12} + D_1$, and $c_{\text{tot}} > 2$ gives the constant $D_2 = 0.8795...$ and the linear bound

$$\Delta_3 \leq \frac{c_{\text{tot}}}{12} + 0.8795... \tag{3.31}$$

We observe that the values of $\Delta_{1,2}$ that maximize Δ_3^+ are degenerate: $\Delta_1 = \Delta_2$.

3.3 Bounds on Δ_n

It should be clear by this point how to extend the proof to higher conformal dimensions. Assuming that the expression (3.28) is defined and nonvanishing in the appropriate range, we can proceed as above to obtain a bound

$$\Delta_n \leq \max(\Delta_{f_1}^+, \Delta_n^+), \tag{3.32}$$

where $\Delta_{f_1}^+$ is given by (3.22) and Δ_n^+ is the largest real root of the polynomial

$$P_n(\Delta_n) \equiv f_3(\Delta_n + \hat{E}_0) - f_1(\Delta_n + \hat{E}_0) F_n(c_{\text{tot}}, \Delta_1, ..., \Delta_{n-1}) \tag{3.33}$$

and is thus a function of $c_{\text{tot}}, \Delta_1, \cdots, \Delta_{n-1}$.

The leading terms in the polynomial with largest root Δ_n^+ are independent of n; therefore the expansion of Δ_n^+ at asymptotically large central charge again goes as $\frac{c_{\text{tot}}}{12}$. Thus it seems reasonable to expect a bound of the same form as before:

$$\Delta_n \leq \Delta_n^+ < \frac{c_{\text{tot}}}{12} + O(1). \tag{3.34}$$

However, there is a potential problem with this argument. For the bounds on Δ_2 and Δ_3, we proved in the Appendices that the functions F_2 and F_3 were positive and well-defined for the relevant ranges of our parameters. This is not the case beginning with the expression F_4. The denominator of F_4 vanishes when the total central charge equals

$$c_{D4} = \frac{2[\sum_{i=1}^3 (-12\pi\Delta_i - \pi + 3)e^{-2\pi\Delta_i} - \pi + 3 + (26\pi - 6)e^{-2\pi} + (3 - 25\pi)e^{-4\pi}]}{\pi(\sum_{i=1}^3 -e^{-2\pi\Delta_i} - 1 + 2e^{-2\pi} - e^{-4\pi})}$$

As before, we extremize this expression over the appropriate ranges of its variables ($0 < \Delta_1 \leq c_{\text{tot}}/12 + \delta_0$, etc.). The largest value of the total central charge for which the denominator of F_4 vanishes is given by

$$c_{D4}^+ = 2.3450... \tag{3.35}$$

Applying the same analysis to the numerator of F_4, we find that the largest value of the total central charge causing it to vanish is

$$c_{N4}^+ = 1.5113... \tag{3.36}$$

Thus for $2 < c_{\text{tot}} < c_{D4}^{+}$,we cannot use these specific methods to set a bound on Δ_4; there is a moduli space where our parameters can fundamentally change the polynomial P_n.

The resolution to this issue is straightforward; we will further restrict the allowed values for the total central charge to $c_{\text{tot}} > \max(c_{D4}^{+}, c_{N4}^{+})$. Allowing c_{tot} to range all the way down to 2.3450... would require an infinite constant; however, a small additional restriction on the range to $c_{\text{tot}} \geq 2.5$ leads to the bound

$$\Delta_4 \leq \frac{c_{\text{tot}}}{12} + 1.0795... \tag{3.37}$$

Further restricting c_{tot} gives a tighter bound; for example, $c_{\text{tot}} \geq 3$ gives $\Delta_4 \leq \frac{c_{\text{tot}}}{12} + 0.6740...$ Similar results can be derived for arbitrary Δ_n using the methods described here. We note that, as before, the values of $\Delta_{1,2,3}$ that saturate the bound (3.37) are degenerate: $\Delta_1 = \Delta_2 = \Delta_3$.

For larger values of n, it can be shown that $c_{Dn}^{+} > c_{Nn}^{+}$ and thus we need only restrict $c_{\text{tot}} > c_{Dn}^{+}$. We can analytically solve for the value of the central charge c_{Dn} which causes the denominator of (3.28) to vanish. The explicit form is a complicated function of $\Delta_1, ..., \Delta_{n-1}$ in terms of Lambert W functions. In Appendix C.1, we derive this expression. In Appendix D, we prove that c_{Dn}^{+} is maximized when $\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1}$; when we maximize c_{Dn} over all of its arguments, it goes for large n as

$$c_{Dn}^{+} \approx \frac{12}{\pi} W_0[(n-1)] \sim \frac{12}{\pi}\log(n), \tag{3.38}$$

where W_0 is the primary branch of the Lambert-W function. Therefore, if we require

$$\log n \lesssim \frac{\pi c_{\text{tot}}}{12} + O(1), \tag{3.39}$$

then F_n will be finite and nonzero. Then an analysis similar to before gives a bound

$$\Delta_n \leq \frac{c_{\text{tot}}}{12} + O(1). \tag{3.40}$$

The $O(1)$ term in expression (3.40) means $O(1)$ in c_{tot}— these subleading terms could have dependence on n that contributes to leading order. For example, if the $O(1)$ term goes as $\log(n)$, then by equation (3.39) we could have contributions as large as $O(c_{\text{tot}})$. Additionally, the specific $O(1)$ term will depend on how we restrict the total central charge. In Appendix E, we show that by considering

$$n \ll e^{\pi c_{\text{tot}}/6} + O(1), \tag{3.41}$$

we can derive a bound on Δ_n for asymptotically large c_{tot} going as

$$\Delta_n \leq \frac{c_{\text{tot}}}{12} + O(1). \tag{3.42}$$

In the limit (3.41), the $O(1)$ term will be 0.4736... and additional corrections will be $O(n c_{\text{tot}} e^{\pi c_{\text{tot}}/6})$. We are already assuming eq. (3.39), so the inequality (3.41) necessarily follows.

Chapter 4

Bound on Δ_n with chiral primary operators

In this chapter, we extend the results of [32, 42] to derive a bound on the conformal dimensions of the lightest few states in general unitary 2D CFTs with discrete spectra and modular invariance. We derive a bound on the conformal dimensions Δ_1 and Δ_2 going as $c_{\text{tot}}/12 + O(1)$. We then prove the inequality $\Delta_n \le c_{\text{tot}}/12 + O(1)$ for large c_{tot} and with appropriate, slightly modified restrictions on n. Earlier proofs assumed that these chiral primaries did not exist; this extension will apply, for example, to CFTs carrying continuous global current algebra symmetries.

4.1 Deriving bound on Δ_1

We again consider a 2D CFT on the torus with modular parameter close to the fixed point of the S-transformation $\tau \equiv (K + i\beta)/2\pi = i$. We have the same S-invariance of the partition function,

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right), \tag{4.1}$$

leading to the same infinite set of differential constraints (3.3). The partition function, however, will differ from the previous cases. We can express the partition function in terms of characters [37]:

$$Z(\tau, \bar{\tau}) = |\eta(\tau)|^{-2} \sum_{(h,\bar{h})\in S} N_{\bar{h}h} \overline{\hat{\chi}_{\bar{h}}(\tau)} \hat{\chi}_h(\tau) \tag{4.2}$$

where $N_{\bar{h}h}$ is the number of primary operators with conformal weights $(h, \bar{h})$ and

$$\overline{\hat{\chi}_{\bar{h}}(\tau)}\hat{\chi}_h(\tau) = \begin{cases} \bar{q}^{-\frac{\bar{c}-1}{24}}(1-\bar{q})q^{-\frac{c-1}{24}}(1-q) & \bar{h}=0, h=0 \\ \bar{q}^{\bar{h}-\frac{\bar{c}-1}{24}}q^{h-\frac{c-1}{24}}(1-q) & \bar{h}>0, h=0 \\ \bar{q}^{\bar{h}-\frac{\bar{c}-1}{24}}(1-\bar{q})q^{h-\frac{c-1}{24}} & \bar{h}=0, h>0 \\ \bar{q}^{\bar{h}-\frac{\bar{c}-1}{24}}q^{h-\frac{c-1}{24}} & \bar{h}>0, h>0 \end{cases} \tag{4.3}$$

We can simplify these expressions using $\hat{E}_0 = \frac{1}{12} + E_0 = \frac{1}{12} - \frac{c_{\text{tot}}}{24}$, $q = \exp(2\pi i\tau) = \exp(-\beta) = \bar{q}$, and $\Delta = h + \bar{h}$ to give

$$\overline{\hat{\chi}_{\bar{h}}(\tau)}\hat{\chi}_h(\tau) = \begin{cases} \exp(-\beta\hat{E}_0)(1-e^{-\beta})^2 & \bar{h}=0, h=0 \\ \exp[-\beta(\Delta+\hat{E}_0)](1-e^{-\beta}) & \bar{h}>0, h=0 \\ \exp[-\beta(\Delta+\hat{E}_0)](1-e^{-\beta}) & \bar{h}=0, h>0 \\ \exp[-\beta(\Delta+\hat{E}_0)] & \bar{h}>0, h>0 \end{cases} \tag{4.4}$$

We arrange the conformal dimensions in increasing order and explicitly count degeneracies : $0 < \Delta_1 \le \Delta_2 \le \cdots$. Then we can express the partition function in terms of Virasoro primaries as the sum of a vacuum contribution and non-vacuum contributions:

$$\begin{aligned} Z(\beta) &= Z_A(\beta) + Z_0(\beta), \\ Z_A(\beta) &\equiv |\eta(i\beta/2\pi)|^{-2} \sum_{A=1} e^{-\beta(\Delta_A+\hat{E}_0)}(1-e^{-\beta})^{\delta_{h_A 0}+\delta_{\bar{h}_A 0}}, \\ Z_0(\beta) &\equiv |\eta(i\beta/2\pi)|^{-2} e^{-\beta\hat{E}_0}(1-e^{-\beta})^2. \end{aligned} \tag{4.5}$$

The first term is the sum over conformal weights with either or both conformal weights being nonzero, and the second term is the unique vacuum contribution with $h = \bar{h} = 0$.

Before continuing, we consider the case of a CFT constructed as tensor product of two (or more, though we consider only two) CFTs. Naively, the partition function of such a theory will not be of the form (4.2). This is because the factor partition functions are written in terms of their respective Virasoro algebras $L_n^{(i)}$, when in the product case we would need to express the partition function in terms of the full Virasoro algebra $L_n^{(1)} + L_n^{(2)}$. There are seemingly problematic states, such as $(L_{-2}^{(1)} - L_{-2}^{(2)})|0\rangle$, not expressible in the form (4.2). This state does not have well-behaved conformal transformation properties, however. It can be thought of as a combination of the linearly independent state $\left(L_{-2}^{(1)} + L_{-2}^{(2)}\right)|0\rangle$ (the product CFT stress tensor and thus descendent of the product CFT vacuum) and $\left(c^2 L_{-2}^{(1)} - c^1 L_{-2}^{(2)}\right)|0\rangle$ (a primary operator), where c^i is the holomorphic central charge of the i^{th} CFT.

Continuing, we apply the differential constraints (3.3) to the partition function (4.5). Following [32] and [42], we abuse notation and introduce the now-modified polynomials $f_p(z)$ defined by

$$(\beta\partial_\beta)^p Z_A(\beta)\bigg|_{\beta=2\pi} = (-1)^p \frac{e^{-2\pi\hat{E}_0}}{\eta(i)^2} \sum_{A=1} e^{-2\pi\Delta_A} f_p(\Delta_A+\hat{E}_0)(1-e^{-2\pi})^{\delta_{h_A 0}+\delta_{\bar{h}_A 0}}. \tag{4.6}$$

The polynomials f_p have been expressed as functions of Δ_A, when in fact they are functions of h_A and $\bar{h}_A$. This is because we are interested in deriving bounds on Δ_A, and explicit dependence on h_A or $\bar{h}_A$ not in the combination $h_A+\bar{h}_A$ only shows up in Kronecker δ's multiplying otherwise constant terms. We simply note and remember that there will be some additional dependence on $h_A, \bar{h}_A$ that we often suppress. The first few poynomials are explicitly

$$\begin{aligned} f_0(z_A) &= 1 \\ f_1(z_A) &= (2\pi z_A) - \frac{1}{2} - \frac{2\pi}{(e^{2\pi}-1)}(\delta_{h_A 0}+\delta_{\bar{h}_A 0}) \\ f_2(z_A) &= (2\pi z_A)^2 - (2\pi z_A)\left(2 + \frac{4\pi}{e^{2\pi}-1}(\delta_{h_A 0}+\delta_{\bar{h}_A 0})\right) + \left(\frac{7}{8} + 2r_{20}\right) \\ &\quad -4\pi\left(\frac{\pi e^{2\pi} - e^{2\pi} + 1}{(e^{2\pi}-1)^2}\right)(\delta_{h_A 0}+\delta_{\bar{h}_A 0}) + \frac{4\pi^2}{(e^{2\pi}-1)^2}(\delta_{h_A 0}+\delta_{\bar{h}_A 0})^2. \end{aligned} \tag{4.7}$$

We likewise define the unmodified polynomials $b_p(z)$ by

$$(\beta\partial_\beta)^p B(\beta)\bigg|_{\beta=2\pi} = (-1)^p \eta(i)^{-2} \exp(-2\pi \hat{E}_0) b_p(\hat{E}_0), \tag{4.8}$$

The polynomials b_p are exactly the same as in the nonchiral case (which is obvious since the unique vacuum contribution does not depend on the presence or lack of chiral primary operators). Explicitly,

$$b_p(z) = f_p(z) - 2e^{-2\pi} f_p(z+1) + e^{-4\pi} f_p(z+2)\bigg|_{h,\bar{h}>0}. \tag{4.9}$$

Using these polynomials, the equations (3.3) for modular invariance of $Z(\beta)$ for odd p now become

$$\sum_{A=1}^{\infty} f_p(\Delta_A + \hat{E}_0)(1-e^{-2\pi})^{\delta_{h_A 0}+\delta_{\bar{h}_A 0}} \exp(-2\pi\Delta_A) = -b_p(\hat{E}_0). \tag{4.10}$$

To clean up the work that follows, we will define

$$(1-e^{-2\pi})^{\delta_{h_A 0}+\delta_{\bar{h}_A 0}} \equiv \Lambda_A.$$

The derivation now proceeds as in [32]. We take the ratio of the $p=3$ and $p=1$ expressions to get

$$\frac{\sum_{A=1}^{\infty} f_3(\Delta_A + \hat{E}_0)\Lambda_A \exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\Lambda_B \exp(-2\pi\Delta_B)} = \frac{b_3(\hat{E}_0)}{b_1(\hat{E}_0)} \equiv F_1$$

$$\Rightarrow \frac{\sum_{A=1}^{\infty}\left[f_3(\Delta_A+\hat{E}_0) - F_1(\hat{E}_0) f_1(\Delta_A+\hat{E}_0)\right]\Lambda_A \exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\Lambda_B \exp(-2\pi\Delta_B)} = 0. \tag{4.11}$$

We obtain a contradiction in this case by assuming $\Delta_1 > \max(\Delta_1^+, \tilde{\Delta}_1^+, \Delta_{f_1}^+)$, where Δ_1^+ is defined as the largest root of the argument of the numerator's sum when both Kronecker δ's vanish, $\tilde{\Delta}_1^+$ is the largest root when one of the Kronecker δ's is nonzero, and f_1^+ is the largest root of f_1. Because $\Delta_A \geq \Delta_1$, this assumption implies that every term in both the numerator and denominator is strictly positive. Then equation (4.11) says that a sum of positive numbers equals zero — an impossibility. Therefore

$$\Delta_1 \leq \max(\Delta_1^+, \tilde{\Delta}_1^+, \Delta_{f_1}^+). \tag{4.12}$$

From the explicit form of $f_1(\Delta + \hat{E}_0)$ in (2.12), we see that

$$\Delta_{f_1}^+ = \frac{c_{\text{tot}}}{24} + \frac{(3-\pi)}{12\pi}. \tag{4.13}$$

From [32], we know that Δ_1^+ is bounded above by

$$\Delta_1^+ \leq \frac{c_{\text{tot}}}{12} + \delta_0 \approx \frac{c_{\text{tot}}}{12} + .4736... \tag{4.14}$$

We will spend the next section trying to simplify our bound by deriving a manageable expression for $\tilde{\Delta}_1^+$.

4.2 Asymptotic and analytic bound on Δ_1

In the limit $c_{\text{tot}} \to \infty$, $\tilde{\Delta}_1^+$ is proportional to c_{tot}, plus corrections of order c_{tot}^0. We thus expand $\tilde{\Delta}_1^+$ as a series at large central charge:

$$\tilde{\Delta}_1^+ \equiv \sum_{a=-1}^{\infty} d_{-a} \left(\frac{c_{\text{tot}}}{24}\right)^{-a}, \text{ such that } P_1(\tilde{\Delta}_1^+) = 0 \tag{4.15}$$

and $\tilde{\Delta}_1^+$ is the largest real value with this property. Substituting equation (4.15) into the explicit form of $P_1(\tilde{\Delta}_1^+) = 0$, the equation to leading order in c_{tot} is:

$$\frac{\pi^3}{1728} d_1(d_1 - 1)(d_1 - 2) = 0 \Rightarrow d_1 = 2. \tag{4.16}$$

Solving to the next order in c_{tot}, we find the expression

$$\frac{\pi^3}{36} d_0 - \frac{\pi^3}{36} \frac{(\delta_{0h} + \delta_{0\bar{h}})}{e^{2\pi} - 1} - \frac{\pi^2}{18} + \frac{\pi^3}{216} \frac{e^{2\pi} - 13}{e^{2\pi} - 1} = 0.$$

$$\Rightarrow d_0 = \frac{(12 - \pi)e^{2\pi} - 12 + 13\pi + 6\pi}{6\pi(e^{2\pi} - 1)} = \delta_0 + \frac{1}{e^{2\pi} - 1} \approx 0.4755...$$

Thus we see that at this order, $\max(\Delta_1^+, \tilde{\Delta}_1^+, \Delta_{f_1}^+) = \tilde{\Delta}_1^+$; for large enough central charge c_{tot}, we can always bound the conformal dimension Δ_1 using the expression

$$\Delta_1 \leq \frac{c_{\text{tot}}}{12} + 0.4755... + O(c_{\text{tot}}^{-1}). \tag{4.17}$$

An absolute bound on Δ_1 can be obtained with additonal work. Following steps similar to those in the appendices of [32], we can show that a least upper linear bound on $\tilde{\Delta}_1^+$ is given by the first two terms of equation (4.17). From Appendix A.5 of [32], we know that Δ_1^+ is bounded above by $\frac{c_{\text{tot}}}{12} + 0.4736...$ Thus the bound (4.12) simplifies to

$$\Delta_1 \leq \frac{c_{\text{tot}}}{12} + 0.4755... \tag{4.18}$$

This is a universal bound, true for all 2D CFTs with modular invariance, discrete spectra, and $c, \tilde{c} > 1$.

4.3 Bound on Δ_2

Following the previous chapter, we extend the methods described above as in [42] to derive bounds on primaries of second-lowest dimension. Following identical steps means that eq. (3.20) becomes

$$\frac{\sum_{A=2}^{\infty} f_3(\Delta_A + \hat{E}_0)\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=2}^{\infty} f_1(\Delta_B + \hat{E}_0)\Lambda_B e^{-2\pi\Delta_B}} = \frac{f_3(\Delta_1 + \hat{E}_0)\Lambda_A e^{-2\pi\Delta_1} + b_3(\hat{E}_0)}{f_1(\Delta_1 + \hat{E}_0)\Lambda_B e^{-2\pi\Delta_1} + b_1(\hat{E}_0)} \equiv F_2(\Delta_1, c_{\text{tot}}).$$

Following work from Appendix A (differing by small constants), it can be shown that F_2 is finite and nonzero for $c, \tilde{c} > 1$ and Δ_1 obeying eq. (4.18). Then

$$\frac{\sum_{A=2}^{\infty}\left[f_3(\Delta_A+\hat{E}_0)-f_1(\Delta_A+\hat{E}_0)F_2\right]\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=2}^{\infty}f_1(\Delta_B+\hat{E}_0)\Lambda_B e^{-2\pi\Delta_B}}=0. \tag{4.19}$$

We define $\Delta_{f_p}^{+}$ to be the largest root of $f_p(\Delta+\hat{E}_0)$ viewed as a polynomial in Δ. The bracketed expression in the numerator is a polynomial cubic in Δ_2; we denote it by $P_2(\Delta_2)$, and define the largest root of P_2 to be $\Delta_2^{+}(c_{\text{tot}}, \Delta_1)$ or $\hat{\Delta}_2^{+}(c_{\text{tot}}, \Delta_1)$.

As before, assuming $\Delta_2 > \max(\Delta_{f_1}^{+}, \Delta_2^{+}, \hat{\Delta}_2^{+})$ means every term in both the numerator and denominator of the left side of equation (4.19) is positive. The left side thus can not be equal to zero, and we have a contradiction. Our assumption is therefore false, and:

$$\Delta_2 \leq \max(\Delta_{f_1}^{+}, \Delta_2^{+}, \hat{\Delta}_2^{+}). \tag{4.20}$$

Motivated by earlier results, we seek a linear bound of the form $\Delta_2 \leq \frac{c_{\text{tot}}}{12} + D_2$, where D_2 is a numerical constant independent of Δ_1. We seek the smallest D_2 such that the inequality is valid for $c_{\text{tot}} > 2$ and for all possible values of Δ_1, h_1, $\tilde{h}_1$, h_2, and $\tilde{h}_2$—unlike previous work, the conformal weights appear explicitly as arguments of Kronecker δ's. We can derive a univeral bound by finding $\max(\Delta_2^{+} - \frac{c_{\text{tot}}}{12}, \tilde{\Delta}_2^{+} - \frac{c_{\text{tot}}}{12})$ over all its dependences for their respective domains. This function attains a global maximum $D_2 \approx 0.5531...$ (for $c_{\text{tot}} \approx 2$, $\Delta_1 \approx 0.2669...$, $\delta_{h_1 0}+\delta_{\tilde{h}_1 0} = 1$, and $\delta_{h_2 0}+\delta_{\tilde{h}_2 0} = 1$). Therefore

$$\Delta_2 \leq \frac{c_{\text{tot}}}{12} + 0.5531... \tag{4.21}$$

This is a weaker bound on Δ_2 than found in the previous chapter; this is expected given that we are now considering 2D CFTs with no conditions on the chirality of primary operators.

4.4 Bound on Δ_n

The extension of these methods to primary operators of higher dimensions is straightforward. A condition that must hold for our proof to hold for Δ_n is that F_n, defined as

$$F_n(\hat{E}_0, \Delta_1, \cdots, \Delta_{n-1}) \equiv \frac{\sum_{i=1}^{n-1} f_3(\Delta_i+\hat{E}_0)\exp(-2\pi\Delta_i)+b_3(\hat{E}_0)}{\sum_{i=1}^{n-1} f_1(\Delta_i+\hat{E}_0)\exp(-2\pi\Delta_i)+b_1(\hat{E}_0)}, \tag{4.22}$$

be well-defined for all relevant values of its arguments. Assuming so, we can proceed as above to obtain a bound

$$\Delta_n \leq \max(\Delta_{f_1}^{+}, \Delta_n^{+}, \tilde{\Delta}_n^{+}), \tag{4.23}$$

where $\Delta_{f_1}^{+}$ is given by (3.22) and $\max(\Delta_n^{+}, \tilde{\Delta}_n^{+})$ is the largest real root of the polynomial

$$P_n(\Delta_n) \equiv f_3(\Delta_n+\hat{E}_0) - f_1(\Delta_n+\hat{E}_0)F_n \tag{4.24}$$

and is thus a function of $c_{\text{tot}}, \Delta_1, \cdots, \Delta_{n-1}$ in an analogous way.

Following Chapter 3, it seems reasonable to expect a bound as before:

$$\Delta_n \leq \Delta_n^+ < \frac{c_{\text{tot}}}{12} + O(1). \tag{4.25}$$

As there, however, the quantity F_n can become undefined for $n \geq 4$. In the case of $n = 4$, we again restrict the allowed values for the total central charge to $c_{\text{tot}} > \max(c_{D4}^+, c_{N4}^+)$. This leads to the bound

$$\Delta_4 \leq \frac{c_{\text{tot}}}{12} + O(1) \tag{4.26}$$

Similar results follow for arbitrary Δ_n using the methods described here.

For larger n, we follow a procedure similar to the one in Chapter 3. In Appendix C.2 we solve for the value of the central charge c_{Dn} which causes the denominator of F_n to vanish, and in Appendix D.2 we show that having all Δ's be degenerate maximizes/universalizes our bound. The result is that we must again require

$$\log n \lesssim \frac{\pi c_{\text{tot}}}{12} + O(1), \tag{4.27}$$

for F_n to be finite and nonzero. Then a similar analysis gives the bound

$$\Delta_n \leq \frac{c_{\text{tot}}}{12} + O(1), \tag{4.28}$$

where $O(1)$ here means with respect to c_{tot}. There still could be subleading corrections that for large enough n contribute to leading order. By requiring

$$n \ll e^{\pi c_{\text{tot}}/6} + O(1),$$

work similar to Appendix E gives the bound

$$\Delta_n \leq \frac{c_{\text{tot}}}{12} + O(1) \tag{4.29}$$

for asymptotically large central charge and where the $O(1)$ term here is a constant with respect to all variables.

Chapter 5

Bounds on number of states and primaries

In this chapter, we turn our attention to bounding the number of states in a given energy range. We show that for a general unitary two-dimensional conformal field theory with discrete spectrum and modular invariance, there are infinitely many constraints on the number of states (primary and descendant) with energy $E \in (\epsilon, \mathcal{E})$, where $\epsilon \sim \mathcal{E} e^{-2\pi\mathcal{E}}$. We also derive a bound on the number of primary states with energy in a similar range for the case where total central charge c_{tot} is large and discuss methods by which these bounds could be improved.

5.1 Polynomial constraints from thermal partition function

Consider again a two-dimensional conformal field theory with a discrete spectrum described by a unitary quantum mechanics. The S-invariance of the conformal field theory partition function resulted in eq. (3.3). For purely imaginary complex structure $\tau = \frac{i\beta}{2\pi}$, the conformal field theory partition function reduces to the thermodynamic partition function, which can be written as

$$Z(\beta) = \text{Tr}\left(e^{-\beta H}\right) = \sum_n \exp(-\beta E_n). \tag{5.1}$$

The E_n are the discrete, real (and possibly degenerate) eigenvalues of the Hamiltonian H of the theory on a circle of length 2π.

Applying eq. (3.3) to eq. (5.1) gives up constraints in terms of polynomial functions of E_n:

$$\sum_n \exp(-2\pi E_n) g_p(E_n) = 0, \quad p \text{ odd} \tag{5.2}$$

where g_p is a p^{th}-order polynomial defined by

$$g_p(E_n) \equiv \exp(2\pi E_n)\left(\beta\partial_\beta\right)^p \exp(-\beta E_n)\Big|_{\beta=2\pi}. \tag{5.3}$$

Some explicit expressions for g_p are

$$g_1(E) = -2\pi E$$

$$g_3(E) = -(2\pi E)^3 + 3(2\pi E)^2 - (2\pi E)$$

We note that the polynomials $g_p(E)$ are related to the Bell polynomials $B_p(x)$ [43]. Defining $x \equiv 2\pi E$, our polynomial constraints (5.2) can thus be expressed as

$$\sum_{n=0} B_p(-x_n) e^{-x_n} = 0, \quad p \text{ odd}. \tag{5.4}$$

5.2 Upper bounds on the number of states

The most straightforward consequence of equation (5.2) is to give an upper bound on the number of states (primary or descendant) with an energy in a given range. From the $p = 1$ equation, it follows that

$$\sum_n E_n \exp(-2\pi E_n) = 0. \tag{5.5}$$

We define energy E_p to be the lowest positive energy level, so that $E_p \geq 0$ and $E_{p-1} < 0$. Then we can split equation (5.5) so that one contribution begins with energy E_p. This gives

$$\sum_{j\geq p} E_j \exp(-2\pi E_j) = \sum_{i=0}^{p-1} |E_i| \exp(2\pi |E_i|). \tag{5.6}$$

We will bound the LHS below and the RHS above. Every term on the RHS is positive, and the largest term happens when $i = 0$. Therefore we have

$$\sum_{i=0}^{p-1} |E_i| \exp(2\pi |E_i|) \leq \sum_{i=0}^{p-1} |E_0| \exp(2\pi |E_0|) = p\frac{c_{\text{tot}}}{24} \exp\left(\frac{\pi c_{\text{tot}}}{12}\right).$$

To bound the LHS, we use the fact the argument of our sum takes its maximum value of e^{-1} when $E_i = (2\pi)^{-1}$. This allows us to split the sum as

$$\sum_{i=p}^{r-1} E_i \exp(-2\pi E_i) + \sum_{j\geq r} E_j \exp(-2\pi E_j),$$

where $E_r \geq (2\pi)^{-1}$ and $E_{r-1} < (2\pi)^{-1}$. We truncate the infinite sum at energy $\mathcal{E}$ — cutting off a sum of positive terms will give us a strictly smaller quantity. The crudest approximation is to drop the first sum completely. Then we can bound the LHS using the fact that every term in the second sum is greater than or equal to $\mathcal{E}\exp(-2\pi\mathcal{E})$. By defining $N_{\mathcal{E}}$ as the number of states with energy $(2\pi)^{-1} \leq E \leq \mathcal{E}$, we have thus derived the inequality

$$N_{\mathcal{E}}\mathcal{E}\exp(-2\pi\mathcal{E}) < p\frac{c_{\text{tot}}}{24} \exp\left(\frac{\pi c_{\text{tot}}}{12}\right). \tag{5.7}$$

Alternatively, we can consider the limit of large $\mathcal{E}$. In this limit all energies between approximately $\mathcal{E}e^{-2\pi\mathcal{E}}$ and $\mathcal{E}$ will contribute terms on the LHS of (5.6) greater than or equal to $\mathcal{E}e^{-2\pi\mathcal{E}}$. If we label the number of energies in this range as $N^+_{\mathcal{E}}$, then we have the inequality

$$N^+_{\mathcal{E}}\mathcal{E}\exp(-2\pi\mathcal{E}) < p\frac{c_{\text{tot}}}{24} \exp\left(\frac{\pi c_{\text{tot}}}{12}\right). \tag{5.8}$$

This inequality allows us to make interesting statements. For example, considering the case of $\mathcal{E} \sim \frac{c_{\text{tot}}}{24}$ (where c_{tot} must therefore be large), we get the result

$$\Rightarrow N^+_{c_{\text{tot}}/24} \leq p \exp\left(\frac{\pi c_{\text{tot}}}{6}\right). \tag{5.9}$$

Thus we have derived an upper bound on the number of states with energy in the range $E \in \left[\frac{c_{\text{tot}}}{24} e^{-\pi c_{\text{tot}}/12}, \frac{c_{\text{tot}}}{24}\right]$. By adding p to both sides, we can get a bound on the total number of states of energy $E < \frac{c_{\text{tot}}}{24}$ (except for the interval $E \in \left[0, \frac{c_{\text{tot}}}{24} e^{-\pi c_{\text{tot}}/12}\right]$):

$$N^*_{c_{\text{tot}}/24} \leq p \left(1 + \exp\left(\frac{\pi c_{\text{tot}}}{6}\right)\right). \tag{5.10}$$

Because equations (5.9) and (5.9) are upper bounds on the number of states for large total central charge, they will also be upper bounds on the number of primary states.

Before proceeding, we turn our attention to the factor p. We are interested in how p depends on c_{tot}, for example, if we consider a theory with only the identity operator and its descendants. We will be interested in studying pure gravity up to as high an energy scale as possible. Given that at $\Delta = c_{\text{tot}}/12$ one must add additional primary states, we thus consider a CFT with only the identity conformal block up to this value of Δ. The number p should count all of the states with energy less than zero; we recall the earlier fact that the number of distinct, linearly independent states at level N is $P(N)$. (We are actually overcounting slightly due to the absence of states at level 1, but this does not affect our conclusion). This means we are interested in counting the number of descendants of the vacuum ($h = 0$) with $E < 0$, or equivalently, with $N < \frac{c}{24}$. We need to calculate

$$p = \sum_{n=0}^{c/24} P(n). \tag{5.11}$$

Considering large c, we use the asymptotic behavior of the partition function [44] to get

$$P(n) \leq P\left(\frac{c}{24}\right) \sim \frac{1}{4\sqrt{3}\left(\frac{c}{24}\right)} e^{\pi\sqrt{\frac{2}{3}\left(\frac{c}{24}\right)}} = \frac{2\sqrt{3}}{c} e^{\pi\sqrt{c}/6}. \tag{5.12}$$

This means in the case that we have only the identity operator as a primary, roughly speaking we have

$$p \lesssim \frac{\sqrt{3}}{12} e^{\pi\sqrt{c}/6}. \tag{5.13}$$

For large central charge, the factor p does not affect the leading behavior in terms of the central charge. Thus in this case, eq. (5.9) goes as

$$\log(N^+_{c_{\text{tot}}/24}) \leq \frac{\pi c_{\text{tot}}}{6} + O(\sqrt{c_{\text{tot}}}). \tag{5.14}$$

5.3 Bounding number of primary operators

We now attempt to derive an upper bound on the number of primary operators with energies that lie within a given range. To do this, we will use S-invariance of the full partition function (4.10) instead of eq. (5.2). We will again focus on the lowest order constraint:

$$\sum_{A=1}^{\infty} f_1\left(E_A + \frac{1}{12}\right) \Lambda_A \exp(-2\pi E_A) = -b_1(\hat{E}_0)\exp(-2\pi E_0), \tag{5.15}$$

where we have used $\Delta_A = E_A - E_0$. As before, it will be convenient to refer to some particular energies. We see the argument of the sum vanishes when $E_i = \frac{1}{4\pi} - \frac{1}{12} + \frac{\delta_{h_A 0}+\delta_{\bar{h}_A 0}}{e^{2\pi}-1}$. We thus define $E_p^+ \equiv \frac{1}{4\pi} - \frac{1}{12} + \frac{1}{e^{2\pi}-1}$, and $E_p^- \equiv \frac{1}{4\pi} - \frac{1}{12}$. Further, we see that the argument of our sum takes its maximum value when $E_i = \frac{3}{4\pi} - \frac{1}{12} + \frac{\delta_{h_A 0}+\delta_{\bar{h}_A 0}}{e^{2\pi}-1}$. We thus also define $E_r^+ \equiv \frac{3}{4\pi} - \frac{1}{12} + \frac{1}{e^{2\pi}-1}$ and $E_r^- \equiv \frac{3}{4\pi} - \frac{1}{12}$.

We now divide equation (5.15) into the sums

$$\sum_{>E_0}^{<E_p^-} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} + \sum_{\geq E_p^-}^{<E_p^+} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} \tag{5.16}$$

$$+ \sum_{\geq E_p^+}^{\infty} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} = -b_1(\hat{E}_0)\exp(-2\pi E_0)$$

The first term is strictly negative by definition; we move it to the RHS. The second term is more complicated. We are interested in deriving a universally true bound using the third sum. To make sure our bound is universal, we want the second sum that we subtract over to be as small (as negative) as possible. This will happen if we evaluate $h_A = \bar{h}_A = 0$ and evaluate every term in the sum at E_p^-. Even then, the most negative that any term in that sum could contribute is still not as much as evaluating any term in the sum at E_0. For now we use this weaker inequality to get

$$\sum_{E_p^+}^{\infty} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} = -b_1(\hat{E}_0)e^{-2\pi E_0}$$

$$- \sum_{E_0}^{E_p^-} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} - \sum_{E_p^-}^{E_p^+} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} \tag{5.17}$$

$$\leq -b_1(\hat{E}_0)e^{-2\pi E_0} + m\left(\frac{\pi c_{\text{tot}}}{12} - \frac{\pi}{6} + \frac{1}{2} + \frac{2\pi}{e^{2\pi}-1}\right)e^{\pi c_{\text{tot}}/12}$$

where m is the number of primary operators with energy greater than the ground state energy and smaller than E_p^+. We also know that

$$-b_1(\hat{E}_0) = \left(\frac{\pi c_{\text{tot}}}{12} - \frac{\pi}{6} + \frac{1}{2}\right)(1-e^{-2\pi})^2 + 4\pi e^{-2\pi}(1-e^{-2\pi})$$

In the limit of large total central charge c_{tot}, then, the RHS of eq. (5.17) will be

$$\leq \frac{\pi c_{\text{tot}}}{12}\left(m + (1-e^{-2\pi})^2\right)e^{\pi c_{\text{tot}}/12}. \tag{5.18}$$

We now turn our attention to the LHS of eq. (5.17). Again, we truncate this infinite sum at some energy $\mathcal{E}$ so that the LHS is

$$\geq \sum_{E_p^+}^{\mathcal{E}} f_1\left(E_A + \frac{1}{12}\right)\Lambda_A e^{-2\pi E_A}.$$

We again divide this sum into pieces, as

$$\sum_{E_p^+}^{E_r^-} f_1\left(E_A+\frac{1}{12}\right)\Lambda_A e^{-2\pi E_A}+\sum_{E_r^-}^{E_r^+} f_1\left(E_A+\frac{1}{12}\right)\Lambda_A e^{-2\pi E_A}+\sum_{E_r^+}^{\mathcal{E}} f_1\left(E_A+\frac{1}{12}\right)\Lambda_A e^{-2\pi E_A} \tag{5.19}$$

The argument of the first sum takes its minimum for $E_A = E_p^+$; the argument of the third sum takes its minimum for $E_A = \mathcal{E}$. Every term in the second sum will evaluate to be larger than if it were evaluated at $\mathcal{E}$, for sufficiently large $\mathcal{E}$. In fact, we will assume $\mathcal{E}$ is large enough that

$$f_1\left(\mathcal{E}+\frac{1}{12}\right)e^{-2\pi\mathcal{E}} \leq f_1\left(E_p^+ + \frac{1}{12}\right)(1-e^{-2\pi})e^{-2\pi E_p^+} \tag{5.20}$$

Defining $N_{\mathcal{E}}^+$ as the total number of primary operators with energies greater than E_p^+ and less than $\mathcal{E}$, we therefore have that the LHS is

$$\geq N_{\mathcal{E}}^+ f_1\left(\mathcal{E}+\frac{1}{12}\right)(1-e^{-2\pi})e^{-2\pi\mathcal{E}} \tag{5.21}$$

Using eqs. (5.21) and (5.18) gives us the inequality

$$N_{\mathcal{E}}^+ f_1\left(\mathcal{E}+\frac{1}{12}\right)(1-e^{-2\pi})e^{-2\pi\mathcal{E}} \leq \frac{\pi c_{\text{tot}}}{12}\left(m+(1-e^{-2\pi})^2\right)e^{\pi c_{\text{tot}}/12}, \tag{5.22}$$

or,

$$N_{\mathcal{E}}^+ \leq \frac{\frac{\pi c_{\text{tot}}}{12}\left(m+(1-e^{-2\pi})^2\right)}{f_1\left(\mathcal{E}+\frac{1}{12}\right)(1-e^{-2\pi})}e^{\pi c_{\text{tot}}/12+2\pi\mathcal{E}}. \tag{5.23}$$

We are already restricting ourselves to the case of large $\mathcal{E}, c_{\text{tot}}$. For $\mathcal{E} = \frac{c_{\text{tot}}}{24}$ we have

$$N_{\frac{c_{\text{tot}}}{24}}^+ \leq \frac{\frac{\pi c_{\text{tot}}}{12}\left(m+(1-e^{-2\pi})^2\right)}{\frac{\pi c_{\text{tot}}}{12}(1-e^{-2\pi})}e^{\pi c_{\text{tot}}/6} \tag{5.24}$$

$$= \left(\frac{m}{1-e^{-2\pi}}+(1-e^{-2\pi})\right)e^{\pi c_{\text{tot}}/6} \approx (m+1)\, e^{\pi c_{\text{tot}}/6}$$

Therefore in the limit of large c_{tot},

$$N_{\frac{c_{\text{tot}}}{24}}^+ \lesssim n_{\frac{c_{\text{tot}}}{24}} e^{\pi c_{\text{tot}}/6}, \tag{5.25}$$

where $N_{\frac{c_{\text{tot}}}{24}}^+$ is the number of primary operators with energy between E_p^+ and $\frac{c_{\text{tot}}}{24}$ and $n_{\frac{c_{\text{tot}}}{24}}$ is the number of primary operators (including the vacuum) with energy less than E_p^+. Note the similarity between this bound and the inequality (5.9).

5.4 Higher-order constraints

We have thus far only considered the lowest order constraints $p = 1$. There are, in fact, infinitely many constraints on the number of states with energy lying in an appropriate range. The order p constraint will involve a degree-p polynomial in E (Δ) for deriving a bound on the number of states (primaries). At the level of $p = 1$, we already had difficulties with Kronecker δ's and splitting up and evaluating sums in order to derive inequalities. Thus for now we will only consider deriving a bound on the number of states. Obviously, however, an upper bound on the number of states will still serve as an upper bound on the number of primaries.

As found earlier, our polynomial constraints can be expressed in terms of Bell polynomials (5.4). The properties of Bell polynomials are well understood. It can be shown that the Bell polynomials obey the recurrence relation

$$B_{n+1}(x) = xB_n(x) + xB_n'(x). \tag{5.26}$$

In particular, every Bell polynomial $B_n(x)$ with $n \geq 1$ has the root $x = 0$. Another important property is that the Bell polynomial $B_n(-x)$ has n distinct, real, non-negative (including zero) roots. This means that a similar procedure to the $p = 1$ case will give an inequality in terms of a degree-p polynomial. One could hope that by combining these constraints, the situation simplies. For example, considering additional constraints could allow us to make statements about intervals that we cannot easily bound. This is an interesting direction for future investigations.

Chapter 6

Gravitational interpretation

In this chapter, we use the AdS_3/CFT_2 correspondence to translate our bounds on state number and conformal dimension in two-dimensional conformal field theories to bounds on the number of microstates and masses for theories of three-dimensional gravity in anti-de Sitter spacetimes (maximally symmetric curved spacetime with cosmological constant $\Lambda < 0$). We will begin with a brief introduction to the relevant features from the AdS_3/CFT_2 correspondence before translating our bounds from the boundary conformal theory to the bulk gravitational theory. This work does not contain a detailed introduction to the AdS/CFT correspondence. For a more complete review, we refer to [45] or [46].

We then consider the specific case of AdS_3 pure gravity and what our bounds can tell us about this three-dimensional theory of quantum gravity. Pure Einstein gravity in three dimensions seems trivial because it has no local propagating degrees of freedom. In n dimensions, the induced metric has $n(n-1)/2$ independent components, and there are n constraint equations that need to be satised by physical solutions (in the ADM formalism, for example, these are the Hamilton and momentum constraints). The number of local physical degrees of freedom, therefore, is $n(n-3)/2$, which obviously vanishes when $n = 3$. In [47, 48], however, it was shown that AdS_3 admits black hole solutions with similar thermodynamical properties to their higher dimensional counterparts. This suggests the study of 2+1 gravity may indeed be quite useful for understanding relevant aspects of more realistic quantum gravity models.

6.1 AdS_3/CFT_2

The study of the asymptotically AdS_3 spacetimes lead to the discovery that the algebra of charges associated with asymptotic spacetime symmetries is given by two copies of the Virasoro algebra [49]. Because the physical states must form a representation of this algebra upon quantization, the quantum theory of asymptotically AdS 2+1 gravity should be a conformal field theory of the corresponding central charge. The obtained value for this Virasoro central charge is

$$c + \bar{c} = \frac{3L}{G_N}, \tag{6.1}$$

where $L = |\Lambda|^{-1/2}$ is the AdS radius and G_N is Newton's constant. This correspondence, along with Cardy's formula (2.49), showed the computation of the black hole entropy to be in agreement with the Bekenstein-Hawking formula [50]. The result of [49] is now seen as a precursor to Maldacena's conjecture [51] of the AdS/CFT correspondence. This conjecture states that there is an exact equivalence between string theory on an AdS background and a conformal field theory on the conformal boundary of AdS. This correspondence is a concrete realization of the holographic principle of 't Hooft and Susskind [52], as it relates a theory in $d+1$ dimensions to a theory in d dimensions.

In addition to matching the total central charge and AdS radius, we wish to match the spectrum of massive bulk objects with the spectrum of boundary primary operators. A primary state corresponds to a state at rest with respect to the global time coordinate of AdS, because its energy cannot be lowered by acting with boost generators. The bulk interpretation of descendants can be interpreted as the original massive state in the bulk with boundary metric excitations added [53]. To be more precise, the states obtained by acting with $L_{-n}, \bar{L}_{-n}$ with $n \geq 2$ correspond to creation operators for quadrupole and higher modes of the metric; these are located at the boundary at spatial infinity and can therefore be thought of as "boundary gravitons" (since as stated earlier, there are no bulk gravitational propagating degrees of freedom). Acting with L_{-1} and $\bar{L}_{-1}$, on the other hand, can be though of as exciting the dipole mode of the metric, which is pure gauge when applied to the vacuum but not pure gauge when applied to a state with a massive object in the bulk to a state of motion with higher energy. Thus we have the correspondence

$$E^{(rest)} = \frac{\Delta}{L}, \tag{6.2}$$

where $E^{(rest)}$ is the rest energy of an object in the bulk of AdS and Δ is the dimension of the primary operator.

6.2 Bounds in gravitational theory

We begin by interpreting our results from Chapter 3. According to eq. (6.2), we can interpret our bounds as saying that the dual gravitational theory, when it exists, must have massive states in the bulk (without boundary excitations) with rest energies $M_n = \Delta_n/L$ satisfying

$$M_n \leq M_n^+ \equiv \frac{1}{L}\Delta_n^+|_{c_{\text{tot}}=\frac{3L}{G_N}}. \tag{6.3}$$

Using our asymptotic bound (3.40), this inequality becomes

$$M_n \leq \frac{1}{4G_N} + \frac{D_n}{L}, \tag{6.4}$$

where D_n is an $O(1)$ or smaller term in c_{tot} and n is constrained appropriately. In the flat-space limit $L \to \infty$, this inequality becomes

$$M_n \leq \frac{1}{4G_N}.$$

Since n can be of exponentially large order in c according to eq.(3.39), this inequality indicates a high density of gravitational microstates of mass $\leq 1/4G_N$. Indeed, the logarithm of the number N of such states should be at least equal to the upper bound on $\log(n)$ of eq.(3.39),

$$\log N \geq \frac{\pi c_{\text{tot}}}{12} + O(1) = \frac{\pi L}{4G_N} + O(1). \tag{6.5}$$

The density of these states should be strongly peaked at the upper limit of the mass range, so this may be interpreted as a lower bound on the entropy of a spinless (2+1)-D black hole of mass $1/4G_N$. The actual entropy of a spinless black hole of this mass is [47, 48, 50]

$$S = \frac{\pi c_{\text{tot}}}{6} = \frac{\pi L}{2G_N}$$

which obeys the bound (6.5).

This discussion extends in a natural way to the results from Chapter 4. Using (6.2), the upper bounds (4.18) and (4.21) become upper bounds of order $1/4G_N$ on the masses of the two lightest states. In the case where one of these primary operators is (anti)holomorphic, our correspondence (6.2) is not valid. This is because such states have a little group different from that of a massive particle in the bulk of AdS. For large L, they can only correspond to massless states without a rest frame or to states that don't propagate in the bulk. Despite being a special case, however, these massless states will trivially satisfy our nonzero upper mass bound.

We can similarly discuss the gravitational interpretation of our results from Chapter 5. We will consider the case of large central charge, meaning the flat space limit $\Lambda \to 0$. For large enough c_{tot}, the bound (5.9) will hold. In the dual gravitational theory (when it exists), this bound becomes an upper bound on the number of gravitational states (including boundary excitations) with mass less than $1/4G_N$

$$N^+_{c_{\text{tot}}/24} \leq \exp\left(\frac{\pi c_{\text{tot}}}{6}\right) O(\exp(\sqrt{c_{\text{tot}}})).$$

As an explicit example, we consider AdS_3 gravity with no other fields. This theory will consist of the identity conformal block — the vacuum and its Virasoro descendants. By the discussions in Chapter 5, the factor d for this theory will not contribute to the leading-order c_{tot} behavior. Combining (5.9) and (6.5) gives that to leading order in c_{tot},

$$\frac{\pi L}{4G_N} + O(1) \leq \log N \leq \frac{\pi L}{2G_N} + O\left(\sqrt{\frac{L}{G_N}}\right). \tag{6.6}$$

Clearly the upper and lower bounds are the same order, but they differ by a factor of two. We are currently employing techniques to push these bounds closer together, but have not yet been able to improve upon eq. (6.6).

Appendix A

Behavior of F_2, F_3

In this appendix, we prove that the functions F_2, F_3 are defined for all relevant values of our parameters. The function F_2 is given by

$$F_2 \equiv \frac{f_3(\Delta_1 + \hat{E}_0)\exp(-2\pi\Delta_1) + b_3(\hat{E}_0)}{f_1(\Delta_1 + \hat{E}_0)\exp(-2\pi\Delta_1) + b_1(\hat{E}_0)} \tag{A.1}$$

and the polynomials f_i and b_i are given in equations (3.12) and (3.14). By inspection, we see that F_2 will only become undefined if the denominator equals zero. The value of the central charge when $f_1(\Delta_1 + \hat{E}_0)\exp(-2\pi\Delta_1) + b_1(\hat{E}_0) = 0$ is

$$c_{D2} = \frac{2}{\pi}\frac{(12\pi\Delta_1 + \pi - 3)e^{-2\pi\Delta_1} + \pi - 3 - 26\pi e^{-2\pi} + 6e^{-2\pi} + 25\pi e^{-4\pi} - 3e^{-4\pi}}{e^{-2\pi\Delta_1} + 1 - 2e^{-2\pi} + e^{-4\pi}} \tag{A.2}$$

The maximum possible value that c_{D2} can take for $\Delta_1 > 0$ is $c^{+}_{D2} = 1.0868...$ This value of the total central charge, however, is outside of the assumed range $c_{\text{tot}} > 2$. Therefore the function F_2 is defined for all relevant values of c_{tot}, Δ_1.

Our proof can also run into problems if F_2 is vanishing for any values of our parameter space. The condition for vanishing F_2 is

$$f_3(\Delta_1 + \hat{E}_0)\exp(-2\pi\Delta_1) + b_3(\hat{E}_0) = 0. \tag{A.3}$$

We once again solve for the total central charge satisfying this equation and label it. This expression can be maximized numerically; it has a maximum value given by $c^{+}_{N2} = 0.9632...$. This value of the total central charge is also outside of the relevant range $c_{\text{tot}} > 2$. Therefore the function F_2 is well-defined and non-vanishing–in fact, positive– for all relevant values of c_{tot}, Δ_1, and our proof by contradiction will be valid.

A similar analysis applies to the function F_3 given by

$$F_3 \equiv \frac{\sum_{i=1}^{2} f_3(\Delta_i + \hat{E}_0)\exp(-2\pi\Delta_i) + b_3(\hat{E}_0)}{\sum_{i=1}^{2} f_1(\Delta_i + \hat{E}_0)\exp(-2\pi\Delta_i) + b_1(\hat{E}_0)}. \tag{A.4}$$

Once again, we are interested in where this function vanishes or becomes undefined. This can be studied by solving for values of the central charge at which either the numerator or denominator vanishes. These solutions will be labeled as c_{N3} and c_{D3}; they are functions of Δ_1 and Δ_2. We maximize c_{N3} and c_{D3} over the allowed range of Δ_1, Δ_2 and find

$$c^{+}_{N3} \approx 1.3929... \tag{A.5}$$

and

$$c^{+}_{D3} \approx 1.8022... \tag{A.6}$$

These values of the central charge, however, are again outside of the relevant range for c_{tot}, since we have restricted our work to $c_{\text{tot}} > 2$. Therefore the function F_3 is defined and positive for all relevant values of c_{tot}, Δ_1, and Δ_2.

Appendix B

The $O(1)$ term in Δ_2^+

In this appendix, we calculate the $O(1)$ term in the expansion of the largest root Δ_2^+ of polynomial $P_2(\Delta_2)$ for asymptotically large total central charge. In the body of the text, we reasoned that the leading coefficient in the large-c_{tot} expansion of Δ_2^+,

$$\Delta_2^+ \equiv \sum_{a=-1}^{\infty} d_{-a}(\Delta_1)\left(\frac{c_{\text{tot}}}{24}\right)^{-a},$$

is $d_1 = 2$. Expanding $P_2(\Delta_2^+) = 0$ to next order in c_{tot}, we find the expression

$$\frac{-1}{e^{-2\pi\Delta_1} + (1-e^{-2\pi})^2}\frac{\pi}{18}(-\pi e^{-2\pi\Delta_1} - 6\pi d_0 e^{-2\pi\Delta_1} - 13\pi e^{-4\pi} + 14\pi e^{-2\pi} - 6\pi d_0 - \pi - 24e^{-2\pi}$$

$$+12e^{-4\pi} + 12e^{-2\pi\Delta_1} + 12 - 6\pi d_0 e^{-4\pi} + 12\pi d_0 e^{-2\pi} - 6\pi\Delta_1 e^{-2\pi*\Delta_1}) = 0.$$

Solving for d_0 gives us

$$d_0(\Delta_1) = \frac{(12-\pi) + (14\pi - 24)e^{-2\pi} + (12 - 13\pi)e^{-4\pi} + (12 - \pi - 6\pi\Delta_1)e^{-2\pi\Delta_1}}{6\pi(e^{-2\pi\Delta_1} + (1-e^{-2\pi})^2)}.$$

To keep our bound universal we should take the maximum possible value of this function. This occurs as $\Delta_1 \to \infty$ meaning $d_0(\Delta_1) \to 0.4736...$—the same constant appearing in the bound on Δ_1. Thus for large enough central charge c_{tot}, we can always bound the conformal dimension Δ_2 using the expression

$$\Delta_2 \le \frac{c_{\text{tot}}}{12} + \delta_0.$$

Appendix C

Condition on c_{tot}, n

C.1 Nonchiral primaries

Here we will sketch the proof of the condition on c_{tot} given by equation (3.38). We begin with the condition that the denominator of F_n vanishes

$$\sum_{A=1}^{n-1} f_1 \left(\Delta_A + \frac{1}{12} - \frac{c_{Dn}^+}{24}\right) e^{-2\pi\Delta_A} - b_1 \left(\frac{1}{12} - \frac{c_{Dn}^+}{24}\right) = 0. \tag{C.1}$$

Upon expansion, this can be rearranged to give

$$\frac{\pi c_{Dn}}{12}\left((1-e^{-2\pi})^2 + \sum_{A=1}^{n-1} e^{-2\pi\Delta_A}\right) =$$

$$\sum_{A=1}^{n-1} 2\pi\Delta_A e^{-2\pi\Delta_A} + \left(\frac{\pi}{6} - \frac{1}{2}\right)\left((1-e^{-2\pi})^2 + \sum_{A=1}^{n-1} e^{-2\pi\Delta_A}\right) - 4\pi e^{-2\pi}(1-e^{-2\pi}).$$

Dividing through by the parenthetical expression on the LHS gives an expression for c_{Dn}.

We wish to consider total central charge larger than c_{Dn} in equation (C.1). We maximize c_{Dn} by differentiating with respect to Δ_i to find critical points. We show in Appendix D that c_{Dn} is maximized when $\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1}$. The value of Δ_i which maximizes c_{Dn} is given by

$$\Delta_i = \frac{1}{2\pi} W_0[A(n-1)] + \frac{1}{2\pi} - \frac{2}{e^{2\pi}-1},$$

$$A \equiv \frac{\exp\left(-\frac{(4\pi e^{-2\pi} + e^{-2\pi} - 1)}{-1+e^{-2\pi}}\right)}{1 - 2e^{-2\pi} + e^{-4\pi}} \approx 0.3780...$$

Substituting this into equation (C.1) gives a complicated expression which may be simplified using the definition of the Lambert-W function

$$z = W_0(z) e^{W_0(z)} \quad \Rightarrow \quad e^{-W_0(z)} = \frac{W_0(z)}{z}.$$

After some algebra, we find the expression

$$c_{Dn}^+ = \frac{12}{\pi}\left(W_0[A(n-1)] + C_1\right), \tag{C.2}$$

$$C_1 \equiv -\frac{4\pi}{e^{2\pi}-1} + \frac{\pi}{6} - \frac{1}{2}.$$

We consider central charge such that $c_{\rm tot} > c_D^+$, and use the fact that $W_0(z) \approx \log(z)$ plus $O(\log(\log(z)))$ corrections. Then for large n only the first term on the RHS of (C.2) will survive, and we deduce

$$c_{Dn}^+ \approx \frac{12}{\pi} W_0[A(n-1)] \sim \frac{12}{\pi}\log(n)$$

as in eq. (3.38)

C.2 Chiral primaries

Here we will sketch the proof of the condition on $c_{\rm tot}$ given by equation (4.27). We begin with the condition that the denominator of F_{n-1} vanishes and that this value is maximized when $2\pi\Delta_1 + \delta_1 = 2\pi\Delta_2 + \delta_2 = \cdots = 2\pi\Delta_{n-1} + \delta_{n-1}$:

$$\frac{\pi c_{Dn}}{12} - \left(\frac{\pi}{6} - \frac{1}{2}\right) = \frac{\sum_{A=1}^{n-1}\left(2\pi\Delta_A + \delta_A\right)\Lambda_A e^{-2\pi\Delta_A} + s_1}{\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} + s_2}$$

$$= \frac{\left(2\pi\Delta_1 + \delta_1\right)\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} e^{-\delta_A} e^{\delta_A} + s_1}{\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} + s_2}$$

$$= \frac{\left(2\pi\Delta_1 + \delta_1\right)e^{-2\pi\Delta_1} e^{-\delta_1}\sum_{A=1}^{n-1}\Lambda_A e^{\delta_A} + s_1}{e^{-2\pi\Delta_1}e^{-\delta_1}\sum_{A=1}^{n-1}\Lambda_A e^{\delta_A} + s_2} = \frac{\left(2\pi\Delta_1 + \delta_1\right)e^{-2\pi\Delta_1}e^{-\delta_1} m + s_1}{e^{-2\pi\Delta_1}e^{-\delta_1} m + s_2}$$

and we use the definitions

$$\delta_A \equiv -\frac{2\pi(\delta_{h_A 0} + \delta_{\bar{h}_A 0})}{e^{2\pi} - 1}, \quad s_1 \equiv -4\pi e^{-2\pi}(1 - e^{-2\pi})$$

$$s_2 \equiv (1 - e^{-2\pi})^2, \quad m \equiv \sum_{A=1}^{n-1}\Lambda_A e^{\delta_A}.$$

The RHS will be maximized for

$$\Delta_1 = \frac{1}{2\pi}W_0(mA) + \frac{B_1}{2\pi},$$

with

$$A \equiv \frac{e^{-\frac{s_1}{s_2} - 1}}{s_2}, \quad B_1 \equiv \frac{s_1}{s_2} + 1 - \delta_1$$

Substituting this back into our expression for the central charge, we find a complicated expression. We simplify it using the definition of the Lambert-W function

$$z = W_0(z)e^{W_0(z)} \Rightarrow e^{-W_0(z)} = \frac{W_0(z)}{z}.$$

After some algebra, we find the largest value of the total central charge causing the denominator to vanish

$$\frac{\pi c_{Dn}^+}{12} = W_0(mA) + R_1 \tag{C.3}$$

where

$$R_1 \equiv \frac{-4\pi}{e^{2\pi}-1} + \left(\frac{\pi}{6} - \frac{1}{2}\right).$$

Let us now turn our attention to the factor

$$m \equiv \sum_{A=1}^{n-1} (1-e^{-2\pi})^{\delta_{h_A 0}+\delta_{\bar{h}_A 0}} \exp\left[-\frac{2\pi}{e^{2\pi}-1}(\delta_{h_A 0}+\delta_{\bar{h}_A 0})\right].$$

How does a term in this sum contribute? If the Kronecker δ's vanish, then the argument of the sum is unity. If the Kronecker δ's evaluate to unity, then the argument of the sum is approximately 0.9864... Since we have $(n-1)$ terms in the sum, we have determined that

$$m = \alpha(n-1), \quad \alpha \in [0.9864, 1].$$

For large arguments of the Lambert-W function, we can use the fact that $W_0(z) \approx \log(z)$, plus $\log(\log(z))$ corrections. For large enough n,the RHS will go as $\log(n)$. We will restrict the total central charge so that $c_{\text{tot}} > c_{Dn}^{+}$, meaning that to leading order we must require

$$c_{\text{tot}} > \frac{12}{\pi}\log(n). \tag{C.4}$$

Appendix D

Degenerate Δ maximizes c_{Dn}

D.1 Nonchiral primaries

In this appendix, we will prove that the value of the central charge $c_{\rm tot}$ causing the denominator of F_{n-1} to vanish is maximized when its arguments are identical. The denominator of F_{n-1} vanishes when

$$\frac{\pi c_{Dn}}{12} - \left(\frac{\pi}{6} - \frac{1}{2}\right) = \frac{\sum_{A=1}^{n-1} 2\pi\Delta_A e^{-2\pi\Delta_A} - 2e^{-2\pi}(1-e^{-2\pi})}{(1-e^{-2\pi})^2 + \sum_{A=1}^{n-1} e^{-2\pi\Delta_A}}$$

or, by appropriate definitions,

$$\tilde{c} = \frac{\sum_{A=1}^{n-1} 2\pi\Delta_A e^{-2\pi\Delta_A} + s_1}{\sum_{A=1}^{n-1} e^{-2\pi\Delta_A} + s_2} \equiv \frac{N}{D}.$$

In some of what follows, we will make use of the fact that $D > 0$ for any values of its arguments (as can be seen from its explicit form).

In order for $\tilde{c}$ to be a maximum when its arguments are identical, we need the Hessian to be negative definite at this value (or equivalently, have all eigenvalues negative). We denote partial derivatives of $\tilde{c}$ with respect to Δ_i as $\tilde{c}_i$. We will need to calculate partial derivatives of N or D with respect to Δ_i:

$$N_i = 2\pi \exp(-2\pi\Delta_i)\,(1-2\pi\Delta_i)\,, \qquad N_{ij} = 0,$$

$$N_{ii} = (2\pi)^2 \exp(-2\pi\Delta_i)\,(-2+2\pi\Delta_i)$$

$$D_i = -2\pi \exp(-2\pi\Delta_i), \qquad D_{ij} = 0,$$

$$D_{ii} = (2\pi)^2 \exp(-2\pi\Delta_i).$$

We then find

$$\tilde{c}_i = \frac{N_i D - D_i N}{D^2}$$

$$= \frac{2\pi e^{-2\pi\Delta_1}}{D^2}\left[(1-2\pi\Delta_i)\left(\sum_{A=1}^{n-1} e^{-2\pi\Delta_A} + s_2\right) + \left(\sum_{A=1}^{n-1} 2\pi\Delta_A e^{-2\pi\Delta_A} + s_1\right)\right].$$

The prefactor is nonvanishing. In order have a critical point, it is necessary and sufficient to have Δ's satisfying the condition

$$2\pi\Delta_i^{crit.} = 1 + \tilde{c}(\Delta_1^{crit.}, \Delta_2^{crit.}, \cdots, \Delta_{n-1}^{crit.}),$$

where we have defined the value of Δ_j giving a critical point as $\Delta_j^{crit.}$. The RHS of this equation will be the same for any value of i on the LHS. This means that critical points will occur when $\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1}$. We will make use of this in detail in Appendix E.

To determine if a critical point is a maximum, we consider the Hessian. This involves taking more partial derivatives; we calculate

$$\tilde{c}_{ii} = \frac{(N_i D - D_i N)_i D^2 - 2DD_i(N_i D - D_i N)}{D^4}.$$

For critical points, the second term vanishes giving

$$\tilde{c}_{ii} \to \frac{(N_{ii} D - D_{ii} N)}{D^2} =$$

$$\frac{(2\pi)^2 e^{-2\pi\Delta_i}}{D^2}\left[(-2+2\pi\Delta_i)\left(\sum_{A=1}^{n-1} e^{-2\pi\Delta_A} + s_2\right) - \left(\sum_{A=1}^{n-1} 2\pi\Delta_A e^{-2\pi\Delta_A} + s_1\right)\right].$$

Using our above condition for a critical point simplifies this expression to

$$\tilde{c}_{ii} = -\frac{(2\pi)^2 e^{-2\pi\Delta_i}}{D} < 0$$

We will also need to calculate mixed partials:

$$\tilde{c}_{ij} = \frac{(N_i D - D_i N)_j D^2 - 2DD_j(N_i D - D_i N)}{D^4},$$

or in the case of a critical point

$$\tilde{c}_{ij} \to \frac{N_i D_j - D_i N_j}{D^2} = \frac{(2\pi)^2 e^{-2\pi\Delta_i} e^{-2\pi\Delta_j}}{D^2}(2\pi\Delta_i - 2\pi\Delta_j).$$

Again using our condition for critical points, we see that all mixed partials will vanish. This means that the Hessian for the case where $\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1}$ is diagonal with purely negative entries; all eigenvalues are negative. Thus by our analysis we conclude that the function c_{Dn} will have a local maximum in the situation where all of its arguments are identical.

D.2 Chiral primaries

The case with chiral primary operators is similar to the case without in Appendix D.1. In this case, the denominator of F_{n-1} vanishes when

$$\frac{\pi c_{Dn}}{12} - \left(\frac{\pi}{6} - \frac{1}{2}\right) = \frac{\sum_{A=1}^{n-1}\left(2\pi\Delta_A - \frac{2\pi(\delta_{h_A 0} + \delta_{\bar{h}_A 0})}{e^{2\pi}-1}\right)\Lambda_A e^{-2\pi\Delta_A} - 2e^{-2\pi}(1-e^{-2\pi})}{(1-e^{-2\pi})^2 + \sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A}}$$

or,

$$\tilde{c} = \frac{\sum_{A=1}^{n-1}\left(2\pi\Delta_A - \frac{2\pi(\delta_{h_A 0} + \delta_{\bar{h}_A 0})}{e^{2\pi}-1}\right)\Lambda_A e^{-2\pi\Delta_A} + s_1}{\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} + s_2} \equiv \frac{N}{D}.$$

Again, $D > 0$ for any values of its arguments.

As before, in order for $\tilde{c}$ to be a maximum when its arguments are identical, we need the Hessian to be negative definite. The partial derivatives are similar to the previous case:

$$N_i = 2\pi\Lambda_i \exp(-2\pi\Delta_i)\left(1 - \left(2\pi\Delta_i - \frac{2\pi(\delta_{h_i 0} + \delta_{\bar{h}_i 0})}{e^{2\pi} - 1}\right)\right), \quad N_{ij} = 0,$$

$$N_{ii} = (2\pi)^2\Lambda_i \exp(-2\pi\Delta_i)\left(-2 + \left(2\pi\Delta_i - \frac{2\pi(\delta_{h_i 0} + \delta_{\bar{h}_i 0})}{e^{2\pi} - 1}\right)\right)$$

$$D_i = -2\pi\Lambda_i \exp(-2\pi\Delta_i), \qquad D_{ij} = 0$$

$$D_{ii} = (2\pi)^2\Lambda_i \exp(-2\pi\Delta_i).$$

We then find

$$\tilde{c}_i = \frac{N_i D - D_i N}{D^2}$$

$$= \frac{2\pi\Lambda_i e^{-2\pi\Delta_1}}{D^2}\left[\left(1 - \left(2\pi\Delta_i - \frac{2\pi(\delta_{h_i 0} + \delta_{\bar{h}_i 0})}{e^{2\pi} - 1}\right)\right)\left(\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} + s_2\right) + \left(\sum_{A=1}^{n-1} 2\pi\Delta_A\Lambda_A e^{-2\pi\Delta_A} + \right.\right.$$

The conditions on the $\Delta^{crit.}$'s become

$$2\pi\Delta_i^{crit.} + \delta_i = 1 + \tilde{c}(\Delta_1^{crit.}, \Delta_2^{crit.}, \cdots, \Delta_{n-1}^{crit.}), \text{ with } \delta_i \equiv -\left(\frac{2\pi(\delta_{h_i 0} + \delta_{\bar{h}_i 0})}{e^{2\pi} - 1}\right).$$

This means that critical points will occur when $2\pi\Delta_1 + \delta_1 = 2\pi\Delta_2 + \delta_2 = \cdots = 2\pi\Delta_{n-1} + \delta_{n-1}$.

To check the Hessian, we calculate

$$\tilde{c}_{ii} \to \frac{(N_{ii}D - D_{ii}N)}{D^2} =$$

$$\frac{(2\pi)^2\Lambda_i e^{-2\pi\Delta_i}}{D^2}\left[(-2 + 2\pi\Delta_i + \delta_i)\left(\sum_{A=1}^{n-1}\Lambda_A e^{-2\pi\Delta_A} + s_2\right) - \left(\sum_{A=1}^{n-1}(2\pi\Delta_A + \delta_A)\Lambda_A e^{-2\pi\Delta_A} + s_1\right)\right].$$

Using our above condition for a critical point simplifies this expression to

$$\tilde{c}_{ii} = -\frac{(2\pi)^2\Lambda_i e^{-2\pi\Delta_i}}{D} < 0$$

The mixed partials at this point are

$$\tilde{c}_{ij} \to \frac{N_i D_j - D_i N_j}{D^2} = \frac{(2\pi)^2\Lambda_i\Lambda_j e^{-2\pi\Delta_i}e^{-2\pi\Delta_j}}{D^2}(2\pi\Delta_i - 2\pi\Delta_j + \delta_i - \delta_j).$$

Again, all mixed partials will vanish. Thus we conclude that the function $\tilde{c}$ (and thus c_{Dn}) will have a local maximum when $2\pi\Delta_1 + \delta_1 = 2\pi\Delta_2 + \delta_2 = \cdots = 2\pi\Delta_{n-1} + \delta_{n-1}$.

Appendix E

Deriving bound on Δ_n

Here we will provide a derivation of the bound (3.42). We define $x \equiv 2\pi(\Delta_1 + \hat{E}_0) - \frac{3}{2}$ in order to depress the cubic polynomial P_n of eq. (3.33) to

$$\begin{aligned} P_n(x) &= x^3 + \hat{C}_1(\hat{E}_0)x + \hat{C}_0(\hat{E}_0)x, \\ \hat{C}_1(\hat{E}_0) &\equiv \hat{C}_0(\hat{E}_0) - \frac{3}{2}, \\ \hat{C}_0(\hat{E}_0) &\equiv -F_n + 6r_{20} - \frac{1}{8}. \end{aligned} \tag{E.1}$$

It is known [54] that the largest real root of a depressed cubic obeys the inequality

$$\begin{aligned} x^+ &\leq \sqrt{\frac{4|\hat{C}_1|}{3}} \cos\left(\frac{\phi}{3} + \frac{2\pi k}{3}\right) \\ &= \frac{2}{\sqrt{3}} \sqrt{\left|F_n - 6r_2 + \frac{13}{8}\right|} \cos\left(\frac{\phi}{3} + \frac{2\pi k}{3}\right), \end{aligned} \tag{E.2}$$

where $|\cos(\phi)| \equiv \sqrt{\frac{-27\hat{C}_0^2}{4\hat{C}_1^3}}$ and $k = 0, 1, 2$.

The difference between the bound on Δ_1 and the bound on Δ_n is the presence of factors of n in terms containing F_n. Therefore we will first consider which limit suppresses this dependence. It can be shown by explicit computation that the function F_n (3.28) has a maximum when $\Delta_1, ..., \Delta_{n-1}$ are degenerate. As we will soon see explicitly (though somewhat apparent from eq. (E.2)), maximizing F_n maximizes the bound on Δ_n. Thus we need to maximize F_n as a function of Δ_1. Differentiating and solving to leading order in c_{tot} for the critical point Δ_1^{max} gives that $\Delta_1^{max} \sim \frac{c_{\text{tot}}}{12}$ plus subleading corrections.

From the definition of F_n, we see that it contains terms depending on n as large as $nc_{\text{tot}}^3 e^{-2\pi\Delta_1} \sim nc_{\text{tot}}^3 e^{-\pi c_{\text{tot}}/6}$. To suppress dependence on n, we therefore impose the condition

$$n \ll c_{\text{tot}}^{-3} e^{\pi c_{\text{tot}}/6}. \tag{E.3}$$

In order for F_n to be nonvanishing and finite in the case of large n, we are already restricting ourselves to the case

$$\log(n) < \frac{\pi c_{\text{tot}}}{12}. \tag{E.4}$$

Thus we will have no issues suppressing these n-dependent terms in F_n as our previous condition on n satisfies this new condition.

Although the argument for a bound going as $c_{\rm tot}/12$ follows immediately, we will continue algebraic manipulations in order to provide justification for previous statements. In the limit (E.3), F_n will be of the form

$$\begin{aligned} F_n &\approx \frac{a_3 c_{\rm tot}^3 + a_2 c_{\rm tot}^2 + a_1 c_{\rm tot} + a_0}{\bar{a}_1 c_{\rm tot} + \bar{a}_0} \\ &= c_{\rm tot}^2 \frac{a_3}{\bar{a}_1}\left(1 + O(c_{\rm tot}^{-1})\right). \end{aligned} \tag{E.5}$$

The a_i and $\bar{a}_j$ are obtained from eq. (3.28) evaluated at $n = 0$, except that $a_0, \bar{a}_0$ contain additional corrections much smaller than $O(1)$. Because $\hat{C}_0$ and $\hat{C}_1$ are just F_n plus constants, in the limit we consider they will be of the same form as above, with a_1 and a_0 replaced by different constants; that is, both $\hat{C}_0$ and $\hat{C}_1$ grow asymptotically like $c_{\rm tot}^2$.

We now turn our attention to the $\cos\phi$ terms in eq. (E.2). From the definition of $\cos\phi$ and considerations of the preceding paragraph, the leading behavior of $\cos(\phi)$ is $O(c_{\rm tot}^{-1})$. By the series expansion of arccosine, we then have $\phi \approx \pm\frac{\pi}{2} + O(c_{\rm tot}^{-1})$. This in turn implies

$$\max\left[\cos\left(\frac{\phi}{3} + \frac{2\pi k}{3}\right)\right] = \frac{\sqrt{3}}{2} + O(c_{\rm tot}^{-1})$$

plus subleading corrections. Then to leading order eq. (E.2) becomes

$$x^+ \leq \sqrt{|F_n|}, \tag{E.6}$$

plus subleading corrections. Given eq. (E.5), the leading term eq. (E.2) is

$$x^+ \leq \frac{\pi c_{\rm tot}}{12} + O(1). \tag{E.7}$$

Finally, the definition of x^+ gives the result

$$\Delta_n^+ \leq \frac{c_{\rm tot}}{12} + O(1). \tag{E.8}$$

As previously stated, this result could have been argued once we placed the appropriate restrictions on n and $c_{\rm tot}$. The details of the preceding paragraph can also be used to justify our assertion that the above bound on Δ_n is attained at the minimum allowed central charge. To see this, consider maximizing the expression $(\Delta_n^+ - \frac{c_{\rm tot}}{12})$. Inspection of eqs. (E.5) and (E.6) shows that the subleading $c_{\rm tot}$ dependence comes from the expansion of the square root of the ratio of polynomials in powers of $c_{\rm tot}^{-1}$. Analysis of this square root shows that it is monotonically decreasing over the allowed range of $c_{\rm tot}$. So to maximize the square root, we should let $c_{\rm tot}$ take its smallest allowed value.

Appendix F

Degenerate Δ maximizes bound

In this appendix, we provide an argument that the bound on Δ_n for ranges we consider achieves a local maximimum when the $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ approach degeneracy. We will consider the case of theories like those found in [32]— no chiral primary operators. The more general case with primary operators follows in a nearly identical way; it is merely more cumbersome.

We will argue that nearly degenerate Δ's will maximize the function F_{n-1}. According to Appendix E, for the limits we consider the function $\Delta_n^+ - \frac{c_{\text{tot}}}{12}$ will take its maximum when $\sqrt{|F_{n-1}|} - \frac{c_{\text{tot}}}{24}$ is maximized. Thus maximizing F_{n-1} will maximize our bound. The quantity F_2 has degenerate Δ's trivially (as there is only Δ_1), thus it seems possible that this condition on the Δ's will give a maximum. It can be shown analytically that for some value of Δ_1, F_2 takes its maximum value. The conditions associated with this are

$$\frac{\partial}{\partial \Delta_1} F_2 \bigg|_{\Delta_1 = \Delta_1^{max}} = 0$$

$$\Leftrightarrow \left(f_3'(\Delta_1^{max} + \hat{E}_0) - 2\pi f_3(\Delta_1^{max} + \hat{E}_0)\right)\left(f_1(\Delta_1^{max} + \hat{E}_0)e^{-2\pi\Delta_1^{max}} + b_1(\hat{E}_0)\right)$$
$$= \left(f_1'(\Delta_1^{max} + \hat{E}_0) - 2\pi f_1(\Delta_1^{max} + \hat{E}_0)\right)\left(f_3(\Delta_1^{max} + \hat{E}_0)e^{-2\pi\Delta_1^{max}} + b_3(\hat{E}_0)\right)$$

and

$$\frac{\partial^2}{\partial \Delta_1^2} F_2 \bigg|_{\Delta_1 = \Delta_1^{max}} < 0$$

$$\Leftrightarrow \left(f_3''(\Delta_1^{max} + \hat{E}_0) - 4\pi f_3'(\Delta_1^{max} + \hat{E}_0) + 4\pi^2 f_3(\Delta_1^{max} + \hat{E}_0)\right)\left(f_1(\Delta_1^{max} + \hat{E}_0)e^{-2\pi\Delta_1^{max}} + b_1(\hat{E}_0)\right)$$
$$< \left(f_1''(\Delta_1^{max} + \hat{E}_0) - 4\pi f_1'(\Delta_1^{max} + \hat{E}_0) + 4\pi^2 f_1(\Delta_1^{max} + \hat{E}_0\right)\left(f_3(\Delta_1^{max} + \hat{E}_0)e^{-2\pi\Delta_1^{max}} + b_3(\hat{E}_0)\right)$$

We will now assume that this fact is true for some finite number of Δ's and see the effect of adding of one more term:

$$F_{k+1} = \frac{f_3(\Delta_k + \hat{E}_0)e^{-2\pi\Delta_k} + N}{f_1(\Delta_k + \hat{E}_0)e^{-2\pi\Delta_k} + D}, \tag{F.1}$$

where N and D are the numerator and denominator respectively of F_k. To see that degenerate Δ's maximize this function, we must check several conditions. The condition that the first derivative with respect to Δ_k vanishes means

$$\left(f_3'(\Delta_k^{max} + \hat{E}_0) - 2\pi f_3(\Delta_k^{max} + \hat{E}_0)\right)\left(f_1(\Delta_k^{max} + \hat{E}_0)e^{-2\pi\Delta_k^{max}} + D^{max}\right)$$

$$= \left(f_1'(\Delta_k^{max} + \hat{E}_0) - 2\pi f_k(\Delta_1^{max} + \hat{E}_0)\right)\left(f_3(\Delta_k^{max} + \hat{E}_0)e^{-2\pi\Delta_k^{max}} + N^{max}\right),$$

where N^{max} and D^{max} are evaluated at the critical point values. Note that the condition for a vanishing first derivative with respect to any of the other Δ's looks the same except we substitute Δ_i^{max} in place of Δ_k^{max}. This condition is of the same form as for F_2, where we know a solution exists. In the case where Δ's are degenerate, it reduces to the case of F_2 differing only by the presence of factors of $(n-1)$. A solution can be easily found to this equation. Thus the case of degenerate Δ's corresponds to a critical point.

To ensure this point is a maximum, we need to consider the second derivatives. We will consider first the case of mixed partials. Taking derivatives of F_{k+1} with respect to Δ_i and Δ_j (including with respect to Δ_k) gives (suppressing $\hat{E}_0$)

$$\frac{\partial^2}{\partial\Delta_i\partial\Delta_j}F_{k+1}\bigg|_{\{\Delta\}=\{\Delta^{max}\}} = \frac{e^{-2\pi\Delta_i}e^{-2\pi\Delta_j}}{\left(f_1(\Delta_k^{max})+D^{max}\right)^2} \times$$

$$\begin{aligned}&[\ \left(\partial_i f_3(\Delta_i^{max}) - 2\pi f_3(\Delta_i^{max})\right)\left(\partial_j f_1(\Delta_j^{max}) - 2\pi f_1(\Delta_j^{max})\right)\\ &-\left(\partial_j f_3(\Delta_j^{max}) - 2\pi f_3(\Delta_j^{max})\right)\left(\partial_i f_1(\Delta_i^{max}) - 2\pi f_1(\Delta_i^{max})\right)\ ].\end{aligned}$$

Clearly for degenerate Δ's, all of the mixed partials will vanish. The expression for a second derivative with respect a particular Δ (suppressing $\hat{E}_0$ once more) is

$$\frac{\partial^2}{\partial\Delta_i^2}F_{k+}\bigg|_{\{\Delta\}=\{\Delta^{max}\}} = \frac{e^{-2\pi\Delta_i}}{\left(f_1(\Delta_k^{max})e^{-2\pi\Delta_k^{max}}+D^{max}\right)^2} \times$$

$$\begin{aligned}&[\ \left(f_3''(\Delta_i^{max}) - 4\pi f_3'(\Delta_i^{max}) + 4\pi^2 f_3(\Delta_i^{max})\right)\left(f_1(\Delta_k^{max})e^{-2\pi\Delta_k^{max}} + D^{max}\right)\\ &-\left(f_1''(\Delta_i^{max}) - 4\pi f_1'(\Delta_i^{max}) + 4\pi^2 f_1(\Delta_i^{max})\right)\left(f_3(\Delta_k^{max})e^{-2\pi\Delta_k^{max}} + N^{max}\right)\ ].\end{aligned}$$

Again, the bracketed expression is of the same form as the condition necessary for F_2. In the case of degenerate Δ's, the expressions become identical save for the presence of some $(n-1)$ factors. And it can be shown in a similar way that this expression is strictly negative.

Thus for the case of degenerate Δ's, the second derivative test shows that F_{k+1} has a local maximum. By the discussions of Appendix E, this corresponds to when $\sqrt{|F_{n-1}|} - \frac{c_{\rm tot}}{24}$ is maximized and thus in the limits we consider when the least upper linear bound $\Delta_n^+ - \frac{c_{\rm tot}}{12}$ is extremized.

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VITA

Joshua D Qualls

Educational Institutions attended and degrees awarded

August 2008 - present
University of Kentucky, pursuing Ph.D. in Physics.

August 2008 - August 2011
University of Kentucky, M. Sc. in Physics.

September 2004 - July 2008
Centre College, B.S. in Physics, Mathematics.

Professional positions held

Teaching Assistant, University of Kentucky, 2010 - 2013

Visiting Professor, Georgetown College, 2011 - 2012

Grader, Georgetown College, 2011 - 2012

Instructor, University of Kentucky, 2011

Scholastic and Professional Honors

Dissertation Year Fellowship, University of Kentucky, 2013

MultiYear Fellowship, University of Kentucky, 2008

Daniel R Reedy Quality Fellowship, University of Kentucky, 2008

Marshall Wilt Physics Prize, Centre College, 2007

Centre College Mathematics Prize, Centre College, 2006

Professional Publications

1. J.D. Qualls, A.S. Shapere, Bounds on Operator Dimensions in 2D Conformal Field Theories, arxiv:1312:0038 [hep-th]