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Combinatorial Potpourri: Permutations, Products, Posets, and Pfaffians

Norman B. Fox
University of Kentucky, bradfox87@gmail.com

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Norman B. Fox, Student

Dr. Richard Ehrenborg, Major Professor

Dr. Peter Perry, Director of Graduate Studies

Combinatorial Potpourri: Permutations, Products, Posets, and Pfaffians

ABSTRACT OF DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Norman Bradley Fox
Lexington, Kentucky

Director: Dr. Richard Ehrenborg, Professor of Mathematics
Lexington, Kentucky 2015

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Combinatorial Potpourri: Permutations, Products, Posets, and Pfaffians

In this dissertation we first examine the descent set polynomial, which is defined in terms of the descent set statistics of the symmetric group $\mathfrak{S}_n$. Algebraic and topological tools are used to explain why large classes of cyclotomic polynomials are factors of the descent set polynomial. Next the diamond product of two Eulerian posets is studied, particularly by examining the effect this product has on their **cd**-indices. A combinatorial interpretation involving weighted lattice paths is introduced to describe the outcome of applying the diamond product operator to two **cd**-monomials. Then the **cd**-index is defined for infinite posets, with the calculation of the **cd**-index of the universal Coxeter group under the Bruhat order as an example. Finally, an extension of the Pfaffian of a skew-symmetric function, called the hyperpfaffian, is given in terms of a signed sum over partitions of n elements into blocks of equal size. Using a sign-reversing involution on a set of weighted, oriented partitions, we prove an extension of Torelli's Pfaffian identity that results from applying the hyperpfaffian to a skew-symmetric polynomial.

KEYWORDS: permutations, posets, **cd**-index, Coxeter groups, hyperpfaffian

Author's signature: Norman Bradley Fox

Date: May 6, 2015

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Dedicated to my parents, Chris and Betsy Fox.

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inspiration and motivation they have given me. The joy they bring to my life has allowed me to endure the stress of graduate school and complete the arduous task of earning my doctorate.

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Chapter 1 introduces many mathematical ideas that are used throughout this dissertation. References to earlier work in these topics will be provided for further reading in these areas.

Chapter 2 includes an examination of the descent set polynomial, which is a polynomial defined by Chebikin, Ehrenborg, Pylyavskyy, and Readdy [10] in terms of the descent set statistics involving permutations in the symmetric group $\mathfrak{S}_n$. Results are provided to explain the existence of large classes of cyclotomic polynomials that are factors of this descent set polynomial. The proofs rely on several different combinatorial, algebraic, and topological tools.

In Chapter 3 we investigate the diamond product of posets, an important operation that corresponds to the Cartesian product of polytopes. In particular we explore the effect of this product on the **cd**-indices of Eulerian posets. Using recursive formulas that were introduced by Ehrenborg and H. Fox [13], a combinatorial interpretation for the diamond product operator applied to **cd**-monomials is provided involving weighted lattice paths.

Continuing with the study of the **cd**-index, Chapter 4 introduces an extension of this polynomial for infinite posets. As an example of an infinite poset that maintains the Eulerian condition on each interval, we examine the **cd**-index of the Bruhat order of the universal Coxeter group, which is generated by involutions with no relations among the generators.

In Chapter 5 we provide an extension of the Pfaffian of a skew-symmetric matrix called the hyperpfaffian. When the hyperpfaffian is evaluated at a skew-symmetric polynomial of a particular degree, we obtain an extension of a classical Pfaffian result by Torelli that involves the Vandermonde determinant. The proof of this formula uses a sign-reversing involution on weighted, oriented partitions.

1.1 Permutations

A *permutation* is a bijection $\pi : [n] \longrightarrow [n]$ where $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Under the composition operation, the set of permutations can be viewed as group, called the symmetric group and denoted $\mathfrak{S}_n$. The symmetric group is an example of a finite Coxeter group, which will be defined in Section 1.8. The notation used for a permutation in this dissertation will be one-line notation $\pi = \pi_1\pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$.

There are many sets and statistics associated with permutations that have been well-studied, including descents, excedances, and inversions. For a permutation $\pi \in \mathfrak{S}_n$, the *descent set*, *excedance set*, and *inversion set* are defined as follows:

$$\begin{aligned}
\mathrm{Des}(\pi) &= \{i \in [n-1] \;:\; \pi_i \;>\; \pi_{i+1}\},\\
\mathrm{Exc}(\pi) &= \{i \in [n] \;:\; \pi_i \;>\; i\},\\
\mathrm{Inv}(\pi) &= \{(i,j) \in [n] \times [n] \;:\; i < j,\ \pi_i > \pi_j\}.
\end{aligned}$$

From these sets, three permutation statistics, or functions from the symmetric group to the nonnegative integers, arise. They are defined as $\mathrm{des}(\pi) = |\,\mathrm{Des}(\pi)|$, $\mathrm{exc}(\pi) = |\,\mathrm{Exc}(\pi)|$, and $\mathrm{inv}(\pi) = |\,\mathrm{Inv}(\pi)|$. Furthermore, the inversion statistic is used to define the *sign* or *signature* of a permutation π, which we will denote by $(-1)^\pi$, where $(-1)^\pi = (-1)^{\mathrm{inv}(\pi)}$.

A classical result in permutation theory is that descents and excedances are equidistributed; i.e., the number of permutations in $\mathfrak{S}_n$ with k descents is the same as those with k excedances. See [9, Theorem 1.36]. The Eulerian number $A(n, k)$ represents the number of permutations in $\mathfrak{S}_n$ with $k-1$ descents, or likewise, $k-1$ excedances.

Focusing our attention on the descent set of a permutation, the descent set statistic $\beta_n(S)$, defined for each set $S \subseteq [n-1]$, is given by

$$\beta_n(S) = |\{\pi \in \mathfrak{S}_n \;:\; \mathrm{Des}(\pi) \;=\; S\}|.$$

Previous results regarding the descent set statistics include work by De Bruijn [11] and Niven [39] that showed the descent set statistics are maximized by the set consisting of either all even positions or all odd positions. These sets correspond to alternating permutations. Also, Ehrenborg and Mahajan [18] showed how to determine the maximum descent set statistic given subsets of a certain size and length. Later on Chebikin, Ehrenborg, Pylyavskyy, and Readdy [10] studied properties of descent set statistics, such as the proportion of odd entries when examining the statistics for each subset of $[n]$. This proportion, defined as

$$\rho(n) = \frac{|S \subseteq [n-1] \;:\; \beta_n(S) \equiv 1 \bmod 2\}|}{2^{n-1}},$$

was found to only depend on the number of 1's in the binary expansion of the integer n. In particular, $\rho(n) = 1/2$ when n has 2 or 3 binary digits, and $\rho(n) = 1$ when n is a power of 2.

There are several useful methods for calculating descent set statistics. First, MacMahon's Multiplication Theorem [36, Article 159] is a formula used to calculate descent set statistics of sets that differ by a single element and is in the form of a recursion using descent set statistics for shorter permutations. Before we state it, we need to define some terminology. We let the interval $[i, j]$ be the set $\{i, i+1, \ldots, j\}$, let $\triangle$ be the symmetric difference of two sets, i.e., $S \triangle T = S \cup T - S \cap T$, and let $S - k$ denote the shifting of S by k, which is the set $\{s - k \;:\; s \in S\}$. Now we can state the theorem as

$$\beta_n(S) + \beta_n(S \triangle \{k\}) = \binom{n}{k} \cdot \beta_k(S \cap [k-1]) \cdot \beta_{n-k}(S \cap [k+1, n-1] - k).$$

A second way to calculate $\beta_n(S)$ is through the use of a triangular array that is formed recursively. This method was introduced by de Bruijn [11] and Viennot [49] with a focus on alternating permutations and later generalized as the boustrophedon transform by Millar, Sloane, and Young [37]. A triangular array $t_{i,j}(S)$ is set up where $S \subseteq [n]$, $0 \le i \le n$, and $0 \le j \le i$. The initial values are given by $t_{0,0} = 1$

and $t_{i,0} = 0$ for $i > 0$. The remaining values in the array are calculated using the recursion

$$t_{i,j} = \begin{cases} t_{i,j-1} + t_{i-1,j-1} & \text{if } i-1, i \in S \text{ or } i-1, i \notin S, \\ t_{i,j-1} + t_{i-1,i-j} & \text{otherwise.} \end{cases}.$$

Informally, the recursive pattern involves adding across the rows of the triangular array, where the entries in S determine along which direction to add. If i is in the set S, addition is performed along the ith row from right to left, whereas we add from left to right if $i \notin S$. The descent set statistic $\beta_n(S)$ is equal to the sum of the bottom row of the array; that is,

$$\beta_n(S) = \sum_{k=0}^{n} t_{n,k}.$$

Example 1.1.1. For $n = 6$ and $S = \{3, 4\}$, the descent set statistic $\beta_6(S)$ is calculated using the following triangular array. Note that the entries in row 3 and 4, corresponding to the set S, are written in reverse order to make the recursive addition more natural.

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 0 && 1 &&&& \\ &&& 0 && 0 && 1 &&& \\ && 1 && 1 && 1 && 0 && \\ & 3 && 2 && 1 && 0 && 0 & \\ 0 && 3 && 5 && 6 && 6 && 6 \end{array}$$

By adding the bottom row, we attain $\beta_6(\{3, 4\}) = 0 + 3 + 5 + 6 + 6 + 6 = 26$.

A third method used to compute $\beta_n(S)$ is through the use of the flag f-vector of the Boolean algebra. This will be explained in Section 1.6.

There are two polynomials of note that encode information relating to the descent set statistics. First, there is a generating function with the descent set statistics as the coefficients, called the *Eulerian polynomial*, which is defined as

$$A_n(t) = \sum_{S \subseteq [n-1]} \beta_n(S) \cdot t^{|S|+1}.$$

Chebikin et al. [10] defined the nth descent set polynomial, where the descent set statistics are instead the exponents of the polynomial, as

$$Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta_n(S)}.$$

The degree of the latter polynomial is given by nth Euler number, which has faster than exponential growth. This leads to quite large degree polynomials for relatively small values of n.

1.2 Cyclotomic polynomials

The *nth cyclotomic polynomial* is defined by

$$\Phi_n(t) = \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} (t - e^{2i\pi\frac{k}{n}}),$$

with the root of each linear factor being a root of unity. It can also be viewed as the unique irreducible polynomial with integer coefficients that divides $t^n - 1$, but not $t^k - 1$ for any $k < n$. Finally, another definition is for $\Phi_n(t)$ to be the minimal polynomial over the field of rational numbers of the primitive nth root of unity $e^{2i\pi/n}$.

The cyclotomic polynomials have many interesting properties and uses within the areas of algebra and combinatorics. Some of these properties stated by Lang in [33] include that $\Phi_n(t)$ has integer coefficients with 1 as the leading coefficient, and that the degree is given by the Euler's totient function, which is the number of positive integers less than or equal to n that are relatively prime to n. Additionally, [33] provides several recursions for the cyclotomic polynomials, including the following:

- For an odd integer $n > 1$, $\Phi_{2n}(t) = \Phi_n(-t)$,
- For a prime p that does not divide a positive integer n, $\Phi_{pn}(t) = \Phi_n(t^p)/\Phi_n(t)$,
- For a prime p and positive integer n in which $p|n$, $\Phi_{pn}(t) = \Phi_n(t^p)$.

As an example of an application of cyclotomic polynomials, see [38] for a description of how these polynomials can be used to prove Dirichlet's Prime Number Theorem, which states that there are infinitely many prime numbers $p \equiv 1 \bmod n$ for every integer $n \ge 1$.

As the minimal polynomial of the primitive nth root of unity $e^{2\pi i/n}$, determining whether $\Phi_n(t)$ is a factor of another polynomial with rational coefficients only requires checking if the polynomial is zero at that particular root of unity. This is described more generally in the following well-known fact from algebra.

Fact 1.2.1. *If $f(t)$ is a polynomial in $\mathbb{Q}[t]$ with $e^{2\pi i/n}$ as a root of multiplicity r then the nth cyclotomic polynomial $\Phi_n(t)$ is a factor of order r of $f(t)$.*

1.3 Simplicial complexes and the Euler characteristic

A *simplicial complex* is a combinatorial structure that can be viewed as an abstract or geometric object. Abstractly, a simplicial complex Δ is a finite collection of sets satisfying if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$ as well. The elements of Δ are called *faces* and maximal faces are known as *facets*. From a geometric perspective, a simplicial complex Δ is a set of simplices, where a simplex is the convex hull of affinely independent points, such that any face of a simplex in Δ is also in Δ, and the intersection of any two simplices in Δ is a face in Δ.

An important tool in the topological study of simplicial complexes is the *Euler characteristic*, denoted $\chi(\Delta)$. We first define the *f-vector* of a simplicial complex to be the set of values $\{f_i\}$ which count the number of i-dimensional faces of the complex Δ. The Euler characteristic is the alternating sum

$$\chi(\Delta) = f_0 - f_1 + f_2 - \cdots .$$

Oftentimes, the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ is used. It includes the empty set in the sum; that is, the reduced Euler characteristic is given by the sum

$$\widetilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + f_2 - \cdots = \chi(\Delta) - 1,$$

where $f_{-1} = 1$ represents the empty face.

When applied to polyhedra, the Euler characteristic always equals 2, resulting in Euler's famous polyhedron formula relating the number of vertices, edges, and faces by $f_0 - f_1 + f_2 = 2$. In general, the Euler-Poincaré formula gives the Euler characteristic in terms of an alternating sum of the *Betti numbers*, where the ith Betti number, denoted as b_i, is the dimension of the ith homology group of a simplicial complex. Informally, b_i counts the number of i-dimensional holes of the simplicial complex. The relation between the f-vector and Betti numbers is

$$f_0 - f_1 + f_2 - \cdots = b_0 - b_1 + b_2 - \cdots .$$

1.4 Posets

A *partially ordered set* (or *poset*) P is a set of elements and an order relation $\leq$ satisfying the reflexivity, antisymmetry, and transitivity properties shown below.

- For all $x \in P$, $x \leq x$.
- If $x \leq y$ and $y \leq x$, then $x = y$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

We say that y *covers* x, denoted $x \prec y$, if $x \leq y$ and there is no $z \in P$ in which $x < z < y$. The relation between x and y in this case is known as a cover relation. The *Hasse diagram* of a poset is a graph whose vertices are the elements of P and whose edges are the cover relations drawn so that if $x < y$ then y is drawn above x. See Figure 1.1 for an example using the Boolean algebra, the poset of all subsets of $[n]$ ordered by inclusion. We use $\hat{0}$ and $\hat{1}$ to denote the unique minimal and maximal elements of the poset, respectively, if such elements exist. The *interval* $[x, y]$ is defined as the set $[x, y] = \{z \in P : x \leq z \leq y\}$. It can be viewed as a subposet of P using the induced order of P. A length n *chain* in the poset is a set of $n+1$ distinct, totally ordered elements in P, meaning that $x_0 < x_1 < \cdots < x_n$. Such a chain is considered *saturated* in the interval $[x, y]$ if $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$. If every saturated chain of P, that is, every chain from a minimal element to a maximal element, has the same length n, then P is said to be of *rank* n. In such a ranked poset, there is a rank function $\rho : P \to \{0, 1, \ldots, n\}$ with the rank of x, denoted $\rho(x)$, defined as the length of any saturated chain from a minimal element of P to x. We also write

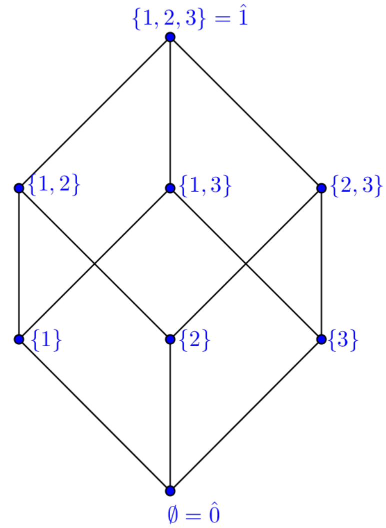

Figure 1.1: The Hasse diagram of the Boolean algebra on three elements.

$\rho(x, y) = \rho(y) - \rho(x)$ to denote the rank difference of x and y. A ranked poset with unique minimal and maximal elements is called a *graded* poset. A poset P is *finite* if the set of elements is finite, whereas if we only assume every interval of P is finite, then P is said to be *locally finite*. For further terminology and examples of posets, see [43, Chapter 3].

An important class of posets are called *lattices*. First, the *join* or the least upper bound of two elements $s, t \in P$ is defined to be the element $u \in P$ such that $s, t \leq u$ and for any other element $v \geq s, t$, we have $v \geq u$ as well. The join is unique by definition, if it exists, and is denoted by $s \vee t$. The *meet* or greatest lower bound of two elements is defined similarly by uniformly exchanging $\leq$ and $\geq$ in the definition of join, and it is denoted by $s \wedge t$. A lattice is a poset in which each pair of elements has a meet and a join. For the Boolean algebra, see Figure 1.1, the join and meet correspond to the union and intersection of sets, respectively.

The *Möbius function* μ is a function defined recursively on intervals of P by $\mu(x, x) = 1$ and for $x \neq y$,

$$\mu(x, y) = \sum_{x \leq z < y} \mu(x, z).$$

When this function is applied to the divisor lattice D_n, which is a poset whose elements are the positive integer divisors of n and where $i \leq j$ if i divides j, the Möbius function becomes the number theoretic Möbius function, with $\mu(x, y) = \mu(y/x)$. Additionally, on the divisor lattice the meet and join of two elements correspond to the least common multiple and greatest common factor functions, respectively.

A poset P is *Eulerian* if it is finite, graded, and if for every interval $[x, y]$, we have $\mu(x, y) = (-1)^{\rho(x,y)}$. Equivalently, every nontrivial interval in an Eulerian poset has the same number of elements of even rank as it does of odd rank. An important example of an Eulerian poset is one associated to a *polytope*, or the convex hull of a finite set of points. The *face lattice* of a convex polytope, denoted $\mathcal{L}(P)$, is a poset formed by using the faces in the polytope as the elements, and the partial order is determined by containment of faces. The face lattice of a polytope is graded with rank function given by $\rho(x) = \dim(x) + 1$. It is an Eulerian poset as well as a lattice. See Figure 1.2 for an example of the face lattice of a pentagonal prism, where $\hat{0}$ represents the empty face and $\hat{1}$ is the entire polyhedron.

There are many operations that are essential to build more complicated posets from simpler posets, such as the Cartesian product, diamond product, star product, pyramid, and prism. For finite posets P and Q, with both posets having unique minimal and maximal elements for the diamond and star products, we define the following:

Cartesian product of P and Q: $P \times Q = \{(p, q) \,:\, p \in P \text{ and } q \in Q\}$
with the order relation given by $(p_1, q_1) \leq_{P\times Q} (p_2, q_2)$ if $p_1 \leq_P p_2$ and $q_1 \leq_P q_2$.

Diamond product of P and Q: $P \diamond Q = (P - \{\hat{0}_P\}) \times (Q - \{\hat{0}_Q\}) \cup \{\hat{0}\}$.

Star product of P and Q: $P * Q = (P - \{\hat{1}_P\}) \cup (Q - \{\hat{0}_Q\})$
with the order relation given by $x \leq_{P*Q} y$ if (i) $x, y \in P$ and $x \leq_P y$,
(ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$.

Pyramid of P: $\text{Pyr}(P) = P \times B_1$.

Prism of P: $\text{Prism}(P) = P \diamond B_2$.

Recall for the last two definitions that B_n is the Boolean algebra on n elements. The pyramid and prism operations on posets are analogous to the pyramid and prism operations on polytopes. The Cartesian and diamond products correspond to polytope operations as well. For an m-dimensional polytope V and n-dimensional polytope W, we define the *Cartesian product* of the polytopes V and W as the $(m+n)$-dimensional polytope

$$V \times W = \{(x_1, \ldots, x_{m+n}) \in \mathbb{R}^{m+n} \,:\, (x_1, \ldots, x_m) \in V, (x_{m+1}, \ldots, x_{m+n}) \in W\}.$$

We define the *free join* of polytopes V and W by first embedding V and W in $\mathbb{R}^{m+n+1}$ by

$$V' = \{(x_1, \ldots, x_m, \underbrace{0 \ldots, 0}_{n}, 0) \in \mathbb{R}^{m+n+1} : (x_1, \ldots x_m) \in V\}$$

and likewise by

$$W' = \{(\underbrace{0, \ldots, 0}_{m}, x_1, \ldots, x_n, 1) \in \mathbb{R}^{m+n+1} : (x_1, \ldots, x_n) \in W\}.$$

Then the free join $V \circledvee W$ is the $(m+n+1)$-dimensional polytope defined as the convex hull of V' and W'. The following proposition from [28] displays the connection between the Cartesian product of posets with the free join of polytopes, along with the connection between the diamond product of posets with the Cartesian product of polytopes.

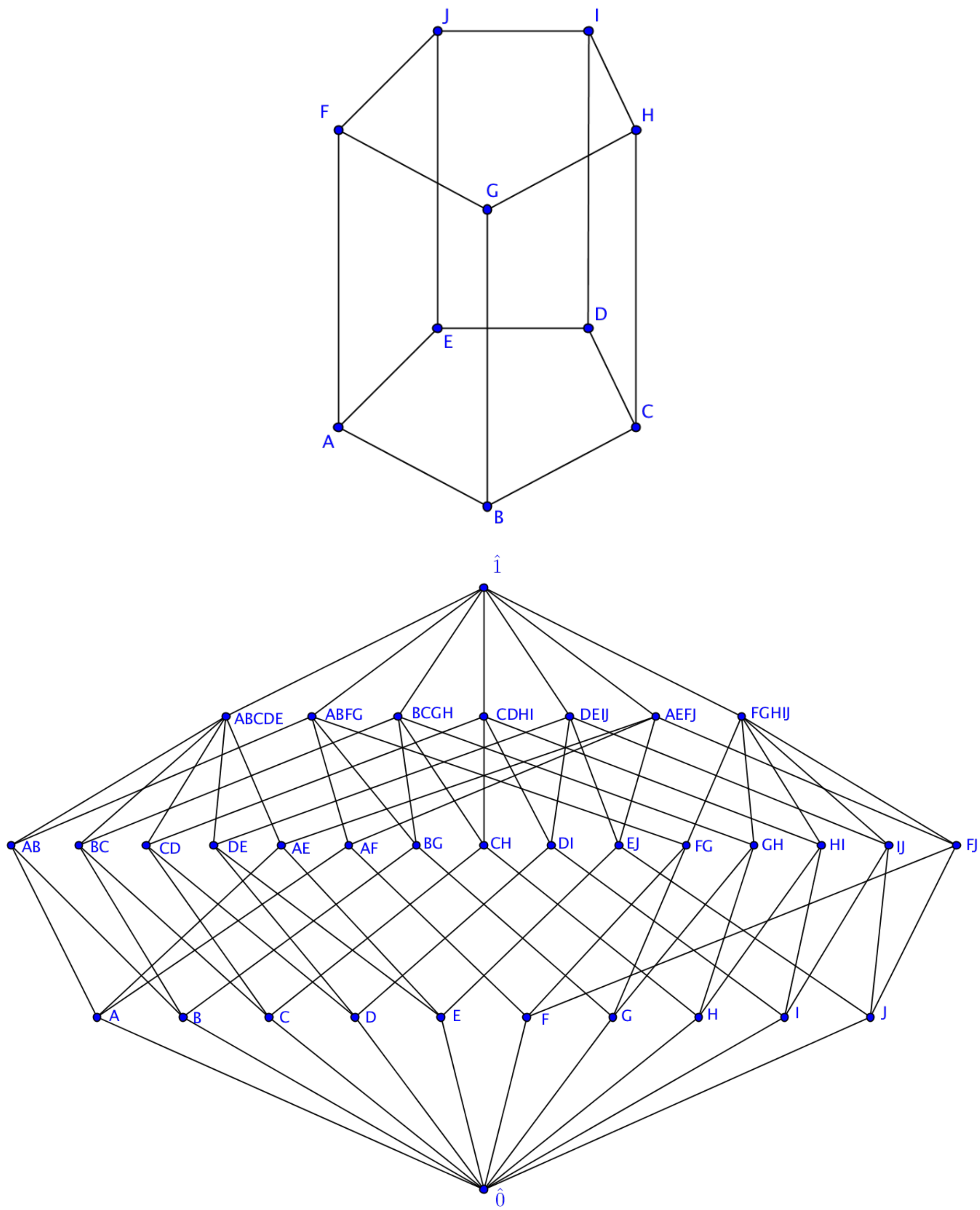

Figure 1.2: The pentagonal prism and its corresponding face lattice.

Proposition 1.4.1 (Kalai)**.** *For two convex polytopes V and W, we have*

$$\mathcal{L}(V \otimes W) = \mathcal{L}(V) \times \mathcal{L}(W)$$
$$\mathcal{L}(V \times W) = \mathcal{L}(V) \diamond \mathcal{L}(W).$$

1.5 The cd-index

The **cd**-index is a non-commutative polynomial which is associated to an Eulerian poset that efficiently encodes enumerative data on chains through that poset. We begin by assuming P is a graded poset of rank $n+1$ with rank function ρ. Recall that this implies P has a unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$.

We want to consider all chains c in the poset P that contain both $\hat{0}$ and $\hat{1}$; hence, they are written as $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$. For each subset $S = \{s_1 < \cdots < s_{k-1}\} \subseteq [n]$, define $f_S(P) = f_S$ to be the number of chains in the poset P whose elements $x_1, \ldots, x_{k-1}$ have ranks that are exactly the elements of the set S; that is, $\{\rho(x_1), \ldots, \rho(x_{k-1})\} = S$. The 2^n values of f_S are collectively known as the *flag f-vector* of P. The *flag h-vector* is defined using the relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T,$$

which by the Möbius inversion theorem is equivalent to

$$f_S = \sum_{T \subseteq S} h_T.$$

Let $\mathbf{a}$ and $\mathbf{b}$ be non-commutative variables. For a subset S of $\{1, \ldots, n\}$, define the **ab**-monomial $u_S = u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Define the **ab**-*index* $\Psi(P)$ of the poset P to be the **ab**-polynomial

$$\Psi(P) = \sum_{S} h_S \cdot u_S,$$

where S ranges over all subsets of $\{1, \ldots, n\}$.

An alternative approach to the definition of the **ab**-index of a graded poset P is through a summation of weighted chains. For each chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in P, we assign a weight $w(c) = z_1 \cdots z_n$, where we define

$$z_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_{k-1})\} \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

The **ab**-index of P is then defined as

$$\Psi(P) = \sum_{c} w(c),$$

where c ranges over all chains in P from $\hat{0}$ to $\hat{1}$.

S	f_S	h_S	u_S	$\mathbf{c}^3$	**dc**	**cd**
$\emptyset$	1	1	**aaa**	1	0	0
$\{1\}$	10	9	**baa**	1	8	0
$\{2\}$	15	14	**aba**	1	8	5
$\{3\}$	7	6	**aab**	1	0	5
$\{1,2\}$	30	6	**bba**	1	0	5
$\{1,3\}$	30	14	**bab**	1	8	5
$\{2,3\}$	30	9	**abb**	1	8	0
$\{1,2,3\}$	60	1	**bbb**	1	0	0

Table 1.1: The flag f-vector, flag h-vector, u_S monomials, and **cd**-monomials needed for calculating the **ab**- and **cd**-indices for the face lattice of the pentagonal prism.

Example 1.5.1. Let P be the face lattice of the pentagonal prism, as illustrated in Figure 1.2. We calculate the flag f-vector and the flag h-vector in Table 1.1. The **ab**-index of the poset is therefore given by $\Psi(P) = \mathbf{aaa} + 9 \cdot \mathbf{baa} + 14 \cdot \mathbf{aba} + 6 \cdot \mathbf{aab} + 6 \cdot \mathbf{bba} + 14 \cdot \mathbf{bab} + 9 \cdot \mathbf{abb} + \mathbf{bbb}$.

Calculating the **ab**-index of an n-dimensional polytope can also be stated in terms of counting flags of faces in the polytope. The flag f-vector entry f_S where $S \subseteq \{0, \ldots n-1\}$ counts chains of faces of increasing dimension $F_1 \subset F_2 \subset \cdots \subset F_k$ with $\dim(F_i) = s_i$. From this perspective, flag vector entries for singleton sets are the standard f-vector of the polytope.

The following result was conjectured by Fine and later proved by Bayer and Klapper [3]. Stanley also gave an elementary proof in [44].

Theorem 1.5.2 (Bayer–Klapper)**.** *The* **ab**-*index* $\Psi(P)$ *of an Eulerian poset* P *is a non-commutative polynomial in* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When written in terms of the non-commutative variables **c** and **d**, with $\deg(\mathbf{c}) = 1$ and $\deg(\mathbf{d}) = 2$, we call $\Psi(P)$ the **cd**-index of the poset P. Observe the same notation is used for the **ab**-index and the **cd**-index. The existence of the **cd**-index is equivalent to the fact that the flag f-vector of an Eulerian poset satisfies the generalized Dehn–Sommerville relations, due to Bayer and Billera in [2]. In [44] Stanley showed that the coefficients of the **cd**-index are nonnegative for face lattices of polytopes, and more generally, regular CW-complexes. For further examples and information on the **cd**-index of posets, see [43, Section 3.17].

An interesting observation that can be made from the example in Table 1.1 is the symmetry that exists within the flag h-vector. In fact, for any Eulerian poset, $h_S = h_{\overline{S}}$ where $\overline{S}$ denotes the complement of S within $[n]$. One advantage of this symmetry is that half of the flag h-vector entries are redundant, allowing data on chains in the poset to be encoded using 2^{n-1} entries rather than the original 2^n entries of the flag f-vector. The **cd**-index goes much further with its efficiency. As seen in the following example, only three coefficients are needed to define the **cd**-index of the pentagonal prism, which we notice is the fourth Fibonacci number f_4. We are using the recursive definition for the nth Fibonacci number as $f_n = f_{n-1} + f_{n-2}$ where $f_1 = f_2 = 1$. In general, there are f_n terms in the **cd**-index of a rank n poset, a much more efficient encoding of the chain data.

Continuing Example 1.5.1, the final three columns of Table 1.1 show the number of each **cd**-monomial needed to write that particular **ab**-monomial in terms of **c** and **d**. Note that the sum of these columns is the flag h-vector column. Thus, the **cd**-index of the pentagonal prism, equivalently its associated face lattice, is

$$\Psi(P) = \mathbf{c}^3 + 8 \cdot \mathbf{dc} + 5 \cdot \mathbf{cd}.$$

In general, the **cd**-index of the prism of an n-gon for $n \geq 2$ is

$$\Psi(\text{prism of an } n\text{-gon}) = \mathbf{c}^3 + (2n-2) \cdot \mathbf{dc} + n \cdot \mathbf{cd}.$$

It has recently been shown in [16] using a more general Euler flag enumeration theory of Whitney stratified spaces that the previous formula holds for $n \geq 1$.

1.6 Descent set statistics and the flag vectors

A third and final method for computing the descent set statistics is via the flag f-vector of the Boolean algebra. For $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$, let $\text{co}(S) = \vec{c} = (c_1, c_2, \ldots, c_{k+1})$ be the associated composition of n. This is a list of positive integers whose sum is n where $c_i = s_i - s_{i-1}$, and where we let $s_0 = 0$ and $s_{k+1} = n$. The flag f-vector entry f_S of the Boolean algebra B_n is given by the multinomial coefficient

$$f_S = \binom{n}{\vec{c}} = \binom{n}{c_1, c_2, \ldots, c_{k+1}},$$

and the descent set statistics are given by inclusion-exclusion

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T. \tag{1.6.1}$$

In other words, the descent set statistics are identical to the flag h-vector for the Boolean algebra B_n.

An efficient encoding of all the flag f-vector entries of the Boolean algebra is obtained by using quasi-symmetric functions. This important class of functions consists of degree-bounded power series in which the coefficient of the monomial $x_{i_1}^{a_1} \cdot x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ is the same for all sets of increasing indices $\{i_1 < i_2 < \cdots < i_k\}$. For a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ let $M_{\vec{c}}$ denote the monomial quasi-symmetric function defined by

$$M_{\vec{c}} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^{c_1} \cdot x_{i_2}^{c_2} \cdots x_{i_k}^{c_k}.$$

The algebra of quasi-symmetric functions is the linear span of the monomial quasi-symmetric functions. Multiplication of monomial quasi-symmetric functions is described in Lemma 3.3 in [12]. Now the quasi-symmetric function of the Boolean algebra is given in [12] by

$$F(B_n) = (x_1 + x_2 + \cdots)^n = M_{(1)}^n = \sum_{\vec{c}} \binom{n}{\vec{c}} \cdot M_{\vec{c}}.$$

Here the flag f-vector entry f_S is the coefficient of the monomial quasi-symmetric function for the associated composition of S. The descent set statistic $\beta_n(S)$ can be calculated using equation (1.6.1).

1.7 Coalgebras

To motivate the definition of a coalgebra, we first define an *algebra* over the field k. An algebra A is a vector space with a product $A \times A \longrightarrow A$, denoted by $\cdot$, and an element 1 in A such that for all $a, b, c \in A$,

(i) the product is distributive: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$,

(ii) the product is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and

(iii) the element 1 is the unit, meaning $1 \cdot a = a \cdot 1$.

A more compact way to define an algebra is to say that it is a vector space with a linear map $\nabla : A \otimes A \to A$, where $A \otimes A$ is the tensor product of A with itself, and a unit $\eta : k \to A$. There are also the requirements that $\nabla \circ (1 \otimes \nabla) = \nabla \circ (\nabla \otimes 1)$ and $\nabla \circ (\eta \otimes 1) = 1 = \nabla \circ (1 \otimes \eta)$, where 1 represents the identity map on A. Hence, we have $\nabla(a \otimes b) = a \cdot b$ and $\eta(c) = c \cdot 1$. With this definition, the distributive law follows from linearity of the map ∇.

A *coalgebra* is the dual of an algebra. It consists of a vector space V and two linear maps: a *coproduct* $\Delta : V \to V \otimes V$ and a *counit* $\epsilon : V \to k$. The analogous two conditions are that $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \epsilon) \circ \Delta = 1 = (\epsilon \otimes 1) \circ \Delta$. The first equation states that the coproduct Δ is *coassociative*.

When applying a coproduct to an element $v \in V$, we represent each pair of elements as $v_{(1)}^{(i)}$ and $v_{(2)}^{(i)}$, resulting in the coproduct of v being $\Delta(v) = \sum_i v_{(1)}^{(i)} \otimes v_{(2)}^{(i)}$. Heyneman and Sweedler [26] introduced the commonly used Sweedler notation to denote the coproduct of v as $\Delta(v) = \sum_v v_{(1)} \otimes v_{(2)}$.

If a vector space V with an associative product ∇ and a coassociative coproduct Δ satisfies the following property, we call the triple (V, ∇, Δ) a *Newtonian coalgebra*:

$$\Delta \circ \nabla = (1 \otimes \nabla) \circ (\Delta \otimes 1) + (\nabla \otimes 1) \circ (1 \otimes \Delta).$$

Using Sweedler notation for the coproduct and the standard $\cdot$ to denote the product, this condition can be rewritten as

$$\Delta(v \cdot w) = \sum_v v_{(1)} \otimes (v_{(2)} \cdot w) + \sum_w (v \cdot w_{(1)}) \otimes w_{(2)},$$

for all $v, w \in V$. Notice that this is a generalization of the Leibniz rule for derivatives.

An example of a Newtonian coalgebra involves the vector space $\mathcal{P}$ spanned by all *types* of graded posets of rank ≥ 1, where the type $\overline{P}$ of a poset P is the collection of all posets that are isomorphic to P. For the product, we use the previously defined star product $P*Q$. The coproduct on $\mathcal{P}$ that gives the Newtonian structure is defined as

$$\Delta(\overline{P}) = \sum_{\hat{0}<x<\hat{1}} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]}.$$

For the original formulation and study of Newtonian coalgebras, see Joni and Rota [27]. See Ehrenborg and Readdy [19] for further examples relating to posets and the **cd**-index. Finally, Ehrenborg and Readdy also studied the homology of certain Newtonian coalgebras in [20].

1.8 Coxeter groups

A *Coxeter group* W is a group generated by a set S with the following relations among the generators:

- All generators are involutions; that is, $s^2 = 1$ for all $s \in S$,
- For each pair of generators s and t, there is a nonnegative integer $m(s,t)$ with $2 \leq m(s,t) \leq \infty$ for which $(st)^{m(s,t)} = 1$.

If $m(s,t) = \infty$, this means the element st has infinite order, implying there is no relation between s and t. The group W paired with the not necessarily unique generating set S is called a *Coxeter system*. All finite Coxeter groups are Euclidean reflection groups, such as the symmetry groups of regular polytopes. Not all Coxeter groups are finite. The *universal Coxeter group* is an infinite Coxeter group. It is defined by setting $m(s,t) = \infty$ for all generators $s \neq t$. We denote the universal Coxeter group by $\mathcal{U}_r$ where r is the size of the generating set S.

A primary example of a finite Coxeter group is the symmetric group $\mathfrak{S}_n$. A generating set is given by $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$. It is known that any permutation can be written as the product of transpositions, more specifically, the adjacent transpositions in S. It is clear that these adjacent transpositions are involutions, satisfying the first set of required relations among the generators. One can easily calculate the values $m(s_i, s_j)$ depending on the difference between i and j. If $|i-j| \geq 2$, then $m(s_i, s_j) = 2$, whereas $m(s_i, s_{i+1}) = 3$.

For additional examples and further information regarding Coxeter groups and their combinatorial aspects, see Björner and Brenti [7].

1.9 The Pfaffian and the exterior algebra

The Pfaffian is a polynomial related to the determinant of a skew-symmetric matrix. Recall that a skew-symmetric matrix is a square matrix A whose transpose is its negative; that is, $-A = A^T$. If A is an $n \times n$ matrix with $A = (a_{i,j})_{1 \leq i,j \leq n}$, then A being skew-symmetric implies that $a_{i,j} = -a_{j,i}$ for all i and j. The determinant of such a matrix is a square of a polynomial in the matrix entries, which is defined as the *Pfaffian*, $\mathrm{Pf}(A)$. More explicitly, for a skew-symmetric matrix A, we have $\mathrm{Pf}(A)^2 = \det(A)$. Since the determinant is 0 for any $n \times n$ skew-symmetric matrix with n being odd, the Pfaffian of that matrix is also 0. Therefore, from this point we assume that n is even.

There is an alternative but equivalent definition of the Pfaffian that resembles the Leibniz formula for calculating determinants since it can be viewed as a sum over all perfect matchings of the complete graph. For a skew-symmetric matrix $A = (a_{i,j})_{1\leq i,j\leq n}$ with n even, the Pfaffian is defined as

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma\in\mathfrak{S}_n} (-1)^\sigma \cdot \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$

where $(-1)^\sigma$ is the sign of the permutation σ.

There is also a recursive definition for the Pfaffian in which we first assume the Pfaffian of the 0×0 matrix is equal to 1. Define the Pfaffian of a skew-symmetric $n \times n$ matrix A to be

$$\mathrm{Pf}(A) = \sum_{i=2}^{n} (-1)^i \cdot a_{1,i} \cdot \mathrm{Pf}(A_{\hat{1},\hat{i}}),$$

where $A_{\hat{1},\hat{i}}$ is the matrix A with the first and ith rows and columns removed.

Finally, there is a fourth definition of the Pfaffian involving the exterior algebra and wedge products. To define the exterior algebra, we begin with the *tensor algebra*, defined as the direct sum of the tensor powers of a vector space V over a field k; that is, $T(V) = k\oplus V\oplus(V\otimes V)\oplus(V\otimes V\otimes V)\oplus\cdots$. Define the ideal I to be generated by all elements of the form $v\otimes v$ for $v \in V$. The *exterior algebra*, denoted by $\Lambda(V)$, is then the quotient algebra of the tensor algebra by the ideal I; that is, $\Lambda(V) = T(V)/I$.

The *wedge product* of two elements in the exterior algebra is $x \wedge y = x \otimes y \bmod I$. Due to taking the quotient with the ideal I, the wedge product has the property $x \wedge x = 0$ for all $x \in V$. Using this fact for the element $x + y$, we have $0 = (x + y) \wedge (x + y) = x \wedge y + y \wedge x$, which implies the anticommutativity property $x \wedge y = -y \wedge x$ for all $x, y \in V$. More generally, if we permute the order of a wedge product of elements, we multiply by the sign of that permutation, as follows. For $t_1, t_2, \ldots, t_n \in V$ we have

$$t_{\pi_1} \wedge t_{\pi_2} \wedge \cdots \wedge t_{\pi_n} = (-1)^\pi \cdot t_1 \wedge t_2 \wedge \cdots \wedge t_n.$$

It possible to use wedge products to define the Pfaffian of an $n\times n$ skew-symmetric matrix where n is even. Consider the following sum of wedge products with the entries from A as coefficients

$$\sum_{1\leq i<j\leq n} a_{i,j} \cdot t_i \wedge t_j.$$

Since any occurrence of the same variable cancels, raising this sum to the $(n/2)$th power results in a polynomial in the $a_{i,j}$'s multiplied by $t_1 \wedge t_2 \wedge \cdots \wedge t_n$. We define the Pfaffian of A as this polynomial divided by $(n/2)!$, as seen below:

$$\left(\sum_{1\le i<j\le n} a_{i,j}\cdot t_i\wedge t_j\right)^{n/2} = (n/2)!\cdot \mathrm{Pf}(A)\cdot t_1\wedge\cdots\wedge t_n.$$

This formulation of the Pfaffian is easy to use in order to extend the Pfaffian to the hyperpfaffian. This will be done in Chapter 5.

The Pfaffian has many useful applications, most notably its use in counting perfect matchings in planar graphs. Research in this area was motivated by counting dimer coverings on a graph, which are equivalent to perfect matchings and represent configurations of diatomic molecules. Temperley and Fisher [46], and independently Kasteleyn [29], developed a method for counting such arrangements on lattice graphs, and later Kasteleyn [30] generalized this for any planar graph by a method now known as the FKT algorithm.

Assuming that G is a finite planar graph, the algorithm described in [30] creates a skew-symmetric matrix D such that the Pfaffian of D is a generating function for dimer coverings. The matrix D is defined as a weighted adjacency matrix of G with edges oriented to provide the skew-symmetric property. Kasteleyn provides a way of orienting these edges, called an admissible orientation, so that every term of the Pfaffian has the same sign. Hence, $|\mathrm{Pf}(A)|$ is the number of perfect matchings in G once the weights are all set to be one since each non-zero term of the Pfaffian corresponds to a perfect matching. This provides an efficient method for enumerating perfect matchings since the Pfaffian can be calculated by taking the square root of the determinant of D.

1.10 Set partitions, integer partitions, and compositions

A *set partition* (or *partition*, if the context is clear) $\pi = \{B_1, B_2, \ldots, B_k\}$ of $[n]$ is a collection of subsets, called *blocks*, such that

- $B_i \neq \emptyset$,
- $B_i \cap B_j = \emptyset$ for $i \neq j$,
- $[n] = \bigcup_{i=1}^{k} B_i$.

Set partitions of $[n]$ into k blocks are counted by the *Stirling numbers of the second kind*, denoted by $S(n,k)$. These numbers can be enumerated by the following explicit formula

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

or by using the recursion

$$S(n,k) = k\cdot S(n-1,k) + S(n-1,k-1),$$

with initial conditions $S(0,0) = 1$ and $S(n,0) = S(0,k) = 0$ for $n, k > 0$. See [43, Section 1.9].

The set of all set partitions of $[n]$ forms a poset called the *partition lattice* Π_n using the order relation $\pi \leq \tau$ if every block of π is contained in a block of τ. In this case we call π a refinement of τ. In the partition lattice, the unique minimal element $\hat{0}$ is the set partition consisting of all singleton blocks $\{\{1\}, \dots, \{n\}\}$ and the unique maximal element $\hat{1}$ is the set partition containing only one block, namely $[n]$.

Not to be confused with set partitions, an *integer partition* (or simply *partition*) $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of a positive integer n is a list of positive integers, called *parts*, for which $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. In this situation, the order of the λ_i parts does not matter. For example, $(1,3)$ and $(3,1)$ are considered to be the same partition of 4.

If we instead consider partitions having the same parts in different order to be distinct, we call these *compositions*. We will write compositions as a vector $\vec{r} = (r_1, r_2, \dots, r_k)$, and again refer to each r_i as a part. Enumerating the compositions of n with exactly k parts is given by the binomial coefficient $\binom{n-1}{k-1}$. See [43, Section 1.2]

A special type of composition is a *weak composition*. This again is a way of writing n as a sum of ordered parts, but where we only require these parts to be nonnegative. To avoid having an infinite number of weak compositions, we require $r_k > 0$. The number of weak compositions of n with exactly k parts is also given by a binomial coefficient $\binom{n+k-1}{k-1}$. Once again, see [43, Section 1.2].

1.11 Sign-reversing involutions

A final combinatorial tool to be discussed are *sign-reversing involutions*. One begins with a signed set X, meaning each element $x \in X$ has a positive or negative sign attached to it by a function $w : X \to \{1, -1\}$. A sign-reversing involution $\phi : X \to X$ is a function satisfying $\phi \circ \phi = \mathrm{id}_X$ and for all $x \in X$, either $\phi(x) = x$ or $w(x) = -w(\phi(x))$. In other words, ϕ either has x as a fixed point or it reverses the sign of x. The usefulness of sign-reversing involutions is that when you sum over the weights of all the elements of X, only the weights of the fixed elements remain; that is, $\sum_{x \in X} w(x) = \sum_{x \in F} w(x)$, where $F = \{x \in X \,:\, \phi(x) = x\}$.

The function w can also be much more complicated than simply assigning a 1 or -1 to an element, as long as the sign is part of the function definition, and ϕ must only change the sign. This makes sign-reversing involutions a great tool to narrow down the sum of the weights to a smaller subset of the signed set since the weights of the non-fixed elements will all cancel out.

The following example of a sign-reversing involution was introduced by Gessel and Viennot [25]. They proved that the number of k-tuples of non-intersecting lattice paths $(w_1, \dots, w_k)$, with w_i having only east and south steps from $(0, a_i)$ to (b_i, b_i) where $0 \leq a_1 < \cdots < a_k$ and $0 \leq b_1 < \cdots < b_k$, is given by the binomial determinant

$$\binom{a_1, \dots, a_k}{b_1, \dots, b_k} = \det\left(\binom{a_i}{b_j}\right)_{1 \leq i,j \leq k}.$$

Here the signed set X consists of pairs $(\sigma; (w_1, \dots, w_k))$ of a permutation $\sigma \in \mathfrak{S}_k$ and a k-tuple of paths such that w_i is a path from $(0, a_i)$ to $(b_{\sigma(i)}, b_{\sigma(i)})$. The sign function is given by $w((\sigma; (w_1, \ldots, w_k))) = (-1)^{\operatorname{inv} \sigma}$. An involution ϕ is constructed that fixes elements in X with non-intersecting paths, but if any paths intersect, a pair (i, j) is strategically selected in which w_i and w_j intersect at a point p. The paths are altered by ϕ so that w_i' follows w_i from $(0, a_i)$ to p then follows w_j to (b_j, b_j) and likewise for w_j'. The other paths are not changed, that is, $w_l' = w_l$ for $l \neq i, j$. The resulting permutation is $\sigma \circ (i, j)$, which causes the weight of $\phi(\sigma; (w_1, \ldots, w_k)) = (\sigma \circ (i, j); (w_1', \ldots, w_k'))$ to be the negative of the weight of the original pair. By adding all weights of elements in X, Gessel and Viennot were able to cancel out the k-tuples with intersecting paths and prove their result.

Chapter 2 Cyclotomic Factors of the Descent Set Polynomial

2.1 Introduction

For a permutation π in the symmetric group $\mathfrak{S}_n$, recall that the descent set of π is the subset of $[n-1] = \{1, 2, \ldots, n-1\}$ given by $\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$. The descent set statistics $\beta_n(S)$ are defined for subsets S of $[n-1]$ by

$$\beta_n(S) = |\{\pi \in \mathfrak{S}_n : \mathrm{Des}(\pi) = S\}| .$$

Chebikin, Ehrenborg, Pylyavskyy and Readdy [10] defined the nth descent set polynomial to be

$$Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta_n(S)}.$$

They observed that this polynomial has many factors that are cyclotomic polynomials. The most common of these cyclotomic polynomials is $\Phi_2 = t + 1$. It is direct that having Φ_2 as a factor implies that the number of subsets of $[n-1]$ having an even descent set statistic is the same as the number of subsets having an odd descent set statistic. Consider the proportion of odd entries among the descent set statistics in the symmetric group $\mathfrak{S}_n$, that is,

$$\rho(n) = \frac{|\{S \subseteq [n-1] : \beta_n(S) \equiv 1 \bmod 2\}|}{2^{n-1}}.$$

Chebikin et al. showed that this proportion depends only on the number of 1's in the binary expansion of n. We quote their paper with Table 2.1. Only the values $2^k - 1$ are included in the table since $\rho(2^k - 1)$ is the same as $\rho(n)$ if n has k 1's in its binary expansion.

Hence when n has two or three 1's in its binary expansion we obtain Φ_2 as a factor in the descent set polynomial $Q_n(t)$, as shown in [10, Theorem 6.1(i)]. Note that the proportion is not known for six or more 1's in the binary expansion.

Chebikin et al. gave more results for cyclotomic factors in the descent set polynomial:

(i) If $n = 2^j \geq 4$, then Φ_4 divides $Q_n(t)$. [10, Theorem 6.1(ii)]

(ii) If $q = p^r$ is an odd prime power with two or three 1's in its binary expansion and $q \neq 3$ or 7, then Φ_{2p} divides $Q_q(t)$. [10, Theorem 6.1(iii)]

(iii) If $q = p^r$ is an odd prime power with two or three 1's in its binary expansion, then Φ_{2p} divides $Q_{2q}(t)$. [10, Theorem 6.1(iv)]

They also found cases when there were double factors in the descent set polynomial:

(iv) If the binary expansion of n has two 1's in its binary expansion and $n > 3$, then Φ_2 is a double factor of $Q_n(t)$. [10, Theorem 7.3]

n	1	3	7	15	31
$\rho(n)$	1	1/2	1/2	$29/2^6$	$3991/2^{13}$

Table 2.1: The proportion $\rho(n)$.

(v) If $n = 2^j \geq 4$ then Φ_4 is a double factor of $Q_n(t)$. [10, Theorem 7.4]

(vi) If $q = p^r$ is an odd prime power and q has two 1's in its binary expansion, then Φ_{2p} is a double factor of $Q_{2q}(t)$. [10, Theorem 7.5]

We continue their work in explaining cyclotomic factors occurring in these polynomials. In Section 2.2 we review some preliminary notions and tools that will help in developing our results. We introduce a simplicial complex in Section 2.3 that determines the parity of the descent set statistics. Namely, the reduced Euler characteristic of an induced subcomplex gives the descent set statistics modulo 2. In Section 2.4 we determine when Φ_{4s} is a factor of $Q_n(t)$ with n being a power of 2 and s is an odd integer. In Section 2.5 we determine when Φ_{2s} is a factor of $Q_n(t)$ with n having two non-zero digits in its binary expansion and s being an odd integer . We prove a multitude of cases in this section when we set s to be a prime number p. Similarly, when n has three non-zero digits in its binary expansion, we develop cases when Φ_{2s}, and likewise Φ_{2p}, is a factor of $Q_n(t)$ in Section 2.6.

We also continue the work on double factors occurring in the descent set polynomial $Q_n(t)$ in Sections 2.7 through 2.9. In fact, the two results (iv) and (vi) both need the condition that the number of 1's in the binary expansion of n is exactly two. Furthermore, the result (vi) applies only (so far) to the five Fermat primes and the prime power 3^2, whereas our results apply when there are two or three 1's in the binary expansion. First, we show in Theorem 2.7.2 that if Φ_2 is a factor of $Q_{2n}(t)$, then it is a double factor. In Theorems 2.8.1 and 2.9.1, we find the double factor Φ_{2p} in $Q_{2q}(t)$ and $Q_{q+1}(t)$ where $q = p^r$ is an odd prime power. The corresponding proofs in [10] depend on substituting values for the variables in the **ab**-index of the Boolean algebra, whereas our proofs rely on evaluating a more general linear function; see Proposition 2.7.1. The underlying reason for these results is that the descent set statistic is straightforward to compute modulo the prime p; see Lemma 2.8.2 and equation (2.9.1).

A summary of cyclotomic factors of $Q_n(t)$ that Chebikin et al. found, as well as which ones were explained by their and our results, can be found in Table 2.6. We end with open questions in the concluding remarks.

A version of this chapter appears in [15].

2.2 Preliminaries

One of the primary tools that we will use to study descent set statistics is MacMahon's Multiplication Theorem [36, Article 159], which relates the descent set statistics of two sets that differ by only one element, stated as

$$\beta_n(S) + \beta_n(S \triangle \{k\}) = \binom{n}{k} \cdot \beta_k(S \cap [k-1]) \cdot \beta_{n-k}(S \cap [k+1, n-1] - k).$$

This result is usually written with the assumption $k \notin S$ and the left-hand side as $\beta_n(S) + \beta_n(S \cup \{k\})$, whereas we find it more convenient to work with the symmetric difference.

Recall that the descent set statistics can also be calculated using the quasi-symmetric function of the Boolean algebra

$$F(B_n) = (x_1 + x_2 + \cdots)^n = M_{(1)}^n = \sum_{\vec{c}} \binom{n}{\vec{c}} \cdot M_{\vec{c}}.$$

Here the multinomial coefficient $\binom{n}{\vec{c}}$ is the flag f-vector of the Boolean algebra for the set associated to the composition $\vec{c}$, and the descent set statistic for this set is given by

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T. \tag{2.2.1}$$

The purpose of quasi-symmetric functions is to allow efficient computations of the flag f-vector modulo a prime p using the classical relation $(x+y)^p \equiv x^p + y^p \bmod p$. Finally, using the inclusion-exclusion in equation (2.2.1), we obtain information about the descent set statistics. Below is a lemma, adapted from Lemma 3.2 in [10], to compute the quasi-symmetric function of the Boolean algebra $F(B_n) = M_{(1)}^n$ modulo a prime.

Lemma 2.2.1. *For p prime and $n = d_1 p^{j_1} + d_2 p^{j_2} + \cdots + d_k p^{j_k}$ with $j_1 > \cdots > j_k \geq 0$, the quasi-symmetric function of the Boolean algebra B_n modulo p is given by $F(B_n) \equiv \prod_{i=1}^{k} M_{(p^{j_i})}^{d_i} \bmod p$.*

Proof. The congruence $(x+y)^{p^m} \equiv x^{p^m} + y^{p^m} \bmod p$ extends to monomial quasi-symmetric functions as $M_{(1)}^{p^m} \equiv M_{(p^m)} \bmod p$. Hence the quasi-symmetric function of Boolean algebra B_n is evaluated as follows:

$$\begin{aligned} F(B_n) = M_{(1)}^{d_1 p^{j_1} + d_2 p^{j_2} + \cdots + d_k p^{j_k}} &= \left(M_{(1)}^{p^{j_1}}\right)^{d_1} \cdot \left(M_{(1)}^{p^{j_2}}\right)^{d_2} \cdots \left(M_{(1)}^{p^{j_k}}\right)^{d_k} \\ &\equiv M_{(p^{j_1})}^{d_1} \cdot M_{(p^{j_2})}^{d_2} \cdots M_{(p^{j_k})}^{d_k} \bmod p. \end{aligned}$$

□

Chebikin et al. defined essential elements in the case of base 2, and we extend this notion to base p for any prime p.

Definition 2.2.2. *Let p be a prime and $1 \leq k \leq n-1$. We say k is* essential *for n in base p if we expand both n and k in base p, that is, $n = \sum_{i \geq 0} n_i \cdot p^i$ and $k = \sum_{i \geq 0} k_i \cdot p^i$ where $0 \leq k_i, n_i < p$, and the inequality $k_i \leq n_i$ holds for all indices i. Otherwise we say k is* non-essential *for n in base p.*

A different way to state that k is essential for n in base p is that when adding k and $n-k$ in base p there are no carries. Directly from this interpretation we have the following natural symmetry:

Lemma 2.2.3. *The element k is essential for n in base p if and only if $n-k$ is essential for n in base p.*

An alternative interpretation is as follows:

Lemma 2.2.4. *The element k is essential for n in base p if and only if $\binom{n}{k} \not\equiv 0 \bmod p$.*

Proof. By Lucas' theorem, see [34, Chapter XXIII, Section 228], we have that

$$\binom{n}{k} \equiv \prod_{i \geq 0} \binom{n_i}{k_i} \bmod p.$$

Observe that for $0 \leq k_i, n_i \leq p-1$ we have that $\binom{n_i}{k_i} \not\equiv 0 \bmod p$ if and only if $k_i \leq n_i$. □

Note that for an element k which is non-essential for n in base p, the previous lemma implies that p divides $\binom{n}{k}$. This allows the following lemma to apply for this integer k when we set the integer m to be the prime p.

Lemma 2.2.5. *Let m and k be positive integers such that $1 \leq k \leq n-1$ and m divides $\binom{n}{k}$. For a subset S of $[n-1]$ the following holds:*

$$\beta_n(S) \equiv -\beta_n(S \triangle \{k\}) \bmod m.$$

Proof. By MacMahon's multiplication theorem we have that

$$\beta_n(S) + \beta_n(S \triangle \{k\}) = \binom{n}{k} \cdot \beta_k(S \cap [k-1]) \cdot \beta_{n-k}(S \cap [k+1, n-1] - k),$$

and the result follows by the assumption that $\binom{n}{k} \equiv 0 \bmod m$. □

For $0 \leq j \leq m-1$ define $a_{m,j}$ to be the number of subsets $S \subseteq [n-1]$ such that $\beta_n(S) \equiv j \bmod m$. Note that we suppress the dependency on n. Furthermore, if m is clear from the context, we simply write a_j.

Lemma 2.2.6. *Let m be a positive integer and $1 \leq k \leq n-1$. If m divides $\binom{n}{k}$ then the equality $a_{m,j} = a_{m,-j}$ holds for all j.*

Proof. By Lemma 2.2.5 we have that $\beta_n(S) \equiv -\beta_n(S \triangle \{k\}) \bmod m$. Hence the map sending S to the symmetric difference $S \triangle \{k\}$ yields a bijection between the sets counted by $a_{m,j}$ and $a_{m,-j}$. □

The following are consequences of Theorem 2.1 in [10], which gives information about the proportion of even or odd descent set statistics $\beta_n(S)$ depending on the number of 1's in the binary expansion of n. We apply their result to achieve equalities involving $a_{m,j}$.

Theorem 2.2.7 (Chebikin et al.). *(a) If n has only one 1 in its binary expansion, i.e., $n = 2^a$, then $\beta_n(S) \equiv 1 \bmod 2$ for all subsets $S \subseteq [n-1]$.*

(b) If n has either two or three 1's in its binary expansion, then there is an identical number of even descent set statistics as there is of odd descent set statistics.

In terms of the proportion introduced in the introduction, we have $\rho(2^a) = 1$, $\rho(2^b + 2^a) = 1/2$ and $\rho(2^c + 2^b + 2^a) = 1/2$ for nonnegative integers $c > b > a$. As a direct corollary we have

Corollary 2.2.8. *Let s be an odd positive integer.*

(a) If n has only one 1 in its binary expansion, then for j even $a_{2s,j} = 0$ holds.

(b) If n has either two or three 1's in its binary expansion, then

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{2s-2} a_{2s,j} = \sum_{\substack{j=0 \\ j \text{ odd}}}^{2s-1} a_{2s,j}.$$

We end with a well-known fact from algebra.

Fact 2.2.9. *If $f(t)$ is a polynomial in $\mathbb{Q}[t]$ with $e^{2\pi i/j}$ as a root of multiplicity r then the jth cyclotomic polynomial $\Phi_j(t)$ is a factor of order r of $f(t)$.*

This follows since the cyclotomic polynomial is the minimal polynomial of $e^{2\pi i/j}$ over the rational field $\mathbb{Q}$.

2.3 The simplicial complex Δ_n

We now introduce a simplicial complex which will encode the descent set statistics modulo 2 via the reduced Euler characteristic. Let Δ_n be a simplicial complex whose vertex set is a subset of $[n-1]$. Let F be a face of Δ_n if there are no carries in base 2 when adding the entries of the associated composition $\text{co}(F) = (c_1, c_2, \ldots, c_{k+1})$, that is, the sum $c_1 + c_2 + \cdots + c_{k+1} = n$.

Notice that $\{i\}$ is a vertex of Δ_n if and only if i is an essential element of n in base 2. In fact, the simplicial complex Δ_n is completely described by the number of 1's in the binary expansion of n. For n with k 1's in its binary expansion, the complex Δ_n is the barycentric subdivision of the boundary of a $(k-1)$-dimensional simplex. A different way to describe it is that Δ_n is the boundary of the dual of the $(k-1)$-dimensional permutahedron.

Theorem 2.3.1. *The quasi-symmetric function of B_n modulo 2 is given by*

$$F(B_n) \equiv \sum_{F \in \Delta_n} M_{\mathrm{co}(F)} \bmod 2.$$

Proof. Write n as a sum of 2-powers, that is, $n = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k}$ where $j_1 > j_2 > \cdots > j_k$. By Lemma 2.2.1 we have have the identity

$$F(B_n) \equiv M_{2^{j_1}} \cdot M_{2^{j_2}} \cdots M_{2^{j_k}} \bmod 2.$$

Now when expanding these k monomial quasi-symmetric functions we obtain a sum over monomial quasi-symmetric functions where the indexing composition has parts consisting of sums of the 2-powers 2^{j_1}, 2^{j_2}, ..., 2^{j_k}. Furthermore, each 2-power can only appear in exactly one part and only once in that part. Also note no composition can be created in two different ways. In the language of the article [21], the partition $\{2^{j_1}, 2^{j_2}, \ldots, 2^{j_k}\}$ is a knapsack partition. Finally, translating the compositions of n into subsets of $[n-1]$ proves the result. □

In other words, the flag f-vector entry $f_S(B_n)$ is odd if and only if S is a face of the complex Δ_n. Let $\Delta_n|_S$ denote the simplicial complex Δ_n restricted to vertex set S, that is,

$$\Delta_n|_S = \{F \subseteq S \,:\, F \in \Delta_n\}.$$

Theorem 2.3.2. *The descent set statistic $\beta_n(S)$ modulo 2 is given by the reduced Euler characteristic of the induced subcomplex $\Delta_n|_S$, that is,*

$$\beta_n(S) \equiv \widetilde{\chi}(\Delta_n|_S) \bmod 2.$$

Proof. By a direct computation we have

$$\begin{aligned}
\beta_n(S) &\equiv \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T(B_n) \\
&\equiv \sum_{T \subseteq S} (-1)^{|T|-1} \cdot f_T(B_n) \\
&\equiv \sum_{T \subseteq S,\, T \in \Delta_n} (-1)^{|T|-1} \\
&\equiv \widetilde{\chi}(\Delta_n|_S) \bmod 2.
\end{aligned}$$

□

2.4 One binary digit

In this section we explore cyclotomic factors in the descent set polynomial $Q_n(t)$ where n is a power of 2; that is, n has one 1 in its binary expansion. First we have a result showing conditions on the values of $a_{m,j}$ when we have a cyclotomic factor in the general nth descent set polynomial. Note that we abbreviate $a_{m,j}$ as a_j.

Lemma 2.4.1. *Let m be an even positive integer. The cyclotomic polynomial Φ_m is a factor of the descent set polynomial $Q_n(t)$ if the following equations hold:*

$$a_j = a_{-j}, \tag{2.4.1}$$

$$a_j = a_{m/2-j}, \tag{2.4.2}$$

for all integers j.

Proof. Consider the primitive mth root of unity $\omega = e^{2i\pi/m}$. In order for Φ_m to be a factor of $Q_n(t)$, we must have $Q_n(\omega) = 0$. Since $\omega^m = 1$, we need to show

$$Q_n(\omega) = \sum_{S\subseteq[n-1]} \omega^{\beta_n(S)} = a_0 + a_1 \cdot \omega + a_2 \cdot \omega^2 + \cdots + a_{m-1} \cdot \omega^{m-1}$$

is zero. By reflection in the real and imaginary axes in the complex plane, we have $\omega^{-j} + \omega^j + \omega^{m/2-j} + \omega^{m/2+j} = 0$, from which the result follows. □

Assume that s is an odd positive integer. We consider which values of s such that the 4sth cyclotomic polynomial Φ_{4s} divides the descent set polynomial $Q_n(t)$ when n is a power of 2.

Theorem 2.4.2. *Let $n = 2^a$ where $a \geq 2$. Assume that s is an odd integer such that s divides the central binomial coefficient $\binom{n}{n/2}$ and s divides $\binom{n}{k}$ for some $k \neq n/2$. Then the cyclotomic polynomial $\Phi_{4s}(t)$ divides the descent set polynomial $Q_n(t)$.*

Proof. Observe that there is one carry in the addition $n/2+n/2 = n$ in base 2. Hence by Kummer's theorem [32, Pages 115–116] 2 is the largest 2-power dividing $\binom{n}{n/2}$. In other words, $\binom{n}{n/2} \equiv 2 \bmod 4$. Combining this with the fact that s divides this central binomial coefficient, we have $\binom{n}{n/2} \equiv 2s \bmod 4s$. MacMahon's multiplication theorem gives that

$$\begin{aligned} \beta_n(S) + \beta_n(S\triangle\{n/2\}) = \binom{n}{n/2} &\cdot \beta_{n/2}(S \cap [1, n/2-1]) \\ &\cdot \beta_{n/2}(S \cap [n/2+1, n-1] - n/2). \end{aligned}$$

Since $\beta_{n/2}$ only takes odd values as shown in Theorem 2.2.7(a), we obtain that

$$\beta_n(S) + \beta_n(S\triangle\{n/2\}) \equiv 2s \bmod 4s.$$

Thus, the statement $\beta_n(S) \equiv j \bmod 4s$ is equivalent to $\beta_n(S\triangle\{n/2\}) \equiv 2s - j \bmod 4s$. In other words, the map $S \longmapsto S\triangle\{n/2\}$ yields a bijection that proves $a_j = a_{2s-j}$ for all j.

Next since the addition $k + (n - k) = n$ in base 2 has at least two carries, we obtain that $2^2 = 4$ divides the binomial coefficient $\binom{n}{k}$. Hence, $4s$ divides $\binom{n}{k}$ and by Lemma 2.2.6 the equality $a_j = a_{-j}$ holds for all j. We now have that both equations (2.4.1) and (2.4.2) from Lemma 2.4.1 hold. Thus, the cyclotomic polynomial Φ_{4s} divides $Q_n(t)$. □

n	s	k	Chebikin et al. statement	Our statement
4	1	1	Thm. 3.5	Thm. 2.4.2
8	1	1	Thm. 3.5	Thm. 2.4.2
8	7	2		Thm. 2.4.2
16	1	1	Thm. 3.5	Thm. 2.4.2
16	5, 11, 13, 55 65, 143, 715	7		Thm. 2.4.2
16	3, 15	5		Thm. 2.4.2
16	39	2		Thm. 2.4.2
32	all the divisors of 17678835	15		Rem. 2.4.3

Table 2.2: Examples of cyclotomic factors of $Q_n(t)$ of the form Φ_{4s} where $n = 2^a$.

Remark 2.4.3. The case $n = 32 = 2^5$ and $k = 15$ is particularly nice. We have that $\binom{32}{16} = 2 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$ and $\binom{32}{15} = 16/17 \cdot \binom{32}{16}$. Hence, for any of the 96 divisors s of $3^2 \cdot 5 \cdot 19 \cdot 23 \cdot 29 \cdot 31$, we obtain the cyclotomic factor Φ_{4s} of $Q_{32}(t)$. Furthermore, we do not obtain any more cyclotomic factors by changing k; that is, all the the odd divisors of $\binom{32}{k}$ for $k \leq 14$ are divisors of $\binom{32}{15}$.

See Table 2.2 for examples of cyclotomic factors of $Q_{2^a}(t)$ that are explained by Theorem 2.4.2, along the k value for which s divides $\binom{2^a}{k}$.

2.5 Two binary digits

Now we state the result that lets us deduce cases when the cyclotomic polynomial Φ_{2s}, where s is an odd positive integer, is a factor of the descent set polynomial $Q_n(t)$ when n has two 1's in its binary expansion.

Theorem 2.5.1. *Let $n = 2^b + 2^a$, where $b > a$, and assume s is an odd positive integer which divides $\binom{n}{2^a}$. Furthermore, assume there is an integer k which is non-essential for n in base 2 (that is, $k \neq 2^a, 2^b$) and such that s divides $\binom{n}{k}$. Then the cyclotomic polynomial Φ_{2s} is a factor of $Q_n(t)$.*

Proof. Since there are no carries in the addition $2^b + 2^a = n$ in base 2, by Kummer's theorem we know that $\binom{n}{2^a}$ is odd. Combining this fact with the congruence modulo s, we obtain $\binom{n}{2^a} \equiv s \bmod 2s$. Therefore, by MacMahon's multiplication theorem, we have that

$$\begin{aligned} \beta_n(S) + \beta_n(S \triangle \{2^a\}) &= \binom{n}{2^a} \cdot \beta_{2^a}(S \cap [2^a - 1]) \cdot \beta_{2^b}(S \cap [2^a + 1, n-1] - 2^a) \\ &\equiv s \bmod 2s, \end{aligned} \tag{2.5.1}$$

since both β_{2^a} and β_{2^b} are odd according to Theorem 2.2.7. Hence, we use the bijective map $S \longmapsto S \triangle \{2^a\}$ to conclude that $a_j = a_{s-j}$ for all j.

Since the addition $k + (n - k)$ has at least one carry in base 2, the binomial coefficient $\binom{n}{k}$ is even. Hence $\binom{n}{k}$ is divisible by $2s$. By Lemma 2.2.6 the inequality $a_j = a_{-j}$ holds for all j. Combining these two equalities using Lemma 2.4.1, the result follows. $\square$

We begin with two remarkable examples.

Remark 2.5.2. Consider the case $n = 18 = 2^4 + 2^1$ and $k = 4$. Note that $\binom{18}{2} = 3^2 \cdot 17 = 153$. Furthermore note that $\binom{18}{4} = 2^2 \cdot 5 \cdot \binom{18}{2}$. Hence for any divisor s of 153 we obtain that the cyclotomic polynomial Φ_{2s} divides the descent set polynomial $Q_{18}(t)$. This argument explains all the cyclotomic factors found in the descent set polynomial $Q_{18}(t)$. See Table 2.6.

Remark 2.5.3. Consider the case $n = 20 = 2^4 + 2^2$ and $k = 6$. Now we have $\binom{20}{4} = 3 \cdot 5 \cdot 17 \cdot 19 = 4845$ and $\binom{20}{6} = 2^3 \cdot \binom{20}{4}$. Hence for any divisor s of 4845 the cyclotomic polynomial Φ_{2s} is a factor in the descent set polynomial $Q_{20}(t)$, explaining all the 16 known cyclotomic factors. See the longest row in Table 2.6.

We now continue to study the case when the integer s is an odd prime p. Recall from Lemma 2.2.4 that k being a non-essential element for n in base p implies that p divides $\binom{n}{k}$. Hence, to satisfy the assumptions in Theorem 2.5.1 for this case, we need to show that 2^a and k are non-essential for n in base p and that k is also non-essential for n in base 2.

Note however that for two relative prime integers p and q, a carry in the addition $k + (n - k) = n$ in base $p \cdot q$ does not imply a carry for this addition in both base p and q. As an example consider the sum $12 + 3 = 15$. In base 15 there is a carry, whereas in base 3 there is no carry. Hence it is difficult to lift results for two primes p and q to their product $p \cdot q$.

The following lemma is useful in determining when 2^a is non-essential for n in base p for p prime in order to apply Theorem 2.5.1. Although rarely cited during the subsequent arguments since we often need the actual value of $i + j \bmod p$ instead of only the fact that it is at least p, it provides reasoning for finding particular values of n.

Lemma 2.5.4. *For $n = 2^a + 2^b$, if $2^a \equiv i \bmod p$ and $2^b \equiv j \bmod p$ where $1 \leq i, j \leq p - 1$ and $i + j \geq p$, then 2^a is non-essential for n in base p.*

Proof. Since $i, j \leq p - 1$ and $i + j \geq p$, we have $i > i + j \bmod p$. Therefore, the last digit of the base p expansion of 2^a is larger than the last digit of the base p expansion of n, causing 2^a to be non-essential for n in base p. $\square$

The following theorems provide conditions for the prime p, the multiplicative order g of 2 in $\mathbb{Z}_p^*$, and the exponents a and b that allow Theorem 2.5.1 to be applied to show that Φ_{2p} is a factor of $Q_n(t)$.

Theorem 2.5.5. *Assume that 2 has order g in the multiplicative group $\mathbb{Z}_p^*$ where g is even. Let $n = 2^b + 2^a$ where we assume $b > a$ and $n \geq 9$. If we have $\{a, b\} \equiv \{0, g/2\} \bmod g$, then 2^a is non-essential for n in base p. Furthermore, the element 7 is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. Since $2^{g/2} \not\equiv 1 \bmod p$ and $(2^{g/2} - 1) \cdot (2^{g/2} + 1) = 2^g - 1 \equiv 0 \bmod p$ we know that $2^{g/2} \equiv -1 \bmod p$ using that p is a prime. Hence the last digits of 2^a and 2^b in their base p expansions are 1 and $p - 1$, in some order. Thus we have $n = 2^b + 2^a \equiv 1 + (p - 1) \equiv 0 \bmod p$; that is, the last digit in the base p expansion of n is 0. Hence 2^a is non-essential in base p.

Notice that 7 has three non-zero digits in its binary expansion compared to only 2 such digits for n, making 7 non-essential for n in base 2. Since the order of 2 in $\mathbb{Z}_7^*$ is 3, which is odd, we have $p \neq 7$. Finally, the last digit of the base p expansion of 7 is non-zero for all odd primes $p \neq 7$. Hence 7 is also non-essential for n in base p, completing the result. □

Remark 2.5.6. The assumption in Theorem 2.5.5 of $n \geq 9$ was needed in order for 7 to always be a non-essential element. Note that the theorem can still be applied when $n = 6$ if $p = 3$. The element 5 is instead chosen as the non-essential element in base 2 and in base p.

Theorem 2.5.7. *Assume that 2 has order g in the multiplicative group $\mathbb{Z}_p^*$ where g is even. Let $n = 2^b + 2^a$ where we assume $b > a$ and $n > 2p - 1$. If we have $a \equiv b \equiv g/2 \bmod g$, then 2^a is non-essential for n in base p. Furthermore, the element $2p - 1$ is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. Similar to part of the previous proof, we have in this case that $2^a \equiv 2^b \equiv 2^{g/2} \equiv p - 1 \bmod p$. Therefore, $n = 2^a + 2^b \equiv (p - 1) + (p - 1) \equiv p - 2 \bmod p$. Thus, the last digit of the base p expansion of n is $p - 2$ while the last digit of the expansion of 2^a is $p - 1$, making 2^a be non-essential for n in base p.

Since $2p - 1$ is odd, the last digit in its base 2 expansion is 1, but the last digit of the base 2 expansion of n is 0 because $a, b \neq 0$. Hence $2p - 1$ is non-essential for n in base 2. Additionally, $2p - 1 \equiv p - 1 > p - 2 \bmod p$, thus it is non-essential for n in base p as well. □

Remark 2.5.8. The equivalence conditions on the exponents in Theorems 2.5.5 and 2.5.7 are not the only such conditions that make the theorem hold true when g is even. There are many such conditions, especially if 2 is a generator of $\mathbb{Z}_p^*$ since the powers of 2 contain every possible non-zero value as the last digit, and all that is needed is for the argument in the proof of 2^a being non-essential for $n = 2^b + 2^a$ in base p is for the sum of these digits to be at least p, as shown in Lemma 2.5.4. In this case of 2 being a generator of $\mathbb{Z}_p^*$ for $p = 2r + 1$, there are exactly $r \cdot (r + 1)$ pairs of possible exponents modulo g that will work. One still needs to find an integer k that is non-essential in base 2 and in base p. Finding this value of k is easy if given a particular pair of n and p values, but this step causes a further generalization of the proof to be difficult.

p	g	$\{a, b\} \bmod g$
3	2	$\{0,1\}, \{1,1\}$
5	4	$\{0,2\}, \{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}, \{3,3\}$
11	10	$\{0,5\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,4\}, \{3,3\}, \{3,5\}, \{3,6\}, \{3,7\}, \{3,8\}, \{3,9\}, \{4,5\}, \{4,6\}, \{4,7\}, \{4,9\}, \{5,5\}, \{5,6\}, \{5,7\}, \{5,8\}, \{5,9\}, \{6,6\}, \{6,7\}, \{6,8\}, \{6,9\}, \{7,7\}, \{7,9\}, \{9,9\}$
17	8	$\{0,4\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,4\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,5\}, \{5,6\}, \{5,7\}, \{6,6\}, \{6,7\}, \{7,7\}$

Table 2.3: Examples of equivalency conditions for small prime numbers.

Table 2.3 includes all of the equivalence conditions modulo the order g for four odd primes that lead to 2^a being non-essential for n in base p. For examples of finding the non-essential k value, see Table 2.4.

Remark 2.5.9. If 2 has multiplicative order g in $\mathbb{Z}_p^*$, then its order G in $\mathbb{Z}_{p^l}^*$ is a divisor of $p^{l-1} \cdot g$. The order g gives the length of the repeating sequence of the last digit of the base p expansions of the powers 2^a. Likewise, the order G gives the length of the repeating sequence of the last l digits of those powers of 2. Similar reasoning to Lemma 2.5.4 applies when adding together any pair of digits together, not just the last digit. Thus, there are equivalences modulo G that cause 2^a to be non-essential for $n = 2^b + 2^a$ in base p because of a carry in one of the last l digits. As an example, when $p = 3$ the order of 2 in $\mathbb{Z}_9^*$ is 6, hence the last two digits of 2^a cycle through the six values 01, 02, 11, 22, 21 and 12 as a increases. Therefore, when $\{a, b\} \equiv \{2, 4\} \bmod 6$, the last two digits of n in base 3 are $11 + 21 \equiv 02$, so the second digit from the right is larger for 2^a than for n, making it non-essential for n in base p.

Theorem 2.5.10. *Let $n = 2^b + 2^a$ where we assume $b > a$ and $n \geq 5$, and also assume that $p > 3$. If we have $a, b \equiv g - 1 \bmod g$ where g is the multiplicative order of 2, then 2^a is non-essential for n in base p. Furthermore, the element 3 is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. If the multiplicative order of 2 is g, then g is the smallest integer for which $2^g \equiv 1 \bmod p$. Thus, $2^a \equiv 2^b \equiv 2^{g-1} > 1 \bmod p$, and $n = 2^a + 2^b \equiv 2^{g-1} + 2^{g-1} \equiv 2^g \equiv 1 \bmod p$. Hence 2^a is non-essential for n in base p because the last digit in its base p expansion is larger than that of n.

The element 3 is non-essential for n in base 2 since our assumption of $n \geq 5$ implies that $b \geq 2$. Because of our assumption that $p > 3$, the element 3 is also non-essential in base p since the last digit of the base p expansion for n is $1 < 3$, concluding the result. □

Note that we omitted $p = 3$ from the previous result because this case was already proven in Theorem 2.5.7 due to the order of 2 being $g = 2$, making $g/2 = g - 1$.

n	s	$\{a,b\}$ mod g	k	Chebikin et al. statement	Our statement
6	3	$\{0,1\}$	5	Thm. 5.6	Rem. 2.5.6
6	5	$\{1,2\}$	3		Rem. 2.5.8
9	3	$\{0,1\}$	7	Thm. 5.5	Thm. 2.5.5
9	9		2		Thm. 2.5.1
10	3	$\{1,1\}$	5		Thm. 2.5.7
10	5	$\{1,3\}$	1	Thm. 5.6	Rem. 2.5.8
10	9		5		Thm. 2.5.1
10	15		3		Thm. 2.5.1
12	3	$\{0,1\}$	7		Thm. 2.5.5
12	5	$\{2,3\}$	3		Rem. 2.5.8
12	11	$\{2,3\}$	2		Rem. 2.5.8
12	55		3		Thm. 2.5.1
12	9, 33, 99		5		Thm. 2.5.1
17	17	$\{0,4\}$	7	Thm. 5.5	Thm. 2.5.5
18	17	$\{1,4\}$	3		Rem. 2.5.8
18	9, 51, 153		4		Rem. 2.5.2
20	3	$\{2,4\}$ mod 6	3		Rem. 2.5.9
20	5	$\{0,2\}$	7		Thm. 2.5.5
20	17	$\{2,4\}$	5		Rem. 2.5.8
20	15, 19, 51, 57, 85, 95, 255, 285, 323, 969, 1615, 4845		6		Rem. 2.5.3
72	3	$\{0,1\}$	7		Thm. 2.5.5
528	31	$\{4,4\}$	3		Thm. 2.5.10
1088	5	$\{2,2\}$	9		Thm. 2.5.7

Table 2.4: Examples of cyclotomic factors of $Q_n(t)$ of the form Φ_{2s}, where the binary expansion of n has two 1's.

Remark 2.5.11. Assuming $p > 3$, if p is a Mersenne prime; that is, p has the form $2^q - 1$ implying that q is also a prime number, the equivalence condition on the exponents in Theorem 2.5.10 is the only such condition modulo g for which Φ_{2p} is a factor of $Q_n(t)$. The first examples of Mersenne primes after 3 are $p = 7$, 31 and 127.

Table 2.4 summarizes particular values of n and s for which Φ_{2s} is a factor of $Q_n(t)$ with n having two binary digits. The fourth column displays an element k that is non-essential for n in base 2 and so that s divides $\binom{n}{k}$. The fifth and sixth columns give references to the statement explaining why it is a factor. For the cases in which s is a prime p, the set of exponents modulo the multiplicative order g of 2 is also listed in the third column. The top portion includes factors that were known by Chebikin et al., although many were left unexplained in their work. The bottom portion displays just a few examples within the infinite classes of factors that are explained by our results that were previously unknown.

2.6 Three binary digits

We now continue to explore cyclotomic factors Φ_{2s}, where s is an odd positive integer, in the descent set polynomial $Q_n(t)$ where n has three 1's in its binary expansion.

Theorem 2.6.1. *Let $n = 2^c + 2^b + 2^a$, where $c > b > a$, and assume s is an odd positive integer which divides the three binomial coefficients $\binom{n}{2^a}$, $\binom{n}{2^b}$, and $\binom{n}{2^c}$. Assume furthermore that there is an element k which is non-essential for n in base 2, that is, $k \notin \{2^a, 2^b, 2^a + 2^b, 2^c, 2^c + 2^a, 2^c + 2^b\}$, and that s divides $\binom{n}{k}$. Then the cyclotomic polynomial Φ_{2s} is a factor of the descent set polynomial $Q_n(t)$.*

Proof. Since there is an element k which is non-essential for n in base 2, we know that 2 divides $\binom{n}{k}$. Thus $2s$ divides $\binom{n}{k}$, and Lemma 2.2.6 gives that $a_j = a_{-j}$ for all j.

Next, our major goal is to show that $a_j = a_{s-j}$. We do that by constructing an involution ϕ on all subsets of $[n-1]$ such that $\beta_n(S) + \beta_n(\phi(S)) \equiv s \bmod 2s$. Hence for every contribution to a_j there is a corresponding contribution to a_{s-j}. The form of the involution ϕ will be $\phi(S) = S \triangle X$ where the subset X depends on how S intersects the four element set $\{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\}$.

Since the elements 2^c and $2^b + 2^a$ are both essential for n in base 2, we apply MacMahon's theorem to get

$$\begin{aligned} \beta_n(S) + \beta_n(S \triangle \{2^b + 2^a\}) &= \binom{n}{2^b + 2^a} \cdot \beta_{2^b+2^a}(S \cap [1, 2^b + 2^a - 1]) \\ &\quad \cdot \beta_{2^c}(S \cap [2^b + 2^a + 1, n-1] - (2^b + 2^a)) \\ &\equiv \begin{cases} 0 & \text{if } |S \cap \{2^a, 2^b\}| = 1, \\ 1 & \text{if } |S \cap \{2^a, 2^b\}| = 0 \text{ or } 2 \end{cases} \bmod 2, \end{aligned}$$

$$\begin{aligned} \beta_n(S) + \beta_n(S \triangle \{2^c\}) &= \binom{n}{2^c} \cdot \beta_{2^c}(S \cap [2^c - 1]) \cdot \beta_{2^b+2^a}(S \cap [2^c + 1, n-1] - 2^c) \\ &\equiv \begin{cases} 0 & \text{if } |S \cap \{2^c + 2^a, 2^c + 2^b\}| = 1, \\ 1 & \text{if } |S \cap \{2^c + 2^a, 2^c + 2^b\}| = 0 \text{ or } 2 \end{cases} \bmod 2, \end{aligned}$$

since the two binomial coefficients $\binom{n}{2^b+2^a} = \binom{n}{2^c}$ are both odd and the descent set statistics involving β_{2^c} are also odd by Theorem 2.2.7 (a). Therefore, the sums of these descent set statistics are determined by the values for $\beta_{2^b+2^a}$, which we examine by considering the complex $\Delta_{2^b+2^a}$ and using Theorem 2.3.2. This complex consists of only of two isolated vertices at 2^b and 2^a. Thus, the induced subcomplex $\Delta_n|_{S \cap [1, 2^b+2^a-1]}$ is a single vertex if $|S \cap \{2^a, 2^b\}| = 1$ with a reduced Euler characteristic of 0. Otherwise, it is two isolated vertices or the empty complex, both of which

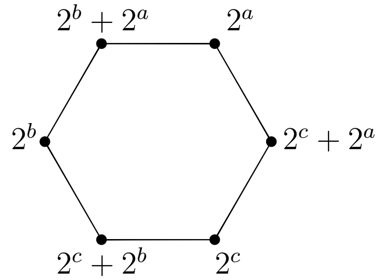

Figure 2.1: The complex Δ_n for $n = 2^c + 2^b + 2^a$. Note that the essential elements are $2^a, 2^b, 2^b + 2^a, 2^c, 2^c + 2^a, 2^c + 2^b$ in base 2, corresponding to the vertices.

have a reduced Euler characteristic of 1 mod 2. The reasoning behind the second sum is identical once the set S is shifted down by 2^c.

Since s divides $\binom{n}{2^c} = \binom{n}{2^a+2^b}$, we have by Lemma 2.2.5 that

$$\beta_n(S) + \beta_n(S\triangle\{2^b + 2^a\}) \equiv \beta_n(S) + \beta_n(S\triangle\{2^c\}) \equiv 0 \bmod s.$$

Combining this with the modulo 2 sums, we have the following results modulo $2s$:

$$\beta_n(S) + \beta_n(S\triangle\{2^b + 2^a\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{2^a, 2^b\}| = 1, \\ s & \text{if } |S \cap \{2^a, 2^b\}| = 0 \text{ or } 2 \end{cases} \bmod 2s, \tag{2.6.1}$$

$$\beta_n(S) + \beta_n(S\triangle\{2^c\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{2^c + 2^a, 2^c + 2^b\}| = 1, \\ s & \text{if } |S \cap \{2^c + 2^a, 2^c + 2^b\}| = 0 \text{ or } 2 \end{cases} \bmod 2s. \tag{2.6.2}$$

We now begin to construct the involution ϕ. Assume that $|S \cap \{2^a, 2^b\}| = 0$ or 2. Then by equation (2.6.1), $\beta_n(S) + \beta_n(S\triangle\{2^b + 2^a\}) \equiv s \bmod 2s$. Hence in this case, let the involution be given by $\phi(S) = S\triangle\{2^b + 2^a\}$.

The symmetric case is as follows. Assume that we have $|S \cap \{2^a, 2^b\}| = 1$ and $|S\cap\{2^c+2^a, 2^c+2^b\}| = 0$ or 2. By equation (2.6.2), $\beta_n(S)+\beta_n(S\triangle\{2^c\}) \equiv s \bmod 2s$. Thus, let the involution be given by $\phi(S) = S\triangle\{2^c\}$.

The case that remains is when the set S satisfies $|S \cap \{2^a, 2^b\}| = 1$ and $|S \cap \{2^c + 2^a, 2^c + 2^b\}| = 1$. By equations (2.6.1) and (2.6.2), and by equation (2.6.1) again, we have the following string of congruences:

$$\beta_n(S) \equiv -\beta_n(S\triangle\{2^b + 2^a\}) \equiv \beta_n(S\triangle\{2^b + 2^a, 2^c\}) \equiv -\beta_n(S\triangle\{2^c\}) \bmod 2s.$$

Observe that these four descent set statistics all have the same parity. In order to determine this parity, we need to consider the complex Δ_n, displayed in Figure 2.1, and then apply Theorem 2.3.2.

We now have four subcases to consider.

- First consider sets S such that $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} = \{2^a, 2^c + 2^a\}$. Note that the four induced subcomplexes $\Delta_n|_S$, $\Delta_n|_{S\triangle\{2^b+2^a\}}$, $\Delta_n|_{S\triangle\{2^c\}}$ and $\Delta_n|_{S\triangle\{2^b+2^a,2^c\}}$ are all contractible and therefore have reduced Euler characteristic 0. Hence in this case $\beta_n(S)$, $\beta_n(S\triangle\{2^b + 2^a\})$, $\beta_n(S\triangle\{2^c\})$ and $\beta_n(S\triangle\{2^b + 2^a, 2^c\})$ are all even.

- Second, when $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} = \{2^b, 2^c + 2^b\}$, by considering the reverse sets of the previous case, the four sets S, $S \triangle \{2^b + 2^a\}$, $S \triangle \{2^c\}$ and $S \triangle \{2^b + 2^a, 2^c\}$ have even descent set statistics because their corresponding induced subcomplexes are contractible.

- Third, consider sets S such that $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} = \{2^a, 2^c + 2^b\}$. Now the four induced subcomplexes $\Delta_n|_S$, $\Delta_n|_{S\triangle\{2^b+2^a\}}$, $\Delta_n|_{S\triangle\{2^c\}}$ and $\Delta_n|_{S\triangle\{2^b+2^a,2^c\}}$ are all homotopy equivalent to two points and hence have reduced Euler characteristic 1. Hence in this case the descent set statistics of the four sets S, $S \triangle \{2^b + 2^a\}$, $S \triangle \{2^c\}$ and $S \triangle \{2^b + 2^a, 2^c\}$ are all odd.

- The fourth and last case is when $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} = \{2^b, 2^c + 2^a\}$. Again, the four induced subcomplexes $\Delta_n|_S$, $\Delta_n|_{S\triangle\{2^b+2^a\}}$, $\Delta_n|_{S\triangle\{2^c\}}$ and $\Delta_n|_{S\triangle\{2^b+2^a,2^c\}}$ are all homotopy equivalent to two points and hence have reduced Euler characteristic 1. Therefore, the descent set statistics of the four sets S, $S \triangle \{2^b + 2^a\}$, $S \triangle \{2^c\}$ and $S \triangle \{2^b + 2^a, 2^c\}$ are all odd.

From these four subcases above we know that $\beta_n(S) \equiv 1 + \beta_n(S \triangle \{2^a, 2^b\}) \bmod 2$. Next, since 2^a and 2^b both satisfy $\binom{n}{2^a} \equiv \binom{n}{2^b} \equiv 0 \bmod s$, we have that $\beta_n(S) \equiv -\beta_n(S \triangle \{2^a\}) \equiv \beta_n(S \triangle \{2^a, 2^b\}) \bmod s$. Combining these two statements and using that $2s$ divides $\binom{n}{k}$ we conclude that

$$\beta_n(S) \equiv s + \beta_n(S \triangle \{2^a, 2^b\}) \equiv s - \beta_n(S \triangle \{2^a, 2^b, k\}) \bmod 2s.$$

Thus, the third and final case of the definition of ϕ is $\phi(S) = S \triangle \{2^a, 2^b, k\}$ when $|S \cap \{2^a, 2^b\}| = 1$ and $|S \cap \{2^c + 2^a, 2^c + 2^b\}| = 1$. This proves the equality $a_j = a_{s-j}$ holds, and hence the theorem follows by Lemma 2.4.1. □

One might ask if it is possible for Φ_{2s} to be a factor of $Q_n(t)$ if the binary expansion of n has more than three binary digits. Although the equations within Lemma 2.4.1 are only sufficient conditions and not necessary conditions for this cyclotomic polynomial to be a factor, it is easy to see why this lemma cannot be used. This is because equal numbers of even and odd descent set statistics are required, as this is implied by the combination of equations (2.4.1) and (2.4.2). Chebikin et al. showed that there are not equal numbers when the binary expansion of n has 4 or 5 digits. It is not known if there is an integer $k > 3$ for which this condition is true when n has k binary digits.

Similar to Remark 2.5.2 and 2.5.3, we have the next remark about $n = 21$ and 22.

Remark 2.6.2. Consider $n = 21 = 2^4 + 2^2 + 1$ and $k = 2$. Observe that $\gcd\left(\binom{21}{16}, \binom{21}{4}, \binom{21}{1}\right) = 21$. Furthermore, observe that $\binom{21}{2}$ is a multiple of 21. Hence we obtain for each divisor s of 21 that the cyclotomic polynomial Φ_{2s} divides $Q_{21}(t)$. Similarly, for $n = 22 = 2^4 + 2^2 + 2^1$ and $k = 3$, we have that $\gcd\left(\binom{22}{16}, \binom{22}{4}, \binom{22}{2}\right) = 77$ divides $\binom{22}{3}$. Hence for each divisor s of 77 we conclude that Φ_{2s} divides $Q_{22}(t)$.

We continue to consider the case when the integer s is an odd prime p. The following theorems give conditions for p and the exponents a, b and c that provide the assumptions made for applying Theorem 2.6.1.

Theorem 2.6.3. *Let $n = 2^c + 2^b + 2^a$ where we assume $c > b > a$, $n \geq 11$, and that the order g of 2 in the multiplicative group $\mathbb{Z}_p^*$ is even. If we have $\{a, b, c\} \equiv \{1, g/2, g/2\} \bmod g$, then 2^c, 2^b, and 2^a are non-essential for n in base p. Furthermore, the element 7 is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. Using the same congruences as in the proof of Theorem 2.5.5, we have

$$n = 2^c + 2^b + 2^a \equiv 2^{g/2} + 2^{g/2} + 2^1 \equiv (p-1) + (p-1) + 2 \equiv 0 \bmod p.$$

Hence the last digit in the base p expansion of n is 0. This makes 2^c, 2^b, and 2^a be non-essential for n in base p since the last digit for these powers of 2 are each greater than 0.

The assumption that $n \geq 11$ implies that $c \geq 3$, hence the number 7 is non-essential for n in base 2. Additionally, the last digit of the base p expansion of 7 is non-zero except when $p = 7$. This case is not included for this theorem since the order of 2 in $\mathbb{Z}_7^*$ is odd. Therefore, 7 is non-essential in base p as well, which concludes the proof of the theorem. □

Theorem 2.6.4. *Let $n = 2^c + 2^b + 2^a$ where $c > b > a$, and assume that p is an odd prime greater than or equal to 5. If $\{a, b, c\} \equiv \{g-2, g-2, g-1\} \bmod g$ where g is the multiplicative order of 2 in $\mathbb{Z}_p^*$, then 2^c, 2^b, and 2^a are non-essential for n in base p. Furthermore, the element 3 is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. We have

$$n = 2^c + 2^b + 2^a \equiv 2^{g-2} + 2^{g-2} + 2^{g-1} \equiv 2^g \equiv 1 \bmod p,$$

hence the last digit of the base p expansion of n is 1. Since we assume $p \geq 5$, we must have $g \geq 3$, hence $2^{g-1} > 2^{g-2} > 1 \bmod p$. Thus, 2^c, 2^b, and 2^a are non-essential for n in base p since the last digit of their base p expansions is larger than 1.

Also, since we assume $p \geq 5$, the last digit of the base p expansion of 3 is greater than 1 as well, making it non-essential for n in base p. The element 3 is also non-essential for n in base 2 since the fact that $g \geq 3$ implies that $2^a \neq 1$. □

Remark 2.6.5. When n has two binary digits, there were many equivalences modulo m on the exponents a and b beyond what could be shown in results that held for all p or for all p with m being even. Likewise, many such equivalences exist in the three binary digit case that cause each of 2^c, 2^b, and 2^a to be non-essential for $n = 2^c + 2^b + 2^a$ in base p. Examples of these equivalences include the following:

- $\{a, b, c\} \equiv \{0, 0, 0\} \bmod 2$ when $p = 3$ since the final digit of n in base 3 is $1 + 1 + 1 \equiv 0 \bmod 3$,
- $\{a, b, c\} \equiv \{1, 2, 4\} \bmod 10$ when $p = 11$ because the last digit of n is $2+4+16 \equiv 0 \bmod 11$,

- $\{a, b, c\} \equiv \{1, 2, 3\} \bmod 12$ when $p = 13$ since the last digit of n in base 13 is $2 + 4 + 8 \equiv 1 \bmod 13$.

Of course, to show that $Q_n(t)$ has Φ_{2p} as a factor, one still needs to find an element k that is non-essential for n in base 2 and p. This is shown for these examples for particular values of n in Table 2.5.

Remark 2.6.6. As with Remark 2.5.9, we can also find equivalences modulo G for the exponents a, b and c when n has three binary digits. As an example, when $p = 3$ there are equivalences such as $\{a, b, c\} \equiv \{3, 4, 5\} \bmod 6$ that cause 2^c, 2^b, and 2^a to be non-essential for n in base p. This exists because the last two digits of n are $22 + 21 + 12 \equiv 02$, whereas the second to last digit of 2^c, 2^b and 2^a is each greater than 0.

Theorem 2.6.7. *Let $n = 2^c + 2^b + 2^a$ where $c > b > a$ and $n > 7$, and assume that $p = 2^e + 2^d + 1$ where $e > d$. If $\{a, b, c\} \equiv \{0, d, e\} \bmod g$ where g is the multiplicative order of 2 in $\mathbb{Z}_p^*$, then 2^c, 2^b, and 2^a are non-essential for n in base p. Furthermore, at least one of the elements 7 or 13 is non-essential for n in both base 2 and base p. Hence Φ_{2p} is a factor of $Q_n(t)$.*

Proof. We have

$$n = 2^c + 2^b + 2^a \equiv 2^e + 2^d + 1 \equiv p \equiv 0 \bmod p,$$

making the last digit in the base p expansion of n be 0. This causes 2^c, 2^b, and 2^a to be non-essential for n in base p since their last digits are 1, 2^d, or 2^e, all of which are greater than $0 \bmod p$.

First consider when $p \neq 7$. In this case, the element 7 is non-essential for n in base p because the last digit in its base p expansion is greater than 0, which is the last digit for n. Since we assume $n > 7$ with three binary digits, the element 7 is also non-essential for n in base 2 since 7 also has three digits in its binary expansion, completing the result in this case.

If we instead assume $p = 7$, then the element 13 is non-essential in base 7 since its base 7 expansion has a 6 as its final digit. The assumption of $n > 7$, the fact that $d = 1$ and $e = 2$, and that $\{a, b, c\} \equiv \{0, d, e\} \bmod 3$ where the 3 is the order of 2 in $\mathbb{Z}_7^*$, result in the smallest such value for n being 14. Since n and 13 each have three binary digits with $n > 13$, we have that 13 is also non-essential for n in base 2, concluding the proof of the theorem. □

The following proposition explains the occurrence of another cyclotomic factor of the form Φ_{2p} which is an outlier compared to other such factors. When $n = 11$ and $p = 3$, observe from the base 2 and base 3 expansions of $11 = 2^3 + 2 + 1 = 3^2 + 2 \cdot 1$ that both 2 and 1 are essential for 11 in base p. Therefore, Theorem 2.6.1 is not applicable. However, Φ_6 is still a factor of the descent set polynomial for $n = 11$, as shown in Proposition 2.6.9. We first need the following lemma to obtain certain descent set statistics modulo 3.

Lemma 2.6.8. *Let R be a subset of the interval $[3,8]$. Then we have the following four evaluations of descent set statistics:*

$$\beta_{11}(R \cup \{1,9\}) \equiv \beta_{11}(R \cup \{2,10\}) \equiv (-1)^{|R|} \bmod 3,$$
$$\beta_{11}(R \cup \{1,10\}) \equiv \beta_{11}(R \cup \{2,9\}) \equiv -(-1)^{|R|} \bmod 3.$$

In particular, all of these values are non-zero modulo 3.

Proof. We consider the quasi-symmetric function of B_{11} modulo 3. Using Lemma 2.2.1, we have

$$\begin{aligned} F(B_{11}) &\equiv M_{(9)} \cdot M_{(1)}^2 \\ &\equiv M_{(9)} \cdot (M_{(2)} + 2M_{(1,1)}) \\ &\equiv M_{(11)} + M_{(9,2)} + M_{(2,9)} + 2M_{(9,1,1)} + 2M_{(10,1)} \\ &\quad + 2M_{(1,9,1)} + 2M_{(1,10)} + 2M_{(1,1,9)} \bmod 3, \end{aligned}$$

where the second and third steps are expanding a product of monomial quasi-symmetric functions; see [12, Lemma 3.3]. Reading off the coefficients of the monomial quasi-symmetric functions, we have the following values for the flag f-vector:

$$f_S \equiv \begin{cases} 1 & \text{if } S = \emptyset,\ \{9\},\ \text{or } \{2\}, \\ 2 & \text{if } S = \{9,10\},\ \{10\},\ \{1,10\},\ \{1\},\ \text{or } \{1,2\}, \\ 0 & \text{otherwise}, \end{cases} \quad \bmod 3.$$

Observe that only eight entries are non-zero modulo 3. Using inclusion-exclusion, the descent set statistic is given by

$$\begin{aligned} \beta_{11}(R \cup \{1,9\}) &\equiv \sum_{T \subseteq R \cup \{1,9\}} (-1)^{|R \cup \{1,9\} - T|} \cdot f_T \\ &\equiv (-1)^{|R \cup \{1,9\}|} \cdot f_\emptyset + (-1)^{|R \cup \{9\}|} \cdot f_{\{1\}} + (-1)^{|R \cup \{1\}|} \cdot f_{\{9\}} \\ &\equiv (-1)^{|R|} \bmod 3. \end{aligned}$$

The three descent set statistics $\beta_{11}(R \cup \{1,10\})$, $\beta_{11}(R \cup \{2,9\})$, and $\beta_{11}(R \cup \{2,10\})$ can be computed similarly. □

Proposition 2.6.9. *The cyclotomic polynomial Φ_6 is a factor of the descent set polynomial $Q_{11}(t)$.*

Proof. Observe from the base 2 and base 3 expansions of 11 that 4 is non-essential for 11 in base 2 and in base 3. Therefore, Lemma 2.2.6 implies that $a_j = a_{-j}$ for all j, or $a_1 = a_5$ and $a_2 = a_4$. We next focus on showing $a_0 = a_3$ before proving $a_j = a_{3-j}$ for all other j.

Similarly to equations (2.6.1) and (2.6.2), since 8 and 3 are essential for 11 in base 2 but non-essential for 11 in base 3, we have

$$\beta_{11}(S) + \beta_{11}(S \triangle \{3\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{1,2\}| = 1, \\ 3 & \text{if } |S \cap \{1,2\}| = 0 \text{ or } 2 \end{cases} \quad \bmod 6,$$

n	s	$\{a, b, c\}$ mod g	k	Chebikin et al. statement	Our statement
11	3	—	4		Prop. 2.6.9
11	11	$\{0, 1, 3\}$	7	Thm. 5.5	Thm. 2.6.7
13	13	$\{0, 2, 3\}$	7	Thm. 5.5	Thm. 2.6.7
14	7	$\{0, 1, 2\}$	13	Thm. 5.6	Thm. 2.6.7
14	13	$\{1, 2, 3\}$	3		Rem. 2.6.5
14	91		3		Thm. 2.6.1
19	19	$\{0, 1, 4\}$	7	Thm. 5.5	Thm. 2.6.7
21	3	$\{0, 0, 0\}$	2		Rem. 2.6.5
21	7	$\{0, 1, 2\}$	13		Thm. 2.6.7
21	21		2		Rem. 2.6.2
22	7	$\{1, 1, 2\}$	3		Thm. 2.6.4
22	11	$\{1, 2, 4\}$	7	Thm. 5.6	Rem. 2.6.5
22	77		3		Rem. 2.6.2
56	3	$\{3, 4, 5\}$ mod 6	3		Rem. 2.6.6
4,108	13	$\{0, 2, 3\}$	7		Thm. 2.6.7
16,576	17	$\{6, 6, 7\}$	3		Thm. 2.6.4
32,802	11	$\{1, 5, 5\}$	7		Thm. 2.6.3

Table 2.5: Examples of cyclotomic factors of $Q_n(t)$ of the form Φ_{2s} where the binary expansion of n has three 1's.

$$\beta_{11}(S) + \beta_{11}(S \triangle \{8\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{9, 10\}| = 1, \\ 3 & \text{if } |S \cap \{9, 10\}| = 0 \text{ or } 2 \end{cases} \mod 6.$$

Assume $S \subseteq [10]$ in which $\beta_{11}(S) \equiv 0 \bmod 3$. As in Theorem 2.6.1, if $|S \cap \{1, 2\}| = 0$ or 2, or if $|S \cap \{9, 10\}| = 0$ or 2, the descent set statistics $\beta_{11}(S)$, $\beta_{11}(S \triangle \{3\})$, $\beta_{11}(S \triangle \{8\})$, and $\beta_{11}(S \triangle \{3, 8\})$ contribute evenly between a_0 and a_3.

On the other hand, if $|S \cap \{1, 2\}| = 1$ and $|S \cap \{9, 10\}| = 1$, then S is one of the four sets in Lemma 2.6.8. Therefore, the descent set statistic of the set S is non-zero modulo 3, and does not contribute to either a_0 or a_3. In conclusion, the only possible sets that do contribute to a_0 and a_3 do so evenly, hence $a_0 = a_3$.

It remains to show $a_1 = a_2$ and $a_4 = a_5$. Since 11 has three digits in its binary expansion, Corollary 2.2.8 gives that $a_0 + a_2 + a_4 = a_1 + a_3 + a_5$. Combining this equality with $a_0 = a_3$, $a_1 = a_5$, and $a_2 = a_4$, it follows that $a_1 = a_2$ and $a_4 = a_5$. Thus, Lemma 2.4.1 implies that Φ_6 is a factor of $Q_{11}(t)$. □

This result is particular to $n = 11$. Attempts to generalize to values of n of the form $2^c + 2 + 1 = p^r + 2$ have so far failed. For these n one can similarly show that $a_0 = a_p$. Unfortunately, this does not imply $a_j = a_{p-j}$, which is in fact not true for all j.

Table 2.5 summarizes particular values of n and s for which Φ_{2s} is a factor of $Q_n(t)$ with n having three binary digits. This was done in Table 2.4 for n with two binary digits.

2.7 The double factor Φ_2 in the descent set polynomial

Our next results are about the occurrence of double factors in the descent set polynomial $Q_n(t)$. Here we sharpen techniques of Chebikin et al. to explain more double factors.

We begin by recalling the **ab**- and the **cd**-index of the Boolean algebra. Let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ denote the polynomial ring in the non-commutative variables **a** and **b**. For S a subset of $[n-1]$ define the **ab**-monomial $u_S = u_1 u_2 \cdots u_{n-1}$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The polynomial $\Psi(B_n)$ given by

$$\Psi(B_n) = \sum_{S \subseteq [n-1]} \beta_n(S) \cdot u_S$$

is the **ab**-index of the Boolean algebra. Recall that it can be written in terms of the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ since the Boolean algebra is an Eulerian poset. For ways to compute $\Psi(B_n)$, refer to Sections 1.5 and 1.6. Also, see [6, Proposition 8.2], which introduces a linear map ω that maps **ab**-monomials into **c**-2**d**-monomials in order to calculate $\Psi(B_n)$. Finally, see [43, Theorem 1.6.3], which uses equivalence classes of min-max trees to generate the **cd**-index of the Boolean algebra.

Define a linear function $\mathcal{L}$ from $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{Z}$ by

$$\mathcal{L}(u_S) = (-1)^{\beta_n(S)},$$

where S is a subset of $[n-1]$ and u_S is the associated **ab**-monomial of degree $n-1$. For an **ab**-monomial u of degree $n-1$, we write $\beta_n(u)$ instead of $\beta_n(S)$, where $u = u_S$.

Proposition 2.7.1. *Let w be a* **cd***-monomial of degree $2n-1$ having j* **d***'s. Then the following evaluation holds:*

$$\mathcal{L}(w) = 2^{2n-j-1} \cdot (1 - 2 \cdot \rho(n)).$$

Proof. Let $u = u_1 u_2 \cdots u_{2n-1}$ be an **ab**-monomial in the expansion of w. Let v be the **ab**-monomial formed by taking the letters in even positions from u, that is, $v = u_2 u_4 \cdots u_{2n-2}$. By Theorem 2.3.2 we have that

$$\beta_{2n}(u) \equiv \widetilde{\chi}(\Delta_{2n}|_S) \equiv \widetilde{\chi}(\Delta_n|_T) \equiv \beta_n(v) \bmod 2,$$

since the two complexes $\Delta_{2n}|_S$ and $\Delta_n|_T$ are identical where $u = u_S$ and $v = u_T$. Furthermore, observe that every **ab**-monomial of degree $n-1$ appears this way.

Given an **ab**-monomial v of degree $n-1$, how many corresponding monomials u can we find within the expansion of the **cd**-monomial w? There are n odd positions in u to fill in. If an odd position is covered by a $\mathbf{d}$ in w, there is a unique way to fill it in. Note that there are $n-j$ odd positions in u associated with $\mathbf{c}$'s in w. Hence there are 2^{n-j} ways to fill in v to obtain an **ab**-monomial u in the expansion of w. Now

$$\begin{aligned}\mathcal{L}(w) &= \sum_u (-1)^{\beta_{2n}(u)} \\ &= 2^{n-j} \cdot \sum_v (-1)^{\beta_n(v)} \\ &= 2^{n-j} \cdot Q_n(-1) \\ &= 2^{2n-j-1} \cdot (1 - 2 \cdot \rho(n)),\end{aligned}$$

where the first sum is over all **ab**-monomials u occurring in the expansion of w and the second sum is over all **ab**-monomials v of degree $n-1$. □

Theorem 2.7.2. *If Φ_2 is a factor of $Q_{2n}(t)$ then Φ_2 is a double factor of $Q_{2n}(t)$.*

Proof. Observe that

$$Q'_{2n}(t) = \sum_S \beta_{2n}(S) \cdot t^{\beta_{2n}(S)-1}.$$

Hence evaluating $Q'_{2n}(t)$ at $t = -1$, we obtain

$$\begin{aligned}Q'_{2n}(-1) &= -\sum_S \beta_{2n}(S) \cdot (-1)^{\beta_{2n}(S)} \\ &= -\mathcal{L}\left(\sum_S \beta_{2n}(S) \cdot u_S\right) \\ &= -\mathcal{L}(\Psi(B_{2n})).\end{aligned}$$

Now if Φ_2 is a factor of $Q_{2n}(t)$, we have $\rho(n) = 1/2$. Since $\Psi(B_{2n})$ can be expressed in terms of the two variables $\mathbf{c}$ and $\mathbf{d}$, we conclude that $\mathcal{L}(\Psi(B_{2n})) = 0$. Thus -1 is a double root of $Q_{2n}(t)$, yielding the conclusion. □

Extending Theorem 7.3 in [10] we have the next result.

Corollary 2.7.3. *If the binary expansion of n has three $1'$s then Φ_2^2 divides $Q_{2n}(t)$.*

2.8 The double factor Φ_{2p} in $Q_{2q}(t)$

Throughout this section, assume q is an odd prime power; that is, $q = p^r$ where p is prime and r is a positive integer.

Observe that by Theorem 6.1, part (iv) in [10] the cyclotomic polynomial Φ_{2p} is a factor of the descent set polynomial $Q_{2q}(t)$. Hence we concentrate on extending Theorem 7.5 from [10] to show in this section that Φ_{2p} is a double factor.

Theorem 2.8.1. *If $\rho(q) = 1/2$, then the cyclotomic polynomial Φ_{2p} is a double factor of the descent set polynomial $Q_{2q}(t)$.*

In order to prove this theorem we introduce two new linear functions $\mathcal{C}$ and $\mathcal{S}$ from **ab**-polynomials of degree $2q-1$ to the real numbers $\mathbb{R}$ by

$$\mathcal{C}(u_S) = \cos(\pi/p \cdot \beta_{2q}(S)), \tag{2.8.1}$$
$$\mathcal{S}(u_S) = \sin(\pi/p \cdot \beta_{2q}(S)). \tag{2.8.2}$$

Our goal is to show that $\mathcal{C}(w) = \mathcal{S}(w) = 0$ for any **cd**-monomial w of degree $2q-1$. We do this by a series of lemmas. First from Corollary 5.3 in [10], we have the following result.

Lemma 2.8.2. *The descent set statistic β_{2q} modulo p is given by*

$$\beta_{2q}(S) \equiv (-1)^{|S-\{q\}|} \bmod p.$$

Proof. Using the fact from Lemma 2.2.1 that $(x_1 + x_2 + \cdots)^{2q} \equiv (x_1^q + x_2^q + \cdots)^2 \equiv M_{(2q)} + 2M_{(q,q)} \bmod p$, it is straightforward to show the result upon inspection of the flag f-vector values and through the use of the inclusion-exclusion formula (2.2.1) for the descent set statistics. □

Lemma 2.8.3. *For any **ab**-monomial u of degree $2q-1$, we have $\mathcal{C}(u) = -\cos(\pi/p) \cdot (-1)^{\beta_{2q}(u)}$.*

Proof. According to Lemma 2.8.2 there are only four possible values for $\beta_{2q}(u)$ mod $2p$. When $\beta_{2q}(u)$ is odd the only two values for $\beta_{2q}(u)$ modulo $2p$ are ± 1, in which case $\mathcal{C}(u)$ is $\cos(\pi/p)$. Similarly, when $\beta_{2q}(u)$ is even it can only take the values $p \pm 1$ modulo $2p$, and hence $\mathcal{C}(u)$ is $-\cos(\pi/p)$. □

Lemma 2.8.4. *If $\rho(q) = 1/2$, then for a **cd**-monomial w of degree $2q-1$ we have $\mathcal{C}(w) = 0$.*

Proof. Assume that the **cd**-monomial w has j **d**'s. By the previous lemma and Proposition 2.7.1, we have

$$\begin{aligned}
\mathcal{C}(w) &= \sum_u \mathcal{C}(u) \\
&= -\cos(\pi/p) \cdot \sum_u (-1)^{\beta_{2q}(u)} \\
&= -\cos(\pi/p) \cdot \mathcal{L}(w) \\
&= -\cos(\pi/p) \cdot 2^{2q-j-1} \cdot (1 - 2 \cdot \rho(q)).
\end{aligned}$$

Since $\rho(q) = 1/2$, we obtain the conclusion $\mathcal{C}(w) = 0$. □

Lemma 2.8.5. *Let u and v be two* **ab***-monomials such that* $\deg(u)+\deg(v) = 2q-2$, *both* $\deg(u)$ *and* $\deg(v)$ *are even, and both* $\deg(u)$ *and* $\deg(v)$ *differ from* $q-1$. *Then* $\mathcal{S}(u \cdot \mathbf{c} \cdot v) = 0$.

Proof. Since $\deg(u) + 1$ is non-essential for $2q$ both in base 2 and in base p, we have by Lemma 2.2.5 that

$$\beta_{2q}(u \cdot \mathbf{a} \cdot v) \equiv -\beta_{2q}(u \cdot \mathbf{b} \cdot v) \bmod 2p.$$

Since sine is an odd function, this identity directly implies $\mathcal{S}(u\cdot\mathbf{a}\cdot v) = -\mathcal{S}(u\cdot\mathbf{b}\cdot v)$. □

Lemma 2.8.6. *Let w be a* **cd***-monomial of degree* $2q-1$ *different from the monomial* $\mathbf{d}^{(q-1)/2}\mathbf{c}\mathbf{d}^{(q-1)/2}$. *Then* $\mathcal{S}(w) = 0$.

Proof. The monomial w has q odd positions and $q-1$ even positions. Since a $\mathbf{d}$ covers both an odd position and an even position, there will always be a $\mathbf{c}$ in an odd position. Unless w is the monomial $\mathbf{d}^{(q-1)/2}\mathbf{c}\mathbf{d}^{(q-1)/2}$ we can find a $\mathbf{c}$ in an odd position different from q. By the previous lemma we know $\mathcal{S}(u \cdot \mathbf{c} \cdot v) = 0$ for all **ab**-monomials u and v, and hence by linearity we conclude $\mathcal{S}(w) = 0$. □

Lemma 2.8.7. *If* $\rho(q) = 1/2$ *then* $\mathcal{S}(\mathbf{d}^{(q-1)/2}\mathbf{c}\mathbf{d}^{(q-1)/2}) = 0$.

Proof. If u is an **ab**-monomial occurring in the expansion of $w = \mathbf{d}^{(q-1)/2}\mathbf{c}\mathbf{d}^{(q-1)/2}$ then it has $q-1$ or q **b**'s. In fact, it has $q-1$ **b**'s in the positions different from the position q since this is the position of the $\mathbf{c}$ in w.

Lemma 2.8.2 implies that $\beta_{2q}(u) \equiv (-1)^{q-1} \equiv 1 \bmod p$ by using the fact that q is odd. Hence we have that $\beta_{2q}(u) \equiv 1$ or $p+1 \bmod 2p$; that is, the value of $\beta_{2q}(u)$ modulo $2p$ only depends on the value modulo 2. Hence, using similar reasoning to the proof of Lemma 2.8.3, we have the sum

$$\begin{aligned}
\mathcal{S}(w) &= \sum_u \mathcal{S}(u) \\
&= \sum_u \sin(\pi/p \cdot \beta_{2q}(u)) \\
&= \sum_u -\sin(\pi/p) \cdot (-1)^{\beta_{2q}(u)} \\
&= -\sin(\pi/p) \cdot \mathcal{L}(w) \\
&= -\sin(\pi/p) \cdot 2^q \cdot (1 - 2 \cdot \rho(q)).
\end{aligned}$$

Since $\rho(q) = 1/2$, we obtain $\mathcal{S}(w) = 0$. □

Proof of Theorem 2.8.1. Observe that

$$\begin{aligned}
e^{\pi \cdot i/p} \cdot Q'_{2q}(e^{\pi \cdot i/p}) &= \sum_S \beta_{2q}(S) \cdot e^{\pi \cdot i/p \cdot \beta_{2q}(S)} \\
&= \sum_S \beta_{2q}(S) \cdot (\mathcal{C}(u_S) + i \cdot \mathcal{S}(u_S))
\end{aligned}$$

$$= (\mathcal{C} + i \cdot \mathcal{S}) \left(\sum_S \beta_{2q}(S) \cdot u_S \right)$$
$$= (\mathcal{C} + i \cdot \mathcal{S})(\Psi(B_{2q})),$$

which vanishes based on the previous lemmas. Hence $e^{\pi \cdot i/p}$ is a root of Q'_{2q}, so $e^{\pi \cdot i/p}$ is a double root of Q_{2q}. □

2.9 The (double) factor Φ_{2p} in $Q_{q+1}(t)$

Let $q = p^r$ be an odd prime power; that is, p is an odd prime and r a positive integer. Now we study the case of the cyclotomic factor Φ_{2p} in $Q_{q+1}(t)$.

Theorem 2.9.1. *If $\rho(q) = 1/2$ then the cyclotomic polynomial Φ_{2p} divides the descent set polynomial $Q_{q+1}(t)$. Furthermore, if $q \equiv 3 \bmod 4$, then Φ_{2p} is a double factor in $Q_{q+1}(t)$.*

We start by explicitly expressing the flag f-vector of the Boolean algebra B_{q+1} modulo p:

$$F(B_{q+1}) \equiv (M_{(1)})^q \cdot M_{(1)} \equiv M_{(q)} \cdot M_{(1)} \equiv M_{(q+1)} + M_{(q,1)} + M_{(1,q)} \bmod p.$$

Hence the flag f-vector $f(S) \equiv 1 \bmod p$ if S is equal to $\emptyset$, $\{1\}$, or $\{q\}$, and zero otherwise. By inclusion-exclusion we obtain that the descent set statistic $\beta_{q+1}(S)$ modulo p is given by

$$\beta_{q+1}(S) \equiv \begin{cases} (-1)^S & \text{if } |S \cap \{1,q\}| = 0, \\ 0 & \text{if } |S \cap \{1,q\}| = 1, \\ -(-1)^S & \text{if } |S \cap \{1,q\}| = 2, \end{cases} \quad \bmod p. \tag{2.9.1}$$

In terms of **ab**-monomials, this result can be stated as $\beta_{q+1}(\mathbf{a} \cdot v \cdot \mathbf{b}) \equiv \beta_{q+1}(\mathbf{b} \cdot v \cdot \mathbf{a}) \equiv 0 \bmod p$ and $\beta_{q+1}(\mathbf{a} \cdot v \cdot \mathbf{a}) \equiv -\beta_{q+1}(\mathbf{b} \cdot v \cdot \mathbf{b}) \equiv (-1)^j \bmod p$ where v is an **ab**-monomial of degree $q-2$ having j **b**'s.

Similar to the previous section, we use two linear functions from **ab**-polynomials of degree q to the real numbers $\mathbb{R}$ defined by

$$\mathcal{C}(u_S) = \cos(\pi/p \cdot \beta_{q+1}(S)), \tag{2.9.2}$$
$$\mathcal{S}(u_S) = \sin(\pi/p \cdot \beta_{q+1}(S)). \tag{2.9.3}$$

Note that these differ slightly from definitions (2.8.1) and (2.8.2) by replacing the descent set statistic β_{2q} by β_{q+1}.

Lemma 2.9.2. *Let w be a **cd**-monomial of degree q beginning or ending with the letter **c**. If $\rho(q+1) = 1/2$ then $\mathcal{C}(w) = 0$.*

Proof. It is enough to consider the case when w begins with a $\mathbf{c}$. Let $u = u_1 u_2 \cdots u_q$ be an $\mathbf{ab}$-monomial in the expansion of w. If u_1 differs from u_q, we have by (2.9.1) that $\beta_{q+1}(u) \equiv 0 \bmod p$. Hence $\mathcal{C}(u) = \cos(\pi/p \cdot \beta_{q+1}(u)) = (-1)^{\beta_{q+1}(u)}$. In the case in which the first and last letter of u are the same, we have that $\beta_{q+1}(u) \equiv \pm 1 \bmod p$ by (2.9.1). Hence $\beta_{q+1}(u)$ takes one of the four values $\pm 1, p \pm 1$ modulo $2p$, and therefore, $\mathcal{C}(u) = \cos(\pi/p \cdot \beta_{q+1}(u))$ takes one of the two values $\pm \cos(\pi/p)$. Note that if $\beta_{q+1}(u)$ is even, then $\beta_{q+1}(u) \equiv p \pm 1 \bmod 2p$ and hence $\mathcal{C}(u) = -\cos(\pi/p)$. Similarly, if $\beta_{q+1}(u)$ is odd we have $\mathcal{C}(u) = \cos(\pi/p)$. To summarize these two cases when $u_1 = u_q$, we have that $\mathcal{C}(u) = -\cos(\pi/p) \cdot (-1)^{\beta_{q+1}(u)}$.

Then we have the sum

$$\mathcal{C}(w) = \sum_{u \,:\, u_1 \neq u_q} (-1)^{\beta_{q+1}(u)} - \cos(\pi/p) \cdot \sum_{u \,:\, u_1 = u_q} (-1)^{\beta_{q+1}(u)},$$

where both summations are over all $\mathbf{ab}$-monomials u in the expansion of w. Let overline denote the involution defined by $\overline{\mathbf{a}} = \mathbf{b}$ and $\overline{\mathbf{b}} = \mathbf{a}$. In each of the sums, also include the term $\overline{u_1} u_2 \cdots u_q$. Since 1 is non-essential for $q+1$ in base 2, we have $\beta_{q+1}(\overline{u_1} u_2 \cdots u_q) \equiv \beta_{q+1}(u) \bmod 2$. Hence both sums will double to give us

$$\begin{aligned} \mathcal{C}(w) &= \frac{1}{2} \cdot \sum_{u} (-1)^{\beta_{q+1}(u)} - \cos(\pi/p) \cdot \frac{1}{2} \cdot \sum_{u} (-1)^{\beta_{q+1}(u)} \\ &= \frac{1}{2} \cdot (1 - \cos(\pi/p)) \cdot \mathcal{L}(w), \end{aligned}$$

where both sums are over all u occurring in the expansion of w. This works since w begins with the letter $\mathbf{c}$. By the assumption $\rho(q+1) = 1/2$, this expression will vanish by Proposition 2.7.1. □

Lemma 2.9.3. *Let w be a $\mathbf{cd}$-monomial of degree q beginning or ending with the letter $\mathbf{d}$. If $\rho(q+1) = 1/2$ and $q \equiv 3 \bmod 4$ then $\mathcal{C}(w) = 0$.*

Proof. Assume that w begins with a $\mathbf{d}$. The proof is the same as the proof of the previous lemma, except that $q \equiv 3 \bmod 4$ implies that 2 is non-essential for $q+1$ in base 2. In the end of the proof when we extend the two sums ranging over $u = u_1 u_2 u_3 \cdots u_q$, also include the terms $\overline{u_1 u_2} u_3 \cdots u_q$. Then both sums become $\mathcal{L}(w)$ and the result follows. □

Lemma 2.9.4. *Let u and v be two $\mathbf{ab}$-monomials such that $\deg(u) + \deg(v) = q - 1$, both $\deg(u)$ and $\deg(v)$ are even, and both $\deg(u)$ and $\deg(v)$ differ from zero. Then $\mathcal{S}(u \cdot \mathbf{c} \cdot v) = 0$.*

Proof. Since $\deg(u) + 1$ is non-essential for $q+1$ both in base 2 and in base p, we have by Lemma 2.2.5 that

$$\beta_{q+1}(u \cdot \mathbf{a} \cdot v) \equiv -\beta_{q+1}(u \cdot \mathbf{b} \cdot v) \bmod 2p.$$

Since sine is an odd function, this identity directly implies $\mathcal{S}(u \cdot \mathbf{a} \cdot v) = -\mathcal{S}(u \cdot \mathbf{b} \cdot v)$. □

Lemma 2.9.5. *Let w be a* **cd***-monomial of degree q different from the monomials* $\mathbf{cd}^{(q-1)/2}$*,* $\mathbf{d}^{(q-1)/2}\mathbf{c}$*, and* $\mathbf{cd}^i\mathbf{cd}^j\mathbf{c}$ *where $i + j = (q-3)/2$. Then $\mathcal{S}(w) = 0$.*

Proof. A monomial w of odd degree q has $(q+1)/2$ odd positions and $(q-1)/2$ even positions. Since a **d** covers both an odd position and an even position, the number of **c**'s in odd positions will be one more than the number of **c**'s in even positions. If the monomial w has a **c** in an odd position i, where $2 \leq i \leq q-1$, then $\mathcal{S}(w)$ vanishes by the previous lemma. Hence, assume that w is a **cd**-monomial with no **c**'s in any odd position between 2 and $q-1$.

Thus, w has either one or two **c**'s in odd positions, that is, in the first position 1 or the last position q. If there is only one **c** in an odd position in w, then w is either the monomial $\mathbf{cd}^{(q-1)/2}$ or the monomial $\mathbf{d}^{(q-1)/2}\mathbf{c}$. If there are two **c**'s in odd positions in w, then there is exactly one **c** in an even position. Thus the monomial w is of the form $\mathbf{cd}^i\mathbf{cd}^j\mathbf{c}$. □

Lemma 2.9.6. *Let w be a* **cd***-monomial of degree q beginning and ending with the letter* **c***. Then $\mathcal{S}(w)$ vanishes. In particular,* $\mathcal{S}(\mathbf{cd}^i\mathbf{cd}^j\mathbf{c}) = 0$ *for $i + j = (q-3)/2$.*

Proof. Let u be an **ab**-monomial occurring in the expansion of w. Observe that if u has the form $\mathbf{a} \cdot v \cdot \mathbf{b}$ or $\mathbf{b} \cdot v \cdot \mathbf{a}$ then $\beta_{q+1}(u) \equiv 0 \bmod p$ by (2.9.1). This implies that $\mathcal{S}(u) = \sin(\pi/p \cdot \beta_{q+1}(u)) = 0$. Hence we have only to consider **ab**-monomials in the expansion of w that begin and end with the same letter. Again by (2.9.1) observe that $\beta_{q+1}(\mathbf{a} \cdot v \cdot \mathbf{a}) \equiv -\beta_{q+1}(\mathbf{b} \cdot v \cdot \mathbf{b}) \bmod p$. Since positions 1 and q are non-essential for $q+1$ in base 2, we have $\beta_{q+1}(\mathbf{a} \cdot v \cdot \mathbf{a}) \equiv \beta_{q+1}(\mathbf{b} \cdot v \cdot \mathbf{b}) \bmod 2$. Combining these two congruences to one statement modulo $2p$, we have $\beta_{q+1}(\mathbf{a}\cdot v\cdot\mathbf{a}) \equiv -\beta_{q+1}(\mathbf{b} \cdot v \cdot \mathbf{b}) \bmod 2p$. This implies that $\mathcal{S}(\mathbf{a} \cdot v \cdot \mathbf{a}) = -\mathcal{S}(\mathbf{b} \cdot v \cdot \mathbf{b})$ and proves the lemma. □

Lemma 2.9.7. *If $\rho(q+1) = 1/2$ and $q \equiv 3 \bmod 4$, then* $\mathcal{S}(\mathbf{d}^{(q-1)/2}\mathbf{c})$ *and* $\mathcal{S}(\mathbf{cd}^{(q-1)/2})$ *vanish.*

Proof. The congruence relation on q implies that 4 divides $q+1$. Hence the element 2 is a non-essential element for $q+1$ in base 2. We will use this fact, together with the facts that 1 and q are also non-essential elements.

By symmetry it is enough to prove the lemma for $w = \mathbf{d}^{(q-1)/2}\mathbf{c}$. Let u be an **ab**-monomial occurring in the expansion of w. Similar to the previous lemma, if u begins and ends with different letters, we have that $\mathcal{S}(u) = 0$. Hence we have that u has the form $\mathbf{ab}\cdot v\cdot\mathbf{a}$ or $\mathbf{ba}\cdot v\cdot\mathbf{b}$. Next we have that $\beta_{q+1}(\mathbf{ab}\cdot v\cdot\mathbf{a}) \equiv -\beta_{q+1}(\mathbf{bb}\cdot v\cdot\mathbf{b}) \equiv \beta_{q+1}(\mathbf{ba}\cdot v\cdot\mathbf{b}) \bmod p$ by (2.9.1). Furthermore, since the three elements 1, 2, and q are non-essential for $q+1$ in base 2, we have that $\beta_{q+1}(\mathbf{ab} \cdot v \cdot \mathbf{a}) \equiv \beta_{q+1}(\mathbf{ba} \cdot v \cdot \mathbf{b}) \bmod 2$. That is, we have $\beta_{q+1}(\mathbf{ab} \cdot v \cdot \mathbf{a}) \equiv \beta_{q+1}(\mathbf{ba} \cdot v \cdot \mathbf{b}) \bmod 2\text{p}$.

Hence these two cases $\mathbf{ab} \cdot v \cdot \mathbf{a}$ and $\mathbf{ba} \cdot v \cdot \mathbf{b}$ are the same, that is,

$$\mathcal{S}(w) = 2 \cdot \sum_{\mathbf{ab}\cdot v\cdot\mathbf{a}} \sin(\pi/p \cdot \beta_{q+1}(\mathbf{ab} \cdot v \cdot \mathbf{a})).$$

The monomial $u = \mathbf{ab} \cdot v \cdot \mathbf{a}$ has $(q-1)/2$ $\mathbf{b}$'s, so $\beta_{q+1}(u) \equiv (-1)^{(q-1)/2} \bmod p$ by equation (2.9.1). By considering the four values $\pm 1, p \pm 1$ of $\beta_{q+1}(u)$ modulo $2p$ we have that

$$\sin(\pi/p \cdot \beta_{q+1}(u)) = -(-1)^{(q-1)/2} \cdot \sin(\pi/p) \cdot (-1)^{\beta_{q+1}(u)}.$$

Hence $\mathcal{S}(w)$ is given by

$$\mathcal{S}(w) = -2 \cdot (-1)^{(q-1)/2} \cdot \sin(\pi/p) \cdot \sum_{\mathbf{ab} \cdot v \cdot \mathbf{a}} (-1)^{\beta_{q+1}(u)}.$$

Again since the elements 1, 2, and q are non-essential for $q+1$ in base 2, we can switch the letters in these places without changing the descent set statistic β_{q+1} modulo 2. Hence we have

$$\mathcal{S}(w) = -\frac{1}{2} \cdot (-1)^{(q-1)/2} \cdot \sin(\pi/p) \cdot \sum_{u} (-1)^{\beta_{q+1}(u)},$$

where the sum is over all $\mathbf{ab}$-monomials u in the expansion of w. By the assumption that $\rho(q+1) = 1/2$ and Proposition 2.7.1, this last sum is zero. □

Proof of Theorem 2.9.1. Observe that

$$\begin{aligned} Q_{q+1}(e^{\pi \cdot i/p}) &= \sum_{u} e^{\pi \cdot i/p \cdot \beta_{q+1}(u)} \\ &= \sum_{u} (\cos(\pi/p \cdot \beta_{q+1}(u)) + i \cdot \sin(\pi/p \cdot \beta_{q+1}(u))) \\ &= (\mathcal{C} + i \cdot \mathcal{S})(\mathbf{c}^q), \end{aligned}$$

since the first two sums are over all $\mathbf{ab}$-monomials of degree q, that is, all the $\mathbf{ab}$-monomials in the expansion of $\mathbf{c}^q$. Finally, the last expression vanishes by Lemmas 2.9.2 and 2.9.5.

With the added assumption $q \equiv 3 \bmod 4$, Lemmas 2.9.2 and 2.9.3 imply that $\mathcal{C}$ applied to any $\mathbf{cd}$-polynomial of degree q vanishes. Similarly, with the assumption, Lemmas 2.9.5 through 2.9.7 imply that $\mathcal{S}$ applied to any $\mathbf{cd}$-polynomial of degree q vanishes. Now we have that

$$e^{\pi \cdot i/p} \cdot Q'_{q+1}(e^{\pi \cdot i/p}) = \sum_{u} \beta_{q+1}(u) \cdot e^{\pi \cdot i/p \cdot \beta_{q+1}(u)} = (\mathcal{C} + i \cdot \mathcal{S})(\Psi(B_{q+1})) = 0,$$

since $\Psi(B_{q+1})$ can be written in terms of the variables $\mathbf{c}$ and $\mathbf{d}$. Thus $e^{\pi \cdot i/p}$ is a double root of $Q_{q+1}(t)$. □

2.10 Concluding remarks

By considering Table 2.6 one sees that there are two unexplained cyclotomic factors. They are Φ_4 and Φ_{28}, both dividing $Q_{14}(t)$. Here it is straightforward to see $a_{4,1} = a_{4,3}$; that is, $Q_{14}(i)$ is a real number. But it remains to find an argument demonstrating that $a_0 = a_2$. Since 4 is a square, the Chinese Remainder Theorem cannot be applied.

n	degree	cyclotomic factors of $Q_n(t)$
3	2	$\boldsymbol{\Phi_2}$
4	5	$\boldsymbol{\Phi_4^2}$
5	16	$\boldsymbol{\Phi_2^2 \cdot \Phi_{10}}$
6	61	$\boldsymbol{\Phi_2^2 \cdot \Phi_6^2 \cdot \Phi_{10}}$
7	272	$\boldsymbol{\Phi_2}$
8	1385	$\boldsymbol{\Phi_4^2 \cdot \Phi_{28}}$
9	7936	$\boldsymbol{\Phi_2^2 \cdot \Phi_6 \cdot \Phi_{18}}$
10	50521	$\boldsymbol{\Phi_2^2 \cdot \Phi_6 \cdot \Phi_{10}^2 \cdot \Phi_{18} \cdot \Phi_{30}}$
11	353792	$\boldsymbol{\Phi_2 \cdot \Phi_6 \cdot \Phi_{22}}$
12	2702765	$\boldsymbol{\Phi_2^2 \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{18} \cdot \Phi_{22}^2 \cdot \Phi_{66} \cdot \Phi_{110} \cdot \Phi_{198}}$
13	22368256	$\boldsymbol{\Phi_2 \cdot \Phi_{26}}$
14	$1.993 \cdot 10^8$	$\boldsymbol{\Phi_2^2} \cdot \Phi_4 \cdot \boldsymbol{\Phi_{14}^2} \cdot \boldsymbol{\Phi_{26}} \cdot \Phi_{28} \cdot \boldsymbol{\Phi_{182}}$
15	$1.904 \cdot 10^9$	–
16	$1.939 \cdot 10^{10}$	$\boldsymbol{\Phi_4^2 \cdot \Phi_{12} \cdot \Phi_{20} \cdot \Phi_{44} \cdot \Phi_{52} \cdot \Phi_{60} \cdot \Phi_{156} \cdot \Phi_{220} \cdot}$ $\boldsymbol{\Phi_{260} \cdot \Phi_{572} \cdot \Phi_{2860}}$
17	$2.099 \cdot 10^{11}$	$\boldsymbol{\Phi_2^2 \cdot \Phi_{34}}$
18	$2.405 \cdot 10^{12}$	$\boldsymbol{\Phi_2^2 \cdot \Phi_6^2 \cdot \Phi_{18} \cdot \Phi_{34} \cdot \Phi_{102} \cdot \Phi_{306}}$
19	$2.909 \cdot 10^{13}$	$\boldsymbol{\Phi_2 \cdot \Phi_{38}}$
20	$3.704 \cdot 10^{14}$	$\boldsymbol{\Phi_2^2 \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{30} \cdot \Phi_{34} \cdot \Phi_{38}^2 \cdot \Phi_{102} \cdot \Phi_{114} \cdot \Phi_{170} \cdot}$ $\boldsymbol{\Phi_{190} \cdot \Phi_{510} \cdot \Phi_{570} \cdot \Phi_{646} \cdot \Phi_{1938} \cdot \Phi_{3230} \cdot \Phi_{9690}}$
21	$4.951 \cdot 10^{15}$	$\boldsymbol{\Phi_2 \cdot \Phi_6 \cdot \Phi_{14} \cdot \Phi_{42}}$
22	$6.935 \cdot 10^{16}$	$\boldsymbol{\Phi_2^2 \cdot \Phi_{14} \cdot \Phi_{22}^2 \cdot \Phi_{154}}$
23	$1.015 \cdot 10^{18}$	–

Table 2.6: Cyclotomic factors of $Q_n(t)$. This table is from Chebikin et al. [10], but the explained factors have been updated. These factors occur in **boldface**. Furthermore the factor Φ_{2860} in $Q_{16}(t)$ has been included, which was missing in the original table. Note that the two factors Φ_4 and Φ_{28} in $Q_{14}(t)$ are still unexplained.

Note that these factors seem to be isolated to $n = 14$ and do not occur among other n with three 1's in their binary expansion up to $n = 23$. Do any other outliers exist beyond this value of n?

Further consideration of Table 2.6 shows that all cyclotomic factors that appear in table with multiplicity have now been explained. Are there other square factors appearing beyond $n = 23$ that have not yet been explained?

Do Theorems 2.8.1 and 2.9.1 apply to infinitely many prime powers? There are only 6 prime powers with two 1's in their binary expansion, but there seems to be an infinite number of primes with three 1's in their binary expansion. See the sequence A081091 in The On-Line Encyclopedia of Integer Sequences. However, this seems to be a hard number theory problem.

Chebikin et al. calculated the proportion for the number of odd entries in the descent set statistics β_n for $n = 1, 3, 7, 15, 31$, that is, for any integer with at most five 1's in its binary expansion. See Table 2.1. Could the topological perspective of Theorem 2.3.2 help to calculate the next case $n = 63$? From this topological viewpoint, is there a classification of simplicial complexes Δ such that exactly half of the induced subcomplexes $\Delta|_S$ have an odd Euler characteristic?

In [10] Chebikin et al. also consider the signed descent set polynomial. This polynomial is defined in terms of signed permutations, which are of the form $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n^{\pm}$ where each $\pi_i \in \{\pm 1, \ldots, \pm n\}$ and $|\pi_1| \cdots |\pi_n|$ is a permutation. The descent set of a signed permutation π is defined as $\{i : \pi_{i-1} > \pi_i\}$ with π_0 assumed to be 0. Then $\beta_n^{\pm}(S)$ denotes the number of signed permutations in $\mathfrak{S}_n^{\pm}$ with descent set S. Finally, the nth signed descent set polynomial is defined to be

$$Q_n^{\pm}(t) = \sum_{S \subseteq [n]} t^{\beta_n^{\pm}(S)}.$$

Chebikin et al. found this polynomial also contains many cyclotomic polynomials. However, they were able to explain the existence of an even smaller percentage of them compared to the descent set polynomial, as seen in Table 3 in [10]. One successful tool they used to examine the descent set statistics for signed permutations was a quasi-symmetric function that encodes the flag f-vector of the cubical lattice C_n, whereas the standard descent set statistics were associated with the Boolean algebra B_n. The cubical lattice is actually obtained by applying the diamond product successively to the Boolean algebra on two elements, that is, $C_n = B_2^{\diamond n}$. This quasi-symmetric function technique was used, for instance, to explain why the cyclotomic polynomial Φ_{4p} for an odd prime p divides $Q_p^{\pm}(t)$. Since that approach was useful for the signed descent set polynomial, can any of our techniques be extended to explain other cyclotomic factors in this polynomial?

Chapter 3 Lattice Path Interpretation of the Diamond Product

3.1 Introduction

The **cd**-index is a polynomial in the non-commutative variables **c** and **d** that efficiently encodes the flag f-vector of an Eulerian poset. One primary family of Eulerian posets consists of the face lattices of convex polytopes. The **cd**-index is a useful invariant for computations, as explicit formulas have been developed to calculate the effect that poset and polytope operations have on the **cd**-index. Polytope operations, or their associated poset operations, that have been studied in [19] include the prism, pyramid, free join, Cartesian product, and truncation of a vertex.

Ehrenborg and Readdy used coalgebraic techniques in [19] to generate expressions for the **cd**-index of polytopes under operations such as the prism of a polytope, or more generally the Cartesian product of polytopes. The corresponding poset operation to this product is the diamond product. Unfortunately, the expressions that were developed were rather complicated and required the use of auxiliary variables **a** and **b**. Ehrenborg and H. Fox [13] (no relation to author) improved upon the earlier work by developing recursive formulas for the bilinear operator that corresponds to the diamond product of posets.

The diamond product operator is nonnegative on **cd**-indices, thus leading to the study of combinatorial interpretations of the resulting coefficients. Slone [42] examined the specific case of the diamond product of two butterfly posets, whose **cd**-indices are simply powers of **c**. He found that one can interpret the polynomial as a weighted sum of lattice paths. In this chapter, a generalization of Slone's lattice path interpretation is given for the diamond product of any two **cd**-monomials in addition to a lattice path interpretation for the product of **ab**-monomials.

In Section 3.2 we discuss the **cd**-index of Eulerian posets and its underlying coalgebraic structure. Section 3.3 includes the definition of the diamond product of two posets, in addition to formulas given by Ehrenborg and Readdy [19] and by Ehrenborg and H. Fox [13] to describe the resulting **ab**- and **cd**-indices when the diamond product is applied. In Section 3.4 we introduce a lattice path interpretation of this product as applied to **ab**-monomials, with Section 3.5 including the corresponding interpretation for **cd**-monomials. Finally, an open problem regarding the Cartesian product of posets is stated in Section 3.6.

A version of this chapter can be found in [23].

3.2 The cd-index and coproducts

Let P be a graded poset of rank $n+1$ with rank function ρ, minimal element $\hat{0}$, and maximal element $\hat{1}$. We recall that the flag f-vector of a poset P consists of entries f_S for $S \subseteq [n]$, which count the number of chains in the poset whose elements $x_1, \ldots, x_{k-1}$ have ranks that are exactly the elements of the set S. Then the flag

h-vector is defined as

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

The **ab**-*index* $\Psi(P)$ of the poset P is the polynomial in the non-commutative variables **a** and **b** given by

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where S ranges over all subsets of $[n]$ and $u_S = u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$.

For an Eulerian poset, that is, a poset whose Möbius function satisfies the relation $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$ for all nontrivial intervals $[x, y]$ in P, the **ab**-index can be written as a **cd**-index in terms of the non-commutative variables **c** and **d**. This is expressed in the following result that was conjectured by Fine and proved by Bayer and Klapper [3].

Theorem 3.2.1 (Bayer–Klapper)**.** *The* **ab**-*index* $\Psi(P)$ *of an Eulerian poset* P *can be rewritten as a non-commutative polynomial in* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$*.*

We now briefly discuss the coalgebraic structures of the **ab**-index and the **cd**-index that were introduced in [19]. It is straightforward to verify that the two coalgebras that are described are both Newtonian.

First, let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ denote the polynomial ring in the non-commutative variables **a** and **b**, where the degree of each variable is one. For the empty word 1, we let $\Delta(1) = 0$. Then for an **ab**-monomial $u = u_1 \cdots u_n$ with $n \geq 1$, define

$$\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n,$$

and extend linearly to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. As examples, $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) = 1 \otimes 1$ and $\Delta(\mathbf{abba}) = 1 \otimes \mathbf{bba} + \mathbf{a} \otimes \mathbf{ba} + \mathbf{ab} \otimes \mathbf{a} + \mathbf{abb} \otimes 1$.

Next consider the subring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ generated by the variables **c** and **d** as defined in Theorem 1.5.2. Once one calculates $\Delta(\mathbf{c}) = \Delta(\mathbf{a} + \mathbf{b}) = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = \Delta(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}) = 1 \otimes \mathbf{c} + \mathbf{c} \otimes 1$, it can be verified that $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ is also a Newtonian coalgebra.

We now define two linear operators on these coalgebras. Define the derivation $G : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ given by the rules

$$G(\mathbf{a}) = \mathbf{b} \cdot \mathbf{a}, \ G(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b},$$

and the product rule

$$G(u \cdot v) = G(u) \cdot v + u \cdot G(v).$$

Since $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{c} \cdot \mathbf{d}$, the operator G becomes a linear operator on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ as well. Let $\mathrm{Pyr} : \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ be the linear operator defined by

$$\mathrm{Pyr}(u) = u \cdot \mathbf{c} + G(u).$$

3.3 The diamond product of posets

Given two graded posets P and Q, the diamond product of P and Q is defined in terms of a Cartesian product as the graded poset $P \diamond Q = (P - \{\hat{0}_P\}) \times (Q - \{\hat{0}_Q\}) \cup \{\hat{0}\}$. This product corresponds to the Cartesian product of polytopes. Kalai showed this in [28], where he stated that the face lattice of the Cartesian product of two polytopes corresponds to the diamond product of their face lattices, that is, $\mathcal{L}(V \times W) = \mathcal{L}(V) \diamond \mathcal{L}(W)$. The diamond product specifically appears when studying the *prism* of a polytope, defined as $\text{Prism}(V) = V \times I$, where I is the unit interval. As stated in Proposition 4.1 of [19], $\mathcal{L}(\text{Prism}(V)) = \mathcal{L}(V) \diamond B_2$.

Because of the importance of the prism operation and the Cartesian product in the study of polytopes, one needs to understand how these operations affect the **ab**- and **cd**-indices of polytopes, and likewise, their associated posets. This leads to the investigation of the **cd**-index of the diamond product of two Eulerian posets. Ehrenborg and Readdy [19] developed a bilinear operator for this purpose. In order to state the formula for this operator, we need to define the mixing operator, which was used in [19] in a formula to calculate the **ab**-index of the Cartesian product of two graded posets. First define the index set

$$I = \{(r, s, n) : r, s \in \{1, 2\}, n \geq 2, n \equiv r + s + 1 \bmod 2\}.$$

The mixing operator $M_{r,s}(u, v, n)$ is then defined for **ab**-monomials u and v and $(r, s, n) \in I$ by the recursion

$$\begin{aligned} M_{1,2}(u, v, 2) &= u \cdot \mathbf{a} \cdot v, \\ M_{2,1}(u, v, 2) &= v \cdot \mathbf{b} \cdot u, \\ M_{1,s}(u, v, n+1) &= \sum_u u_{(1)} \cdot \mathbf{a} \cdot M_{2,s}(u_{(2)}, v, n), \\ M_{2,s}(u, v, n+1) &= \sum_v v_{(1)} \cdot \mathbf{b} \cdot M_{1,s}(u, v_{(2)}, n). \end{aligned}$$

Next define the algebra map $\kappa : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by assigning $\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}$ and $\kappa(\mathbf{b}) = 0$. Using the mixing operator, the map κ, and coproducts, Ehrenborg and Readdy introduced the following formula to calculate the **ab**-index of the diamond product of two graded posets. Properties, examples, and recurrences for this operator are in Section 6 of [13] and Section 10 of [19].

Proposition 3.3.1 (Ehrenborg–Readdy)**.** *Given two graded posets P and Q, let $u = \Psi(P)$ and $v = \Psi(Q)$. The* **ab***-index of $P \diamond Q$ is given by*

$$\Psi(P \diamond Q) = \kappa(u) \cdot \kappa(v) + \sum_u \kappa(u_{(1)}) \cdot \kappa(v) \cdot \mathbf{b} \cdot u_{(2)} + \sum_v \kappa(u) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)}$$
$$+ \sum_u \sum_v \kappa(u_{(1)}) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \left(\sum_{(r,s,n) \in I} M_{r,s}(u, v, n) \right).$$

The formula described in Proposition 3.3.1 is denoted by the bilinear operator $N(u, v)$ in the paper [13]. We use the diamond product $u \diamond v$ to simplify the notation. Note that for two polytopes V and W, the **ab**-index of the Cartesian product $V \times W$ can also be expressed using this operator as

$$\Psi(V \times W) = \Psi(V) \diamond \Psi(W).$$

The following statements made by Ehrenborg and H. Fox in [13] give useful properties and a recursive formula for calculating the diamond product of two **ab**-polynomials, as well as extending the operator for **cd**-polynomials. Proposition 3.3.3 is a reformulated version of Proposition 7.6 in [13]. Likewise, Proposition 3.3.4 is a reformulation of Theorem 7.1 in [13], as was shown in Corollary 2.3.7 in [42].

Corollary 3.3.2 (Ehrenborg–H. Fox)**.** *For any* **ab**- *or* **cd**-*polynomials u, v, and w, the following identities are satisfied:*

$$u \diamond 1 = u,$$
$$u \diamond v = v \diamond u,$$
$$u \diamond (v \diamond w) = (u \diamond v) \diamond w.$$

Proposition 3.3.3 (Ehrenborg–H. Fox)**.** *For any* **ab**-*polynomials u and v, the diamond product satisfies the following recursions:*

$$u \diamond (v \cdot \mathbf{a}) = (u \diamond v) \cdot \mathbf{a} + \sum_u (u_{(1)} \diamond v) \cdot \mathbf{a} \cdot \mathbf{b} \cdot u_{(2)}, \tag{3.3.1}$$

$$u \diamond (v \cdot \mathbf{b}) = (u \diamond v) \cdot \mathbf{b} + \sum_u (u_{(1)} \diamond v) \cdot \mathbf{b} \cdot \mathbf{a} \cdot u_{(2)}. \tag{3.3.2}$$

Proposition 3.3.4 (Ehrenborg–H. Fox)**.** *For any* **cd**-*polynomials u and v, the diamond product satisfies the following recursions:*

$$u \diamond (v \cdot \mathbf{c}) = (u \diamond v) \cdot \mathbf{c} + \sum_u (u_{(1)} \diamond v) \cdot \mathbf{d} \cdot u_{(2)}, \tag{3.3.3}$$

$$u \diamond (v \cdot \mathbf{d}) = (u \diamond v) \cdot \mathbf{d} + \sum_u (u_{(1)} \diamond v) \cdot \mathbf{d} \cdot \mathrm{Pyr}(u_{(2)}). \tag{3.3.4}$$

3.4 Lattice path interpretation for ab-monomials

Before introducing the lattice path interpretation for the diamond product of **cd**-monomials, we first introduce a similar interpretation for the diamond product of two **ab**-monomials. Define the set of lattice paths Ω as words in the non-commutative letters **D**, **R**, and **U**, where **D** is degree 2, and **R** and **U** are each degree 1. The letters correspond to the lattice path steps as follows

$$\text{Right} : \mathbf{R} = (1, 0), \;\; \text{Up} : \mathbf{U} = (0, 1), \;\; \text{and Diagonal} : \mathbf{D} = (1, 1).$$

Let $\Omega(p, q)$ be the set of lattice paths using only these 3 steps from $(0, 0)$ to (p, q) which do not contain **UR** as a contiguous subword, that is, as a factor.

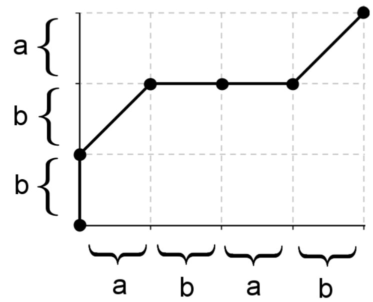

Figure 3.1: The lattice path **UDRRD** $\in \Omega(4,3)$ labeled by the words $u = \mathbf{abab}$ and $v = \mathbf{bba}$.

For a given pair of **ab**-monomials u and v of degrees p and q, respectively, consider lattice paths in $\Omega(p,q)$ in which the axes are labeled by the words u and v, as shown by the example in Figure 3.1. We now define a weight function for such paths based on this labeling.

Definition 3.4.1. *For $p' \leq p$ and $q' \leq q$, define $\mathrm{wt}_{u,v} : \Omega(p',q') \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ to be the multiplicative map, taking concatenation to be the product, determined by the following rules:*

$$\mathrm{wt}_{u,v}(\boldsymbol{R}) = \begin{cases} \mathbf{a} & \textit{if above an } \mathbf{a} \textit{ label} \\ \mathbf{b} & \textit{if above a } \mathbf{b} \textit{ label,} \end{cases}$$

$$\mathrm{wt}_{u,v}(\boldsymbol{U}) = \begin{cases} \mathbf{a} & \textit{if to the right of an } \mathbf{a} \textit{ label} \\ \mathbf{b} & \textit{if to the right of a } \mathbf{b} \textit{ label,} \end{cases}$$

$$\mathrm{wt}_{u,v}(\boldsymbol{D}) = \begin{cases} \mathbf{a}\cdot\mathbf{b} & \textit{if to the right of an } \mathbf{a} \textit{ label} \\ \mathbf{b}\cdot\mathbf{a} & \textit{if to the right of a } \mathbf{b} \textit{ label.} \end{cases}$$

For the example path in Figure 3.1, we have $\mathrm{wt}_{\mathbf{abab},\mathbf{bba}}(\mathbf{UDRRD}) = \mathbf{bbabaab}$.

For a given **ab**-monomial u of degree p, define $\tau(u) \in \Omega(p,0)$ as the word $\tau(u) = \mathbf{R}^{\deg(u)}$. Now that we have notation for creating horizontal paths, we give the interpretation for the diamond product of two **ab**-monomials as a sum of weighted lattice paths.

Theorem 3.4.2. *For any two* **ab***-monomials u and v of degree p and q, respectively, the* **ab***-polynomial $u \diamond v$ is given by the sum*

$$u \diamond v = \sum_{P \in \Omega(p,q)} \mathrm{wt}_{u,v}(P).$$

Proof. To keep the notation simpler, we will leave out the dependency on u and v of the weight function. The proof of this theorem is by induction on the degree q of the monomial v. For the base case, we assume that q is 0, making $v = 1$, and that

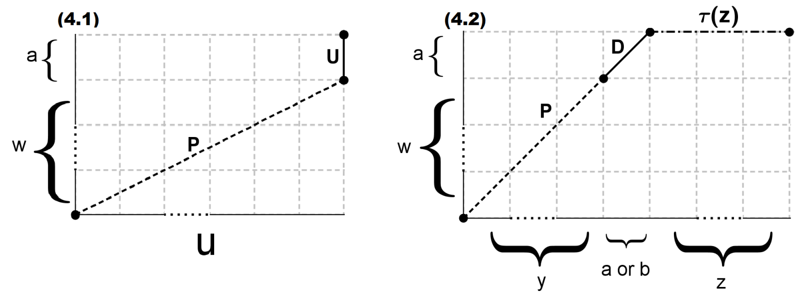

Figure 3.2: Illustrations of the lattice paths described in Case 1.

the degree p of u is any nonnegative integer. The diamond product $u \diamond 1$ is u, and the only lattice path in $\Omega(p, 0)$ is the horizontal path $\tau(u)$ of length p. The weight of this path $\tau(u)$ is $\mathrm{wt}(\tau(u)) = u$ since it is only $\mathbf{R}$ steps along the labels of u. Thus the base case of the theorem is true.

Suppose the statement is true for any two words of degree p' and q' where $p' \leq p$ and $q' < q$. The proof is split into two cases depending on the last letter of v.

Case 1: We first assume that the last letter of v is $\mathbf{a}$, or $v = w \cdot \mathbf{a}$. According to Equation (3.3.1), we have

$$u \diamond (w \cdot \mathbf{a}) = (u \diamond w) \cdot \mathbf{a} + \sum_{u} (u_{(1)} \diamond w) \cdot \mathbf{a} \cdot \mathbf{b} \cdot u_{(2)}.$$

By induction, the first term is

$$(u \diamond w) \cdot \mathbf{a} = \sum_{P \in \Omega(p, q-1)} \mathrm{wt}(P \cdot \mathbf{U}). \tag{3.4.1}$$

Since the final $\mathbf{U}$ step is to the right of an $\mathbf{a}$ label, $\mathbf{a}$ is the correct weight for this step. Figure 3.2 gives an illustration of the lattice paths in equation (3.4.1) as well as the next equation.

For the terms that result from the coproduct, we observe that the cases of u being broken apart by the coproduct at either an $\mathbf{a}$ or a $\mathbf{b}$ are identical. We assume that either $u = y \cdot \mathbf{a} \cdot z$ or $u = y \cdot \mathbf{b} \cdot z$ where y is of degree i. Hence, we have $u_{(1)} \otimes u_{(2)} = y \otimes z$ in each case since $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) = 1 \otimes 1$. This gives the term

$$(y \diamond w) \cdot \mathbf{a} \cdot \mathbf{b} \cdot z = \sum_{P \in \Omega(i, q-1)} \mathrm{wt}(P \cdot \mathbf{D} \cdot \tau(z)). \tag{3.4.2}$$

Notice that the weight of a $\mathbf{D}$ step does not depend on the label below that step. Rather, it only depends on the label on the vertical axis. Since this $\mathbf{D}$ step is to the right of the $\mathbf{a}$ label that ends the word v, its weight is $\mathbf{a} \cdot \mathbf{b}$, which matches the left side of equation (3.4.2).

Since we only consider lattice paths without consecutive $\mathbf{UR}$ steps, every lattice path in $\Omega(p, q)$ must end in a $\mathbf{U}$ step or end in a $\mathbf{D}$ step followed by a horizontal path. The paths contained within equation (3.4.1) correspond to the paths ending

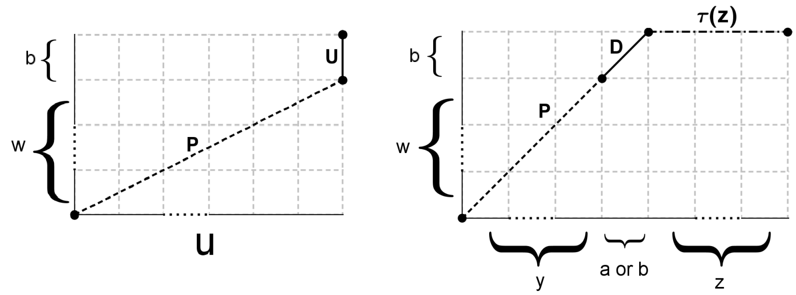

Figure 3.3: Illustrations of the lattice paths described in Case 2.

in **U**, and the remaining possible paths with the **D** step are found in equation (3.4.2). Thus $\Omega(p,q)$ decomposes into a disjoint union of lattice paths as

$$\begin{aligned}\Omega(p,q) =& \{P\cdot\mathbf{U} : P\in\Omega(p,q-1)\}\\ &\dot{\cup}\{P\cdot\mathbf{D}\cdot\tau(z) : P\in\Omega(i,q-1), u=y\cdot\mathbf{a}\cdot z \text{ or } u=y\cdot\mathbf{b}\cdot z\},\end{aligned}$$

completing the proof if v ends with the letter **a**.

Case 2: If we instead assume that $v = w\cdot\mathbf{b}$, then Equation (3.3.2) gives us

$$u\diamond(w\cdot\mathbf{b}) = (u\diamond w)\cdot\mathbf{b} + \sum_u (u_{(1)}\diamond w)\cdot\mathbf{b}\cdot\mathbf{a}\cdot u_{(2)}.$$

This second situation follows nearly identically to the first case from this point. This is because the lattice paths ending in **U** would have **b** as the weight for this final step since it would be to the right of a **b** label. Additionally, the **D** step in lattices paths ending in a **D** step followed by a horizontal path will contribute a weight of $\mathbf{b}\cdot\mathbf{a}$ since this step will also be to the right of the final **b** label. Illustrations of the lattice paths for this case are shown in Figure 3.3. This second case concludes the proof of the theorem. □

3.5 Lattice path interpretation for cd-monomials

To try to give a better understanding of the recursive formulas given in (3.3.3) and (3.3.4) that Ehrenborg and H. Fox developed for the diamond product of two **cd**-polynomials, Slone examined in [42] the specific case of the diamond product of the form $\mathbf{c}^p\diamond\mathbf{c}^q$. He was able to interpret the coefficients of the resulting **cd**-polynomial using weighted lattice paths.

Concentrating on the diamond product of powers of **c**, or $\mathbf{c}^p\diamond\mathbf{c}^q$, Slone defined the set of lattice paths Λ as words in the non-commutative letters **D**, **R**, and **U**, in which **D** has degree 2 whereas **R** and **U** both have degree 1. As defined in the **ab**-index case, these letters correspond to lattice path steps as follows:

$$\text{Right} : \mathbf{R} = (1,0),\ \text{Up} : \mathbf{U} = (0,1),\ \text{and Diagonal} : \mathbf{D} = (1,1).$$

Let $\Lambda(p,q)$ be the set of lattice paths using only these 3 steps from $(0,0)$ to (p,q) which do not contain $\mathbf{UR}$ as a contiguous subword. Note that labeling the axes, as was done in the **ab**-index case, is not necessary here since each letter in the **cd**-monomials is a $\mathbf{c}$. Define $\mathrm{wt} : \Lambda(p,q) \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ to be the multiplicative map, taking concatenation to be the product, determined by $\mathrm{wt}(\mathbf{D}) = 2\mathbf{d}$ and $\mathrm{wt}(\mathbf{R}) = \mathrm{wt}(\mathbf{U}) = \mathbf{c}$. The main result of Slone's work on the diamond product is the following statement, which is Proposition 2.4.2 in [42].

Proposition 3.5.1 (Slone)**.** *For any nonnegative integers p and q, the* **cd***-polynomial* $\mathbf{c}^p \diamond \mathbf{c}^q$ *is given by the sum*

$$\mathbf{c}^p \diamond \mathbf{c}^q = \sum_{P \in \Lambda(p,q)} \mathrm{wt}(P).$$

Now we extend Slone's interpretation to look beyond the case of **cd**-monomials consisting of powers of $\mathbf{c}$ to the diamond product of any two **cd**-monomials. Define the set of lattice paths Γ as words in the non-commutative letters $\mathbf{R}$, $\mathbf{U}$, $\mathbf{D}$, $\overline{\mathbf{R}}$, and $\overline{\mathbf{U}}$. We consider $\mathbf{R}$ and $\mathbf{U}$ to be degree 1, and $\mathbf{D}$, $\overline{\mathbf{R}}$, and $\overline{\mathbf{U}}$ to be degree 2. The letters correspond to the steps

$$\text{Right: } \mathbf{R} = (1,0), \text{ Up: } \mathbf{U} = (0,1), \text{ Diagonal: } \mathbf{D} = (1,1),$$
$$\text{Double Right: } \overline{\mathbf{R}} = (2,0), \text{ and Double Up: } \overline{\mathbf{U}} = (0,2).$$

Let $\Gamma(p,q)$ be the set of all lattice paths from the origin to (p,q) using the 5 steps described above and which do not contain consecutive $\mathbf{UR}$, $\mathbf{U}\overline{\mathbf{R}}$, $\overline{\mathbf{U}}\mathbf{R}$, or $\overline{\mathbf{U}}\,\overline{\mathbf{R}}$ steps.

We now restrict this set to a particular subset $\Gamma(u,v)$ given two **cd**-monomials u and v with the degrees of the monomials being p and q, respectively. This subset within $\Gamma(p,q)$ requires that the word and its corresponding lattice path adhere to the following four rules, where we label the horizontal axis by the word u and likewise label the vertical axis by v, as shown in Figure 3.1. This is similar to the labels used earlier with **ab**-monomials except that the $\mathbf{d}$ label covers two units on the axis. In the example, we have $u = \mathbf{ddcc}$ and $v = \mathbf{cdc}$; hence, the degrees are $p = 6$ and $q = 4$, with the lattice path $\mathbf{DR}\overline{\mathbf{R}}\mathbf{DDU}$ being shown.

The rules for a word $P \in \Gamma(p,q)$ to be in $\Gamma(u,v)$ are as follows:

1. No $\mathbf{U}$ step is allowed at the bottom of a $\mathbf{d}$ label on the vertical axis.

2. Although an $\mathbf{R}$ step is allowed along the first part of a $\mathbf{d}$ label on the horizontal axis, two consecutive $\mathbf{R}$ steps along such a $\mathbf{d}$ label are not allowed.

3. A $\overline{\mathbf{U}}$ step is only allowed at the bottom of a $\mathbf{d}$ label on the vertical axis, and similarly, an $\overline{\mathbf{R}}$ step is only allowed at the left of a $\mathbf{d}$ label on the horizontal axis.

4. If a $\mathbf{D}$ step is at the bottom of a $\mathbf{d}$ label on the vertical axis, then the steps $\mathbf{DR}$ above a $\mathbf{d}$ label on the horizontal axis and within the top half of this $\mathbf{d}$ label on the vertical axis are not allowed.

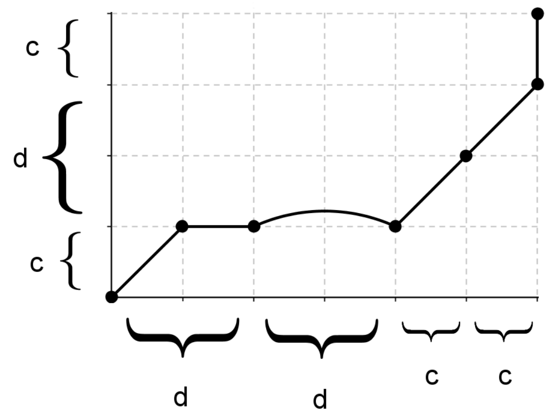

Figure 3.1: The lattice path $\mathbf{DR\overline{R}DDU} \in \Gamma(\mathbf{ddcc}, \mathbf{cdc})$.

The following definition gives the method of weighting the steps of the lattice paths in $\Gamma(u, v)$ for **cd**-monomials u and v to obtain the **cd**-index of the diamond product, albeit the choice of coefficient for weight of the **D** steps becomes complicated.

Definition 3.5.2. *For u' an initial subword of u, that is, u can be factored as $u = u' \cdot u''$, and v' an initial subword of v, define $\mathrm{wt}_{u,v} : \Gamma(u', v') \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ to be the multiplicative map determined by*

$$\mathrm{wt}_{u,v}(\boldsymbol{R}) = \mathrm{wt}_{u,v}(\boldsymbol{U}) = \mathbf{c}, \quad \mathrm{wt}_{u,v}(\overline{\boldsymbol{R}}) = \mathrm{wt}_{u,v}(\overline{\boldsymbol{U}}) = \mathbf{d}, \quad \mathrm{wt}_{u,v}(\boldsymbol{D}) = k\mathbf{d},$$

where depending on the location of a diagonal step $\boldsymbol{D}$, the scalar k is given by

$$k = \begin{cases} 2 & \text{if above a } \mathbf{c} \text{ label and to the right of either a } \mathbf{c} \text{ label or the} \\ & \text{bottom of a } \mathbf{d} \text{ label} \\ 2 & \text{if above the first part of a } \mathbf{d} \text{ label, to the right of a } \mathbf{c} \text{ label,} \\ & \text{and followed by a } \boldsymbol{U} \text{ step, a } \overline{\boldsymbol{U}} \text{ step, or a } \boldsymbol{D} \text{ step} \\ 2 & \text{if above the first part of a } \mathbf{d} \text{ label, to the right of the bottom} \\ & \text{of a } \mathbf{d} \text{ label, and followed by a } \boldsymbol{U} \text{ step} \\ 1 & \text{otherwise.} \end{cases}$$

Note that this weight function matches Slone's weight function when we restrict our view to lattice paths in $\Gamma(\mathbf{c}^p, \mathbf{c}^q) = \Lambda(p, q)$, because the coefficient of a **D** step will always be 2 in this situation.

Example 3.5.3. One can compute the diamond product of **cd** and **dc** as

$$\begin{aligned} \mathbf{cd} \diamond \mathbf{dc} = 3\mathbf{cddc} + \mathbf{ccdcc} + \mathbf{ccdd} + \mathbf{cdcc} + 2\mathbf{cdcd} \\ + 2\mathbf{ddcc} + 4\mathbf{dcdc} + 2\mathbf{dccd} + 4\mathbf{ddd}. \end{aligned}$$

There are 13 lattice paths in $\Gamma(\mathbf{cd}, \mathbf{dc})$, which are shown in Figure 3.2. Note that none of the paths begin with **U** as required by rule 1 since the word **dc** begins with **d**.

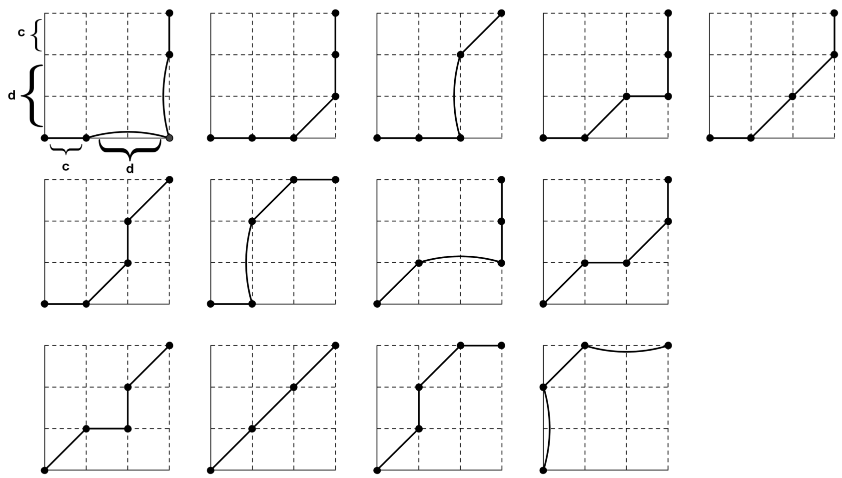

Figure 3.2: The lattice paths in $\Gamma(\mathbf{cd}, \mathbf{dc})$.

Path	$\mathbf{R\overline{R}\,\overline{U}U}$	$\mathbf{RRDUU}$	$\mathbf{RR\overline{U}D}$	$\mathbf{RDRUU}$	$\mathbf{RDDU}$
Weight	$\mathbf{cddc}$	$\mathbf{ccdcc}$	$\mathbf{ccdd}$	$\mathbf{cdccc}$	$\mathbf{cddc}$

Path	$\mathbf{RDUD}$	$\mathbf{R\overline{U}DR}$	$\mathbf{D\overline{R}UU}$	$\mathbf{DRDU}$
Weight	$2\mathbf{cdcd}$	$\mathbf{cddc}$	$2\mathbf{ddcc}$	$2\mathbf{dcdc}$

Path	$\mathbf{DRUD}$	$\mathbf{DDD}$	$\mathbf{DUDR}$	$\mathbf{\overline{U}D\overline{R}}$
Weight	$2\mathbf{dccd}$	$2\mathbf{ddd}$	$2\mathbf{dcdc}$	$2\mathbf{ddd}$

Table 3.1: The weights of the lattice paths in $\Gamma(\mathbf{cd}, \mathbf{dc})$

Additionally, due to rule 4, the path **DDRU** is omitted. The terms of $\mathbf{cd} \diamond \mathbf{dc}$ can be obtained by adding the weights of the paths as defined in Definition 3.5.2. Some of the paths, such as **DRDU** and **DUDR**, give the same term of $\mathbf{cd} \diamond \mathbf{dc}$, leading to only 9 terms from the 13 lattice paths. The paths are shown in Figure 3.2, and their corresponding weights are given in Table 3.1.

Before we state the main result, we first define a map to create horizontal paths that will be useful in its proof, as was done with the map τ in the **ab**-index case. Define π such that for a given **cd**-monomial u, $\pi(u)$ is the word in $\Gamma(u, 1)$ resulting from replacing each **c** in u with the step **R** and each **d** with the step $\overline{\mathbf{R}}$. This map will be important in the proof of Theorem 3.5.4 since rules 2 and 3 imply that $\pi(u)$ is the only valid horizontal path along a portion of the horizontal axis labeled by u.

Theorem 3.5.4. *For any two* **cd***-monomials* u *and* v*, the* **cd***-polynomial* $u \diamond v$ *is given by the sum*

$$u \diamond v = \sum_{P \in \Gamma(u,v)} \mathrm{wt}_{u,v}(P).$$

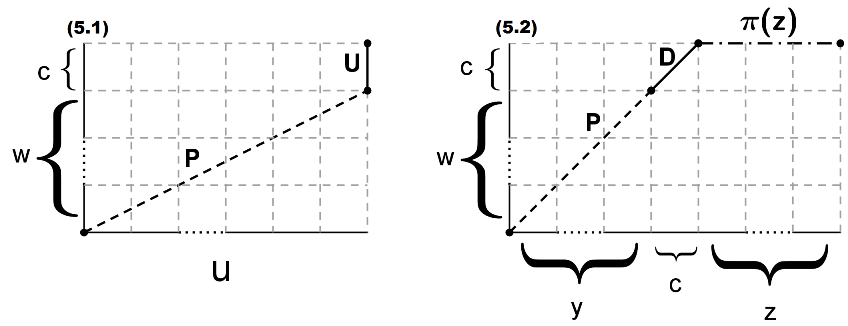

Figure 3.3: Illustrations of the lattice paths described in Subcases (i) and (ii) of Case 1.

Proof. Again to simplify notation, the dependency of the weight function on the words u and v will be omitted. We will prove this result using induction on the degree q of v. For the base case when $q = 0$ and the degree of u is any nonnegative integer p, we have that $v = 1$. The diamond product $u \diamond 1$ is simply u, and the only lattice path in $\Gamma(u, 1)$ is $\pi(u)$, the horizontal path along the labels from u. The fact that $\mathrm{wt}(\pi(u)) = u$ shows that the base case is true.

Suppose the statement is true for any two words of degree p' and q' where $p' \leq p$ and $q' < q$. We will break up the proof for $u \diamond v$ according to the final letter of v.

Case 1: Assume $v = w \cdot \mathbf{c}$. Due to equation (3.3.3), we have

$$u \diamond (w \cdot \mathbf{c}) = (u \diamond w) \cdot \mathbf{c} + \sum_u (u_{(1)} \diamond w) \cdot \mathbf{d} \cdot u_{(2)}.$$

We now examine four subcases, each of which has a set of lattice paths that corresponds to either the first term of the previous equation or to a collection of terms from the coproduct.

Subcase (i): By induction, the first term is

$$(u \diamond w) \cdot \mathbf{c} = \sum_{P \in \Gamma(u,w)} \mathrm{wt}(P \cdot \mathbf{U}). \tag{3.5.1}$$

An illustration of the lattice paths in equation (3.5.1) as well as the next equation can be seen in Figure 3.3.

For the remaining terms that result from the coproduct, we must separately examine the subcases of u being broken apart by the coproduct at either a $\mathbf{c}$ or $\mathbf{d}$.

Subcase (ii): If the coproduct breaks up the monomial u at a $\mathbf{c}$, we assume $u = y \cdot \mathbf{c} \cdot z$; thus, u splits such that $u_{(1)} \otimes u_{(2)} = 2 \cdot y \otimes z$. This gives the term

$$(y \diamond w) \cdot 2\mathbf{d} \cdot z = \sum_{P \in \Gamma(y,w)} \mathrm{wt}(P \cdot \mathbf{D} \cdot \pi(z)). \tag{3.5.2}$$

Since the $\mathbf{D}$ step is above the $\mathbf{c}$ label that is between y and z and to the right of a $\mathbf{c}$ label at the end of the word v, the weight of this step is correctly $2\mathbf{d}$.

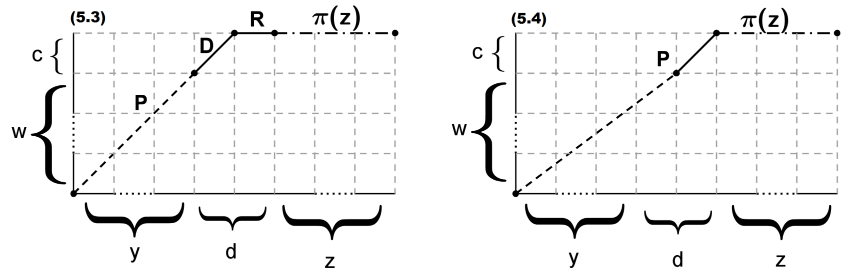

Figure 3.4: Illustrations of the lattice paths described in Subcases (iii) and (iv) of Case 1.

If u is instead broken up at a $\mathbf{d}$, we assume $u = y \cdot \mathbf{d} \cdot z$; thus, u splits as $y \otimes \mathbf{c} \cdot z + y \cdot \mathbf{c} \otimes z$. This leads to two terms that represent our last two subcases.

Subcase (iii): The first of these terms is

$$(y \diamond w) \cdot \mathbf{d} \cdot \mathbf{c} \cdot z = \sum_{P \in \Gamma(y,w)} \mathrm{wt}(P \cdot \mathbf{D} \cdot \mathbf{R} \cdot \pi(z)). \tag{3.5.3}$$

Although the $\mathbf{D}$ step is above the first part of a $\mathbf{d}$ label and to the right of a $\mathbf{c}$ label, 1 is the correct coefficient of the weight of this $\mathbf{D}$ step since it is not followed by a $\mathbf{U}$, $\overline{\mathbf{U}}$, or $\mathbf{D}$ step. The lattice paths described in equation (3.5.3) and the following equation can be seen in Figure 3.4.

Subcase (iv): The other term we get is

$$(y \cdot \mathbf{c} \diamond w) \cdot \mathbf{d} \cdot z = \sum_{P' \in \Gamma(y \cdot \mathbf{c}, w)} \mathrm{wt}(P') \cdot \mathbf{d} \cdot z = \sum_{\substack{P \in \Gamma(y \cdot \mathbf{d}, v) \\ \text{P ends with } \mathbf{D}}} \mathrm{wt}(P \cdot \pi(z)). \tag{3.5.4}$$

First, note that the $\mathbf{D}$ step that is appended to P' to create P has the correct coefficient 1 since it is above the second half of a $\mathbf{d}$ label. As we switch labels from $y \cdot \mathbf{c}$ to $y \cdot \mathbf{d}$, it is important to notice that the coefficient of a $\mathbf{D}$ step above this $\mathbf{c}$ label does not change. The only scenario in which it could change is if it was to the right of the bottom of a $\mathbf{d}$ label and was not followed by a $\mathbf{U}$ step, but this is impossible because a $\mathbf{U}$ step would be required to move vertically through the top half of the $\mathbf{d}$ label, concluding this final subcase.

To avoid the subwords $\mathbf{UR}$ and $\mathbf{U}\overline{\mathbf{R}}$, every lattice path in $\Gamma(u, w \cdot \mathbf{c})$ must either end in a $\mathbf{U}$ step, which is described in Subcase (i), or end in a $\mathbf{D}$ step followed by a horizontal path to the point (p, q). The paths within the three types of terms resulting from the coproduct in Subcases (ii), (iii), and (iv) cover all possible ways for this $\mathbf{D}$ step to occur, either above a $\mathbf{c}$ label or above one of the two parts of a $\mathbf{d}$ label. Thus $\Gamma(u, w \cdot \mathbf{c})$ decomposes as the disjoint union

$$\begin{aligned} \Gamma(u, w \cdot \mathbf{c}) = & \{P \cdot \mathbf{U} : P \in \Gamma(u, w)\} && (3.5.5) \\ & \dot\cup \{P \cdot \mathbf{D} \cdot \pi(z) : P \in \Gamma(y, w), u = y \cdot \mathbf{c} \cdot z\} && (3.5.6) \\ & \dot\cup \{P \cdot \mathbf{D} \cdot \mathbf{R} \cdot \pi(z) : P \in \Gamma(y, w), u = y \cdot \mathbf{d} \cdot z\} && (3.5.7) \end{aligned}$$

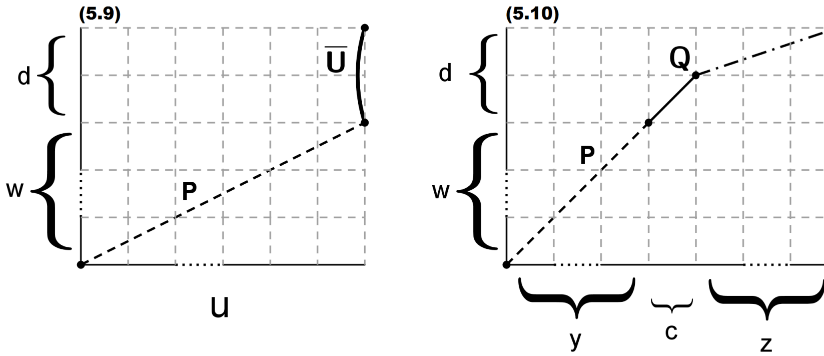

Figure 3.5: Illustrations of the lattice paths described in Subcases (i) and (ii) of Case 2.

$$\dot{\cup}\{P \cdot \pi(z) : P \in \Gamma(y \cdot \mathbf{d}, v), P \text{ ends in } \mathbf{D}, u = y \cdot \mathbf{d} \cdot z\}, \tag{3.5.8}$$

where the set (3.5.5) is from equation (3.5.1), (3.5.6) from (3.5.2), (3.5.7) from (3.5.3), and (3.5.8) from (3.5.4). This concludes the proof for this case.

Case 2: Assume $v = w \cdot \mathbf{d}$. By applying equation (3.3.4), we have

$$u \diamond (w \cdot \mathbf{d}) = (u \diamond w) \cdot \mathbf{d} + \sum_{u} (u_{(1)} \diamond w) \cdot \mathbf{d} \cdot \mathrm{Pyr}(u_{(2)}).$$

We again break down the terms in this equation into four subcases and describe the sets of lattice paths corresponding to those terms.

Subcase (i): The first term, by induction, gives us

$$(u \diamond w) \cdot \mathbf{d} = \sum_{P \in \Gamma(u,w)} \mathrm{wt}(P \cdot \overline{\mathbf{U}}). \tag{3.5.9}$$

See an illustration of the lattice paths in equation (3.5.9) and the next equation in Figure 3.5.

As was done in Case 1, we separate the remaining terms from the coproduct depending on whether u is broken up at a $\mathbf{c}$ or $\mathbf{d}$.

Subcase (ii): If the monomial u is broken up at a $\mathbf{c}$, we assume $u = y \cdot \mathbf{c} \cdot z$; hence, u splits into $u_{(1)} \otimes u_{(2)} = 2 \cdot y \otimes z$ as it did in Case 1. This gives the term

$$(y \diamond w) \cdot 2\mathbf{d} \cdot \mathrm{Pyr}(z) = \sum_{\substack{P \in \Gamma(y,w), Q \in \Gamma(\mathbf{c} \cdot z, \mathbf{d}) \\ Q \text{ begins with } \mathbf{D}}} \mathrm{wt}(P \cdot Q). \tag{3.5.10}$$

The $2\mathbf{d}$ is the weight of the $\mathbf{D}$ step that it is above the $\mathbf{c}$ label since it is to the right of the bottom of a $\mathbf{d}$ label, so it remains to show that $\mathrm{Pyr}(z)$ gives the weights of the steps that follow the $\mathbf{D}$ step in the path Q. Since this step is at the bottom part of a $\mathbf{d}$ label on the vertical axis, rule 4 causes any path with $\mathbf{DR}$ along any $\mathbf{d}$ label to be invalid. Due to consecutive $\mathbf{UR}$ and $\mathbf{U}\overline{\mathbf{R}}$ steps not being allowed, there also cannot be any path with a $\mathbf{U}$ step, except possibly as the final step. Thus these paths only

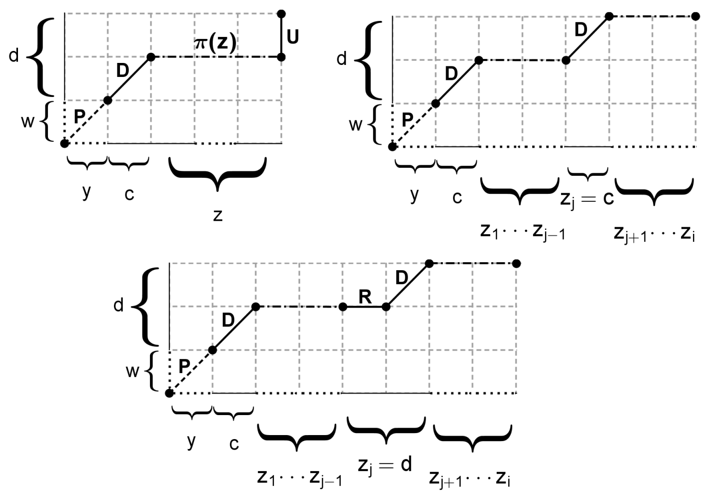

Figure 3.6: Illustrations of the lattice paths Q within Subcase (ii) of Case 2.

have horizontal steps with a **U** step at the end, or they only have horizontal steps with the exception of one **D** step, either above a **c** label or following an **R** step along a **d** label.

Recall that

$$\text{Pyr}(z) = z \cdot \mathbf{c} + G(z).$$

The first term is

$$z \cdot \mathbf{c} = \text{wt}(\pi(z) \cdot \mathbf{U}),$$

corresponding to the horizontal path with **U** appended to the end. Illustrations for this and the following two descriptions of the possible options for the path Q are shown in Figure 3.6.

Since G is a derivation, we apply the product rule to $z = z_1 \cdots z_i$ to get

$$G(z) = \sum_{j=1}^{i} z_1 \cdots z_{j-1} \cdot G(z_j) \cdot z_{j+1} \cdots z_i.$$

If $z_j = \mathbf{c}$, we have

$$\begin{aligned} z_1 \cdots z_{j-1} \cdot G(z_j) \cdot z_{j+1} \cdots z_i &= z_1 \cdots z_{j-1} \cdot \mathbf{d} \cdot z_{j+1} \cdots z_i \\ &= \text{wt}(\pi(z_1 \cdots z_{j-1}) \cdot \mathbf{D} \cdot \pi(z_{j+1} \cdots z_i)), \end{aligned}$$

corresponding to the paths where the **D** step is above a **c** label. The weight of this step has coefficient 1 since it is along the top half of a **d** label on the vertical axis. On the other hand, if $z_j = \mathbf{d}$, we have

$$\begin{aligned} z_1 \cdots z_{j-1} \cdot G(z_j) \cdot z_{j+1} \cdots z_i &= z_1 \cdots z_{j-1} \cdot \mathbf{c} \cdot \mathbf{d} \cdot z_{j+1} \cdots z_i \\ &= \text{wt}(\pi(z_1 \cdots z_{j-1}) \cdot \mathbf{R} \cdot \mathbf{D} \cdot \pi(z_{j+1} \cdots z_i)), \end{aligned}$$

corresponding to the paths with **RD** steps above the **d** label, where the coefficient of the weight of the **D** step is again 1 by the same reasoning. Therefore, Pyr(z) gives the

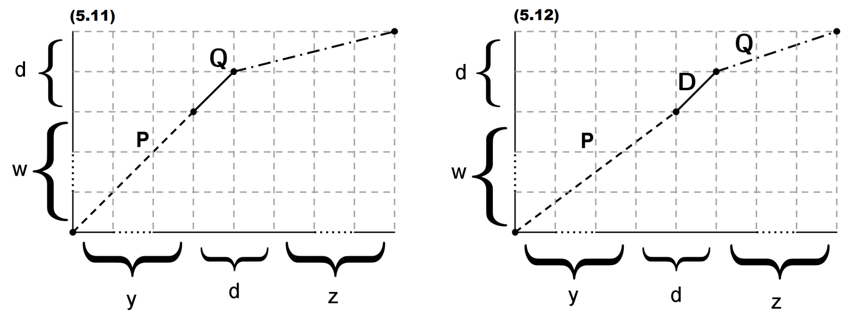

Figure 3.7: Illustrations of the lattice paths described in Subcases (iii) and (iv) of Case 2.

correct paths that combine with the initial $\mathbf{D}$ step to make up the paths Q, proving equation (3.5.10) and concluding Subcase (ii).

If u is broken up by the coproduct at a $\mathbf{d}$, we assume $u = y \cdot \mathbf{d} \cdot z$, and we have that u splits as $y \otimes \mathbf{c} \cdot z + y \cdot \mathbf{c} \otimes z$. This gives two terms, each representing a subcase of Case 2.

Subcase (iii): The first of the two terms is

$$(y \diamond w) \cdot \mathbf{d} \cdot \operatorname{Pyr}(\mathbf{c} \cdot z) = \sum_{\substack{P \in \Gamma(y,w), Q \in \Gamma(\mathbf{d}\cdot z, \mathbf{d}) \\ Q \text{ begins with } \mathbf{D}}} \operatorname{wt}(P \cdot Q). \tag{3.5.11}$$

The $\mathbf{d}$ is the correct weight of the first $\mathbf{D}$ step in Q since it cannot be followed by a $\mathbf{U}$ step. Otherwise, the path would be invalid since it would have $\mathbf{UR}$ or $\mathbf{U}\overline{\mathbf{R}}$ as a subword. $\operatorname{Pyr}(\mathbf{c} \cdot z)$ gives the weights of the steps following this $\mathbf{D}$ step in the paths Q due to an argument analogous to the one used in the previous subcase, because treating the second half of the $\mathbf{d}$ label on the horizontal axis as a $\mathbf{c}$ label does not change any of the weights of these paths. Illustrations of the lattice paths in equation (3.5.11) and the following equation can be found in Figure 3.7.

Subcase (iv): The second term from this situation is

$$\begin{aligned}(y \cdot \mathbf{c} \diamond w) \cdot \mathbf{d} \cdot \operatorname{Pyr}(z) &= \left(\sum_{P' \in \Gamma(y \cdot \mathbf{c}, w)} \operatorname{wt}(P') \right) \cdot \mathbf{d} \cdot \operatorname{Pyr}(z) \\ &= \sum_{\substack{P \in \Gamma(i+1, q-2), Q \in \Gamma(p-i-2, 1) \\ P \cdot \mathbf{D} \cdot Q \in \Gamma(y \cdot \mathbf{d} \cdot z, w \cdot \mathbf{d})}} \operatorname{wt}(P \cdot \mathbf{D} \cdot Q). \end{aligned} \tag{3.5.12}$$

Here, we are assuming the degree of y is i; hence, the degree of z is $p - i - 2$. Note that the path P' does not have its weight changed as it becomes the path P when the $\mathbf{c}$ label is switched to become the first half of a $\mathbf{d}$ label. This is true since the only possible difference could be the coefficient of a $\mathbf{D}$ step above the final $\mathbf{c}$ label. However, this coefficient will not change since it must be followed by a $\mathbf{U}$ or $\overline{\mathbf{U}}$ step if the $\mathbf{D}$ step is not the final step in P', or it is followed by a $\mathbf{D}$ step if it is the final step in P'. The coefficient of 1 is correct for the $\mathbf{D}$ step between the paths P and Q since it is above the second part of a $\mathbf{d}$ label. Although it is not possible to partition the labels in order to have the correct weights when writing P and Q as elements of $\Gamma(x, x')$ for some $\mathbf{cd}$-monomials x and x' as was done in the previous cases, it is still clear that the contribution that Q makes to the weight is $\mathrm{Pyr}(z)$, similarly to the last two subcases. This concludes the final subcase of Case 2.

The lattice paths in $\Gamma(u, w \cdot \mathbf{d})$ must either end in a $\overline{\mathbf{U}}$ step, as described in Subcase (i), or by rule 1 and the restriction of avoiding consecutive $\overline{\mathbf{U}}\mathbf{R}$ and $\overline{\mathbf{U}}\overline{\mathbf{R}}$ steps, there must be two $\mathbf{D}$ steps to the right of the last $\mathbf{d}$ label of $v = w \cdot \mathbf{d}$ with horizontal paths between and after these steps. The three types of terms from the coproduct consist of all ways for these $\mathbf{D}$ steps to occur, with the three types being distinguished by whether the first $\mathbf{D}$ step is above a $\mathbf{c}$ label in Subcase (ii), the first part of a $\mathbf{d}$ label in Subcase (iii), or the second part of a $\mathbf{d}$ label in Subcase (iv). Therefore, $\Gamma(u, w \cdot \mathbf{d})$ decomposes as the disjoint union

$$\begin{aligned}
\Gamma(u, w \cdot \mathbf{d}) = \; & \{P \cdot \overline{\mathbf{U}} : P \in \Gamma(u, w)\} && (3.5.13)\\
& \dot{\cup}\, \{P \cdot Q : P \in \Gamma(y, w), Q \in \Gamma(\mathbf{c} \cdot z, \mathbf{d}), Q \text{ begins with } \mathbf{D},\\
& \quad u = y \cdot \mathbf{c} \cdot z\} && (3.5.14)\\
& \dot{\cup}\, \{P \cdot Q : P \in \Gamma(y, w), Q \in \Gamma(\mathbf{d} \cdot z, \mathbf{d}), Q \text{ begins with } \mathbf{D},\\
& \quad u = y \cdot \mathbf{d} \cdot z\} && (3.5.15)\\
& \dot{\cup}\, \{P \cdot \mathbf{D} \cdot Q : P \in \Gamma(i+1, q-2), Q \in \Gamma(p-i-2, 1),\\
& \quad P \cdot \mathbf{D} \cdot Q \in \Gamma(y \cdot \mathbf{d} \cdot z, w \cdot \mathbf{d}), u = y \cdot \mathbf{d} \cdot z\}, && (3.5.16)
\end{aligned}$$

where the set (3.5.13) is from equation (3.5.9), (3.5.14) is from (3.5.10), (3.5.15) is from (3.5.11), and (3.5.16) is from (3.5.12). This decomposition gives us the proof for the case of v ending in a $\mathbf{d}$, concluding the proof of the theorem. □

3.6 Concluding remarks

The effect on the $\mathbf{cd}$-index of a second important operation on posets was studied in [13] and [19]. This operation is the Cartesian product of posets, which we recall from Section 1.4 to be defined for posets P and Q as $P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$. As the diamond product of posets is related to the Cartesian product of polytopes, the Cartesian product of posets is connected to the *free join* of polytopes, for which we recall the following definition. If V is an m-dimensional polytope and W is an n-dimensional polytope, then embed V and W in $\mathbb{R}^{m+n+1}$ by

$$V' = \{(x_1, \ldots, x_m, \underbrace{0 \ldots, 0}_{n}, 0) \in \mathbb{R}^{m+n+1} : (x_1, \ldots x_m) \in V\}$$

and likewise by

$$W' = \{(\underbrace{0, \ldots, 0}_{m}, x_1, \ldots, x_n, 1) \in \mathbb{R}^{m+n+1} : (x_1, \ldots, x_n) \in W\}.$$

Then the free join $V \circledvee W$ is the $(m+n+1)$-dimensional polytope defined as the convex hull of V' and W'. Kalai [28] observed that the face lattice of the free join of two polytopes is the Cartesian product of the two face lattices, i.e., for two polytopes V and W we have $\mathcal{L}(V \circledvee W) = \mathcal{L}(V) \times \mathcal{L}(W)$. Ehrenborg and Readdy [19] introduced a bilinear operator from $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \times \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$, called the mixing operator M, which was defined in Section 3.3, in order to study the **ab**-index of the Cartesian product of posets, or likewise the **ab**-index of the free join of polytopes. The following proposition describes how to calculate this **ab**-index using the mixing operator.

Proposition 3.6.1 (Ehrenborg–Readdy)**.** *Let P and Q be two posets. Then*

$$\Psi(P \times Q) = \sum_{(r,s,n) \in I} M_{r,s}(\Psi(P), \Psi(Q), n).$$

As with the diamond product operator, Section 6 of [13] and Section 10 of [19] give properties and recurrences for this operator. By reformulating Theorem 5.1 in [13], we have the following recursions for the Cartesian product of posets.

Proposition 3.6.2 (Ehrenborg–H. Fox)**.** *For any* **cd***-polynomials u and v, the Cartesian product satisfies the following:*

$$u \times (v \cdot \mathbf{c}) = (u \times v) \cdot \mathbf{c} + v \cdot \mathbf{d} \cdot u + \sum_{u} (u_{(1)} \times v) \cdot \mathbf{d} \cdot u_{(2)},$$
$$u \times (v \cdot \mathbf{d}) = (u \times v) \cdot \mathbf{d} + v \cdot \mathbf{d} \cdot \mathrm{Pyr}(u) + \sum_{u} (u_{(1)} \times v) \cdot \mathbf{d} \cdot \mathrm{Pyr}(u_{(2)}).$$

The recurrence is very similar to that of the diamond product; however, the degree of $M(u, v)$ is one higher than the degree of $u \diamond v$, and there is an additional term that does not occur within the diamond product recursion. Is there a similar lattice path interpretation for this product? Even a good interpretation for the easier case of $\mathbf{c}^m \times \mathbf{c}^n$ or the Cartesian product of **ab**-monomials is currently unknown.

Recently Carl Lee (personal communication) found an equation that relates the free join and Cartesian product of polytopes, which also involves the pyramid and prism operations. It is stated in the following lemma in terms of the analogous poset operations. Together with Ehrenborg, the author used a chain counting argument to show it is true for the **cd**-indices of the posets.

Lemma 3.6.3 (Lee)**.** *For two posets P and Q, we have*

$$\Psi(P \times Q) = \Psi(\mathrm{Pyr}(P) \diamond Q) + \Psi(P \diamond \mathrm{Pyr}(Q)) - \Psi(\mathrm{Prism}(P \diamond Q)).$$

If one could develop lattice path interpretations for the three simpler terms on the right-hand side of 3.6.3, it would allow us to have an interpretation for the Cartesian product $P \times Q$.

A different approach to studying how flag f-vectors change during poset operations such as the Cartesian product and diamond product is by using quasi-symmetric functions. The quasi-symmetric function of a poset is multiplicative with respect to

Cartesian product; see [12, Proposition 4.4]. Similarly, the type B quasi-symmetric function of a poset is multiplicative with respect to the diamond product; see [22, Theorem 13.3]. Could this approach be helpful in gaining a better understanding of these product operators?

Chapter 4 The Infinite cd-index and the Universal Coxeter Group

4.1 Introduction

The **cd**-index is a polynomial that encodes data on chains through an Eulerian poset P, which is finite, graded, and the Möbius function applied to each nontrivial interval $[x, y]$ is given by $\mu(x, y) = (-1)^{\rho(x,y)}$. This polynomial has been well-studied, but the focus of previous work has been with only finite posets. The **cd**-index can be extended to infinite posets which are locally finite, Eulerian, and have an upper bound for each pair of elements. A class of this type of poset, called level Eulerian posets, was introduced and studied by Ehrenborg, Hetyei, and Readdy [17].

Our focus will be on a second example of an infinite Eulerian poset, known as the universal Coxeter group. In general, Coxeter groups are generated by involutions, often with a relation between each pair of generators s and t in which $(st)^{m(s,t)} = 1$ for some positive integer $m(s, t)$. If each of these exponents $m(s, t)$ is finite, then the group itself is finite. However, when we remove these relations or make $m(s, t) = \infty$ for every pair, the group is known as a universal Coxeter group. This group is infinite, but there is a partial order that can be placed on it that results in each interval having the Eulerian property, and hence allows us to study its **cd**-index.

In Section 4.2 we extend the flag vectors and the **ab**-index for infinite, but locally finite posets. We continue in Section 4.3 by introducing and proving the existence of an extension of the **cd**-index to Eulerian infinite posets. Section 4.4 contains the definition of the k-vector of a poset, which will be helpful in calculating the coefficients of **cd**-monomials. We reintroduce in Section 4.5 the definitions of Coxeter groups and the universal Coxeter group, along with describing the strong Bruhat order that generates a partial order on the words in that group. Section 4.6 includes an in-depth study of universal Coxeter group under the Bruhat order, including the development of generating functions that contain enumerative data on the number of words for which a particular element is a subword. We compute the coefficients of some monomials in the **cd**-index of the universal Coxeter group in Section 4.7 and conclude the chapter by making a conjecture about the coefficients of a general monomial.

4.2 Infinite posets

Recall that a poset P is *locally finite* if for all $x, y \in P$ such that $x \leq y$, we have that the interval $[x, y]$ has a finite cardinality. We call a ranked, locally finite poset P *Eulerian* if the Möbius function is given by $\mu(x, y) = (-1)^{\rho(x,y)}$ for every nontrivial interval $[x, y]$. Lastly, we call a poset *confluent* if for all pairs $x, y \in P$ there exists exists an upper bound z, that is, an element z such that $x, y \leq z$.

Let P be an infinite, locally finite, and ranked poset with minimal element $\hat{0}$ and rank function $\rho : P \longrightarrow \mathbb{N}$ such that $\rho(\hat{0}) = 0$. Our interest is to study flag or chain enumeration in infinite posets. We first need the extra condition that for all positive integers k, the number of elements of rank k is finite. In other words, the preimage $\rho^{-1}(k)$ is finite. We denote the cardinality of $\rho^{-1}(k)$ by f_k. Note that this condition implies that the poset is locally finite. In what follows we restrict our attention to infinite posets that have a finite set of elements at each rank.

We extend the classical notion of the flag f-vector as follows. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a finite subset of the positive integers $\mathbb{P}$, define f_S to be the number of chains in P that visit the ranks in S, that is,

$$f_S = \{(x_1, x_2, \ldots, x_k) \ : \ \rho(x_i) = s_i\}.$$

Observe that the cardinality of f_S is indeed finite since it has the upper bound of $f_{s_1} \cdot f_{s_2} \cdots f_{s_k}$. Define the flag f-vector to be the infinite vector $(f_S)_S$ where S ranges over all finite subsets of the positive integers. Similarly define the flag h-vector entry h_S for S a finite subset of $\mathbb{P}$ by the inclusion-exclusion formula and its inverse

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} h_T.$$

The flag h-vector is the infinite vector $(h_S)_S$ where S ranges over all finite subsets of the positive integers.

Example 4.2.1. Let P be the infinite butterfly poset, which is the poset with $f_k = 2$ for all positive integers k and where each element covers all the elements in the rank below. This poset is also described as the strong Bruhat order of the infinite dihedral group. Here the flag f-vector and the flag h-vector are given by

$$f_S = 2^{|S|} \quad \text{and} \quad h_S = 1.$$

Our next goal is to extend the notion of the **ab**-index to infinite posets. For S a subset of the positive integers $\mathbb{P}$, we define the infinite **ab**-monomial $u_S = u_1 u_2 \cdots$, where $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}$ otherwise.

When we would like to describe a finite **ab**-monomial of degree n, we let $u_S^{(n)}$ denote the finite product $u_1 u_2 \cdots u_n$. In order to write infinite words, we let the power v^∞ denote the infinite product $v \cdot v \cdots$.

Let AB denote the set of all infinite monomials in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$, and let $AB^{=\infty}$ denote the set of all monomials with an infinite number of $\mathbf{b}$'s. Let $\mathbb{Z}[AB]$ and $\mathbb{Z}[AB^{=\infty}]$ denote all formal sums of monomials in AB, and respectively, in $AB^{=\infty}$. Finally, define the quotient $\mathcal{A} = \mathbb{Z}[AB]/\mathbb{Z}[AB^{=\infty}]$. In effect, we are setting each monomial with an infinite number of $\mathbf{b}$'s to be zero. A linear basis of $\mathcal{A}$ is given by all the monomials in AB with a finite number of $\mathbf{b}$'s.

Define the **ab**-index of an infinite poset P to be

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where the sum is over all finite subsets S of $\mathbb{P}$. Observe that $\Psi(P)$ lies within the space $\mathcal{A}$ since each set S being finite implies there is a finite number of $\mathbf{b}$'s in each monomial.

Example 4.2.2. The sum $\sum_S u_S$, where S ranges over all finite subsets of the positive integers $\mathbb{P}$, factors as follows:

$$\sum_S u_S = (\mathbf{a} + \mathbf{b})^\infty = \mathbf{c}^\infty.$$

Hence the infinite butterfly poset of Example 4.2.1 has the **ab**-index $\mathbf{c}^\infty$ since $h_S = 1$ for this poset.

This last example is our motivating example for defining the **cd**-index for infinite Eulerian posets.

4.3 Extending the cd-index to infinite posets

First we define the extension of polynomials in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Similar to the definitions of AB and $AB^{=\infty}$, we define the sets of infinite monomials CD and $CD^{=\infty}$ in the variables $\mathbf{c}$ and $\mathbf{d}$. Note that we restrict our attention to monomials with only a finite number of $\mathbf{d}$'s. Now define $\mathcal{C}$ to be the quotient $\mathbb{Z}[CD]/\mathbb{Z}[CD^{=\infty}]$. We embed $\mathcal{C}$ into $\mathcal{A}$ by the map $\mathbf{c} \longmapsto \mathbf{a} + \mathbf{b}$ and $\mathbf{d} \longmapsto \mathbf{ab} + \mathbf{ba}$.

The following theorem is an extension of the classical result of Bayer–Klapper [3].

Theorem 4.3.1. *The* **ab***-index of an Eulerian confluent infinite poset belongs to the image of the quotient space* $\mathcal{C} = \mathbb{Z}[CD]/\mathbb{Z}[CD^{=\infty}]$.

Proof. We begin by creating an infinite chain $\{w_o < w_1 < \cdots\}$ in the poset P. First we set $w_0 = \hat{0}$. For each rank $k \geq 1$, since P is confluent, we can find an element w_k that is greater than all the elements of rank k and the element w_{k-1}. Note that $\{w_0 < w_1 < w_2 < \cdots\}$ is indeed a chain. Now we want a particular chain $\{z_0 \prec z_1 \prec z_2 \prec \cdots\}$ such that $\rho(z_j) = j$. To create this chain, for each w_k of rank $\rho(w_k) = j$, we set $z_j := w_k$. We then fill in the remaining elements in the z-chain with elements at each rank between every pair w_k and w_{k+1}. This can be done since P is Eulerian and hence ranked.

Let S be a finite subset of positive integers such that $\max(S) = k$. Consider the sequence of flag f-vector entries

$$f_S([\hat{0}, z_j]) \qquad j = k+1, k+2, \ldots. \tag{4.3.1}$$

This sequence will stabilize and become constant since the interval $[\hat{0}, z_j]$ will contain all the elements of rank k once we reach $z_j = w_k$. Similarly, the sequence of flag h-vectors

$$h_S([\hat{0}, z_j]) \qquad j = k+1, k+2, \ldots \tag{4.3.2}$$

will stabilize since it is a linear combination of sequences of the form (4.3.1).

Now consider a **cd**-monomial w of degree k that ends with the variable **d**. Yet again, we consider a sequence as j tends to infinity, but this time it is a sequence of **cd**-coefficients

$$[w \cdot \mathbf{c}^{j-1-k}]\Psi([\hat{0}, z_j]) \qquad j = k+1, k+2, \ldots. \tag{4.3.3}$$

Recall that the coefficient of the **cd**-monomial $w \cdot \mathbf{c}^{j-1-k}$ is a linear combination of flag h-vectors h_S entries where $\max(S) \leq k-1$. (Note that the maximum of the empty set is considered to be $-\infty$.) Hence the sequence in (4.3.3) will stabilize.

Define an infinite **cd**-polynomial by setting

$$\Phi = \mathbf{c}^\infty + \sum_w \left(\lim_{j \longrightarrow \infty} [w \cdot \mathbf{c}^{j-1-\deg(w)}]\Psi([\hat{0}, z_j]) \right) \cdot w \cdot \mathbf{c}^\infty, \tag{4.3.4}$$

where the sum is over all finite **cd**-monomials ending with the variable **d**. We claim that the **cd**-polynomial Φ when expanded into an infinite **ab**-polynomial will be the **ab**-index of the infinite poset P.

If S is the empty set then $u_S = \mathbf{a}^\infty$. Note that the only contribution to the term $\mathbf{a}^\infty$ is 1 from the monomial $\mathbf{c}^\infty$, which is equal to $h_\emptyset = 1$. Next consider the case where we assume S is non-empty and $\max(S) = k$. Now the coefficient of the infinite monomial u_S is given by

$$[u_S]\Phi = 1 + [u_S] \sum_w \left(\lim_{j \longrightarrow \infty} [w \cdot \mathbf{c}^{j-1-\deg(w)}]\Psi([\hat{0}, z_j]) \right) \cdot w \cdot \mathbf{c}^\infty. \tag{4.3.5}$$

Since w ends with a **d**, when expanding $w \cdot \mathbf{c}^\infty$ there will be either a **b** in position $\deg(w) - 1$ or in position $\deg(w)$. Hence, if $\deg(w) > k$ there will be no contribution to the term u_S. Thus the sum in (4.3.5) reduces to a finite sum, allowing us to change the order between the sum and the limit, as follows.

$$\begin{aligned}
[u_S]\Phi &= 1 + [u_S] \sum_{w : \deg(w) \leq k} \left(\lim_{j \longrightarrow \infty} [w \cdot \mathbf{c}^{j-1-\deg(w)}]\Psi([\hat{0}, z_j]) \right) \cdot w \cdot \mathbf{c}^\infty z \\
&= \lim_{j \longrightarrow \infty} \left([u_\emptyset]\mathbf{c}^\infty + [u_S] \sum_{w : \deg(w) \leq k} [w \cdot \mathbf{c}^{j-1-\deg(w)}]\Psi([\hat{0}, z_j]) \cdot w \cdot \mathbf{c}^\infty \right) \\
&= \lim_{j \longrightarrow \infty} [u_S] \sum_{v : \deg(v) = k} [v \cdot \mathbf{c}^{j-1-k}]\Psi([\hat{0}, z_j]) \cdot v \cdot \mathbf{c}^\infty,
\end{aligned} \tag{4.3.6}$$

where the last sum is over all **cd**-monomials v of degree k. The monomial $v = \mathbf{c}^k$ comes from $[u_\emptyset]\mathbf{c}^\infty$, whereas the other monomials are formed as $v = w \cdot \mathbf{c}^{k-\deg(w)}$. We can now restrict our attention to the first $j-1$ letters in the infinite monomials. Recall that $u_S^{(m)}$ is an **ab**-monomial of degree m. Then we have

$$[u_S]\Phi = \lim_{j \longrightarrow \infty} [u_S^{(j-1)}] \sum_{v : \deg(v) = k} [v \cdot \mathbf{c}^{j-1-k}]\Psi([\hat{0}, z_j]) \cdot v \cdot \mathbf{c}^{j-1-k}. \tag{4.3.7}$$

Since the **ab**-monomial $u_S^{(j-1)}$ factors as $u_S^k \cdot \mathbf{a}^{j-1-k}$, we can extend the sum to all **cd**-monomials y of degree $j-1$, not only those of the form $v \cdot \mathbf{c}^{j-1-k}$. Hence the sum expands as follows, which can be simplified

$$\begin{aligned}[u_S]\Phi &= \lim_{j\longrightarrow\infty} [u_S^{(j-1)}] \sum_{y:\deg(v)=j-1} [y]\Psi([\hat{0}, z_j]) \cdot y \\ &= \lim_{j\longrightarrow\infty} [u_S^{(j-1)}]\Psi([\hat{0}, z_j]) \\ &= \lim_{j\longrightarrow\infty} h_S([\hat{0}, z_j]). \end{aligned} \tag{4.3.8}$$

But this limit stabilizes to be $h_S(P)$, thus proving that the expansion of Φ is indeed the **ab**-index of the infinite poset P. Therefore, we have that the **cd**-index exists. □

4.4 Explicit formulas for coefficients

In order to calculate the **cd**-index for an infinite Eulerian poset, we will define a third vector associated to P, which we call the k-vector. Before doing so, we first need some additional definitions relating sets and **cd**-monomials in $\mathcal{C}$. First, we call a set S *sparse* if $\{i, i+1\} \not\subseteq S$ for all i. The idea of using the sparse k-vector is due to Billera, Ehrenborg, and Readdy [5].

Definition 4.4.1. *For sparse subsets $S = \{s_1 < s_2 < \cdots < s_m\}$ and $T = \{t_1 < t_2 < \cdots < t_m\}$ of $\mathbb{P}$, we define the following four notions:*

*(a) Define $w(T)$ as the **cd**-monomial $w(T) = \mathbf{c}^{t_1-1}\mathbf{d}\mathbf{c}^{t_2-t_1-2}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{t_m-t_{m-1}-2}\mathbf{c}^{\infty}$.*

*(b) For a **cd**-monomial $w = \mathbf{c}^{i_1}\mathbf{d}\mathbf{c}^{i_2}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{i_m}\mathbf{d}\mathbf{c}^{\infty}$, we say w covers S, denoted by $w \sim S$, if*

$$S \subseteq \{j_1, j_1+1, j_2, j_2+1, \ldots, j_m, j_m+1\}$$

where $j_1 = i_1 + 1$ and $j_{k+1} = j_k + i_{k+1} + 2$ for $k = 1, \ldots, m-1$.

(c) Define the relation $S \preceq T$ if $1 \leq s_1 \leq t_1$ and $t_{k-1} + 1 < s_k \leq t_k$ for all $k = 2, \ldots, m$. Note that this relation is not a partial order as it is not transitive.

(d) Let ΣT denote the sum of the elements of T, that is, $\sum_{k=1}^{m} t_k$.

Definition 4.4.2. *For an infinite poset P and a sparse subset S of size m, define the k-vector of P at S by*

$$k_S = \sum_{w\sim S} [w]\Psi(P),$$

*where we sum over all **cd**-monomials that cover S.*

Note that not only can we define a **cd**-monomial $w(T)$ for each sparse set T, but we can also write any **cd**-monomial w as $w(T)$ for some sparse set T. Hence, we could alternatively define the k-vector as

$$k_S = \sum_{w(T)\sim S} [w(T)]\Psi(P).$$

Also, using the definitions above, observe that if $w(T) \sim S$, then for each $k = 1, \ldots, m$, we get $t_k = s_k$ or $s_k - 1$.

Theorem 4.4.3. *The coefficients of the* **cd***-monomials with* m **d***'s can be expressed by the following alternating sum of* k*-vectors* k_U *where* $|U| = m$

$$[w(T)]\Psi(P) = \sum_{U \preceq T} (-1)^{\Sigma T - \Sigma U} \cdot k_U$$

Proof. It suffices to show

$$k_S = \sum_{w(T) \sim S} [w(T)]\Psi(P) = \sum_{w(T) \sim S} \sum_{U \preceq T} (-1)^{\Sigma T - \Sigma U} \cdot k_U.$$

If $U = S$, then S is the largest (in terms of $\preceq$) sparse set of size m for which $w(S)$ covers S. Hence $T = S$, and we get that the coefficient of k_S on the right-hand side is 1.

If $S = \{1, 3, 5, \ldots, 2m - 1\}$, S is the only set such that $w(S) \sim S$, and S is also the only set in which $S \preceq S$. Then in this case, the only term in the sum is k_S, completing this case.

If S is not the minimal sparse set, we need to show that the coefficient for k_U is 0 for all $U \neq S$. In this situation, $u_i < s_i$ for some i. We wish to consider the smallest such i. For each T in which $w(T) \sim S$ and $U \preceq T$, if $u_{i+1} - s_i > 1$, then T can still be a sparse set with the choice of $t_i = s_i$ or $s_i - 1$. This choice gives a 1 and -1 as the coefficient for k_U for each T since the ΣT values would differ by 1. These two terms will cancel then, leaving a coefficient of 0 for k_U.

If $u_{i+1} - s_i \leq 1$, then if some j exists in which $u_{j+1} - s_j > 1$, we get a choice of t_j as either s_j or $s_j - 1$, once again giving the coefficient for k_U as 0.

If no such j value exists, then $u_m \neq s_m$. Otherwise, $u_m - s_{m-1} > 1$ since S is sparse. In this final case, there is a choice for t_m as s_m or $s_m - 1$, causing the coefficient on k_U to be 0. □

There are the following relations connecting the k-vector and the f-vector, as stated by Billera, Ehrenborg, and Readdy as equations (6.1) and (6.2) in [5]

$$f_S = \sum_{U \subseteq S} 2^{|S-U|} \cdot k_U, \tag{4.4.1}$$

$$k_S = \sum_{U \subseteq S} (-2)^{|S-U|} \cdot f_U, \tag{4.4.2}$$

where S is a sparse subset of $\mathbb{P}$.

4.5 Coxeter groups and the strong Bruhat order

Recall that a Coxeter system (S, W) is a pair with S a set of generators, W the group generated by S, and having the following relations:

- All generators have order 2; that is, $s^2 = 1$ for all $s \in S$,
- For every pair of generators s and t, there exists a nonnegative integer $m(s,t)$ with $2 \leq m(s,t) \leq \infty$ such that $(st)^{m(s,t)} = 1$.

Note that if $m(s,t) = \infty$, this means that the element st has infinite order, or in other words, there is no relation between s and t.

Elements in a Coxeter group W are written as words with the letters being generators from S. The length of a word $w \in W$, denoted by $\ell(w)$, is the smallest k such that w can be written as a product $s_1 \cdot s_2 \cdots s_k$. A word $s_1 s_2 \cdots s_k$ is called a reduced expression if $\ell(s_1 s_2 \cdots s_k) = k$.

The strong Bruhat order, also known as simply the Bruhat order, is an order relation defined on a Coxeter group as follows. Let v and w be two elements in the Coxeter group. Define $v \leq w$ if there is a reduced word $s_1 s_2 \cdots s_k$ for w such that v is the product of a subword of this expression; that is, one can write $v = s_{i_1} s_{i_2} \cdots s_{i_j}$ where $1 \leq i_1 < i_2 < \cdots < i_j \leq k$. Observe that the identity element 1, or the empty product, is the minimal element $\hat{0}$ of the Bruhat order, and that the rank function is provided by the length of each word.

We will investigate the class of Coxeter groups known as the universal Coxeter group in which $m(s,t) = \infty$ for all $s \neq t$. We assume that there are r generators, i.e., $|S| = r$, and we will denote this group by $\mathcal{U}_r$. Since there are no relations on the generators other than the fact that each is an involution, we only consider reduced, or valid, words which are of the form $v = s_1 s_2 \cdots s_k$ where each $s_i \in S$ and every pair of consecutive generators is distinct, that is, $s_i \neq s_{i+1}$ for $1 \leq i \leq k-1$.

The Bruhat order of a Coxeter group is an Eulerian poset, as shown by Verma [48]. This implies that the **cd**-index can be calculated for the Bruhat order of finite Coxeter groups, which has been previously studied by Reading [40]. A more general nonhomogeneous polynomial called the complete **cd**-index has also been computed by Blanco [8] for certain Coxeter groups, including the dihedral Coxeter group.

4.6 The Bruhat order of $\mathcal{U}_r$

Consider the universal Coxeter group $\mathcal{U}_r$ with generating set S of cardinality r. To examine the flag f-vector of the Bruhat order of this group, we define $\eta_n(v)$ to be the number of words w of length n having the word v as a subword.

Example 4.6.1. We have that $\eta_n(1)$ is given by $r \cdot (r-1)^{n-1}$ for $n \geq 1$ and 1 for $n = 0$. This is true since we are simply enumerating the number of words of length n with no consecutive repeated letters.

Lemma 4.6.2. *Assume that we have a universal Coxeter group with at least three generators, i.e., $r \geq 3$. Given words $v = s_1 \cdots s_{i-1} \cdot t \cdot s_{i+1} \cdots s_k$ and $w = s_1 \cdots s_{i-1} \cdot$*

$u \cdot s_{i+1} \cdots s_k$ in $\mathcal{U}_r$ with $t \neq u$, we have the equality $\eta_n(v) = \eta_n(w)$ for all $n > k$; that is, the number of words of length n containing v equals the number of words of length n containing w.

Proof. Let z be a word of length $n > k$ that contains v.

Assume that $i \neq 1, n$. Find the first s_{i-1} in z such that $z = \alpha \cdot s_{i-1} \cdot \beta$ where α contains $s_1 \cdots s_{i-2}$. Within β, find the first s_{i+1} after the first t such that $\beta = \gamma \cdot s_{i+1} \cdot \zeta$ with ζ containing $s_{i+2} \cdots s_k$. Within γ, which contains at least one t, uniformly exchange the t's and u's. We call the resulting word $\bar{\gamma}$.

Then define $\bar{z} := \alpha \cdot s_{i-1} \cdot \bar{\gamma} \cdot s_{i+1} \cdot \zeta$. This is a valid word because the only way two letters could repeat is if the first or last letters of $\bar{\gamma}$ are s_{i-1} or s_{i+1}, respectively. However, this would mean that the first or last letters of γ are also s_{i-1} or s_{i+1}. However, since $s_{i-1}, s_{i+1} \neq t$ or u, this implies that $z = \alpha \cdot s_{i-1} \cdot \gamma \cdot s_{i+1} \cdot \zeta$ would also not be a valid word, which is a contradiction. Therefore, $\bar{z}$ is a word of length n containing w since $\bar{\gamma}$ now contains at least one u.

Assume now that $i = 1$; that is, the first letters of v and w are t and u, respectively. Find the first s_2 in z such that $z = \alpha \cdot s_2 \cdot \beta$ where β contains $s_3 \cdots s_k$. Then we can define $\bar{z} := \bar{\alpha} \cdot s_2 \cdot \beta$ where we create $\bar{\alpha}$ by uniformly exchanging the t's and u's. As in case 1, since $s_2 \neq t$ or u, the word $\bar{z}$ is still valid, has length n, and contains w.

The last case is $i = n$, which means that t is the last letter of the word v. The argument is symmetric to the second case, and hence is omitted.

Applying the analogous process to the word $\bar{z}$ containing w will reverse the swaps that were made to give back the word z which contains v. This gives a bijection between words of length n containing v and words of length n containing w. Thus, we have $\eta_n(v) = \eta_n(w)$. □

Lemma 4.6.3. *Assume that we have a universal Coxeter group with at least three generators, that is, $r \geq 3$. Let v and w be words of length k. Starting with v, we can change one letter at a time to transform v into w with every intermediate step being a valid word.*

Proof. We will use induction on the length k. The induction base is $k = 1$, which is straightforward.

Assume now that $k \geq 2$. For our induction hypothesis, we assume any word of length $k-1$ can be transformed one letter at a time to any other word of length $k-1$.

Let $v = x_1 \cdot x_2 \cdots x_k$ and $w = y_1 \cdot y_2 \cdots y_k$. Using our induction hypothesis, the word $x_2 \cdots x_k$ can be transformed into $y_2 \cdots y_k$. To transform v to w, we perform the same steps starting with v instead of $x_2 \cdots x_k$ with some additional steps inserted along the way. For each intermediate word $z = s \cdot t \cdot z_3 \cdots z_k$ in which the next step is to change the second letter from t to s, we must first insert a step to change the first letter from s to u, where $u \neq s, t$. Note that such a u exists as long as $r \geq 3$. This will make every intermediate word a valid word. After reaching $s \cdot y_2 \cdots y_k$, we must also add the final step of swapping the first letter s to y_1 if $s \neq y_1$. This results in the word w. □

Proposition 4.6.4. *Given any two words v and w of length k in $\mathcal{U}_r$, $\eta_n(v) = \eta_n(w)$ for any length $n > k$.*

Proof. Observe that the result is direct for $r = 2$. For $r \geq 3$ it follows by combining Lemmas 4.6.2 and 4.6.3. □

In order to determine $\eta_n(v)$, it is more practical to consider the number of words of length n not having the word v as a subword. We will also describe these as words w that avoid v. To do this, define the generating function $F(v)$ by

$$F(v) \quad = \quad \sum_{v \not\leq w} x^{\ell(w)}.$$

Therefore, $[x^n]F(v)$, which denotes the coefficient of x^n in $F(v)$, is the number of words of length n that do not have v as a subword. Hence $\eta_n(v)$ is given by

$$\eta_n(v) = r \cdot (r-1)^{n-1} - [x^n]F(v). \tag{4.6.1}$$

Example 4.6.5. For a generator s of the group $\mathcal{U}_r$, we have

$$F(s) = 1 + (r-1) \cdot x \cdot \frac{1}{1-(r-2)x} = \frac{1+x}{1-(r-2)x}.$$

Either we have the empty word 1, or we are selecting a word using the alphabet $S - \{s\}$, which consists of $r - 1$ letters. Then we use the same argument as in Example 4.6.1.

Directly from Proposition 4.6.4 and equation (4.6.1), we have the next result.

Corollary 4.6.6. *For any words v and w of the same length, we have $F(v) = F(w)$.*

In order to find an explicit expression for the generating function $F(v)$, we introduce the generating function $F(v, s)$ which restricts the sum in $F(v)$ to words beginning with the generator s, that is,

$$F(v, s) = \sum_{w} x^{\ell(w)},$$

where the sum is over all words w such that $v \not\leq w$ and w begins with the letter s. Note that this definition can also be stated as

$$F(v, s) = \sum_{v \not\leq sw} x^{\ell(sw)} = x \cdot \sum_{v \not\leq sw} x^{\ell(w)}, \tag{4.6.2}$$

where w ranges over all words not beginning with s.

Example 4.6.7. For two different generators s and t of the group $\mathcal{U}_r$ we have

$$F(s, t) = x \cdot \frac{1}{1-(r-2)x},$$

since the first letter is t, and because we have $r - 2$ choices for each of the remaining letters.

Now we introduce recurrence relations that are satisfied by the generating functions $F(v)$ or $F(v,s)$. These will be essential in developing an explicit expression for $F(v)$, and hence $\eta_n(v)$.

Proposition 4.6.8. *The generating functions $F(v)$ and $F(v,s)$ satisfy the recurrences*

$$\begin{aligned} F(v) &= 1+\sum_{s\in S} F(v,s), && (4.6.3)\\ F(sv,s) &= F(v,s), && (4.6.4)\\ F(sv,t) &= x\cdot\left(1+\sum_{q\in S-\{t\}} F(sv,q)\right). && (4.6.5)\end{aligned}$$

Proof. The recurrence (4.6.3) is straightforward to verify since a word is either empty or begins with one of the generators in S. The second recurrence (4.6.4) follows since a word starting with s, say sw, avoids a valid word sv if and only if sw avoids v. The final recurrence (4.6.5) is due to the fact that a word tw avoids sv if and only if w is the empty word 1 or w both avoids sv and begins with a letter different from t. The factor of x comes from concatenation of t to the beginning of w. See equation (4.6.2). □

The following equality will be useful in the proof of an upcoming improvement to the recursion formula for $F(sv,t)$.

Lemma 4.6.9. *The following identity holds between the given two $r-1$ by $r-1$ determinants:*

$$\det\begin{pmatrix} 1 & -x & \cdots & -x\\ -x & 1 & \cdots & -x\\ \vdots & \vdots & \ddots & \vdots\\ -x & -x & \cdots & 1\end{pmatrix} = (1-(r-2)\cdot x)\cdot\det\begin{pmatrix} 1 & -x & \cdots & -x\\ 1 & 1 & \cdots & -x\\ \vdots & \vdots & \ddots & \vdots\\ 1 & -x & \cdots & 1\end{pmatrix}.$$

Proof. Use column reduction on the first determinant by adding columns 2 through $r-1$ to the first column. Then every entry in the first column is now $1-(r-2)\cdot x$, which can be factored out. □

Proposition 4.6.10. *For two generators s and t with $s\neq t$, we have the recurrence*

$$F(sv,t)=\frac{x\cdot(1+F(v,s))}{1-(r-2)x}.$$

Proof. For each generator $t\in S-\{s\}$, we have from the recurrence (4.6.5) that

$$F(sv,t)=x\cdot\left(1+\sum_{q\in S-\{t\}} F(sv,q)\right).$$

giving us a system of $r-1$ equations. By reordering terms, we have for each $t \in S-\{s\}$

$$F(sv,t) - x \cdot \sum_{q \in S-\{s,t\}} F(sv,q) = x \cdot (1 + F(sv,s)).$$

We can write this system in matrix form as $Ay = b$ where the rows and columns are indexed by the set $S-\{s\}$. The matrix A is the first matrix occurring in Lemma 4.6.9. The column vector y has entries $F(sv,q)$ for $q \in S-\{s\}$. Finally, the column vector b has $x \cdot (1 + F(sv,s))$ in every entry.

By Cramer's Rule,

$$F(sv,t) = \frac{\det(A_t)}{\det(A)},$$

where A_t is the matrix A where we replace the column corresponding to the generator t with the column vector b. Without loss of generality we may assume that t corresponds to the first column. Since each entry of b is the same, we can factor this entry out from the first column of the determinant of A_t to obtain $\det(A_t) = x \cdot (1 + F(sv,s)) \cdot \det(B)$, where B is second matrix occurring in Lemma 4.6.9. Hence, applying Lemma 4.6.9 gives us

$$F(sv,t) = \frac{x \cdot (1 + F(sv,s))}{1 - (r-2)x}.$$

To complete the proof, use equation (4.6.4). □

Corollary 4.6.11. *For any word v not beginning with s, we have the recurrence*

$$F(sv) = \frac{(x+1) \cdot (1 + F(v,s))}{1 - (r-2)x}.$$

Proof. We have

$$\begin{aligned} F(sv) &= 1 + F(sv,s) + \sum_{t \neq s} F(sv,t) \\ &= 1 + F(v,s) + \sum_{t \neq s} \frac{x \cdot (1 + F(v,s))}{1 - (r-2)x} \\ &= 1 + F(v,s) + (r-1) \cdot \frac{x \cdot (1 + F(v,s))}{1 - (r-2)x}, \end{aligned}$$

where the first step is recurrence (4.6.3), and second step is recurrence (4.6.4) and Proposition 4.6.10. The result follows by simplifying. □

In order to use this recurrence to find a simple closed form for the generating function $F(v)$, we need to introduce a change in variables. We substitute the variable z which is given by

$$z = \frac{x}{1 - (r-2)x}.$$

Proposition 4.6.12. *Let v be a word of length k such that v starts with s, and assume t is a generator different from s. Then the following equations hold for the generating functions:*

$$F(v) = (1 + (r-1) \cdot z) \cdot \frac{1 - z^k}{1 - z}, \tag{4.6.6}$$

$$F(v, s) = z \cdot \frac{1 - z^{k-1}}{1 - z}, \tag{4.6.7}$$

$$F(v, t) = z \cdot \frac{1 - z^k}{1 - z}. \tag{4.6.8}$$

Proof. We prove these three identities by induction on the length k. In the base case of $k = 1$, Equations (4.6.6) and (4.6.8) are Examples 4.6.5 and 4.6.7, respectively. Finally, equation (4.6.7) states that $0 = 0$, completing the induction basis.

Next we do the induction step. Assume that the three identities hold for words of length less than k. Write the word as $v = su$, where u has length $k - 1$ and begins with a letter different from s. From the final step of the proof of Corollary 4.6.11,

$$\begin{aligned} F(su) &= 1 + F(u, s) + (r-1) \cdot \frac{x \cdot (1 + F(u, s))}{1 - (r-2)x} \\ &= \left(1 + (r-1) \cdot \frac{x}{1 - (r-2)x}\right) \cdot (1 + F(u, s)) \\ &= (1 + (r-1) \cdot z) \cdot \frac{1 - z^k}{1 - z}, \end{aligned}$$

where in the last step we used $1 + F(u, s) = (1 - z^k)/(1 - z)$ by applying (4.6.8) to the length $k - 1$ word u.

Next we have

$$F(su, s) = F(u, s) = z \cdot \frac{1 - z^{k-1}}{1 - z},$$

by (4.6.8) applied to the word u.

Lastly,

$$\begin{aligned} F(su, t) &= \frac{x \cdot (1 + F(u, s))}{1 - (r-2)x} \\ &= z \cdot (1 + F(u, s)) \\ &= z \cdot \frac{1 - z^k}{1 - z}, \end{aligned}$$

completing the induction step. □

Observe that when we expand (4.6.6) as a formal power series, it simplifies to the following polynomial in the variable z:

$$F(v) = 1 + r \cdot z + r \cdot z^2 + \cdots + r \cdot z^{k-1} + (r-1) \cdot z^k. \tag{4.6.9}$$

Using this closed form expression for $F(v)$, we can now determine a formula for the coefficients of $F(v)$. We begin by introducing the sequence

$$a_i = \frac{r(r-2)^i - (-1)^i}{r-1},$$

which is also given by the second order recursion $a_i = (r-3)\cdot a_{i-1} + (r-2)\cdot a_{i-2}$ with initial conditions $a_1 = r-1$ and $a_0 = 1$. Also observe that $a_i + a_{i+1} = r(r-2)^i$.

Theorem 4.6.13. *For any word v of length k, the number of words of length $n > k$ not containing v as a subword is given by*

$$[x^n]F(v) = (r-2)^{n-k}\cdot\sum_{i=1}^{k} a_i \cdot \binom{n}{k-i}.$$

Proof. We use the explicit expression (4.6.9) for $F(v)$, replacing z with $x/(1-(r-2)x)$ and expanding each term as

$$z^k = \left(\frac{x}{1-(r-2)x}\right)^k = \sum_{i\geq k}\binom{i-1}{k-1}\cdot(r-2)^{i-k}\cdot x^i.$$

We get an x^n term from the expansion of each term $rz, rz^2, \ldots, rz^{k-1}$, and $(r-1)z^k$. Collecting these terms gives that

$$\begin{aligned}[x^n]F(v) \;=\; & r\left[\binom{n-1}{0}(r-2)^{n-1} + \binom{n-1}{1}(r-2)^{n-2} + \cdots\right. \\ & \left. + \binom{n-1}{k-2}(r-2)^{n-k+1}\right] + (r-1)\binom{n-1}{k-1}(r-2)^{n-k}.\end{aligned}$$

We now show that the proposed coefficient is equivalent to this coefficient. By applying Pascal's identity, we have

$$\begin{aligned}(r-2)^{n-k}\cdot\sum_{i=1}^{k} a_i\cdot\binom{n}{k-i} &= (r-2)^{n-k}\cdot\sum_{i=1}^{k} a_i\cdot\left(\binom{n-1}{k-i} + \binom{n-1}{k-i-1}\right) \\ &= (r-2)^{n-k}\cdot\left(a_1\cdot\binom{n-1}{k-1} + \sum_{i=1}^{k-1}(a_i + a_{i+1})\binom{n-1}{k-i-1}\right)\end{aligned}$$

Using $a_i + a_{i+1} = r\cdot(r-2)^i$ and $a_1 = r-1$, we get

$$\begin{aligned}&= (r-2)^{n-k}\left((r-1)\binom{n-1}{k-1} + \sum_{i=1}^{k-1} r(r-2)^i\binom{n-1}{k-i-1}\right) \\ &= (r-1)\cdot\binom{n-1}{k-1}\cdot(r-2)^{n-k} + r\cdot\sum_{i=1}^{k-1}(r-2)^{n-k+i}\cdot\binom{n-1}{k-i-1}\end{aligned}$$

$$= (r-1) \cdot \binom{n-1}{k-1} \cdot (r-2)^{n-k} + r \cdot \left[\binom{n-1}{k-2} \cdot (r-2)^{n-k+1} + \binom{n-1}{k-3} \cdot (r-2)^{n-k+2} + \cdots + \binom{n-1}{0} \cdot (r-2)^{n-1} \right]$$

Reordering the terms gives us the same formula as calculated by expanding the closed formula for $F(v)$. □

4.7 The cd-index of $\mathcal{U}_r$

We wish to calculate the coefficients of the **cd**-index, but we first must consider if $\mathcal{U}_r$ has the necessary properties for Theorem 4.3.1 to imply the existence of this polynomial. First, $\mathcal{U}_r$ is locally finite since the number of elements of rank n was shown to be $r \cdot (r-1)^{n-1}$ in Example 4.6.1. It is also has the Eulerian property with regards to its Möbius function, as shown by Verma [48]. Finally, we need to show this poset is confluent by providing an upper bound for any two words v and w in $\mathcal{U}_r$. If the last letter of v and the first letter of w differ, then the word $v \cdot w$ gives an upper bound. If these two letters are each s, then the word $v \cdot t \cdot w$ for some generator $t \neq s$ is an upper bound. Thus, $\mathcal{U}_r$ is confluent, and hence the **cd**-index exists by Theorem 4.3.1.

In order to compute the **cd**-index of the Bruhat order of the universal Coxeter group $\mathcal{U}_r$, we will use Theorem 4.4.3 and equation (4.4.2), but we first need to compute the flag f-vector. We take advantage of the fact that the number of elements of rank n above a particular element only depends on the length of that element, as shown by $\eta_n(v) = \eta_n(w)$ from Lemma 4.6.4. To count chains in the Bruhat order of $\mathcal{U}_r$, we successively multiply formula (4.6.1) with the coefficients of $F(v)$ being supplied by Theorem 4.6.13. This can be seen in the following equation. If $S = \{s_1 < s_2 < \cdots < s_m\}$, then

$$f_S = r \cdot (r-1)^{s_1 - 1} \prod_{j=2}^{m} \left(r \cdot (r-1)^{s_j - 1} - (r-2)^{s_j - s_{j-1}} \left(\sum_{i=1}^{s_{j-1}} a_i \cdot \binom{s_j}{s_{j-1} - i} \right) \right). \tag{4.7.1}$$

We next use the previous equation to calculate the coefficients of the **cd**-index $\Psi(\mathcal{U}_r)$ for certain monomials with only 1 or 2 **d**'s.

Example 4.7.1. *(a) For the* **cd***-monomial* $\mathbf{c}^n\mathbf{d}\mathbf{c}^\infty$ *in* $\Psi(\mathcal{U}_r)$*, we have* $\mathbf{c}^n\mathbf{d}\mathbf{c}^\infty = w(T)$ *in which* $T = \{n+1\}$*; thus,*

$$\begin{aligned} [\mathbf{c}^n\mathbf{d}\mathbf{c}^\infty] &= \sum_{i=1}^{n+1} (-1)^{n+1-i} \cdot k_i \\ &= \sum_{i=1}^{n+1} (-1)^{n+1-i} \cdot (f_i - 2) \\ &= \left(\sum_{i=1}^{n+1} (-1)^{n+1-i} \cdot f_i \right) - 1 + (-1)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{n+1}(-1)^{n+1-i}\cdot r\cdot(r-1)^{i-1}\right) - 1 + (-1)^{n+1}\\
&= r\cdot\frac{(r-1)^{n+1}-(-1)^{n+1}}{(r-1)-(-1)} - 1 + (-1)^{n+1}\\
&= (r-1)^{n+1} - 1
\end{aligned}
$$

(b) For the **cd***-monomial* $\mathbf{dc}^n\mathbf{dc}^\infty$ *in* $\Psi(\mathcal{U}_r)$*, we have* $\mathbf{dc}^n\mathbf{dc}^\infty = w(T)$ *in which* $T = \{1, n+3\}$*; therefore,*

$$
\begin{aligned}
[\mathbf{dc}^n\mathbf{dc}^\infty] &= \sum_{i=3}^{n+3}(-1)^{n+3-i}\cdot k_{1,i}\\
&= \sum_{i=3}^{n+3}(-1)^{n+3-i}\cdot(f_{1,i}-2f_i-2f_1+4)\\
&= \sum_{i=3}^{n+3}(-1)^{n+3-i}\cdot\Bigg[r\cdot\left(r\cdot(r-1)^{i-1}-(r-2)^{i-1}\cdot a_1\binom{i}{0}\right)\\
&\quad - 2r\cdot(r-1)^{i-1} - 2r + 4\Bigg]\\
&= \left(\sum_{i=3}^{n+3}(-1)^{n+3-i}\cdot(r^2-2r)\cdot(r-1)^{i-1}\right)\\
&\quad - \left(\sum_{i=3}^{n+3}(-1)^{n+3-i}\cdot r\cdot(r-1)\cdot(r-2)^{i-1}\right) + \mathcal{O}(1)\\
&= r\cdot(r-2)\cdot(r-1)^2\left(\frac{(r-1)^{n+1}-(-1)^{n+1}}{(r-1)-(-1)}\right)\\
&\quad - r\cdot(r-1)\cdot(r-2)^2\left(\frac{(r-2)^{n+1}-(-1)^{n+1}}{(r-2)-(-1)}\right) + \mathcal{O}(1)\\
&= (r-2)\cdot(r-1)^{n+3} - (-1)^{n+1}\cdot(r-2)\cdot(r-1)^2 - r\cdot(r-2)^{n+3}\\
&\quad + (-1)^{n+1}\cdot r\cdot(r-2)^2 + \mathcal{O}(1)\\
&= (r-2)\cdot(r-1)^{n+3} - r\cdot(r-2)^{n+3} + \mathcal{O}(1)
\end{aligned}
$$

where the Big O notation is used with regards to r *being a fixed constant and* n *varying.*

(c) *For the* **cd**-*monomial* $\mathbf{cdc}^n\mathbf{dc}^\infty$*, in* $\Psi(\mathcal{U}_r)$*, we have* $\mathbf{cdc}^n\mathbf{dc}^\infty = w(T)$ *in which* $T = \{2, n+4\}$*; therefore,*

$$
\begin{aligned}
[\mathbf{cdc}^n\mathbf{dc}^\infty] &= \sum_{i=1}^{2}(-1)^{2-i} \cdot \sum_{j=4}^{n+4}(-1)^{n+4-j}k_{i,j} \\
&= \sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot k_{2,j} - \sum_{j=4}^{n+4}(-1)^{n+4-j}k_{1,j} \\
&= \sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot (f_{2,j} - 2f_j - 2f_2 + 4) \\
&\quad - \sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot (f_{1,j} - 2f_j - 2f_1 + 4) \\
&= \sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot \Bigg[r \cdot (r-1) \cdot \Big(r \cdot (r-1)^{j-1} - (r-2)^{j-2} \\
&\quad \cdot \left(a_1 \cdot \binom{j}{1} + a_2 \cdot \binom{j}{0}\right)\Big) - 2r \cdot (r-1) + 4\Bigg] \\
&\quad - \sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot \left[r \cdot \left(r \cdot (r-1)^{j-1} - (r-2)^{j-1} \cdot a_1 \cdot \binom{j}{0}\right) - 2r + 4\right] \\
&= \left(\sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot (r^2 \cdot (r-2)) \cdot (r-1)^{j-1}\right) \\
&\quad - \Bigg(\sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot r \cdot (r-1) \cdot (r-2)^{j-2} \\
&\quad \cdot \Big((r-1) \cdot j + (r-3) \cdot (r-1) + (r-2)\Big)\Bigg) + \mathcal{O}(1) \\
&\quad + \left(\sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot r \cdot (r-1) \cdot (r-2)^{j-1}\right) + \mathcal{O}(1) \\
&= r \cdot (r-2) \cdot (r-1)^{n+4} - (-1)^{n+1} r \cdot (r-2) \cdot (r-1)^3 \\
&\quad - \left(\sum_{j=4}^{n+4}(-1)^{n+4-j} \cdot r \cdot (r-1)^2 \cdot (r-2)^{j-2} \cdot j\right) \\
&\quad - ((r-3) \cdot (r-1) + (r-2)) \cdot r \cdot (r-2)^{n+3} \\
&\quad - (-1)^{n+1}\left((r-3) \cdot (r-1) + (r-2)\right) \cdot r \cdot (r-2)^2 - r \cdot (r-2)^{n+4} \\
&\quad + (-1)^{n+1} r \cdot (r-2)^3 + \mathcal{O}(1) \\
&= r \cdot (r-2) \cdot (r-1)^{n+4} - r \cdot (r-1) \cdot (r-3) \cdot (r-2)^{n+3} - 2r \cdot (r-2)^{n+4}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\sum_{j=4}^{n+4} (-1)^{n+4-j} \cdot r \cdot (r-1)^2 \cdot (r-2)^{j-2} \cdot j \right) + \mathcal{O}(1) \\
&= r \cdot (r-2) \cdot (r-1)^{n+4} - r \cdot (r-1) \cdot (r-3) \cdot (r-2)^{n+3} - 2r \cdot (r-2)^{n+4} \\
&\quad - r \cdot (r-1) \cdot (r-2)^{n+3} \cdot (n+5) + r \cdot (r-2)^{n+4} + \mathcal{O}(1)
\end{aligned}
$$

where the final line comes from the following:

$$
\begin{aligned}
&\sum_{j=4}^{n+4} (-1)^{n+4-j} \cdot r \cdot (r-1)^2 \cdot (r-2)^{j-2} \cdot j \\
&\quad = \frac{r \cdot (r-1)^2}{(r-2)} \left(\sum_{j=4}^{n+4} (-1)^{n+4-j} (r-2)^{j-1} \cdot j \right) \\
&\quad = \frac{r \cdot (r-1)^2}{(r-2)} \left(\frac{d}{dr} \left(\sum_{j=4}^{n+4} (-1)^{n+4-j} (r-2)^{j} \right) \right) \\
&\quad = \frac{r \cdot (r-1)^2}{(r-2)} \left(\frac{d}{dr} \left(\frac{(r-2)^{n+5}}{r-1} + \mathcal{O}(1) \right) \right) \\
&\quad = \frac{r \cdot (r-1)^2}{(r-2)} \left(\frac{(r-1) \cdot (n+5) \cdot (r-2)^{n+4} - (r-2)^{n+5}}{(r-1)^2} \right) + \mathcal{O}(1) \\
&\quad = r \cdot (r-1) \cdot (r-2)^{n+3} \cdot (n+5) - r \cdot (r-2)^{n+4} + \mathcal{O}(1)
\end{aligned}
$$

From these examples, we see that the order of the coefficients is a power of $r-1$. If we write the general **cd**-monomial that contains m **d**'s as $\mathbf{c}^{\alpha_1}\mathbf{d}\mathbf{c}^{\alpha_2}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{\alpha_m}\mathbf{d}\mathbf{c}^{\infty}$, we hope to determine the order of the coefficients in terms of the exponents α_i. We currently have the following conjecture regarding this order.

Conjecture 4.7.2. *In the Bruhat order of the universal Coxeter group $\mathcal{U}_r$, there is a constant C depending upon r such that the order of the coefficients of the infinite* **cd***-index is given by*

$$
[\mathbf{c}^{\alpha_1}\mathbf{d}\mathbf{c}^{\alpha_2}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{\alpha_m}\mathbf{d}\mathbf{c}^{\infty}] = C \cdot (r-1)^{\alpha_1+\cdots+\alpha_m} + O((r-2+\varepsilon)^{\alpha_1+\cdots+\alpha_m}).
$$

Chapter 5 The Hyperpfaffian and Extending Torelli's Identity

5.1 Introduction

The Pfaffian of a skew-symmetric matrix is commonly defined as the square root of its determinant. Note that if the order of this matrix is odd, then the determinant vanishes, and the Pfaffian is zero. Hence we assume that the order is even. Similar to the determinant (of any square matrix) being expressed as a sum over all perfect matchings of the complete bipartite graph, the Pfaffian has an explicit expression as a sum over all perfect matchings of the complete graph.

Barvinok [1] extended the notion of the Pfaffian to the hyperpfaffian. Instead of considering matchings of the complete graph, consider set partitions of the set $[n] = \{1, 2, \ldots, n\}$ into blocks of equal size k. Let $\Pi_{n,k}$ denote the set of such partitions. Furthermore, let k be an even integer and n a multiple of k. Let f be a k-ary skew-symmetric function defined on the set $[n]^k$. Note that these functions are extensions of matrices since an $n \times n$ matrix is a 2-ary function defined on $[n]^2$, where the input into the function gives the row and column of the matrix. For a k-element subset $B = \{b_1 < b_2 < \cdots < b_k\}$ of $[n]$ write $f(B) = f(b_1, b_2, \ldots, b_k)$. Lastly, define the sign $(-1)^\tau$ of a partition $\tau = \{B_1, B_2, \ldots, B_{n/k}\}$ in $\Pi_{n,k}$ to be the sign of the permutation $b_{1,1}, b_{1,2}, \ldots, b_{1,k}, b_{2,1}, \ldots, b_{2,k}, b_{3,1}, \ldots, b_{n/k,k}$, where the ith block is given by $B_i = \{b_{i,1} < b_{i,2} < \cdots < b_{i,k}\}$. Then the hyperpfaffian is defined by

$$\mathrm{Pf}(f) = \sum_{\tau} (-1)^\tau \cdot \prod_{i=1}^{n/k} f(B_i), \tag{5.1.1}$$

where the sum is over all partitions $\tau = \{B_1, B_2, \ldots, B_{n/k}\}$ in $\Pi_{n,k}$; see [1, Section 3].

In the case when the function f is a skew-symmetric polynomial in k variables of degree $k/2 \cdot (n-1)$, we can evaluate the hyperpfaffian; see Theorem 5.4.1. The result is the Vandermonde product multiplied by an expression of the coefficients of the polynomial f. We prove this using a sign-reversing involution that cancels all of the terms except those corresponding to the Vandermonde determinant. The proof can be made completely combinatorial by combining the last step with Ira Gessel's sign-reversing involution in his proof of the Vandermonde identity [24]. In the classical Pfaffian case, that is, when $k = 2$, our identity yields a nice expression, generalizing an identity due to Torelli [47].

In the last section we state some open questions about the hyperpfaffian, among them what other identities it satisfies.

A version of this chapter appears in [14].

5.2 The hyperpfaffian in connection with the exterior algebra

To give more motivation for the hyperpfaffian, we introduce the exterior algebra. Recall that f is a skew-symmetric function if for all permutations σ in $\mathfrak{S}_k$ we have that

$$f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)}) = (-1)^\sigma \cdot f(i_1, i_2, \ldots, i_k),$$

where $(-1)^\sigma$ denotes the sign of the permutation σ. Observe that if two of the entries $i_1, i_2, \ldots, i_k$ are equal then $f(i_1, i_2, \ldots, i_k) = 0$.

Let Λ denote the exterior algebra in the variables $t_1, t_2, \ldots, t_n$. For $S = \{s_1 < s_2 < \cdots < s_m\}$ a subset of $[n]$, let t_S denote the exterior or wedge product $t_S = t_{s_1} \wedge t_{s_2} \wedge \cdots \wedge t_{s_m}$. Observe that for two sets S and T that share at least one element we have that $t_S \wedge t_T = 0$. Also note that if at least one of the two sets S and T has even cardinality, then the elements t_S and t_T commute, that is, $t_S \wedge t_T = t_T \wedge t_S$. Furthermore, let $f(S)$ denote the function value $f(s_1, s_2, \ldots, s_k)$.

Luque and Thibon expressed the hyperpfaffian in terms of the exterior algebra [35, Equation (79)]. We include a proof for completeness.

Proposition 5.2.1 (Luque–Thibon)**.** *The hyperpfaffian of the skew-symmetric function f defined on the set $[n]^k$ is the unique scalar given by the equation*

$$\left(\sum_S f(S) \cdot t_S\right)^{n/k} = (n/k)! \cdot \mathrm{Pf}(f) \cdot t_{[n]}, \tag{5.2.1}$$

where the sum is over all k-element subsets of the set $[n]$.

Proof. Begin by noting that the sign of a partition $\tau = \{B_1, B_2, \ldots, B_{n/k}\}$ is the unique scalar $(-1)^\tau$ such that $t_{B_1} \wedge t_{B_2} \wedge \cdots \wedge t_{B_{n/k}} = (-1)^\tau \cdot t_{[n]}$. Now expand the power in equation (5.2.1) to obtain that

$$\left(\sum_S f(S) \cdot t_S\right)^{n/k} = \sum_{B_1} \cdots \sum_{B_{n/k}} f(B_1) \cdots f(B_{n/k}) \cdot t_{B_1} \cdots t_{B_{n/k}},$$

where each sum on the right-hand side is over all k-element subsets of $[n]$. Observe that the product in the exterior algebra is zero if two of the sets have a common element. Hence the sum reduces to sum over all ordered partitions of $[n]$. Ordered here refers to the set of blocks having a linear order. Given a partition in $\Pi_{n,k}$, there are $(n/k)!$ ways to obtain an ordered partition. Hence the sum reduces to $(n/k)! \cdot t_{[n]}$ times the right-hand side of equation (5.1.1), proving the result. □

Lemma 5.2.2. *Let f be a skew-symmetric function on the set $[n]^k$, and let σ be a permutation on the set $[n]$. Then the function $g(i_1, i_2, \ldots, i_k) = f(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$ is skew-symmetric, and the two hyperpfaffians differ by the sign $(-1)^\sigma$, that is, $\mathrm{Pf}(g) = (-1)^\sigma \cdot \mathrm{Pf}(f)$.*

Proof. It is straightforward to observe that g is skew-symmetric. It is enough to prove the identity for the adjacent transposition $\sigma = (j, j+1)$. Let $u_i = t_{\sigma(i)}$, that is, a reordering of the basis of the exterior algebra. We claim that $g(\sigma(S)) \cdot u_{\sigma(S)} = f(S) \cdot t_S$. If neither j and $j+1$ belong to the set S, there is nothing to prove. If only one of them belongs to S, then yet again there is nothing to prove. Finally, if both j and

$j+1$ belong to S, we have that $g(\sigma(S)) = -f(S)$ and $u_{\sigma(S)} = -t_S$, and the two signs cancel. Hence the two sums $\sum_S g(S) \cdot u_S$ and $\sum_S f(S) \cdot t_S$ are equal. Now the result follows from the definition of the hyperpfaffian and that $u_{[n]} = -t_{[n]}$. □

For more information regarding the hyperpfaffian and its applications, see Redelmeier [41].

5.3 Preliminaries

A weak composition $\vec{r}$ of an integer m is a vector $(r_1, r_2, \ldots, r_k)$ whose entries are nonnegative integers and their sum is m. The entries are called parts. For a composition $\vec{r}$ into k parts we let $x^{\vec{r}}$ denote the monomial $x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$. Furthermore, let the symmetric group $\mathfrak{S}_k$ act on compositions into k parts by reordering the parts.

Let $f(x_1, x_2, \ldots, x_k)$ be a homogeneous polynomial of degree $k/2 \cdot (n-1)$. The polynomial f can be expressed as

$$f(x_1, x_2, \ldots, x_k) = \sum_{\vec{r}} a_{\vec{r}} \cdot x^{\vec{r}},$$

where the sum is over all weak compositions $\vec{r}$ of $k/2 \cdot (n-1)$ into k parts. Furthermore, assume that the polynomial f is skew-symmetric, which implies that the coefficients satisfy the equation $a_{\sigma \circ \vec{r}} = (-1)^{\sigma} \cdot a_{\vec{r}}$.

Let $\Gamma_{n,k}$ denote the set of increasing weak compositions of $k/2 \cdot (n-1)$ into k distinct parts; that is, the set $\Gamma_{n,k}$ is given by

$$\Gamma_{n,k} = \left\{ (r_1, r_2, \ldots, r_k) \in \mathbb{N}^k \;:\; 0 \leq r_1 < r_2 < \cdots < r_k, \; \sum_{i=1}^{k} r_i = k/2 \cdot (n-1) \right\}.$$

Hence we can write the skew-symmetric polynomial f on the form

$$f(x_1, x_2, \ldots, x_k) = \sum_{\vec{r} \in \Gamma_{n,k}} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} \cdot a_{\vec{r}} \cdot x^{\sigma \circ \vec{r}}. \tag{5.3.1}$$

We define an oriented partition to be a partition where each block is endowed with a linear order. Let $T_{n,k}$ denote the set of all oriented partitions ρ of the set $[n]$ where each block has cardinality k; that is, for an oriented partition $\rho = \{C_1, C_2, \ldots, C_{n/k}\}$, each block C_i is an ordered list $C_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,k})$.

Observe that the number of oriented partitions is given by $|T_{n,k}| = (k!)^{n/k} \cdot |\Pi_{n,k}| = n!/(n/k)!$. This can be directly observed by taking a permutation on n elements and dividing into n/k blocks of size k. Permuting the n/k blocks yields the same oriented partition. Also observe that since k is even, all the $(n/k)!$ permutations yielding the same oriented partition have the same sign. We define this sign to be the sign of the oriented partition, denoted $(-1)^{\rho}$. More explicitly, the sign of ρ is given by the sign of the permutation

$$\pi(\rho) = c_{1,1}, \ldots, c_{1,k}, \;\; c_{2,1}, \ldots, c_{2,k}, \;\; c_{3,k}, \ldots, c_{n/k,k}. \tag{5.3.2}$$

By removing the linear order on each block from the oriented partition ρ, we obtain a partition τ. We note that the sign of the oriented partition ρ and the sign of the partition τ are related by

$$(-1)^{\rho} = (-1)^{\tau} \cdot (-1)^{\sigma_1} \cdot (-1)^{\sigma_2} \cdots (-1)^{\sigma_{n/k}}, \tag{5.3.3}$$

where σ_i is the permutation on the set $\{c_{i,1}, c_{i,2}, \ldots, c_{i,k}\}$ that orders the ith block, that is, $\sigma_i(c_{i,1}) < \sigma_i(c_{i,2}) < \cdots < \sigma_i(c_{i,k})$.

Let $R_{n,k}$ denote the collection of sets of size n/k of compositions in $\Gamma_{n,k}$, where all the parts of the compositions are distinct. Let $\beta = \{\vec{r}_1, \ldots, \vec{r}_{n/k}\}$ denote such a set in $R_{n,k}$. Observe that the sum of all the entries of the compositions is given by $n/k \cdot k/2 \cdot (n-1)$ which is the sum $0 + 1 + \cdots + (n-1)$. Hence we conclude that the underlying parts of the compositions of β are the integers 0 through $n-1$. Thus we view β as an oriented set partition of the elements $\{0, \ldots, n-1\}$ into n/k blocks of size k in which each block is a composition in $\Gamma_{n,k}$. Define the sign of $\beta = \{\vec{r}_1, \ldots, \vec{r}_{n/k}\} \in R_{n,k}$ with $\vec{r}_i = (r_{i,1}, \ldots, r_{i,k})$, denoted by $(-1)^{\beta}$, to be the sign of the permutation

$$\pi(\beta) = r_{1,1}, \ldots, r_{1,k},\ \ r_{2,1}, \ldots, r_{2,k},\ \ r_{3,k}, \ldots, r_{n/k,k},$$

where $\pi(\beta)$ is a permutation of the elements $\{0, 1, \ldots, n-1\}$.

5.4 Main Theorem

Using the skew-symmetric polynomial given in equation (5.3.1), we have the following identity.

Theorem 5.4.1. *The hyperpfaffian* $\mathrm{Pf}(f(x_S))$ *of order* n *is the Vandermonde product multiplied by a signed sum of products of coefficients* $a_{\vec{r}}$*:*

$$\mathrm{Pf}(f(x_S))_{S \in \binom{[n]}{k}} = \left(\sum_{\beta} (-1)^{\beta} \cdot \prod_{i=1}^{n/k} a_{\vec{r}_i} \right) \cdot \prod_{1 \le i < j \le n} (x_j - x_i),$$

where the sum ranges over all partitions β *in* $R_{n,k}$*.*

Example 5.4.2. When $n = 12$ and $k = 4$, there are 32 oriented partitions in $R_{12,4}$. The coefficient in Theorem 5.4.1 is in this case given by

$$\begin{aligned}
&a_{0,1,10,11}a_{2,3,8,9}a_{4,5,6,7} + a_{0,1,10,11}a_{2,4,7,9}a_{3,5,6,8} + a_{0,1,10,11}a_{2,5,6,9}a_{3,4,7,8}\\
&+a_{0,1,10,11}a_{2,5,7,8}a_{3,4,6,9} + a_{0,2,9,11}a_{1,3,8,10}a_{4,5,6,7} + a_{0,2,9,11}a_{1,4,7,10}a_{3,5,6,8}\\
&+a_{0,2,9,11}a_{1,5,6,10}a_{3,4,7,8} - a_{0,2,9,11}a_{1,6,7,8}a_{3,4,5,10} + a_{0,3,8,11}a_{1,2,9,10}a_{4,5,6,7}\\
&+a_{0,3,8,11}a_{1,4,7,10}a_{2,5,6,9} + a_{0,3,8,11}a_{1,5,6,10}a_{2,4,7,9} + a_{0,3,8,11}a_{1,5,7,9}a_{2,4,6,10}\\
&+a_{0,3,9,10}a_{1,2,8,11}a_{4,5,6,7} - a_{0,3,9,10}a_{1,4,6,11}a_{2,5,7,8} - a_{0,3,9,10}a_{1,6,7,8}a_{2,4,5,11}\\
&+a_{0,4,7,11}a_{1,2,9,10}a_{3,5,6,8} + a_{0,4,7,11}a_{1,3,8,10}a_{2,5,6,9} + a_{0,4,7,11}a_{1,5,6,10}a_{2,3,8,9}\\
&+a_{0,4,8,10}a_{1,3,7,11}a_{2,5,6,9} + a_{0,4,8,10}a_{1,5,7,9}a_{2,3,6,11} + a_{0,5,6,11}a_{1,2,9,10}a_{3,4,7,8}\\
&+a_{0,5,6,11}a_{1,3,8,10}a_{2,4,7,9} + a_{0,5,6,11}a_{1,4,7,10}a_{2,3,8,9} + a_{0,5,6,11}a_{1,4,8,9}a_{2,3,7,10}\\
&-a_{0,5,7,10}a_{1,2,8,11}a_{3,4,6,9} + a_{0,5,7,10}a_{1,4,6,11}a_{2,3,8,9} + a_{0,5,7,10}a_{1,4,8,9}a_{2,3,6,11}\\
&+a_{0,5,8,9}a_{1,3,7,11}a_{2,4,6,10} + a_{0,5,8,9}a_{1,4,6,11}a_{2,3,7,10} + a_{0,5,8,9}a_{1,4,7,10}a_{2,3,6,11}\\
&-a_{0,6,7,9}a_{1,2,8,11}a_{3,4,5,10} - a_{0,6,7,9}a_{1,3,8,10}a_{2,4,5,11}.
\end{aligned}$$

Let $W_{n,k}$ be the set of oriented partitions $\rho = \{C_1, C_2, \ldots, C_{n/k}\}$ on the set $[n]$ with a composition $\vec{w}_i = (w(c_{i,1}), \ldots, w(c_{i,k})) \in \Gamma_{n,k}$, referred to as the weight vector, assigned to each block $C_i = (c_{i,1}, \ldots, c_{i,k})$. We define the following notions for such a weighted oriented partition ρ. Let $(-1)^\rho$ be the *sign* of ρ, defined as in the previous section by the sign of the permutation $\pi(\rho)$ from equation (5.3.2). Let the *coefficient* $c(\rho)$ denote the product $\prod_{i=1}^{n/k} a_{\vec{w}_i}$ determined by the weight vectors of ρ. Lastly, let $w(\rho)$ denote the *monomial* $\prod_{i=1}^{n/k} x_{C_i}^{\vec{w}_i}$ where $x_{C_i}^{\vec{w}_i} = \prod_{j=1}^{k} x_{c_{i,j}}^{w(c_{i,j})}$.

Lemma 5.4.3. *The following expansion holds for the hyperpfaffian:*

$$\operatorname{Pf}\left(f(x_S)\right)_{S\in\binom{[n]}{k}} = \sum_{\rho\in W_{n,k}} (-1)^\rho \cdot c(\rho) \cdot w(\rho).$$

Proof. By applying equation (5.3.1) to equation (5.1.1), we have

$$\operatorname{Pf}\left(f(x_S)\right)_{S\in\binom{[n]}{k}} = \sum_{\tau\in\Pi_{n,k}} (-1)^\tau \cdot \prod_{i=1}^{n/k} \left(\sum_{\vec{r}\in\Gamma_{n,k}} \sum_{\sigma\in\mathfrak{S}_k} (-1)^\sigma \cdot a_{\vec{r}} \cdot x_{B_i}^{\sigma\circ\vec{r}} \right).$$

Using the distributive law, expand the above product. We obtain an oriented, weighted partition ρ for each term by orienting the elements in each block $B_i \in \tau$ by increasing size of the exponents of their associated variables. The composition $\vec{r}$ corresponds to the choice of weight vector for each block, and the permutation σ will undo the orientation of the block to properly assign the weights as exponents. Multiplying the sign of σ for each block with the sign of τ gives the sign of ρ as described in equation (5.3.3) because for the block B_i, we have $\sigma = \sigma_i^{-1}$.

□

Let $W_{n,k}^r$ denote the subset of $W_{n,k}$ with repeated weights, and let $W_{n,k}^d$ denote the complement, that is, partitions with distinct weights. We now create a sign-reversing involution ϕ for the set $W_{n,k}^r$ to narrow our focus to only partitions with distinct weights. Given a partition ρ in $W_{n,k}^r$, let (i, j) be the lexicographically smallest pair of elements in $[n]$ in which $w(i) = w(j)$. Define $\phi(\rho)$ by swapping i and j, while leaving the weight vector for each block and the orientation unchanged.

Lemma 5.4.4. *The function ϕ is a sign-reversing involution on the set $W_{n,k}^r$ which does not change the coefficient nor the monomial. That is, for an oriented partition $\rho \in W_{n,k}^r$ we have that $\phi^2(\rho) = \rho$, $c(\phi(\rho)) = c(\rho)$, $w(\phi(\rho)) = w(\rho)$, but $(-1)^{\phi(\rho)} = -(-1)^\rho$.*

Proof. By definition, it follows that ϕ is an involution, and that it leaves the coefficient and the monomial of ρ unchanged. To see that ϕ is sign-reversing, consider the consequences of swapping i and j within the permutation $\pi(\rho)$. We get that $\pi(\phi(\rho)) = (i\ j) \circ \pi(\rho)$; hence, the transposition changes the sign of the corresponding permutation as ϕ is applied. Thus $(-1)^{\phi(\rho)} = -(-1)^\rho$. □

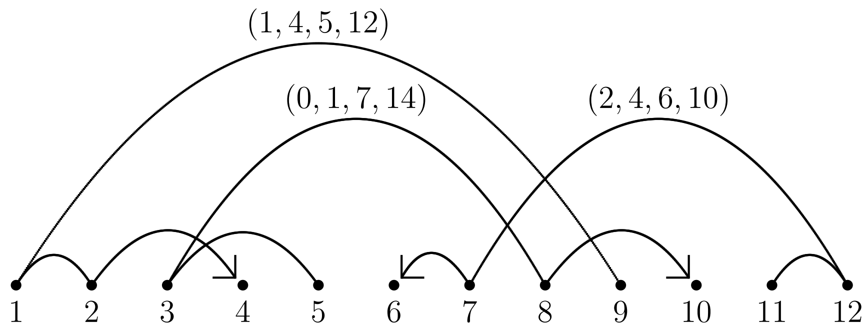

Figure 5.1: The oriented partition $\{(9,1,2,4),(5,3,8,10),(11,12,7,6)\}$. Note that it has a negative sign. The labels yields the monomial $x_9^1 x_1^4 x_2^5 x_4^{12} \cdot x_5^0 x_3^1 x_8^7 x_{10}^{14} \cdot x_{11}^2 x_{12}^4 x_7^6 x_6^{10}$ and the coefficient $a_{1,4,5,12} \cdot a_{0,1,7,14} \cdot a_{2,4,6,10}$.

Observe that the weighted oriented partition in Figure 5.1 has non-distinct powers for the pair of variables x_1 and x_{12} and the pair x_3 and x_9, of which the pair $(1, 12)$ is lexicographically least. Hence this weighted oriented partition cancels with the oriented partition $\{(9,12,2,4),(5,3,8,10),(11,1,7,6)\}$ with the same weight vector.

We now concentrate on weighted oriented partitions where the weights are distinct, that is, the set $W_{n,k}^d$. Note that this implies that the weights are 0 through $n-1$, allowing us to narrow our focus to weight vectors that make up an oriented partition in $R_{n,k}$.

For a weighted oriented partition ρ in $W_{n,k}^d$, let σ be the unique permutation such that $w(\rho) = x_1^{\sigma_1 - 1} x_2^{\sigma_2 - 1} \cdots x_n^{\sigma_n - 1}$. Furthermore, let $\beta \in R_{n,k}$ be the set of weight vectors assigned to the blocks of ρ. Observe that this describes a bijection between $W_{n,k}^d$ and the Cartesian product of the symmetric group $\mathfrak{S}_n$ and the weight vectors $R_{n,k}$.

Lemma 5.4.5. *The sign of a weighted oriented partition ρ in $W_{n,k}^d$ factors as $(-1)^\rho = (-1)^\beta \cdot (-1)^\sigma$.*

Proof. Define the permutation $\pi(\beta)'$ on $[n]$ such that $\pi(\beta)'_i = \pi(\beta)_i + 1$. Since $(-1)^{\pi(\beta)'} = (-1)^{\pi(\beta)}$, it is enough to observe that the permutation $\pi(\beta)'$ factors as $\sigma \circ \pi(\rho)$. □

Proof of Theorem 5.4.1. By combining Lemmas 5.4.3 through 5.4.5, we have that

$$\begin{aligned}\mathrm{Pf}\left(f(x_S)\right)_{S \in \binom{[n]}{k}} &= \sum_{\rho \in W_{n,k}^d} (-1)^\rho \cdot c(\rho) \cdot w(\rho) \\ &= \left(\sum_{\beta} (-1)^\beta \cdot \prod_{i=1}^{n/k} a_{\vec{r}_i} \right) \cdot \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \cdot x_1^{\sigma_1 - 1} \cdots x_n^{\sigma_n - 1},\end{aligned}$$

where the last sum is the Vandermonde determinant, which is equal to the Vandermonde product. □

An algebraic proof of Theorem 5.4.1 is as follows. Note that when setting two of the variables x_i and x_j equal, the hyperpfaffian vanishes by Lemma 5.2.2. Hence as a polynomial in x_1 through x_n, the Vandermonde product divides the hyperpfaffian. However, the two sides have same degree $n/k \cdot k/2 \cdot (n-1) = \binom{n}{2}$, and hence are equal up to a constant. By considering the coefficient of the term $x_2 \cdots x_{n-1}^{n-2} x_n^{n-1}$, we obtain the constant $\sum_\beta (-1)^\beta \cdot \prod_{i=1}^{n/k} a_{\vec{r}_i}$.

Finally, observe that when the polynomial f is replaced with a polynomial of degree less than $k/2 \cdot (n-1)$, the hyperpfaffian will be zero. This can be seen in two ways. The only polynomial of degree less than $\binom{n}{2}$ which is divisible by the Vandermonde product is the zero polynomial. Alternatively, the sign-reversing involution has no fixed points; that is, it cancels all the terms.

5.5 Application to the classical Pfaffian

Let us now focus on the $k = 2$ case. In this case the oriented partitions devolve into directed matchings, and the compositions in $\Gamma_{n,2}$ have two parts with the sum $n - 1$. Hence, they have the form $(i, n-1-i)$ from $i = 0, 1, \ldots, n/2 - 1$. This leads to the skew polynomial f having the following form:

$$f(x, y) = \sum_{i=0}^{n-1} a_i \cdot x^i y^{n-1-i},$$

where $a_{n-1-i} = -a_i$. Here we abbreviate the coefficients $a_{i,n-1-i}$ as simply a_i. Since the only oriented partition in $R_{n,2}$ is $\{(0, n-1), (1, n-2), \ldots, (n/2-1, n/2)\}$, which has the sign $(-1)^{2\binom{n/2}{2}} = 1$, Theorem 5.4.1 reduces to the following corollary.

Corollary 5.5.1. *The Pfaffian* $\mathrm{Pf}(f(x_i, x_j))$ *of order* n *is the product of the first* $n/2$ *of the coefficients* a_i *multiplied by the Vandermonde product:*

$$\mathrm{Pf}\left(f(x_i, x_j)\right)_{1 \le i < j \le n} = \prod_{i=0}^{n/2-1} a_i \cdot \prod_{1 \le i < j \le n} (x_j - x_i).$$

As a corollary we have the following identity due to Torelli [47]; see also [31, Equation (4.6)].

Corollary 5.5.2 (Torelli)**.** *When the skew-symmetric polynomial is the function* $f(x, y) = (y - x)^{n-1}$*, the Pfaffian is given by*

$$\mathrm{Pf}\left(f(x_i, x_j)\right)_{1 \le i < j \le n} = (-1)^{\binom{n/2}{2}} \cdot \prod_{i=0}^{n/2-1} \binom{n-1}{i} \cdot \prod_{1 \le i < j \le n} (x_j - x_i).$$

It is enough to observe that $a_i = (-1)^i \cdot \binom{n-1}{i}$.

5.6 Counting with the hyperpfaffian

Let G be a planar graph on the vertex set $[n]$. For k even and that divides n, we define a k-covering of G to be to be a spanning subgraph of G with each component consisting of k vertices which is able to be covered with a 2-covering, also known as a dimer covering or a perfect matching. We can represent a k-covering by a partition $\tau \in \Pi_{n,k}$ so that for each block $B = \{i_1, \ldots, i_k\}$, the subgraph of G induced by the vertices in B is connected, and this subgraph admits a perfect matching $M_B = \{\{i_{j_1}, i_{j_2}\}, \ldots, \{i_{j_{k-1}}, i_{j_k}\}\}$ such that each pair is an edge in G. Define $N^k(G)$ to be the number of $k-$coverings of G.

When focusing on the $k = 2$ case in which the partitions τ are perfect matchings of G, the problem at hand has been well-studied and is known as the dimer problem. By Kasteleyn's Theorems 1 and 2 in section 2V of [30], it is possible to orient the edges of G and assign to it a skew-symmetric matrix D in which $\mathrm{Pf}(D) = N^2(G)$. Note that these theorems are stated using $|\mathrm{Pf}(D)|$. However, Kasteleyn goes on to describe a way to reorient certain edges to avoid the need for the absolute value operation. Once this orientation is placed on G to create the directed edge set $E(G)$, the matrix D is defined as a $(-1, 0, 1)$-matrix with the following entries

$$D_{i,j} = s(i, j) := \begin{cases} 1 & \text{if } e = i \to j \in E(G), \\ -1 & \text{if } e = j \to i \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

In order for $\mathrm{Pf}(D) = \sum_M (-1)^{\pi(M)} \prod_{(i,j)\in M} s(i, j) = N^2(G)$, where the sum is over all perfect matchings M of G, we have

$$(-1)^{\pi(M)} \prod_{(i,j)\in M} s(i, j) = 1, \tag{5.6.1}$$

for each perfect matching M.

For a general k, we wish to create a similar matrix whose hyperpfaffian is equal to the number of k-coverings. First, we define the sign of a block $B = \{i_1, \ldots, i_k\}$ with matching M_B as described above as $s(B) = (-1)^B \cdot s(i_{j_1}, i_{j_2}) \cdots s(i_{j_{k-1}}, i_{j_k})$ where $(-1)^B$ is the sign of the permutation $\pi(B) = \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_{j_1} & i_{j_2} & \cdots & i_{j_k} \end{pmatrix} \in \mathfrak{S}_k$. We have the following result relating the signs of the permutations that correspond to the partition and matchings.

Lemma 5.6.1. *For a partition $\tau \in \Pi_{n,k}$ in which each block B admits a matching M_B which together form the matching M on $[n]$, the following equation holds:*

$$(-1)^{\pi(M)} = (-1)^{\tau} \cdot \left(\prod_{B\in\tau} (-1)^B \right).$$

Proof. If we label the blocks $B_1, \dots, B_{n/k}$, then for each block we define the permutation $\pi^*(B_i) \in \mathfrak{S}_n$ by applying $\pi(B_i)$ to the elements in the block B_i and the identity permutation on the other $n-k$ elements. Since $(-1)^{\pi(B_i)} = (-1)^{\pi^*(B_i)}$, it is enough to observe that $\pi(M) = \pi^*(B_1) \circ \cdots \circ \pi^*(B_{n/k}) \circ \pi(\tau)$. □

Define the k-dimensional $n \times \cdots \times n$ matrix A by defining the entry $A_B = s(B)$. Note that this matrix is skew-symmetric because for every $\sigma \in \mathfrak{S}_k$, since $\pi_{\sigma(B)} = \pi(B) \circ \sigma$, we have $s(\sigma(B)) = (-1)^\sigma s(B)$. Applying the hyperpfaffian to this matrix A gives the following theorem.

Theorem 5.6.2. *For a planar graph G, the Pfaffian of the associated k-dimensional matrix A gives the number of k-coverings of G, that is,*

$$\mathrm{Pf}(A) = N^k(G).$$

Proof. By definition,

$$\mathrm{Pf}(A) = \sum_\tau (-1)^\tau \cdot \left(\prod_{B \in \tau} s(B) \right),$$

where the sum is over all partitions $\tau \in \Pi_{n,k}$. Notice that blocks in which the induced subgraph of G on those vertices is not connected, as well as blocks that do not admit a matching of edges in $E(G)$, create a zero as the corresponding matrix entry. Thus, the sum can be viewed as only being over partitions τ where each block B generates a connected induced subgraph and admits a matching M_B; hence, each τ corresponds to a k-covering. This gives

$$\begin{aligned} \mathrm{Pf}(A) &= \sum_\tau (-1)^\tau \cdot \prod_{B \in \tau} \left((-1)^B \cdot \prod_{(i,j) \in M_B} s(i,j) \right) \\ &= \sum_\tau (-1)^\tau \cdot \left(\prod_{B \in \tau} (-1)^B \right) \left(\prod_{(i,j) \in M} s(i,j) \right), \end{aligned}$$

where $M = \{M_B\}_{B \in \tau}$ is a matching of $[n]$. Then we have

$$\begin{aligned} \mathrm{Pf}(A) &= \sum_\tau (-1)^\tau \cdot \left(\prod_{B \in \tau} (-1)^B \right) \cdot (-1)^M \cdot \left((-1)^M \cdot \prod_{(i,j) \in M} s(i,j) \right) \\ &= \sum_\tau (-1)^\tau \cdot \left(\prod_{B \in \tau} (-1)^B \right) \cdot (-1)^M, \end{aligned}$$

due to equation (5.6.1). By Lemma 5.6.1, we see that these signs multiply to be one, giving us the desired result. □

The previous theorem allows the hyperpfaffian to be used to count k-coverings of planar graphs, similar to how the Pfaffian is used to count perfect matchings. The downside, however, is that the Pfaffian of a skew-symmetric matrix can be calculated efficiently using the determinant, but there is not an efficient way to calculate the hyperpfaffian of a k-dimensional skew-symmetric matrix.

5.7 Concluding remarks

Benjamin and Dresden [4] gave a combinatorial proof of the Vandermonde identity differing from that of Gessel [24]. Their combinatorial interpretation involved counting rows of cards that each possess a particular value and suit, where the suit for each row is determined by a permutation. They used a sign-reversing involution on the opposite side than Gessel. Is it possible to prove Corollary 5.5.1 or more generally, Theorem 5.4.1, by a similar technique?

What other identities does the hyperpfaffian satisfy? See Knuth [31] and Tanner [45] for the expansion for products of two overlapping Pfaffians, and for applications of this identity. Can any of these results be generalized for hyperpfaffians? One such example is the following identity for compositions of the hyperpfaffians, proved by Luque and Thibon [35].

Theorem 5.7.1 (Luque–Thibon)**.** *Let k, n, and p be three even positive integers such that n is a multiple of k and p is a multiple of n. Let f be a skew-symmetric k-ary function on the set $[p]$. Define an n-ary function g by the hyperpfaffian of order n, that is,*

$$g(i_i, \ldots, i_n) = \mathrm{Pf}(f)_{(i_1,\ldots,i_n)}.$$

Then the hyperpfaffian of order p of the function g is given by a constant times the hyperpfaffian of f order p, that is,

$$\mathrm{Pf}(g) = \frac{1}{(p/n)!} \cdot \binom{p/k}{n/k, \ldots, n/k} \cdot \mathrm{Pf}(f),$$

where there are p/n instances of n/k in the multinomial coefficient.

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Vita

Norman B. Fox

- Education
  - 2015: Ph.D. University of Kentucky (expected)
  - 2014: Graduate Certificate in College Teaching and Learning, University of Kentucky
  - 2011: M.A. University of Kentucky
  - 2009: B.A. Transylvania University
- Professional Experience
  - Teaching Assistant, University of Kentucky, 2009–2015
  - Research Assistant, University of Kentucky, Summer 2012–2014
- Publications
  - "A Sign-reversing Involution for an Extensions of Torelli's Pfaffian Identity," *Discrete Mathematics* **332** (2014), 69–74, with Richard Ehrenborg.
  - "The Descent Set Polynomial Revisited," to appear in *European Journal of Combinatorics*, with Richard Ehrenborg. http://dx.doi.org/10.1016/j.ejc.2015.03.027
  - "A Lattice Path Interpretation of the Diamond Product," submitted to *Annals of Combinatorics.*
  - "The Infinite **cd**-index and the Extreme Coxeter Group," in preparation, with Richard Ehrenborg.
- Selected Presentations
  - "The Descent Set Polynomial Revisited," Joint Mathematics Meetings, January 2015.
  - "Cyclotomic Factors of the Descent Set Polynomial," Discrete CATS Seminar, University of Kentucky, October 2014.
  - "A Lattice Path Interpretation of the Diamond Product," AMS Southeastern Sectional Meeting, University of Tennessee, March 2014.
  - "An Extension of Torelli's Pfaffian Identity," Graduate Student Combinatorics Conference, University of Minnesota, April 2013.
  - "The **cd**-index of Bruhat Intervals," Discrete CATS Seminar, University of Kentucky, November 2012.

  - "An Extended Euler Poincaré Theorem," Topology Seminar, University of Kentucky, October 2011
  - "Dynamic Equations on Time Scales," poster presentation, Joint Mathematics Meetings, January 2008

- Awards
  - Assocation of Emeriti Faculty 2013 Endowed Doctoral Fellowship Award, University of Kentucky, 2013
  - College of Arts & Sciences Certificate for Outstanding Teaching, University of Kentucky, 2013
  - Wimberly C. Royster Outstanding Teaching Assistant Award, University of Kentucky, 2013
  - Van Meter Fellowship, University of Kentucky, Fall 2009–Spring 2011
  - William T. Young Scholarship, Transylvania University, Fall 2005–Spring 2009
  - Robert C. Byrd Scholarship, Fall 2005–Spring 2009