COMMONWEALTH OF KENTUCKY

DEPARTMENT OF TRANSPORTATION

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January 21, 1977

MEMORANDUM: G. F. Kemper
State Highway Engineer
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SUBJECT: Research Report 464; "Stiffness of Solid-Liquid Mixtures; Theoretical Considerations;" KYHPR-64-20; HPR-1(8), Part II.

KYHPR-64-20 has been a long-time, involved study (entitled "Flexible Pavement Study Using Viscoelastic Principles"). Phase I was devoted to the flow properties (rheology) of asphalt cements. Progress reports on Phase I were issued in 1964 and 1967 *(1, 2)*. Phases III, IV, and V were directed toward rationalization of pavement design criteria. In these latter phases, temperature-deflection and deflection-strain relationships were derived, ambient temperatures were analyzed, and mechanistic models and analogies of empirical design procedures were developed. Phase II was conspicuously by-passed. As originally conceived, it was to be an in-depth study of the basic, physical properties of asphaltic concrete and the influence of the asphaltic binder. The objective was to correlate, and thus predict, the viscoelastic properties of materials in a pavement structure and, thereby, to determine the applicability of elastic theory to pavement design. In the interim, developments from research elsewhere provided the inputs sought in Phase II; and we proceeded toward the larger objective. Phase II was deleted, with FHWA approval, July 25, 1969.

In Phase I, asphalts were chosen on the basis of foreknowledge of their performance and research histories. Some had been studied previously by us *(1)* and by others *(2)*. Those were the asphalts reported in 1964 *(3)* and 1967 *(4)*. Before concluding Phase I, it seemed appropriate to investigate a more random set of asphalts -- more especially those supplied to construction projects in Kentucky. The Division of Materials furnished an array of samples taken from construction projects during one construction season. A side benefit from this set of samples was foreseen in connection with the then-imminent, national trend toward specifying asphalts according to viscosity grades.
(Note: Viscosity-grading was adopted April 21, 1971; Special Provision No. 91; amended January 18, 1973, -- S.P. No. 91-A)

Report 333 covered the aforementioned set of construction samples only -- and supplemented the two, previous reports. The analyses, there, were simplified considerably.

The Phase II question was: How do properties of the asphalt affect the properties of an asphaltic concrete in a pavement structure? From elastic theory analyses, we deduced that a typical value of Young's Modulus of bituminous concrete at 64°F was 480,000 psi. Having determined the viscosity-temperature relationships for asphalts, it should be possible to relate the stiffness or modulus of asphaltic concrete to the stiffness (or viscosity) of the asphalt cement. Although we were aware of difficulties other have encountered in solving this problem, we have endeavored to synthesize an equation which will fulfill the original objectives of Phase II.

An authority [5] on this subject stated several years ago: "...the stiffness of a bitumen-mineral mixture is determined only by the stiffness of the bitumen alone and the volume concentration of the mineral aggregate." The Einstein and Eilers-Van Dyck (or Van Dijck) equations fail in the regions of high concentrations of aggregates [6]. The Einstein equation is:

$$\eta = \eta_0 (1 + 2.5 \ C_v)$$

where η = viscosity of mixture,
η_0 = viscosity of solvent or liquid and
C_v = volume concentration of solids.

Eilers' and Van Dijck's equation is:

$$S_m/S_b = [1 + 1.25 \ C_v/(1 - 1.28 \ C_v)]^2$$

where S_m = stiffness of the mixture and
S_b = stiffness of bitumen.

Heukelom and Klomp [7] modified previous, empirical equations, as follows:

$$S_{mix}/S_{bit} = [1 + 2.5 \ C_v/n(1 - C_v)]^n$$

where S = stiffness = $3G^*$ (G^* = complex shear modulus),
C_v = volume concentration of aggregate,
n = $0.83 \log (S_{agg}/S_{bit})$,
S_{agg} = 6.7×10^6 psi = E_{agg}, and
n = $0.83 (6.83 - \log S_{bit})$.

Huekelom also cited an empirical equation derived by Nijboer which related stiffness of a mixture to the Marshall stability test, as follows:

$$S_{60°C; \ 4 \ sec.} \ (\text{in } kg/cm^2) = 1.6 \times \text{stability (kg)/flow (mm)}$$

or

$$S_{140°F; \ 4 \ sec.} (lbs/in.^2) = 5.3 \times \text{stability (lbs)/ flow (in.)}$$

The Heukelom-Klomp equation could be employed with some minor adjustments to obtain desired values. However, there was a more challenging aspect of the problem: that was to find a prototype, mechanistic equation which yields at least a first approximation of the stiffness or rigidity of asphaltic concretes.

McLeod's [8] graphical representation of the Heukelom equation is shown herein.
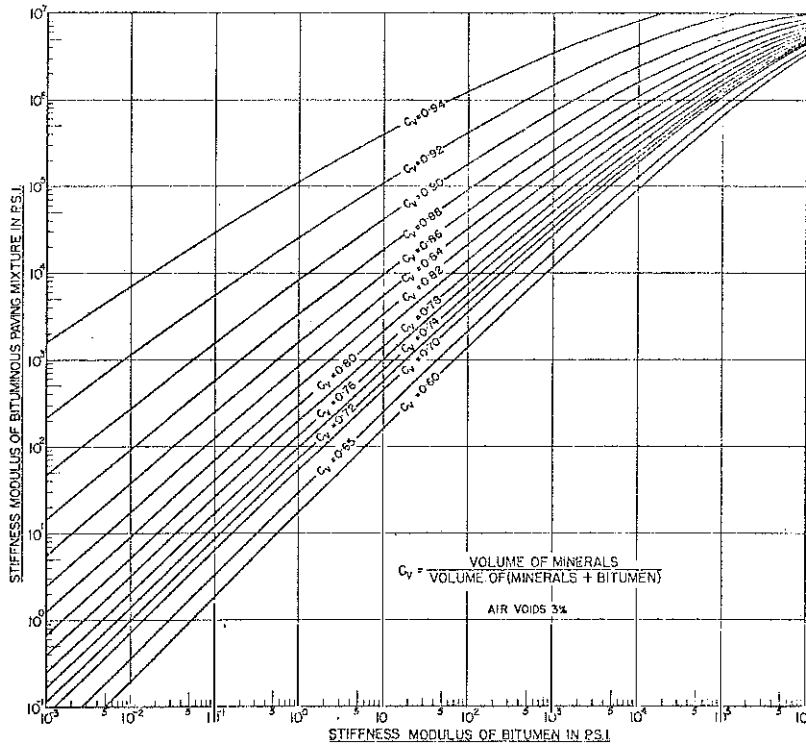
Fig. L. Relationships between Moduli of Stiffness of Asphalt
Cements and of Paving Mixtures Containing the Same
Asphalt Cements (Based on Heukelom and Klomp).

Report 464, submitted herewith, presents a mechanistic-type derivation which appears to fulfill the stated objective. The whole study is, therefore, concluded -- including Phase II, which was canceled or voided quite some time ago. It seemed that Report 333 should await the issuance of a companion report to explain, in some degree, the significance, continuity and application of the combined work.

Stiffness of a viscous material is related to Young's modulus of elasticity by the expression,

$$E = 3\eta/t$$

where η = viscosity and
 t = time.

Difficulties arise in deriving a rational equation for the modulus of elasticity of blends or mixtures of elastic solids. Simply adding together strain energies yields

$$1/2 \; \epsilon_c^2 E_c A_c l_c = 1/2 \; \epsilon_s^2 E_s l_s A_s + 1/2 \; \epsilon_{s2}^2 E_{s2} A_{s2} l_{s2}.$$

where ϵ = engineering strain,
 E = Young's modulus
 A = area,
 l = length
and $\quad \epsilon_c^2 E_c V_c = \epsilon_s^2 E_s V_s + \epsilon_{s2}^2 E_{s2} V_{s2}$

where $V_c = V_{s1} + V_{s2}$
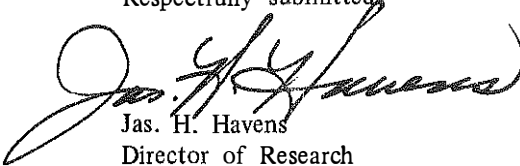$$ $V_{s1}/V_c = C_v; \; V_{s2}/V_c = 1 - C_v$

Unfortunately, it appears that this seemingly simple derivation cannot be extended.

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Respectfully submitted,

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Director of Research

JHH: gd
Attachments
cc's: Research Committee

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16. Abstract

A rational approach to the solution of the stiffness of solid-liquid mixtures is presented. The stiffness of such mixtures is dependent on the stiffness of the viscous medium and the volume concentration and elastic modulus of the solid portion. Finally, the general solution is applied, in particular, to bitumen-aggregate mixtures; and the results are compared to experimental data.

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Research Report
464

STIFFNESS OF SOLID-LIQUID MIXTURES:
THEORETICAL CONSIDERATIONS

KYHPR 64-20; HPR-1(8), Part II
Final Report

by

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in cooperation with
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U.S. Department of Transportation

January 1977

STIFFNESS OF SOLID-LIQUID MIXTURES: THEORETICAL CONSIDERATIONS

It has been stated by Van der Poel [1] in a paper on the deformational characteristics of bitumen and bitumen-mineral mixtures that the stiffness of mixtures is determined only by the stiffness of the bitumen and the volume concentration of the mineral aggregate (assuming stresses and strains are small). Figure 1 is a summary of his data showing this relationship.

The curves in Figure 1 appear to converge at a point where the stiffnesses of both the bitumen and the mixture are equal to approximately 6.89×10^{10} N/m^2 (6.89×10^{10} kPa). Van der Poel says that value represents the stiffness (elastic modulus) of the mineral aggregate. Therefore, it would appear that, as the stiffness of the bitumen approaches the modulus of the aggregate, the deflection in the aggregate will become significant and play an important role in determining the stiffness of the mixture.

This relationship or behavior can be visualized as shown in Figure 2. The stiffness of the liquid medium in Figure 2(A) is simply defined as

$$S = (F/\pi R^2)/\epsilon_v, \qquad 1$$

where
S = stiffness.
F = force,
R = radius,
ϵ_v = d_v/l_o = strain,
d_v = deflection, and
l_o = initial total height of specimen.

In Figure 2(B), the stiffness of the solid-liquid mixture is defined as

$$S = (F/\pi R^2)/\epsilon_m, \qquad 2$$

where
ϵ_m = $(d_v + d_s)/l_o$ = strain in the mixture,
d_v = deflection in the liquid phase,
d_s = deflection in the solid phase, = $[(F/\pi R^2)/M]$ $[l_o - h_o]$, and
M = elastic modulus of the solid phase.

Thus, to find the stiffness of both models in Figure 2, it is necessary to know or assume the modulus of the solid phase and, also, to know the deflection in the liquid phase after some time, t, for any set of conditions (load, area, and viscosity). These models present conditions closely approximating those encountered in parallel plate plastometers (see Figure 3). Therefore, the problem is to find the equation of motion of two parallel plates, each moving towards the other under the action of an applied force, with a liquid medium between the plates. This problem has been solved by Stefan [2], and the solution has been successfully applied to the parallel plate plastometer by Dienes and Klemm [3].

The general equation of motion of the Newtonian fluid of viscosity η is given in cylindrical coordinates by the following:

$$-\partial p/\partial r + \eta\nabla^2 v_r = \rho\ \partial v_r/\partial t,$$

$$(-1/r)(\partial p/\partial\theta) + \eta\nabla^2 v_\theta = p\ \partial v_\theta/\partial t, \text{ and} \qquad 3$$

$$-\partial p/\partial z + \eta\nabla^2 v_z = \rho\ \partial v_z/\partial t,$$

where
p = fluid pressure,
v = velocity,
ρ = density, and
∇ = vector operator.

To solve these equations, a number of assumptions must be made as follows:

1. Let the circular parallel plates in Figure 3 be the planes z = 0 and z = h.
2. Let the ratio of h/r be small, that is, the velocity, v_z, normal to the plates is negligible compared to the radial velocity, v_r. Therefore, $v_z = 0$.
3. Because of circular symmetry, $v_\theta = 0$.
4. No slippage at the plates. Therefore, $v_r = 0$ at z = 0 and z = h.
5. Assume steady state flow. Therefore, $\partial v_r/\partial t = 0$.

With the above assumptions, the equation of motion reduces to the following form:

$$\partial p/\partial r = \eta\ \partial^2 v_r/\partial z^2 \qquad 4$$

Integrating Equation 4 twice with respect to z and inserting the previously stated boundary conditions, $v_r = 0$ at z = 0 and z = h, gives

$$v_r = (1/2\ \eta)(\partial p/\partial r)(z^2 - zh). \qquad 5$$

From Equation 5, it is obvious that in any plane of angle θ the velocity in the fluid varies parabolically with z.

The flow, U, per unit arc length between the planes z = 0 and z = h is

$$U = \int_o^h v_r dz. \qquad 6$$

Substituting from Equation 5 into Equation 6,

$$U = (1/2\eta)(\partial p/\partial r) \int_o^h (z^2 - zh)dz \qquad 7$$

$$= -(h^3/12\eta)(\partial p/\partial r). \qquad 8$$

Due to the action of a force, plate z = h moves toward plate z = 0 at the rate of dh/dt. This will cause the volume of the elemental prism, dr rdθ h, to change at the rate of dr rdθ dh/dt. Because the liquid is assumed here to be incompressible, the net rate of outward radial flow must equal the decrease of the volume of the element. Therefore,

$$- dr\, rd\theta\, dh/dt = \partial\, [(rd\theta\, U)dr]/\partial r. \qquad 9$$

Substituting Equation 8 into Equation 9 and performing the necessary manipulations gives

$$r(12\eta/h^3)(dh/dt) = \partial(r\, \partial p/\partial r)/\partial r. \qquad 10$$

Intregrating twice with respect to r gives

$$p = r^2(3\eta/h^3)(dh/dt) + C'\ln r + C, \qquad 11$$

where C' and C are constants of integration. Since p must be finite for r = 0, C' must vanish. At the outer boundary of the specimen, r = R, the pressure in the fluid must be in equilibrium with atmospheric pressure, p_0. Therefore,

$$C = p_0 - R^2(3\eta/h^3)(dh/dt) \qquad 12$$

and Equation 11 becomes

$$p = (-3\eta/h^3)(R^2 - r^2)(dh/dt) + p_0. \qquad 13$$

The total force acting on the plate in the positive z direction is

$$2\pi \int_0^h p\, rdr. \qquad 14$$

The total force acting on the plate in the opposite direction is

$$F + \int_0^h p_0\, rdr, \qquad 15$$

where F is the applied, external force. Substituting for p in Expression 14 and equating to Expression 15 gives

$$F + 2\pi \int_0^h p_0\, rdr = -2\pi \int_0^h (3\eta/h^3)(dh/dt)(R^2 - r^2)\, rdr + 2\pi \int_0^h p_0\, rdr, \qquad 16$$

and

$$F = -2\pi\, (dh/dt)(3\eta/h^3) \int_0^h (R^2 - r^2)\, rdr. \qquad 17$$

Inasmuch as the liquid completely fills the space between the plates, then r can be assumed equal to R at all times.

Intregating Equation 17 with respect to r gives

$$F = - (3\pi\eta R^4/2h^3)(dh/dt). \qquad 18$$

Equation 18 integrated with respect to t gives

$$1/h^2 = (4Ft/3\pi\eta R^4) + C, \qquad 19$$

where C is the constant of integration. If z = h_0 at time zero, then z = h at any time t can be found from

$$(3\pi R^4/4)\, [(1/h^2) - (1/h_0^2)] = F(t)/\eta, \qquad 20$$

or

$$h = [3\pi\eta R^4 h_0^2/(4Fth_0^2 + 3\pi\eta R^4)]^{1/2}. \qquad 21$$

To find the stiffness of a solid-liquid mixture, Equation 21 must first be solved to find the stiffness of the liquid phase alone (C_v = 0) as in Figure 2(A). Therefore, given force, F; time, t; viscosity, η; radius, R; and initial height of the liquid phase, h_0; substituting into Equation 21 will give the height of the liquid phase, h_a, after time t. In Figure 2(A) l_0 and h_0 are equivalent; therefore, h_a subtracted from l_0 or h_0 gives the deflection, d_v. Thus,

$$d_v/l_0 = \epsilon_v.$$

Substituting ϵ_v into Equation 1 gives the stiffness of the liquid phase.

To solve for the stiffness of the mixture, Equation 21 must be solved using Figure 2(B). In this case, the volume concentration of the solid, C_v, equals $(l_0-h_0)/l_0$; therefore,

$$h_0 = l_0(1 - C_v). \qquad 22$$

Using the same conditions of force, time, viscosity, and radius as previously used and solving Equation 21 using h_0 as defined in Equation 22 gives a new height, h_b, for the liquid phase of Figure 2(B). Thus, the deflection in the liquid phase is given by

$$d_v = h_0 - h_b. \qquad 23$$

Again, the deflection in the solid phase is given by

$$d_s = [(F/\pi R^2)/M]\, [l_0 - h_0]. \qquad 24$$

Therefore, the engineering strain in the mixture can be calculated from

$$\epsilon_m = (d_v + d_s)/l_o. \qquad 25$$

Substituting Equation 25 into Equation 2 gives the stiffness of the mixture.

This solution is general and should be applicable to solving for the stiffness of many different systems, including solid-solid or liquid-liquid systems (assuming no movement at the interface between the two liquid media). It appears, therefore, to have applications to many types of industrial and engineering problems. This could include slurries (solid particles in liquids), emulsions, or liquids that are composed of two or more other liquids. This solution also could be used to solve such esoteric problems as the stiffness of blood (solids in a liquid).

The general solution of solid-liquid mixtures can now be applied to the original problem of bitumen-aggregate mixtures. Figures 8 and 9 of Report No. 333 (pages 8 and 9) give the apparent viscosities of several asphalts as a function of temperature $(1/^oR)$. The apparent viscosity (steady-state viscosity) was calculated as described in the body of that report and as shown there in Figure 7. For any set of conditions, this viscosity can be employed to solve for the stiffness of any of the asphalts tested in that study by using Equation 21 and Figure 2(A). Therefore, the stiffness of any mixture, using any of the asphalts of that study, can also be found from Equation 21 and Figure 2(B), assuming the modulus of the aggregate is known.

To test this hypothesis, a number of calculations were made using the viscosities of Figures 8 and 9 of Report No. 333 and assuming the modulus of the aggregate was 3.45×10^{10} N/m^2 (3.45×10^{10} kPa).

Results are shown here as dashed lines in Figure 1. The calculated values correspond closely with Van der Poel's experimental data. This appears to support his conclusion that the stiffness of a bitumen-aggregate mixture is not a function of the type of bitumen, type of test, or time of loading, but is a function of the bitumen stiffness and the volume concentration of the aggregate. However, one other parameter should be added; i.e., the modulus of the aggregate.

The solution of this problem should be very helpful to the design and analysis of pavements, in particular. For instance, if the properties of a mixture are known, such as the viscosity-temperature relationship of the asphalt and the percent, by volume, of the aggregate, the designer then has the tools available to obtain the stiffness modulus of the mixture. This value can then be used to calculate stress distributions within the asphalt-bound layers of a flexible pavement for any combination of load, time, and temperature. This stiffness modulus could be used with a number of theories such as linear elastic layer theory, linear viscoelastic layer theory, or finite elements.

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C_v
0.88
0.81
0.70
0.50
0.30
0.00

.88
.81
.70
.50
.30

THEORY — — —

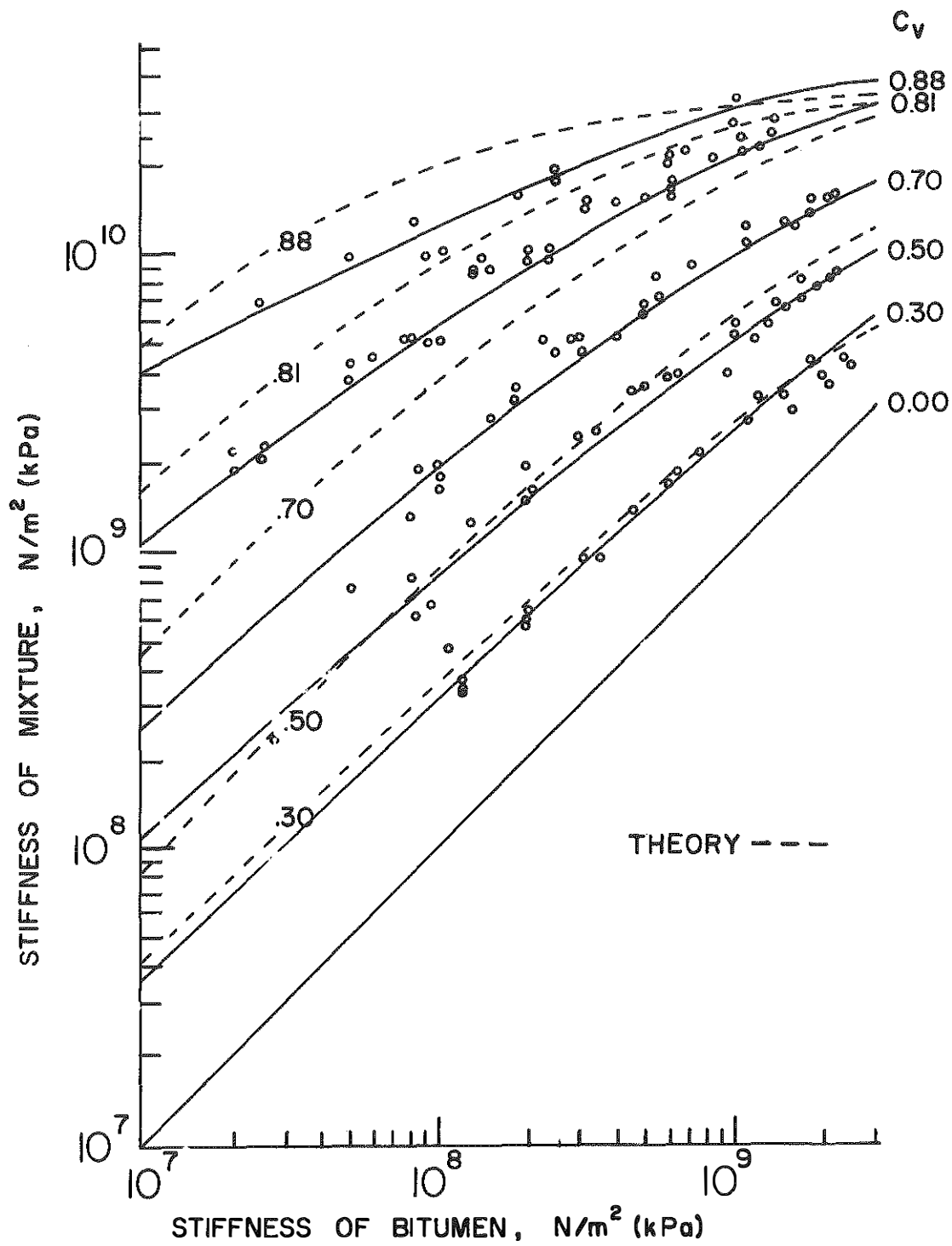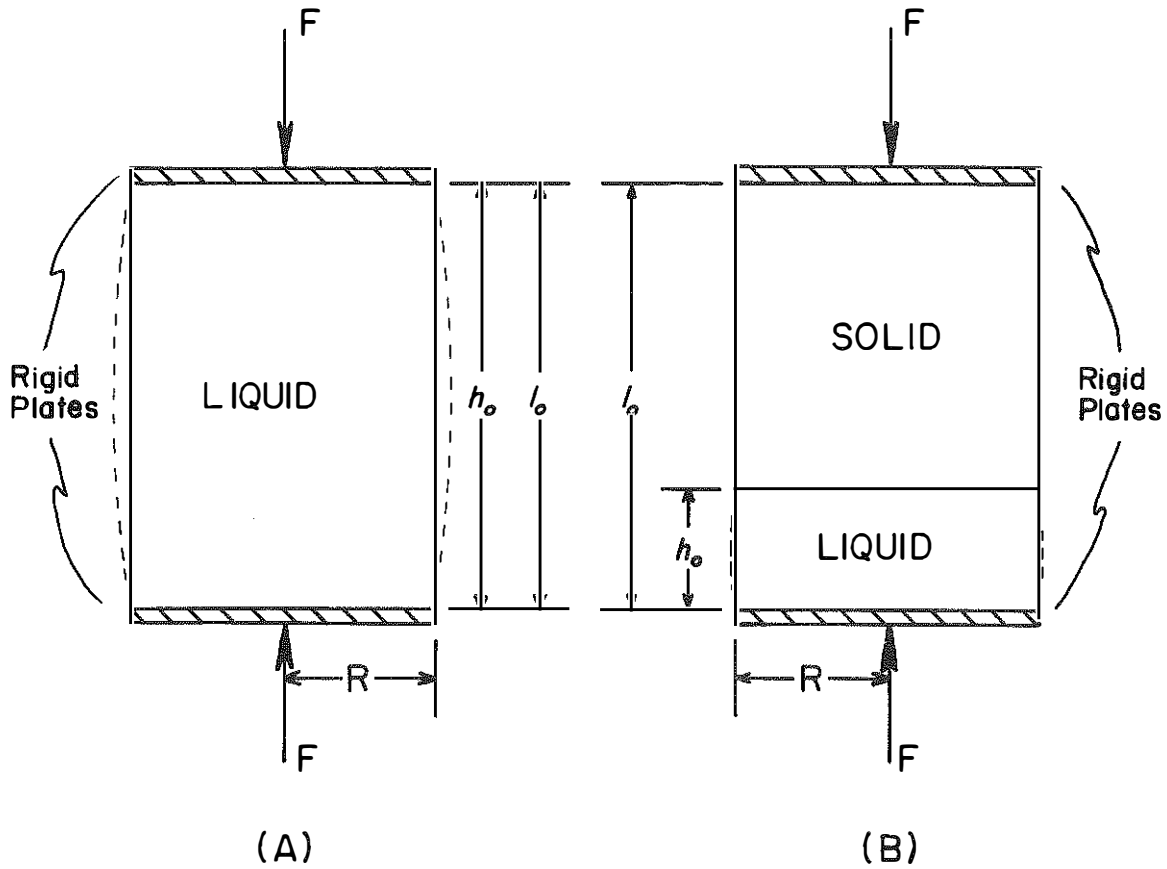STIFFNESS OF MIXTURE, N/m² (kPa)

STIFFNESS OF BITUMEN, N/m² (kPa)

Figure 1.   Relationships between Stiffness of the Mixture and Stiffness of Bitumen.

F = FORCE

R = RADIUS

l_o = TOTAL INITIAL HEIGHT OF SPECIMEN

h_o = INITIAL HEIGHT OF LIQUID PHASE = $(1-C_v)\, l_o$

C_v = VOLUME CONCENTRATION OF SOLID = $\dfrac{l_o - h_o}{l_o}$

$1 - C_v$ = VOLUME CONCENTRATION
OF LIQUID MEDIUM
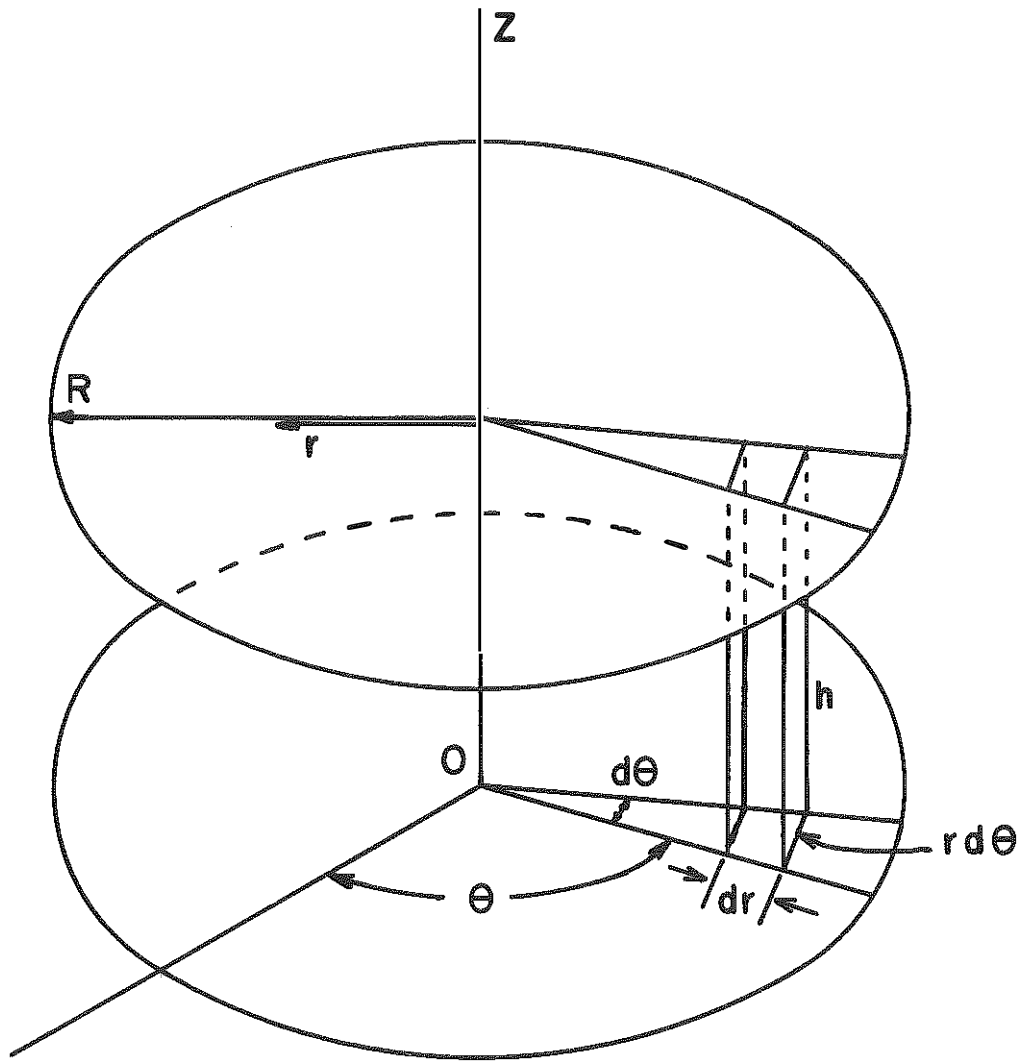
Figure 2.   Basic Relationships.

Figure 3. Parallel Plate Plastometer.