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Spin Cobordism and Quasitoric Manifolds

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SPIN COBORDISM AND WEDGE QUASITORIC MANIFOLDS

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Clinton Monroe Hines
Lexington, Kentucky

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Lexington, Kentucky 2014

ABSTRACT OF DISSERTATION

SPIN COBORDISM AND WEDGE QUASITORIC MANIFOLDS

This dissertation demonstrates a procedure to view any quasitoric manifold as a "minimal" sub-manifold of an ambient quasitoric manifold of codimension two via the wedge construction applied to the quotient polytope. These we term wedge quasitoric manifolds. We prove existence utilizing a construction on the quotient polytope and characteristic matrix and demonstrate conditions allowing the base manifold to be viewed as dual to the first Chern class of the wedge manifold. Such dualization allows calculations of KO characteristic classes as in the work of Ochanine and Fast. We also examine the Todd genus as it relates to two types of wedge quasitoric manifolds. Background matter on polytopes and toric topology, as well as spin and complex cobordism are provided.

KEYWORDS: quasitoric manifolds, toric topology, wedge polytope, complex projective spaces, Bott manifolds, connected sum, spin cobordism, complex cobordism, Chern class, Hirzebruch genera, Todd genus

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Date: 08. May 2014

SPIN COBORDISM AND WEDGE QUASITORIC MANIFOLDS

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This dissertation is dedicated to my wife Kathleen for her constant love and support.

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0.1 Introduction

We begin with a brief introduction to the materials presented throughout this dissertation and a listing of main results. Our main purpose is the pursuit of the topological structure of spin quasitoric manifolds through calculations in terms of their combinatorial data. There is a substantial body of work and quite a lot is known about the topological structure of quasitoric manifolds in terms of *complex cobordism* via omniorientations and their subsequent stably complex structures (see [4], [3] and [18]). Moreover we have the following.

Theorem 0.1.1 (Buchstaber, Panov and Ray [5], [4])**.** *In dimensions* > 2*, every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

Much less is currently known about these manifolds in terms of *spin cobordism* which requires the calculation of cohomology classes as well as KO-characteristic classes. Such KO-characteristic classes have been more difficult to quantify. In particular we have been interested in developing a method for viewing any spin quasitoric manifold as a codimension two submanifold of an ambient quasitoric manifold in such a way that allows the calculation of KO-characteristic classes as in [10].

Quasitoric manifolds are a generalization of smooth projective toric varieties. As manifolds they enjoy several properties (such as invariance under the connected sum) that make them useful in algebraic topology and complex cobordism theory but retain a simple combinatorial description that allows for combinatorial calculations and formulae [19]. Somewhat more formally a quasitoric manifold is any $2n$-dimensional manifold equipped with a "nice" action of the n-torus so that the quotient of this action reduces the manifold to a simple polytope.

Facets of this quotient polytope play an important role in the structure of each of these manifolds and correspond directly to codimension two submanifolds termed *facial submanifolds*. As in the theory of toric varieties each of quasitoric manifolds codimension two facial submanifolds give rise to *facet vectors*. The quotient polytope and corresponding facet vectors provide a complete combinatorial description of any given quasitoric manifold.

The topological and combinatorial relationships between quasitoric manifolds and each of their codimension two facial submanifolds has been well studied. Less well known are procedures for producing ambient quasitoric manifolds of codimension two for any given quasitoric manifold. To this end we began an investigation of the combinatorial wedge.

At the conclusion of Chapter 1 we introduce the wedge polytope Construction 1.4.1 due to Klee and Walkup [12]. This we apply in Chapter 2 to quasitoric manifolds forming what we refer to as *wedge quasitoric manifolds*. This construction and application are of growing interest in the field of toric topology, see Choi and Park [6].

We prove existence of such manifolds given specific requirements on the characteristic matrix in Theorem 2.3.2 formulating at alternate proof to that given by Choi and Park in [6] utilizing polytopal constructions rather than simplicial complexes. In Section 2.3 we use Theorems 2.3.2 and 2.3.6 to show Corollary 2.3.7 which states that "the connected sum of wedge quasitoric manifolds is a wedge quasitoric manifold." In the conclusion of Chapter 2 we use Theorem 2.3.2 and Theorem 2.4.1 to show Corollary 2.4.4 which states that "there exists a wedge quasitoric manifold over any Bott manifold (or tower) which is itself a Bott manifold."

These results concerning the connected sum and Bott Manifolds serve throughout this dissertation as a proof of concept allowing numerous $2n$-dimensional examples of wedge quasitoric manifolds which are not themselves algebraic toric varieties (the connected sum usually destroys algebraicity [3]). For further reading concerning complex cobordism classes of manifolds that are connected *and algebraic* consult Wilfong [25].

Speaking less formally there appears to be a general feeling in the literature of toric topology that results concerning connected sums of Bott manifolds most likely apply to most if not all quasitoric manifolds [14]. This most likely stems from Buchstaber and Panov's proof of the Theorem 0.1.1 which utilizes the connected sum of certain Bott manifolds.

Chapter 3 includes our main results concerning the wedge polytope and the Todd genus. This research grew out of an investigation into the alpha invariant which we now realize vanishes for all quasitoric manifolds [24]. Using a fixed point formula due to Panov [20] we proved that the canonical wedge 2.3.3 preserves the Todd genus (see Theorem 3.4.4) while the Todd genus of the reverse wedge 2.3.4 vanishes (see Theorem 3.4.5).

Chapter 4 features our main results concerning the wedge polytope construction and spin quasitoric manifolds. Specifically we demonstrate criteria for a spin quasitoric manifold to be viewed as dual to the first Chern class of its canonical wedge (see Theorem 4.4.4). This setup should allow the calculation of KO-characteristic classes for all spin quasitoric manifolds satisfying 4.7.

Chapter 1 Preliminaries

Here in Chapter 1 we introduce the wedge polytope Construction 1.4.1 due to Klee and Walkup [12]. This we apply in Chapter 2 to quasitoric manifolds forming what we refer to as *wedge quasitoric manifolds*. For more details related to the wedge operation as it is applied to quasitoric manifolds see Choi and Park [6].

1.1 Polytopes

A simple polytope P^n is the intersection of an arrangement of m closed half-spaces (given by m defining hyperplanes) in an n-dimensional vector space V so that the defining hyperplanes meet in general position [4]. We assume that the intersection of these hyperplanes is non-empty and bounded in defining P^n. See [4] for a discussion of *virtual polytopes* a generalization which is useful in discussing hyperplane arrangements and in particular *the moment angle manifold* 4.2.

We further insist that the number of half-spaces is minimal in that we have no extraneous (unnecessary) bounding hyperplanes. More formally we may write the following.

Definition 1.1.1 ([3]). *A convex polyhedron P is a non-empty intersection of finitely many half-spaces in some $\mathbb{R}^n$:*

$$P := \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq -b_i, i = 1, 2, \dots, m\},$$

where $a_i \in (\mathbb{R}^n)^*$ are some linear functions and $b_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. A (convex) polytope is a bounded convex polyhedron.

A polytope is then said to be *simple* when the defining hyperplanes for these halfspaces are in general position. A set of $m > n$ hyperplanes $\langle a_i, x\rangle \geq -b_i, i = 1, 2, \dots, m$ is in *general position* if no point belongs to more than n hyperplanes. This ensures that in any vertex of a simple polytope is the intersection of precisely n facets.

Example 1.1.2 (simplices, [4]). *The standard n-simplex Δ^n is the polytope defined by the half-spaces*

$$H_i = \begin{cases} \{x : \langle e_i, x\rangle \geq 0\} & \text{for } 1 \leq i \leq n \\ \{x : \langle a_{n+1}, x\rangle + 1 \geq 0\} & \text{for } i = n + 1 \end{cases}$$

in $\mathbb{R}^n$*; where* $a_{n+1} = (-1, \dots, -1)$*. Its vertices are the* $n+1$ *points* $0, e_1, \dots, e_n$*. Without the defining hyperplane* H_{n+1} *we in fact have the positive cone* $\mathbb{R}^n_{\geq 0}$*.*

A supporting hyperplane of P^n is an affine hyperplane H which intersects P^n and for which the polytope is contained in one of the two closed half-spaces determined by the hyperplane.

Definition 1.1.3 ([4])**.** *Non-empty intersections of the form* $P^n \cap H$ *for any supporting hyperplane* H *are termed* faces *of the polytope* P^n*. Zero-dimensional faces are termed* vertices *and one-dimensional faces are* edges*. We term the* $(n-1)$*-dimensional faces of* P^n facets *and label them* F_i *for each integer* $(1 \leq j \leq m)$*.*

The faces in all dimensions of a polytope P^n form a *poset* (a finite partially ordered set) under inclusion. This is referred to as the *face poset* associate to the polytope [3]. Two polytopes are *combinatorially equivalent* if and only if their face posets are isomorphic.

Definition 1.1.4. *A* combinatorial polytope *is a class of combinatorially equivalent convex polytopes.*

For any two simple polytopes P_1 and P_2 the *product* $P_1 \times P_2$ is a simple polytope as well [3]. Another method of combining simple polytopes is the *connected sum*. Given any two simple polytopes of the same dimension we may (informally) think of the connected sum as "cutting off" a single vertex from each polytope and "glueing" the polytopes together after a projective transformation has been made to place the two in line with one another [3].

The (formal) construction for the connected sum of simple polytopes requires a polyhedral template Γ^n an intersection of n half-spaces in $\mathbb{R}^n$. This template may be achieved by an embedding of the standard simplex Δ^{n-1} in the subspace $\{x : x_1 = 0\}$ of $\mathbb{R}^{n-1}$, and then taking cartesian products with the first coordinate axis [5]. Each facet of Γ^n is then of the form $G_r = \mathbb{R} \times D_r$, for $1 \leq r \leq n$ are the facets of Δ^{n-1}. Each of Γ^n and G_r are divided into positive and negative halves depending on the value of coordinate x_1.

We now describe the full procedure as follows.

Construction 1.1.5 (Connected sum of simple polytopes, [3], [5])**.** *Let* P^n *and* Q^n *be simple polytopes with distinguished vetices* v *and* w *respectively. We insist on an ordering of the facets of each vertex. Denote* E_r *for the facets meeting at* v *in* P^n *and* F_r *for the facets meeting at* w *in* Q^n *for each* $1 \leq r \leq n$*. Additionally, we denote the complimentary sets of facets* C_v *and* C_w *(i.e. those in* P^n *not incident to*

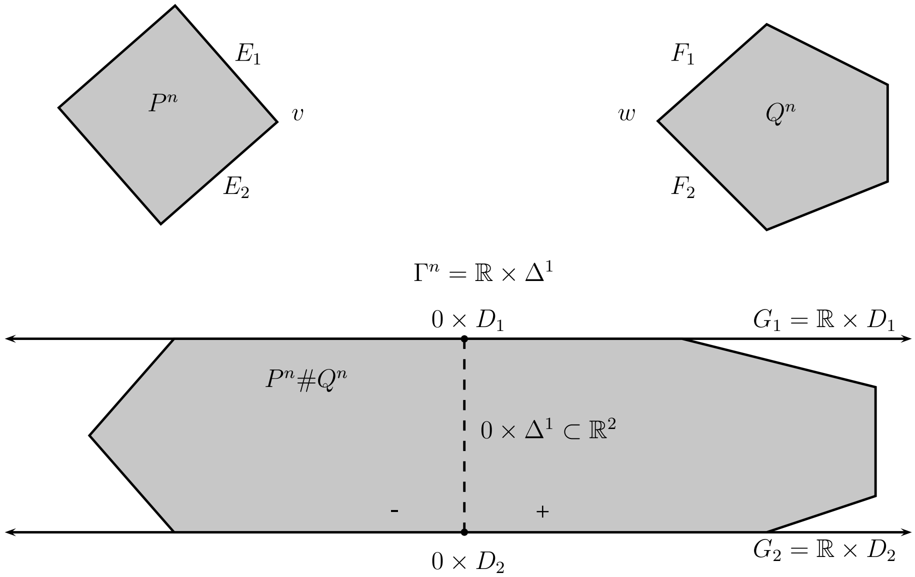


Figure 1.1: The Connected Sum of Simple Polytopes

v and similarly for Q^n). We then define a projective transformation φ_P mapping v to $x_1 = +\infty$, and embedding P^n in Γ^n such that

1. *the hyperplane defining E_r is identified with the hyperplane G_r for each $1 \leq r \leq n$ and*

2. *that the images of hyperplanes defining C_v meet in the negative half plane.*

We define a projective transformation φ_Q similarly identifying defining hyperplanes for F_r with those for G_r and insisting that the images of hyperplanes defining C_w meet within the positive half plane. We define the connected sum $P^n \#_{v,w} Q^n$ of P^n at v and Q^n at w *to be the simple convex n-polytope determined by the images of the hyperplanes defining C_v and C_w as well as those defining G_r for each $1 \leq r \leq n$. This we define only up to combinatorial equivalence. We simply write $P^n \# Q^n$ whenever our choice of vertices is clear or irrelevant. Figure 1.1 demonstrates the (formal) connected sum of the combinatorial square with the combinatorial pentagon.*

1.2 Quasitoric Manifolds

Let $T^n := (S^1)^n$ be the compact n-dimensional torus, and let T_i denote the i^{th} coordinate sub-torus. We refer to the representation of T^n by diagonal matrices in

$U(n)$ as the *standard* action on $\mathbb{C}^n$ [3]. This action $T^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ may be viewed in terms of coordinate multiplication as

$$(t_1, \ldots, t_n) \times (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$$

where each $t_k = e^{2\pi i s_k}$ is a unit element in $\mathbb{C}$. Such an action leaves fixed the magnitude of any element $z \in \mathbb{C}^n$ and so the orbit space is the positive cone ${\mathbb{R}_+}^n$.

Let M^{2n} be a smooth $2n$-dimensional manifold, then an action of T^n on M^{2n} is said to be *locally standard* if it is locally homeomorphic to this standard action [3]. Let P^n be a combinatorial simple polytope.

Definition 1.2.1. *A smooth manifold M^{2n} equipped an action of the n-torus is said to be a* Quasitoric Manifold over P^n *provided*

1. *the action is locally standard and*
2. *there is a projection $\pi : M^{2n} \rightarrow P^n$ constant over T^n-orbits for which each k-dimensional orbit maps to a point interior to a k-dimensional face of the quotient polytope P^n, for each integer k over $0 \leq k \leq n$.*

We will refer to the two conditions stated above from here on as *the Davis, Januszkiewicz conditions*. From this second condition we see immediately that the action of T^n must be fixed over the pre-images of the vertices of P^n and yet the action is free over the interior of the quotient polytope. Let the facets of T^n be labeled $F_1, F_2, \ldots, F_m$. Each set $\pi^{-1}(int(F_i))$ contain T^n-orbits each with the same isotropy subgroup.

Definition 1.2.2. *Let $T(F_i)$ denote the isotropy subgroup corresponding to F_i for each i, $0 \leq i \leq m$ and define $M_i^{2(n-1)} = \pi^{-1}(F_i)$ as the* facial sub-manifold of M^{2n} *corresponding to facet F_i.*

Each facial submanifold $M_i^{2(n-1)}$ is a $2(n-1)$-dimensional quasitoric manifold with respect to the action of the $(n-1)$-torus $\frac{T^n}{T(F_i)}$ and the corresponding one dimensional isotropy subgroups $T(F_i)$ may be written as

$$T(F_i) = \left(e^{2\pi i \lambda_{i1} \phi}, e^{2\pi i \lambda_{i2} \phi}, \ldots, e^{2\pi i \lambda_{in} \phi}\right) \tag{1.1}$$

with the help of primitive vectors $\lambda_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})^T \in \mathbb{Z}^n$. These *facet vectors* λ_i are determined uniquely up a choice of sign which we make arbitrarily in most cases.

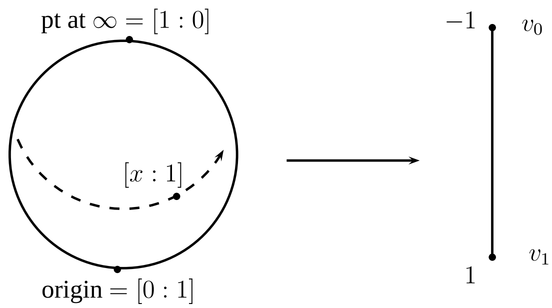


Figure 1.2: $\mathbb{C}P^1$

Example 1.2.3. *We begin with the simplest of quasitoric manifolds, two (real) dimensional complex projective space, $\mathbb{C}P^1$. First we invoke a coordinate description of the total space. Using equivalence classes for sets of ordered pairs of complex values we define*

$$\mathbb{C}P^1 = \{[x:y] | x, y \in \mathbb{C} \text{ with } x \text{ and } y \text{ not both zero}\} = \frac{(\mathbb{C}^2)^*}{\sim}$$

where the relation $\sim$ yields $[x:y] = [\zeta x : \zeta y]$ for any non-zero $\zeta \in \mathbb{C}$.

Such *homogeneous coordinates* are often used to describe projective spaces. With such coordinates in mind if we assume for the moment that $y \neq 0$ then we may write any such coordinate as the pair $[z, 1]$ where $z = \frac{x}{y}$. Then the only other coordinate left among such pairs is $[x, 0]$ which may be written $[1, 0]$. This allows us to view $\mathbb{C}P^1$ as the one point compactification of the complex plane or in simpler terms the two sphere. The value $[1, 0]$ shown as the "north pole" in Figure 1.2 is the ideal point at infinity while $[0, 1]$ the "south pole" corresponds to the origin.

We now describe $\mathbb{C}P^1$ as a quasitoric manifold under the standard action of the compact torus. For any unit norm element of the complex plane α we define an action of alpha on $\mathbb{C}P^1$ via $\alpha \cdot [x:y] = [\alpha x : y]$. This is a locally standard action of the one torus as it is locally isomorphic to the standard action of the circle on the complex plane given by such products $e^{2\pi i t} z$. If we consider $\alpha = e^{2\pi i t}$ then for any "nonpolar" element we achieve a free circle action $\alpha \cdot [z:1] = [e^{2\pi i t} z : 1]$ winding these element around the sphere longitudinally as shown in Figure 1.2.

The action is fixed however at the poles since

$$\alpha \cdot [0:1] = [e^{2\pi i t} 0 : 1] = [0:1] \text{ and}$$
$$\alpha \cdot [1:0] = [e^{2\pi i t} 1 : 0] = [e^{2\pi i t} : 0] = [1:0]$$

It is clear then topologically that the quotient of the sphere $\mathbb{C}P^1$ by this circle action is the line segment Δ^1 wherein the endpoints are the images of our fixed points $[0:1]$ and $[1:0]$. We have then the requisite projective map π mapping down to a simple polytope Δ^1 defined in such a way that the projection is constant over the orbits. Further since the map is free over the interior and fixed at the endpoints Davis and Januszkiewicz's second condition holds as well.

As they are fixed points the isotropy subgroup corresponding to $\pi^{-1}(v_i)$ is the entire circle for $i = 0, 1$. So indeed we may select primitive vectors -1 and 1 as the generators for the isotropy subgroups of $\pi^{-1}(v_0)$ and $\pi^{-1}(v_1)$, respectively. By convention this choice of facet vectors corresponds to the inner normal vectors for a geometric realization of the polytope as shown in Figure 1.2.

Example 1.2.4. *Proceeding now with $\mathbb{C}P^2$ we determine the total space and projective map similarly. Close attention is paid to the role of facial submanifolds which are themselves copies of $\mathbb{C}P^1$. Once again relying on homogeneous coordinates we define*

$$\mathbb{C}P^2 = \{[x:y:z] | x, y, z \in \mathbb{C} \text{ with } x,\ y \text{ and } z \text{ not simultaneously zero}\} = \frac{(\mathbb{C}^3)^*}{\sim}$$

wherein the equivalence classes $[x:y:z]$ are distinct only up to scalar multiples by any element of $\mathbb{C}$. Next, for any pair of unit circle elements of the complex plain (α, β) we achieve an action of the compact two-torus on $\mathbb{C}P^2$ by defining

$$(\alpha, \beta) \cdot [x:y:z] = [\alpha x : \beta y : z].$$

By a similar argument to Example 1.2.3, we have fixed points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. These last two fixed points are contained in the set $Z_1 = \{[0:y:1]\}$ for which we see the action $(\alpha, \beta) \cdot [0:y:1]$ must be trivial in the first coordinate. This yields a one dimensional isotropy subgroup. We often choose to view $\mathbb{C}P^2$ and other such smooth projective toric varieties via geometric representations as shown in Figure 1.3. Here we highlight the facet vectors as the inward facing normals to the three edges (facets) of the 2-simplex.

A Combinatorial Formulation of Quasitoric Manifolds

Quasitoric manifolds have been defined so far in terms of topological data given by way of a projective map and a suitable action of the n-Torus yielding a simple polytope as its quotient space. There is an equivalent description in terms of *combinatorial data* wherein a simple polytope P^n with m facets is paired with an $n \times m$ integral matrix Λ for which each column of Λ is associated a particular facet of P^n.

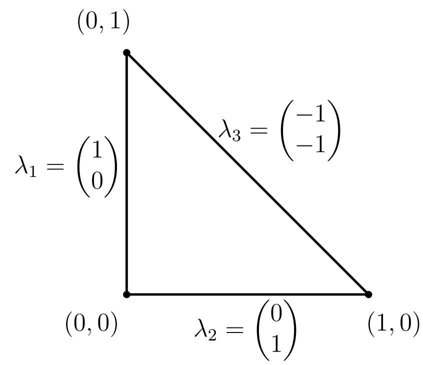


Figure 1.3: $\mathbb{C}P^2$

Such a pairing defines a quasitoric manifold, written $M^{2n} := (P^n, \Lambda)$ provided Λ satisfies a matrix minor condition which will be shown to be equivalent to the Davis, Januszkiewicz conditions described above. First we define Λ_v to be the $n \times n$ minor of Λ obtained by including only those columns of λ associated to facets incident to a vertex $v \in P^n$.

Theorem 1.2.5 (Buchstaber, Panov [3])**.** *Let P^n be a simple polytope with m facets and Λ an $n * m$ characteristic matrix then*

$$M^{2n} := \frac{T^n \times P^n}{\sim}$$

is a quasitoric manifold provided we have for each vertex $v \in P^n$

$$\det \Lambda_v = \pm 1.$$

These *characteristic pairs* $M^{2n} := (P^n, \Lambda)$ are in fact, in bijective correspondence with ψ-equivariant equivalence classes of quasitoric manifolds, where ψ is any automorphism of the n-torus [3].

Further, given any quasitoric manifold M^{2n} over a simple polytope P^n we may choose a preferred facet ordering and an *initial* vertex of the polytope so that the characteristic matrix Λ is of the form:

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda_{1,n+1} & \lambda_{1,n+2} & \cdots & \lambda_{1,m} \\ 0 & 1 & & 0 & \lambda_{2,n+1} & \lambda_{2,n+2} & \cdots & \lambda_{2,m} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_{n,n+1} & \lambda_{n,n+2} & \cdots & \lambda_{n,m} \end{pmatrix} \qquad (1.2)$$

insisting that the facets intersecting to form this initial vertex correspond to standard basis vectors for facet vectors λ_1 through λ_n. The sub-matrix $S = (\lambda_{i,j})$ for $j = n+1, \ldots, m$ will be referred to as the *reduced characteristic submatrix* of Λ.

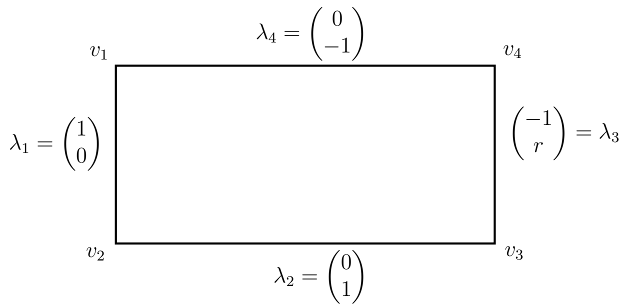


Figure 1.4: The Hirzebruch Surface

Example 1.2.6. *The Hirzebruch Surface is a four dimensional quasitoric manifold* $\mathbb{H}_r = (\mathbb{I}^2, \Lambda)$ *for any* $r \in \mathbb{Z}$ *where* $\mathbb{I}^2$ *is a combinatorial square and*

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & r & -1 \end{pmatrix} \tag{1.3}$$

Here the first and third columns of Λ *correspond to opposing sides as do the second and fourth, as is shown in Figure 1.4.*

Connected Sums of Quasitoric Manifolds

A procedure for taking the connected sum of quasitoric manifolds is described by Buchstaber and Panov in [5] and extended by Buchstaber, Panov and Ray in [4].

Definition 1.2.7. *Given quasitoric manifolds* $M_1^n = (P_1^n, \Lambda_1)$ *and* $M_2^n = (P_2^n, \Lambda_2)$ *we define the* connected sum $M_1^n \# M_2^n$ *to be the quasitoric manifold corresponding to the characteristic pair* $(P_1^n \# P_2^n, \Lambda)$ *where* Λ *is obtained from those facet vectors of* Λ_1 *and* Λ_2 *remaining under the connected sum procedure applied to the polytopes.*

We now illustrate the procedure with the following.

Example 1.2.8. *The connected sum* $\mathbb{C}P^2 \# \mathbb{C}P^2 = (\Delta^2 \# \Delta^2, \Lambda)$ *is shown in Figure 1.5.*

This particular example may now be used to demonstrate the realization of a refined characteristic submatrix given any quasitoric manifold. In the connected sum shown in Figure 1.5 we have the characteristic matrix

$$\Lambda = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

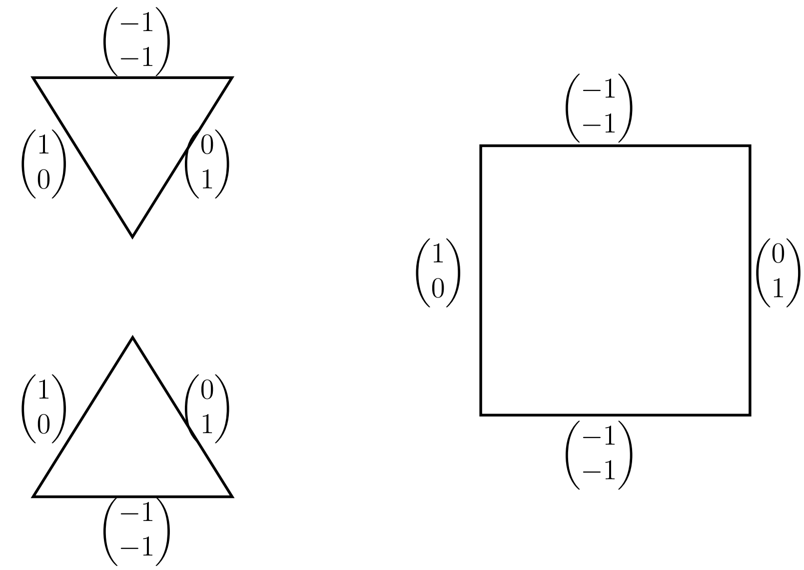


Figure 1.5: $\mathbb{C}P^2\#\mathbb{C}P^2$

where no selection of intersecting facets correspond to the standard basis vectors for $\mathbb{R}^2$. However if we select an initial vertex say v as shown in Figure 1.6 we may affect a change of basis fixing vector $(1,0)^T$ swapping vectors $(-1,-1)^T$ with $(0,1)^T$ to achieve the characteristic matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

which is now of the form shown by Equation 1.2.

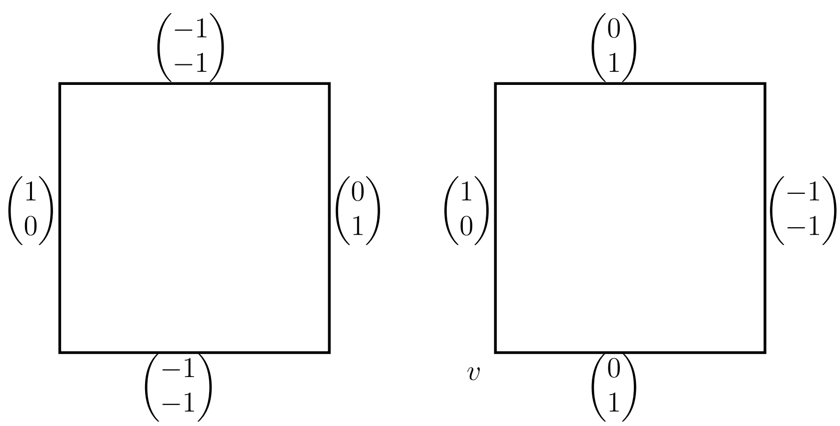


Figure 1.6: Refinement of $\mathbb{C}P^2\#\mathbb{C}P^2$

1.3 Bott Towers and Bott Manifolds

Definition 1.3.1 (Masuda, Panov 2007 [16])**.** *A* Bott Tower of height n *is a sequence of manifolds* $\left(B^{2k} : k \le n\right)$ *such that* $B^2 = \mathbb{C}P^1$ *and* $B^{2k} = P\left(\underline{\mathbb{C}} \oplus \xi_{k-1}\right)$ *for* $1 < k \le n$ *where*

1. *$P(\cdot)$ denotes complex projectivization*
2. *ξ_{k-1} is a complex line bundle over $B^{2(k-1)}$ and*
3. *$\underline{\mathbb{C}}$ is a trivial line bundle.*

In particular we have bundle(s) $B^{2n} \to B^{2(n-1)} \to \cdots \to B^4 \to \mathbb{C}P^1 \to pt$ with fibre $\mathbb{C}P^1$.

Remark. *Consider again the Hirzebruch Surface $\mathbb{H}_r \to \mathbb{C}P^1 \to pt$*

Theorem 1.3.2 (Masuda, Panov 2007 [16])**.** *Any Bott Tower is representable as $M^{2n} = (\mathbb{I}^n, \Lambda)$, a quasitoric manifold over a cube where the characteristic matrix is of the form:*

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \lambda_{1,2} & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \lambda_{1,3} & \lambda_{2,3} & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 & \vdots & & \ddots & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda_{1,2n} & \lambda_{2,2n} & \cdots & \lambda_{n-1,2n} & -1 \end{pmatrix}$$

Here we may insist that the refined characteristic submatrix be lower triangular with values -1 along the main diagonal.

Generalized Bott Towers

We now extend our interest to a larger class of manifolds termed *Generalized Bott Manifolds.*

Definition 1.3.3 ([16], [8])**.** *A* generalized Bott Tower *is a sequence of manifolds* $\left(B^{2k} : k \le n\right)$ *such that* $B^2 = \mathbb{C}P^1$ *and* $B^{2k} = P\left(\underline{\mathbb{C}} \oplus \xi_{k-1}\right)$ *for* $1 < k \le n$ *where*

1. *$P(\cdot)$ denotes complex projectivization*
2. *$\xi_{k-1}^{n_i}$ is the Whitney sum of n_i complex line bundles over $B^{2(k-1)}$ and*

3. $\underline{\mathbb{C}}$ *is a trivial line bundle.*

Here we have bundle(s) $B^{2n} \to B^{2(n-1)} \to \cdots \to B^4 \to \mathbb{C}P^1 \to pt$ with fibre(s) $\mathbb{C}P^{n_i}$.

1.4 The Wedge Polytope Construction

Now given any quasitoric manifold M^{2n} we wish to construct an ambient quasitoric manifold M^{2n+2} so that the original is contained as a codimension 2 sub-quasitoric manifold. For this relationship to hold we must first formulate a relationship among the base polytopes so that the polytope corresponding to the sub-manifold appears as a facet (codimension one face) of the larger polytope. The absolute minimal way in which this could occur would be with the cone over the original polytope. Consider again Examples 1.2.3 and 1.2.4. Such a construction however does not always preserve the property of being a simple polytope which is a necessary condition for the existence of a quasitoric manifold. A minimal procedure for developing an ambient polytope which is still combinatorially simple is given by the wedge polytope construction.

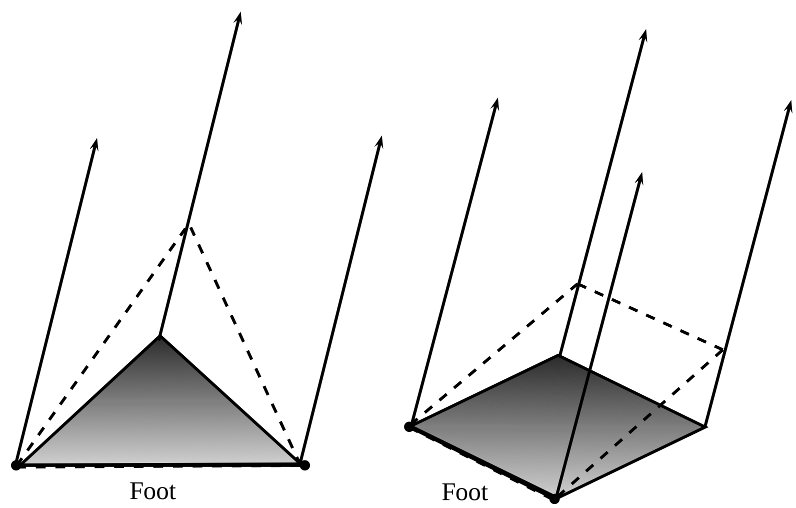


Figure 1.7: Wedge Over a Combinatorial Triangle and Square

Wedging a Polytope

Construction 1.4.1 (Klee and Walkup, [12])**.** *Let P^n be an n-dimensional polytope with m facets and let f be a facet. The* wedge over P with foot f, *written P_f^w is* *the $(n+1)$-dimensional space given by the intersection of a cylinder over P call it $(C = P \times [0, \infty))$ with a half-space H given by a defining hyperplane so that*

1. *$H \cap P = f$ and*

2. *H intersects the interior of C*

The wedge over any polytope may be taken in a number of equivalent ways. The wedge may be performed by taking the cone over a polytope and cutting a hyperplane through cone intersecting one particular facet of the base, in this case the foot. Then one selects an intersection with the half-space containing the base of the cone as shown in Figure 1.8. Using the cone in this manner is equivalent (topologically) to Construction 1.4.1 which utilizes a product with $[0, \infty)$. Think of the apex as being an ideal point at infinity. Also, somewhat less formally we may take the wedge by

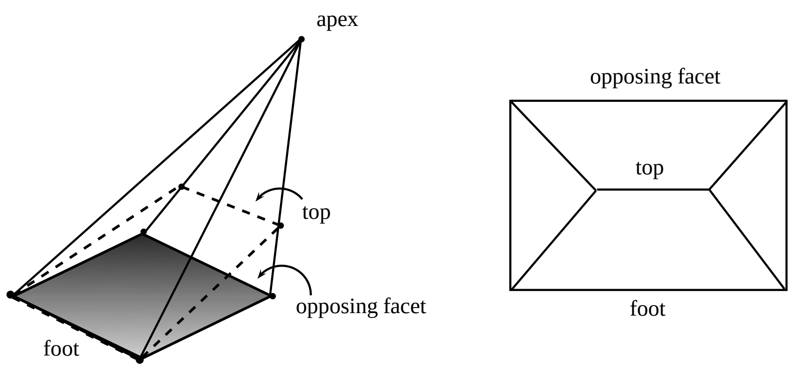


Figure 1.8: An Equivalent Formulation for the Wedge

first forming the cone over the polytope and then perturbing one of the defining hyperplanes to force all of them into general position. When it is clear or irrelevant what facet over which the wedge is taken we will refer to the wedge over P^{2n} with foot f as P^w. In Figure 1.13 the wedge has been taken over the "bottom" facet of the three cube. All four vertices not incident to this bottom face are doubled in the resulting wedge. This is then a wedge polytope with $2 \times (8-4) + 4 = 12$ vertices.

Remark. *The wedge Construction 1.4.1 is in fact a* minimal *(in terms of the number of facets of the resulting polytope) procedure for producing a* simple *polytope which contains the given base polytope as a facet. This construction will always provide exactly facet (that which corresponds to the base). Several examples are illustrated by Figure 1.9.*

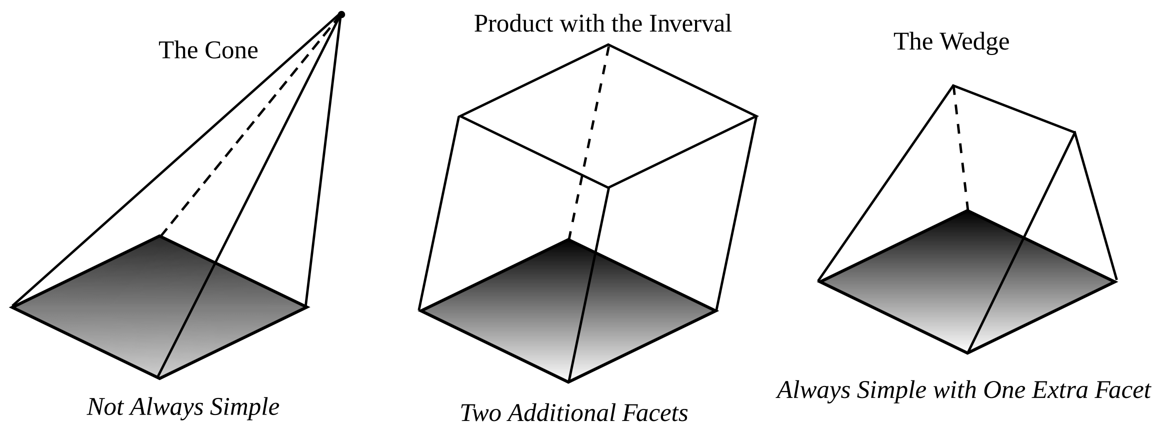


Figure 1.9: Several Polytopes Containing the Combinatorial Square as a Facet.

Theorem 1.4.2 (Klee and Walkup, [12])**.** *The wedge over P^n with foot f is an $(n+1)$-dimensional Polytope with $m+1$ facets. When P^n is a simple polytope, P^w will be a simple polytope with vertices equaling*

$$2 \times (n[P^n] - n[f]) + n[f]$$

in number, thus doubling the number of vertices not incident to foot f.

Consider now the n-simplex Δ^n and notice that any given facet F of Δ^n contains all vertices save one. Select any facet F as the foot of a wedge and consider the opposing vertex. This opposing vertex is duplicated by the wedge construction yielding the cone over n-simplex and the following corollary.

Corollary 1.4.3. *The wedge over any simplex is the next higher dimensional simplex i.e.*

$$(\Delta^n)^w = \left(\Delta^{n+1}\right).$$

Example 1.4.4. *As we see in Figure 1.10, when one forms the wedge over Δ^1 choosing v_1 as the foot the opposing vertex v_0 is duplicated by the process which we then may label v_2. Forming the wedge over Δ^2 we may choose the foot to be the facet formed by $v_1 v_2$ and once again duplicating the initial vertex v_0. We may continue this process ad infinitum verifying Corollary 1.4.3.*

Combinatorially equivalent copies of the polytope P itself appear as two facets within the wedge. Both as the "base" $P \times 0$, which as a facet of P^w we will refer to

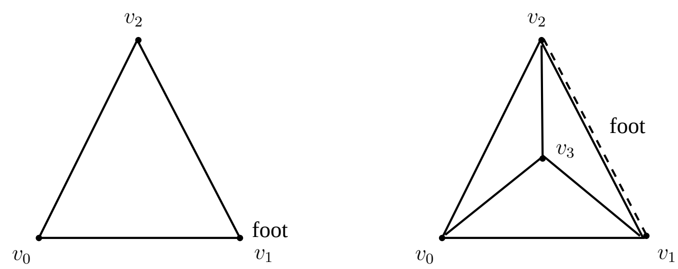


Figure 1.10: Wedges over Δ^1, and Δ^2

simply as F_0, and the facet corresponding to the defining hyperplane T. We will refer to it as the "top" and frequently it will appear in the wedge as facet F_n within our constructions to follow. Another aspect to notice for the wedge construction regards a relationship between facets of the underlying polytope and those of the wedge.

Remark. *Each facet in the base polytope corresponds exactly to a particular lateral facet of the wedge polytope. Specifically if f is a facet of P then F is the lateral facet of P^w corresponding to f exactly when $F \cap P = f$*

Wedges over cubes

As all Bott Towers are Quasitoric Manifolds over an n-cubes, it is necessary to first consider the wedge polytope over any cube. Forming the cone over the n-cube adds a vertex a. When we wish to form wedge over the cube we select a facet (often corresponding to the first column of a certain characteristic matrix) and perturb it forcing all of the facets in the cone over the n-cube into general position. This shatters the apex into 2^{n-1} "top" vertices and forms the wedge over the n-cube.

Example 1.4.5. *As a primary example we now revisit the wedge over the combinatorial square $\mathbb{I}^2$. We label the vertices of the base polytope: $00, 01, 10,$ and 11. Facets of the base polytope follow the convention that edges such as f_{1*} contain vertices 10 and 11 as shown, the star indicating that it contains vertices of the polytope with either a one or zero in the second slot. Choosing f_{1*} as the foot we note its opposite f_{0*} and form the wedge.*

As indicated in Theorem 1.4.2, vertices not incident to the foot f_{1*} are duplicated by the wedge procedure. As is the case in all n-cubes, each vertex is contained in either a facet or its opposite, hence all of the vertices in f_{0*} here are duplicated and labeled as the top vertices 20 and 21 as pictured. We now consider the opposing

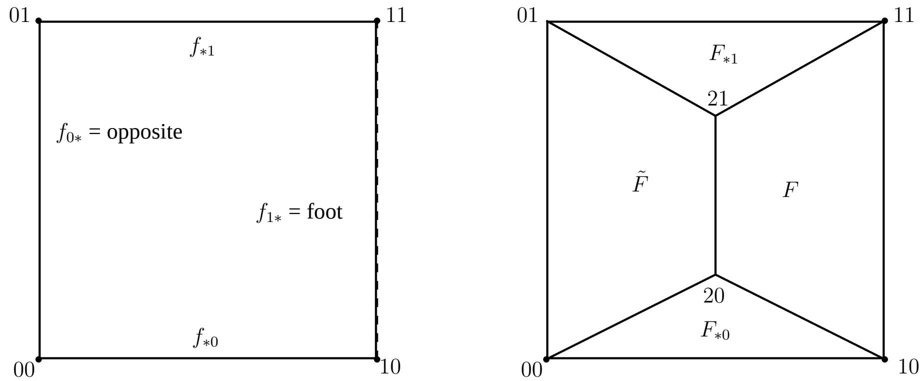


Figure 1.11: Wedge over the square

lateral facets F and $\tilde{F}$ corresponding respectively to the lateral facet over the foot f_{1*} and its opposite f_{0*} from the base polytope. Clearly the lateral facets F and $\tilde{F}$ each contain these "top" vertices. Lastly, we note the opposing lateral facets F_{*0} and F_{*1} precisely determine these top vertices via the following intersections

$$20 = F_{*0} \cap F \cap \tilde{F}$$
$$21 = F_{*1} \cap F \cap \tilde{F}.$$

The following property of wedges over the n-cube will allow us to form a wedge quasitoric manifold over any Bott tower.

Lemma 1.4.6. *Each of the 2^{n-1} top vertices in the wedge over an n-cube is contained in the lateral facet corresponding to the foot and the lateral facet over its opposite. These top vertices are the intersection of these particular lateral facets F and $\tilde{F}$ respectively with $n-1$ other non-opposing lateral facets.*

Let us consider for a moment lateral facet F_{*0} from Figure 1.12. We note that the top vertex is (exactly in this case) the intersection of the lateral facets over the foot and its opposing lateral facet. The same situation holds for facet F_{*1} or indeed any particular cross-section between the two.

Viewing Figure 1.12 as a generic cross-section of the wedge over the square and comparing with $(\Delta^1)^w = \Delta^2$ (see Figure 1.10) demonstrates an equivalence between the wedge over the square and the product of the interval with the wedge over the interval. Specifically, we see the combinatorial equivalence:

$$\left(\mathbb{I}^2\right)^w = \left(\mathbb{I}^1\right)^w \times \mathbb{I} = \Delta^2 \times \mathbb{I}. \tag{1.4}$$

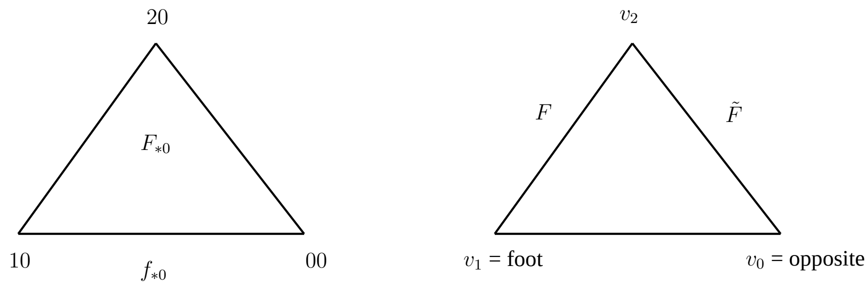


Figure 1.12: Wedge Over the Interval

Remark. *It is worth mentioning at this point that an alternate choice for the foot (say f_{*1}) of the wedge would have given:*

$$\left(\mathbb{I}^2\right)^w = \mathbb{I} \times \left(\mathbb{I}^1\right)^w = \mathbb{I} \times \Delta^2 \tag{1.5}$$

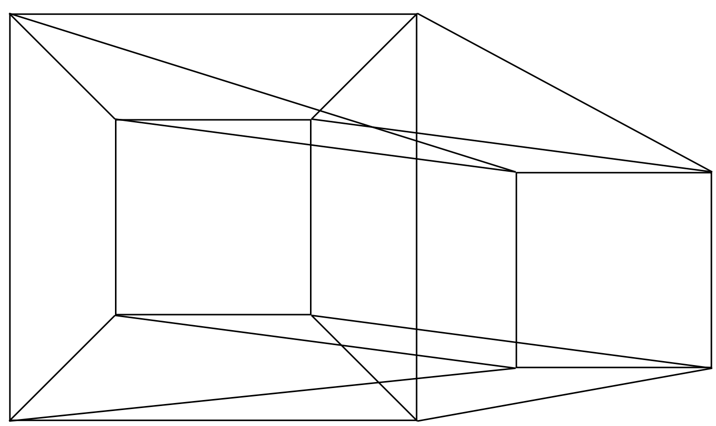


Figure 1.13: Wedge Over a Three Cube

Example 1.4.7. *As a final example we exhibit the wedge over the three cube shown in Figure 1.13 and carefully labeled in Figure 1.14. Here the foot is the bottom facet of the three cube f_{1**}. The vertices contained in its opposite facet f_{0**} have been duplicated and are shown to the right in the figure.*

The triangle given by vertices 000, 100 and 200 encapsulates the wedge operation for the whole. We see that wedge over the one cube (or if one prefers the interval) giving us Δ^2 but crossed with the interval in two other dimensions. This indeed indicates a general fact regarding the wedge over cubes, that is:

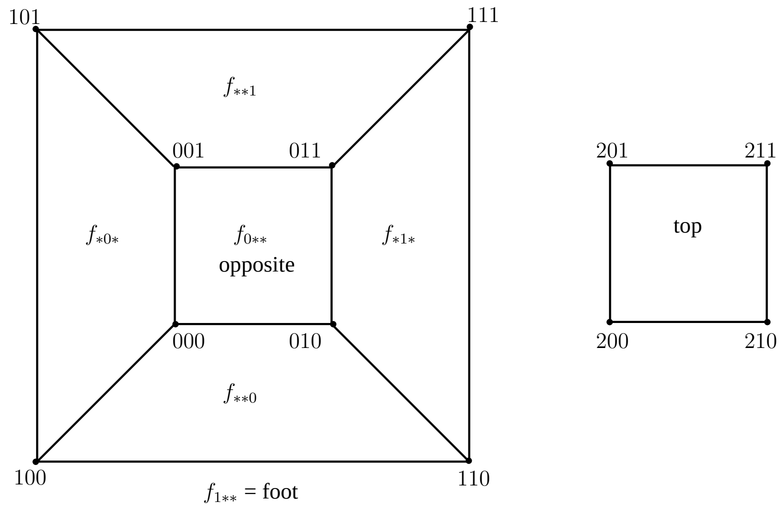


Figure 1.14: Three Cube Wedge Diagram

Lemma 1.4.8. *The wedge over an n-cube is "half" an $(n+1)$-cube:*

$$(\mathbb{I}^n)^w = \mathbb{I}^w \times \mathbb{I}^{n-1} = \Delta^2 \times \mathbb{I}^{n-1}.$$

This brings us to the proof of Lemma 1.4.6.

Proof. Let $\mathbb{I}^n$ be an n-cube. We label its vertices as the n-tuples $0\cdots0$, through $1\cdots1$. Then we take the wedge canonically choosing the foot to be facet $f_{1*\cdots*}$, forcing its opposite to be $f_{0*\cdots*}$ and duplicating all 2^{n-1} vertices that begin with a zero, and labeling them $20\cdots0$, through $21\cdots1$. It is then clear that each of these top vertices is contained in the lateral facet F over the foot $f_{1*\cdots*}$ and its opposite $\tilde{F}$ over $f_{0*\cdots*}$.

We see in fact, as indicated in our previous example, the triangle given by vertices: $0\cdots0$, $10\cdots0$, and $20\cdots0$ scaled out by the interval in 2^{n-1} independent directions. Indeed we achieve a copy of Δ^2 for each triple $0a_2\cdots a_n$, $1a_2\cdots a_n$, and $2a_2\cdots a_n$. This proves our lemma and justifies the equation given by Lemma 1.4.8. ☐

The Wedge over a Product of Simplices

The Bott Towers previously described are generalized by Bott manifolds. Bott manifolds are quasitoric manifolds with some additional structure concerning the characteristic matrix. While Bott towers are QTMs over cubes Bott manifolds are QTMs

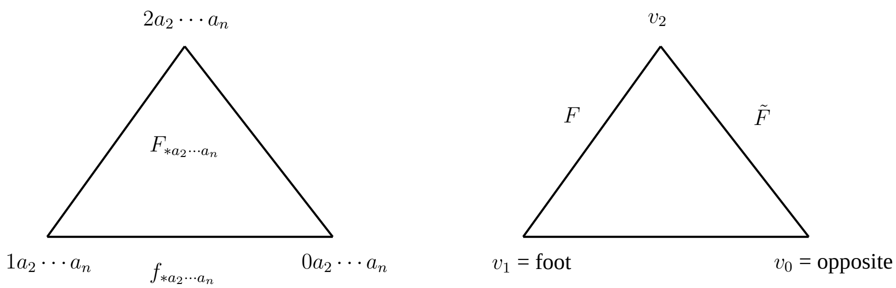


Figure 1.15: Wedge over the n-cube

over products of simplices as shown by Choi, Masuda and Suh [7]. We will define a Bott manifold over a product of simplices with characteristic matrix Λ as $M = (\Pi_{i=1}^{m} \Delta^{n_i}, \Lambda)$. In the interest of creating wedge quasitoric manifolds over any Bott manifold, we must first discuss the wedge over such a product of simplices $P = \Pi_{i=1}^{m} \Delta^{n_i}$.

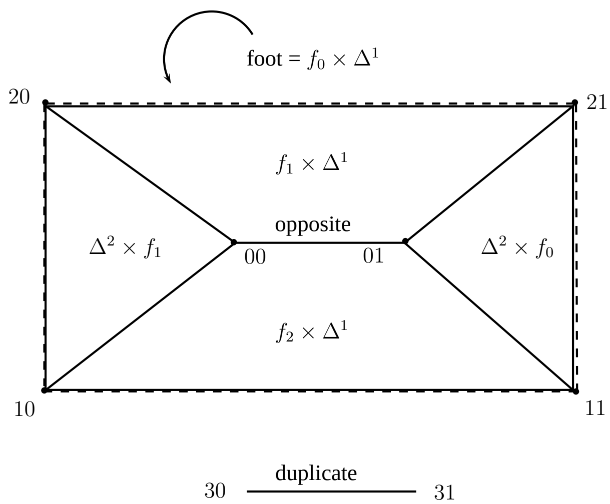


Figure 1.16: A Wedge over $\Delta^2 \times \Delta^1$

Example 1.4.9. *Consider first the smallest nontrivial case* $P = \Delta^2 \times \Delta^1$*. This is equivalent to, if you will recall, the wedge over the combinatorial square.* P^w *is itself not well defined without a particular choice for the foot. In Figure 1.16 the foot* $f_0 \times \Delta^1$ *is chosen forcing the two vertices opposing this to be duplicated. Consider now the full skeleton depicting the wedge over* $\Delta^2 \times \Delta^1$ *with foot* $f_0 \times \Delta^1$ *shown in*

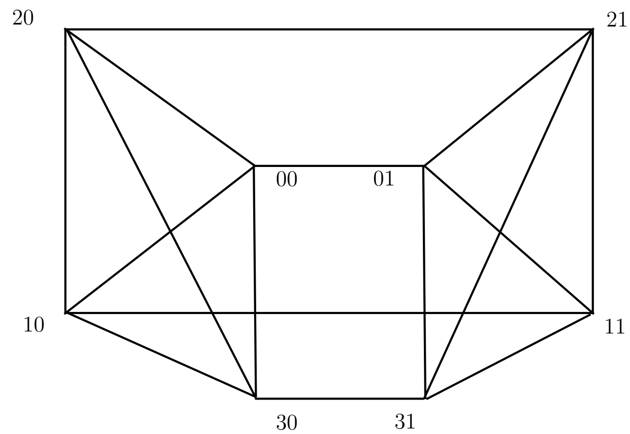


Figure 1.17: $(\Delta^2 \times \Delta^1)^w_{f_0 \times \Delta^2} = \Delta^3 \times \Delta^1$

Figure 1.17. We see the complete graph on four vertices crossed with the interval. This would then indicate that the wedge is indeed $\Delta^3 \times \Delta^1$.

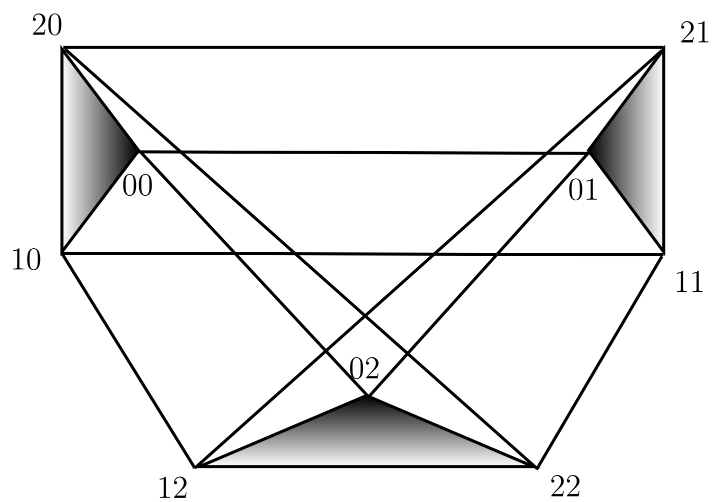


Figure 1.18: $(\Delta^2 \times \Delta^1)^w_{\Delta^2 \times 0} = \Delta^2 \times \Delta^2$

Example 1.4.10. *We may however choose an entirely different facet for the foot of the wedge. Consider instead the wedge over* $\Delta^2 \times \Delta^1$ *with foot* $\Delta^2 \times f_0$ *as shown in Figure 1.18. An entirely different wedge polytope is then achieved. Here the foot is* *a copy of* Δ^2 *as is the facet opposing it. Hence three vertices are duplicated by the wedge process and the vertices of Figure 1.18 have been labeled accordingly.*

We will make the canonical choice for the foot of the wedge over a product of simplices $\Pi_{i=1}^m \Delta^{n_i}$ to be $f_0 \times \Pi_{i=2}^m \Delta^{n_i}$ where f_0 here indicates the facet opposing vertex 0 in Δ^{n_1}. And so unless otherwise indicated

$$\left(\prod_{i=1}^m \Delta^{n_i}\right)^w = \left(\prod_{i=1}^m \Delta^{n_i}\right)^w_{f_0 \times \Pi_{i=2}^m \Delta^{n_i}}$$

With this particular choice in mind we offer the following.

Theorem 1.4.11. *The wedge over a polytope which is a product of simplicial complexes is itself a product of simplicial complexes. Specifically,*

$$\left(\prod_{i=1}^m \Delta^{n_i}\right)^w = \Delta^{n_1+1} \times \left(\prod_{i=2}^m \Delta^{n_i}\right).$$

Chapter 2 Wedge Quasitoric Manifolds

In this chapter we prove the existence of wedge quasitoric manifolds given specific requirements on the characteristic matrix in Theorem 2.3.2. We construct an alternate proof to that given by Choi and Park in [6] utilizing polytopal constructions rather than simplicial complexes. In Section 2.3 we use Theorems 2.3.2 and 2.3.6 to show Corollary 2.3.7 which states that "the connected sum of wedge quasitoric manifolds is a wedge quasitoric manifold." In the conclusion we use Theorem 2.3.2 and Theorem 2.4.1 to show Corollary 2.4.4 which states that "there exists a wedge quasitoric manifold over any Bott manifold (or tower) which is itself a Bott manifold."

2.1 Quasitoric Manifolds and the Wedge

An Algorithmic Approach

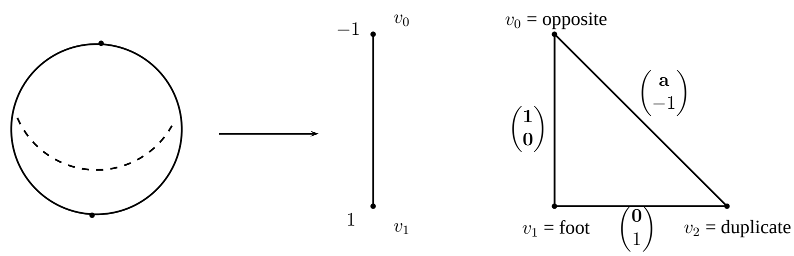


Figure 2.1: Wedge Over $\mathbb{C}P^1$

Example 2.1.1. *Beginning again with the simplest example of a quasitoric manifold over a line segment, we reconsider $\mathbb{C}P^1$ as first shown in Example 1.2.3. We develop now an approach to view it as a codimension two submanifold of an ambient QTM. We have $M^2 = \mathbb{C}P^1$ given by the one simplex Δ^1 and characteristic matrix $\Lambda = \begin{bmatrix} 1 & -1 \end{bmatrix}$. We make the canonical choice for the foot of the wedge to be v_1 (opposite vertex zero), and corresponding to the facet vector in the initial column of the characteristic matrix, in this case column vector (1). We now note that the wedge over Δ^1 is the triangle displayed in Figure 2.1. The wedge construction from Section 1.4 provides a polytope P^w one dimension higher with one extra facet which itself corresponds to the original polytope P.*

We wish to view $\mathbb{C}P^1 = (\Delta, \Lambda)$ as a codimension two facial sub-manifold of an ambient quasitoric manifold M^w. To this end we invoke a characteristic pair

(Δ^2, Λ^w) *where*

$$\Lambda^w = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & 1 & -1 \end{pmatrix} \tag{2.1}$$

These new entries shown in bold extend the original facet vectors in a reasonable way. Specifically here the initial column for the wedge characteristic matrix corresponds to the new facet introduced by the wedge construction. We will make the canonical choice that it be given the value $\mathbf{e_1}$ in $n+1$ dimensions. If such a quasitoric manifold exists we would be able to write it in refined form and thus we insist that this second facet here corresponding to the lateral facet over the foot be $\mathbf{e_2} = (0, 1)^T$.

Lastly, though this method of construction certainly may not be unique and may not in general provide a quasitoric manifold depending on our choice for this final column. There is in this particular example an obvious choice which will suffice, see Example 1.2.4. We currently provide a variable entry $\mathbf{a}$ in the column corresponding to the lateral facet above v_0.

In order to determine precisely when such selections may provide quasitoric manifolds we must apply Theorem 1.2.5 to this wedge characteristic matrix. Thus we must solve the system of equations:

$$\Lambda^w_{v_0} = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix} = \pm 1$$

$$\Lambda^w_{v_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pm 1$$

$$\Lambda^w_{v_2} = \begin{pmatrix} 0 & a \\ 1 & -1 \end{pmatrix} = \pm 1.$$

It is clear upon immediate inspection that we have already $\Lambda^w_{v_0} = \pm 1 = \Lambda^w_{v_1}$ owing to the fact that $\mathbb{C}P^1$ is itself a quasitoric manifold. This leaves only $\Lambda^w_{v_2} = \pm 1$ which is solved trivially by a particular choice $a = -1$. This choice of values gives us:

$$\Lambda^w = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

clearly forming $\mathbb{C}P^2$ as an ambient quasitoric manifold containing $\mathbb{C}P^1$.

Example 2.1.2. *We may similarly form the wedge quasitoric manifold over* $\mathbb{C}P^2 = (\Delta^3, \Lambda)$ *(see Example 1.2.4) by defining characteristic pair* (Δ^3, Λ^w) *where*

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{2.2}$$

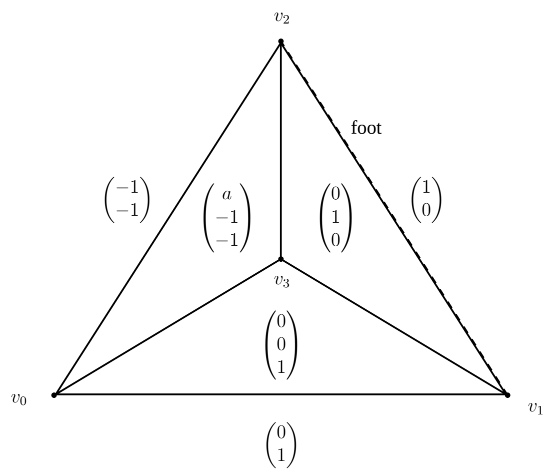


Figure 2.2: Wedge Over $\mathbb{C}P^2$

wherein the matrix minor condition holds immediately for all vertices save that which has been duplicated under the wedge construction $(\Delta^1) = \Delta^2$.

Again in this case we may make the selection $a = -1$ which would force

$$\Lambda^w_{v_3} = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \pm 1.$$

giving us $\mathbb{C}P^3$ as an ambient quasitoric manifold containing $\mathbb{C}P^2$.

It is in fact the case in general that the matrix minor condition from Theorem 1.2.5 will hold for those minors associated to vertices incident to the base of the wedge, the original polytope P^{2n}.

Wedge Quasitoric Manifolds

Given a quasitoric manifold, described as in Theorem 1.2.5, $M^{2n} = (P^{2n}, \Lambda)$ where P^{2n} is a simple polytope with m facets and Λ an $n * m$ characteristic matrix of facet vectors, we now define precisely what is meant by an ambient quasitoric manifold.

Definition 2.1.3. *A* wedge quasitoric manifold over M^{2n} *is any quasitoric manifold determined by the characteristic pair* $M^w = (P^w, \Lambda)$ *where* P^w *is a wedge over* P^{2n}

and Λ^w *is a corresponding characteristic matrix given by*

$$\Lambda^w = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_m} \\ \mathbf{0} & 1 & & 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,m} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \cdots & 0 & 1 & \lambda_{n,1} & \lambda_{n,2} & \cdots & \lambda_{n,m} \end{pmatrix}.$$

In particular here whenever λ_i is a facet vector for f_i in P^{2n}, for each $i = 1, \ldots, m$, we insist that $\lambda_i^w = (\mathbf{a_i}, \lambda_{1i}, \ldots, \lambda_{ni})^T$ is a facet vector corresponding to the lateral facet F_i of P^w. Lastly, we insist that the new initial column $(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})^T$ be assigned to the facet in P^w corresponding to the original polytope P^{2n} under the wedge construction. It is this last choice in particular that allows the following lemma.

Lemma 2.1.4. *For any wedge quasitoric manifold* M^w *over* $M^{2n} = (P^{2n}, \Lambda)$ *the matrix minor condition is satisfied for each* $v \in P^w$ *corresponding to a vertex from the original polytope* P^{2n}*.*

Proof. We defer the proof for this particular lemma for now. It is proved fully in case 1 of Theorem 2.3.2. ☐

Thus in order to generate a wedge quasitoric manifold over any QTM we need only verify that the matrix minor condition from Theorem 1.2.5 is satisfied for those $(n[P^n] - n[F_1])$ vertices which have been duplicated under the wedge construction.

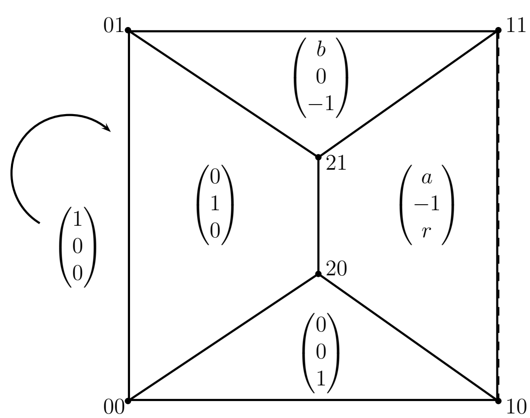


Figure 2.3: A Wedge QTM over $\mathbb{H}_r$

Example 2.1.5. *We now revisit the Hirzebruch Surface first introduced in Example 1.2.6. As previously discussed it is a quasitoric manifold given by the pair* $(\mathbb{I}^2, \Lambda)$ *where*

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & r & -1 \end{pmatrix}$$

In order to construct a wedge quasitoric manifold over $\mathbb{H}_r$ *we note* $(\mathbb{I}^2)^w = \Delta^2 \times \mathbb{I}$ *and define the characteristic pair* $(\Delta^2 \times \mathbb{I}, \Lambda^w)$ *where*

$$\Lambda^w = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{a} & \mathbf{b} \\ \mathbf{0} & 1 & 0 & -1 & 0 \\ \mathbf{0} & 0 & 1 & r & -1 \end{pmatrix}$$

The wedge over a combinatorial square $\Delta^2 \times \mathbb{I}$ shown in Figure 1.7, introduces $2 = (n[P^n] - n[F])$ new vertices namely 20, and 21. Any wedge quasitoric manifold over $\mathbb{H}_r$ must satisfy the following matrix minor restrictions:

$$\Lambda^w_{v_{20}} = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & -1 \\ 0 & 1 & r \end{pmatrix} = \pm 1$$
$$\Lambda^w_{v_{21}} = \begin{pmatrix} 0 & a & b \\ 1 & -1 & 0 \\ 0 & r & -1 \end{pmatrix} = \pm 1.$$

These restrictions immediately result in the equations

$$a = \pm 1 = -a - br \tag{2.3}$$

which are solved by

$$a = -1$$
$$b = 0.$$

Finally, setting

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & r & -1 \end{pmatrix}$$

yields a wedge quasitoric manifold $\mathbb{H}^w_r = (\Delta^2 \times \mathbb{I}, \Lambda^w)$ over $\mathbb{H}_r$.

Remark. *It is worth noting that the restrictions given from Equation 2.3 may in fact be solved by several sets of values. For instance Equation 2.3 may be solved by* $a = 1$ *and* $b = \frac{-2}{r}$ *or* $a = -1$ *and* $b = \frac{2}{r}$*, provided* r *is divisible by* 2*. See Dobrinskaya's work on the classification of quasitoric manifolds over a given polytope [9]. As will soon be apparent, in the case of Bott towers and manifolds a canonical choice may be made here which will always lead to the existence of a quasitoric manifold of a certain type.*

2.2 Wedge QTMs over Bott Towers

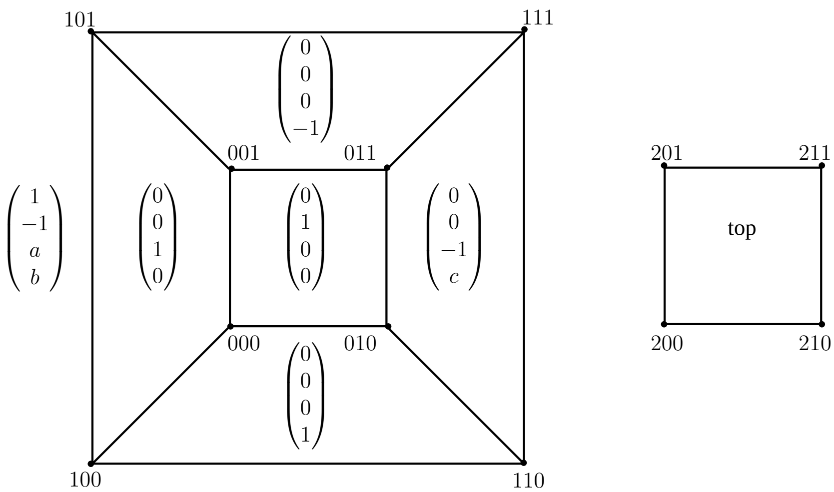


Figure 2.4: Wedge Over B_3

Example 2.2.1. *We begin with the wedge quasitoric manifold over a stage three Bott tower* $B_3 = (\mathbb{I}^3, \Lambda)$*. Where from Theorem 1.3.2, we may insist that* Λ *is of the form*

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & a & -1 & 0 \\ 0 & 0 & 1 & b & c & -1 \end{pmatrix}$$

Taking the wedge over the three cube as shown in Figure 1.14 we recall that $\mathbb{I}^3 =$

$\Delta^2 \times \mathbb{I}^2$, and so we define $B_3^w = (\Delta^2 \times \mathbb{I}^2, \Lambda^w)$ where

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & -1 & 0 \\ 0 & 0 & 0 & 1 & b & c & -1 \end{pmatrix}$$

One may then quickly verify the following matrix minors associated to these top vertices:

$$|\Lambda^w_{200}| = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{vmatrix} = \pm 1$$

$$|\Lambda^w_{201}| = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & b & -1 \end{vmatrix} = \pm 1$$

$$|\Lambda^w_{210}| = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & a & -1 \\ 0 & 1 & b & c \end{vmatrix} = \pm 1$$

$$|\Lambda^w_{211}| = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & a & -1 & 0 \\ 0 & b & c & -1 \end{vmatrix} = \pm 1$$

Hence this is indeed a quasitoric manifold by Theorem 1.2.5. Notice in particular that facet vectors $(1, -1, a, b)^T$ and $(0, 1, 0, 0)^T$ corresponding to the lateral facets over the foot and its opposite, respectively, appear in each of these minors.

Theorem 2.2.2. *There is a wedge quasitoric manifold over any Bott tower.*

Proof. Let M be a Bott Tower over an n-cube $\mathbb{I}^n$. The characteristic matrix may then be written in the form:

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \lambda_{1,2} & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \lambda_{1,3} & \lambda_{2,3} & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 & \vdots & & \ddots & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda_{1,2n} & \lambda_{2,2n} & \cdots & \lambda_{n-1,2n} & -1 \end{pmatrix}$$

Lemma 1.4.8 implies that the wedge over the n-cube is the simple polytope $\Delta^2 \times \mathbb{I}^{n-1}$. We now define the characteristic pair $(\Delta^2 \times \mathbb{I}^{n-1}, \Lambda^w)$ where we insist that wedge characteristic matrix is of the form

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & 0 & \lambda_{1,2} & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \lambda_{1,3} & \lambda_{2,3} & \ddots & \ddots & \vdots \\ 0 & & & & 0 & \vdots & & \ddots & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda_{1,2n} & \lambda_{2,2n} & \cdots & \lambda_{n-1,2n} & -1 \end{pmatrix}$$

Let us label the columns here $\mathbf{e_1}, \mathbf{e_2} \ldots, \mathbf{e_n}, \mathbf{e_{n+1}}$ followed by $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \ldots, \boldsymbol{\lambda_n}$. With the exception of the new initial column $\mathbf{e_1}$ which corresponds to the "bottom" face of the wedge (the original n-cube), each of the columns of Λ^w is paired by a 1 and -1 in the same dimension or row. These are namely $\mathbf{e_{i+1}}$ paired with $\boldsymbol{\lambda_i}$ for each $i = 1, 2, \ldots n$. Each of these columns correspond to a lateral facet that stems from opposing facet in the original n-cube. To show $(\Delta^2 \times \mathbb{I}^{n-1}, \Lambda^w)$ is indeed a quasitoric manifold, we must now verify that each of these new 2^{n-1} top vertices in the wedge n-cube satisfies the matrix minor condition of Theorem 1.2.5.

Let v be a top vertex of $(\Delta^2 \times \mathbb{I}^{n-1})$. From Lemma 1.4.6 we see that the lateral facet over the foot and the lateral facet above its opposite each contains v. In this case the corresponding facet vectors are $\mathbf{e}_2$ and λ_1, respectively. These columns then by definition must be included in the matrix minor associated to v. By Lemma 1.4.6 the remaining $n-1$ columns of the matrix minor must then include either of the associated pairs $\mathbf{e_{i+1}}$ or $\boldsymbol{\lambda_i}$ whose lateral facets form the intersection determining v.

The matrix minor of Λ_v^w associated to v will be of the form

$$\Lambda_v^w = \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_{1,2} & \pm 1 & 0 & \cdots & 0 \\ 0 & \lambda_{1,3} & \lambda & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \pm 1 & 0 \\ 0 & \lambda_{1,2n} & \lambda & \cdots & \lambda & \pm 1 \end{pmatrix} \tag{2.4}$$

Trivially then we see that:

$$|\Lambda_v^w| = \pm 1 \tag{2.5}$$

and hence $(\Delta^2 \times \mathbb{I}^{n-1})$ is an ambient quasitoric manifold containing the original Bott tower as a codimension 2 subquasitoric manifold. □

2.3 Existence of a Wedge QTM over any Quasitoric Manifold

We've shown that there exist wedge QTMs over Bott Towers and Manifolds but this may be proved in more generality. By way of demonstrating a more general proof we begin with our standard motivational example, again revisiting the Hirzebruch surface.

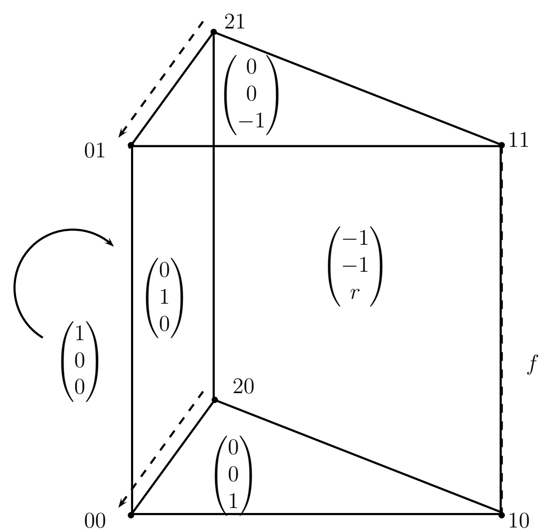


Figure 2.5: A Wedge QTM over $\mathbb{H}_r$, Revisited

Example 2.3.1. *Recall that*

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & r & -1 \end{pmatrix}$$

yields a wedge quasitoric manifold $\mathbb{H}_r^w = (\Delta^2 \times \mathbb{I}, \Lambda^w)$ *over* $\mathbb{H}_r$. *In particular for any of the vertices* $00, 10, 01$ *and* 11 *of the base polytope we have the associated matrix minors*

$$|\Lambda_{00}^w| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = |\Lambda_{00}| = \pm 1$$

$$|\Lambda_{01}^w| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = |\Lambda_{01}| = \pm 1$$

$$|\Lambda_{10}^w| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & r \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & r \end{vmatrix} = |\Lambda_{10}| = \pm 1$$

$$|\Lambda_{11}^w| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & r \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -1 & r \end{vmatrix} = |\Lambda_{11}| = \pm 1$$

expanding the determinant for each along the first column. On the other hand for vertices 20 *and* 21, *we expand along the top row to get*

$$|\Lambda_{20}^w| = \begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & r \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = |\Lambda_{00}| = \pm 1$$

$$|\Lambda_{21}^w| = \begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & r \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = |\Lambda_{01}| = \pm 1.$$

Here the matrix minors fall back on those of other vertices from which these were duplicated by the wedge construction.

Theorem 2.3.2 (see also Choi and Park [6] using simplicial complexes)**.** *Let* $M^{2n} = (P^{2n}, \Lambda)$ *be a quasitoric manifold over simple polytope* P^n *containing* m *facets. Set* $M^w = \left(P_f^w, \Lambda^w\right)$ *where* P_f^w *is the wedge polytope over* P^n *with foot* f *corresponding* *to the* $n+1$*th column of matrix* Λ*. Then*

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \lambda_{1,n+1} & \lambda_{1,n+2} & \cdots & \lambda_{1,m} \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ 0 & & & & 0 & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \lambda_{n,n+1} & \lambda_{n,n+2} & \cdots & \lambda_{n,m} \end{pmatrix} \tag{2.6}$$

or equivalently,

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n+1} & \lambda_{1,n+2} & \cdots & \lambda_{1,m} \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{n,1} & \cdots & \cdots & \lambda_{n,n+1} & \lambda_{n,n+2} & \cdots & \lambda_{n,m} \end{pmatrix}$$

gives a wedge quasitoric manifold over M^{2n}*.*

Proof. Let us denote the lateral facet over the foot f by F and the lateral facet over the base of the polytope by B. Let v be a vertex of the wedge polytope P^w_f.

- Case 1: *vertex* v *is incident to the base* B*.* In this case let us say that $v = B \cap F_{i_1} \cap \cdots \cap F_{i_n}$ and so the matrix minor associated to v is of the form

$$\begin{aligned} |\Lambda^w_v| &= \begin{vmatrix} 1 & * & * & \cdots & * \\ 0 & \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\ &= \begin{vmatrix} \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots \\ \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\ &= |\Lambda_v| = \pm 1 \end{aligned}$$

  Here the starred entries are all zero save one possible entry of -1. In either case, expansion along the first column simplifies the determinant to that shown. Geometrically, we see this as the matrix minor associated to our vertex $v = f_{i_1} \cap \cdots \cap f_{i_n}$ as a vertex of the original polytope P^{2n}. We further note these starred may be any particular value proving in full Lemma 2.1.4.

- Case 2: *vertex v is incident to the facet F and not B.* We insist in this case that the vertex v not be contained in B otherwise our cases would overlap and the vertex would be incident to the foot $f = F \cap B$. In this case the vertex v has been duplicated by the wedge construction from a vertex b of the base polytope B.

  Let us assume that the foot $F = f_{i_k}$ and so we may write $v = f_{i_0} \cap \cdots \cap f_{i_n}$ then indeed the corresponding point on the base is $b = B \cap f_{i_0} \cap \cdots \cap \widehat{f_{i_k}} \cap \cdots \cap f_{i_n}$ intersecting B instead of $F = f_{i_k}$. And so as a vertex of the original polytope P we may simply write $b = f_{i_0} \cap \cdots \cap \widehat{f_{i_k}} \cap \cdots \cap f_{i_n}$. Therefore we have the following calculation

$$
\begin{aligned}
|\Lambda_v^w| = &\begin{vmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \lambda_{i_0,1} & \lambda_{i_1,1} & \cdots & \lambda_{i_k,1} & \cdots & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ \lambda_{i_0,n} & \cdots & \cdots & \lambda_{i_k,n} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\
=&(-1)^k \begin{vmatrix} \lambda_{i_0,1} & \lambda_{i_1,1} & \cdots & \widehat{\lambda_{i_k,1}} & \cdots & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ \lambda_{i_0,n} & \cdots & \cdots & \widehat{\lambda_{i_k,n}} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\
=&(-1)^k |\Lambda_b| = \pm 1.
\end{aligned}
$$

  In this case we expand the determinant along the top row to achieve the product of $(-1)^k$ with the matrix minor associated to b as a vertex of the original polytope P^n.

In each case we have verified the matrix minor condition of Theorem 1.2.5 for the wedge demonstrating a wedge QTM over any quasitoric manifold. □

Definition 2.3.3. *There are in fact potentially many wedge quasitoric manifolds over any given QTM see (Dobrinskaya [9]). We will take the wedge given by Theorem 2.3.2 to be the* canonical wedge quasitoric manifold. *Unless otherwise stated wedge quasitoric manifolds will always be taken in this fashion with characteristic matrix of type 2.6.*

Definition 2.3.4. *We take the wedge over any quasitoric manifold* $M^{2n} = (\Lambda, P^n)$ *by forming* P^w *and setting*

$$\Lambda^{rw} = \begin{pmatrix} -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n+1} & \lambda_{1,n+2} & \cdots & \lambda_{1,m} \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{n,1} & \cdots & \cdots & \lambda_{n,n+1} & \lambda_{n,n+2} & \cdots & \lambda_{n,m} \end{pmatrix} \tag{2.7}$$

We refer to the resulting manifold $M^{rw} := (\Lambda^{rw}, P^w)$ *as the* reverse wedge over M^{2n}. *It is almost identical to the canonical wedge except we take* -1 *rather than* 1 *as the top left entry.*

Corollary 2.3.5 (to Theorem 2.3.2)**.** M^{rw} *yields a wedge quasitoric manifold over* M^{2n}*.*

Proof. It is immediate that

$$|\Lambda_v^{rw}| = -|\Lambda_v^w| = \pm 1$$

for any $v \in P^n$. Further,

$$|\Lambda_v^{rw}| = |\Lambda_v^w| = \pm 1$$

for any $v \in P^w \backslash P^n$. □

Remark. *One may apply a change of basis to the matrix* Λ_v^{rw} *at any point to place it in "reduced form" with a clearly defined initial vertex and reduced characteristic submatrix as demonstrated in Equation 1.2.*

Wedge Quasitoric Manifolds and the Connected Sum

Theorem 2.3.6. *For any simple polytopes* P^n *and* Q^n *we have the following*

$$(P\#Q)^w = P^w\#Q^w.$$

This immediately give us the following corollary to Theorem 2.3.2.

Corollary 2.3.7. *The connected sum of wedge quasitoric manifolds is a wedge quasitoric manifold (though it is not necessarily the canonical or reverse wedge). Specifically for any quasitoric manifolds* M_1 *and* M_2*, the manifold* $M_1^w\#M_2^w$ *is a wedge quasitoric manifold.*

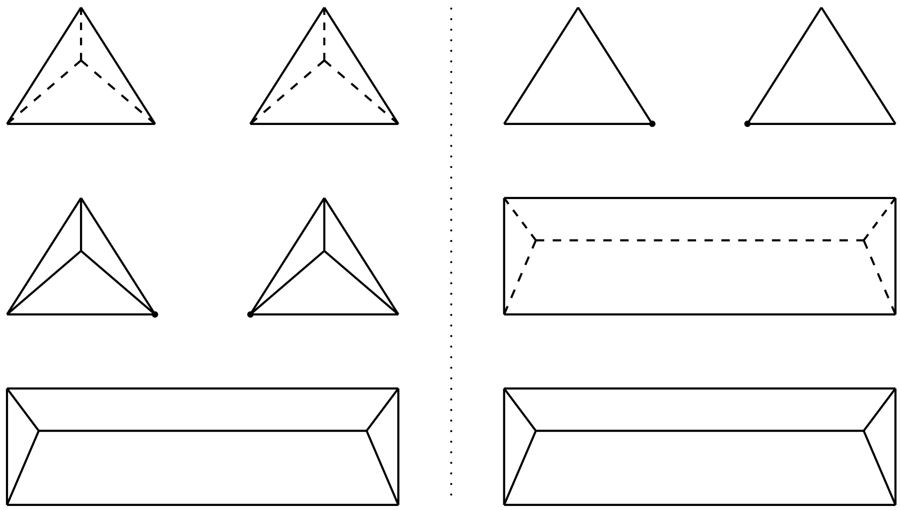


Figure 2.6: $(\Delta^2)^w \# (\Delta^2)^w = (\Delta^2 \# \Delta^2)^w$

2.4 Wedge Bott Manifolds

In this section we consider the wedge construction applied to Bott manifolds. Recall that Bott manifolds generalize Bott towers in that the fibre over any stage is $\mathbb{C}P^{n_i}$ rather than simply $\mathbb{C}P^1$ (see Definition 1.3.3). They must as quasitoric manifolds be defined over products of n-simplices rather than simply n-cubes. These manifolds share however a similar form for their refined characteristic submatrix as Bott towers 1.3.2. Choi, Masuda and Suh demonstrate the following.

Theorem 2.4.1 (Choi, Masuda and Suh [7])**.** *Any Generalized Bott Manifold may be represented by a quasitoric manifold over a product of simplices $\prod_{i=1}^{d} \Delta^{n_i}$ with a corresponding refined characteristic sub-matrix of the form*

$$\Lambda' = \left[\begin{array}{cccccc} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda_{1,n+1} & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ \hline \lambda_{k,n+1} & \lambda_{k,n+2} & \ddots & -1 & & 0 \\ \hline \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & -1 & 0 \\ \lambda_{d,n+1} & \lambda_{d,n+2} & \cdots & & \lambda_{d,d-1} & -1 \end{array}\right] \tag{2.8}$$

where for each entry of row k we have $\lambda_{k,j} \in \mathbb{Z}^{n_k}$ and $n = \sum_{i=1}^{m} n_i$. Here 0 and -1 represent the appropriate size n_i-vectors of repeated entries 0 and -1 respectively. Further, such quasitoric manifolds may be realized as Bott manifolds.

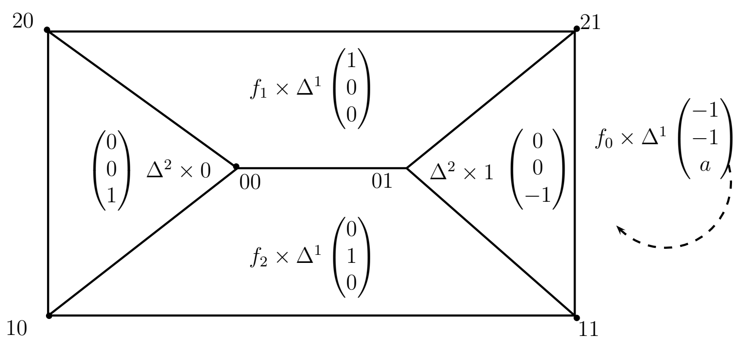


Figure 2.7: Bott Manifold over $\Delta^2 \times \Delta^1$

Example 2.4.2. *A Bott Manifolds over $\Delta^2 \times \Delta^1$ will have characteristic matrices of the form*

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & a & -1 \end{pmatrix}$$

as shown by the facet vectors in Figure 2.7. Here we note that the refined charactersistic submatrix is then of the form

$$\Lambda = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ a & -1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix}$$

where entries in the top row are 2-vectors.

In Figure 2.8 we show the canonical wedge of a Bott manifold over $\Delta^2 \times \Delta^1$.

Example 2.4.3. *We now consider a wedge QTM over the Bott manifold $M = (\Delta^2 \times \Delta^1, \Lambda)$ given in Example 2.4.2. The wedge charcteristic matrix is*

$$\Lambda^w = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & a & -1 \end{bmatrix}$$

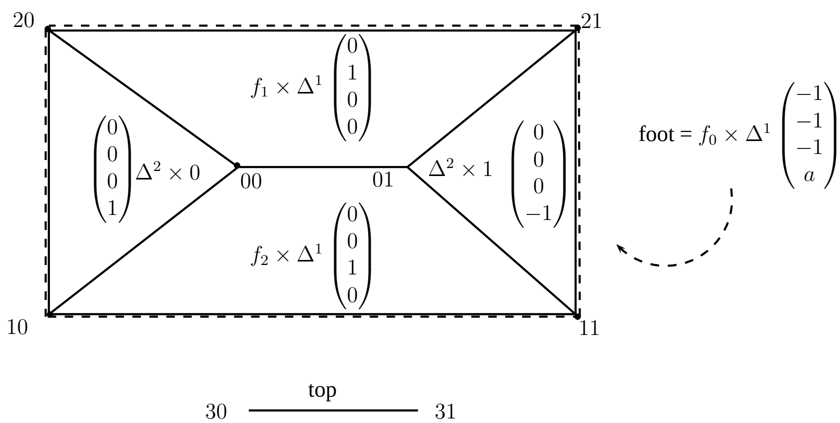


Figure 2.8: Wedge Bott Manifold over $\Delta^2 \times \Delta^1$

as shown by the facet vectors in Figure 2.8. Note that vector $e^1 = (1, 0, 0, 0)^T$ *corresponds to the interior of the base polytope. We now have a refined charactersistic submatrix of the form*

$$\Lambda = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ -1 & 0 \\ a & -1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix}$$

where entries in the top row are 3*-vectors. Further, we note that the wedge poytope is* $(\Delta^2 \times \Delta^1)^w = \Delta^3 \times \Delta^1$ *as indicated by Theorem 1.4.11. Theorem 2.3.2 ensures that* $M^w = \left((\Delta^2 \times \Delta^1)^w, \Lambda^w\right)$ *is in fact a (canonical) wedge quasitoric manifold and Theorem 2.4.1 implies that this is a Bott manifold.*

One immediate result of the canonical wedge construction given by Theorem 2.3.2 is that we may construct wedge QTMs over Bott manifolds. Recall that,

$$\left(\prod_{i=1}^{m} \Delta^{n_i}\right)^w = \Delta^{n_1+1} \times \left(\prod_{i=2}^{m} \Delta^{n_i}\right)$$

from Theorem 1.4.11. The wedge procedure on the polytope increases the dimension by one on the initial simplex so that $n_1 \to n_1 + 1$ and entries of the top row of the characteristic submatrix are now in $\mathbb{Z}^{n_1+1}$. For Λ^w we get what appears to be exactly the same matrix as 2.8 but with 0 and -1 now as $n_1 + 1$-vectors. Thus we have proved the following.

Corollary 2.4.4 (to Theorems 1.4.11, 2.3.2 and 2.4.1)**.** *There exists a wedge QTM over any Bott Manifold (or Tower) which is itself a Bott Manifold.*

Chapter 3 Hirzebruch Genera of Wedge Quasitoric Manifolds

Chapter 3 includes our main results concerning the wedge polytope and the Todd genus. Using a fixed point formula due to Panov [20] we proved that the canonical wedge 2.3.3 preserves the Todd genus (see Theorem 3.4.4) while the Todd genus of reverse wedge 2.3.4 vanishes (see Theorem 3.4.5).

3.1 Stably Complex Structures

Definition 3.1.1 (Buchstaber, Panov [3])**.** *A* stably complex structure *on a manifold M is given by a complex structure on the vector bundle $\tau(M) \oplus \mathbb{R}^k$ for some k, where $\tau(M)$ is the tangent bundle of M and $\mathbb{R}^k$ is a trivial real k-dimensional bundle over M.*

In answering a (quasi)toric version of Hirzebruchs famous question "Which complex cobordism classes in Ω^U contain connected smooth algebraic varieties?" Buchstaber, Panov and Ray develop additional structure for quasitoric manifolds in order to represent cobordism classes (see Theorem 0.1.1). This added structure will be referred to as an *omniorientation*, a combinatorial description for canonical stably complex structure. Consider a quasitoric manifold $\pi : M^{2n} \to P^{2n}$ with characteristic map Λ. We specify first an orientation of $\mathbb{R}^n$. This provides an orientation for P^n which in turn provides an orientation for for the manifold M^{2n} since the torus T^n is oriented (as the standard subgroup in $\mathbb{C}^n$).

Definition 3.1.2. *A quasitoric manifold M^{2n} is said to be* omnioriented *given an orientation of the manifold M^{2n} and fixed orientations of each facial submanifold $M_i = \pi^{-1}(F_i) = M^{2(n-1)}$.*

Lemma 3.1.3. *A choice of omniorientation for M^{2n} is equivalent to a choice of orientation for P^n together with an unambiguous choice of facet vectors.*

Remark. *An orientation for P^n is specified by orienting the ambient space $\mathbb{R}^n$. This we consider fixed unless otherwise stated.*

An omnioriented quasitoric manifold may then be described via a characteristic matrix for which a distinct choice of facet vector directions has been assigned. Such a matrix is then referred to as a *dicharacteric matrix* (or *directed characteristic matrix*). An omnioriented quasitoric manifold is then defined for any given *dicharacteristic pair* $M^{2n} := (P^n, \Lambda)$.

The following theorem provides an initial description of the resulting stably complex structure.

Theorem 3.1.4 (Buchstaber, Ray 2001 [5]). *Every omniorientation of a quasitoric manfiold M^{2n} determines a stably complex structure on it by means of the following isomorphism of a real $2m$-bundles:*

$$\tau\left(M^{2n}\right) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m$$

where m denotes the number of facets in the quotient polytope.

In Chapter 4 we will determine the bundles ρ_i explicitly via what is known as the *moment angle complex* (see Section 4.2).

3.2 Hirzebruch Genera of Quasitoric Manifolds

Buchstaber and Panov provide methods for calculating various cobordism invarients. Here we provide their methods and combinatorial formulae and will subsequently extend these to wedge quasitoric manifolds.

Definition 3.2.1 (Buchstaber, Panov 2002 [3]). *The* Hirzebruch genus associated with the series

$$Q(x) = 1 + \sum q_k x^k, q \in \mathbb{Q}$$

is the ring homomorphism $\varphi_Q : \Omega^U \to \mathbb{Q}$ that to each cobordism class $[M^{2n}] \in \Omega^W_{2n}$ assigns the value given by the formula

$$\varphi_Q[M^{2n}] = \left(\prod_{i=1}^{n} Q(x_i), \langle M^{2n} \rangle\right)$$

where M^{2n} is a smooth manifold whose stable tangent bundle $\tau\left(M^{2n}\right)$ is a complex bundle with complete Chern class in cohomology

$$c(\tau) = 1 + c_1(\tau) + \cdots + c_n(\tau) = \prod_{i=1}^{n} (1 + x_i)$$

and $\langle M^{2n} \rangle$ is the fundamental class in homology.

For details concerning the complex cobordism ring Ω^U please see the introductory materials provided at the start of Chapter 4 and in particular the Definition 4.1.2.

Definition 3.2.2 (Buchstaber, Panov 2002 [3]). *The* χ_y-genus *is the Hirzebruch genus associated with the series*

$$Q(x) = \frac{x\left(1 + ye^{-x(1+y)}\right)}{1 - e^{-x(1+y)}},$$

where $y \in \mathbb{R}$ *is a parameter.*

In particular when $y = 0$ we achieve the *Todd Genus* 3.3. If $y = +1$ this is the *L-genus* or *signature* and when $y = -1$ we get the *n-th Chern number*. Combinatorial formulae for each were developed by Panov and Buchstaber in terms of the so called *edge vectors* and more specifically, the *sign* and *index* of vertices.

The Sign and Index of a Vertex

Suppose M^{2n} is an omnioriented quasitoric manifold with quotient polytope P^n. The vertices of P^n will play a critical roll in the calculation of particular invariants of M^{2n}. Calculation of the χ_y genus will depend not only on our global orientation of M^{2n} but also upon local orientations near each vertex. We now introduce a local construction for a canoncial orientation near each vertex of the quotient polytope.

Construction 3.2.3 ([3]). *Let* v *be a vertex of* P^n. *Since the polytope is simple we may express* v *as the intersection of* n *facets and we write*

$$v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

For each facet F_{i_k} *we have a unique edge incident to* v, *call it* E_k, *which does not lie entirely within the facet; in particular* $E_k \cap F_{i_k} = v$. *Let* e_k *be the direction vector along edge* E_k *with origin* v. *These* $e_1, \ldots, e_n$ *form a basis of* $\mathbb{R}^n$ *which may be positively or negatively oriented with respect to our orientation of the polytope (given by a fixed orientation of* $\mathbb{R}^n$ *see Lemma 3.1.3 and the associated remark). We insist throughout this Chapter on a local ordering (near each vertex* v*) of the facets and thus edges and ultimately the vectors* $e_1, \ldots, e_n$ *so that this orientation is positive i.e.*

$$\det\left[e_1 \ldots e_n\right] = 1.$$

These vectors are then referred to as local basis vectors near v.

Definition 3.2.4 (Dobrinskaya [9], Panov [20], [3]). *The* sign of a vertex $v = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n}$ *is*

$$\sigma(v) := \det \Lambda_v = \left|\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}\right|.$$

Note that $\det \Lambda_v$ is calculated with local facet ordering indicated by Lemma 3.1.3. It does not necessarily match with the matrix minors first examined in Theorem 1.2.5. The sign is now destinguished in an appropriate manner given the local orientation near a particular vertex provided by our local basis vectors $e_1, \ldots, e_k$.

Recall our definition of the isotropy subgroups corresponding to the facial submanifolds of M^{2n}, the one-dimensional subgroups $T(F_i)$ as in Definition 1.2.2 where

$$T(F_i) = \left(e^{2\pi i \lambda_{i1}\phi}, e^{2\pi i \lambda_{i2}\phi}, \ldots, e^{2\pi i \lambda_{in}\phi}\right).$$

Such isotropy subgroups allowed us to define the facet vectors λ_i. We now consider the isotropy subroups of the n-torus acting on M^{2n} which correspond to the edges of the polytope P^n.

Definition 3.2.5. *Let E be an edge of P^n. The set $\pi^{-1}(intE)$ determines a two dimensional sub-manifold of M^{2n}. The action of the n-torus on this sub-manifold yields an isotropy subgroup of T^n which we denote $T(E)$. It is indeed an $(n-1)$-dimensional subtorus which may be written*

$$T(E) := \left\{\left(e^{2\pi i\varphi_1}, e^{2\pi i\varphi_2}, \ldots, e^{2\pi i\varphi_n}\right) \in P^n : \mu_1\varphi_1 + \cdots + \mu_n\varphi_n = 0\right\}$$

for some integers $\mu_1, \ldots, \mu_n$. Here $\mu = (\mu_1, \ldots, \mu_n)^T$ is referred to as the edge vector corresponding to E.

These edge vectors μ at v are primitive vectors in the dual lattice $(\mathbb{Z}^n)^*$ and are determined (up to sign) by the facet vectors of Λ_v. The signs and edge vectors themselves may be determined by the following lemma.

Lemma 3.2.6 (Buchstaber, Panov [3])**.** *For each vertex $v \in P^n$, the signs of the edge vectors $\mu_1, \ldots \mu_n$ meeting at v can be chosen in such a way that the $n \times n$-matrix $M_v := (\mu_1, \ldots, \mu_n)$ satisfies the equation*

$$M_v^T \cdot \Lambda_v = \mathbb{I}.$$

Here, in fact the edge vectors $\mu_1, \ldots \mu_n$ form a conjugate basis to facet vectors $\lambda_{i_1}, \ldots \lambda_{i_n}$ near v and in particular we note $\sigma(v) := \det \Lambda_v = \det M_v$ allowing us to use either when calculating signs.

Remark (Geometric Interpretation of the sign)**.** *We have an orientation given by (conjugate) basis $\mu_{i_1}, \ldots, \mu_{i_n}$ and one given by the local basis vectors from Construction 3.2.3. Thus in particular,*

$$\sigma(v) = \begin{cases} 1 & \text{when these orientations agree} \\ -1 & \text{if they do not.} \end{cases}$$

Definition 3.2.7. *Let $\nu \in \mathbb{Z}^n$ be a primitive vector such that $\nu \cdot \mu \neq 0$ for all edge vectors μ of omnioriented quasitoric manifold M^{2n}. We define the* index of a vertex *$v \in P^n$ as follows*

$$ind_\nu v = \{\#k : \mu_k \cdot \nu < 0\}$$

over all μ_k edge vectors of vertex v. This is, simply put, the number of negative scalar products of ν with the edge vectors of v.

3.3 The Todd Genus

Theorem 3.3.1 (Panov [19] [20])**.** *The Todd genus of an omnioriented quasitoric manifold can be calculated as*

$$\mathrm{td}\left(M^{2n}\right) = \sum_{v \in P^{2n} : ind_\nu(v)=0} \sigma(v)$$

where the sum is taken over all vertices of index zero.

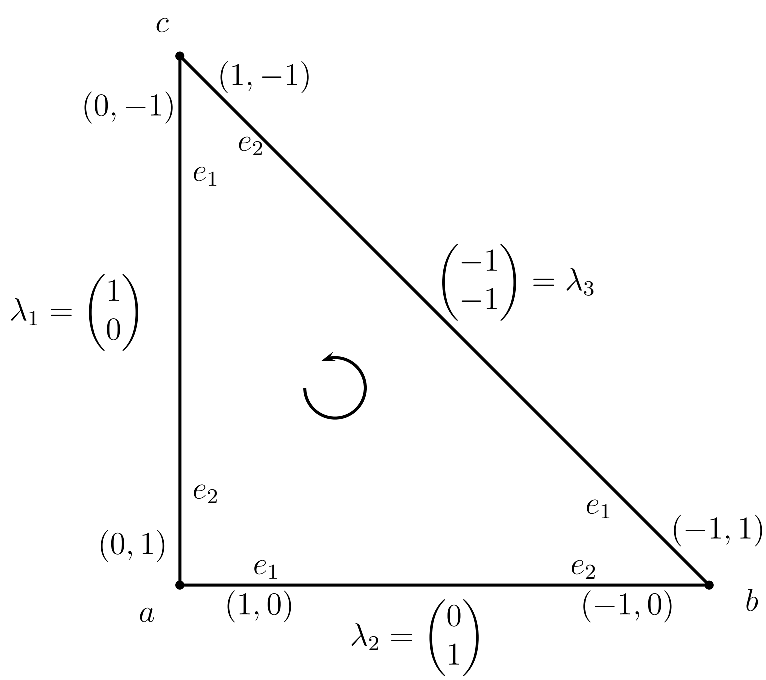


Figure 3.1: $\mathbb{C}P^2$ with Edge, Facet and Basis Vectors

Example 3.3.2. *We begin with the example $\mathbb{C}P^2$ veiwed as a toric variety with stably complex structure implied by the standard complex structure on $\mathbb{C}P^2$. As a toric variety it arises from the simplex Δ^2 on vertices $a = (0,0)$, $b = (1,0)$ and $c = (0,1)$ within integer lattice $\mathbb{Z}^2$. The edge vectors for $\mathbb{C}P^2$ above may be calculated via Lemma 3.2.6. Consider the vertex b. The edge vectors at b are given by the inverse matrix $(\Lambda_b)^{-1}$. Specifically we have*

$$M_b\Lambda_b = \begin{matrix} \mu_1 \\ \mu_2 \end{matrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{matrix} \begin{matrix} \lambda_2 & \lambda_3 \end{matrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \end{matrix} = \mathbb{I}$$

where vector e_1 is assigned to the edge opposite the facet corresponding to λ_2 and vector e_2 is assigned to the edge opposite the facet corresponding to λ_3.

Computing the Todd Genus for $\mathbb{C}P^2$ we see that

$$\sigma(a) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$
$$\sigma(b) = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 1, \text{ and}$$
$$\sigma(c) = \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

taking special note of the local ordering near each vertex of the edges (and thus facet vectors) given by Construction 3.2.3. To compute the index at each vertex we choose a primitive vector $\nu = (1,2)$ and find $ind_\nu(a) = 0$, $ind_\nu(b) = 1$ and $ind_\nu(c) = 2$ counting negative scalar products with the edge vectors. Therefore, we have $\operatorname{td}(\mathbb{C}P^2) = 1$.

Remark (Panov [20])**.** *The edge vectors for smooth projective toric varieties will always point out of their associated vertices and along edges i.e. $\mu_k = e_k$. So, in fact given our local orientations described in Construction 3.2.3 we must have $\sigma(v) = 1$ for any vertex v of a smooth projective toric variety. Further, in the case of smooth projective toric variety M_P there is only one vertex of index 0. Thus, $\operatorname{td}(M_P) = 1$ a well known result.*

Now we present a computation for the Todd Genus of the connected sum $\mathbb{C}P^2\#\mathbb{C}P^2$ and we will in Example 3.4.3 demonstrate the extension of this combinatorial formula to the wedge.

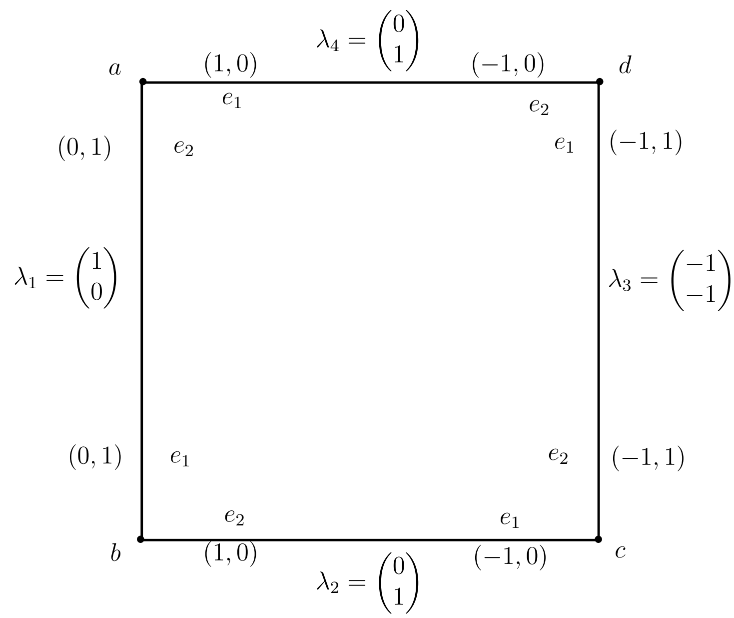


Figure 3.2: $\mathbb{C}P^2\#\mathbb{C}P^2$ with Edge, Facet and Basis Vectors

Example 3.3.3. *Recall the quasitoric manifold $\mathbb{C}P^2\#\mathbb{C}P^2$ as introduced in Example 1.2.8 which we picture here in Figure 3.2. Here the edge vectors and their signs are calculated as in Lemma 3.2.6. We find the index of each vertex using the primitive vector $\nu = (1,2)$ (recalling that it is the number of positive scalar products with ν). So calculating we see that,*

$$ind_\nu a = 0 = ind_\nu b$$
$$ind_\nu c = 1 = ind_\nu d.$$

Since a and b are the only vertices of index zero the Todd Genus will be $\sigma(a) + \sigma(b)$ and hence

$$\mathrm{td}\left(\mathbb{C}P^2\#\mathbb{C}P^2\right) = \sigma(a) + \sigma(b) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2.$$

Remark. *This is expected since connected sum is additive and $\mathbb{C}P^2$ is toric. So,*

$$\mathrm{td}\left(\mathbb{C}P^2\#\mathbb{C}P^2\right) = \mathrm{td}\left(\mathbb{C}P^2\right) + \mathrm{td}\left(\mathbb{C}P^2\right) = 1 + 1 = 2.$$

3.4 Hirzebruch Genera and the Wedge

We now develop a reasonable way to extend the combinatorial formula for the Todd genus for the base manifold to the wedge. As shown in Section 1.4 the wedge over

$\mathbb{C}P^n$ is $\mathbb{C}P^{n+1}$ so we in fact notice that

$$\text{td}\,(\mathbb{C}P^n) = 1 = \text{td}\left(\mathbb{C}P^{n+1}\right) = \text{td}\,((\mathbb{C}P^n)^w)$$

as they are smooth projective toric varieties. It will be shown in Theorem 3.4.4 that these match in general for the canonical wedge (Definition 2.3.3). First we must canonically extend the idea of local orientations near vertices of the base polytope (as in Construction 3.2.3) to the wedge.

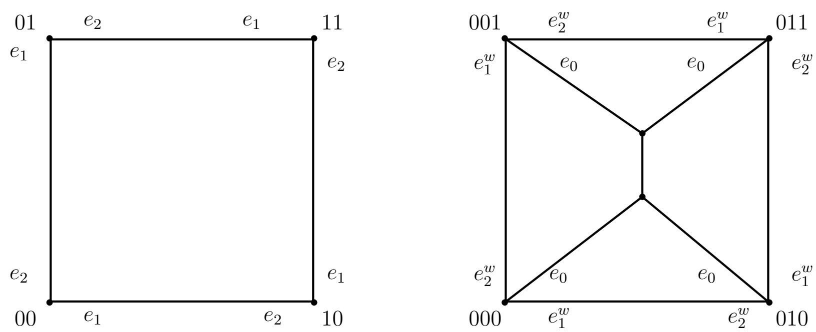


Figure 3.3: Local Basis Vectors for the Wedge over the Square

The following construction allows us to extend the local basis vectors near vertices of the base polytope to those of the wedge.

Construction 3.4.1. *Let v be a vertex of P^n. Since the polytope is simple we may express v as the intersection of n facets and we write*

$$v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

We apply Construction 3.2.3 to P^n and get the local basis vector matrix $E_v = [e_1, \ldots, e_n]$ where $\det E_v = 1$ (as a positively oriented basis of $\mathbb{R}^n$). We form the wedge P^w which we note is an $n+1$ dimensional polytope, so we have one new edge at v which we label $e_0 = (1, *, \ldots, *)^T$. For the other n edges we choose $e_1^w, \ldots, e_n^w$ from $e_1, \ldots, e_n$ by including a new initial coordinate entry of 0. Now computing we get

$$\det\,[e_0, e_1^w, \ldots, e_n^w] = \det \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & E_v & \\ * & & & \end{array} \right) = \det\,[e_1, \ldots, e_n] = 1$$

from expansion of the determinant along the top row. Hence $E_v^w = [e_0, e_1^w, \ldots, e_n^w]$ is a positively oriented basis of $\mathbb{R}^{n+1}$ that agrees in the base polytope with E_v. For any vertices not incident to the base of the wedge we simply apply Construction 3.2.3.

We move our attention now to the sign of vertices which may be calculated with respect to these local orientations.

Lemma 3.4.2. *Let $v \in P^n$ be vertex of the base polytope. The sign of v as a vertex of P^n is the same as its sign as a vertex for the canonical wedge i.e. $\sigma^w(v) = \sigma(v)$.*

Proof. Let vertex $v \in P^n$ and form the wedge P^w. Since v is incident to the base of the wedge $F_0 = P^n$, we have λ_0 as the initial column of Λ_v^w and so

$$\begin{aligned}\sigma^w(v) = |\Lambda_v^w| &= \begin{vmatrix} 1 & * & * & \cdots & * \\ 0 & \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\ &= \begin{vmatrix} \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots \\ \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{vmatrix} \\ &= |\Lambda_v| = \sigma(v).\end{aligned}$$

Recall the local ordering at v within P^w is determined by Construction 3.4.1 which ensures λ_0 corresponds to the "initial facet" opposite edge e_0. □

Todd Genus and the Canonical Wedge Construction

We extend Panov's combinatorial formula for the Todd genus (see Theorem 3.3.1) to the wedge over any base manifold. We first consider the wedge over $\mathbb{C}P^2 \# \mathbb{C}P^2$. The foot and wedge dicharacteristic matrix Λ^w are chosen canonically as in Theorem 2.3.2 and the edge vectors are calculated as in Lemma 3.2.6. Specifically, we consider the polytope $\Delta^2 \times \mathbb{I}$ and dicharacteristic matrix

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

as in Figure 3.4 where our new initial column $\lambda_0 = (1, 0, 0)^t$ is associated with the facet given by vertices *abcd* i.e. the base polytope.

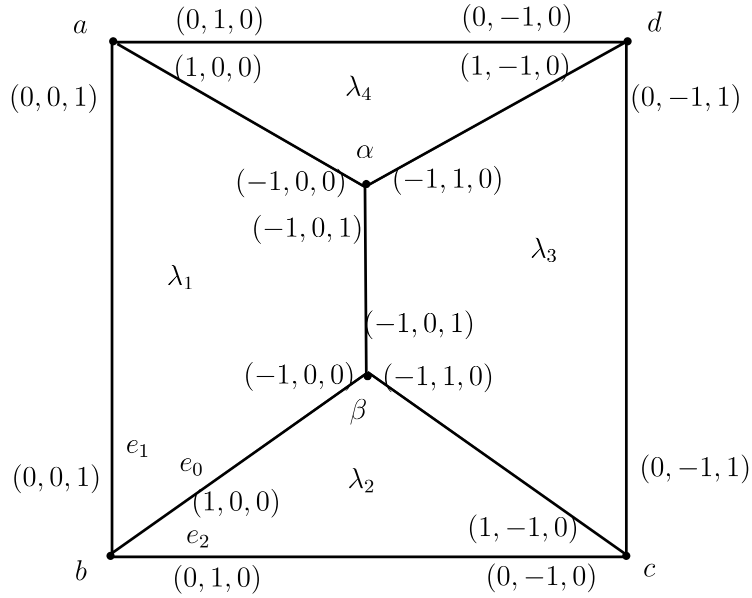


Figure 3.4: $(\mathbb{C}P^2\#\mathbb{C}P^2)^w$ with Edge, Basis and Facet Vectors

Example 3.4.3. *We proceed by calculation of the Todd genus for* $(\mathbb{C}P^2\#\mathbb{C}P^2)^w$ *via the index and edge vectors as shown in Figure 3.4. Here however for our calculations we must insist on using primitive integer vector* $\nu^w = (3|\nu) = (3, 1, 2)$*. So calculating the number of negative scalar products with* ν^w *we have,*

$$\begin{aligned} ind_{(\nu^w)}a &= 0 = ind_{(\nu^w)}b \\ ind_{(\nu^w)}c &= 1 = ind_{(\nu^w)}d \\ ind_{(\nu^w)}\alpha &= 3 = ind_{(\nu^w)}\beta. \end{aligned}$$

Since a *and* b *are still the only vertices of index zero the Todd Genus will be* $\sigma(a)+\sigma(b)$ *and hence*

$$\operatorname{td}\left((\mathbb{C}P^2\#\mathbb{C}P^2)^w\right) = \sigma(a) + \sigma(b) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2.$$

We take note in these last examples that the index of vertices in the wedge incident to the original polytope have the remains unchanged by the wedge construction. This is due to the determination of edge vectors from Lemma 3.2.6 as we shall see in the proof of Theorem 3.4.4. The only remaining obstruction to maintaining the same

Todd genus for the wedge as the base manifold is the possible introduction of new vertices of index zero. Here we see the particular edge vector $(-1, 0, 0)$ along edges incident to both vertex α and β forcing the index of both of our new vertices to be strictly positive. Thus the Todd genus remains unchanged.

Theorem 3.4.4. *The Todd genus of an omnioriented quasitoric manifold is the same as its canonical wedge i.e.*

$$\operatorname{td}\left(M^{2n}\right) = \operatorname{td}\left(M^{w}\right).$$

Proof. Let $M^{2n} = (P^n, \Lambda)$ be an omnioriented quasitoric manifold with wedge M^w canonically chosen as in Theorem 2.3.2. Specifically, given dicharacteristic matrix $\Lambda = (\lambda_{ij})$ we have,

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda_{1,1} & \lambda_{2,1} & \cdots & \lambda_{n+1,1} & \cdots & \cdots & \lambda_{m,1} \\ \vdots & \vdots & & & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots & & & \vdots \\ 0 & \lambda_{1,n} & \cdots & \cdots & \lambda_{n+1,n} & \cdots & \cdots & \lambda_{m,n} \end{pmatrix}$$

where the new initial column corresponding to the base of the wedge is referred to as facet vector λ_0. We assume the existence of edge vectors at each vertex of M^{2n} and M^w determined as in Lemma 3.2.6 and a primitive vector ν satisfying $\nu \cdot \mu \neq 0$ for any edge vector μ of M^{2n}.

Let v be a vertex of M^w. First, we need to show that $ind_{(\nu^w)} v = ind_\nu v$ that is, the index of v as a vertex of P^w is the same as the index of v as a vertex of P^{2n}. We fix a value x, the maximum in absolute value over all entries of ν or M_v^w and let $\nu^w = (\nu_0|\nu) \in \mathbb{Z}^{n+1}$ be a primitive vector such that $\nu_0 > nx^2$. Simply put, ν^w is the vector formed by a maximum value ν_0 followed by the entries of ν. We can assume ν_0 is taken sufficiently large so that $\nu^w \cdot \mu^w \neq 0$ for any edge vector μ^w of M^w. We now consider the following two cases.

- **Case 1**: $v \in P^n$.
  First we label edge vectors of the base manifold M^{2n} at v via the matrix

  $$M_v = (\Lambda_v)^{-1} = (\mu_{ij})$$

  for i and j between 1 and n. The rows form our edge vectors μ_1 through μ_n.

  Now vertex v is incident to the base of the wedge which for clarity we refer to as facet F_0 of P^w. By assumption vertex v is contained in the facet of

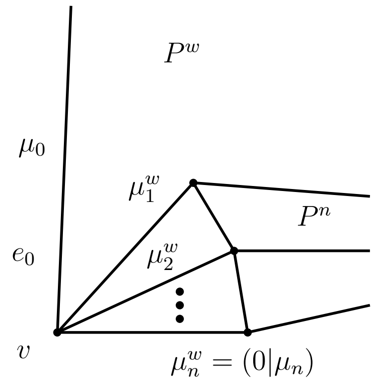


Figure 3.5: Edge Vectors of the Canonical Wedge

P^w corresponding to vector λ_0. Hence Λ_v^w obtained from the dicharacteristic matrix Λ^w must contain λ_0 and is of the form

$$\Lambda_v^w = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{pmatrix}$$

where at most one of the starred entries may be -1 and the rest zero, as first demonstrated in the proof of Theorem 2.3.2. Note that λ_0 must in fact be in this first column since the base of the polytope is opposite the local basis vector e_0, see Construction 3.4.1. Now, from Lemma 3.2.6 and the formation of the inverse of matrix Λ_v^w we must have an edge vector matrix at v of the form

$$M_v^w = (\Lambda_v^w)^{-1} = \begin{pmatrix} 1 & \mu_{1,0} & \mu_{2,0} & \cdots & \mu_{n,0} \\ 0 & \mu_{1,1} & \mu_{2,1} & \cdots & \mu_{n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \mu_{1,n} & \cdots & \cdots & \mu_{n,n} \end{pmatrix}$$

where rows of this matrix form edge vectors nearly identical to those of the base manifold save a new initial entry. Specifically, we have $\mu_i^w = (0|\mu_i)$ i.e. a zero followed by the entries from μ_i for $1 \leq i \leq n$ and so

$$\nu^w \cdot \mu_i^w = (\nu_0|\nu) \cdot (0|\mu_i) = \nu \cdot \mu_i \tag{3.1}$$

for all $1 \leq i \leq n$, due to annihilation in the first coordinate entry. Hence,

$$ind_{(\nu^w)} v \leq ind_\nu v.$$

Consider now edge vector $\mu_0 = (1, \mu_{0,0}, \mu_{2,0}, \cdots, \mu_{n,0})$ corresponding to the unique edge opposite facet F_0. It lies along the single "new" edge e incident to v created by the formation of the wedge (the remaining edge vectors μ_0 through μ_n lie within the original polytope P^n as illustrated in Figure 3.5). Calculating we see that

$$\begin{aligned}\nu^w \cdot \mu_0^w &= (\nu_0|\nu) \cdot (1|\mu_i) \\ &= \nu_0 + \nu \cdot \mu_0 \\ &> nx^2 + \nu \cdot \mu_0 \\ &\geq 0\end{aligned}$$

recalling x as the maximum in absolute value over all entries of ν or M_v^w. Our extra wedge from the edge begins with the value 1 while the initial entry ν_0 of ν^w has been selected so large that its dot product with μ_0 cannot contribute an additional negative scalar products to the index.

Given this last dot product positive and Equation 3.1 we have finally

$$ind_{(\nu^w)} v = ind_\nu v + 0 = ind_\nu v.$$

- **Case 2**: $v \in P^w \backslash P^n$.
  Vertex v is a vertex of the wedge not incident to the base, i.e. it is a new vertex created by the formation of the wedge. All vetices of the wedge are contained in either the base or the lateral facet over the foot. Since v is not in the base it must be in this lateral facet and so Λ_v^w must be of the form

$$\Lambda_v^w = \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_k,1} & \cdots & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_k,n} & \cdots & \cdots & \lambda_{i_n,n} \end{pmatrix}$$

and thus

$$M_v^w = (\Lambda_v^w)^{-1} = \begin{pmatrix} \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n,1} \\ \vdots & & & \vdots \\ \mu_{0,k-1} & \mu_{1,k-1} & \cdots & \mu_{n,k-1} \\ -1 & 0 & \cdots & 0 \\ \mu_{0,k-1} & \mu_{1,k-1} & \cdots & \mu_{n,k-1} \\ \vdots & & & \vdots \\ \mu_{0,n} & \mu_{1,n} & \cdots & \mu_{n,n} \end{pmatrix}$$

where we label row k as μ_k. Calculating we see that

$$\nu^w \cdot \mu_k = -\nu_0 < 0.$$

Therefore

$$ind_{(\nu^w)} v > 0.$$

From case 1 we see that the wedge construction preserves the index for vertices incident to the base as it was shown for the sign in Lemma 3.4.2. Case 2 indicates that the wedge construction creates no "new" vertices of index zero. Thus given the formula for the Todd genus of a quasitoric manifold given by Theorem 3.3.1 we compute

$$\begin{aligned}
\operatorname{td}(M^w) &= \sum_{v \in P^w : ind_{(\nu^w)}(v)=0} \sigma^w(v) \\
&= \sum_{v \in P^{2n} : ind_{\nu}(v)=0} \sigma(v) + \sum_{v \in P^w \setminus P^n : ind_{(\nu^w)}(v)=0} \sigma^w(v) \\
&= \operatorname{td}\left(M^{2n}\right) + 0 \\
&= \operatorname{td}\left(M^{2n}\right).
\end{aligned}$$

Therefore the canonical wedge preserves the Todd genus. □

Todd Genus and the Reverse Wedge Construction

We will now recall an alternate method for taking the wedge of a quasitoric manifold.

Theorem. *A wedge quasitoric manifold is formed over any quasitoric manifold $M^{2n} = (\Lambda, P^n)$ by forming P^w and setting*

$$\Lambda^{rw} = \begin{pmatrix} \mathbf{-1} & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda_{1,1} & \lambda_{2,1} & \cdots & \lambda_{n+1,1} & \cdots & \cdots & \lambda_{m,1} \\ \vdots & \vdots & & & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots & & & \vdots \\ 0 & \lambda_{1,n} & \cdots & \cdots & \lambda_{n+1,n} & \cdots & \cdots & \lambda_{m,n} \end{pmatrix} \tag{3.2}$$

We refer to the resulting manifold $M^{rw} := (\Lambda^{rw}, P^w)$ as the reverse wedge over *M^{2n}.*

This is the *reverse wedge* quasitoric manifold as given by Definition 2.3.4. Remarkably the change is sign of one entry in the characteristic matrix result in a drastic change for value of the Todd genus as shown in the following.

Theorem 3.4.5. *Let M^{2n} be an omnioriented quasitoric manifold with reverse wedge M^{rw} then*

$$\mathrm{td}\,(M^{rw}) = 0.$$

Proof. As in the proof of Theorem 3.4.4 we first determine a suitable primitive vector $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$ where $\nu_0 > nx^2$ such that x is the maximum in absolute value over all entries of ν_k for $1 \leq k \leq n$ and all entries of M_v^{rw}. We will show that the index of each vertex of M^{rw} is strictly positive thereby ensuring that the Todd genus will be zero. Let v be a vertex of M^{rw}.

- **Case 1**: $v \in P^n$.
  We know that v is incident to the base of the wedge so Λ_v^{rw} is of the form

$$\Lambda_v^{rw} = \begin{pmatrix} -1 & * & * & \cdots & * \\ 0 & \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_n,n} \end{pmatrix}$$

  and so likewise we have

$$M_v^{rw} = (\Lambda_v^{rw})^{-1} = \begin{pmatrix} -1 & \mu_{1,0} & \mu_{2,0} & \cdots & \mu_{n,0} \\ 0 & \mu_{1,1} & \mu_{2,1} & \cdots & \mu_{n,1} \\ \vdots & \vdots & & & \vdots \\ 0 & \mu_{1,n} & \cdots & \cdots & \mu_{n,n} \end{pmatrix}$$

  where we label the initial edge vector $\mu_0 = (-1, \mu_{1,0}, \mu_{2,0}, \cdots, \mu_{n,0})$. From this one vector we see that $\nu \cdot \mu_0 < 0$ from the definition of ν and so $ind_\nu v > 0$.

- **Case 2**: $v \in P^w \backslash P^n$.
  In this case Λ_v^{rw} is of the form

$$\Lambda_v^{rw} = \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \lambda_{i_1,1} & \lambda_{i_2,1} & \cdots & \lambda_{i_k,1} & \cdots & \cdots & \lambda_{i_n,1} \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ \lambda_{i_1,n} & \cdots & \cdots & \lambda_{i_k,n} & \cdots & \cdots & \lambda_{i_n,n} \end{pmatrix}$$

and thus

$$M_v^{rw} = (\Lambda_v^{rw})^{-1} = \begin{pmatrix} \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n,1} \\ \vdots & & & \vdots \\ \mu_{0,k-1} & \mu_{1,k-1} & \cdots & \mu_{n,k-1} \\ -1 & 0 & \cdots & 0 \\ \mu_{0,k-1} & \mu_{1,k-1} & \cdots & \mu_{n,k-1} \\ \vdots & & & \vdots \\ \mu_{0,n} & \mu_{1,n} & \cdots & \mu_{n,n} \end{pmatrix}$$

where we label row k as μ_k. Again, calculating we see

$$\nu \cdot \mu_k = -\nu_0 < 0.$$

Therefore, $ind_\nu v > 0$.

Since the index is positive for all vertices the Todd genus vanishes for M^{rw}. $\square$

Chapter 4 Spin Cobordism

Chapter 4 features our main results concerning the wedge polytope construction and spin quasitoric manifolds. Specifically we demonstrate criteria for a spin quasitoric manifold to be viewed as dual to the first Chern class of its canonical wedge (see Theorem 4.4.4). This setup should allow the calculation of KO-characteristic classes for spin quasitoric manifolds satisfying 4.7 as in [10].

For more details concerning complex cobordism please see [22], [21] and [18]. For more details concerning spin cobordism please see [15], [1], and [17].

4.1 The Complex Cobordism Ring

Definition 4.1.1. *Two closed smooth manifolds M_1 and M_2 are* cobordant *if their disjoint union $M_1 \bigsqcup M_2$ forms the boundary of an $(n+1)$-dimensional smooth compact manifold-with-boundary.*

Cobordism forms an equivalence relation among such manifolds, the equivalence classes for which we term *cobordism classes*. Now given any two complex manifolds each of dimension $2n$ (such as any quasitoric manifold) there cannot exist a complex manifold of (real) dimension $2n+1$ for which the two could disjointly bound. The purpose of the stably complex structures first introduced in Section 3.1 is to surmount this obstacle.

We weaken the requirement of a complex structure and recall the following.

Definition (Buchstaber, Panov [3])**.** A stably complex structure on a manifold M *is given by a complex structure on the vector bundle $\tau(M) \oplus \mathbb{R}^k$ for some k, where $\tau(M)$ is the tangent bundle of M and $\mathbb{R}^k$ is a trivial real k-dimensional bundle over M.*

When M is itself a complex manifold then it possesses the *canonical stably complex structure* $(M, \tau(M))$.

Definition 4.1.2. *The set of stably complex cobordism classes may be equipped with the structure of a graded ring via the operations product and disjoint union. This we call the* complex cobordism ring *and write Ω^U.*

4.2 Facial Bundles, Chern Classes and the Moment Angle Manifold

Unless otherwise stated all quasitoric manifolds in this chapter will be considered omnioriented. We insist then on specific choices of sign corresponding to the omniorientation for all facet vectors in any given (directed) characteristic matrix (or dicharacteristic matrix).

We now recall Theorem 3.1.4.

Theorem. *Every omniorientation of a quasitoric manfiold M^{2n} determines a stably complex structure on it by means of the following isomorphism of a real $2m$-bundles:*

$$\tau\left(M^{2n}\right) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m$$

The bundles ρ_i are 2-dimensional complex line bundles over M^{2n} and may be defined via what is known as the moment-angle manifold. They each restrict to their corresponding facial sub-manifolds as the normal bundle of $M_i \subset M^{2n}$ [18]. Thus they are referred to as the *facial bundles* of the given quasitoric manifold. These bundles are necessary in describing the resulting complex cobordism ring of the manifold.

Theorem 4.2.1 (Buchstaber, Panov [3] [5])**.** *The cohomology ring of M^{2n} is given by*

$$H^*\left(M^{2n}, \mathbb{Z}\right) = \frac{\mathbb{Z}[x_1, x_2, \ldots, x_m]}{I_P + J_\Lambda} = \frac{\mathbb{Z}\left(P^n\right)}{J_\Lambda}$$

where $x_i = c_1(\rho_i)$ is the first Chern class of the facial bundle ρ_i for each $i = 1, 2, \ldots, m$. These are 2-dimensional cohomology classes Poincaré dual to the facial sub-manifolds $M_i = M_i^{2(n-1)}$.

The ideal I_P is determined by the nonfaces of P^n, forming the Stanley-Reisner ring $\mathbb{Z}\left(P^n\right) = \mathbb{Z}[x_1, x_2, \ldots, x_m]/I_P$. The ideal J_Λ is determined from each row of Λ forming the additive relations

$$x_i = -\lambda_{i,n+1}x_{n+1} - \cdots - \lambda_{i,m}x_m \tag{4.1}$$

for each $1 \leq i \leq n$.

The Moment Angle Manifold

We will now determine the bundles ρ_i from Chapter 3 explicitly in terms of the moment angle manifold. We closely follow the work of Panov in [18] for further reading consult [3] and [4].

First we must recall our definition of polytope as given by a compact intersection of finitely many half-spaces in some $\mathbb{R}^n$:

$$P := \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq -b_i, i = 1, 2, \dots, m\},$$

where $a_i \in (\mathbb{R}^n)^*$ are some linear functions and $b_i \in \mathbb{R}$, $i = 1, 2, \dots, m$.

Set $A_P = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = (a_{ij})$ an $m \times n$ matrix, and set $b_P = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$. Then P may be written

$$P = \{x : A_P x + b_P \geq 0\}.$$

We consider now the m-torus with representation

$$T^m = \{(t_1, t_2, \dots, t_m) = \left(e^{2\pi\varphi_1}, \dots, e^{2\pi\varphi_1}\right) \in \mathbb{C}^m; \varphi_i \in \mathbb{R}\}$$

and the standard T^m-action on $\mathbb{C}^m$;

$$(t_1, \dots, t_m) \cdot, (z_1, \dots, z_m) = (t_1 z_1, \dots, t_m z_m)$$

with orbit space $\mathbb{R}^m_{\geq} = \{|z_1|^2, \dots, |z_m|^2\}$.

We define Z_P from the pullback diagram

$$\begin{array}{ccc} Z_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \rho \\ P & \xrightarrow[i_P]{} & \mathbb{R}^m_{\geq} \end{array}$$

where $\rho((z_1, \dots, z_m)) = (|z_1|^2, \dots, |z_n|^2)$ denotes the orbit projection. Z_P is a T^m-space and i_Z a T^m-equivariant embedding.

We refer to Z_P as the *moment angle manifold* corresponding to P.

Theorem 4.2.2 ([18], [3], [4])**.** *Let $M^{2n} = (P^n, \Lambda)$ be a quasitoric manifold. Then*

$$M^{2n} = Z_P/K \text{ where } K = \ker\left(\Lambda : T^m \to T^n\right).$$

Also for each facial submanifold $M_i \subset M^{2n}$ of M^{2n} we have

$$M_i = \pi^{-1}(F_i) = Z_{F_i}/K \text{ for } 1 \leq i \leq m.$$

Now we set

$$Z_P \times_K \mathbb{C}_i = \{(z, w) : z \in Z_p, w \in \mathbb{C}_i\}/\sim$$

where $(z, w) \sim (zt^{-1}, tw)$ for every $t \in K$. Then we achieve complex line bundles

$$\rho_i : Z_P \times_K \mathbb{C}_i \to M \tag{4.2}$$

over M whose restriction to M_i is the normal bundle of the inclusion $M_i \hookrightarrow M$. As first indicated in Chapter 3 these *facial bundles* establish the stably complex structure for each (omnioriented) quasitoric manifold (see Theorem 3.1.4).

4.3 Spin Manifolds

Chapter 3 and Section 4.2 laid the ground work for interpreting (omnioriented) quasitoric manifolds in terms of the complex cobordism. The following section reinterprets these results for those quasitoric manifolds with (additional) *spin structure* in terms of *spin cobordism* which we now seek to define.

Definition 4.3.1 (Lawson and Michelsohn [15])**.** *Let X be a paracompact Hausdorff space and G a topological group. A* principle G-bundle *over X is a fibre bundle $\pi : E \to X$ together with a continuous right action of G on E which preserves the fibres acting freely and transitively on them. Thus, the fibres are exactly the orbits of G.*

Definition 4.3.2 (Lawson and Michelsohn [15])**.** *The group $Spin_n$ we identify as the double cover of SO_n exhibited here in the short exact sequence*

$$0 \to \mathbb{Z}/2 \to Spin_n \longrightarrow SO_n \to 1$$

where $\gamma : Spin_n \longrightarrow SO_n$ represents the universal covering homomorphism for all $n \geq 3$.

Theorem 4.3.3 (Lawson and Michelsohn [15] thoerem 1.2, page 79)**.** *A manifold M is orientable if and only if its first Stiefel-Whitney class satisfies $w_1(M) = 0$.*

An orientable manifold is spin provided it admits a spin structure on its tangent bundle. More precisely we state the following.

Definition 4.3.4 (Atiyah, Hirzebruch [1])**.** *Let X be a compact oriented differentiable n-dimensional manifold (without boundary) on which a Riemannian metric is introduced. Let Q be the principle tangent SO_n-bundle of X together with a covering map $\pi : P \to Q$ (of degree 2) such that the diagram*

$$\begin{array}{ccccc} P \times Spin_n & \longrightarrow & P & & \\ \pi \times \gamma \big\downarrow & & \pi \big\downarrow & \searrow & \\ & & & & X \\ & & & \nearrow & \\ Q \times SO_n & \longrightarrow & Q & & \end{array}$$

commutes, where $\gamma : Spin_n \longrightarrow SO_n$ represents the universal covering homomorphism. This principle $Spin(_n)$-bundle P over X together with the covering map π are termed a spin-structure of *X.*

Definition 4.3.5 (Lawson and Michelsohn [15] page 85)**.** *A* spin manifold *is an oriented Riemannian manifold with a spin structure on its tangent bundle.*

Theorem 4.3.6 ([Milnor [17] and [2], Lawson and Michelsohn [15] page 86 Theorem 2.1)**.** *An oriented manifold M is spin if and only if its second Stiefel-Whitney class satisfies $w_2(M) = 0$.*

Definition 4.3.7. *The set of cobordism classes of spin manifolds may be equipped with the structure of a graded ring via the operations product and disjoint union. This we call the* spin cobordism ring *and write Ω^{spin}.*

Recall that our goal has been to develop a method for viewing any spin quasitoric manifold as a codimension two sub-manifold of an ambient quasitoric manifold. It is worth noting that within the required set-up for dualization as in [10] the ambient manifold should in general be what is termed a spin^c-manifold. All quasitoric manifolds are spin^c-manifold owing to the existence of stably complex structure (Wiemeler [23] page 14).

A *spin^c-structure* on a manifold is similar to spin structures as above except we use the spin^c group as defined by the short exact sequence

$$1 \to \mathbb{Z}/2 \to Spin_n^c \longrightarrow SO_n \times U_1 \to 1.$$

Alternatively, we may state the following.

Definition 4.3.8 (Lawson and Michelsohn [15] page 390)**.**

$$Spin^c(n) \equiv Spin(n) \times_{\mathbb{Z}/2} U_1.$$

Definition 4.3.9 (Lawson and Michelsohn [15] page 391)**.** *Let Q be a principal SO_n-bundle over X and let P be a principle $Spin^c_n$-bundle over Q. Finally, let S be a principle U_1-bundle over M^{2n}, all so that the diagram*

$$\begin{array}{ccccc} P \times Spin^c_n & \longrightarrow & P & & \\ \pi^c \times \gamma^c \downarrow & & \pi^c \downarrow & \searrow & \\ & & & & X \\ & & & \nearrow & \\ (Q \times_{M^{2n}} S) \times (SO_n \times U_1) & \longrightarrow & Q \times_{M^{2n}} S & & \end{array}$$

commutes, where $\gamma^c : Spin^c_n \longrightarrow SO_n \times U_1$ represents a two-fold covering. This principle $spin^c_n$-bundle P over X together with the covering map π^c are termed a spin^c_n-structure *on X.*

Definition 4.3.10 (Lawson and Michelsohn page 391 [15])**.** *An oriented Riemannian manifold with a $spin^c$-structure on its tangent bundle is called a* spin^c-manifold*.*

Spin Quasitoric Manifolds

We present now an immediate corollary to Theorem 4.3.6. Since the reduction mod 2 of the first Chern class is w_2, we have the following.

Corollary 4.3.11. *A quasitoric manifold M^{2n} is spin if and only if its first Chern class satisfies $c_1(M) \equiv 0 \bmod 2$.*

Complex projective spaces $\mathbb{C}P^n$ are spin if and only if n is odd. We now include a result of Kuroki which distinguishes the existence of spin quasitoric manifolds from criteria in terms of only combinatorial data.

Theorem 4.3.12 (Kuroki, [13])**.** *A quasitoric manifolds $M^{2n} = M(P^n, \Lambda)$ is spin if and only if*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1 \pmod 2$$

for each facet vector $\lambda \in \Lambda$.

Example 4.3.13. *Odd dimensional complex projective spaces are spin. As indicated at the start of Chapter 2 we have* $\mathbb{C}P^n = \Delta^n, \Lambda_n$ *where*

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & & 0 & -1 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

We see that all the columns sums are odd if and only if n *is odd. Hence* $\mathbb{C}P^n$ *is spin if and only if* n *is odd.*

4.4 Spin Quasitoric Manifolds and the Wedge Construction

Theorem 4.4.1. *Let* M^{2n} *be a quasitoric manifold and let* M^w *be the associated canonical wedge 2.3.2.* M^{2n} *is dual mod 2 to the first Chern class of* M^w *if and only if* M^{2n} *is spin.*

Proof. Let $M = M^{2n}$ be a spin quasitoric manifold given by (di)-characteristic pair (P^n, Λ) with canonically chosen wedge M^w. Let $x_k = c_1(\rho_k)$ where the bundles ρ_k are defined as in Section 4.2 over the wedge quasitoric manifold M^w. Here each is determined by a facial submanifold of the wedge. Specifically, we require that x_0 be determined via the inclusion $M^{2n} \subset M^w$. Then

$$\begin{aligned} c_1\left(M^w\right) = x_0 + & x_1 + x_2 + \cdots x_n \equiv x_0 \bmod 2 \\ \Leftrightarrow \qquad & x_1 + x_2 + \cdots x_n \equiv 0 \bmod 2 \\ \Leftrightarrow \qquad & x_1' + x_2' + \cdots x_n' \equiv 0 \bmod 2 \\ \Leftrightarrow \qquad & M^{2n} \text{ is spin.} \end{aligned}$$

Whereas the classes $x_k = c_1(\rho_k) \in H^2\left(M^w, \mathbb{Z}\right)$ are associated with facial sub-manifolds over the lateral facets within the wedge, we have corresponding classes $x_k' = c_1(\rho_k) \in H^2\left(M^{2n}, \mathbb{Z}\right)$ associated with the facial sub-manifolds corresponding the "original" facets themselves. The sum $x_1 + x_2 + \cdots x_n$ vanishes precisely when $x_1' + x_2' + \cdots x_n'$ does because of the structure of the wedge characterstic matrix and the associated additive relations given by 4.1 in Theorem 4.2.1. With respect to the classes x_k for only $k = 1, 2, \dots m$ the wedge characterstic matrix

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \lambda_{n+1,1} & \lambda_{n+2,1} & \cdots & \lambda_{m,1} \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ 0 & & & & 0 & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \lambda_{n+1,n} & \lambda_{n+2,n} & \cdots & \lambda_{m,n} \end{pmatrix}$$

provides exactly the same additive relation 4.1 as those for the classes x_k' given by the original matrix Λ. Specifically, these are

$$x_i = -\lambda_{i,n+1}x_{n+1} - \cdots - \lambda_{i,m}x_m \text{ and}$$
$$x_i' = -\lambda_{i,n+1}x_{n+1}' - \cdots - \lambda_{i,m}x_m'$$

for each $1 \leq i \leq n$, given by Λ^w and Λ respectively .

□

Theorem 4.4.1 is not true in more generality than mod 2 unless we include positive/negative scalars on each of the columns of the wedge characteristic matrix as we see in example 4.4.3. This brings us to the following.

Question 4.4.2. *Which spin quasitoric manifolds are dual to the first Chern class their canonical wedges?*

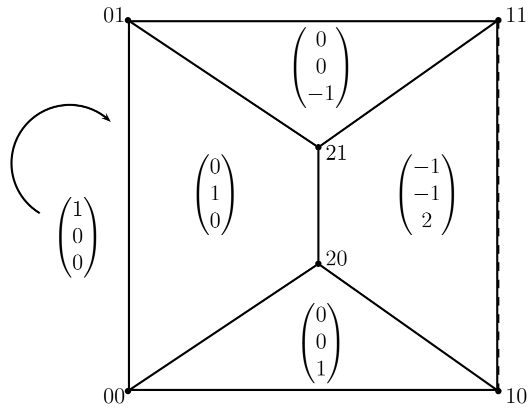
More formally, and with regards to the canonical wedge construction, we ask for which spin quasitoric manifolds M^{2n} do we have

$$c_1(M^w) = x_0$$

where the class x_0 is Poincaré dual to the base manifold M^{2n}? We recall now the wedge quasitoric manifold over the Hirzebruch surface $\mathbb{H}_2^4$. Here we have a wedge characteristic matrix

$$\Lambda^w = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & 1 & 0 & -1 & 0 \\ \mathbf{0} & 0 & 1 & 2 & -1 \end{pmatrix}.$$

We are interested in spin cobordism rather than complex and the wedge construction allows for the conjugation of any or all of these bundles ρ_i via the negation of particular columns in the corresponding wedge characteristic matrix. We may for



instance choose a scalar α_4 which negates the final column. Negation of any particular column of a characterstic matrix corresponds to the negation of the associated facial bundles ρ_i, see Panov [18] remark 5.8 page 17.

We now have the wedge characteristic matrix

$$\Lambda^w = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \bar{\rho}_4 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & +\mathbf{1} \end{pmatrix}. \tag{4.3}$$

The ideal J_{Λ^w} is determined from each row of Λ^w yielding the additive relations

$$x_i = -\lambda_{i,n+1}x_{n+1} - \cdots - \lambda_{i,m}x_m$$

for each $1 \leq i \leq n$. In particular we have equivalences $x_1 = x_3$ and $x_2 = -2x_3 - x_4$. We may then calculate the first Chern class of the wedge manifold

$$\begin{aligned} c_1\left(\mathbb{H}_2^w\right) &= x_0 + x_1 + x_2 + x_3 + x_4 \\ &= x_0 + (x_3) + (-2x_3 - x_4) + x_3 + x_4 \\ &= x_0 = c_1\left(\rho_0\right). \end{aligned}$$

We wish to determine when conjugation of bundles ρ_i or equivalently the negation of columns of the characteristic matrix may allow the sum $x_1+x_2+\cdots+x_n$ to vanish. Consider first the canonical wedge characteristic matrix

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & \lambda_{14} & \lambda_{15} \\ 0 & 0 & 1 & 0 & \lambda_{24} & \lambda_{25} \\ 0 & 0 & 0 & 1 & \lambda_{34} & \lambda_{35} \end{pmatrix} \tag{4.4}$$

Consider then the additive relations given by ideal J in Theorem 4.2.1.

$$\begin{aligned} x_1 &= -\lambda_{14}x_4 - \lambda_{15}x_5 \\ x_2 &= -\lambda_{24}x_4 - \lambda_{25}x_5 \\ x_3 &= -\lambda_{34}x_4 - \lambda_{35}x_5. \end{aligned}$$

So now,

$$\begin{aligned} c_1(M^w) &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ &= x_0 + (-\lambda_{14}x_4 - \lambda_{15}x_5) + (-\lambda_{24}x_4 - \lambda_{25}x_5) + (-\lambda_{34}x_4 - \lambda_{35}x_5) + x_4 + x_5 \\ &= x_0 + (1 - \lambda_{14} - \lambda_{24} - \lambda_{34})x_4 + (1 - \lambda_{15} - \lambda_{25} - \lambda_{35})x_5 \end{aligned}$$

If we include a scalar on each column given by $\alpha_i \in \{\pm 1\}$ then we have

$$\Lambda^w_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & -\alpha_4 & 0 \\ 0 & \alpha_1 & 0 & 0 & \alpha_4\lambda_{14} & \alpha_5\lambda_{15} \\ 0 & 0 & \alpha_2 & 0 & \alpha_4\lambda_{24} & \alpha_5\lambda_{25} \\ 0 & 0 & 0 & \alpha_3 & \alpha_4\lambda_{34} & \alpha_5\lambda_{35} \end{pmatrix} \tag{4.5}$$

and we get

$$\begin{aligned} x_1 &= -\alpha_1\alpha_4\lambda_{14}x_4 - \alpha_1\alpha_5\lambda_{15}x_5 \\ x_2 &= -\alpha_2\alpha_4\lambda_{24}x_4 - \alpha_2\alpha_5\lambda_{25}x_5 \\ x_3 &= -\alpha_3\alpha_4\lambda_{34}x_4 - \alpha_3\alpha_5\lambda_{35}x_5 \end{aligned}$$

Hence,

$$\begin{aligned} c_1(M) &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ &= x_0 + (1 - \alpha_1\alpha_4\lambda_{14} - \alpha_2\alpha_4\lambda_{24} - \alpha_3\alpha_4\lambda_{34})x_4 + (1 - \alpha_1\alpha_5\lambda_{15} - \alpha_2\alpha_5\lambda_{25} - \alpha_3\alpha_5\lambda_{35})x_5 \end{aligned}$$

Therefore we may achieve $c_1(M) = x_0$ provided a solution to

$$\begin{aligned} \alpha_1\alpha_4\lambda_{14} + \alpha_2\alpha_4\lambda_{24} + \alpha_3\alpha_4\lambda_{34} &= 1 \\ \alpha_1\alpha_5\lambda_{15} + \alpha_2\alpha_5\lambda_{25} + \alpha_3\alpha_5\lambda_{35} &= 1 \end{aligned}$$

or more simply

$$\begin{aligned} \alpha_1\lambda_{14} + \alpha_2\lambda_{24} + \alpha_3\lambda_{34} &= \alpha_4 \\ \alpha_1\lambda_{15} + \alpha_2\lambda_{25} + \alpha_3\lambda_{35} &= \alpha_5 \end{aligned} \tag{4.6}$$

in $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ with each $\alpha_i = \{1, -1\}$.

Example 4.4.3. *Consider now the connected sum of* $\mathbb{C}P^3$ *with itself. Let* $M^6 = \mathbb{C}P^3 \# \mathbb{C}P^3 = (\Delta^2 \times \mathbb{I}, \Lambda)$ *(see Example 1.2.4 and Section 2.3) where*

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

We may then form the canonical wedge and choose a wedge characteristic matrix of the form

$$\Lambda^w = \begin{pmatrix} 1 & 0 & 0 & 0 & \alpha_4 & 0 \\ 0 & \alpha_1 & 0 & 0 & \alpha_4\lambda_{14} & \alpha_5\lambda_{15} \\ 0 & 0 & \alpha_2 & 0 & \alpha_4\lambda_{24} & \alpha_5\lambda_{25} \\ 0 & 0 & 0 & \alpha_3 & \alpha_4\lambda_{34} & \alpha_5\lambda_{35} \end{pmatrix}$$

for any $\alpha \in \{\pm 1\}^5$*. Substituting into Equations 4.6 we achieve the system of equations*

$$-\alpha_1 - \alpha_2 - \alpha_3 = \alpha_4$$
$$-\alpha_1 - \alpha_2 - \alpha_3 = \alpha_5.$$

This system contains a number of solutions. In particular, we may set $\alpha_1 = \alpha_2 = 1$*,* $\alpha_3 = \alpha_4 = \alpha_5 = -1$ *so that*

$$\begin{aligned} c_1(M^w) &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ &= x_0 + [\alpha_4 - \alpha_1\lambda_{14} - \alpha_2\lambda_{24} - \alpha_3\lambda_{34}]\, x_4 + [\alpha_5 - \alpha_1\lambda_{15} - \alpha_2\lambda_{25} - \alpha_3\lambda_{35}]\, x_5 \\ &= x_0 + [-1 - 1(-1) - 1(-1) - (-1)(-1)]\, x_4 + [-1 - 1(-1) - 1(-1) - (-1)(-1)]\, x_5 \\ &= x_0 + 0 \cdot x_4 + 0 \cdot x_5 = x_0. \end{aligned}$$

Spin Quasitoric Manifolds and Wedge Dualization

We now establish our main theorem.

Theorem 4.4.4. *Let* $M^{2n} = (P^n, \Lambda)$ *be a quasitoric manifold. If there exists a solution in* $\alpha = (\alpha_1, \cdots, \alpha_m)$ *to the* $m - n$ *equations*

$$\begin{aligned} \alpha_1\lambda_{1(n+1)} + \alpha_2\lambda_{2(n+1)} + \cdots + \alpha_n\lambda_{n(n+1)} &= \alpha_{n+1} \\ \alpha_1\lambda_{1(n+2)} + \alpha_2\lambda_{2(n+2)} + \cdots + \alpha_n\lambda_{n(n+2)} &= \alpha_{n+2} \\ &\vdots \\ \alpha_1\lambda_{1m} + \alpha_2\lambda_{2m} + \cdots + \alpha_n\lambda_{nm} &= \alpha_m \end{aligned} \tag{4.7}$$

where $\alpha_i = \pm 1$ for each $1 \le i \le m$ then their exists a wedge quasitoric manifold with a corresponding wedge characteristic matrix Λ^w_α so that M^{2n} is dual to the first Chern class of this ambient manifold.

Proof. We wish to determine when selective conjugation of bundles ρ_i (or equivalently the negation of certain columns of the characteristic matrix) may allow the sum $x_1 + x_2 + \cdots + x_n$ of cohomology classes in the wedge to vanish. Consider first the canonical wedge characteristic matrix

$$\Lambda^w = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & \lambda_{1(n+1)} & \lambda_{1(n+2)} & \cdots & \lambda_{1m} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 & \lambda_{n(n+1)} & \lambda_{n(n+2)} & \cdots & \lambda_{nm} \end{pmatrix}$$

Consider the additive relations 4.1 given by ideal J in Theorem 4.2.1.

$$\begin{aligned} x_1 &= -\lambda_{1(n+1)}x_{n+1} - \lambda_{1(n+2)}x_{(n+2)} - \cdots - \lambda_{1m}x_m \\ &\vdots \\ x_n &= -\lambda_{n(n+1)}x_{n+1} - \lambda_{n(n+2)}x_{(n+2)} - \cdots - \lambda_{nm}x_m. \end{aligned}$$

So,

$$\begin{aligned} c_1(M^w) &= x_0 + x_1 + \cdots + x_n + x_{n+1} + \cdots + x_m \\ &= x_0 + \left(-\lambda_{1(n+1)}x_{n+1} - \lambda_{1(n+2)}x_{(n+2)} - \cdots - \lambda_{1m}x_m\right) + \\ &\quad \cdots + \left(-\lambda_{n(n+1)}x_{n+1} - \lambda_{n(n+2)}x_{(n+2)} - \cdots - \lambda_{nm}x_m\right) + x_{n+1} + \cdots + x_m \\ &= x_0 + \left(1 - \lambda_{1(n+1)} - \lambda_{2(n+1)} - \cdots - \lambda_{n(n+1)}\right) x_{n+1} + \\ &\quad \cdots + \left(1 - \lambda_{1m} - \lambda_{2m} - \cdots - \lambda_{nm}\right) x_m. \end{aligned}$$

Now consider a scalar $\alpha_i \in \{\pm 1\}$ on each column of the canonical wedge characteristic matrix so that now

$$\Lambda^w_\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\alpha_{n+1} & 0 & \cdots & 0 \\ 0 & \alpha_1 & & 0 & \alpha_{n+1}\lambda_{1(n+1)} & \alpha_{n+2}\lambda_{1(n+2)} & \cdots & \alpha_m\lambda_{1m} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \alpha_n & \alpha_{n+1}\lambda_{n(n+1)} & \alpha_{n+2}\lambda_{n(n+2)} & \cdots & \alpha_m\lambda_{nm} \end{pmatrix}$$

and we get

$$\begin{aligned} \alpha_1 x_1 &= -\alpha_{n+1}\lambda_{1(n+1)}x_{n+1} - \cdots - \alpha_m\lambda_{1m}x_m \\ &\vdots \\ \alpha_n x_n &= -\alpha_{n+1}\lambda_{n(n+1)}x_{n+1} - \cdots - \alpha_m\lambda_{nm}x_m \end{aligned}$$

or more simply

$$
\begin{aligned}
x_1 &= -\alpha_1\alpha_{n+1}\lambda_{1(n+1)}x_{n+1} - \cdots - \alpha_1\alpha_m\lambda_{1m}x_m \\
&\vdots \\
x_n &= -\alpha_n\alpha_{n+1}\lambda_{n(n+1)}x_{n+1} - \cdots - \alpha_n\alpha_m\lambda_{nm}x_m.
\end{aligned}
$$

So,

$$
\begin{gathered}
c_1(M^w) = x_0 + \big(1 - \alpha_1\alpha_{n+1}\lambda_{1(n+1)} - \cdots - \alpha_n\alpha_{n+1}\lambda_{n(n+1)}\big)\, x_{n+1} + \\
\cdots + (1 - \alpha_1\alpha_m\lambda_{1m} - \cdots - \alpha_n\alpha_m\lambda_{nm})\, x_m.
\end{gathered}
$$

Therefore we may achieve $c_1(M^w) = x_0$ with $M^{2n} \subset M^w = (P^w, \Lambda^w_\alpha)$ provided there exists any solution to

$$
\begin{aligned}
\alpha_1\alpha_{n+1}\lambda_{1(n+1)} + \cdots + \alpha_n\alpha_{n+1}\lambda_{n(n+1)} &= 1 \\
&\vdots \\
\alpha_1\alpha_m\lambda_{1m} + \cdots + \alpha_n\alpha_m\lambda_{nm} &= 1
\end{aligned}
$$

or more simply

$$
\begin{aligned}
\alpha_1\lambda_{1(n+1)} + \alpha_2\lambda_{2(n+1)} + \cdots + \alpha_n\lambda_{n(n+1)} &= \alpha_{n+1} \\
\alpha_1\lambda_{1(n+2)} + \alpha_2\lambda_{2(n+2)} + \cdots + \alpha_n\lambda_{n(n+2)} &= \alpha_{n+2} \\
&\vdots \\
\alpha_1\lambda_{1m} + \alpha_2\lambda_{2m} + \cdots + \alpha_n\lambda_{nm} &= \alpha_m.
\end{aligned}
$$

for $\alpha = (\alpha_1, \cdots, \alpha_m)$ with $\alpha_i = \pm 1$ for each $1 \le i \le m$. $\square$

Example 4.4.5. *We consider now a non-example, the Hirzebruch surface $\mathbb{H}_{10} = (\mathbb{I}^2, \Lambda)$ where*

$$
\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 10 & -1 \end{pmatrix}.
$$

Consider then any wedge quasitoric manifold given by $(\Delta^2 \times \mathbb{I}, \Lambda^w_\alpha)$ so that

$$
\Lambda^w_\alpha = \begin{pmatrix} 1 & 0 & 0 & -\alpha_3 & 0 \\ 0 & \alpha_1 & 0 & -\alpha_3 & 0 \\ 0 & 0 & \alpha_2 & 10\alpha_3 & -\alpha_4 \end{pmatrix}.
$$

From 4.2.1 we achieve then the system of equations

$$\begin{aligned} -\alpha_1 x_1 + 10\alpha_2 x_2 &= \alpha_3 x_3 \\ -\alpha_2 x_2 &= \alpha_4 x_4. \end{aligned}$$

This system contains no solution in $\alpha \in \{\pm 1\}^4$. *Cohomology class* x_2 *cannot be killed off in the sum* $x_1 + x_2 + x_3 + x_4$ *in order to yield* $c_1(\mathbb{H}_{10}^w) = x_0$. *Hence* $\mathbb{H}_{10}$ *is not realizable as dual to the first Chern class of the canonical wedge manifold.*

Appendices

Using Macaulay2 to Calculate Wedge QTMs

We now demonstrate how open source mathematics software Macaulay2 [11] may be used to calculate wedge quasitoric manifolds. In our first example we revisit the Hirzebruch surface to demonstrate how the software may be used to generate $\mathbb{H}_r^w = (P^w, \Lambda^w)$ as shown above.

```
Macaulay2, version 1.4
   with packages:  ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
                             PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage"Polyhedra";

i2 :  P = hypercube 2


i3 :  halfspaces (P)
```

$$o3 = \left(\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$o3$: Sequence

```
i4 :  vertexFacetMatrix(P)
```

$$o4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$o4$: Matrix

```
i5 :  M1 = vertices P
```

$$o5 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$o5$: Matrix

```
i6 :  M2 = M1||matrix{{0,0,0,0}};
```

```
i7 :  M3 = M2 | matrix{{0},{0},{1}};
i8 :  CP = convexHull(M3)

i9 :  (HS,v) = halfspaces (CP)
```

$$o9 = \left(\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$o9$: Sequence

```
i10 :  entries HS
```

$$o10 = \{\{0,0,-1\},\{-1,0,1\},\{1,0,1\},\{0,-1,1\},\{0,1,1\}\}$$

$o10$: List

```
i11 :  WHS = matrix{{0,0,-1},{-1,0,1},{1,0,1},{0,-1,1},{0,1,2}}
```

$$o11 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$o11$: Matrix

```
i12 :  WP = intersection (WHS,v);
i13 :  H1 = (vertices(WP))
```

$$o13 = \begin{pmatrix} -1 & 1 & -1 & 1 & -\frac{1}{3} & \frac{1}{3} \\ -1 & -1 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$o13$: Matrix

```
i14 :  VFM = vertexFacetMatrix(WP)
```

$$o14 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 1 & 0 & 1 & 1 \\ 6 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$o14$: Matrix

```
i15 :  R = QQ[a..r]
```

$o15 = R$

$o15$: PolynomialRing

```
i16 :  L = matrix{{0,1,-1,0,0},{0,0,r,-1,1},{1,a,c,d,b}}
```

$$o16 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & r & -1 & 1 \\ 1 & a & c & d & b \end{pmatrix}$$

$o16$: Matrix

```
i17 :  v5 = det (L_{1,3,4})
```

$o17 = -b - d$

$o17$: R

```
i18 :  v6 = det (L_{2,3,4})
```

$o18 = b + d$

$o18$: R

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