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INTRODUCTION

Predicting the rate of consolidation of saturated clays has been one of the most difficult problems in geotechnical engineering. Terzaghi's theory of consolidation *(7)* has been the basis for predicting rates of consolidation settlement. The theory uses many simplifying assumptions to obtain the following differential equation describing the dissipation of excess hydrostatic pressures, u, in one-dimensional flow:

$$[k(1 + e)/a_v \gamma_w]\ \partial^2 u/\partial z^2 = \partial u/\partial t \qquad 1$$

where z = coordinate in vertical direction,

u = excess pore pressure,

t = time,

k = permeability in the vertical direction,

e = void ratio,

a_v = coefficient of compressibility, and

γ_w = unit weight of water.

Terzaghi's nondimensional solution to this differential equation takes the following form:

$$U = 1 - (8/\pi^2) \sum_{N=0}^{\infty} ((2N + 1)^{-2} \exp[-(2N + 1)^2\ \pi^2 T_v/4]) \qquad 2$$

where U = degree of consolidation = $1 - u/u_o$,

u = excess pore pressure,

u_o = initial excess pore pressure, and

T_v = dimensionless time factor = $C_v t/H^2$, 3

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where H = length of drainage path in z direction,

t = time from beginning of loading, and

C_v = coefficient of consolidation = $k(1 + e)/a_v\gamma_w$.

Methods of Analysis

Various procedures have been developed to determine the coefficient of consolidation, C_v, from conventional laboratory consolidation tests. Taylor *(6)* and Casagrande *(1)* developed empirical, graphical procedures for obtaining C_v, while Naylor and Doran *(4)* developed an analytical method based on an iterative method of successive approximations. Figure 1 shows the essential elements of the well known Taylor square-root-of-time method. If a clear linear trend is present in the initial portion of the deflection-square-root-of-time data, Taylor's method usually provides a good estimation of d_0, the dial reading at the beginning of primary consolidation. However, if substantial secondary compression has occurred when U equals 90 percent, the determined deflection reading corresponding to this point will yield an inaccurate estimate of the dial reading at the end of primary consolidation, d_{100}, and thus the coefficient of consolidation, C_v.

In contrast, the Casagrande logarithm-of-time method as shown in Figure 2 usually provides a good estimation of a suitable value of d_{100}. The initial deflection reading, d_0, determined from this method will closely compare to the d_0 value obtained from the square-root-of-time method provided the initial portion of the deflection-logarithm of time curve conforms approximately to a parabola.

The Naylor-Doran analytical method is based on the fact that, if Equation 2 is plotted with the natural logarithm of (1 - U) versus the dimensionless time factor, T_v, then a straight line is obtained for the portion of the curve lying between 60 and 80 percent consolidation ($0.2 < 1 - U < 0.4$). A straight line ensues because the higher order terms appearing in Equation 2 may be neglected beyond 60 percent consolidation and the effects of secondary consolidation are negligible before 80 percent consolidation. Figure 3 illustrates the theoretical relationship between the time factor, T_v, and logarithm of (1 - U) for the Naylor-Doran analytical method. In this method, d_0 and d_{100} are assumed and then corrected by an iterative process of successive approximations that narrows the differences between the assumed and calculated values of d_0 and d_{100} until they are sufficiently small. Although the Naylor-Doran analytical method may yield the most reliable values for d_0 and d_{100} *(2)*, the method has largely remained a research tool because the iterative procedures required are too involved for manual application.

Purpose and Scope of Paper

Reduction in the amount of work involved in analyzing time-dependent consolidation data has been accomplished by computer application of the three methods discussed above. The first attempt at this type of solution was by Murray *(3)* in 1970. A more comprehensive and basically different approach was developed by the senior author in July of 1977. The computer program which resulted from this effort determines the following basic parameters using one or all of the methods under discussion: the coefficient of consolidation; the compression ratios, i.e., ratios of primary, initial, and secondary compression to total compression; the coefficients of permeability for primary and total compression; and the coefficient of secondary compression, C_α. Analyses of data from each load increment are plotted with a summary of results. In addition, plot summaries superimposing all C_v values and superimposing the values for the coefficient of secondary compression and secondary compression ratio are provided. Development of the computer program was prompted by the need for a rapid computational and plotting algorithm which could be used in a data acquisition scheme. The program was written in Fortran IV and developed on the IBM 370/165 and the Calcomp 663 drum plotted.

METHOD OF SOLUTION

Application of the three methods involves using a linear least-squares analysis to fit a straight line through sample groups of data and statistical procedures to evaluate the representativeness of the sample-data groups. The linear least-squares analysis yields values for the slope, m, and intercept, b, of the straight-line equation

$$Y = m X + b. \qquad 4$$

The least-squares definition of the slope (regression coefficient) is defined *(5)* as

$$m = [((N \sum_{i=1}^{N} X_i Y_i) - (\sum_{i=1}^{N} X_i \sum_{i=1}^{N} Y_i)] / [(N \sum_{i=1}^{N} X_i^2 - (\sum_{i=1}^{N} X_i)^2]. \qquad 5$$

The least-squares definition of the intercept is

$$b = [(\sum_{i=1}^{N} Y_i \sum_{i=1}^{N} X_i^2) - (\sum_{i=1}^{N} X_i \sum_{i=1}^{N} X_i Y_i)] / [N \sum_{i=1}^{N} X_i^2 - (\sum_{i=1}^{N} X_i)^2] \qquad 6$$

where X_i and Y_i are individual abscissae and ordinate values, respectively, for the data points and N is the number of data points.

Statistical procedures used to evaluate the representativeness of the least-square fits on sample groups of data are based on the use of the unbiased standard error of the estimate, S_e, which is simply the standard deviation of the residuals. The residuals are the deviations of the actual ordinate values, Y_i, from the predicted ordinate values, Y_p; i.e., $Y_i - Y_p$. The following equation is used to calculate the unbiased standard error of the estimate *(5)*:

$$S_e = [(\sum_{i=1}^{N} Y_i^2 - b \sum_{i=1}^{N} Y_i - m \sum_{i=1}^{N} X_i Y_i)/(N - 2)]^{1/2} \qquad 7$$

Using only the term S_e given by Equation 7 to select the most representative linear portion of the data will lead to problems similar to the one shown in Figure 4. There, a group of closely spaced data points has the smallest value of S_e, but those points do not provide a suitable representation of the linear portion of the primary compression curve. The problem shown in Figure 4 occurs because the definition of S_e does not distinguish among the representativeness of sample-data groups on the basis of the amount of curve length covered, but rather the definition considers only the amount of vertical scatter about the least-squares linear fit of the data. Figures 5a and b illustrate this property of S_e. In both figures, the scatter of the data about the fitted straight lines is essentially the same, as indicated by the calculated values of S_e and the appearance of the data distribution. Therefore, using the definition of S_e, a line twice as long as another and having the same scatter band will have nearly the same value of S_e but should be considered twice as significant.

To rectify these deficiencies inherent to the standard error of the estimate, the concept of statistical weighting factors is introduced. These weighting factors modify the absolute amount of scatter, S_e, so that those properties that distinguish among the significance of S_e in different sample-data groups are considered. The statistical weighting factors are formulated to account for the length and position significance of the sample-data group being analyzed. The meaning of length significance was described above. Position significance uses the slope properties of a line representation of a sample-data group to indicate whether or not this group is located in the most significant portion of a given data curve. The procedure for using length and position weighting factors in conjunction with S_e is described below for both the square-root-of-time and logarithm-of-time methods.

Square Root of Time

For the square-root-of-time method, the length weighting factor, WF_L, gives greater significance to data spanning a larger ordinate distance. This is done by defining WF_L as

$$WF_L = VHAT/YBOT \qquad (8)$$

where VHAT is the total vertical distance spanned by the square-root-of-time data set and YBOT is the ordinate distance spanned by a sampled portion of that data, as shown in Figure 6. As YBOT approaches zero, the length weighting factor, WF_L, approaches infinity. When S_e is multiplied by a large value of WF_L, the large value of the resultant product will indicate a definite lack of significance in the sample-data group that spans ordinate distance YBOT. In contrast, as YBOT approaches the total vertical distance spanned by the square-root-of-time data set, WF_L will approach unity. When S_e is multiplied by a value of WF_L approaching unity, the resultant product will be small and indicate greater significance.

The position weighting factor, WF_p, is used in a similar way as described above to establish the position significance of a value of S_e. The factor WF_p adjusts the values of S_e by using the slope characteristics of each sample-data group to give added significance to those sample-data groups located at the earlier portions of the square-root-of-time data curve. The position weighting factor is formulated as

$$WF_P = \quad SHAT/SBOT \qquad 9$$

where SHAT is the maximum slope possible in the square-root-of-time data set and SBOT is the least-squares slope of the sample-data group. These parameters are shown in Figure 6. As the value of WF_P approaches one, the more significant is the value of S_e associated with the sample-data group.

The total adjustment of S_e for the length and position significance of sample-data groups from the square-root-of-time curve is defined as

$$S_e \text{ (adjusted)} = ABS(WF_P) \; . \; ABS(WF_L) \; . \; S_e. \qquad 10$$

Logarithm of Time

Procedures for modifying S_e in the logarithm-of-time method are very similar to the ones presented above. The formulation of the statistical weighting factors, WF_L and WF_P, for the logarithm-of-time sample-data groups is determined by whether the line representation of the primary or secondary compression data is being sought.

Considering the primary portion of the logarithm-of-time compression curve, the length and position weighting factors are formulated in a manner almost identical eto the method used to formulate the weighting factors for the square-root-of-time sample-data groups. The length weighting factor, WF_L, is formulated to give more significance to the sample-data groups spanning a large ordinate distance, as given previously in Equation 8. Next, the postion weighting factor, WF_p, is expressed in Equation 9 where SHAT is now the slope of the line extending between the first and last data points of the logarithm-of-time curve and SBOT is the least-squares slope of the sample-data group under consideration. Figure 7 illustrates the physical meaning of these two variables. The total adjustment of the value of S_e for sample-data groups in the primary compression curve is accomplished by substituting the new definitions of WF_L and WF_p into the general expression of Equation 10. There are two advantages of using the two weighting factors in this prescribed manner. First, any sample-data group composed of data points having the same or nearly the same ordinate values will be given less significance by the length weighting factor. These types of data points will either not lie in the steep portion of the primary compression curve or be spurious data caused by reading errors in the deflection values and(or) frictional effects in the equipment. Second, the use of least-squares slope values from the sample-data groups as an indication of position significance gives added significance to sample-data groups having larger slopes, such as those near the region of inflection in the logarithm-of-time curve.

In the secondary portion of the logarithm-of-time curve, the length and position factors used to determine the significance of S_e are formulated in a manner different than any of the procedures previously presented. In this case, the length weighting factor is based on abscissa (horizontal) rather than ordinate (vertical) distance. The position weighting factor is used to give significance to those data groups nearest the end of the logarithm-of-time curve. The expression for the length weighting factor is

$$WF_L = \quad ABS(XHAT/XBOT), \qquad 11$$

where XHAT and XBOT are defined as in Figure 8. Defining WF_L according to the abscissa distances spanned by the sample-data group utilizes that flatter nature of the secondary compression curve to avoid giving undue length significance to sample-data groups encompassing portions of the primary compression curve.

Next, to give position significance to the value of S_e associated with the flattest portion of the secondary compression curve, the value of S_e associated with a sample-data group having a small slope is given additional weight by defining the position weighting factor, WF_P, as

$$WF_P = \quad ABS(STOP/SHAT), \qquad 12$$

where STOP and SHAT are also defined in Figure 8. The smaller the slope of the sample-data group, STOP, the greater the position significance of this data group. The value of WF_P is prevented from having a value less than 0.2 so that undue significance is not given to sample-data groups having least-squares slopes approaching zero.

The value of S_e is given length and position significance by using the new definitions of WF_L and WF_P in Equation 10. If the value of WF_P is allowed to go to zero for sample-data groups having least squares slopes approaching zero, the adjusted value of S_e would become zero and fail to reflect the length and scatter properties of the linear representation of the sample-data group. In addition, values of WF_P less than 0.2 have been found to generally indicate the presence of spurious data.

In the computer application of the Naylor-Doran method, the statistical weighting factors described above are not used. Rather, only the linear least-squares portion of this algorithm is used.

PROCEDURES UNIQUE TO COMPUTER SOLUTION

Three types of procedures are unique to the computer solution. First, there are those procedures which search for the linear portions of the square-root-of-time and logarithm-of-time compression curves by selecting sample-data groups which are statistically evaluated for their linear representativeness. Next, there are those iterative procedures associated with the Naylor-Doran analytical method. Finally, there are the procedures which extend the effectiveness of the Naylor-Doran analytical method as far as error convergence and the calculation of the coefficient of consolidation, C_v, are concerned.

Square-Root-of-Time Search Procedures

In the analysis of the square-root-of-time data, a set of search ordinate values is prepared to provide the framework within which the sample-data groups are to be selected. The procedure used to select the search ordinate values is illustrated in Figure 9. The search ordinate values are used to define the ordinate interval from which data points are sampled. The ordinate intervals used for data sampling are varied in size and location by selecting different sequences of search ordinate values. In Figure 9, the sequencing of the search ordinate values created ordinate sampling intervals of sizes 1 and 2, at different initial ordinate offset locations of A, B, and C. The ordinate sampling interval of size 1 illustrated in Figure 9 samples data between every other search ordinate value as it is moved down the square-root-of-time curve in the vertical direction from initial offset locations A and B. Similarly, the ordinate sampling interval is increased to size 2 by using a sequence of search ordinate values spaced further apart than the sampling interval of size 1. Starting from initial offset locations, A, B, and C, the ordinate sampling interval of size 2 is moved down the square-root-of-time curve in the vertical direction. Other passes are made through the data as the ordinate sampling intervals are increased in size. Each pass starts from various initial offset locations to obtain sample-data groups that cover a wide range of length and position combinations.

Logarithm-of-Time Search Procedures

Sample-data groups are selected from the primary and secondary portions of the curve using sampling intervals that move in the positive horizontal direction. The computer algorithm distinguishes between the primary and secondary portions of the curve by using the previously described statistical weighting factors to produce the minimum adjusted values of S_e in conjunction with the procedures discussed below to generate the horizontal search intervals.

Primary Compression

For the primary compression curve, abscissa sampling-intervals of varying widths 'B' are used to select the sample-data groups as shown in Figure 10. The abscissa sampling-interval 'B' is set to a large value during the first pass through the entire logarithm-of-time data set. Additional sample-data groups containing fewer data points are selected with successively smaller abscissa sampling-intervals of width 'B' in the vicinity having the maximum slope. Of the sample-data groups within these smaller sampling intervals 'B', the one having the minimum adjusted value of S_e is selected to represent the linear portion of primary compression.

Secondary Compression

Sample-data groups are generated from the secondary compression curve using a slightly different abscissa sampling procedure. These sample data groups are selected by progressively increasing the abscissa length of the sampling interval 'A_i' backward from an initial point near the end of the secondary compression curve until 'A_i' reaches the beginning of the logarithm-of-time curve. New initial points are progressively chosen at earlier points on the secondary compression data curve to allow the newly selected sample-data groups to be unaffected by any nonrepresentative data points at the end of secondary compression. For example, at LLL = 1 in Figure 11, the last data point will serve as the initial starting point for the backward selection of the abscissa sampling-intervals. At LLL = 2, the last data point is ignored and the next to last data point is used as the starting point for the backward selection of the abscissa sampling intervals. This same sort of reasoning is followed for values of LLL = 3, 4, and up to 30 percent of the total number of data points, where the last three, two, and one less than the number corresponding to 30 percent of the total number of data points are ignored during the selection of additional sample-data groups. The sample-data group having the minimum adjusted value of S_e is selected to provide the linear representation of the secondary compression data.

Naylor-Doran Method

The second set of procedures unique to the computer solution are the interative procedures characterizing the Naylor-Doran analytical method for determining the precise values of d_0 and d_{100}. In the Naylor-Doran method, one minus the average degree of consolidation, (1 - U), is defined as

$$(1 - U) = (d - d_{100})/(d_0 - d_{100}), \qquad 13$$

where d is the particular deflection reading and d_0 and d_{100} are the assumed values for the deflection readings corresponding to zero percent and 100 percent consolidation, respectively. The term (1 - U) is calculated for each deflection reading, d, and the errors in the assumed values for d_{100} and d_0 are then evaluated from the $\log_e$ (1 - U)-versus-time relationship.

The error in d_{100}, err_{100}, is calculated from

$$err_{100} = 0.4\ (AX/BX - 1)/(1 - 2AX/BX), \qquad 14$$

where AX = slope of $\log_e(1 - U)$-versus-time relationship at 60-percent average degree of consolidation and

BX = slope of $\log_e$ (1 - U)-versus-time relationship at 80-percent average degree of consolidation.

This error in d_{100} is used to correct the assumed value of d_{100} to obtain a new assumed value as

$$d_{100}(new) = (d_{100} - err_{100} d_0)/(1 - err_{100}). \qquad 15$$

The values of (1 - U) are recalculated and the error err_{100} is reevaluated until err_{100} becomes less than five percent.

Once a reasonably accurate estimate of d_{100} has been obtained, the error in d_0, err_0, is evaluated using

$$err_0 = \log_e (8/\pi^2) - AC = -(0.21 + AC) \qquad 16$$

where AC is the intercept of the least squares straight line portion of the $\log_e$ (1 - U)-versus-t relation. The error in d_0, err_0, is then used to adjust the assumed value of d_0 according to

$$d_0(new) = (d_0 + err_0\ d_{100})/(1 + err_0). \qquad 17$$

When the err_0 is less than five percent, the (1 - U) values are reevaluated and the process of computing the values of d_0, d_{100}, and (1 - U) is repeated for greater precision. Other details of the interative procedure proposed by Naylor and Doran are presented elsewhere *(4)*.

Special Extensions of Naylor-Doran Analytical Method

The values of d_{100} and d_0 as obtained by Equations 15 and 17 generally converge to minimum values which oscillate between plus and minus values of one to five percent. True convergence below five percent is often not obtained. The errors in d_{100} and d_0 can be further reduced to around 0.05 percent using an interpolative procedure that calculates intermediate values for d_{100} and d_0 from those values of deflection corresponding to the oscillating plus-and-minus values of error.

Calculation of Coefficient of Consolidation, C_v

After the Naylor-Doran analytical method has been used, C_v can be determined by three different equations. Naylor and Doran *(4)* used only the following version of Equation 3 to determine C_v, or

$$C_v = 0.565H^2/t_{80} \qquad 18$$

where t_{80}, the time of 80-percent consolidation, was obtained manually from the graph of $\log_e$ (1 - U)-versus-time relationship at U equal 0.80, and the value of 0.565 is taken as the dimensionless time factor corresponding to 80-percent consolidation.

A second way of determining C_v is based on Equation 2 and the fact that for degrees of consolidation, U, greater than 60 percent the first term solution (N = 0) is sufficiently accurate in the following form:

$$1 - U = (8/\pi^2)\exp(-\pi^2 T_V/4). \qquad 19$$

Taking the natural logarithm of both sides of Equation 19 yields the linear relationship

$$\log_e(1 - U) = \log_e(8/\pi^2) - \pi^2 T_v/4. \qquad 20$$

Sustituting in Equation 3 for the time factor, T_v, Equation 20 becomes

$$\begin{aligned}\log_e(1 - U) &= \log_e(8/\pi^2) - \pi^2 C_v t/4H^2 \qquad 21\\ &= -0.21 - \pi^2 C_v t/4H^2,\end{aligned}$$

where H is the length of the drainage path. The coefficient of consolidation, C_v, can be determined for (1 - U) equal to 0.2 once the time at 80-percent consolidation, t_{80}, is known. The value for t_{80} is determined using the equation of the linear portion of the $\log_e$ (1 - U)-versus-time relationship as follows:

$$t_{80} = (\log_e(0.2) - AC)/UTSLOP, \qquad 22$$

where AC is the intercept and UTSLOP is the slope of the line. Substituting t_{80} from Equation 22 into Equation 21 and solving for C_v yields

$$\begin{aligned}C_v &= [4H^2(-0.21 - \log_e 0.2)]/\pi^2 t_{80} \qquad 23\\ &= 0.5672\ H^2/t_{80}.\end{aligned}$$

Using a value of t_{80} in Equation 23 that has been computed from the line representation of the entire linear portion of the $\log_e$ (1 - U)-versus-time relation smooths out any variation discrete data points may have in the vicinity of U equal to 0.8.

The coefficient of consolidation, C_v, obtained by either Equation 18 or 23 is susceptible to errors associated with t_{80} as a consequence of the effects of an error in the intercept value, AC, of Equation 22. The computer program does not use the value of t_{80} obtained from Equation 22 to determine the coefficient of consolidation, C_v. However, t_{80} can be used with Equation 23 to serve as a check on the computer program value of C_v which is determined according to the procedure described below.

Another way of determining C_v is to take the derivative of Equation 21 with respect to time and use the slope, UTSLOP, of the linear portion of the $\log_e$ (1 - U)-versus-time curve as follows:

$$\text{UTSLOP} = d[\log_e (1 - U)]/dt = -\pi^2 C_v/(4H^2). \qquad 24$$

Solving for C_v yields

$$C_v = (-4H^2/\pi^2)\, d[\log_e (1 - U)]/dt. \qquad 25$$

Values of C_v obtained from Equations 23 and 25 are the same only if the intercept of the straight-line portion of the experimental $\log_e$ (1 - U)-versus-time curve is equal to -0.21. When the intercept is not equal to -0.21, the values of C_v obtained from Equation 23 will differ slightly from the values determined by Equation 25. The question naturally arises; which equation is the best to use? Equation 25 is a better procedure to determine C_v for two reasons. First, Equation 25 is independent of the initial deflection reading, d_0, and consequently, the equation is independent of the intercept of the linear portion of the $\log_e$ (1 - U)-versus-time curve. This results from the removal of the intercept and time terms from Equation 21 upon differentiation. Therefore, C_v is dependent only on the slope of the linear portion of the $\log_e$ (1 - U)-versus-time relationship. Second, making C_v dependent only on the slope of the linear portion of the $\log_e$ (1 - U)-versus-time relationship conforms to the inherent meaning of the coefficient of consolidation because C_v describes only the rate of the consolidation process and does not yield any information on the path of this process as contained in the time and intercept components of Equation 21. For these reasons, the slope equation given by Equation 25 is used to calculate the value for C_v after the Naylor-Doran method is applied.

In an attempt to avoid using discrete points from the $\log_e$ (1 - U)-versus-time relationship to calculate C_v, since these points are subject to experimental error, Murray *(3)* used the same relationship given in Equation 25. In addition, as shown above, using Equation 25 has many other clear advantages.

NUMERICAL EXAMPLES

To establish the capabilities and credibility of the computer program, data published and manually analyzed by Naylor and Doran *(4)* are compared to results obtained from the computer program. Naylor and Doran analyzed the example data set using the square-root-of-time, logarithm-of-time, and their proposed methods. Results obtained by Naylor and Doran for d_0 and d_{100} are compared in Table 1 with those obtained from the computer program for 90-, 50-, and 80-percent consolidation, respectively. The computer program plots the graphical results of each analysis as shown in Figures 12a, b, and c and provides a detailed summary of the data and computed numerical quantities as shown in Figure 13. All three methods give nearly the same results for the coefficient of consolidation, C_v.

An example of typical computer program results for a complete set of consolidation data is given in Tables 3 and 4. The data were obtained from a consolidation test on a specimen of remolded kaolinite. All three methods yield nearly the same values of d_0 for most load increments. For the values of d_{100}, there is generally greater consistency between the square-root-of-time and Naylor-Doran methods than with values obtained from the logarithm-of-time method, which gives slightly larger values of d_{100} during compression.

CONCLUSIONS

1. The computer program is extremely effective in the reduction, plotting, and analysis of time-dependent consolidation data and reduces considerably the time required to analyze such data.

2. The standard error of estimate, S_e, cannot be used alone to select suitable linear representations for various portions of the data curves. The newly developed procedures for adjusting the statistical scatter, that is, the standard error of estimate, have been found to be very effective in establishing the significance of this scatter as far as finding suitable linear representations for various portions of the curves.

3. The computer program is a suitable adjunct to any data acquisition scheme involving conventional, time-dependent consolidation data.

4. The computer program can be readily adapted for use with cathode ray plotters such as the Tektronix Model 4012 or any other peripheral computer equipment compatible with the IBM 370/165 computer.

5. Before the computer program can successfully analyze a given set of conventional consolidation data using either the square-root-of-time, the logarithm-of-time, or the Naylor-Doran methods, the graphical representation of the laboratory data should conform approximately to the traditional curve shape associated with each particular method. Traditional curve shapes associated with each of the three methods are shown in Figures 1, 2, and 3.

6. All three methods to determine C_v give essentially the same values of d_0.

7. Values of d_{100} determined by the square-root-of-time and Naylor-Doran methods are usually

reasonably close and, consequently, give similar values for the coefficient of consolidation, C_v. The logarithm-of-time method tends to give values of d_{100} that produce slightly lower values of C_v.

8. The simplifying assumption made by Naylor and Doran that the values of d_{100} and d_0 are independent of each other is not completely valid. However, in practice it has proven to be a highly useful assumption that enables the development of closed-form iterative solutions for errors in d_{100} and d_0 and the correction of these errors.

9. The precision which can be obtained in the Naylor-Doran method should not be considered to indicate the accuracy which is obtained. This accuracy has yet to be determined.

NOTATION

AC - least squares intercept of straight-line portion of $\log_e$ (1 - U)-versus-time relationship

a_v - coefficient of compressibility

AX - slope of $\log_e$ (1 - U)-versus-time relationship at 60-percent consolidation

b - least-squares definition of intercept

BX - slope of $\log_e$ (1 - U)-versus-time relationship at 80-percent consolidation

C_v - coefficient of consolidation

d_0, d_{50}, d_{90}, d_{100} - deflection readings corresponding to zero-, 50-, 90- and 100-percent consolidation

e - void ratio

err_0 - error in d_0

err_{100} - error in d_{100}

H - length of drainage path in z direction

k - permeability in vertical direction

LLL - starting point for backward generation of secondary compression search intervals

m - least-squares definition of slope

N - number of data points

SBOT - least-squares slope of sample-data group in the primary compression portion of logarithm-of-time data set

S_e - standard error of estimate

SHAT - either maximum slope in square-root-of-time data set or slope of line between first and last points of logarithm-of-time data set

STOP - least-squares slope of sample-data group in secondary compression curve for the logarithm-of-time data set

t - elapsed time

T_v - dimensionless time factor

t_{50}
t_{80} - elapsed times to 50, 80-, and 90-percent consolidation
t_{90}

u - excess pore pressure

U - degree of consolidation

u_o - initial excess pore pressure

UTSLOP - least-squares slope of data between 60- and 80-percent consolidation of $\log_e$ (1 - U)-versus-time relationship

VHAT - vertical distance spanned by data in square-root-of-time and logarithm-of-time data sets

WF_L - length weighting factor

WF_P - position weighting factor

X - horizontal coordinate axis

XBOT - abscissa distance spanned by sample-data group

X_i - abscissa value of a data point

Y - vertical coordinate axis

YBOT - ordinate distance spanned by sample-data group.

Y_i - ordinate value of a data point

z - coordinate in vertical direction

γ_w - unit weight of water

REFERENCES

1. Casagrande, A.; and Fadum, R. E.; *Notes on Soil Testing for Engineering Purposes,* Harvard University, Graduate School of Engineering, Soil Mechanics Series No. 8, No. 268, 1939.
2. Leonards, G. A.; **Foundation Engineering,** McGraw-Hill, New York, 1960.
3. Murray, R. T.; *The Analysis of Consolidation Test Data by Computer,* Road Research Laboratory, Report LR 344, Crowthorne, Berkshire, 1970.
4. Naylor, A. J.; and Doran, J. G.; *Precise Determination of Primary Consolidation,* **Proceedings,** Second International Conference on Soil Mechanics, Vol 1, Rotterdam, 1948.
5. Nie, N. H.; Hull, C. H.; Jenkins, J. G.; Steinbrenner, K.; and Bent, D. H.; **Statistical Package for the Social Sciences,** 2nd Edition, McGraw-Hill, New York, 1975.
6. Taylor, D. W.; *Research on Consolidation of Clays,* Serial 82, Massachusetts Institute of Technology, Department of Civil and Sanitary Engineering, August 1942.
7. Terzaghi, K.; **Theoretical Soil Mechanics,** John Wiley & Sons, Inc., New York, 1943.

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TABLE 1. INITIAL AND FINAL VALUES OF PRIMARY CONSOLIDATION OBTAINED BY MANUAL VERSUS COMPUTER APPLICATION OF THREE METHODS (TEST DATA FROM NAYLOR AND DORAN *(4)*).

	INITIAL VALUES, d_0			FINAL VALUES, d_{100}		
ANALYSIS	TAYLOR	CASAGRANDE	NAYLOR-DORAN	TAYLOR	CASAGRANDE	NAYLOR-DORAN
Naylor-Doran	1940	(1940)	1952	1151	1166	1161
Computer	1928	(1928)	1957	1151	1168	1175

TABLE 2. VALUES OF t_{90}, t_{50}, AND t_{80} FOR 90-, 50-, AND 80-PERCENT CONSOLIDATION, RESPECTIVELY, BY MANUAL VERSUS COMPUTER APPLICATION OF THREE METHODS.

	TIME (IN MINUTES)	
TIME	NAYLOR-DORAN (MANUALLY)	COMPUTER PROGRAM
t_{90}	140.4	145.9
t_{50}	30.3	31.9
t_{80}	88.5	83.6

TABLE 3. COMPUTER RESULTS FOR INITIAL AND FINAL VALUES OF PRIMARY CONSOLIDATION BY THREE METHODS (CONSOLIDATION TEST ON REMOLDED KAOLINITE).

LOAD INCREMENT Tsf (kPa)	INITIAL VALUES, d_0			FINAL VALUES, d_{100}		
	TAYLOR	CASAGRANDE	NAYLOR-DORAN	TAYLOR	CASAGRANDE	NAYLOR-DORAN
0.25 (23.95)	0.1794	0.1794	0.1795	0.1812	0.1823	0.1813
0.50 (47.9)	0.1841	0.1836	0.1840	0.1873	0.1877	0.1872
1.00 (95.8)	0.1902	0.1902	0.1899	0.2001	0.2008	0.1995
2.00 (191.6)	0.2038	0.2043	0.2039	0.2254	0.2283	0.2260
4.00 (383.2)	0.2297	0.2300	0.2289	0.2574	0.2569	0.2573
8.00 (766.4)	0.2647	0.2641	0.2642	0.2912	0.2925	0.2915
16.00 (1532.8)	0.2981	0.2982	0.2975	0.3296	0.3329	0.3320
1.00 (95.8)	0.3366	0.3378	0.3357	0.3143	0.3040	0.3133

TABLE 4. COMPUTER RESULTS FOR C_V BY THREE METHODS (CONSOLIDATION TEST ON REMOLDED KAOLINITE).

LOAD INCREMENT Tsf (kPa)	TAYLOR C_V ft^2/DAY (m^2/DAY)	CASAGRANDE C_V ft^2/DAY (m^2/DAY)	NAYLOR-DORAN C_V ft^2/DAY (m^2/DAY)
0.25 (23.95)	5.97 (0.5550)	1.957 (0.1819)	5.224 (0.4854)
0.50 (47.9)	0.644 (0.0599)	0.721 (0.0669)	0.736 (0.0684)
1.00 (95.8)	0.347 (0.0323)	0.307 (0.0285)	0.405 (0.0376)
2.00 (191.6)	0.264 (0.0245)	0.200 (0.0186)	0.243 (0.0226)
4.00 (383.2)	0.286 (0.0266)	0.295 (0.0274)	0.300 (0.0278)
8.00 (766.4)	0.415 (0.0385)	0.404 (0.0375)	0.416 (0.0386)
16.00 (1532.8)	0.698 (0.0648)	0.520 (0.0484)	0.565 (0.0525)
1.00 (95.8)	0.995 (0.0925)	0.458 (0.0425)	0.820 (0.0762)

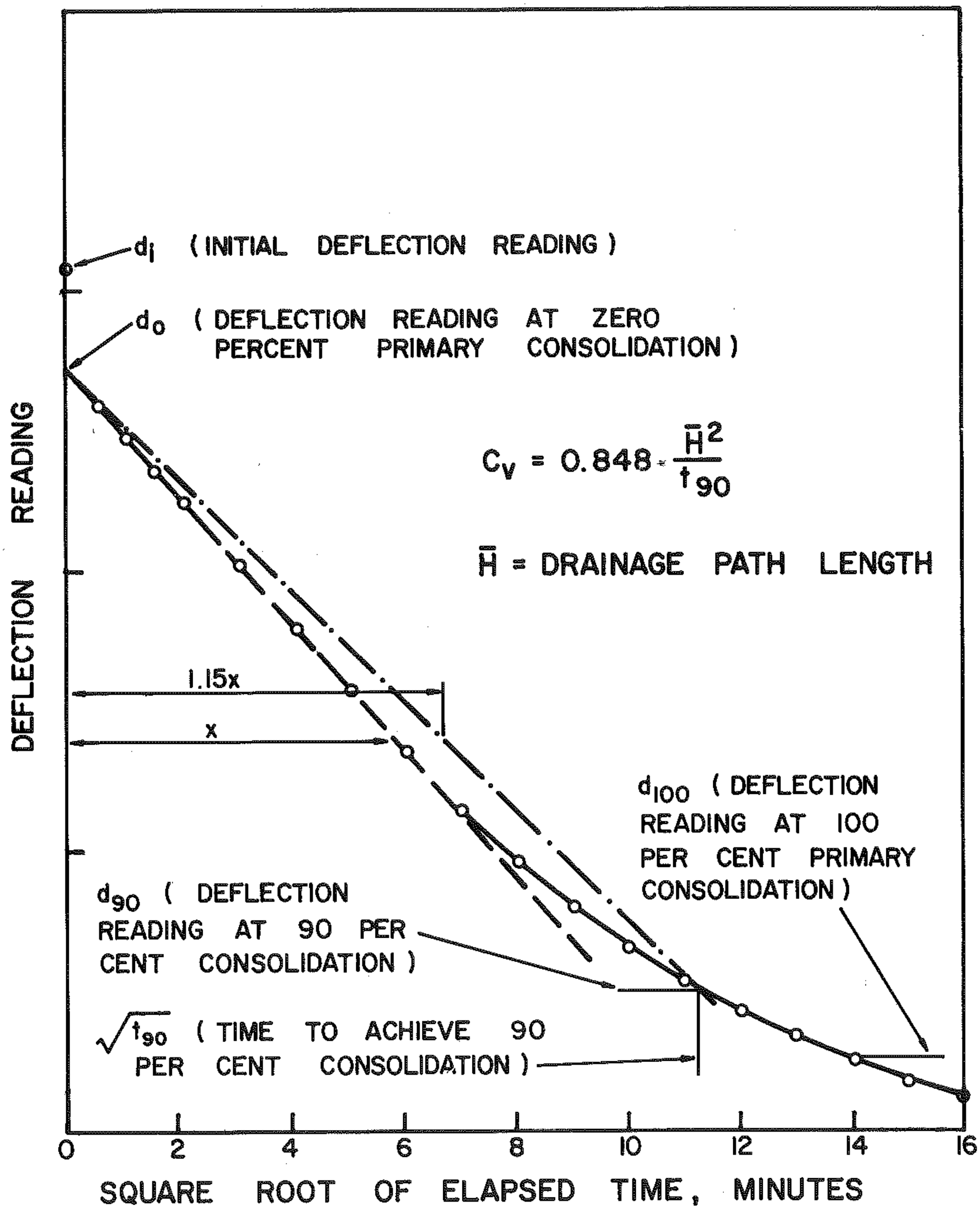


Figure 1. Taylor Square-Root-of-Time Method.

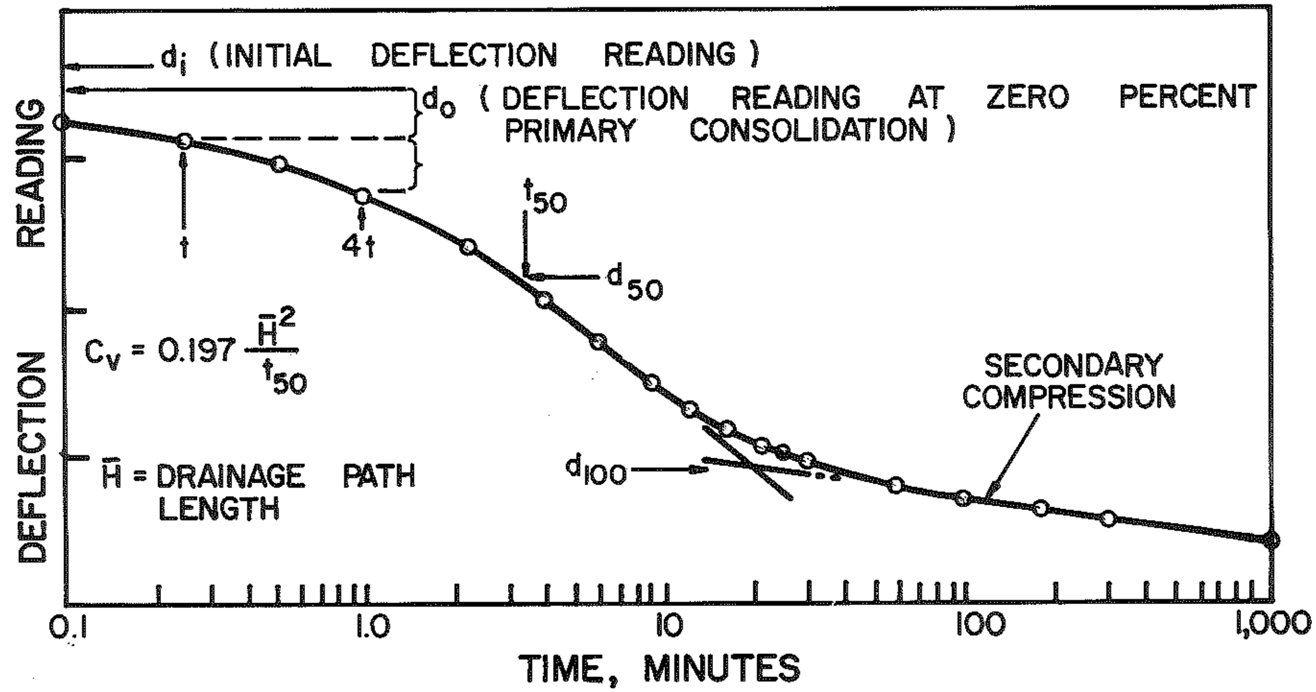
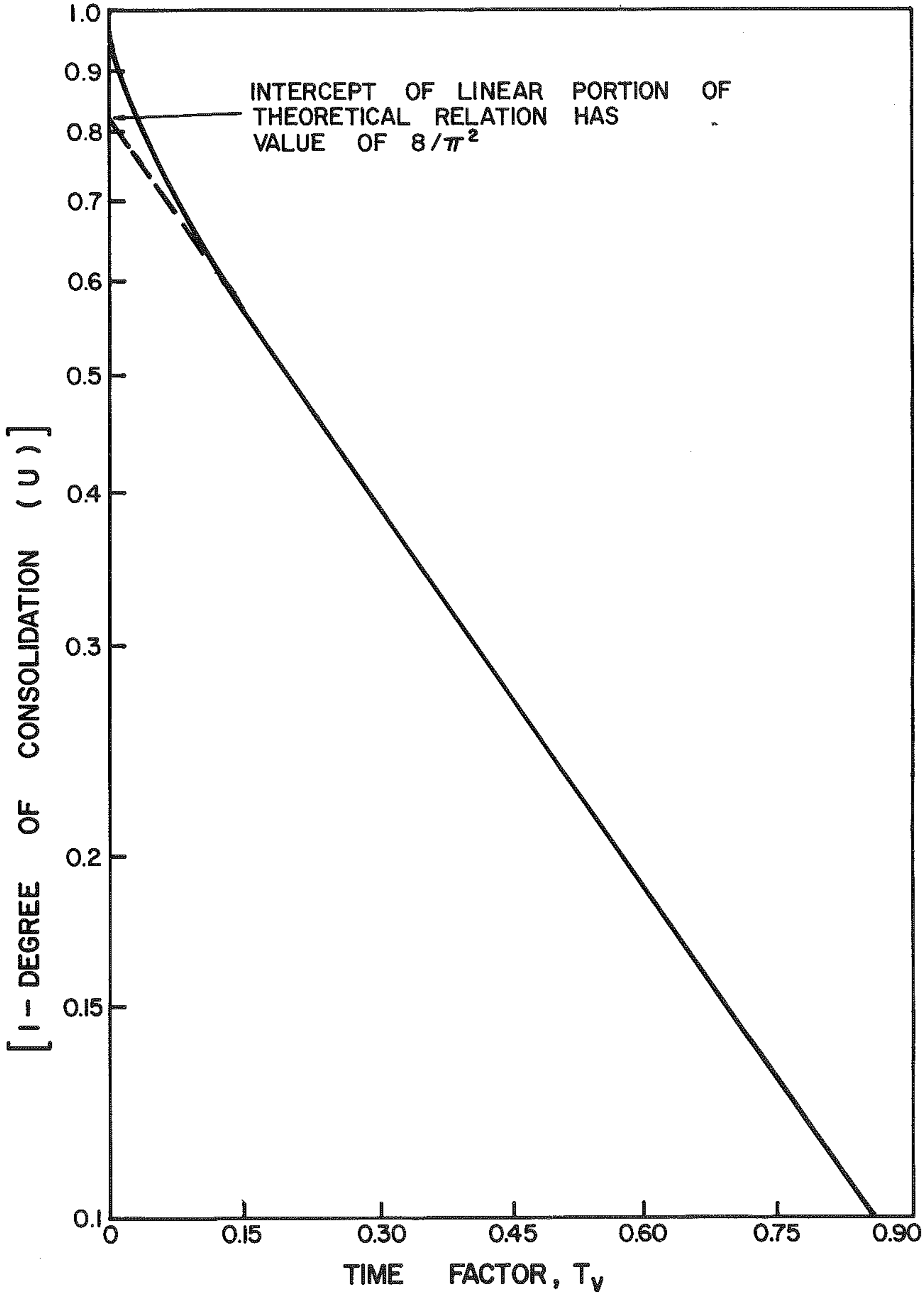


Figure 2. Casagrande Logarithm-of-Time Method.

Figure 3. Theoretical Relationship between Time Factor, T_V, and Natural Logarithm of (1 - U).



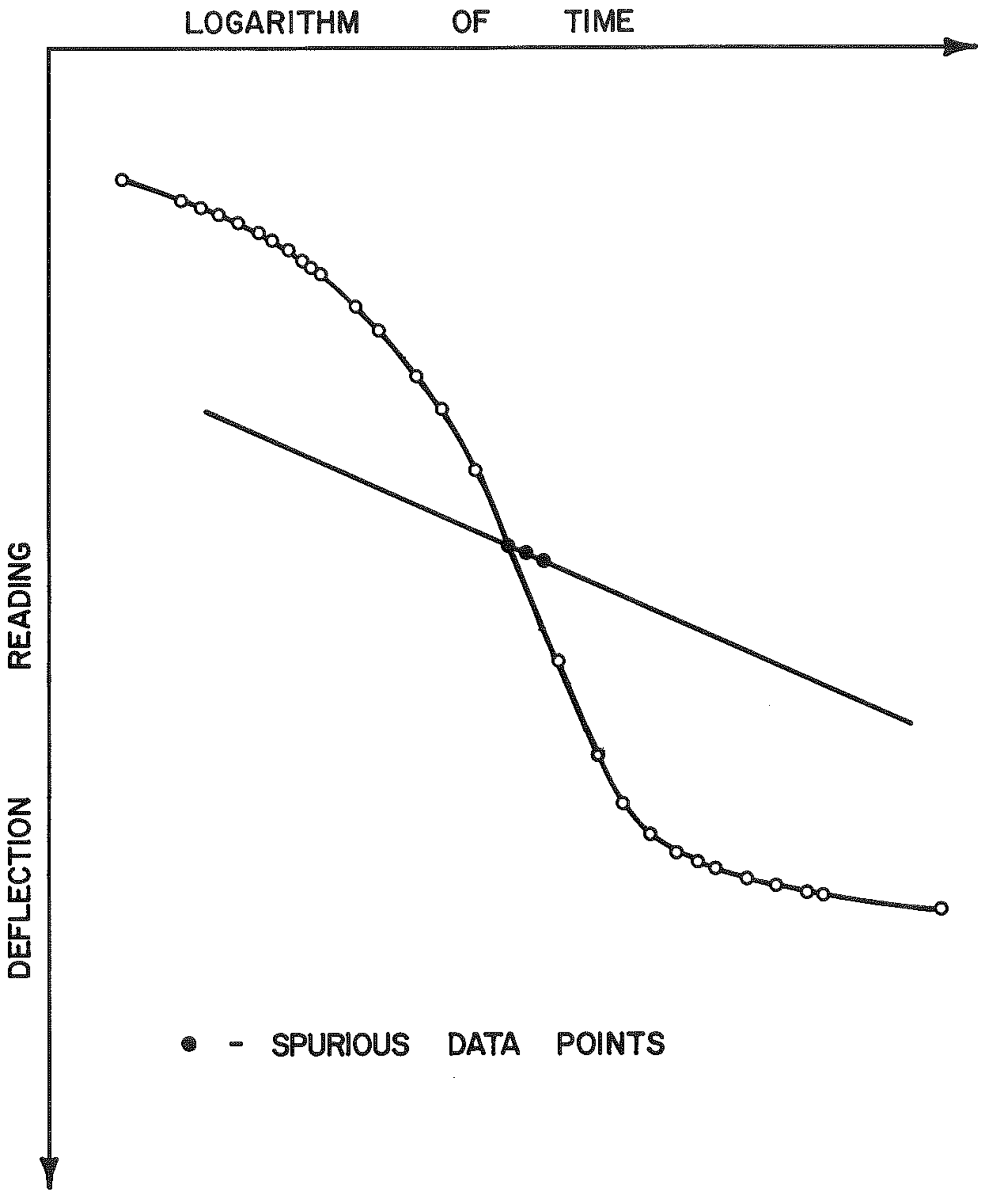
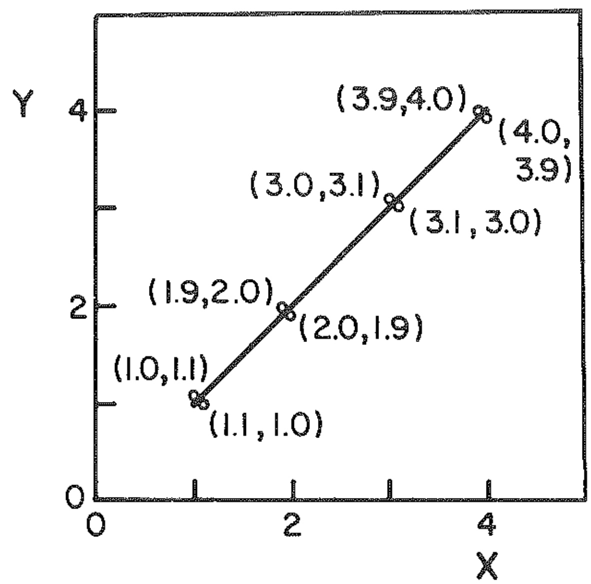
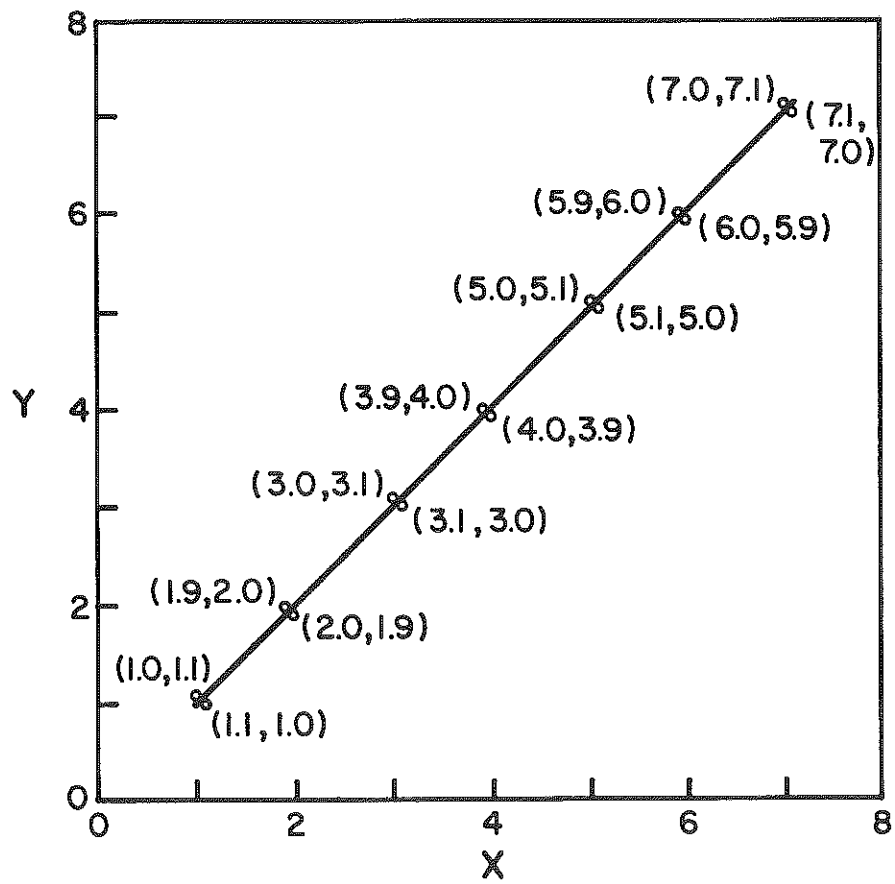


Figure 4. Inappropriate Representation of the Linear Portions of Consolidation Data Using only the Unbiased Standard Error of Estimate, S_e.



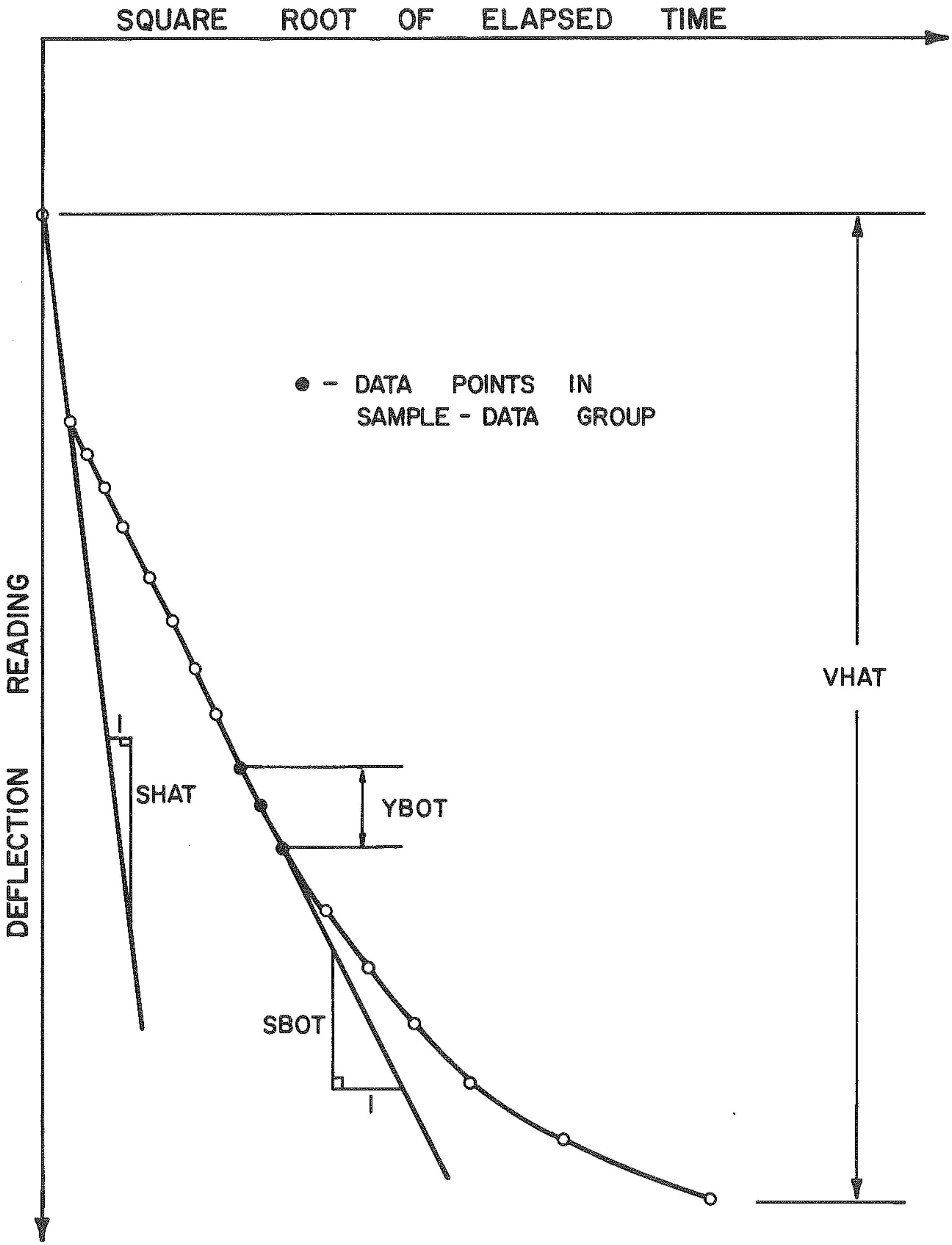
STANDARD ERROR OF ESTIMATE, S_e = 0.11535



STANDARD ERROR OF ESTIMATE, S_e = 0.10798

Figure 5. Standard Error of the Estimate, S_e, Essentially Unaffected by Curve Length: (a) S_e = 0.11535; (b) S_e = 0.10798.

Figure 6. Components of Length and Position Weighting Factors Used in Square-Root-of-Time Analysis.



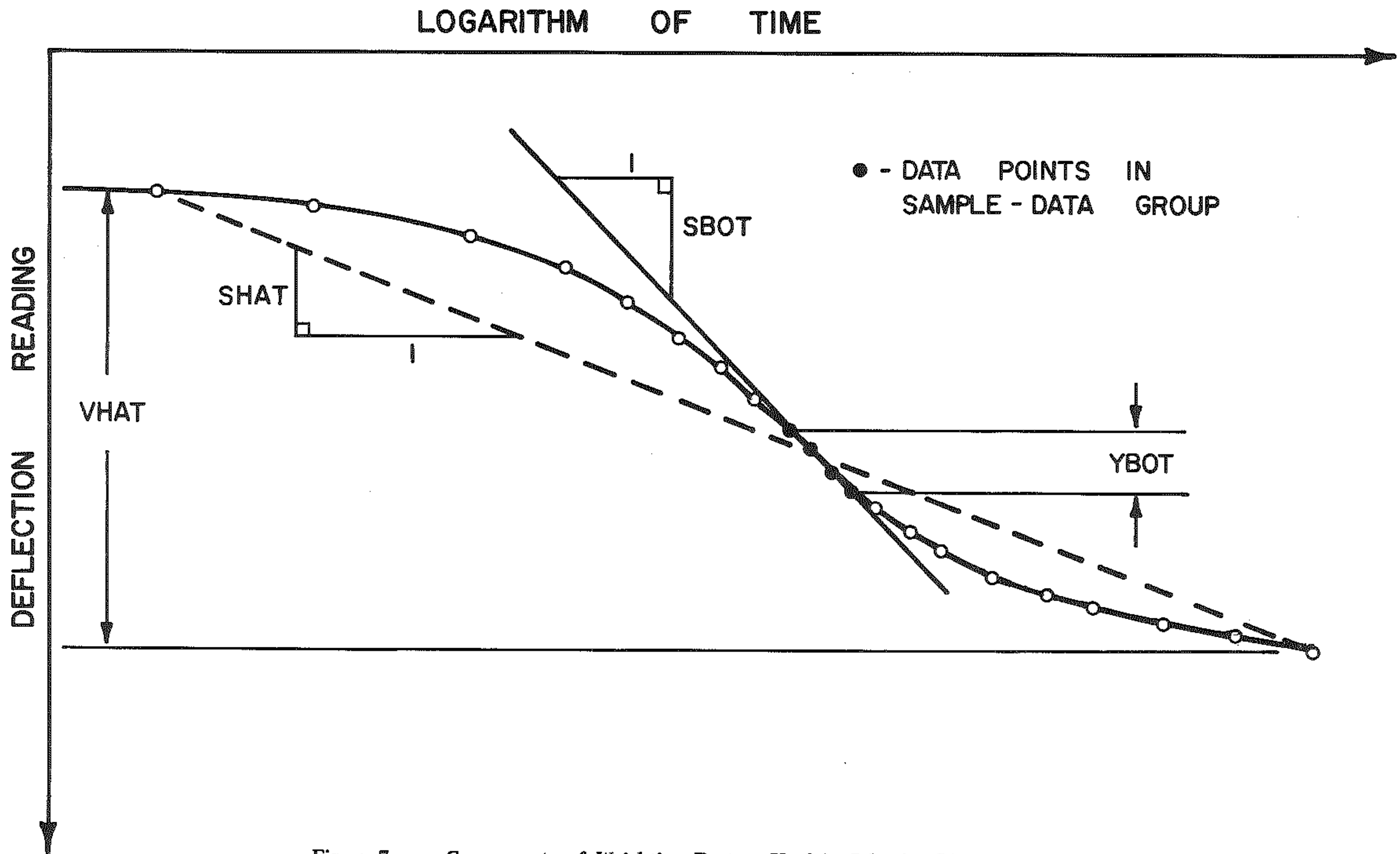


Figure 7. Components of Weighting Factors Used in Selecting Linear Portion of Primary Compression on Logarithm-of-Time Curve.

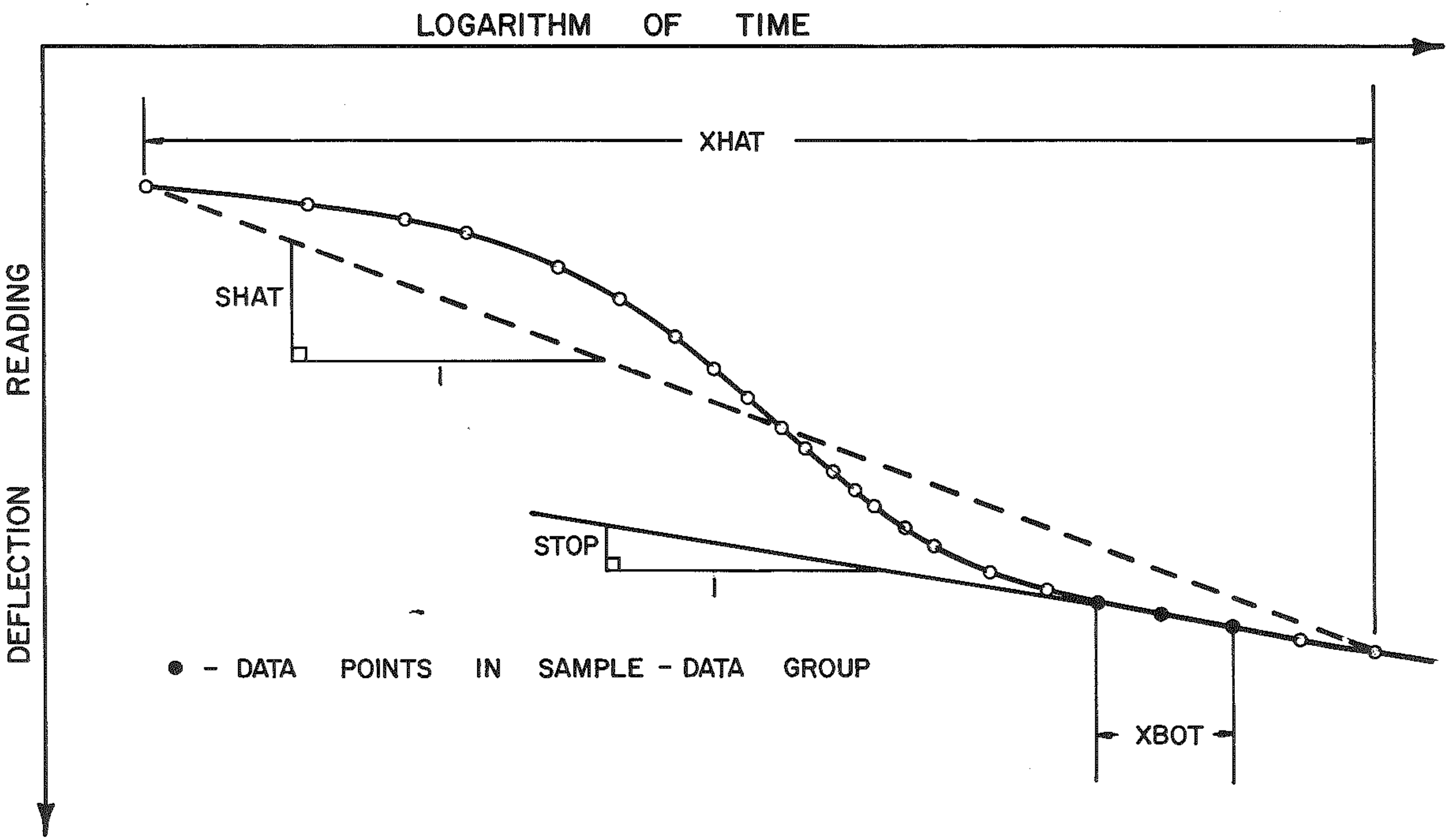


Figure 8. Components of Weighting Factors Used in Selecting Linear Representation of Secondary Compression on Logarithm-of-Time Curve.

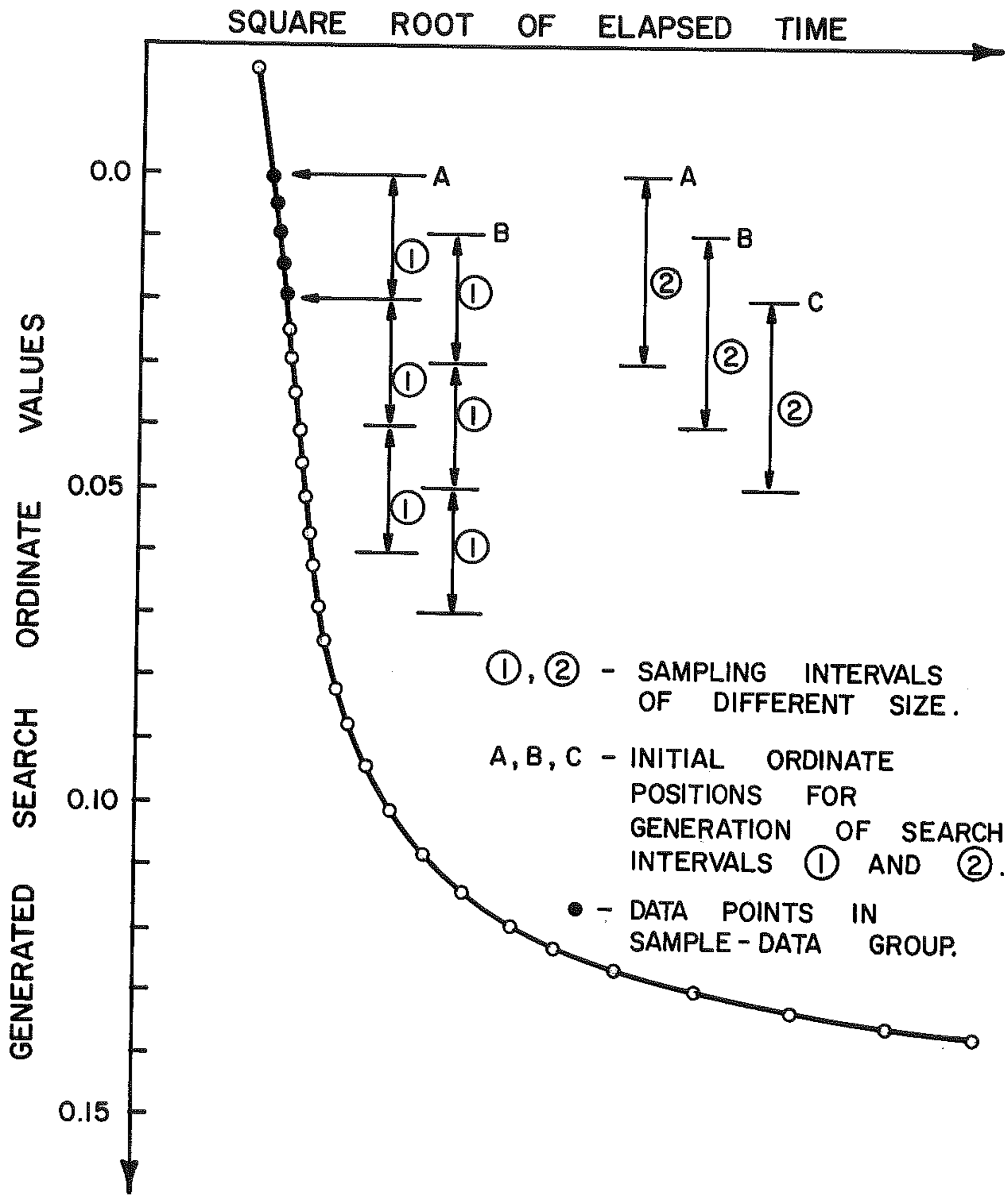


Figure 9. Search Procedures for Selecting Linear Portion of Square-Root-of-Time Curve.

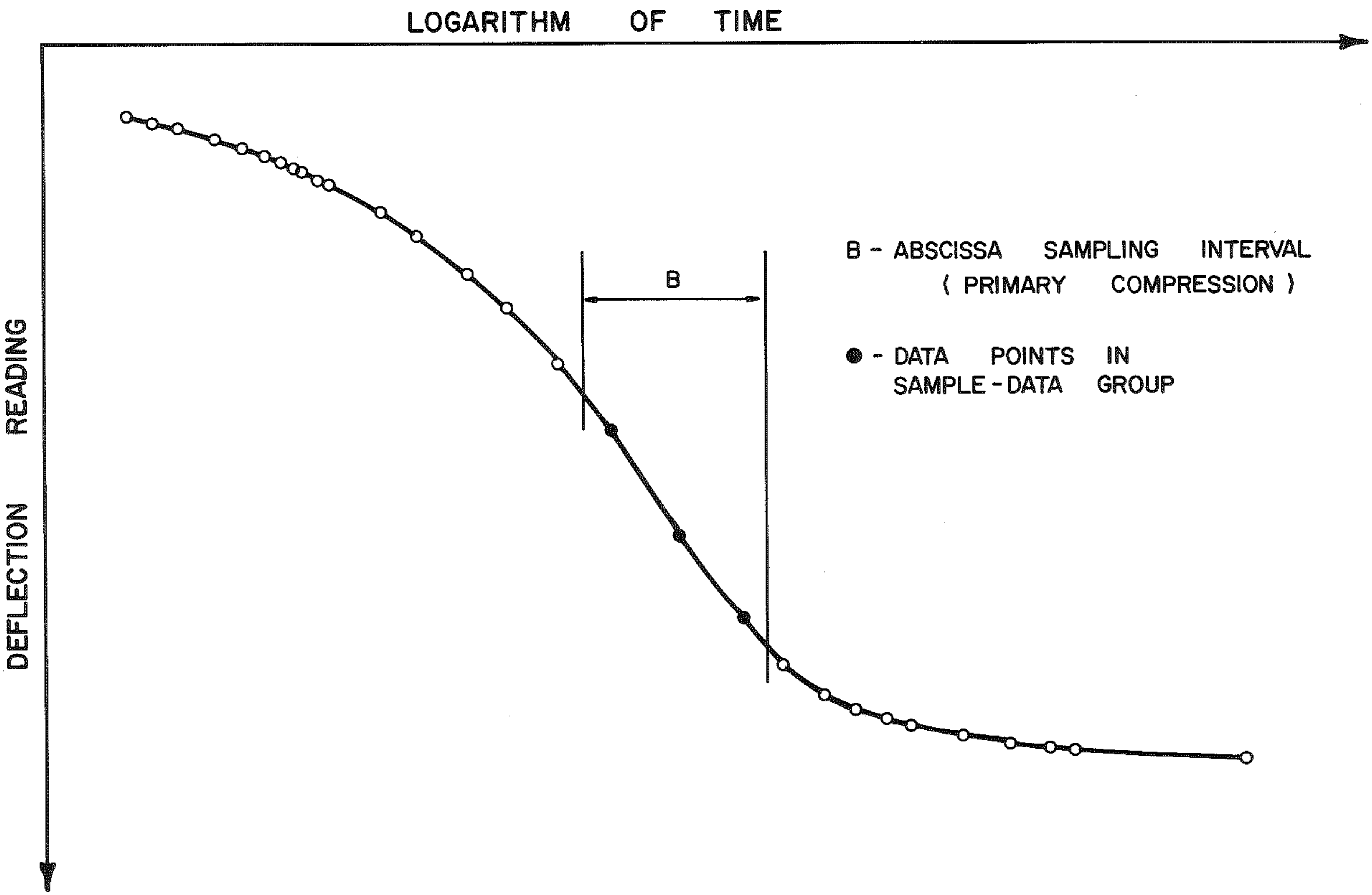


Figure 10. Search Procedures Used in Selecting Linear Portion of Primary Compression on Logarithm-of-Time Curve.

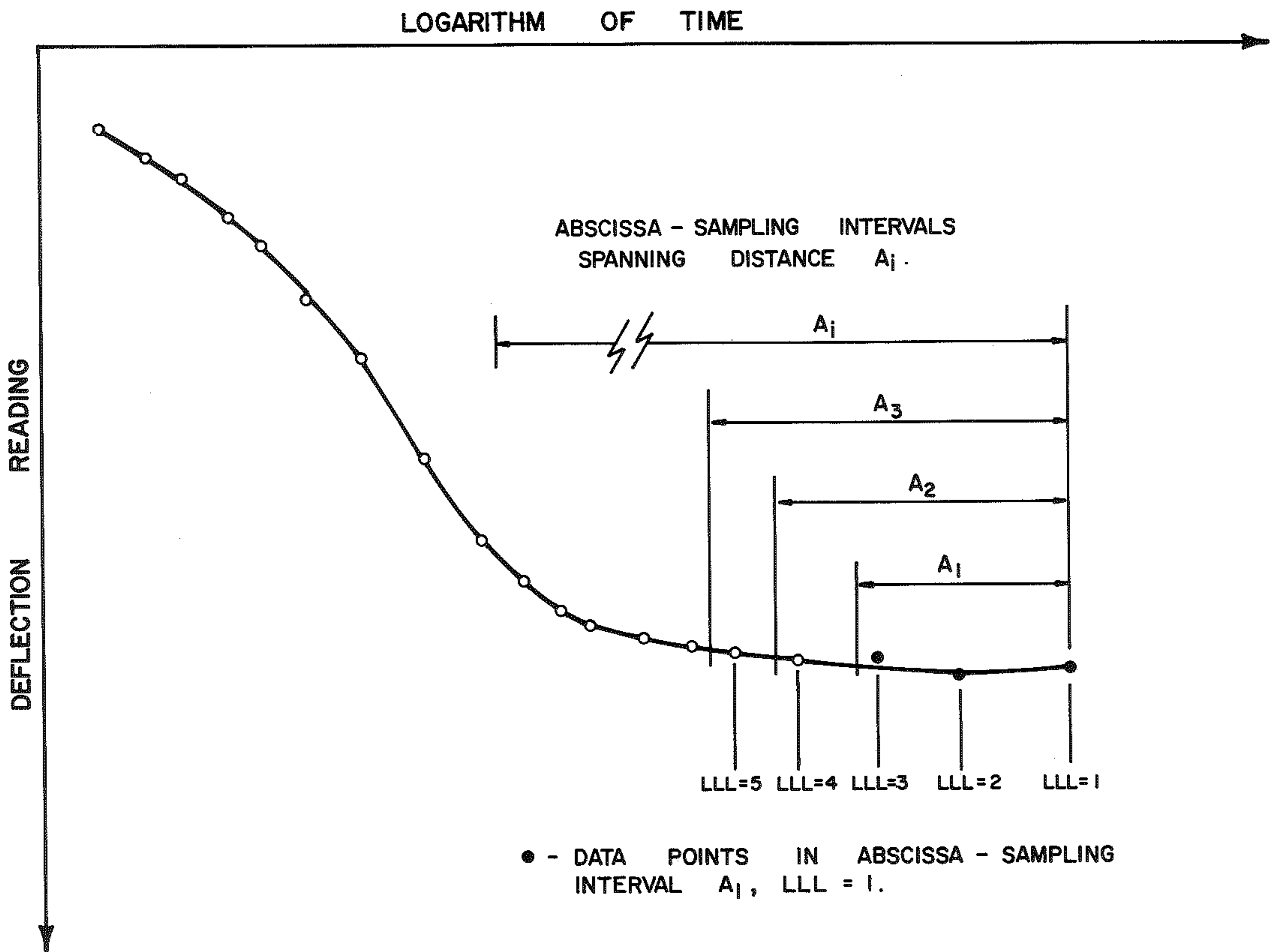


Figure 11. Search Procedures Used in Selecting Linear Representation of Secondary Compression on Logarithm-of-Time Curve.

NAYLOR-DORAN EXAMPLE 1948

C_V = 0.45 SQ.M./Y (0.013 SQ.FT./D) T_{90} = 145.904 MIN

PERMEABILITY, K = 0.4142×10^{-7} CM./SEC. D_0 =-4.8976 MM. (-0.1928 IN.)

K-PRIMARY = 0.4021×10^{-7} CM./SEC. D_{100} =-2.9225 MM. (-0.1151 IN.)

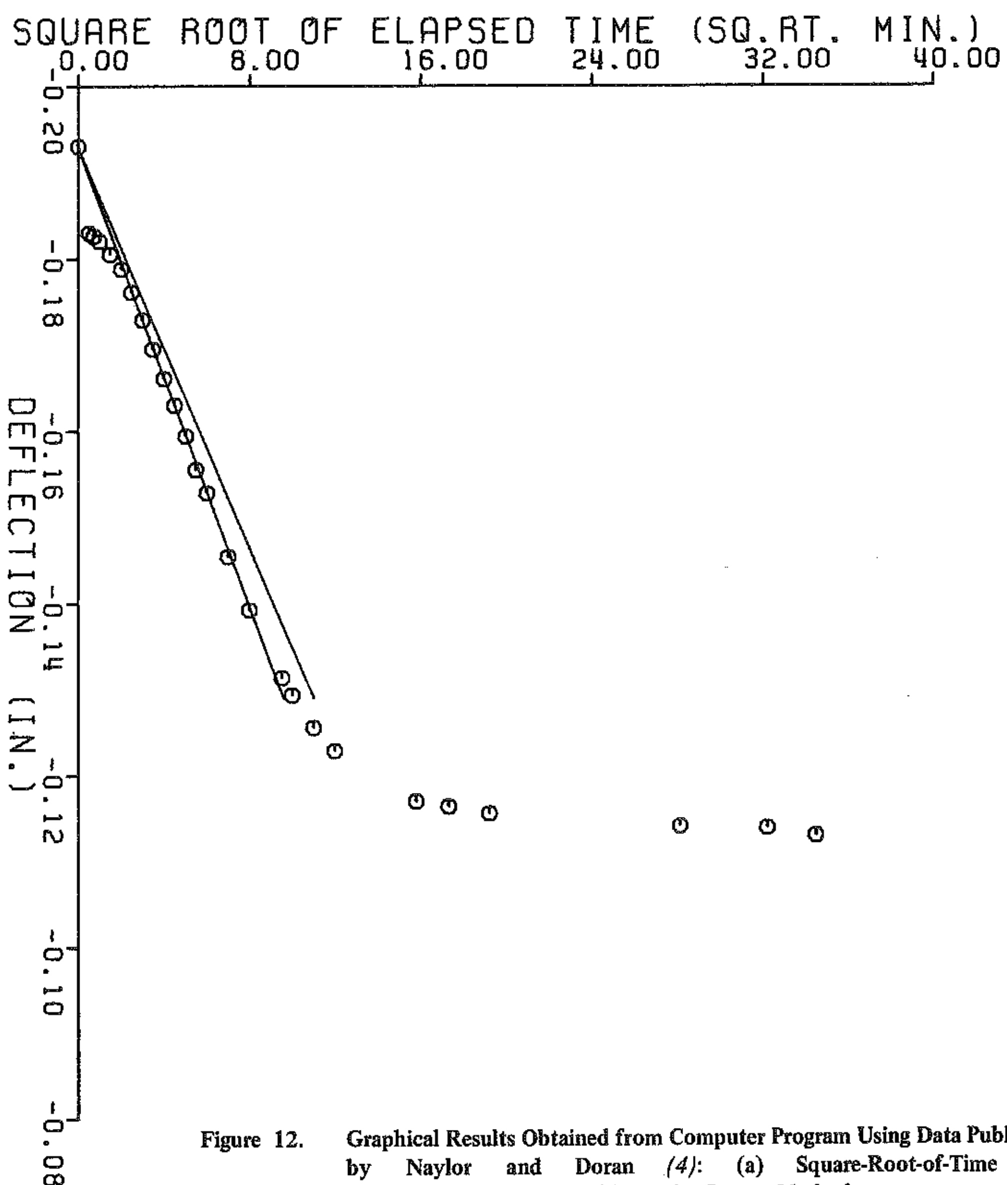


Figure 12. **Graphical Results Obtained from Computer Program Using Data Published by Naylor and Doran** *(4)*: **(a) Square-Root-of-Time (b) Logarithm-of-Time, and (c) Naylor-Doran Methods.**

NAYLOR-DORAN EXAMPLE 1948

C_V = 0.48 SQ.M./Y (0.014 SQ.FT./D) T_{50} = 31.853 MIN

PERMEABILITY, K = 0.4416*10^{-7} CM./SEC. D_0 =-4.8976 MM. (-0.1928 IN.)

K-PRIMARY = 0.4189*10^{-7} CM./SEC. D_{100} =-2.9681 MM. (-0.1168 IN.)

C-SUB-ALPHA = 0.00825

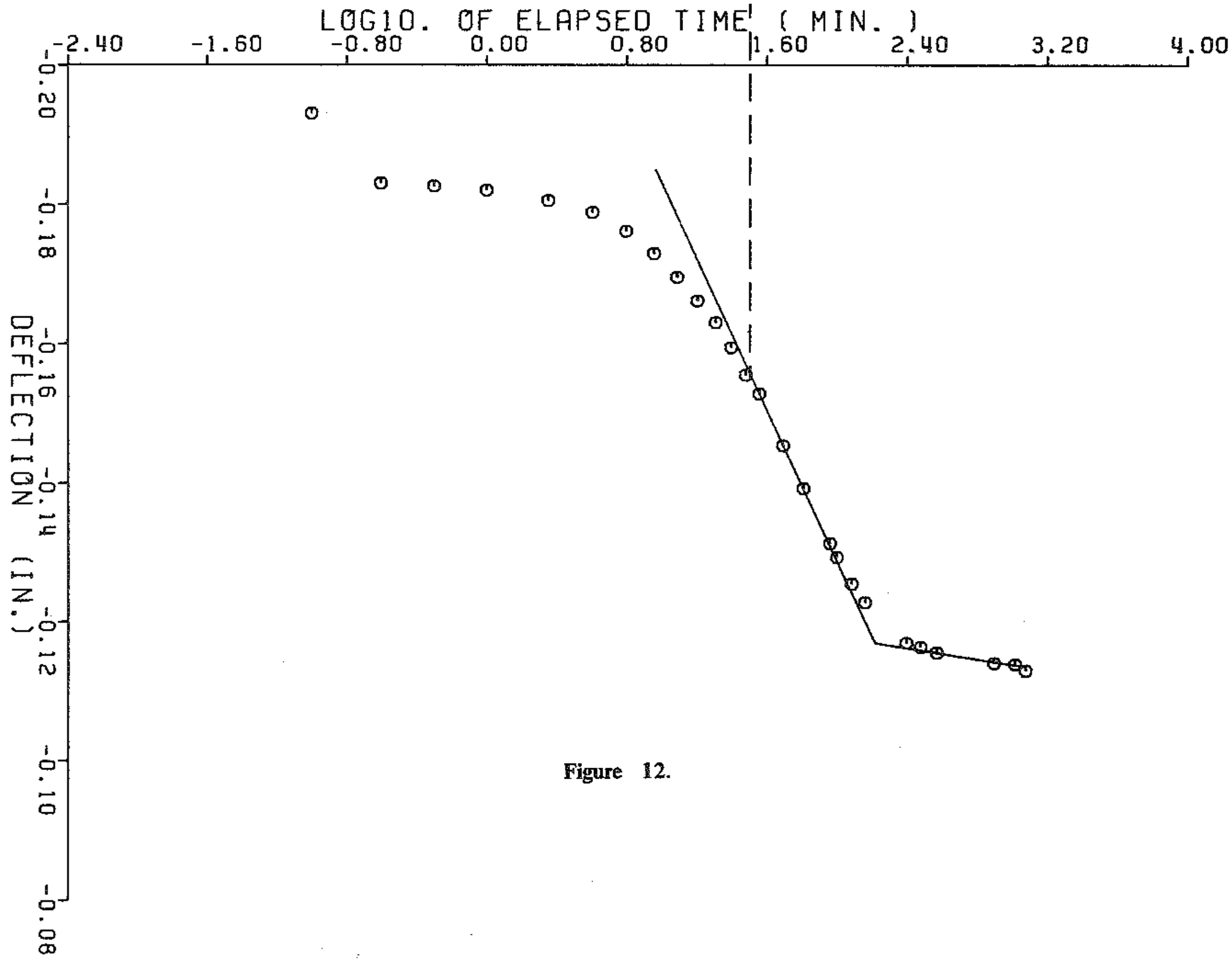


Figure 12.

NAYLOR-DORAN EXAMPLE 1948

C_V = 0.53 SQ.M./Y (0.016 SQ.FT./D)

T_{80} = 83.601 MIN

PERMEABILITY, K = 0.4835×10^{-7} CM./SEC.

D_0 =-4.9711 MM. (-0.1957 IN.)

K-PRIMARY = 0.4723×10^{-7} CM./SEC.

D_{100} =-2.9839 MM. (-0.1175 IN.)

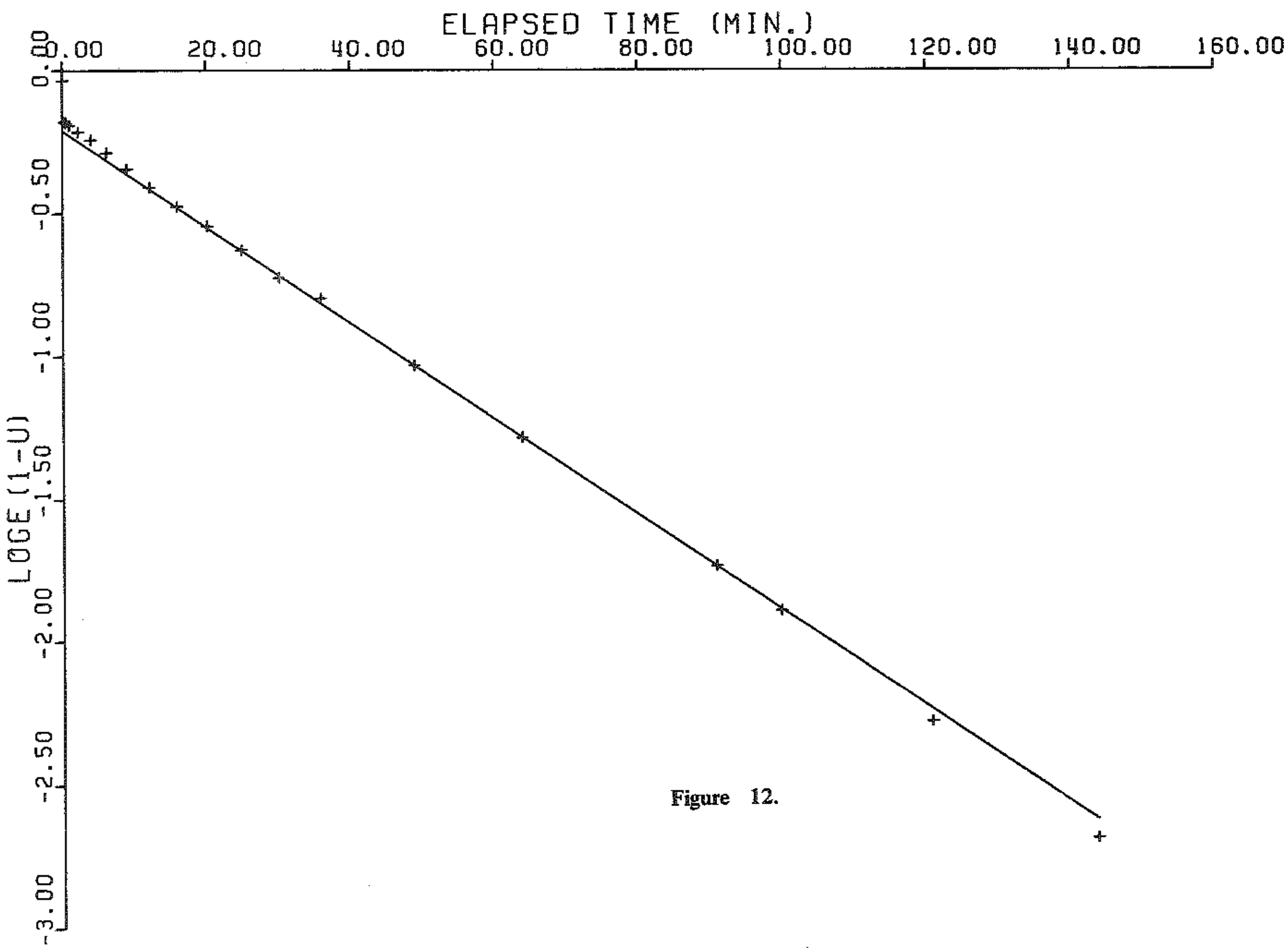


Figure 12.

TEST NO. 1 NAYLOR-DORAN EXAMPLE 1948

DOUBLE DRAINAGE

	THICKNESS	PRESSURE INCREMENT
INITIAL	25.4000MM. (1.00000IN.)	0.0 KPA/SQ.M. (0.0 T/SQ.FT.)
FINAL	23.3654MM. (0.91990IN.)	27.30KPA/SQ.M. (0.2850T/SQ.FT.)

	TAYLOR'S SQUARE ROOT TIME	CASAGRANDE'S LOGARITHM TIME	NAYLOR - DORAN ANALYTICAL METHOD
	T90 =145.90 MIN.	T50 = 31.85 MIN.	T80 = 83.60 MIN.
	D90 = -0.1228 IN.	D50 = -0.1548 IN.	D80 = -0.1331 IN.
CONSOLIDATION COEFFICIENT	0.45SQ.M/Y (0.013SQ.FT./D)	0.48SQ.M/Y (0.014SQ.FT./D)	0.53SQ.M/Y (0.016SQ.FT./D)
DIAL RDG. AT ZERO % PRIMARY	-4.8976MM. (-0.1928IN.)	-4.8976MM. (-0.1928IN.)	-4.9711MM. (-0.1957IN.)
DIAL RDG. AT 100 % PRIMARY	-2.9225MM. (-0.1151IN.)	-2.9661MM. (-0.1168IN.)	-2.9838MM. (-0.1175IN.)
EST. PRIMARY CONSOLIDATION	1.9751MM. (0.0778IN.)	1.9315MM. (0.0760IN.)	1.9873MM. (0.0782IN.)
COMPRESSIBILITY, M-SUB-V	2.9337 SQ.M./MPA (0.2811 SQ.FT./TON)	CALPHA = 0.00825	-
PRIMARY COMP. RATIO	0.9708	0.9494	0.9768
SECONDARY COMP. RATIO	0.0270	0.0484	0.0571
INITIAL COMP.RATIO	0.0023	0.0023	-0.0339
PERMEABILITY COEFFICIENT	.41E-07 CM./SEC.	.44E-07 CM./SEC.	.48E-07CM./SEC.
K-PRIMARY CONSOLIDATION	.40E-07 CM./SEC.	.42E-07 CM./SEC.	.47E-07CM./SEC.

ELAPSED TIME	DEFLECTION	ELAPSED TIME	DEFLECTION
0.0998 MIN.	-0.19300 IN. (-4.9022 MM.)	36.00 MIN.	-0.15280 IN. (-3.8811 MM.)
0.2500 MIN.	-0.18300 IN. (-4.6482 MM.)	49.00 MIN.	-0.14540 IN. (-3.6932 MM.)
0.5000 MIN.	-0.18260 IN. (-4.6380 MM.)	64.00 MIN.	-0.13920 IN. (-3.5357 MM.)
1.0000 MIN.	-0.18200 IN. (-4.6228 MM.)	91.00 MIN.	-0.13130 IN. (-3.3350 MM.)
2.2500 MIN.	-0.18050 IN. (-4.5847 MM.)	100.00 MIN.	-0.12930 IN. (-3.2842 MM.)
4.0000 MIN.	-0.17880 IN. (-4.5415 MM.)	121.00 MIN.	-0.12550 IN. (-3.1877 MM.)
6.2500 MIN.	-0.17610 IN. (-4.4729 MM.)	144.00 MIN.	-0.12280 IN. (-3.1191 MM.)
9.0000 MIN.	-0.17290 IN. (-4.3917 MM.)	250.00 MIN.	-0.11690 IN. (-2.9693 MM.)
12.2500 MIN.	-0.16950 IN. (-4.3053 MM.)	300.00 MIN.	-0.11630 IN. (-2.9540 MM.)
16.0000 MIN.	-0.16610 IN. (-4.2189 MM.)	370.00 MIN.	-0.11550 IN. (-2.9337 MM.)
20.2500 MIN.	-0.16300 IN. (-4.1402 MM.)	790.00 MIN.	-0.11400 IN. (-2.8956 MM.)
25.0000 MIN.	-0.15940 IN. (-4.0488 MM.)	1038.00 MIN.	-0.11380 IN. (-2.8905 MM.)
30.2500 MIN.	-0.15550 IN. (-3.9497 MM.)	1190.00 MIN.	-0.11290 IN. (-2.8677 MM.)

Figure 13. Computer Printout.

KEY WORDS: Computers, Consolidation, Laboratory Test, Data Analysis, Time-Rate of Strain, Coefficient of Consolidation, Statistical Analysis, Graphical Constructions, Iterative Procedures.

ABSTRACT: A computerized, statistical algorithm has been developed for analyzing the time-dependent properties of conventional consolidation test data. The Taylor square-root-of-time and Casagrande logarithm-of-time methods are analytically represented along with a modified version of the Naylor-Doran method. The coefficient of consolidation, C_v, and values of deflection corresponding to the beginning and end of primary consolidation are determined by each of the three methods. The algorithm is written in Fortran IV for use with the IBM 370/165 computer program and is extremely effective in the reduction of time-dependent consolidation data.