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Unimodality Questions in Ehrhart Theory

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DISSERTATION

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ABSTRACT OF DISSERTATION

Unimodality Questions in Ehrhart Theory

An interesting open problem in Ehrhart theory is to classify those lattice polytopes having a unimodal h^*-vector. Although various sufficient conditions have been found, necessary conditions remain a challenge. Highly-structured polytopes, such as the polytope of real doubly-stochastic matrices, have been proven to possess unimodal h^*-vectors, but the same is unknown even for small variations of it.

In this dissertation, we mainly consider two particular classes of polytopes: reflexive simplices and the polytope of *symmetric* real doubly-stochastic matrices. For the first class, we discuss an operation that preserves reflexivity, integral closure, and unimodality of the h^*-vector, providing one explanation for why unimodality occurs in this setting. We also discuss the failure of proving unimodality in this setting using weak Lefschetz elements. With the second class, we prove partial unimodality results by examining their toric ideals and using a correspondence between these and regular triangulations of the polytopes. Lastly, we describe the computational methods used to help develop these results. Several software programs were used, and the code has proven useful outside of the main focus of this work.

KEYWORDS: Ehrhart theory, unimodal sequence, lattice polytope, reflexive polytope, integrally closed, toric ideal, weak Lefschetz property

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Chapter 1 Introduction

1.1 Lattice Polytopes

We begin by describing objects that have been studied since antiquity. Although simply described, there is an astounding amount of deep mathematics required to answer basic questions surrounding them.

Definition 1.1.1. A *magic square* of size n and order r is an $n \times n$ square array of nonnegative integers such that each row and each column sums to r.

Perhaps the most famous magic square is the Lo Shu, which is said to have been found on the shell of a turtle after a massive flood in China:

4	9	2
3	5	7
8	1	6

Figure 1.1: The Lo Shu magic square

In this example, the entries are all distinct, positive, and the sums of the diagonals add to the same sum as the rows and columns. These extra properties cause the literature to be a bit inconsistent in terminology: What we call a magic square is sometimes referred to as a *semi-magic square*, e.g. in [14]. Our definition follows that of [36].

Combinatorists often like to ask "How many X are there?" This situation is no different: deep mathematics have developed based on the question "How many magic squares of a fixed size n and order r exist?" If we let $\mathcal{M}_n$ be the function defined on $\mathbb{Z}_{\geq 0}$ whose evaluation at r gives the number of magic squares of size n and order r, this question is equivalent to asking for the value of $\mathcal{M}_n(r)$. When $n = 1$, the solution is trivial: $\mathcal{M}_1(r) = 1$.

When $n = 2$, there are $r + 1$: one for each $k \in \{0, \ldots, r\}$; see Figure 1.2. Once $n \geq 3$, it is not quite so easy to determine how many magic squares of order r exist, although it has been shown that

$$\mathcal{M}_3(r) = \frac{1}{8}r^4 + \frac{3}{4}r^3 + \frac{15}{8}r^2 + \frac{9}{4}r + 1,$$

k	$r-k$
$r-k$	k

Figure 1.2: An arbitrary 2×2 magic square

and it is not quite as obvious as to why this is. In [2] it was conjectured that $\mathcal{M}_n(r)$ is always a polynomial in r of degree $(n-1)^2$.

Counting the number of magic squares of size n and order r is equivalent to determining how many elements in the r^{th} dilate of the set

$$B_n := \left\{ (x_{ij}) \in \mathbb{R}^{n \times n}_{\geq 0} \middle| \sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j, \sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \right\}$$

have all entries in $\mathbb{Z}$. In [2] it was proven that B_n can be described as the convex hull of the $n \times n$ permutation matrices; i.e., let $\{P_1, \ldots, P_{n!}\}$ be the set of permutation matrices. Then

$$B_n = \left\{ \sum_{i=1}^{n!} \lambda_i P_i \middle| \lambda_i \geq 0 \text{ for all } i, \sum_{i=1}^{n!} \lambda_i = 1 \right\}.$$

This result, known as the Birkhoff-von Neumann Theorem, is the key to proving the above conjecture, whose proof then quickly follows. It also motivates the following more general definitions.

Definition 1.1.2. A *lattice* N is a subgroup of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^k$ for some k. A (convex) *polytope* is the convex hull of a finite number of points in $\mathbb{R}^n$. A polytope is called *lattice* if every vertex lies in N.

When given an infinite sequence of rational numbers, it is common to try to find a generating function for the sequence; i.e., a closed form of the formal power series whose coefficients are the given sequence. In our situation, we end up with

$$\begin{aligned}
H(\mathcal{M}_1;x) &= \sum_{r\geq 0}\mathcal{M}_1(r)x^r \\
&= \sum_{r\geq 0} x^r \\
&= \frac{1}{1-x}; \\
H(\mathcal{M}_2;x) &= \sum_{r\geq 0}\mathcal{M}_2(r)x^r \\
&= \sum_{r\geq 0}(r+1)x^r \\
&= \frac{1}{(1-x)^2}; \\
H(\mathcal{M}_3;x) &= \sum_{r\geq 0}\mathcal{M}_3(r)x^r \\
&= \sum_{r\geq 0}\left(\frac{1}{8}r^4+\frac{3}{4}r^3+\frac{15}{8}r^2+\frac{9}{4}r+1\right)x^r \\
&= \frac{1+x+x^2}{(1-x)^5}.
\end{aligned}$$

Thinking of B_n as a polytope, it can be seen that these power series are encoding the number of lattice points in dilations of the polytope. We will return to magic squares in the third chapter.

1.2 Ehrhart Theory

For a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$ of dimension d, consider the counting function $|m\mathcal{P} \cap \mathbb{Z}^n|$, where $m\mathcal{P}$ is the m-th dilate of $\mathcal{P}$. Figure 1.3 provides a concrete example of the points being counted by $|m\mathcal{P} \cap \mathbb{Z}^n|$. The *Ehrhart series* of $\mathcal{P}$ is

$$E_{\mathcal{P}}(t) := 1 + \sum_{m\in\mathbb{Z}_{\geq 1}} |m\mathcal{P}\cap\mathbb{Z}^n| t^m \, .$$

Combining two well-known theorems due to Ehrhart [16] and Stanley [33], there exist values $h_0^*,\ldots,h_d^* \in \mathbb{Z}_{\geq 0}$ with $h_0^*=1$ such that

$$E_{\mathcal{P}}(t) = \frac{\sum_{j=0}^{d} h_j^* t^j}{(1-t)^{d+1}} \, .$$

We say the polynomial $h^*_{\mathcal{P}}(t) := \sum_{j=0}^{d} h^*_j t^j$ is the *h^*-polynomial* of $\mathcal{P}$ (sometimes referred to as the δ-polynomial of $\mathcal{P}$) and the vector of coefficients $h^*(\mathcal{P})$ is the *h^*-vector* of $\mathcal{P}$. That $E_{\mathcal{P}}(t)$ is of this rational form with $h^*_{\mathcal{P}}(1) \neq 0$ is equivalent to $|m\mathcal{P} \cap \mathbb{Z}^n|$ being a polynomial function of m of degree d; the non-negativity of the h^*-vector is an even stronger property. The h^*-vector of a lattice polytope $\mathcal{P}$ is a fascinating partial invariant. Obtaining a general understanding of h^*-vectors of lattice polytopes and their geometric/combinatorial implications is currently of great interest.

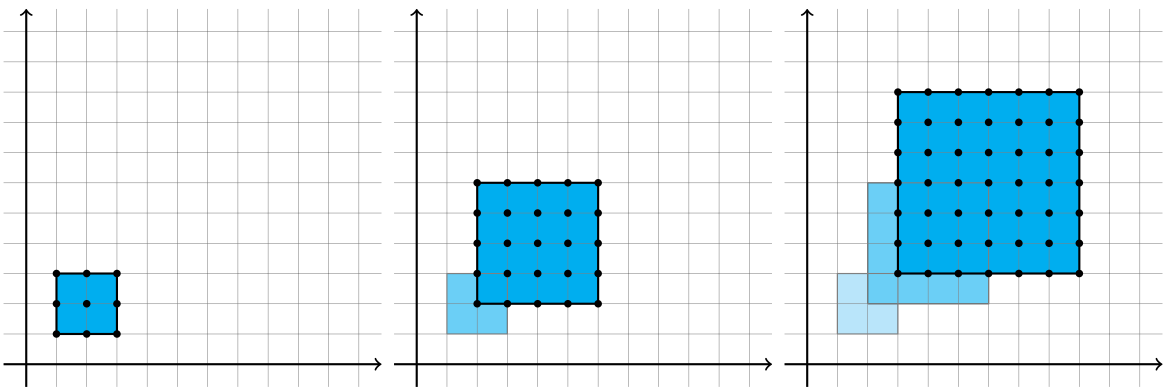

Figure 1.3: A lattice square and two of its dilations

Note that, since the Ehrhart series may be written as a rational function, the counting function $\mathcal{L}_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^n|$ can be written as the polynomial

$$\mathcal{L}_{\mathcal{P}}(m) = h^*_0 \binom{m+d}{d} + h^*_1 \binom{m+d-1}{d} + \cdots + h^*_{d-1} \binom{m+1}{d} + h^*_d \binom{m}{d}.$$

From this expansion, one can easily compute that the number of lattice points in $\mathcal{P}$ is $h^*_1 + d + 1$. Using other results, namely Ehrhart-Macdonald reciprocity, it can be shown that if $\deg h^*_{\mathcal{P}}(t) = k$, then the smallest positive integer dilate of $\mathcal{P}$ containing an interior lattice point is $d - k + 1$, and h^*_k is the number of lattice points in the interior of this scaling. However, such nice combinatorial results do not exist for general choices of h^*_i.

Recent work has focused on determining when $h^*(\mathcal{P})$ is unimodal, that is, when there exists some k for which $h^*_0 \leq \cdots \leq h^*_k \geq \cdots \geq h^*_d$. One reason combinatorists are interested in unimodality results is that their proofs often point to interesting and unexpected properties of combinatorial, geometric, and algebraic objects. In particular, symmetric h^*-vectors play a key role in Ehrhart theory through their connection to reflexive polytopes, defined below.

There are many interesting techniques for studying symmetric unimodal sequences, using tools from analysis, Lie theory, algebraic geometry, etc. [35].

1.3 Techniques for Proving Unimodality of the h^*-vector

Triangulations

There are multiple ways one may attempt to prove the unimodality of the h^*-vector of a lattice polytope. A common notion underlying many of these methods is the idea of decomposing a polytope into simplices in a controlled manner. The structure of this decomposition then manifests in multiple ways, one of which being a unimodal h^*-vector. The following definitions and examples make these ideas more precise.

Definition 1.3.1. A *supporting hyperplane* of a polytope $\mathcal{P}$ is hyperplane such that $\mathcal{P}$ lies in one of the closed half spaces determined by H. The *faces* of $\mathcal{P}$ consist of $\mathcal{P}$ itself and any intersection of $\mathcal{P}$ with a supporting hyperplane (including the empty set). The faces of dimension one less than $\dim \mathcal{P}$ are called *facets*.

There is much research that comes from the simple question: Given a certain polytope $\mathcal{P}$, how can we describe the faces? The Birkhoff-von Neumann theorem is a significant result, and that only sought to describe the *vertices* of a polytope. Another result in this direction relates to Gelfand-Tsetlin polytopes, which simply asked if their vertices were integral [15] (The answer, by the way, is “Not always.”).

Definition 1.3.2. A *triangulation* of a polytope $\mathcal{P}$ is a finite collection of simplices $T = \{T_i\}$ such that: $\cup T_i = \mathcal{P}$; for every $T_i, T_j \in T$, the intersection $T_i \cap T_j$ is a common face of each; and if $T_i \in T$, then every face of T_i is in T as well.

Definition 1.3.3. A lattice simplex is *unimodular* if it has smallest possible volume in the lattice. Equivalently, the simplex is unimodular if, for any ordering $\{v_0, \ldots, v_n\}$ of its vertices, the set $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ is a basis of the lattice. A triangulation T of $\mathcal{P}$ is *unimodular* if every simplex in T is unimodular.

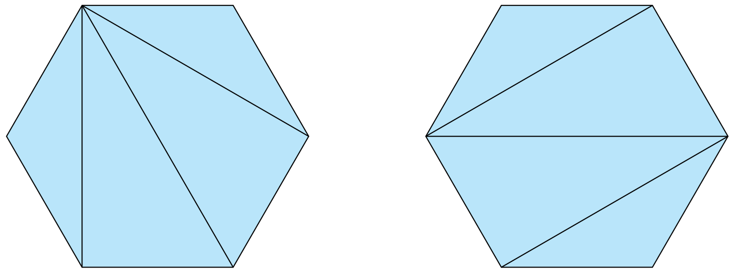

Figure 1.4: Two triangulations of a regular hexagon

In two dimensions, Pick's theorem proves that any triangle in $\mathbb{R}^2$ such that the only points of $\mathbb{Z}^2$ it contains are its vertices must be unimodular. However, this does not generalize to higher dimensions: the Reeve tetrahedron $\mathcal{R}_h = \operatorname{conv}\{0, e_1, e_2, e_1+e_2+he_3\} \subseteq \mathbb{R}^3$, $h \in \mathbb{Z}_{>1}$ has only its vertices as lattice points, but has volume $h/6$. Its vertices generate the lattice $\mathbb{Z}^2 \oplus h\mathbb{Z}$ instead of $\mathbb{Z}^3$.

Example 1.3.4. If $\mathcal{P} \cap \mathbb{Z}^n = \{v_0, \ldots, v_k\}$, choose real numbers $w_0, \ldots, w_k$ and form the polytope

$$\operatorname{conv}\{(v_0, w_0), (v_1, w_1), \ldots, (v_k, w_k)\} \subseteq \mathbb{R}^{n+1}.$$

Project each facet whose normal vector has a negative last coordinate back to $\mathbb{R}^n$. If the image of each projection is a simplex, then the collection of projections form the maximal simplices of a triangulation of $\mathcal{P}$. If this triangulation agrees with a given triangulation T of $\mathcal{P}$, then T is called a *regular* triangulation.

Figure 1.4 provides examples of triangulations, while Figure 1.5 demonstrates the construction of a regular triangulation.

Example 1.3.5. Let $\tau = \{v_1, \ldots, v_k\}$ be an ordering of the vertices of a lattice polytope $\mathcal{P}$. The *reverse lexicographic triangulation*, or *pulling triangulation*, with respect to τ, which we denote $pull_\tau(\mathcal{P})$ is as follows. Let $\mathcal{F}$ denote the polytopal complex consisting of all faces of $\mathcal{P}$. If $\mathcal{F}$ is a single vertex point v, then $pull_\tau(\mathcal{F}) = \{v\}$. Otherwise,

$$pull_\tau(\mathcal{F}) = pull_\tau(\mathcal{F}\backslash v_i) \cup \left(\bigcup_F \{\operatorname{conv}\{\{v_i\} \cup G\} \,|G \in pull_\tau(\mathcal{F}(F)) \cup \{\emptyset\}\} \right)$$

where the union is over facets of the maximal faces of $\mathcal{F}$ containing v_i, but the facets themselves do not contain v_i. We call τ *compressed* if the pulling triangulation with respect

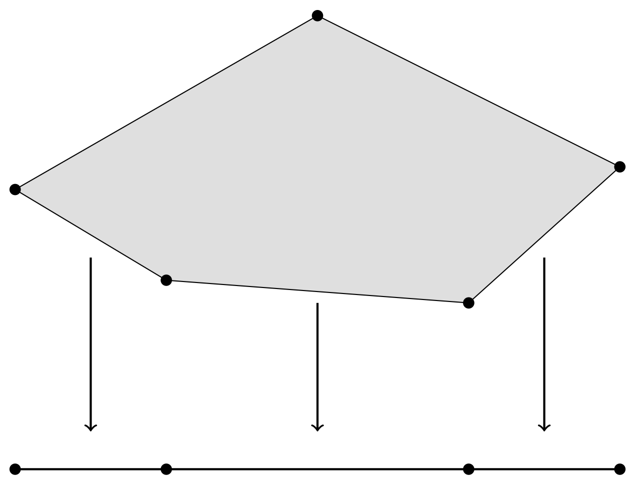

Figure 1.5: A triangulation of an interval of length 4 obtained as a projection of the lower hull of a higher-dimensional polytope

to this ordering is unimodular. If the reverse lexicographic triangulation with respect to every possible τ is unimodular, then we called $\mathcal{P}$ itself *compressed*.

The definitions of regular triangulations and reverse lexicographic triangulations are both fairly cumbersome, but there are miraculous algebraic interpretations of these triangulations that simplify much of the work. More of this perspective will be explored in Section 3.3.

The g-theorem

Triangulations of polytopes are desirable in part because they can be considered combinatorially as geometric realizations of the following structures, which appear frequently in mathematics.

Definition 1.3.6. An *abstract simplicial complex* Δ on the finite vertex set V is a nonempty subset of 2^V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called *faces*, and the *dimension* of a face σ is $|\sigma| - 1$. By convention, $\dim \emptyset := -1$. The *dimension* of a simplicial complex is $\max\{\dim \sigma | \sigma \in \Delta\}$.

Simplicial complexes and their geometric realizations appear in many forms. They are useful in topology since they may be used to approximate a more complicated space, allowing simplified computation of, say, (co)homology. Simplicial complexes may encode structure of posets through the order complex. Stanley examined non-faces of simplicial complexes to study what later became known as Stanley-Reisner rings, which connects the study of faces of simplicial complexes with the Hilbert series of associated quotient algebras.

Definition 1.3.7. For a $(d-1)$-dimensional simplicial complex Δ, let f_i denote the number of i-dimensional faces of Δ. The *f-vector* of Δ is $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$. The *$h$-vector* of Δ is the vector of coefficients $(h_0, \ldots, h_d)$ in the expansion

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{j=0}^{d} h_j x^{d-j}.$$

The h-vector of a simplicial complex is a natural sequence to study because it appears when studying the Hilbert series of the corresponding Stanley-Reisner ring. One of the most well-known results involving the h-vector is a characterization of h-vectors for simplicial polytopes; that is, polytopes whose facets are all simplices.

Theorem 1.3.8 (Billera, Lee [8, 9], Stanley [34])**.** The sequence $(h_0, h_1, \ldots, h_d)$ is the h-vector of a simplicial polytope if and only if

1. $h_i = h_{d-i}$ for all $i = 0, \ldots, \lfloor d/2 \rfloor$, and

2. the sequence $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence.

In particular, the h-vector of a simplicial d-polytope is symmetric and unimodal.

Although we do not discuss M-sequences here, their important property for this discussion is that they are sequences of *nonnegative integers*, hence the unimodality. Thus, one way to prove that an h^*-vector is unimodal is to prove that it is also the h-vector of a simplicial polytope. While this may seem just as difficult, there have been results that make such a verification more tractable.

Definition 1.3.9. Suppose a subset S of vertices of a lattice polytope $\mathcal{P}$ satisfy

1. The set $\operatorname{conv}\{S\}$ is a simplex, and

2. Each facet of $\mathcal{P}$ contains all but one element of S.

Then $\operatorname{conv}\{S\}$ is called a *special simplex.*

Athanasiadis introduced special simplices to prove that the Birkhoff polytope has a unimodal h^*-vector. Along the way, he proved the following theorem.

Theorem 1.3.10 (Athanasiadis [3])**.** Suppose $\mathcal{P}$ is a lattice polytope and $\tau = \{v_k, v_{k-1}, \ldots, v_1\}$ is an ordering of its vertices such that

1. τ is compressed and

2. for some n, $\{v_1, \ldots, v_n\}$ is the vertex set of a special simplex.

Then the h^*-vector of $\mathcal{P}$ is the h-vector of a simplicial polytope.

This theorem was the starting point for what became Theorem 3.2.6, a very useful result for proving unimodality of h^*-vectors. Other methods available involve shelling orders [33], studying the roots of the h^*-polynomial, the γ-vector, or log-concavity of the h^*-vector [10].

Chapter 2 Special Classes of Lattice Polytopes

2.1 Reflexive Polytopes

Definition 2.1.1. A lattice polytope $\mathcal{P}$ is called *reflexive* if $0 \in \mathcal{P}^\circ$ and its *(polar) dual*

$$\mathcal{P}^\Delta := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in \mathcal{P}\}$$

is also a lattice polytope. A lattice translate of a reflexive polytope is also called reflexive.

Reflexive polytopes have been the subject of a large amount of recent research [4, 5, 7, 11, 20, 23, 28, 30]. It is known from work of Lagarias and Ziegler [27] that there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension, with one reflexive in dimension one, 16 in dimension two, 4 319 in dimension three, and 473 800 776 in dimension four according to computations by Kreuzer and Skarke [26]. The number of five-and-higher-dimensional reflexives is unknown. One of the reasons reflexives are of interest is the following.

Theorem 2.1.2 (Hibi, [23])**.** A d-dimensional lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(\mathcal{P})$ satisfies $h^*_i = h^*_{d-i}$ for all $i = 0, \ldots, d$.

Hibi [22] conjectured that every reflexive polytope has a unimodal h^*-vector. Counterexamples to this were found in dimensions 6 and higher by Mustaţă and Payne [28, 30]. However, Hibi and Ohsugi [29] also asked whether or not every normal reflexive polytope has a unimodal h^*-vector; we consider the related question for integrally closed reflexives, where integral closure is defined as follows.

Definition 2.1.3. A lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is *integrally closed* if, for every $x \in m\mathcal{P} \cap \mathbb{Z}^n$, there exist $x_1, \ldots, x_m \in \mathcal{P} \cap \mathbb{Z}^n$ such that $x = x_1 + \cdots + x_m$.

While the terms integrally closed and normal are often used interchangably, these are not synonymous [19]. The counterexamples found by Mustaţă and Payne are not normal, hence not integrally closed. It remains to be seen whether or not every integrally closed reflexive polytope has a unimodal h^*-vector. A stronger open question is whether or not being integrally closed is alone sufficient to imply unimodality [31]. One condition that forces a lattice polytope $\mathcal{P}$ to be integrally closed is if $\mathcal{P}$ admits a unimodular triangulation; the latter condition has been shown to imply unimodality in the reflexive case by Athanasiadis [3] and Bruns and Römer [12].

Construction 2.1.4 (Conrads, [13])**.** Let $Q = (q_0, \ldots, q_n)$ be a sequence of positive, increasing integers such that

$$\gcd(Q) = 1 \text{ and } q_i \left| \sum_{j=0}^{n} q_j \right. \text{ for all } i. \tag{2.1}$$

Furthermore, define

$$\begin{aligned} d_{i \ldots j} &:= \gcd(q_i, \ldots, q_j) \\ d_{i \ldots \hat{k} \ldots j} &:= \gcd(q_i, \ldots, \hat{q_k}, \ldots, q_j) \\ d_i &:= d_{0 \ldots \hat{i} \ldots n}, \end{aligned}$$

where $\hat{q_k}$ means to *not* include q_k. Form the matrix

$$\left(c_{ij}\right) := \begin{pmatrix} \frac{q_0}{d_{01}} & c_{12} & c_{13} & \cdots & c_{1(n-1)} & c_{1n} \\ 0 & \frac{d_{01}}{d_{0\ldots2}} & c_{23} & \cdots & c_{2(n-1)} & c_{2n} \\ 0 & 0 & \frac{d_{0\ldots2}}{d_{0\ldots3}} & \cdots & c_{3(n-1)} & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{d_n}{d_{0\ldots n}} \end{pmatrix},$$

where the c_{ij} are determined in the following way: for fixed $j \in \{2, \ldots, n\}$ recursively construct c_{ij} for $i = j-1, \ldots, 1$ by setting c_{ij} to be the smallest nonnegative integer such that

$$c_{ij} q_i + \sum_{k=i+1}^{j} c_{kj} q_k \in d_{0\ldots i}\mathbb{Z}.$$

Labeling the rows of the above matrix by $r_1, \ldots, r_n$, define one additional vector as

$$r_0 := -\sum_{i=1}^{n} \frac{q_i}{q_0} r_i.$$

Then the vectors $r_0, \ldots, r_n$ form the vertices of the unique reflexive simplex (up to isomorphism) of type $(Q, 1)$.

Ranging over all possible $(n+1)$-tuples, one obtains all reflexive simplices of type $(Q, 1)$. Obtaining reflexive simplices of type (Q, λ) for $\lambda > 1$ requires additional work, and these details are contained in [13].

2.2 A New Construction

We follow the notation of [6] and define the relevant operation on polytopes that we will consider.

Definition 2.2.1. Suppose $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ are lattice polytopes. Call $\mathcal{P} \oplus \mathcal{Q} := \operatorname{conv}\{\mathcal{P} \cup \mathcal{Q}\}$ a *free sum* if, up to unimodular equivalence, $\mathcal{P} \cap \mathcal{Q} = \{0\}$ and the affine spans of $\mathcal{P}$ and $\mathcal{Q}$ are orthogonal coordinate subspaces of $\mathbb{R}^n$.

Example 2.2.2. The Reeve tetrahedron $\mathcal{R}_h$ *cannot* be expressed as a free sum when $h > 1$; if it could, then the lattice generated by $\mathcal{R}_h$ would be $\mathbb{Z}^3$. However, it only generates $\mathbb{Z}^2 \times h\mathbb{Z}$.

Example 2.2.3. The d-cross-polytope, given by $\operatorname{conv}\{e_1, \ldots, e_d, -e_1, \ldots, -e_d\} \subset \mathbb{R}^d$, is a d-fold free sum of $[-1, 1]$.

As with normality and integral closure, one must be cautious when discussing free sums; different authors sometimes use different definitions, and the validity of results may change based on which definition is used. The definition above is useful due to the following result.

Theorem 2.2.4. [6, Corollary 3.4] If $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ are reflexive polytopes such that $0 \in \mathcal{P}^\circ$ and $\mathcal{P} \oplus \mathcal{Q} = \operatorname{conv}\{\mathcal{P} \cup \mathcal{Q}\}$ is a free sum, then

$$h^*_{\mathcal{P}\oplus\mathcal{Q}}(t) = h^*_{\mathcal{P}}(t) h^*_{\mathcal{Q}}(t).$$

Our next proposition provides a method for producing reflexive simplices from pairs of lower-dimensional reflexive simplices.

Proposition 2.2.5. Suppose $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ are full-dimensional simplices with $0 \in \mathcal{P}$ and $\{v_0, \ldots, v_m\}$ denoting the vertices of $\mathcal{Q}$. Then for each $i = 0, 1, \ldots, m$ the polytope formed by

$$\mathcal{P} *_i \mathcal{Q} := \operatorname{conv}\{(\mathcal{P} \times 0^m) \cup (0^n \times \mathcal{Q} - v_i)\} \subseteq \mathbb{R}^{n+m}$$

is a free sum and is itself a simplex. Moreover, if $0 \in \mathcal{P}^\circ$ and $\mathcal{P}$ and $\mathcal{Q}$ are both reflexive, then $\mathcal{P} *_i \mathcal{Q}$ is also reflexive.

Proof. Since each of $\mathcal{P}$ and $\mathcal{Q} - v_i$ are full-dimensional, their affine spans are orthogonal subspaces of $\mathbb{R}^{n+m}$. Moreover, their intersection is 0, so the operation gives a free sum. By Theorem 2.2.4, the denominator of $E_{\mathcal{P}*_i\mathcal{Q}}(t)$ as a rational function is of degree $n+m+1$, so $\dim(\mathcal{P} *_i \mathcal{Q}) = n+m$. Because $\mathcal{P} *_i \mathcal{Q}$ is the convex hull of $n+m+2$ distinct point, but one vertex lies inside $\mathcal{P} \times 0^m$, it can be expressed as a convex hull of at most $n+m+1$ points. Thus $\mathcal{P} *_i \mathcal{Q}$ is a simplex.

Now we assume that both $\mathcal{P}$ and $\mathcal{Q}$ are reflexive. Noting that $E_{\mathcal{Q}-v_i}(t) = E_{\mathcal{Q}}(t)$, Theorem 2.2.4 tells us that the numerator of $E_{\mathcal{P}*_i\mathcal{Q}}(t)$ as a rational function has degree $n+m$. This polynomial also has symmetric coefficients, since it is the product of polynomials that each have symmetric coefficients. A well-known result in Ehrhart theory tells us that the smallest dilate of $\mathcal{P} *_i \mathcal{Q}$ containing an interior lattice point is $\dim(\mathcal{P} *_i \mathcal{Q}) - (n+m-1) = 1$. Thus, by Theorem 2.1.2, the constructed simplex must be reflexive. $\square$

Geometrically, applying this operation to reflexive simplices corresponds to fixing $\mathcal{P}$ and translating $\mathcal{Q}$ so that their intersection point is a vertex of $\mathcal{Q}$ and the unique interior point of $\mathcal{P}$.

An important property of the $*_i$ operation is that, under appropriate constraints, it preserves being integrally closed.

Theorem 2.2.6. If $\mathcal{P}$ and $\mathcal{Q}$ are any integrally closed simplices with $0 \in \mathcal{P}^\circ$ and $\mathcal{P}$ reflexive, then $\mathcal{P} *_i \mathcal{Q}$ is integrally closed.

Proof. Since $\mathcal{P} *_i \mathcal{Q}$ is a free sum, we may assume that $\mathcal{P}$ and $\mathcal{Q}$ intersect at the origin with $\mathcal{P} \subseteq \mathbb{R}^n \times 0^m$ and $\mathcal{Q} \subseteq 0^n \times \mathbb{R}^m$.

By definition, the convex hull of $\mathcal{P}$ and $\mathcal{Q}$ is the set of points representable as

$$\sum_{i=1}^{r} \alpha_i p_i + \sum_{j=1}^{s} \beta_j q_j$$

where $p_i \in \mathcal{P}, q_j \in \mathcal{Q}$ for each i, j, and the α_i, β_j are nonnegative numbers whose total sum is 1. Form the points

$$u = \frac{1}{\sum_{j=1}^{r} \alpha_j} \left(\sum_{i=1}^{r} \alpha_i p_i \right), v = \frac{1}{\sum_{k=1}^{s} \beta_k} \left(\sum_{l=1}^{s} \beta_l q_l \right).$$

Then $u \in \mathcal{P}$ and $v \in \mathcal{Q}$. Setting $t = \sum_{i=1}^{r} \alpha_i$, their convex sum

$$\left(\sum_{i=1}^{r} \alpha_i \right) u + \left(\sum_{j=1}^{s} \beta_j \right) v = tu + (1-t)v$$

is in $\mathcal{P} \oplus \mathcal{Q}$, and, in particular, is in $t\mathcal{P} \times (1-t)\mathcal{Q}$. Therefore the free sum is covered by sets of this form for $0 \leq t \leq 1$.

For the last step, let $(p, q) \in t\mathcal{P} \times (m-t)\mathcal{Q}$ where m is a positive integer, $p \in t\mathcal{P}$, and $q \in (m-t)\mathcal{Q}$. Since $\mathcal{P}$ is reflexive, p lies on the boundary of some integer scaling of $\mathcal{P}$, thus we may assume t is an integer. Hence q is an integer scaling of $\mathcal{Q}$. By the integral closure of $\mathcal{P}$ and $\mathcal{Q}$, there are t lattice points of $\mathcal{P}$ summing to p and $m-t$ lattice points of $\mathcal{Q}$ summing to q. These summands are all contained in $\mathcal{P} *_i \mathcal{Q}$, hence it is integrally closed. □

This brings us to our main observation.

Corollary 2.2.7. If $\mathcal{P}$ and $\mathcal{Q}$ are integrally closed, reflexive simplices with $0 \in \mathcal{P}^\circ$, then so is $\mathcal{P} *_i \mathcal{Q}$ for each i. If, in addition, $h^*(\mathcal{P})$ and $h^*(\mathcal{Q})$ are unimodal, then so is $h^*(\mathcal{P} *_i \mathcal{Q})$.

Proof. Integral closure follows from Theorem 2.2.6, and reflexivity follows from Proposition 2.2.5. By Theorem 2.2.4 and [35, Proposition 1], which states that the product of two polynomials with symmetric unimodal coefficients has these same properties, the last claim holds. □

We end this section by noting that the conclusions of Proposition 2.2.5 and Theorem 2.2.6 still hold when "simplex" is replaced with "polytope;" adaptations of their proofs are straightforward. However, there is no classification for arbitrary reflexive polytopes by type in the manner that we discuss in the next section. Regardless, this gives one reason why a reflexive polytope may have a unimodal h^*-vector. We remark that it is not clear what the relationship is between polytopes formed when using the $*_i$ construction on different vertices of the second operand, and it is not easy to identify geometrically that a reflexive polytope decomposes as a free sum.

2.3 Searching for Non-unimodal Examples

If one wishes to search for an example of an integrally closed, reflexive polytope with a non-unimodal h^*-vector, then it is natural to first rule out those polytopes obtained as a result of Corollary 2.2.7. As mentioned earlier, reflexive simplices are a class one might focus on when searching for such a polytope. It is helpful in this case to consider how Construction 2.1.4 interacts with the free sum operation.

Given a reflexive simplex, the type $Q = (q_0, \ldots, q_n)$ can be found by setting

$$q_i = \left| \det \begin{pmatrix} | & | & \cdots & | & \cdots & | \\ v_0 & v_1 & \cdots & \widehat{v_i} & \cdots & v_n \\ | & | & \cdots & | & \cdots & | \end{pmatrix} \right| .$$

Note that reordering Q corresponds to performing this same process to a unimodularly equivalent simplex. Thus, we may assume that Q is nondecreasing. Setting $\lambda = \gcd(q_0, \ldots, q_n)$ and $Q_{red} = \frac{1}{\lambda}Q$, the reflexive simplex will have type (Q_{red}, λ). Recall that simplices of type $(Q, 1)$ are exactly those that satisfy the conditions of Construction 2.1.4, and denote these simplices by Δ_Q. The remaining simplices are found by performing various additional operations on the Δ_Q.

For any reflexive simplex, we call Q_{red} the *reduced weight* of the simplex, and a simplex with this reduced weight has the property that

$$\sum_i \frac{q_i}{\sum_\beta q_\beta} v_i = 0 \, .$$

This follows from scaling the equality

$$\sum_i q_i v_i = 0 \, ,$$

which itself follows from Cramer's rule. Note that because there are $n+1$ of the v_i's in n-dimensional space, the coefficients of the above sum are uniquely determined up to scaling. Thus, the Q_{red} vector of a reflexive simplex is the particular choice of coefficients for this sum that satisfies the divisibility condition (2.1).

Example 2.3.1. The weight $Q = (1, 1, 1, \ldots, 1) \in \mathbb{Z}^{n+1}$ corresponds to the polytope

$$\Delta_Q = \operatorname{conv}\left\{ e_1, \ldots, e_n, -\sum_i e_i \right\},$$

which is often called the *standard reflexive simplex of minimal volume.* Note that the sum of these vertices, each weighted by 1, is equal to zero. It is well known that one can demonstrate that this polytope is integrally closed by showing that it has a unimodular triangulation, specifically the triangulation whose facets consist of those simplices that are the convex hull of the origin and all but one of the vertices of Δ_Q.

This $*_i$ operation has a corresponding interpretation in terms of the types of the summands.

Theorem 2.3.2. If $\mathcal{P} = \operatorname{conv}\{v_0, \ldots, v_n\} \subseteq \mathbb{R}^n$ and $\mathcal{Q} = \operatorname{conv}\{w_0, \ldots, w_m\} \subseteq \mathbb{R}^m$ are full-dimensional reflexive simplices of types $((p_0, \ldots, p_n), \lambda)$ and $((q_0, \ldots, q_m), \mu)$, respectively, then $\mathcal{P} *_i \mathcal{Q}$ is a reflexive simplex of type

$$\left(\frac{1}{d}(q_i p_0, q_i p_1, \ldots, q_i p_n, s q_0, s q_1, \ldots, \widehat{s q_i}, \ldots, s q_m), d \right),$$

where $s = \sum_{j=0}^{n} p_j$ and $d = \gcd(q_i, \sum_{j=0}^{n} p_j)$.

Proof. For notational convenience, we identify $\mathcal{P}$ and $\mathcal{Q}$ with their embeddings in $\mathbb{R}^{n+m}$. Before the embedding, we know from the weights of $\mathcal{P}$ and $\mathcal{Q}$ that

$$\sum_{j=0}^{n} \frac{p_j}{\sum p_\alpha} v_j = 0 \text{ and } \sum_{k=0}^{m} \frac{q_k}{\sum q_\beta} w_k = 0\,.$$

After the embedding, the translation of $\mathcal{Q}$ in $\mathbb{R}^{n+m}$ results in

$$\sum_{k=0}^{m} \frac{q_k}{\sum q_\beta}(w_k - w_i) = -w_i\,.$$

Therefore, on the vertices of the free sum, we see

$$\begin{aligned} -w_i &= \frac{q_i}{\sum q_\beta}(w_i - w_i) + \sum_{\substack{k=0\\k\neq i}}^{m} \frac{q_k}{\sum q_\beta}(w_k - w_i) \\ &= \sum_{j=0}^{n} \left(\frac{q_i}{\sum q_\beta} \cdot \frac{p_j}{\sum p_\alpha} \right) v_j + \sum_{\substack{k=0\\k\neq i}}^{m} \frac{q_k}{\sum q_\beta}(w_k - w_i)\,, \end{aligned}$$

giving us the unique interior point of the simplex. Thus, Q_{red} for $\mathcal{P} *_i \mathcal{Q}$ is given by a scaling of the vector

$$\left(\frac{q_i}{\sum q_\beta} \cdot \frac{p_0}{\sum p_\alpha}, \frac{q_i}{\sum q_\beta} \cdot \frac{p_1}{\sum p_\alpha}, \ldots, \frac{q_i}{\sum q_\beta} \cdot \frac{p_n}{\sum p_\alpha}, \frac{q_0}{\sum q_\beta}, \frac{q_1}{\sum q_\beta}, \ldots, \widehat{\frac{q_i}{\sum q_\beta}}, \ldots, \frac{q_m}{\sum q_\beta} \right).$$

Scaling this vector by $(\sum p_\alpha)(\sum q_\beta)$ and dividing by $\gcd(q_i, \sum_{j=0}^{n} p_j)$, we obtain an integer vector that satisfies (2.1). Thus, this is our desired Q_{red}. To find the full Q vector for $\mathcal{P} *_i \mathcal{Q}$, we first translate the polytope by w_i so that the interior vertex is zero, then compute determinants as described at the beginning of the section. Since the determinant of the matrix formed by $v_1 + w_i, v_2 + w_i, \ldots, v_n + w_i, w_0, w_1, \ldots, \widehat{w_i}, \ldots, w_m$ (where all vectors are considered to be embedded in $\mathbb{R}^{n+m}$) is equal to $q_i p_0$, this determines the type vector for $\mathcal{P} *_i \mathcal{Q}$, and completes our proof. $\square$

Thus, one way to search for examples of integrally closed reflexive simplices with non-unimodal h^*-vectors is to generate Q-vectors for the polytopes, then reduce the Q-vectors under consideration using Theorem 2.3.2 before testing Δ_Q for integral closure and unimodality. This operation is particularly helpful when a simplex has type $(Q_{red}, 1)$, since it is the only simplex of that type. For example, $\Delta_{(1,1,2)}$ can be decomposed as $\Delta_{(1,1)} *_0 \Delta_{(1,1)}$, since we know the $*_0$ operation provides a reflexive simplex of type $((1,1,2),1)$, and there is only one of this type. However, there may be multiple simplices of type (Q_{red}, λ) when $\lambda > 1$, no longer guaranteeing that a simplex decomposes in a particular way. An example would be $((1,2,3,3,9),2)$; there are two simplices of this type, but only one of them can be of the form $\Delta_{(1,2,3)} *_1 \Delta_{(1,2,3)}$. In this case, more checks are needed to identify which simplex decomposes as a free sum.

Unfortunately, while the free sum operation produces a large number of reflexive polytopes, it appears that these might be rare among the reflexive polytopes with unimodal

h^*-vectors. For example, when we randomly generated 1100 eight-dimensional integrally-closed reflexive simplices, all of them had unimodal h^*-vectors, yet none of their type vectors split in the manner given in Theorem 2.3.2. Given that there are nearly 500 million reflexive polytopes of dimension 4, and the number of reflexive polytopes in higher dimensions is unknown, 1100 in dimension 6 is a very small sample size. In dimension 6 there are 2676 reflexive simplices of type $(Q, 1)$ alone (verified experimentally), so 1100 such simplices in dimension 8 is a small sample even among simplices. It would be interesting to know more about the reflexive simplices formed via the free sum operation in comparison to the family of all reflexive simplices.

2.4 The Weak Lefschetz Property

In this section, we show that a natural approach inspired by commutative algebra fails to establish unimodality for integrally closed reflexive simplices in general. For any lattice simplex $\mathcal{P} \subseteq \mathbb{R}^n$ with vertices $\{v_0, \ldots, v_n\}$, recall that there is an associated semigroup algebra given by

$$\mathbb{C}[\mathcal{P}] := \mathbb{C}[x^a z^m | a \in m\mathcal{P} \cap \mathbb{Z}^n]$$

where $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. The Ehrhart series $E_{\mathcal{P}}(t)$ coincides with the Hilbert series of $\mathbb{C}[\mathcal{P}]$. Thus, the numerator of $E_{\mathcal{P}}$ corresponds to the Hilbert series of $\mathbb{C}[\mathcal{P}]/J$, where $J = (\theta_0, \ldots, \theta_n)$ is a 0-dimensional ideal whose generators are degree 1 – more specifically, J is a linear system of parameters (l.s.o.p.). There is a canonical choice of J when $\mathcal{P}$ is a simplex,

$$R_{\mathcal{P}} := \mathbb{C}[\mathcal{P}]/(x^{v_0} z, \ldots, x^{v_n} z) \ ,$$

graded by the exponent on z. Here, h_k^* is equal to the number of lattice points satisfying $\sum c_i = k$ in the *fundamental parallelepiped* $\Pi(\mathcal{P})$ defined by

$$\Pi(\mathcal{P}) := \left\{ \sum_{i=0}^{n} c_i (v_i, 1) \middle| 0 \leq c_i < 1 \right\} \subset \mathbb{R}^{n+1} \ .$$

The study of Hilbert functions gives a method for establishing unimodality of $h^*(\mathcal{P})$ in this context.

Definition 2.4.1. A linear form $l \in R_{\mathcal{P}}$ is called a *weak Lefschetz element* if the multiplication map

$$\times l : [R_{\mathcal{P}}]_i \to [R_{\mathcal{P}}]_{i+1}$$

has maximal rank, that is, is either injective or surjective, for each i.

By Remark 3.3 of [21], if $R_{\mathcal{P}}$ has a weak Lefschetz element, then the Hilbert series has unimodal coefficients in its numerator, and therefore so does $E_{\mathcal{P}}(t)$. Experimental data suggests that a weak Lefschetz element exists for many instances of $R_{\mathcal{P}}$ when $\mathcal{P}$ is an integrally closed reflexive simplex, but we will now show that such an element need not exist using the choice of J above.

Proposition 2.4.2. For every $d \geq 3$, there exists a d-dimensional integrally closed reflexive simplex Δ_Q such that R_{Δ_Q} does not admit a weak Lefschetz element when $J = (x^{v_0}z, \ldots, x^{v_n}z)$.

Proof. For fixed d, let $Q = (1, d, \underbrace{d+1, \ldots, d+1}_{d-1 \text{ times}})$. Then Q defines a reflexive simplex

$$\Delta_Q = \operatorname{conv}\left\{e_1, \ldots, e_d, (-d, -d-1, \ldots, -d-1)^T\right\}.$$

Consider the cone consisting of all rays from the origin though a point in $(\Delta_Q, 1) \subseteq \mathbb{R}^{d+1}$. Elements of this cone with last coordinate m are in bijection with points of $m(\Delta_Q)$ by projection onto the first d coordinates. Additionally, the cone has hyperplane description given by $Ax \geq 0$, where A is the $(d+1) \times (d+1)$ matrix

$$\frac{1}{d(d+1)} \begin{pmatrix} d^2 & -d & -d & \cdots & -d & d \\ -d-1 & d^2-1 & -d-1 & \cdots & -d-1 & d+1 \\ -d-1 & -d-1 & d^2-1 & \cdots & -d-1 & d+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d-1 & -d-1 & -d-1 & \cdots & d^2-1 & d+1 \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

Thus, there are $d(d+1)$ lattice points in $\Pi(\Delta_Q)$. For each $r \in \{1, \dots, d-1\}$, form the vectors

$$\begin{aligned}
v_{0,r} &= (0, 0, \dots, 0, r) \\
v_{1,r} &= (-1, -1, \dots, -1, r) \\
\vdots \quad & \quad \vdots \\
v_{r-1,r} &= (-r+1, -r+1, \dots, -r+1, r) \\
v_{r,r} &= (-r, -r, \dots, -r, r) \\
v_{r+1,r} &= (-r+1, -r, \dots, -r, r) \\
v_{r+2,r} &= (-r, -r-1, \dots, -r-1, r) \\
\vdots \quad & \quad \vdots \\
v_{d,r} &= (-d+2, -d+1, \dots, -d+1, r) \\
v_{d+1,r} &= (-d+1, -d, \dots, -d, r)
\end{aligned}$$

There are $d+2$ of these for each r, and along with the zero vector and $(-d+1, -d, \dots, -d, d)$ we have $(d-1)(d+2)+2 = d(d+1)$ total vectors, which we claim to be all of the lattice points in $\Pi(\Delta_Q)$.

To make this easier, we first show that Δ_Q is integrally closed. Observe that every vector $v_{i,r}$ can be written as a sum of vectors $v_{j,r-1} + v_{k,1}$ in the following way. We assume $r \geq 2$. When $i > r$, we may let $j = i$ and $k = 0$; when $i < r$ we may let $j = i$ and $k = 1$; when $i = r$ we may use $j = r-1$ and $k = 1$. Thus, by induction on r, every lattice point in $\Pi(\Delta_Q)$ is a sum of elements satisfying $r = 1$. Since every lattice point in the cone over Δ_Q is a sum of lattice points that are either ray generators or fundamental parallelepiped points, we conclude that Δ_Q is integrally closed.

To see that the lattice points of $\Pi(\Delta_Q)$ are precisely those described above, we show that all are obtained as a linear combination of the ray generators with coefficients less than one. It is straightforward using the matrix above to show that all coefficients of the ray generators are less than $\frac{1}{d}$ when representing the lattice points in $\Pi(\Delta_Q)$ with $r = 1$; integral closure then ensures that the coefficients of all points for $r \in \{2, \dots, d-1\}$ will be bounded by $\frac{d-1}{d}$. Then one only needs to check the coefficients on the vector $(-d+1, -d, \dots, -d, d)$. These verifications are also straightforward, and the details are omitted.

Now we must verify that no potential weak Lefschetz element is injective from $[R_{\Delta_Q}]_1$ to $[R_{\Delta_Q}]_2$. A weak Lefschetz element would be of the form

$$\sum_{i=0}^{d+1} a_i x^{v_{i,1}} z$$

where a_i are field elements. The map from $[R_{\Delta_Q}]_1$ to $[R_{\Delta_Q}]_2$ induced by multiplication by this element is representable as the matrix

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_4 & a_3 & 0 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ a_5 & a_4 & 0 & 0 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_d & a_{d-1} & 0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\ a_{d+1} & a_d & 0 & 0 & 0 & \cdots & 0 & a_1 & a_0 \end{pmatrix}$$

where the columns are indexed by the degree 1 elements in the order $v_{0,1}, \ldots, v_{d+1,1}$ and similarly for the rows with the degree 2 elements. This is a triangular matrix with a zero on the diagonal, so it cannot have full rank regardless of the values of the a_i. Therefore, the map is not injective and there is no weak Lefschetz element in R_{Δ_Q}. □

Despite the non-existence of a weak Lefschetz element for this choice of J, the h^*-vectors of these simplices are easily computed and found to be of the form $(1, d+2, d+2, \ldots, d+2, 1)$. Thus, unimodality still holds for this family, indicating that unimodality, if it holds in general for integrally closed reflexive simplices, is a consequence of some subtle properties of these polytopes.

It is very important to note that although there was not a weak Lefschetz element of the quotient of $\mathbb{C}[\mathcal{P}]$ by $(x^{v_0}z, \ldots, x^{v_n}z)$, there may very well be a weak Lefschetz element when taking the quotient by different choices of J. For example, the choice $J = (x^{v_0}z, \ldots, x^{v_n}z + z)$ appears to induce algebras R_{Δ_Q} with the weak Lefschetz property when $Q = (1, d, d +$

$1, \ldots, d+1$). Indeed, experimental data suggests that when J is generated by generic linear forms, a weak Lefschetz element *always* exists. More information about experimental data is discussed in Chapter 4.

Chapter 3 A Return to Magic Squares

3.1 Background

Since the 1950s, much has become known about the Ehrhart theory of the Birkhoff polytope, B_n. Stanley [32] was able to prove that B_n has the h^*-vector of a reflexive polytope, hence $h^*(B_n)$ is symmetric. Later, Athanasiadis introduced in [3] the notion of a "special simplex" embedded in a polytope. Using this idea, he was able to prove that the h^*-vector of B_n is the same as the combinatorial h-vector for the boundary of a simplicial polytope. The g-theorem proves that these vectors are unimodal.

However, little is known about the polytope Σ_n obtained by intersecting B_n with the hyperplanes $x_{ij} = x_{ji}$ for all i, j, that is, by requiring the corresponding matrices to be symmetric. Nothing is new when $n \leq 2$, but complications arise once $n \geq 3$ since the vertices of Σ_n are no longer always integral. They are contained in the set

$$L_n = \left\{ \frac{1}{2}(P + P^T) | P \in \mathbb{R}^{n\times n} \text{ is a permutation matrix} \right\},$$

but L_n is not necessarily equal to the vertex set of Σ_n. When a polytope has rational vertices, the Ehrhart series of the polytope, while still able to be defined, comes in a different form.

Let den $\mathcal{P}$ denote the least common multiple of the denominators appearing in the coordinates of the vertices of $\mathcal{P}$. Then there exist values $h_0^*, \ldots, h_k^* \in \mathbb{Z}_{\geq 0}$ with $h_0^* = 1$ such that

$$E_{\mathcal{P}}(t) = \frac{\sum_{j=0}^k h_j^* t^j}{(1 - t^{\text{den } \mathcal{P}})^{d+1}} .$$

Although $E_{\mathcal{P}}(t)$ may have cancellation when written as a rational function, we refer to its h^*-vector as the coefficients in the unreduced form. A description of the vertices and a generating function for the number of them can be found in [37]. The h^*-vector of Σ_n is known to be symmetric [36] and $E_{\Sigma_n}(t)$ has been computed in a reduced form for some small n [38], but it is still unknown whether the h^*-vector is always unimodal in this case.

Definition 3.1.1. Denote by S_n the polytope containing all real $n \times n$ symmetric matrices with nonnegative entries such that every row and column sum is 2. That is, S_n is the dilation of Σ_n by two.

Fortunately, some information about Σ_n is retained by S_n, a polytope that is combinatorially equivalent but with integral vertices. Figure 3.1 provides a realization of S_3 as a three-dimensional polytope. This is the first n for which S_n differs from the second scaling of the Birhkoff polytope itself, and is the largest n which has a realization in three dimensions or fewer.

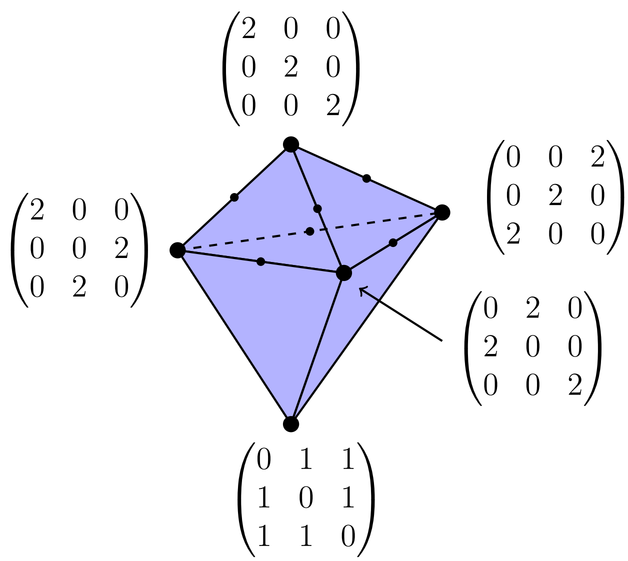

Figure 3.1: Second dilate of the polytope of real doubly-stochastic symmetric matrices.

A common way to prove that the h^*-vector of a polytope $\mathcal{P}$ is unimodal is to examine the toric ideal $I_{\mathcal{P}}$ of the polytope and Gröbner bases of the toric ideals, notions which will be discussed more precisely in Section 3.3. The main purpose of this chapter is to examine what happens when trying to prove that $h^*(S_n)$ is unimodal by adapting the techniques used to prove that $h^*(B_n)$ is unimodal. Key ingredients of proving unimodality of $h^*(B_n)$ are that B_n is integrally closed and that for a certain class of orderings on the variables of I_{B_n}, the initial ideal of I_{B_n} is generated by squarefree monomials. In this direction, we will show the following.

Theorem 3.1.2. For all n, let I_{S_n} denote the toric ideal of S_n. The following properties hold:

1. For any term ordering, every element of the reduced Gröbner basis $\mathcal{G}$ of I_{S_n} with respect to this order consists of binomials, one monomial of which is squarefree.

2. For any term ordering, every variable in I_{S_n} appears in a degree-two binomial in $\mathcal{G}$.

3. There exists a class of term orders $\prec_{S_n}$ for which the initial term of each degree-two binomial in $\mathcal{G}$ is squarefree.

4. For the term orders $\prec_{S_n}$, the initial term $\text{in}_{\prec_{S_n}}(g)$ of each $g \in \mathcal{G}$ is cubefree; that is, $\text{in}_{\prec_{S_n}}(g)$ is not divisible by t_i^3 for any variable t_i appearing in g.

3.2 Basic Properties, Symmetry, and Integral Closure

Although relatively little has been established about the Ehrhart theory of S_n, it has still been studied and some basic information is known. For Σ_n, the degrees of the constituent polynomials of its Ehrhart quasipolynomial are known.

Theorem 3.2.1 (Theorem 8.1, [24]). The Ehrhart quasipolynomial of Σ_n is of the form $f_n(t) + (-1)^t g_n(t)$, where $\deg f(t) = \binom{n}{2}$ and

$$\deg g_n(t) = \begin{cases} \binom{n-1}{2} - 1 & \text{if } n \text{ odd} \\ \binom{n-2}{2} - 1 & \text{if } n \text{ even} \end{cases}.$$

Stanley first proved that the above degrees are upper bounds and conjectured equality [32], and the conjecture was proven using analytic methods. These degrees provide an upper bound on the degree of $h^*_{\Sigma_n}(t)$; we will provide exact degrees later. Since the Ehrhart series of S_n, as a formal power series, consists of the even-degree terms of the monomials appearing in $E_{\Sigma_n}(t)$, we get $\mathcal{L}_{S_n}(t) = f_n(2t) + g_n(2t)$.

The defining inequalities of our polytopes will be helpful in some contexts. For S_n, these are

$$\begin{aligned} x_{ij} &\geq 0 \text{ for all } 1 \leq i \leq j \leq n, \\ x_{ij} &= x_{ji} \text{ for all } 1 \leq i < j \leq n, \\ \sum_{i=1}^{n} x_{ij} &= 2 \text{ for each } j = 1, \ldots, n. \end{aligned}$$

The first set of inequalities provided indicate that the facet-defining supporting hyperplanes of S_n are $x_{ij} = 0$: if any of these are disregarded, the solution set strictly increases in size.

Additionally, by knowing that the vertices of B_n are the permutation matrices, it follows very quickly that B_n is integrally closed. It is not so simple for S_n, since it contains lattice points that are not vertices. In this case, it is helpful to interpret the lattice points of S_n as certain incidence matrices of graphs.

Proposition 3.2.2. For all n, S_n is integrally closed.

Proof. This result can be seen as a corollary of Petersen's 2-factor theorem. For any $m \in \mathbb{Z}_{\geq 0}$, each lattice point $X = (x_{ij}) \in mS_n$ can be interpreted as the incidence matrix of an undirected m-regular multigraph G_X on distinct vertices $v_1, \ldots, v_n$, with loops having degree 1. We first observe that the total number of loops will be even: if there were an odd number of loops, consider the graph with the loops removed. The sum of degrees of the vertices in the resulting graph would be odd, which is an impossibility.

Denote by $V_{odd}(G_X)$ the vertices of G_X with an odd number of loops, and write $|V_{odd}(G_X)| = 2mt + s$, where t, s are nonnegative integers and $s < 2m$. Note in particular that s will be even. Construct a new graph G_Y with vertex set $V(G_Y) = \{v_1, \ldots, v_n, w_0, w_1, \ldots, w_t\}$ with the same edges as in G_X with the following modifications:

1. For each $v_i \notin V_{odd}(G_X)$, v_i will have $\frac{1}{2}x_{ii}$ loops in G_Y.

2. For each $v_i \in V_{odd}(G_X)$, v_i will have $\frac{1}{2}(x_{ii} - 1)$ loops and an edge between v_i and the lowest-indexed w_j such that $\deg w_j < 2m$.

3. Vertex w_t will have $\frac{1}{2}(2m - s)$ loops.

This new graph will be $2m$-regular, now counting loops as degree 2; see Figure 3.2 for an example on four vertices.

Thus, by Petersen's 2-factorization theorem, G_Y can be decomposed into 2-factors. Hence the matrix Y corresponding to G_Y will decompose as the sum of $Y_1, \ldots, Y_m$, each summand a lattice point of mS_{n+t+1}.

Now we must "undo" the changes we made to G_X to obtain the desired sum. Index the rows and columns by $\{v_1, \ldots, v_n, w_0, w_1, \ldots, w_t\}$. Each edge $v_i w_j$ will appear in some Y_k as

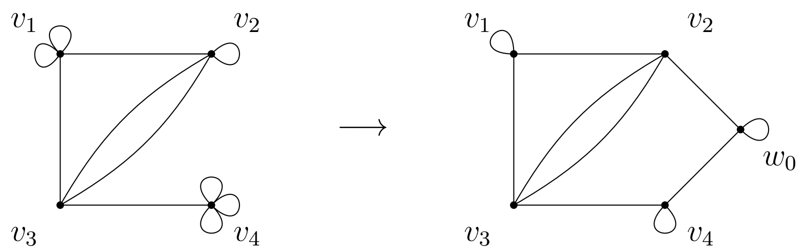

Figure 3.2: Altering a graph to be $2m$-regular

a 1 in positions (v_i, w_j) and (w_j, v_i). Replace these entries with 0 and add 1 to entry (v_i, v_i). Denote by X_k the submatrix of Y_k consisting of rows and columns indexed by $v_1, \dots, v_n$ after any appropriate replacements have been made. Each replacement preserves the sum of row/column v_i, and applying this to each Y_k leaves any entry (v_i, w_j) as 0, so each X_k is a lattice point of S_n. Thus $X = \sum X_k$, as desired. $\square$

A second necessary ingredient in proving that $h^*(B_n)$ is unimodal is proving that it has the following property.

Definition 3.2.3. For a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denote by $k[\mathcal{P}]$ the semigroup algebra

$$k[\mathcal{P}] := k[x^a z^m | a \in m\mathcal{P} \cap \mathbb{Z}^{n+1}] \subseteq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}, z].$$

Then $\mathcal{P}$ is called *Gorenstein* if $k[\mathcal{P}]$ is Gorenstein. More specifically, $\mathcal{P}$ is *Gorenstein of index* r if there exists a (necessarily unique) monomial $x^c z^r$ for which

$$k[\mathcal{P}^\circ] \cong (x^c z^r) k[\mathcal{P}].$$

Gorenstein polytopes generalize reflexive polytopes; reflexive polytopes are exactly the Gorenstein polytopes of type 1. Moreover, they retain the symmetry of the h^*-vector.

Theorem 3.2.4 (see [36])**.** The h^*-vector of a lattice polytope $\mathcal{P}$ is symmetric if and only if $\mathcal{P}$ is Gorenstein.

It is worth noting that reflexive polytopes were not introduced as special cases of Gorenstein polytopes. This fact was only determined later, leading to Theorem 2.1.2

Having the hyperplane description of a polytope can make it easier to determine if it is Gorenstein, as evidenced by the following lemma.

Lemma 3.2.5 (Lemma 2(ii), [12])**.** Suppose $\mathcal{P}$ has irredundant supporting hyperplanes $l_1, \ldots, l_s \geq 0$, where the coefficients of each l_i are relatively prime integers. Then $\mathcal{P}$ is Gorenstein (of index r) if and only if there is some $c \in r\mathcal{P} \cap \mathbb{Z}^n$ for which $l_i(c) = 1$ for all i.

Generally, proving the unimodality of an h^*-vector is a challenging task. There are more techniques available, though, if we have a Gorenstein polytope; that is, if the semigroup algebra $k[\mathcal{P}]$ is Gorenstein.

Lemma 3.2.6 (Corollary 7, [12])**.** Suppose $\mathcal{P} \subseteq \mathbb{R}^n$ is an integrally closed Gorenstein polytope with irredundant, integral supporting hyperplanes $l_1, \ldots, l_s$, and contains lattice points $v_0, \ldots, v_k$. If these points form a k-dimensional simplex and $l_i(v_0 + \cdots + v_k) = 1$ for each i, then $\mathcal{P}$ projects to an integrally closed reflexive polytope $\mathcal{Q}$ of dimension $n - k$ with equal h^*-vector.

Theorem 3.2.7. S_n is Gorenstein if and only if n is even. When $n = 2k$, S_n is Gorenstein of type k, and $h^*(S_n)$ is the h^*-vector of a reflexive polytope of dimension $2k^2 - 2k + 1$. Hence, $\deg h^*_{S_n}(t) = 2k^2 - 2k + 1$.

Proof. By Lemma 3.2.5 and knowing the facet description of S_n, we can see that the polytope is Gorenstein by choosing integer matrices of S_n whose sum is the all-ones matrix. When n is odd, this is impossible: such a matrix has an odd line sum, whereas any sum of matrices in S_n has even line sum.

Let $n = 2k$. For each $i \in \{1, 2, \ldots, k-1\}$, construct a matrix

$$\begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ a_{n-1} & a_0 & a_{n-1} & \cdots & a_3 & a_2 \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_4 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}$$

by setting $a_i = a_{n-i} = 1$ and $a_j = 0$ for all $j \neq i$. Construct one additional matrix by setting $a_0 = a_k = 1$ and $a_j = 0$ for all $j \neq 0, k$. Each of the k matrices are symmetric and have pairwise disjoint support by construction. These are therefore vertices of a simplex of dimension $k-1$, and Lemma 3.2.6 provides the reflexivity result. $\square$

Note that this is not the only class of simplices satisfying the conditions of Lemma 3.2.5 contained in S_n for even n; others may be found. It may be interesting to ask how many such distinct simplices in S_n exist.

Example 3.2.8. For $n = 6$, we construct the special simplex described above. It has three vertices, which are

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3.2.9. If $n = 2k+1$, then the first scaling of S_n containing interior lattice points is $\left(\frac{n+1}{2}\right) S_n$. Specifically, the number of interior lattice points in this scaling is the number of symmetric permutation matrices; i.e. the number of involutions of the set $\{1, 2, \ldots, n\}$. Thus, $\deg h^*_{S_n}(t) = 2k^2$.

Proof. For an interior point, each matrix entry must be positive. However, the matrix of all 1s does not work since this results in an odd line sum. Thus there must be a 2 in each row and column as well. Thus by subtracting the all-1s matrix, each lattice point corresponds to a symmetric permutation matrix, that is, an involution. The line sum for the interior lattice points will be $n+1$, and we remember that the line sums of matrices in S_n is 2.

By Theorem 1.5 of [33],

$$E_{(S_n)^\circ}(t) = (-1)^{\binom{n}{2}} E_{S_n}\left(\frac{1}{t}\right).$$

When expanded as a power series, the lowest-degree term will be $t^{(\binom{n}{2}+1)-d}$, where $d = \deg h^*_{S_n}(t)$. The degree of $h^*_{S_n}(t)$ follows. $\square$

With this information, we can deduce the degrees of $h^*_{\Sigma_n}(t)$ for each n.

Proposition 3.2.10. For all n, $h^*(S_n)$ consists of the even-indexed entries of $h^*(\Sigma_n)$. Thus, if n is even, then $\deg h^*_{\Sigma_n}(t) = 2(\deg h^*_{S_n}(t))$, and if n is odd, then $\deg h^*_{\Sigma_n}(t) = 2(\deg h^*_{S_n}(t))+1$.

Proof. As power series, the coefficient of t^m in $E_{S_n}(t)$ is the same as the coefficient of t^{2m} in $E_{\Sigma_n}(t)$. Recalling Theorem 3.2.1, this gives

$$E_{\Sigma_n}(t) = E_{S_n}(t^2) + t\sum_{m\geq 0} f(m)t^{2m}$$

for some polynomial f. So, as rational functions, the first summand of the above will have entirely even-degree terms in the numerator and the same denominator as the rational form of $E_{\Sigma_n}(t)$. Thus, the second summand, when written to have a common denominator as the first summand, will have entirely odd-degree terms in its numerator. Therefore, $h^*(S_n)$ consists of the even-indexed entries of $h^*(\Sigma_n)$.

Since $h^*(S_n)$ is symmetric for even n only, and by Proposition 3.2.9, the degrees of $h^*(\Sigma_n)$ follow. $\square$

3.3 Toric Ideals and Regular, Unimodular Triangulations

For a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ let $\mathcal{P} \cap \mathbb{Z}^n = \{a_1, \dots, a_s\}$. We define the *toric ideal* of $\mathcal{P}$ to be the kernel of the map

$$\pi : T_{\mathcal{P}} = k[t_1, \dots, t_s] \to k[\mathcal{P}],$$

where $\pi(t_i) = (\prod x^{a_i})\, z$, using the multivariate notation. This ideal we denote $I_{\mathcal{P}}$. Because the lattice points of S_n correspond to matrices, it will sometimes be more convenient to use the indexing

$$T_{S_n} = k[t_A | A \in S_n \cap \mathbb{Z}^{n\times n}] \text{ and } k[S_n] = k[x^A z^m | A \in mS_n \cap \mathbb{Z}^{n\times n}],$$

where we now use

$$x^A z^m = \prod_{0\leq i,j\leq n} x_{ij}^{a_{i,j}} z^m$$

with $A = (a_{i,j})$. Thus $\pi : T_{S_n} \to k[S_n]$ is given by $\pi(t_M) = x^M z$.

The toric ideal of a polytope has been widely studied, in large part for its connections to triangulations of the polytope. Various properties of the initial ideal of $I_{\mathcal{P}}$ are equivalent to corresponding properties of the triangulation, with perhaps the most well-known connection being the following result.

Theorem 3.3.1 (Theorem 8.9, [39])**.** Given a monomial ordering $\prec$ on $T_{\mathcal{P}}$, the initial ideal $\mathrm{in}_{\prec}(I_{\mathcal{P}})$ is squarefree if and only if the corresponding regular triangulation of $\mathcal{P}$ is unimodular.

In general, $\mathrm{in}_{\prec_{\mathrm{rlex}}}(I_{\mathcal{P}})$ cannot be guaranteed to be squarefree. This does not rule out the existence of $\mathrm{in}_{\prec_{\mathrm{rlex}}}(I_{\mathcal{P}})$ being squarefree for *some* ordering of their lattice points, though this may require much more work; the generators of a toric ideal are notoriously difficult to compute in general. Fortunately, the following ordering we place on S_n provides enough structure to prove the existence of a regular, unimodular triangulation.

Definition 3.3.2. We place a total order $<_{S_n}$ on the lattice points of S_n by setting $M <_{S_n} N$ if M contains more 2s in its entries than N, and by then taking a linear extension. This induces a graded reverse lexicographic monomial order $\prec_{S_n}$ on the variables of T_{S_n}, specifically $t_M \prec_{S_n} t_N$ if $M <_{S_n} N$.

We are now ready to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. First, let $\mathcal{G}$ be the reduced Gröbner basis of I_{S_n} with respect to any ordering. It is known to consist of binomials itself. Suppose $\mathcal{G}$ has a binomial $u-v$ with both terms containing squares, and $\pi(u) = \pi(v) = x^A z^k$. Note in particular that the variables in u and v are distinct. Suppose t_M and t_N are the variables in the separate terms with powers greater than 1. Then $\pi(t_M t_N)$ is the average of the points corresponding to $\pi(t_M^2)$ and $\pi(t_N^2)$, thus is subtractable from A. By the integral closure of S_n, there is some third monomial b such that $\pi(t_M t_N b) = x^A z^k$. So $u - t_M t_N b$ is in I_{S_n}; however, we can factor out t_M from this to get $u - t_M t_N b = t_M(u_1 - u_2)$. We may similarly factor t_N from $v - t_M t_N b$ to get $t_N(v_1 - v_2)$, which must also be in I_{S_n}. Therefore $u_1 - u_2$ and $v_1 - v_2$ must be in I_{S_n} themselves, and $u - v$ can be written as

$$u - v = u - t_M t_N b + t_M t_N b - v = t_M(u_1 - u_2) - t_N(v_2 - v_1)$$

which contradicts $\mathcal{G}$ being reduced. Therefore no binomial in $\mathcal{G}$ can have both terms containing a square.

For the second property, we must show that, for any lattice point $M \in S_n$, we can find a second lattice point $N \in S_n$ such that $M + N$ can be represented in a second, distinct sum. Since these are degree 2, the relation must be recorded in I_{S_n}, meaning both terms appear individually in $\mathcal{G}$ (even if not as part of the same binomial). While this can be proven in terms of matrices, it will be easier to work in terms of graph labelings.

As we saw in Proposition 3.2.2, each lattice point $M \in S_n$ corresponds to a 2-factor G_M, a covering of n vertices so that each vertex is incident to two edges. Thus for each 2-factor G_M, we want to find a second 2-factor G_N such that $G_M \cup G_N$ can be written as a union of 2-factors, each distinct from both G_M and G_N. Each covering is a disjoint union of two possible connected components: first, a path, possibly of length 0, whose endpoints also have loops; second, a k-cycle for some $k \leq n$. This allows us to break the remainder of the proof into three cases.

First suppose G_M contains a path $v_1, v_2, \ldots, v_k$, $k > 1$, with loops at its endpoints. Set G_N to be the graph agreeing with G_M except on these vertices. Here we place a single loop on each of v_1 and v_k, an edge between these two vertices, and two loops on each of $v_2, \ldots, v_{k-1}$. The union $G_M \cup G_N$ can be decomposed appropriately as a cycle $v_1, v_2, \ldots, v_k, v_1$ and as two loops on each vertex.

Next suppose that G_M contains no such paths but does contain a cycle $v_1, v_2, \ldots, v_k, v_1$ for some $k \geq 2$. Let G_N be the cover with two loops on each v_i. Then $G_M \cup G_N$ decomposes as the path $v_1, \ldots, v_k$ with a loop on v_1 and v_k as one covering and the other covering as the edge v_1, v_k with loops v_1, v_1 and v_k, v_k along with two loops on all other vertices.

If G_M does not fit into either of the previous cases, then its connected components all consist of two loops on each of the n vertices. Form a new graph G' by setting it equal to G_M, except for two distinct vertices, v_1 and v_2. Instead, place two edges between v_1 and v_2. Then G' is also a 2-factor, and $G' = G_N$ for some lattice point $N \in S_n$. Moreover, the entries of both M and N consist of only zeros or twos, so their average $A = \frac{1}{2}(M + N)$ is a lattice point of S_n distinct from both M and N. So, $G_M \cup G_N = G_A \cup G_A$. This covers all cases, so the corresponding M will always appear in a degree-two binomial of $\mathcal{G}$.

We restrict to the order $\prec_{S_n}$ and fix this order for the remainder of the proof. For the third property, consider $t_M t_N - t_X t_Y \in \mathcal{G}$. Since we know one of the monomials must be squarefree, it is enough to check the case when the other monomial is a square square. Suppose $M = N$. This can only occur if M is not a vertex; hence, M is the midpoint of X and Y. Thus if any entries of M are 2, the corresponding entries of X and Y must also be 2. Since X and Y are distinct, though, they have distinct support. This implies that some entry of M is 1, which arises from one of the corresponding entries of X and Y being 0 and the other being 2. So, one of X or Y will contain more twos than M, giving us $\mathrm{in}_{\prec_{S_n}}(t_M t_N - t_X t_Y) = -t_X t_Y$.

Lastly, consider an arbitrary binomial $u - v$ of degree k from $\mathcal{G}$. If the initial term is the squarefree term, then it is certainly cubefree. Otherwise, the binomial is of the form $t_{A_1}^{a_1} \cdots t_{A_r}^{a_r} - t_{B_1} \cdots t_{B_k}$, with $\mathrm{in}_{\prec_{S_n}}(u - v) = u = t_{A_1}^{a_1} \cdots t_{A_r}^{a_r}$ and each $a_i \geq 1$. Since we are using the order $\prec_{S_n}$, one of the variables of $v = t_{B_1} \cdots t_{B_k}$ is less than all variables in u; without loss of generality, assume this variable is t_{B_1}.

Choose a nonzero entry of B_1. There will be some variable t_{M_1} such that $M_1 \in \{A_1, \ldots, A_r\}$ and M_1 is also nonzero in the same position. Now, choose a nonzero entry of B_1 such that the position is zero in A_1. Then we know there is some variable t_{M_2} such that $M_2 \in \{A_1, \ldots, A_r\} \setminus \{M_1\}$ and M_2 is nonzero in this new position. Repeating this process gives a monomial $t_{M_1} \cdots t_{M_s}$ such that $M = M_1 + \cdots + M_s$ is nonzero whenever B_1 is nonzero. If there are any positions that are 2 in B_1 and 1 in M, then square a variable of $t_{M_1} \cdots t_{M_s}$ whose corresponding matrix is nonzero in that position. Repeat on distinct variables if necessary.

The resulting monomial, which we will call m_1, is cubefree, and there is some second monomial m_2 such that $m_1 - t_{B_1} m_2 \in I_{S_n}$. Because t_{B_1} was chosen to be less than all the variables $t_{A_1}, \ldots, t_{A_r}$, we know that $\mathrm{in}_{\prec_{S_n}}(m_1 - t_{B_1} m_2) = m_1$, which divides $t_{A_1}^{a_1} \cdots t_{A_r}^{a_r}$ Since our chosen binomial is in a reduced Gröbner basis, the two must be equal. Therefore, every initial term of a binomial in $\mathcal{G}$ is cubefree. $\square$

If the initial terms can be proven to be squarefree, then the following results follow.

Conjecture 3.3.3. S_n has a regular, unimodular triangulation, hence $h^*(S_n)$ is unimodal when n is even.

The last part of the previous proof adapts the method used in Theorem 14.8 of [39] to show that I_{B_n} has a squarefree initial ideal for any reverse lexicographic ordering. However, we cannot continue to adapt this proof so simply at this point: although one of the matrices A_j in a variable of the initial term may be nonzero in a same position that B_1 is, the entry may be 1 in A_j and 2 in B_1, and there is no indication that any other variable corresponds to a matrix with a nonzero entry in the same position.

3.4 Future Directions, Questions, and Conjectures

Experimental data and the results we have shown lead to some natural questions and conjectures.

Conjecture 3.4.1. Let $\mathcal{G}$ be the reduced Gröbner basis of I_{S_n}, and let $g \in \mathcal{G}$ with $\deg g \geq 3$.

1. The matrix corresponding to g does not have a block form. In other words, the corresponding graph is connected.

2. The matrix corresponding to g has a decomposition into lattice points of S_n such that one summand consists of only ones and zeros.

If the second part of this conjecture holds, then Conjecture 3.3.3 holds as well.

To prove that an initial term of a binomial is squarefree, one strategy is to prove that both monomials are squarefree. While this is a stronger result, it may rely less on the monomial order than simply proving the initial term is squarefree. We propose a monomial order that is a refinement of $\prec_{S_n}$ and appears to hold this behavior.

Conjecture 3.4.2. Set $t_M > t_N$ if the matrix M contains more twos than N. If neither contains a two, then set $t_M > t_N$ if M contains more zeros. This refinement induces an order such that $\mathcal{G}$ consists of binomials of degree at most $n-1$, and the binomials of degree greater than 2 are squarefree in both terms.

Another modification that can be made to Σ_n is the following. Define by P_n the convex hull of the lattice points in Σ_n. In general, P_n is neither Gorenstein nor integrally closed. However, based on experimental data, we conjecture the following.

Conjecture 3.4.3. For all n, $h^*(P_n)$ is unimodal.

Many methods for showing unimodality aim to show that the h^*-vector of a polytope is the same as the h-vector of a simplicial polytope, but another approach is necessary for P_n, as well as S_n for odd n.

Instead of looking at all lattice points of S_n, one can form triangulations using only the vertices. These will not be unimodular triangulations, but they might lead to something interesting.

Conjecture 3.4.4. For $n \geq 2$, any reverse lexicographic initial ideal of the toric ideal I_{S_n} (no new vertices) is n-free and is generated by monomials of degree $3(n-2)$.

The conjecture is experimentally true for $n = 3$ by an exhaustive search. Higher dimensions result in exponentially increasing numbers of vertices, vastly increasing the computational difficulty of experimentation.

Chapter 4 Experimental Methods and Results

4.1 Integral Closure

Determining when a polytope is integrally closed is generally a highly nontrivial task. If the polytope possesses a unimodular triangulation (or even a unimodular cover), then integral closure clearly holds. However, wthout knowledge of additional structure, proving the integral closure of a lattice polytope is difficult. Fortunately, if one wants to test conjectures through computer programs, there are multiple choices available. Tests for this work were done mainly inside of *Macaulay2* [18], using the *Normaliz* interface. Explicitly checking h^*-vectors was often performed using *LattE* [25].

The complete lists of vectors in $\mathbb{Z}^n_{>0}$, $n \leq 7$, satisfying the conditions from Construction 2.1.4 were produced using *GAP* [17] by Jack Schmidt, to whom we are very grateful. From these vectors we used Construction 2.1.4 to produce all reflexive simplices of type $(Q, 1)$. Those vectors corresponding to integrally closed simplices were then collected, the complete lists of which are available at

$$https://sites.google.com/site/rtda223/research.$$

Table 4.1 lists the number of reflexive simplices of type $(Q, 1)$ for low dimensions, as well as how many of them are integrally closed.

Dimension	*# reflexive simplices of type* $(Q, 1)$	*# integrally closed*
3	14	14 (100%)
4	147	113 (77%)
5	3462	1124 (33%)
6	294134	2676 (<1%)

Table 4.1: The number of integrally closed reflexive simplices for low dimensions.

4.2 The Weak Lefschetz Property

As we saw in Chapter 2, not every integrally closed reflexive simplex $\mathcal{P}$ induces a weak Lefschetz element in $\mathbb{C}[\mathcal{P}]$ using the standard choice of l.s.o.p. Nevertheless, using this choice is successful for a highly significant portion of such simplices when using a generic linear combination of degree-1 elements. Table 4.2 describes vectors $Q = (q_0, \ldots, q_n)$ corresponding to integrally closed reflexive simplices such that the canonical choice of linear system of parameters does not induce a generic weak Lefschetz element of such a form. Due to computational limitations, there were restrictions on vectors Q that were tested.

Dimension	*# simplices tested*	*# without generic WLE using standard l.s.o.p*
3	14	1 (7.1%)
4	83	1 (1.2%)
5	312	31 (9.9%)
6	577	21 (3.6%)

Table 4.2: Integrally closed reflexive simplices whose canonical choice of linear system of parameters does not induce a generic weak Lefschetz element.

Although these are small samples, this data suggests that the canonical choice of l.s.o.p. is more successful for simplices of even dimension. The reason for this phenomenon is currently completely opaque.

Since every reflexive polytope contains the origin, there will always be a minimal generator of $\mathbb{C}[\mathcal{P}]$ corresponding to it. There is no single way to choose how to incorporate this generator into the canonical l.s.o.p., but simply choosing a second l.s.o.p.

$$(x^{r_1}z, \ldots, x^{r_n}z, x^{r_0}z + z),$$

where the r_i are the vertices as they appear in Construction 2.1.4, has been effective experimentally. Table 4.3 describes how many of the same simplices as before do not induce a weak Lefschetz element using this second l.s.o.p. Proving that the resulting quotient algebras do have a weak Lefschetz element is more difficult since the correspondence between basis elements of the algebra and the lattice points of the fundamental parallelepiped is not as clear.

Dimension	*# simplices tested*	*# without generic WLE using second l.s.o.p*
3	14	0 (0%)
4	83	0 (0%)
5	312	13 (4.2%)
6	577	9 (1.6%)

Table 4.3: Integrally closed reflexive simplices whose second choice of linear system of parameters does not induce a generic weak Lefschetz element.

4.3 Lexicographic Toric Ideals

One may also approach the unimodality of h^*-vectors for integrally closed, reflexive simplices by examining their toric ideals, as we did in Section 3.3. This time, however, there is no hope for any reverse lexicographic orderings to work in full generality; if a polytope contains at least four collinear lattice points on an edge, then *some* initial term of a Gröbner basis element will not be squarefree. A more successful ordering to use is lexicographic, although the "correct" variation is not entirely obvious.

For example, let $Q = (1, 2, 9, 24, 36)$. By ordering the lattice points of Δ_Q using the standard lexicographic order, the reduced Gröbner basis of the toric ideal is not generated by binomials that all have squarefree initial terms. However, ordering the lattice points in a different way does result in such a reduced Gröbner basis. Similarly to the existence of a weak Lefschetz element for some ideal J in the previous section, it appears that for each integrally closed reflexive simplex, there is *some* lexicographic initial ideal that is squarefree.

Table 4.3 describes a sample of how many integrally closed reflexive simplices fail to posses a regular unimodular triangulation using the standard choice of lexicographic order.

Dimension	*# simplices tested*	*# without a regular unimodular triangulation*
3	14	1 (7%)
4	91	24 (26%)
5	312	84 (27%)
6	577	97 (17%)

Table 4.4: Integrally closed reflexive simplices with unimodular lexicographic triangulations.

4.4 Applications to Other Areas

Although the code was originally developed to explore the questions asked in Chapter 2, it was written with enough generality to be used for many other polytopes. In fact, it was used heavily when exploring symmetric magic squares, which led to the main results of Chapter 3. The code has already found use outside of the settings for which it was written and is expected to continually demonstrate its use as a helpful computational tool.

In May 2014, Alexandersson made an initial preprint of [1] available. This article discusses various properties of certain Gelfand-Tsetlin polytopes, including reverse lexicographic triangulations and partial orders on the polytopes. It originally included a conjecture that one of these classes is integrally closed. To either disprove or gather evidence to support this conjecture, we examined the following polytope.

Definition 4.4.1. Let $Part(n,d)$ be the convex hull of weak integer partitions of n into d parts. That is,

$$Part(n,d) := \operatorname{conv}\left\{(x_1,\ldots,x_d) \in \mathbb{Z}^d_{\geq 0} | x_d \geq x_{d-1} \geq \cdots \geq x_1 \geq 0 \text{ and } \sum_{i=1}^{d} x_i = n\right\}.$$

Although $Part(n,d)$ was computationally verified to be integrally closed for many small values of n and d, it is *not* integrally closed when $n = 18$ and $d = 9$: the partition $(6,6,6,6,4,4,2,1,1)$ is in $2Part(18,9)$, using the convex combination

$$\begin{aligned}(6,6,6,6,4,4,2,1,1) &= \frac{1}{2}((2,2,2,2,2,2,2,2,2) + (3,3,3,3,2,2,2,0,0) \\ &= (3,3,3,3,3,3,0,0,0) + (4,4,4,4,1,1,0,0,0)).\end{aligned}$$

However, it is straightforward to check that this cannot be written as a sum of two lattice points of $Part(18,9)$.

After this finding was communicated to Alexandersson, he was able to use it to construct a counterexample of his conjecture. The article was consequently revised to explicitly include the non-integrally closed Gelfand-Tsetlin polytope found from $Part(18,9)$.

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