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A Characterization of Serre Classes of Reflexive Modules Over a Complete Local Noetherian Ring

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A CHARACTERIZATION OF SERRE CLASSES OF REFLEXIVE MODULES
OVER A COMPLETE LOCAL NOETHERIAN RING

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Casey Ryan Monday
Lexington, Kentucky

Director: Dr. Edgar Enochs, Professor of Mathematics
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ABSTRACT OF DISSERTATION

A CHARACTERIZATION OF SERRE CLASSES OF REFLEXIVE MODULES OVER A COMPLETE LOCAL NOETHERIAN RING

Serre classes of modules over a ring R are important because they describe relationships between certain classes of modules and sets of ideals of R. We characterize the Serre classes of three different types of modules. First we characterize all Serre classes of noetherian modules over a commutative noetherian ring. By relating noetherian modules to artinian modules via Matlis duality, we characterize the Serre classes of artinian modules. A module M is reflexive with respect to E if the natural evaluation map from M to $M^{\nu\nu}$ is an isomorphism where $M^{\nu\nu} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$. When R is complete local and noetherian, take E as the injective envelope of the residue field of R. The main result provides a characterization of the Serre classes of reflexive modules over a complete local noetherian ring. This characterization depends on an ability to "construct" reflexive modules from noetherian modules and artinian modules. We find that Serre classes of reflexive modules over a complete local noetherian ring are in one-to-one correspondence with pairs of collections of prime ideals which are closed under specialization.

KEYWORDS: Serre class, reflexive module, Matlis duality, complete local ring, local nilpotence

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Date: 02. May 2014

This is dedicated to my husband, my parents and the memory of my grandparents.

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Chapter 1 Introduction

1.1 Motivation

Serre classes have held the interest of researchers since their introduction in [13] by Jean-Pierre Serre. This is due to their relationship with localization which is explained by Gabriel in [6]. The task of classifying all Serre classes of R-modules for a general ring R remains unfinished. To quote Walker and Walker ([14]), "It is a hopeless task to attempt to characterize all Serre classes for an arbitrary R." Indeed, this is a daunting task, but significant progress has been made.

Gabriel characterized Serre classes of left $R-$modules which are closed under arbitrary direct sums. In 1972, Walker and Walker [14] built upon Gabriel's work and –assuming the ring commutative and noetherian– characterized the Serre classes of modules whose socles are essential. Their approach was to first study characterizations of more general classes such as additive classes and bounded, complete additive classes.

In 2000, Mark Hovey and Karen Collins of Wesleyan University claimed, but never published, the characterization of Serre classes of noetherian modules over a general commutative ring. In 2003 Manuel López classified all Serre classes of artinian modules over a complete local noetherian ring using their results. His result relies also on the work of Lam in [10] which describes a bijection between Serre classes of noetherian modules and artinian modules.

The work included in this thesis classifies all Serre classes of noetherian modules over a commutative noetherian ring and all Serre classes of artinian modules with the additional assumption that the base ring is complete. We use different techniques than those mentioned above. The culmination of ideas presented in this paper provides a characterization of Serre classes of reflexive modules over a commutative local noetherian ring which is also complete. This characterization will rely on the ability to "build" reflexive modules from noetherian modules and artinian modules. Throughout, arguments will rely on properties of Matlis duality, bijections formed between Serre classes and sets of certain ideals, and the relationship between modules and their Matlis duals.

Many results in this paper require familiarity with the dual of an $R-$module. In the study of duality, objects of interest include those which are self-dual, dual-preserving or even dual-reversing. We will focus on the properties of reflexive modules. These modules are isomorphic to their double dual (the dual of their dual) via the canonical map which we define in the following section. Results on duality can be traced back to Gauss's work which studied the set of homomorphisms from a finite abelian group G to the quotient $\mathbb{Q}/\mathbb{Z}$. Gauss defined this set of homomorphisms as the dual of G. Gauss used the fact that every finite abelian group is reflexive to prove many of his number theory results.

Finite dimensional vector spaces provide another arena for studying duality. In 1974 Halmos wrote about the nature of this duality and gave an argument that any vector space of finite dimension is reflexive. So that we can become familiar with reflexive modules, a proof of this fact is included in chapter two.

After studying the duality of vector spaces, a natural next step is considering duality for modules. In particular, the algebraic dual of a left $R-$module M is the set of module homomorphisms from M to R, which we represent by $\mathrm{Hom}(M, R)$. This dual is a right $R-$module, but not every M is reflexive. The question of which M are reflexive comes up in commutative algebra.

In the case where R is complete, local and noetherian, we consider the Matlis dual ([11]). This is the dual which considers $M^{\nu} = \mathrm{Hom}(M, E)$ as the dual of M with respect to E. Here E is the injective envelope of $R/\mathfrak{m}$ and $\mathfrak{m}$ is the unique maximal ideal of R. We will be interested in using Matlis duality to better understand the relationship between noetherian modules and artinian modules. In turn, this will help us in our study of reflexive modules. Matlis duality is also related to Macaulay duality which concerns graded polynomial rings. In particular, Macaulay duality is used to understand properties of Gorenstein ideals.

More recent work in duality can be attributed to Grothendieck who studied a local dual related to the Matlis dual. In this work there is the notion of a dualizing module. This module is useful because certain finitely generated modules are reflexive with respect to their dualizing module. In fact, if dualizing modules are in R, R must be Cohen-Macaulay. Not all Cohen-Macaulay rings have dualizing modules, but they do if they are complete and local. In [5], Enochs, López-Ramos and Torrecillas showed that if a ring R admits two Matlis dualizing modules, M and N, then an $R-$module is reflexive with respect to M if and only if it is reflexive with respect to N. Further, they found that the Matlis dualizing modules exist in bijective correspondence with invertible $(R, R)-$bimodules.

Grothendieck extended the study of duality to complexes (chains of modules with their module homomorphisms). In particular, if C and D are complexes of modules, we can form $\mathrm{Hom}(C, D)$ with the natural homomorphism $\mathrm{Hom}(\mathrm{Hom}(C, D), D)$. As isomorphisms of complexes are rare, Grothendieck instead considered isomorphisms of homologies. Homologies are quotient modules formed by specific kernels and images of the homomorphisms from the complexes at each step. Grothendieck described situations in which these homologies are isomorphic. Grothendieck's duality extends to sheaves and complexes of sheaves. Another duality which also concerns sheaves is called Serre duality. While this duality is named for the same Serre of the Serre classes studied here, we will not be concerned with sheaves or Serre duality.

A recent and interesting result in [2] described by Belshoff, Enochs and Garcia Rozas also considers Matlis duality and gives a classification of reflexive modules, M, where R is only assumed commutative and noetherian. This result gives that M is reflexive with respect to Matlis duality if and only if M has a finitely generated submodule, S so that M/S is artinian and $R/\operatorname{ann}(M)$ is a complete and semi-local ring. (Note that the **annihilator** of an element $x \in M$ is $\operatorname{ann}(x) = \{r \in M | rx = 0\}$.) The S and M/S referred to in this work explain how reflexive modules can be built up by noetherian modules and artinian modules. A similar construction is explained in chapter six which allows us to characterize Serre classes of reflexives over a complete local noetherian ring.

In [15], Xue generalized the results in [2] by showing an $R-$module M is reflexive if and only if $R/\operatorname{ann}(M)$ is linearly compact and M has a finitely generated submodule S so that M/S is finitely cogenerated. Xue explains that the Prüfer group (described below) is linearly compact but lacks the additional assumption on submodules. By noticing the Prüfer group is not reflexive, we see linear compactness is not enough to guarantee reflexivity.

For each characterization result in this dissertation, we will be concerned with different types of filters. We will consider modules with a finite filtration, and our characterization of Serre classes will depend on a filter which is closed under specialization. In the recent work of Kameyama ([8]), Matlis duality is extended to filtered noetherian rings. In this situation, the filter is the one associated with the $\mathfrak{m}-$adic topology described here in chapter five. Specifically, Kameyama showed that Matlis duality extends to filtered pseudocompact algebras. In the case where this algebra fits under some additional restraints, Xue also examined local cohomology. Cohomology is the algebraic dual of homology.

In this dissertation we will be primarily concerned with the Hom functor. In recent work by Kubic, Leamer and Sather-Wagstaff ([9]), it is assumed that R is commutative local and noetherian. The authors show that if M_1 is an artinian $R-$module and M_2 is a noetherian $R-$module, then $\operatorname{Hom}(M_1, M_2)$ has finite length. They also give results on the Ext and Tor functors. The relationship they describe between Ext and Tor over the completion of R is similar to a result included in chapter four for the duals of noetherian and artinian modules. Kubik, Leamer and Sather-Wagstaff also show that when M_1 is artinian and M_2 is Matlis reflexive, then $\operatorname{Ext}^i_R(M_1, M_2)$, $\operatorname{Ext}^i_R(M_2, M_1)$ and $\operatorname{Tor}^R_i(M_1, M_2)$ are each Matlis reflexive.

Serre classes are also relevant to other areas of current research such as the work in [7]. Here Garkusha and Prest give a bijection between thick subcategories of perfect complexes and the Serre classes of finitely presented modules. In this paper we will not address complexes, homology, cohomology or the Ext and Tor functors. However, their relationship to Serre classes and duality highlights many avenues for extending the efforts which follow.

1.2 Notation

Throughout this paper, we assume R to be commutative unles otherwise stated. In this section we introduce basic definitions, notation and ideas which will be needed throughout the paper. We start by describing both the dual of a module and the dual of a linear $R-$module homomorophism.

Duals

When R is a commutative ring and M, N are $R-$modules, we define $\mathrm{Hom}_R(M, N)$ as the set of all $R-$linear module homomorphisms from M to N. Now, if M_1, M_2, N are $R-$modules, an element $f \in \mathrm{Hom}_R(M_1, M_2)$ gives a natural linear homomorphism from $\mathrm{Hom}_R(M_2, N)$ to $\mathrm{Hom}_R(M_1, N)$ by mapping an $h \in \mathrm{Hom}_R(M_2, N)$ to $h \circ f$. We write $M^\nu = \mathrm{Hom}_R(M, N)$ for the dual of M with respect to N and write $h \circ f$ as f^ν. We will also define linear map to mean linear $R-$module homomorphism.

Definition 1.1. *By the* ***canonical homomorphism*** *from M to $M^{\nu\nu}$, we mean the homomorphism which takes $x \in M$ to the map in $M^{\nu\nu}$ which evaluates a homomorphism in M^ν at the element x.*

We define M to be a **reflexive** $R-$module if the canonical homomorphism is an isomorphism. Throughout this dissertation, we will refer to the canonical homomorphism by ϕ where $\phi(x) = (\varphi \mapsto \varphi(x))$ where $x \in M$ and $\varphi \in M^\nu$.

Exact Sequences

Given linear maps $f : M \to N$ and $g : N \to P$ between $R-$modules M, N and P, we say $M \xrightarrow{f} N \xrightarrow{g} P$ is an **exact sequence** if $\mathrm{Im}(f) = \ker(g)$. This definition can be extended to longer sequences –even infinite sequences.

Definition 1.2. *We call an exact sequence of the form $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ a* ***short exact sequence****.*

Note that based on the definition of an exact sequence, we have f is injective and g is surjective. It follows easily from the definition that given any submodule $S \subset M$, $0 \to S \hookrightarrow M \xrightarrow{p} M/S \to 0$ is short exact where $\hookrightarrow$ represents the inclusion map and p represents the projection $p(x) = x + S$ for any $x \in M$.

We say that two short exact sequences $0 \to M_1 \xrightarrow{f} N_1 \xrightarrow{g} P_1 \to 0$ and $0 \to M_2 \xrightarrow{f} N_2 \xrightarrow{g} P_2 \to 0$ are isomorphic if we have isomorphisms $\sigma : M_1 \to M_2$ and $\varphi : P_1 \to P_2$ so that the diagram formed by joining the two short exact sequences via σ and φ is commutative. Note that by the short five lemma, this makes $N_1 \cong N_2$. The following property of short exact sequences will be quite useful to us throughout this dissertation, as it is often preferable to reduce arguments on general submodules to arguments on submodules and their quotients.

Lemma 1.3. *Any short exact* $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ *is isomorphic to a short exact sequence of the form* $0 \to S \hookrightarrow M \xrightarrow{p} M/S \to 0$ *where* S *is a submodule of* N.

Proof. Two applications of the first isomorphism theorem for modules give $M \cong f(M) \subset N$ and $P \cong N/\ker(g) = N/f(M)$. In the following diagram, let id represent the identity map. Notice the diagram is commutative as g is an isomorphism on $N/\ker(g) = N/\ker(f)$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\cong"'] & N \arrow[r, "g"] \arrow[d, "id"'] & P \arrow[d, "\cong"'] \\ f(M) \arrow[r, "i"] & N \arrow[r, "p"] & N/f(M) \end{tikzcd}$$

□

Simple Modules and Module Length

A left $R-$module, M is said to be **simple** if $M \neq 0$ and if the only submodules of M are 0 and M itself. One standard result of this definition is given below as a remark. Another result is that if M is simple, then M is isomorphic to R/I for some maximal left ideal I of R.

Remark 1.4. If S and T are two simple left $R-$modules and if $f : S \to T$ is linear, then either $f = 0$ or f is an isomorphism.

Together, this remark and the fact above show that if R is a commutative local ring, any two simple $R-$modules are isomorphic to each other. This relies on the fact that, in a local ring, there is a unique maximal ideal, $\mathfrak{m}$.

Definition 1.5. *If* M *is a left* $R-$*module, a* ***finite filtration*** *of* M *is a finite increasing sequence of modules* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *where* $n < \infty$.

The module M is said to be a **module of finite length** if there is such a filtration where each of the quotients M_i/M_{i-1} is a simple module. The result that if M has finite length that any submodule $S \subset M$ and its quotient M/S also have finite length is standard. An equivalent description of modules of finite length will be needed for our final result, and is introduced in the next chapter.

It can be shown by induction that any two different finite filtrations of the same module with simple quotients must have the same length. The unique length of these filtrations is called the length of M and we denote it by $\ell(M)$. It is a standard result that any vector space V over a field F has finite length if and only if it has finite dimension and that $\ell(V) = \dim(V)$.

Given a filtration of M and any submodule $S \subset M$, considering the filtration built by submodules of the form $S \cap M_i$, we can show that any submodule S also has finite length. Additionally, considering the filtration formed by submodules $(M_i + S)/S$, we get that M/S has finite length. The more interesting result about modules of finite length is that $\ell(M) = \ell(S) + \ell(M/S)$.

Noetherian and Artinian Modules

We define an $R-$module, M to be **noetherian** if it satisfies the ascending chain condition (ACC) on submodules. This means that for every increasing sequence $S_0 \subset S_1 \subset S_2 \cdots$ of submodules of M, there exists some $n \geq 0$ so that $S_m = S_n$ whenever $m \geq n$. An equivalent definition is that any submodule $S \subset M$ is finitely generated.

Analogously, we define a module to be **artinian** if it satisfies the descending chain condition (DCC) on submodules. In other words, for every decreasing sequence $\cdots \subset S_2 \subset S_1 \subset S_0$ of submodules of M, there exists some $n \geq 0$ so that $S_m = S_n$ whenever $m \geq n$. If M is an artinian module, any descending chain of submodules of M must stabilize, and as a result we will have a simple submodule of M.

Remark 1.6. If M is a nonzero artinian module, then M contains some submodule $S \subset M$ which is simple.

Injective Envelopes

For Matlis duality, we allow our ring to be local and take the dual of a module with respect to the injective envelope of the residue field of R. We will first describe what it means for a module to be injective, and develop Matlis duality more fully in chapter 2.

Definition 1.7. *A module E is said to be* ***injective*** *if for M, N, left $R-$modules, and $f : M \to N$ an injective homomorphism, $g : M \to E$ a homomorphism, there exists $h : N \to E$ such that $h \circ f = g$. In other words, there exists an h which makes the following diagram commutative.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ E & \end{tikzcd}$$

While the definition of a projective module is dual to that of an injective module, we will not consider the former in this paper. As explained by Eisenbud in [4], "the theory is not dual at all". To better understand injective modules, some well-known properties are included here. One property of injective modules is that direct sums of finitely many injective modules is injective, but infinite direct sums of injective modules may not be injective. In fact, a ring is left noetherian if and only if arbitrary direct sums of injective modules are injective (Bass-Papp Theorem [1][12]). Direct summands of injective modules are always injective as are arbitrary products of injectives. Lam includes in [10] proof that if a direct product of modules is injective, then each module from the product is injective. Yet another useful property of injective modules is Baer's Criterion which we state in the following remark([4]).

Remark 1.8 (Baer's Criterion)**.** A module M over R is injective if and only if every module homomorphism from an ideal $I \subset R$ to M can be extended to a homomorphism from R to M.

Denote the set of prime ideals of a commutative ring R by $\mathrm{Spec}(R)$ which stands for **spectrum**. Matlis showed that for a commutative noetherian ring R every injective $R-$module can be written uniquely, up to isomorphism, as the direct sum of indecomposable injective modules. Each indecomposable injective from the direct sum is isomorphic to an injective envelope of R/P where $P \in \mathrm{Spec}(R)$. In other words, every injective E can be written uniquely as $E = \bigoplus_i E_i$ where $E_i \cong R/P$ and $P \in \mathrm{Spec}(R)$. Furthermore, Matlis's Theorem gives that $\mathrm{Spec}(R)$ is in bijective correspondence with these indecomposable injectives [11].

Definition 1.9. *E is said to be an **essential extension** of M if $M \subset E$ and for every $H \subset E$, $H \cap M = \{0\}$ implies that $H = \{0\}$. We can also say that M is an **essential submodule** of E, or simply M is **essential in** E.*

We define $E(M)$ as an **injective envelope** of M if $E(M)$ is an essential extension of M and $E(M)$ is injective. One example of an essential extension is found by letting $R = \mathbb{Z}$. Note that a $\mathbb{Z}$ module is injective if and only if it is divisible. Next show that $\mathbb{Q}$ is divisible. After checking that $\mathbb{Q}$ is an essential extension of $\mathbb{Z}/(0)$, conclude $E(\mathbb{Z}) = \mathbb{Q}$.

It can also be shown that if $p \in \mathbb{Z}$ is a prime, then $E(\mathbb{Z}/(p)) = \mathbb{Z}(p^\infty)$, the Prüfer $p-$group. More concretely, $\mathbb{Z}(p^\infty) = \left\{ \frac{m}{p^k} + \mathbb{Z} | m, k \in \mathbb{Z} \text{ and } \; k \geq 0 \right\}$. The only possibilities for R/P given $R = \mathbb{Z}$ are $\mathbb{Z}$ and $\mathbb{Z}/(p)$, so every indecomposable injective over $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ by Matlis's Theorem. Incidentally, the module $\mathbb{Z}(p^\infty)$ is an example of an artinian module which is not noetherian.

For the main result of this dissertation, we will be interested in a ring, R, which is local. A **local** ring is a ring with a unique maximal ideal $\mathfrak{m}$. Up to isomorphism, this ring has a unique simple module which we denote $\kappa = R/\mathfrak{m}$. This is the residue field of R and we denote its injective envelope by $E = E(\kappa)$. As Lam points out in [10], E is the $\mathfrak{m}-$adic completion of R with respect to $\mathfrak{m}$.

Chapter 2 Preliminary Results

As mentioned in chapter one, the duality of vector spaces has been an important motivation for studying later dualities. We include here some basic, yet important results on the duality of vector spaces as well as properties of the Matlis dual. We will also introduce Serre classes and give several examples. In this section we introduce local nilpotence and use this concept to prove a version of the Krull Intersection Theorem where the R is local.

Duality of Vector Spaces

Below, let V be a vector space over a fixed field F. Define the vector space $\mathrm{Hom}_F(V, F)$ as the dual of V and denote it by V^*. Then any element of V^* is a linear map $\sigma : V \to F$, regarding F as a vector space over itself. It can be shown that if $v_1, v_2, \ldots, v_n$ is a base for a finite dimensional V, then $v_1', v_2', \ldots, v_n'$ such that $v_i'(v_j) = 1$ if $i = j$ and $v_i'(v_j) = 0$ if $i \neq j$ is a base for V^*. This shows that $\dim(V) = \dim(V^*)$.

Notice that the canonical map $V \to V^{**}$ is an injection because if $\phi(v) = 0$ for some nonzero $v \in V$, then $f(v) = 0$ for all $f \in \mathrm{Hom}_F(V, F)$. In particular, considering each v_i' gives $v = 0$. Additionally, if $\dim(V) < \infty$ we can show the canonical injection is an isomorphism due to the fact that $\dim(V) = \dim(V^*) = \dim(V^{**})$. Note that this shows V is reflexive. In fact, we have the following lemma which characterizes all reflexive vector spaces.

Lemma 2.1. *A vector space V is reflexive if and only if* $\dim(V) < \infty$.

Proof. By the arguments above, we need only show if $\dim(V) = \infty$, then V is not reflexive. To simplify notation, let $v_1, v_2, \ldots$ be a countable base for V. Defining $v_i' \in V^*$ as above does not give a spanning set of V^*, so does not form a base of V^*. (Note that even if the basis were uncountable, this argument still holds, only the notation would change.) If the v_i' spanned V^*, then any $\sigma \in V^*$ could be written as a linear combination of the v_i'. Consider the map $\sigma : V \to F$ where $\sigma(v_i) = 1$ for all i and write $\sigma = \alpha_1 v_1' + \alpha_2 v_2' + \cdots + \alpha_m v_m'$ for a finite collection of m possibly re-indexed v_i' and each $\alpha_i \in F$. Notice that if v_j' is not one of the $v_1', v_2', \ldots, v_m'$, then $\sigma(v_j) = 0$, but also $\sigma(v_j) = 1$. This is a contradiction for all but finitely many basis elements of V.

Notice this implies a basis of V^* must contain an element from outside the span of the v_i'. This allows us to define a map $\psi \in V^{**}$ where $\psi(v_i') = 0$ for all i and $\psi \neq 0$. Recall we are using ϕ to represent the canonical homomorphism from V to V^{**} and suppose there exists some $0 \neq v \in V$ so that $\phi(v) = \psi$. Notice $0 = \psi(v_i') = \phi(v)(v_i')$ for all i which implies $v = 0$. This is a contradiction. □

2.1 Properties of the Dual

The following properties of the dual of a module will eventually be extended to the Matlis dual mentioned earlier. For now, we consider a module duality where the ring is only assumed commutative.

Lemma 2.2. *When M_1, M_2, M_3, N are $R-$modules and $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are linear maps, $(g \circ f)^\nu = f^\nu \circ g^\nu$.*

Proof. Let $h \in \operatorname{Hom}_R(M_3, N)$. By definition, $(g \circ f)^\nu(h) = h \circ (g \circ f)$. Notice that $(f^\nu \circ g^\nu)(h) = f^\nu(h \circ g) = (h \circ g) \circ (f) = h \circ (g \circ f)$ making the two maps identical. $\square$

Remark 2.3. If $f : M_1 \to M_2$ is the zero map, then $f^\nu : M_2^\nu \to M_1^\nu$ is also zero.

Remark 2.4. Let $f : M_1 \to M_2$ be a surjection. Then f^ν is injective.

Lemma 2.5. *If $S \subset M$ and $0 \to S \xrightarrow{i} M \xrightarrow{p} M/S \to 0$ is the standard short exact sequence formed where i is the canonical injection and p is the canonical surjection, then*

$$0 \to (M/S)^\nu \xrightarrow{p^\nu} M^\nu \xrightarrow{i^\nu} S^\nu \tag{2.1}$$

is exact.

Proof. Notice $p \circ i$ is the zero map. By 2.3 $i^\nu \circ p^\nu = 0$. This means $\operatorname{Im}(p^\nu) \in \ker(i^\nu)$. We now show p^ν is injective and $\ker(i^\nu) \in \operatorname{Im}(p^\nu)$. If $\varphi \in (M/S)^\nu$ so that $p^\nu(\varphi) = 0$, 2.4 and the surjectivity of p conclude p^ν is injective.

Now, let $\varphi \in \ker(i^\nu)$. Then $\varphi \in M^\nu$ and $i^\nu(\varphi) = \varphi \circ i = 0$. Notice that because p is surjective, for any $m + S \in M/S$, there exists some $m \in M$ mapping to it. Define $\psi : M/S \to N$ by $\psi(m + S) = \varphi(m)$. This ψ proves that $\varphi \in \operatorname{Im}(p^\nu)$, as $\psi \in (M/S)^\nu$ and $\varphi = p^\nu(\psi)$. We need only check that ψ is well-defined.

$$\begin{tikzcd}
S \arrow[r, "i"] \arrow[dr, "\varphi\circ i"'] & M \arrow[r, "p"] \arrow[d, "\varphi"] & M/S \arrow[dl, "\psi"] \\
& N &
\end{tikzcd}$$

This map is well-defined because if $m_1 + S = m_2 + S$, then $m_1 - m_2 \in S = \ker(p) = \operatorname{Im}(i)$. Since $\varphi \circ i = 0$, $\varphi(m_1 - m_2) = 0 \Rightarrow \varphi(m_1) = \varphi(m_2)$. The existence of this ψ proves the exactness of the sequence of the duals in (2.1). $\square$

We would like to extend 2.5 and get that $0 \to (M/S)^\nu \xrightarrow{p^\nu} M^\nu \xrightarrow{i^\nu} S^\nu \to 0$ is short exact. However, to get i^ν surjective we need the module N above to be injective. The next theorem shows why considering duality with respect to an injective module is important.

Theorem 2.6. *When N is injective, the standard short exact sequence formed by the duals and dual maps of submodules and quotient modules of M, i.e.,*

$$0 \to (M/S)^\nu \xrightarrow{p^\nu} M^\nu \xrightarrow{i^\nu} S^\nu \to 0$$

is short exact.

Proof. By 2.5 we only need to show that i^ν is surjective. So we want to show that given $\varphi \in S^\nu$, there exists some $\psi \in M^\nu$ so that $i^\nu(\psi) = \varphi$. This amounts to showing that there is some $\psi : M \to N$ that makes the following diagram commutative.

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\varphi"'] & M \arrow[d, "\psi"] \\ & N \end{tikzcd}$$

This is clear by the injectivity of i and the fact that N is an injective module. $\square$

Lemma 2.7. *With f linear, and ϕ_{M_1} and ϕ_{M_2} the canonical homomorphisms between M_1 and $M_1^{\nu\nu}$, and respectively M_2 and $M_2^{\nu\nu}$, the following diagram commutes.*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\phi_{M_1}"'] & M_2 \arrow[d, "\phi_{M_2}"] \\ M_1^{\nu\nu} \arrow[r, "f^{\nu\nu}"] & M_2^{\nu\nu} \end{tikzcd}$$

Proof. Let $m_1 \in M_1$. Notice $(\phi_{M_2} \circ f)(m_1)$ sends m_1 to the homomorphism in $M_2^{\nu\nu}$ which evaluates any $\varphi \in M_2^\nu$ at $f(m_1)$. We write $\varphi(f(m_1))$ as our result. Considering $(f^{\nu\nu} \circ \phi_{M_1})(m_1)$ takes m_1 to the map in $M_2^{\nu\nu}$ which performs $f^{\nu\nu}((\phi_{M_1})(m_1))$ on $\varphi \in M_2^\nu$. Recalling the definition of the dual of a map, we see that $[f^{\nu\nu} \circ (\phi_{M_1})(m_1)] \circ (\varphi) = [(\phi_{M_1})(m_1) \circ f^\nu] \circ (\varphi)$. Applying the definition again, we get $[(\phi_{M_1})(m_1)] \circ (\varphi \circ f)$. This gives $\varphi(f(m_1))$ as our result, and we conclude that the diagram is commutative. $\square$

2.2 Matlis Duality

Theorem 2.6 gives motivation for studying duals taken with respect to injective modules. Matlis duality, named after Eben Matlis, is such a dual. Included here are several useful features of the Matlis dual. In what follows, let R be a commutative ring, E an injective module and for an $R-$module M, let M^ν denote the dual with respect to E. Ultimately, Matlis duality refers to duality with respect to $E = E(\kappa)$ for local rings, but the following results are true for any injective E. With this new notation, we rephrase the previous theorem as a remark.

Remark 2.8. If $0 \to S \to M \to M/S \to 0$ is the exact sequence formed by a submodule $S \subset M$ where the canonical maps are implied, then

$$0 \to (M/S)^\nu \to M^\nu \to S^\nu \to 0$$

is exact where $(-)^\nu$ represents the dual of $-$ with respect to E.

Remark 2.9. With the same hypothesis as 2.8, by again taking duals, the following is an exact sequence.

$$0 \to S^{\nu\nu} \to M^{\nu\nu} \to (M/S)^{\nu\nu} \to 0$$

As mentioned in the first chapter, an $R-$ module M is **reflexive** if the canonical homomorphism between M and $M^{\nu\nu} = \mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E)$ is an isomorphism.

Theorem 2.10. *The following diagram is commutative and if S and M/S are reflexive, then so is M.*

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, "\phi_S"'] & M \arrow[r] \arrow[d, "\phi"'] & M/S \arrow[r] \arrow[d, "\phi_{M/S}"'] & 0 \\
0 \arrow[r] & S^{\nu\nu} \arrow[r] & M^{\nu\nu} \arrow[r] & (M/S)^{\nu\nu} \arrow[r] & 0
\end{tikzcd}$$

Proof. The diagram is commutative by 2.7. If S and M/S are reflexive then ϕ_S and $\phi_{M/S}$ are isomorphisms.

We first prove that $\phi : M \to M^{\nu\nu}$ is injective and use the following section of the diagram for reference:

$$\begin{tikzcd}
S \arrow[r, "i"] \arrow[d, "\cong"'] & M \arrow[r, "p"] \arrow[d, "\phi"'] & M/S \arrow[d, "\cong"'] \\
S^{\nu\nu} \arrow[r, "i^{\nu\nu}"] & M^{\nu\nu} \arrow[r, "p^{\nu\nu}"] & (M/S)^{\nu\nu}
\end{tikzcd}$$

Suppose $\phi(x) = 0$ for some $x \in M$. Notice $\phi_{M/S}(p(x)) = p^{\nu\nu}(\phi(x)) = p^{\nu\nu}(0) = 0$. That $\phi_{M/S}$ is an isomorphism implies $p(x) = 0$, hence $x \in \ker(p)$. Now, $\ker(p) = \mathrm{Im}(i)$ which implies $x \in \mathrm{Im}(i)$ thus landing $x \in S$. Now, $i^{\nu\nu}(\phi_S(x)) = \phi(i(x)) = \phi(x) = 0$. Because $i^{\nu\nu} \circ \phi_S$ is an injection, we see that $x = 0$ and conclude ϕ is injective as desired.

We now prove that ϕ is surjective. Let ψ be any map in $M^{\nu\nu}$. We see that $p^{\nu\nu}(\psi) \in (M/S)^{\nu\nu}$. By $\phi_{M/S}$ an isomorphism and p surjective, there exists $x \in M$ so that $\phi_{M/S}(p(x)) = p^{\nu\nu}(\psi)$. But $p^{\nu\nu}(\phi(x)) = \phi_{M/S}(p(x))$, so $p^{\nu\nu}(\phi(x)) = p^{\nu\nu}(\psi)$. This implies $p^{\nu\nu}(\phi(x) - \psi) = 0$, placing $\phi(x) - \psi \in \ker(p^{\nu\nu}) = \mathrm{Im}(i^{\nu\nu})$. This implies that there exists some $\varphi \in S^{\nu\nu}$ so that $i^{\nu\nu}(\varphi) = \phi(x) - \psi$. By the surjectivity of ϕ_S, there exists an $s \in S$ so that $i^{\nu\nu}(\phi_S(s)) = \phi(x) - \psi$. Notice that $\phi(i(s)) = i^{\nu\nu}(\phi_S(s)) = \phi(x) - \psi$. This implies $\phi(x - s) = \psi$, thus proving the surjectivity of ϕ. $\square$

Our goal is to have a converse of 2.10. Then we will know given M is reflexive, so are S and M/S for any submodule $S \subset M$. It turns out this holds for faithfully injective modules.

Definition 2.11. *An injective $R-$module, E is **faithfully injective** if for every $R-$module, N, $\mathrm{Hom}(N,E) = 0$ implies $N = 0$.*

Remark 2.12. Let E and N be $R-$modules. If E is faithfully injective, and $y \in N$ and $y \neq 0$ then there exists some homomorphism $\sigma : N \to E$ so that $\sigma(y) \neq 0$.

Lemma 2.13. *Let E and N be $R-$modules. E is faithfully injective if and only if the canonical homomorphism $\phi : M \to M^{\nu\nu}$ is injective.*

Proof. Let $x \in M$ be such that $\phi(x) = 0$. By the definition of ϕ, x maps to $\psi \in M^{\nu\nu}$ where ψ is the map which evaluates any $\sigma \in M^{\nu}$ at x. So, $\phi(x) = 0$ implies $\sigma(x) = 0$ for all $\sigma \in M^{\nu}$. By 2.12 this implies $x = 0$. Hence the canonical homomorphism is injective.

Now, let ϕ be injective and $M \neq 0$. Choose some nonzero $y \in M$ and define $\phi(y) = \psi$. By the injectivity of ϕ, $\psi \neq 0$. This implies there exists some $\varphi \in M^{\nu}$ so that $\psi(\varphi) \neq 0$. Since $\psi(\varphi) = \varphi(y)$ we have $\varphi(y) \neq 0$ implying $\varphi \neq 0$ and that E is faithfully injective. $\square$

We are interested in local rings and would like to extend the 2.12 to $E = E(\kappa)$. If R has finite length as a module over itself it is true that E is faithfully injective.

Lemma 2.14. *In a local ring, R which has finite length as a module over itself, $E = E(\kappa)$ is faithfully injective.*

Proof. By the definition of a faithfully injective module, it is enough to show that $N^{\nu} = \operatorname{Hom}_R(N, E) \neq 0$ for any nonzero $R-$module, N. Notice there exists a finitely generated $M \subset N$. We have $M \cong R^n/S$ for some $n \geq 1$ and $S \subset R^n$ implying M has finite length. Suppose $\ell(M) = n$ and $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ for some $n > 0$ where each M_i/M_{i-1} is simple. In particular this makes M_1 simple, thus $M_1 \cong \kappa$. Consider $M_1 \cong \kappa \hookrightarrow E$ and notice that because E is an injective module and $M_1 \hookrightarrow M$ is an injective homomorphism, there exists a nonzero $h \in \operatorname{Hom}_R(M, E)$ so that the implied diagram between the three modules commutes. Clearly this means $\operatorname{Hom}_R(N, E) \neq 0$. $\square$

In fact, if the ring is local, $E = E(\kappa)$ is faithfully injective if and only if $(R/\mathfrak{m})^{\nu} \neq 0$. One direction is easily realized, but showing given $(R/\mathfrak{m})^{\nu} \neq 0$ that E is faithfully injective takes some work. Below we show that given R is a local ring, then $E = E(\kappa)$ is faithfully injective. In fact, we prove a stronger result: given any nonzero element of an $R-$module, M, there exists a map so that the image of that element is nonzero.

Lemma 2.15. *In a local ring R, $E = E(\kappa)$ is faithfully injective by the following:*

If $M \neq 0$ is any $R-$module then there is a nonzero, linear map $\sigma : M \to E$ which maps a given nonzero element of M to a nonzero element of E.

Proof. Consider first any nonzero cyclic submodule $C \subset R$ with generator $x \neq 0$. Mapping any $r \in R$ to rx, we see that $R/\operatorname{ann}(x) \cong C$. We must have $\operatorname{ann}(x) \subseteq \mathfrak{m}$. If $\operatorname{ann}(x)$ is a maximal ideal, then $\operatorname{ann}(x) = \mathfrak{m}$. If $\operatorname{ann}(x) \neq \mathfrak{m}$, then $\operatorname{ann}(x)$ is a subset of $\mathfrak{m}$ since R is a local ring. (Otherwise $\operatorname{ann}(x) = R$ making $C \cong 0$.) This gives a map

$$C \xrightarrow{\cong} R/\operatorname{ann}(x) \xrightarrow{g} \kappa$$

$$rx \mapsto r + \operatorname{ann}(x) \mapsto r + \mathfrak{m}$$

which is well-defined since $\operatorname{ann}(x) \subseteq \mathfrak{m}$. Because g is nonzero and linear, our composition defines a map $h : C \to \kappa$ which is nonzero and linear. We know any nonzero $R-$module, M, has a nonzero cyclic submodule, making the following a commutative diagram:

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d, "h"] \arrow[dd, bend right, "\varphi"'] & M \arrow[ddl, dashed, "\sigma"] \\
\kappa \arrow[d, hook] & \\
E &
\end{tikzcd}$$

The map $\varphi : C \to E$ is a nonzero homomorphism and we use the natural injection from C to M and the fact that E is injective to give σ. By the existence of σ from above where C is the cyclic submodule generated by y, $\varphi(y) \neq 0$, otherwise φ is the zero map. Since the diagram commutes, $\sigma(y) \neq 0$. ☐

Extending this property from rings which have finite length to more general local rings will help to characterize Serre classes of finitely generated modules. In chapter five we will want to extend another useful property of rings with finite length to more general local rings. This will not be possible without requiring that the ring be complete as well.

2.3 Reflexivity of the Dual of Reflexives

When M is known to be reflexive, notice that M^ν is isomorphic to $M^{\nu\nu\nu}$. It is tempting to assume this makes M^ν reflexive. We must instead examine more carefully the reason the canonical map $M^\nu \to M^{\nu\nu\nu}$ is an isomorphism. First, we will need some notation and a lemma.

In the arguments below, assume M reflexive and let:

$$\begin{aligned}
&x \text{ be any element of } M \\
&\sigma \text{ be any element of } M^\nu \\
&\psi \text{ be any element of } M^{\nu\nu} \\
&\beta \text{ be any element of } M^{\nu\nu\nu}
\end{aligned}$$

We denote by ϕ_M the canonoical homomorphism

$$\phi_M : M \to M^{\nu\nu}$$

which maps $\phi_M(x)(\sigma) = \sigma(x)$. In other words $\phi_M = (x \mapsto (\sigma \mapsto \sigma(x))$. An analogous definition gives us $\phi_{M^\nu} : M^\nu \to M^{\nu\nu\nu}$. By definition of the dual of a map, we also get $(\phi_M)^\nu : M^{\nu\nu\nu} \to M^\nu$ defined by $\phi_M^\nu(\beta) = \beta \circ \phi_M$.

Lemma 2.16. *If $f : M \to N$ is an isomorphism for $R-$modules M and N, then $f^\nu : N^\nu \to M^\nu$ is an isomorphism.*

Proof. Recall by the definition of f^ν that it maps an $h \in N^\nu$ to $h \circ f$. To show $\ker(f^\nu) = 0$, suppose $h \circ f(x) = 0$ for all $x \in M$. Because f is injective, $f(x) = 0$ if and only if $x = 0$. This implies h evaluates the image of f as zero. Since f is surjective, $h(N) = 0$ which proves $h = 0$.

Now, because f is injective there exists $f^{-1} : N \to M$. Let $g \in M^\nu$. We would like to find some element of N^ν so that when composed with f and evaluated at any x we get $g(x)$. Notice $g \circ f^{-1}$ is such a function. □

Theorem 2.17. *Given M is reflexive, M^ν is also reflexive.*

Proof. First we show that:

$$\phi_M^\nu \circ \phi_{M^\nu} = id_{M^\nu} \tag{2.2}$$

By definition of ϕ_M^ν, we see $\phi_M^\nu(\phi_{M^\nu}(\sigma)) = \phi_{M^\nu}(\sigma) \circ \phi_M$. After noticing these are both maps from M to E, we would like to see that the latter is exactly σ. By definition, $\phi_{M^\nu}(\sigma)(\psi) = \psi(\sigma)$ for any $\psi \in M^{\nu\nu}$. So, $\phi_{M^\nu}(\sigma) \circ \phi_M(x) = \sigma(x)$. Hence (2.2) holds.

Now, if M is reflexive, ϕ_M is an isomorphism, so ϕ_M^ν is an isomorphism by 2.16. By (2.2) we know ϕ_M^ν has an inverse in ϕ_{M^ν} which makes ϕ_{M^ν} an isomorphism as well. Hence M^ν is reflexive. □

The following properties of modules and their duals are helpful, but are not difficult to verify. They are presented without proof. We will need these basic results in chapter five when we change from local rings which have finite length as modules over themselves to more general local rings.

Remark 2.18. In a local ring, $\kappa \cong \kappa^\nu$. (This also implies the dual of a simple module is simple and the dual of a module of length 1 is a module of length 1.)

Remark 2.19. By 1.4 a module of length 1 is reflexive. Hence κ is reflexive.

Remark 2.20. The dual of a module of length n is of length n. (Use induction with 2.18 as a base case.)

2.4 Serre Classes

Serre classes of modules over an arbitrary ring R are useful in part because they describe the relationships between certain classes of modules and sets of ideals of R. However, this relationship was not the reason for their introduction in 1953 (see [13]). Serre used these classes –which were later named after him– to study the homotopy groups of spheres. After introducing some useful properties of Serre classes, our focus will be to characterize the Serre classes of three types of modules over a specific type of ring.

Definition 2.21. *A class of modules, $\mathcal{S}$, is called a* ***Serre class*** *if $\mathcal{S}$ is non-empty and satisfies the property that for a short exact sequence of modules*

$$0 \to M \to N \to P \to 0$$

$N \in \mathcal{S}$ if and only if $M \in \mathcal{S}$ and $P \in \mathcal{S}$ where M, N, P are modules of an abelian category $\mathcal{C}$ of which $\mathcal{S}$ is a full subcategory.

The following remark follows easily by considering $0 \to M' \to M \to 0 \to 0$ where M and M' are $R-$modules.

Remark 2.22. If $M \cong M'$ and $M \in \mathcal{S}$ then $M' \in \mathcal{S}$.

Remark 2.23. Let S be a class of modules, and let $M \in \mathcal{S}$. When proving that $\mathcal{S}$ is a Serre class, it suffices to show the following:

(i) if $S \subset M$, then $S, M/S \in \mathcal{S}$

(ii) if $0 \to N_1 \to N \to N_2 \to 0$ is a short exact sequence of $R-$modules with $N_1, N_2 \in \mathcal{S}$, then $N \in \mathcal{S}$.

Proof. This is clear by the definition of a Serre class and 1.3. □

In the final chapter we will be interested in Serre classes of reflexive modules. In chapter six we will show that the reflexive modules over a specific type of ring form a Serre class. We can give examples of Serre classes over a more general ring here. For a general ring R, we show below that left noetherian and left artinian $R-$modules form a Serre class.

Lemma 2.24. *Let R be a ring and let $\mathcal{S}$ consist of left noetherian modules. $\mathcal{S}$ is a Serre class.*

Proof. Using the technique in 2.23, let $S \subset M$ and notice any submodule of S is a submodule of M. Hence S must be finitely generated implying $S \in \mathcal{S}$. Considering M/S and any ascending chain of submodules $T_1/S \subset T_2/S \cdots \subset M/S$, note that $T_1 \subset T_2 \subset \cdots \subset M$ is an ascending chain of submodules of M which must stabilize. This implies $T_1/S \subset T_2/S \cdots \subset M/S$ must stabilize. This implies $M/S \in \mathcal{S}$.

Now, suppose $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence with $M_1, M_2 \in \mathcal{S}$. Notice M_1 is isomorphic to a submodule of M and M_2 is isomorphic to the quotient M/S, so consider instead the sequence $0 \to S \hookrightarrow M \xrightarrow{\pi} M/S \to 0$ where π is the projection map. Consider the ascending chain of submodules $T_1, T_2, \cdots \subset M$ and the submodule $D_i = T_i \cap S$. The D_i form an ascending chain in S which must stabilize, say, at D_n. Notice the $\pi(T_i)$ form an ascending chain in M/S which must stabilize, say, at $\pi(T_m)$. Define $N = \max\{n, m\}$ and consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & D_N \arrow[r] \arrow[d, "="'] & T_n \arrow[r] \arrow[d] & \pi(T_N) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & D_{N+1} \arrow[r] & T_{N+1} \arrow[r] & \pi(T_{N+1}) \arrow[r] & 0
\end{tikzcd}$$

By the Snake Lemma, $T_N = T_M$ for all $M \geq N$. □

Lemma 2.25. *Let R be a ring and let $\mathcal{T}$ consist of left artinian modules. $\mathcal{T}$ is a Serre class.*

Proof. The proof is entirely similar to above, only we use the descending chain condition and rather than consider $T_i \cap S$, consider $(T_i + S)/S$. $\square$

Another consequence of Serre classes following easily from the definition is that the intersection of two Serre classes is again a Serre class. The following result gives that the class of modules of finite length is a Serre class.

Lemma 2.26. *An $R-$module, M is of finite length if and only if it is both noetherian and artinian.*

Proof. First we show if M is noetherian and artinian, then M has finite length. Since M is artinian, there exists some $M_1 \subset M$ so that M_1 is simple by 1.6. Now, if M/M_1 is simple, $\ell(M) = 1$, which is finite. If not, there is a module M_2 so that $0 \subset M_1 \subset M_2 \subset M$. As needed, insert more submodules until each M_{i+1}/M_i is simple. This chain must stabilize, as M is finitely generated. Hence, M has finite length.

Now we show if M has finite length, then M is noetherian (artinian). If M has finite length and we pick any submodule $S \subset M$, since $\ell(S) < \ell(M)$, we see that any descending (ascending) chain stabilizes due to strict containment of submodules in the chain and the fact that lengths of modules are integers. Hence, every submodule satisfies DCC (ACC). $\square$

Remark 2.27. Let R be a ring and let $\mathcal{S}$ consist of modules of finite length. $\mathcal{S}$ is a Serre class.

2.5 An Application of Local Nilpotence

In this section, we use the concept of local nilpotence to prove a version of the Krull-Intersection theorem that will be useful for our final result. Some proofs of the Krull Intersection theorem appeal to the Artin-Rees lemma (see [4]). Since we only need this theorem for local rings, we take a different approach.

Lemma 2.28. *Suppose R is a local ring and $P \in \mathrm{Spec}(R)$. Also let $x \in E(R/P)$ and $r \in P$. There exists an $n \geq 1$ so that $r^n x = 0$.*

*(In other words, multiplication by such an r is **locally nilpotent** on E.)*

Proof. Define $\overline{E} = E \oplus E \oplus E \oplus \cdots$ and $\overline{p} = (R/P) \oplus (R/P) \oplus (R/P) \oplus \cdots$. Notice that the map from $\overline{E}$ to itself defined by

$$(x_1, x_2, x_3, \ldots, 0, 0, 0, \ldots) \mapsto (x_1, x_2 + rx_1, x_3 + rx_2, \ldots)$$

is a homomorphism which fixes $\overline{p}$, making it an isomorphism. The injectiveness of this map is realized by the injectiveness of the restriction to $\overline{p}$ and the fact that $\overline{p}$ is essential in $\overline{E}$. The map is surjective because $\mathrm{Im}(\overline{p}) \subset \mathrm{Im}(\overline{E}) \subset \overline{E}$. By the first isomorphism theorem, $\mathrm{Im}(\overline{E}) \cong \overline{E}$ making $\mathrm{Im}(\overline{E})$ an injective module. Since it's injective envelope is the smallest injective module containing $\overline{p}$, $\mathrm{Im}(\overline{E}) = \overline{E}$ meaning the map is surjective. Hence there exists $(x_1, x_2, x_3, \dots, 0, 0, 0, \dots)$ so that $(x_1, x_2 + rx_1, x_3 + rx_2, \dots) = (x, 0, 0, 0, \dots)$ which shows that for some $n \geq 1$, $r^n x = 0$. $\square$

Theorem 2.29. *With the same requirements above, there is an* $n \geq 1$ *so that* $P^n x = 0$.

Proof. We must find $n \geq 1$ so that the product of any n elements of P together with x is zero. R noetherian implies that any $r \in P$ is a linear combination of some given $r_1, \dots, r_m$ elements of P, where $m < \infty$. By 2.28, since each $r_i \in P$, there exists $j_i \geq 1$ so that $r_i^{j_i} x = 0$.

Defining $j = \max\limits_{i=1}^{m} j_i$, and letting $n = m(j-1)+1$, it is clear that each term of the product of any n many elements of P will have a factor of r_i^j for at least one $i \in \{1, \dots, m\}$, and so multiplies x to zero. This, in turn, makes the product of any n elements of P with x zero. $\square$

Krull Intersection Theorem

We now use the two results above to prove a different version of the Krull Intersection Theorem. First, recall 2.15 says for any nonzero element $x \in M$ where M is an $R-$module, there is a nonzero, linear map $\sigma : M \to E$ so that $\sigma(x) \neq 0$.

Theorem 2.30 (Krull Intersection)**.** *Let* R *be a local ring with maximal ideal* $\mathfrak{m}$*. Then* $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$.

Proof. Choose any nonzero $r \in \cap_{n=1}^{\infty} \mathfrak{m}^n$. By 2.15, considering R as a module over itself, there exists a nonzero, linear $\sigma : R \to E$ so that $\sigma(r) \neq 0$, and σ is determined completely by $\sigma(1) = x$.

Since $\mathfrak{m}$ is prime, by 2.29, $\exists \hat{n} \geq 1$ so that $\mathfrak{m}^{\hat{n}} x = 0$. Now, because $r \in \cap_{n=1}^{\infty} \mathfrak{m}^n$, $r \in \mathfrak{m}^{\hat{n}}$ for this particular $\hat{n}$ associated to x. This means $r \in \mathrm{ann}(x)$. However, if $\sigma(1) = x \neq 0$, $\sigma(r) = r\sigma(1) = rx \neq 0$. This is a contradiction. $\square$

Now that we have completed the necessary background, we will begin the work of classifying Serre classes. In chapter three we do this for noetherian modules. The goal is to extend these results, but we will first need to look at the relationship noetherian modules have with other types of modules.

Chapter 3 The Characterization of Serre Classes of Noetherian Modules

Here we characterize all Serre classes of finitely generated modules over a given commutative noetherian ring R. This characterization is given by a bijection between the Serre classes and the defining sets of ideals of R described below. We will use this characterization along with the results of chapter four (i.e., the characterization of Serre classes of artinian modules) to characterize the Serre classes of reflexive modules in chapter six.

3.1 A Defining Set of Ideals

The defining set of ideals is composed of prime ideals. These ideals have a special relationship with respect to containment which we will now examine.

Lemma 3.1. *Let x be a nonzero element of M. There is an $r \in R$ so that* $\operatorname{ann}(rx)$ *is a prime ideal of R.*

Proof. Notice $\operatorname{ann}(x) \neq R$ since $x \neq 0$ and $1 \in R$. If $\operatorname{ann}(x)$ is prime, choose $r = 1$. If $\operatorname{ann}(x)$ is not prime, let $rs \in \operatorname{ann}(x)$ where $r \notin \operatorname{ann}(x)$ and $s \notin \operatorname{ann}(x)$. We know $\operatorname{ann}(rx) \neq R$ since if $1 \in \operatorname{ann}(rx), rx = 0$ which implies $r \in \operatorname{ann}(x)$. We also know $s \in \operatorname{ann}(rx)$ since $s(rx) = srx = rsx = 0$, yet $s \notin \operatorname{ann}(x)$. So we have:

$$\operatorname{ann}(x) \subsetneq \operatorname{ann}(rx) \subsetneq R$$

Now, if $\operatorname{ann}(rx)$ is prime, we have found our r. If $\operatorname{ann}(rx)$ is not prime, there exists some $r' \in R$ so that $\operatorname{ann}(x) \subsetneq \operatorname{ann}(rx) \subsetneq \operatorname{ann}(r'rx)$. Because R is noetherian, this chain must stabilize. $\square$

Lemma 3.2. *Let M be a finitely generated $R-$module. Then M has a filtration*

$$0 = M_0 \subset M_2 \subset \cdots \subset M_n = M$$

for some $n \geq 0$ such that for each i with $0 \leq i < n$ we have that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Proof. As in the proof of 2.15, $R/\operatorname{ann}(x) \cong Rx$. So, by the above, we see that M has a submodule isomorphic to $R/\mathfrak{p}$ where $\mathfrak{p}$ is prime. Choose some nonzero $x \in M$. By 3.1, there is an $r \in R$ so that $\operatorname{ann}(rx)$ is prime. Define $x_1 = rx$, $\mathfrak{p}_1 = \operatorname{ann}(x_1)$. Define also $M_1 = Rx_1 \cong R/\mathfrak{p}_1$. If $M/M_1 = 0$, we have our (very short) filtration.

If $M/M_1 \neq 0$ we can find some $0 \neq y \in M/M_1$. Using 3.1 again, there is an $r \in R$ so that $x_2 = ry$ and $\operatorname{ann}(x_2) = \mathfrak{p}_2$. So to keep track, $Rx_2 \subset M/M_1$ meaning Rx_2 looks like some M_2/M_1 where $M_1 \subset M_2 \subset M$.

Define M_2 so that $M_1 \subset M_2$ so that $Rx_2 \cong M_2/M_1$. Because $Rx_2 \cong R/\mathfrak{p}_2$, we have continued our filtration. We can continue in this way using 3.1. Since M is finitely generated, our filtration will stabilize at some $n < \infty$. $\square$

Lemma 3.3. *Let $\mathfrak{p}$ be a prime ideal and let $M = R/\mathfrak{p}$.*

(i) If $x \in M$ and $x \neq 0$, $\operatorname{ann}(x) = \mathfrak{p}$.

(ii) If we have a filtration of M as above, $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{p} \subset \mathfrak{p}_i$ for $1 \leq i \leq n$.

Proof. First we show $\mathfrak{p} \subseteq \operatorname{ann}(x)$. Let $p \in \mathfrak{p}$. Notice $x = r + \mathfrak{p}$ where $r \in R$ and $r \notin \mathfrak{p}$. Now, $px = p(r + \mathfrak{p}) = \mathfrak{p}$ which implies $\mathfrak{p} \subset \operatorname{ann}(x)$. We now show $\operatorname{ann}(x) \subseteq \mathfrak{p}$. Let $y \in \operatorname{ann}(x)$. Then $y(r + \mathfrak{p}) = 0 \Rightarrow yr \in \mathfrak{p}$. Because $\mathfrak{p}$ is prime, $r \in \mathfrak{p}$ which concludes $\operatorname{ann}(x) = \mathfrak{p}$.

We now prove the second claim above. Let M have a filtration as above. Recall $\mathfrak{p}_1 = \operatorname{ann}(x_1)$ where $0 \neq x_1 \in M$, hence $\operatorname{ann}(x) = \mathfrak{p}$ by the first part of 3.3, so by definition we get $\mathfrak{p} = \mathfrak{p}_1$.

Now we will show $\mathfrak{p} \subset \mathfrak{p}_i$. Let $q \in \mathfrak{p}_1$ and notice q annihilates x_1. Consider any $x_i, 1 \leq i \leq n$ as determined in 3.2. We have $x_i \in M/M_{i-1}$, so $x_i = m + M_{i-1}$ where $m \in M$ and $m \notin M_{i-1}$. Then $qx_i = q(m + M_{i-1}) = qm + M_{i-1}$. Notice $m \neq 0$, so by the first part of 3.3 $\operatorname{ann}(m) = \mathfrak{p}$ which implies $qm = 0$, so $qm \in M_{i-1}$, meaning q annihilates x_i. □

Notice that the above result means we could begin our filtration of such an $M = R/\mathfrak{p}$ with different x_1, but all will have $\mathfrak{p}$ as their annihilator. The M_i formed may be different, but we still have the property that $\mathfrak{p} \subset \mathfrak{p}_\mathfrak{i}$ no matter what the filtration.

Given that $\mathcal{S}$ is a Serre class, let $\mathcal{F}$ be the set of prime ideals, $\mathfrak{p}$, of R so that $R/\mathfrak{p} \in \mathcal{S}$. Our goal is now to find all such sets $\mathcal{F}$ that come from some Serre class $\mathcal{S}$ in the category of finitely generated modules. We then set up a bijective correspondence between the Serre classes $\mathcal{S}$ and these sets $\mathcal{F}$ of prime ideals of R to give the complete characterization of finitely generated modules over a commutative noetherian ring.

These sets, $\mathcal{F}$ were studied by Gabriel in [6] and also used by Walker and Walker in [14]. Gabriel described when such sets, $\mathcal{F}$ form a Serre class, but finding a bijection between the two took more work. Rather than immediately using these sets $\mathcal{F}$ to study Serre classes, their approach was to first look at more general classes and the sets $\mathcal{F}$ that could be mapped bijectively to them. They began characterizing additive classes, then bounded, complete additive classes and eventually characterize what the sets $\mathcal{F}$ looked like for some Serre classes. In particular, they described such sets $\mathcal{F}$ for classes of modules with essential socles, which we will discuss in chapter six.

3.2 A Characterizing Bijection

Remark 3.4. If $\mathcal{F}$ is as above for some Serre class $\mathcal{S}$ in the category of finitely generated modules over R, given $\mathfrak{p} \in \mathcal{F}$ and $\mathfrak{p} \subset \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Spec}(R)$, we have that $\mathfrak{q} \in \mathcal{F}$.

To realize this, consider the short exact sequence $0 \to \mathfrak{q}/\mathfrak{p} \to R/\mathfrak{p} \to R/\mathfrak{q} \to 0$. By 2.24 $\mathfrak{q}/\mathfrak{p} \in \mathcal{S}$, and $\dfrac{R/\mathfrak{p}}{\mathfrak{q}/\mathfrak{p}} \in \mathcal{S}$. Since $\dfrac{R/\mathfrak{p}}{\mathfrak{q}/\mathfrak{p}} \cong R/\mathfrak{q}$, 2.22 shows $R/\mathfrak{q} \in \mathcal{S}$, meaning $\mathfrak{q} \in \mathcal{F}$.

The property described in 3.4 means that $\mathcal{F}$ is a **filter** which is closed under "specialization". We now show through the following two lemmas that given a filter $\mathcal{F} \subset \operatorname{Spec}(R)$, we can generate a set $\mathcal{S}$ which is a Serre class. The generation of this set $\mathcal{S}$ indicates a bijection thus giving our characterization of all Serre classes of finitely generated modules over R.

Lemma 3.5. *Let $\mathcal{F} \subset \operatorname{Spec}(R)$ be a filter. This means that when $\mathfrak{p} \in \mathcal{F}$ and $\mathfrak{p} \subset \mathfrak{q}$ for $\mathfrak{q} \in \operatorname{Spec}(R)$, then $\mathfrak{q} \in \mathcal{F}$. Define $\mathcal{S}$ to be the set of finitely generated modules M such that M has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_{i+1}/M_i \cong R/\mathfrak{p}$ for $\mathfrak{p} \in \mathcal{F}$. Given any submodule $A \subset M$, if A and its quotient M/A are elements of $\mathcal{S}$, we have that $M \in \mathcal{S}$.

Proof. We quickly check that M is finitely generated by noticing if $\{x_1, \ldots, x_m\}$ generates A and $\{y_1 + A, \ldots, y_n + A\}$ generates M/A, any $y \in M$ is such that $y - (\alpha_1 y_1 + \cdots + \alpha_n y_n) \in A$ where $\alpha_i \in R$ for $1 \leq i \leq n$. This implies $y = \alpha_1 y_1 + \cdots + \alpha_n y_n + \beta_1 x_1 + \cdots + \beta_m x_m$ for $\beta_i \in R$, showing that M is finitely generated. Because $A, M/A \in \mathcal{S}$, there exist filtrations $A = A_n \supset A_{n-1} \supset \cdots \supset A_1 \supset A_0 = 0$ and $M/A = T_k/A \supset T_{k-1}/A \supset \cdots \supset T_1/A \supset T_0/A = 0$ where $A \subset T_j, 0 \leq j \leq k$, and

$$A_{i+1}/A_i \cong R/\mathfrak{p}_{i+1} \text{ with } \mathfrak{p}_{i+1} \in \mathcal{F}, \text{ for } \ 0 \leq i \leq n-1 \tag{3.1}$$

$$\frac{T_{j+1}/A}{T_j/A} \cong T_{j+1}/T_j \cong R/\mathfrak{q}_{j+1} \text{ with } \mathfrak{q}_{j+1} \in \mathcal{F}, \text{ for } \ 0 \leq j \leq k-1 \tag{3.2}$$

So, we have $M = T_k \supset T_{k-1} \supset \cdots \supset T_1 \supset T_0 = A = A_n \supset A_{n-1} \supset \cdots \supset A_0 = 0$ and noting (3.1),(3.2), conclude $M \in \mathcal{S}$. ∎

Lemma 3.6. *Let $\mathcal{F} \subset \operatorname{Spec}(R)$ be a filter. Define $\mathcal{S}$ to be the set of finitely generated modules M so that M has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_{i+1}/M_i \cong R/\mathfrak{p}$ for $\mathfrak{p} \in \mathcal{F}$. Given $M \in \mathcal{S}$, any submodule $A \subset M$ and its quotient, M/A are also elements of $\mathcal{S}$.

Proof. Because $M \in \mathcal{S}$, $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ where $M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1}$ and $\mathfrak{p}_{i+1} \in \mathcal{F}$ for $0 \leq i \leq n$. It is clear that both A and M/A are finitely generated, being submodules and quotient modules of M. We look for our filtration of A which will show $A \in \mathcal{S}$.

Notice $0 = (M_0 \cap A) \subset (M_1 \cap A) \subset \cdots \subset (M_n \cap A) = A$. While this does not directly give us the necessary filtration of A, it will give rise to one. Consider first the following map:

$$\begin{aligned} M_{i+1} \cap A/M_i \cap A &\xrightarrow{f} M_{i+1}/M_i \\ x + (M_i \cap A) &\mapsto x + M_i \end{aligned}$$

where $x \in M_{i+1} \cap A$. Notice that $\ker(f)$ is the set of elements in $(M_{i+1} \cap A)/(M_i \cap A)$ which map into M_i. If $x + (M_i \cap A)$ maps into M_i, then $x \in M_i$. Since $x \in M_{i+1} \cap A$, $x \in M_i \cap A$ implying $\ker(f) = 0$. This gives $M_{i+1} \cap A/M_i \cap A$ is isomorphic to a submodule of M_{i+1}/M_i, which we know is isomorphic to $R/\mathfrak{p}_{i+1}$ by $M \in \mathcal{S}$.

Now, if $M_1 \cap A \cong R/\mathfrak{p}$, where $\mathfrak{p} \in \mathcal{F}$, we continue. If not, we have $M_1 \cap A$ is isomorphic to a submodule of $R/\mathfrak{p}_1$. Use 3.2 to find a filtration of $M_1 \cap A$ with subquotients isomorphic to $R/\mathfrak{q}$ where $\mathfrak{q} \in \operatorname{Spec}(R)$. By 3.3 these $\mathfrak{q}$ contain $\mathfrak{p}_1$.

Insert these new submodules given by the filtration of $M_1 \cap A$ into our filtration as needed. Continue this process for each $M_i \cap A$, noticing in each case these $\mathfrak{q} \in \mathcal{F}$ because $\mathcal{F}$ is a filter. This gives the necessary filtration of A, showing $A \in \mathcal{S}$. Lastly we show that $M/A \in \mathcal{S}$. Notice that

$$0 = (M_0 + A)/A \subset (M_1 + A)/A \subset \cdots \subset (M_{n-1} + A)/A \subset (M_n + A)/A = M/A$$

is a filtration of M/A. While its subquotients may not subscribe to our requirements for $\mathcal{S}$, we can use a similar process to that above by defining the map g below:

$$\begin{aligned} M_{i+1}/M_i &\xrightarrow{g} (M_{i+1} + A)/(M_i + A) \\ x + (M_i) &\mapsto x + (M_i + A) \end{aligned}$$

Notice that g is surjective, so we have:

$$(M_{i+1} + A)/(M_i + A) \cong \frac{M_{i+1}/M_i}{\ker(g)} \cong \frac{R/\mathfrak{p}_{i+1}}{\ker(g)}$$

We see that each $(M_{i+1} + A)/(M_i + A)$ is isomorphic to a quotient of $R/\mathfrak{p}_{i+1}$. Call this quotient N and notice that $\mathfrak{p}_{i+1}$ annihilates any element of N. Similar to above, if $(M_1 + A)/A \cong R/\mathfrak{p}_1$, continue. If not, use 3.2 to find a filtration of $(M_1 + A)/A$ with subquotients isomorphic to $R/\mathfrak{q}, \mathfrak{q} \in \operatorname{Spec}(R)$. By 3.3 and the fact that $\mathcal{F}$ is a filter, $q \in \mathcal{F}$. Continue to add more modules into our filtration of M/A as needed using this method and we see that $M/A \in \mathcal{S}$. $\square$

Theorem 3.7. *Let $\mathcal{F} \subset \operatorname{Spec}(R)$ be a filter. Define $\mathcal{S}$ to be the set of finitely generated modules M so that M has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_{i+1}/M_i \cong R/\mathfrak{p}$ for $\mathfrak{p} \in \mathcal{F}$. This $\mathcal{S}$ is a Serre class.

Proof. Recall that any short exact sequence of $R-$modules $0 \to M_1 \to M \to M_2 \to 0$ is isomorphic to a short exact sequence $0 \to A \to M \to M/A \to 0$ where A is a submodule of M and M/A is its corresponding quotient module. By 3.5 and 3.6, our claim is proved. $\square$

We now write $\mathcal{S} \Rightarrow \mathcal{F}$ to mean that $\mathcal{F}$ is the set of prime ideals associated with a given Serre class $\mathcal{S}$. Write $\mathcal{F} \Rightarrow \mathcal{S}$ to indicate $\mathcal{S}$ is the Serre class associated with a given set of prime ideals $\mathcal{F}$. We will show that there is a bijective correspondence between these two sets.

Theorem 3.8. *With $\mathcal{S}$ and $\mathcal{F}$ as above:*

(i) If $\mathcal{S} \Rightarrow \mathcal{F}$ and if $\mathcal{F} \Rightarrow \mathcal{S}'$, then $\mathcal{S} = \mathcal{S}'$.

(ii) If $\mathcal{F} \Rightarrow \mathcal{S}$ and if $\mathcal{S} \Rightarrow \mathcal{F}'$, then $\mathcal{F} = \mathcal{F}'$.

Proof. We first prove that $\mathcal{S} \subseteq \mathcal{S}'$. If $M \in \mathcal{S}$, then $M = M_n \supset \cdots \supset M_1 \supset M_0 \supset 0$ where each M_{i+1}/M_i is isomorphic to $R/\mathfrak{p}$. This implies $p \in \mathcal{F}$. We know that if $M' \in \mathcal{S}'$ there is a similar chain of submodules forming a filtration of M' and the exact same $\mathcal{F}$ is prescribed up front. Clearly M is of this type.

We now prove that $\mathcal{S}' \subseteq \mathcal{S}$. Let $M' \in \mathcal{S}'$. Then $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_n = M'$ where $M'_i/M'_{i-1} \cong R/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \mathcal{F}$. But $\mathfrak{p}_i \in \mathcal{F} \Rightarrow R/\mathfrak{p}_i \in \mathcal{S}$. This implies M'_i/M'_{i-1} is isomorphic to some element of $\mathcal{S}$ and since $\mathcal{S}$ is a Serre class, we have $M'_i/M'_{i-1} \in \mathcal{S}$ for each i. We will now use induction to show $M = M'_n \in \mathcal{S}$. Begin by noting noting $M'_0 = 0 \in \mathcal{S}$. We use the short exact sequence below, the fact that $M'_n = M'$ and that $M'_{n-1} \in \mathcal{S}$ by our induction assumption to finish our claim.

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0$$

Next we show $\mathcal{F} = \mathcal{F}'$. Notice that $\mathcal{F} \subseteq \mathcal{F}'$ because if $\mathfrak{p} \in \mathcal{F}$, $R/\mathfrak{p} \in \mathcal{S}$ which implies $\mathfrak{p} \in \mathcal{F}'$. Letting $\mathfrak{p} \in \mathcal{F}'$ implies $R/\mathfrak{p} \in \mathcal{S}$. Given any filtration of $R/\mathfrak{p}$, $0 = M_0 \subset M_1 \subset \cdots \subset M_n = R/\mathfrak{p}$, notice $M_1 \cong R/\mathfrak{p}_1$ which means $\mathfrak{p}_1 \in \mathcal{F}$. By 3.3 $\mathfrak{p} = \mathfrak{p}_1$, thus proving $\mathfrak{p} \in \mathcal{F}$. $\square$

The result described in 3.8 was announced by Mark Hovey and Karen Collins of Wesleyan University without publication. Since Hovey and Collins didn't publish their result, we included this proof for completeness. It seems likely that the techniques they used were similar. Manuel López, a student of Hovey and Collins, used their result in his thesis where he characterized Serre classes of artinian modules. In the following chapter, we will also characterize Serre classes of artinian modules, only using different techniques. We will need to use the fact that the Serre classes of finitely generated modules exist in bijective correspondence with the sets $\mathcal{F}$ where $\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p}$ is a finitely generated $R-$ module$\}$ and $\mathcal{F}$ is closed under specialization.

Chapter 4 A Noetherian-Artinian Relationship

In this chapter we will define a bijective correspondence between submodules of M and M^ν when M is a reflexive module over a complete and local ring R. We show that M satisfies the ascending chain condition (M is noetherian) if and only if M^ν satisfies the descending chain condition (M^ν is artinian). We will then have an analogous result which says M is artinian if and only if M^ν is noetherian. In chapter three, we characterized all Serre classes of noetherian modules, and we now use that result to characterize Serre classes of artinian modules.

4.1 A Bijective Correspondence

In what follows, we build an argument to explain the relationship between the chain conditions on an $R-$module M and the chain conditions on its dual. We use interchangeably the phrases "M satisfies the ascending chain condition (ACC) on its submodules" with "M is noetherian" and "M satisfies the descending chain condition (DCC) on its submodules" with "M is artinian". At first, M is allowed to be any module over our usual R. Then, we use the fact that when M is reflexive, so is M^ν (see 2.17) to get a bijective correspondence between submodules. We extend this result to show that M satisfies ACC (DCC) if and only if M^ν satisfies DCC (ACC).

Now, let M be any $R-$module with $S \subset M$ a submodule. Define, as we did previously, p as the canonical surjection $M \to M/S$. By 2.8, $p^\nu : (M/S)^\nu \to M^\nu$ is an injection. Denote the image of p^ν by ${}^\perp S$, and notice ${}^\perp S$ is a submodule of M^ν. In fact, this ${}^\perp S$ will be the submodule of M^ν hinted at above which forms half of the bijective correspondence. Before defining this bijection, we need to obtain a better understanding of ${}^\perp S$.

Lemma 4.1. *With the notation defined above:*

$${}^\perp S = \{\varphi \in M^\nu | \varphi(S) = 0\}$$

Proof. We first show that any element, φ, of ${}^\perp S$ is such that $\varphi(s) = 0$ for all $s \in S$. Since ${}^\perp S = \mathrm{Im}(p^\nu)$, $\varphi \in^\perp S$ implies $\varphi = p^\nu(\psi) = \psi \circ p$ where $\psi \in (M/S)^\nu$. Notice $\psi \circ p \in M^\nu$ and consider $\varphi(s) = \psi(p(s))$. Since $s \in S$, $p(s) = 0$ and we conclude $\varphi(s) = 0$.

Now, if $\varphi \in M^\nu$ and $\varphi(s) = 0$ for all $s \in S$, notice $\ker(p) = S \subset \ker(\varphi)$. Define $\psi : M/S \to E$ by $x + S \mapsto \varphi(x)$, where $x \in M$. This ψ is well-defined as $\varphi(S) = 0 = \varphi(S)$. Clearly $\varphi = \psi \circ p$ which means $\varphi \in^\perp S$. □

Now that we have a more concrete understanding of ${}^\perp S$, we can construct a similar set to complete our proposed bijection between submodules of M and submodules of M^ν. Notice that the map $M^\nu \times M \to R$ which takes (φ, x) to $\varphi(x)$ helps us understand why we used the notation for ${}^{\perp}S$: it is the set of all homomorphisms in M^{ν} which are orthogonal to S. Now, given $T \subset M^{\nu}$, define $T^{\perp}$ to consist of all the $x \in M$ so that $\varphi(x) = 0$ for all $\varphi \in T$. We formalize this definition in the following remark.

Remark 4.2. As described above:

$$T^{\perp} = \{x \in M | \varphi(x) = 0 \text{ for all } \varphi \in T\}$$

Notice that the above remark holds true for any subset T of M^{ν}, but we will be especially interested in those T which are submodules of M^{ν}. In order to prove the injectivity of our bijection between S and ${}^{\perp}S$, we will need the following results. For the first, recall, as in the proof of 2.15 that for any element $x \in M$, Rx is isomorphic to $R/\operatorname{ann}(x)$. We now want to use this fact to show that there exists some $\varphi \in M^{\nu}$ so that the evaluation of this φ at our nonzero x is also nonzero. Recall first 2.15 which states that in a local ring, E is faithfully injective. We use this to better fit our current needs as shown below.

Lemma 4.3. *If x is a nonzero element of M, then there is a homomorphism $\varphi \in M^{\nu}$ so that $\varphi(x) \neq 0$.*

Proof. Consider the diagram below where the map from Rx to E is defined by the composition of the nonzero map found in 2.15 and the injection of $R/\mathfrak{m}$ into its injective envelope.

$$\begin{tikzcd} Rx \arrow[r, hook] \arrow[d] & M \arrow[dl, dashed, "\varphi"] \\ E & \end{tikzcd}$$

Because E is injective and the map from Rx to E is nonzero, there exists a nonzero $\varphi : M \to E$ so that $\varphi(x)$ is nonzero. $\square$

Lemma 4.4. *Let $S \subset M$ be a submodule of M with $x \in M$ and $x \notin S$. There exists a $\varphi \in M^{\nu}$ so that $\varphi(S) = 0$, but $\varphi(x) \neq 0$.*

Proof. Notice by our assumption if $x \neq 0$, $x + S$ is also nonzero as an element of M/S. Noting that M/S is an $R-$module, we apply 4.3 to find $\bar{\varphi} : M/S \to E$ so that $\bar{\varphi}(x) \neq 0$. Consider the composition $M \xrightarrow{p} M/S \xrightarrow{\bar{\varphi}} E$. Define $\varphi := \bar{\varphi} \circ p$ and notice that $\varphi(x) \neq 0$. Also, $\varphi(s) = \bar{\varphi}(0) = 0$ for any $x \in S$. $\square$

Lemma 4.5. *If S is any submodule of the $R-$module M, then $({}^{\perp}S)^{\perp} = S$.*

Proof. By our definitions in 4.1 and 4.2, we see that:

$$({}^{\perp}S)^{\perp} = \{x \in M | \varphi(x) = 0 \text{ for all } \varphi \in M^{\nu} \text{ where } \varphi(S) = 0\}$$

We need to show that $\varphi(x) = 0$ for all $\varphi \in M^{\nu}$ provided $\varphi(S) = 0$ precisely when $x \in S$. Clearly if $x \in S$ and φ is of this type, then $\varphi(x) = 0$, giving $S \subseteq ({}^{\perp}S)^{\perp}$. Now, suppose there exists some $x \in ({}^{\perp}S)^{\perp}$ where $x \notin S$, and $\varphi(x) = 0$ for all $\varphi \in M^{\nu}$ where $\varphi(S) = 0$. By 5.15, this is impossible. $\square$

Lemma 4.6. *Consider the correspondence described above which maps submodules of M to submodules of M^ν described by $S \mapsto^\perp S$. This map is injective.*

Proof. Let ${}^\perp S_1 = {}^\perp S_2$ where ${}^\perp S_1, {}^\perp S_2 \in \mathrm{Im}(M)$. Notice that

$$({}^\perp S_1)^\perp = \{x \in M | \varphi(x) = 0 \text{ for all } x \in^\perp S_1\} = \{x \in M | \varphi(x) = 0 \text{ for all } x \in^\perp S_2\} = ({}^\perp S_1)^\perp$$

where the middle equality is clear because ${}^\perp S_1 =^\perp S_2$. Using 4.5 upon noticing $({}^\perp S_1)^\perp = ({}^\perp S_2)^\perp$, we conclude $S_1 = S_2$ meaning our correspondence is injective. □

Notice that so far, we have not needed M reflexive. To prove the injectivity of the map $T \mapsto T^\perp$ needed for our correspondence, we will need this assumption. Recall that by 2.17 if M is reflexive, then so is M^ν. Now, just as with reflexive modules, we can "identify" M with $M^{\nu\nu}$ through the canonical isomorphism. Looking back on the previous arguments in this chapter, we see that we can apply the same procedures that we did on the map $M \times M^\nu \to R$, to the map $M^{\nu\nu} \times M^\nu \to R$. This is, in essence, repeating the procedure with M^ν in the place of M. Because M is reflexive, identifying M with $M^{\nu\nu}$ gives us a map from $M \times M^\nu \to R$. It turns out by a simple check that this map is defined by $(x, \varphi) \mapsto \varphi(x)$.

First we let $(\tau, \varphi) \in M^{\nu\nu} \times M^\nu$ and note that this element should map to $\tau \circ \varphi$. Because $M \cong M^{\nu\nu}$ via the canonical isomorphism, ϕ, there exists a unique $x \in M$ corresponding to τ. This defines our identified map from $M \times M^\nu \to R$ by $(\phi^{-1}(\tau), \varphi) \mapsto \phi^{-1}(\tau)(\varphi)$ which is $\varphi(x)$ based on the definition of ϕ. Now that we realize our two maps $M^\nu \times M \to R$ and $M \times M^\nu \to R$ are defined in the same way, we refer to the map from here on as $M^\nu \times M \to R$.

Lemma 4.7. *When M is reflexive, the map $T \to T^\perp$ we get from $M^\nu \times M \to R$ is an injection.*

Proof. Let $T_1^\perp = T_2^\perp$. By definition we see that ${}^\perp(T_1)^\perp =^\perp (T_2)^\perp$. Because M^ν is a reflexive module in its own right, we can easily use 5.15 to prove a similar version of 4.5 (ie: that ${}^\perp(T^\perp) = T$). This implies $T_1 = T_2$. (Note that we needed a reflexive M to properly consider the map $M^\nu \times M \to R$.) □

Lemma 4.8. *Let M be a reflexive $R-$module. There is a bijective correspondence between the submodules of M and the submodules of M^ν.*

Proof. Define $\mathcal{S}$ as the set of all submodules of M and $\mathfrak{T}$ as the set of all submodules of M^ν. As suggested above, define our bijection $\psi : \mathcal{S} \to \mathfrak{T}$ by $S \mapsto^\perp S$ where $S \in \mathcal{S}$. By 4.6, we have that $S \mapsto^\perp S$ is injective. We will show ψ has an inverse and its surjectivity will be clear by 4.7. Notice that $\bar{\psi} : \mathfrak{T} \to \mathcal{S}$ defined by $T \mapsto T^\perp$ suggested above is an inverse for ψ. This is because $\bar{\psi} \circ \psi(S) = \bar{\psi}({}^\perp S) = ({}^\perp S)^\perp = S$ for all $S \in \mathcal{S}$ (by 4.5). Also, $\psi \circ \bar{\psi}(T) = \psi(T^\perp) =^\perp (T^\perp) = T$ for all $T \in \mathfrak{T}$ by the proof of 4.7. Now that ψ has an inverse, notice ψ is surjective, because if $T \in \mathfrak{T}$, there exists some $\bar{\psi}(T) \in \mathcal{S}$ so that $\psi(\bar{\psi}(T)) = T$. We now have a bijection between M and M^ν when M is reflexive. □

Along with this bijection we also find that certain chain conditions on M imply certain chain conditions on M^ν and vice versa. In order to realize this property, we need the following, more specific result related to 4.8.

Remark 4.9. If S_1, S_2 are submodules of M and T_1, T_2 are the corresponding submodules of M^ν (via the bijection described in 4.8), then $S_1 \subset S_2$ if and only if $T_1 \supset T_2$.

Proof. Both directions are clear simply by the assumed containment and the definition of the sets mapped to under the bijection. $\square$

Theorem 4.10. *Let M be a reflexive $R-$module. M satisfies ACC (DCC) on its submodules if and only if M^ν satisfies DCC (ACC) on its submodules.*

Proof. Both directions of both cases are easily realized using 4.9. $\square$

4.2 Serre Classes of Artinian Modules

We will now characterize all Serre classes of artinian modules over a complete local noetherian ring using the results of chapter four and the bijection developed above. We know that there exists a bijective correspondence between Serre classes of noetherian modules and the subsets $\mathcal{F} \subset \operatorname{Spec} R$ which are closed under specialization. We also know if M is reflexive then M satisfies 4.10.

Given any class $\mathcal{S}$ of modules, denote by $\mathcal{S}^\nu$ the set of all M^ν where $M \in \mathcal{S}$. We will begin with some basic properties of $\mathcal{S}^\nu$. Then we will show that there is a bijection between the Serre classes, $\mathcal{T}$, of artinian modules and certain subsets of $\operatorname{Spec}(R)$ which will give our characterization.

Lemma 4.11. *If $\mathcal{S}$ is any Serre class of reflexive modules, then $\mathcal{S}^\nu$ is also a Serre class of reflexive modules.*

Proof. By our definition of $\mathcal{S}^\nu$ and the fact that Serre classes are closed under isomorphism, first note that $\mathcal{S}^{\nu\nu} = \mathcal{S}$ where $\mathcal{S}$ is any class of reflexive modules.

Now, let $M_1^\nu, M_3^\nu \in \mathcal{S}^\nu$ and let $0 \to M_1^\nu \to M \to M_3^\nu \to 0$ be a short exact sequence. Then by definition of $\mathcal{S}^\nu$, we must have $M_1, M_3 \in \mathcal{S}$. Notice that $0 \to M_3^{\nu\nu} \to M^\nu \to M_1^{\nu\nu} \to 0$ is short exact by 2.8. By M_1, M_3 reflexive, we have another short exact sequence $0 \to M_3 \to M^\nu \to M_1 \to 0$. Because $\mathcal{S}$ is a Serre class, we have $M^\nu \in \mathcal{S}$. By the definition of $\mathcal{S}^\nu$, this implies $M^{\nu\nu} \cong M \in \mathcal{S}^\nu$.

Now, let $M^\nu \in \mathcal{S}^\nu$. We need to show that if $0 \to S \to M^\nu \to M/S \to 0$ is short exact, then both S and M/S are elements of $\mathcal{S}^\nu$. We have the commutative diagram below from 2.8 and the fact that M is reflexive. ($M^\nu \in \mathcal{S}^\nu$ implies that M is reflexive by definition of $\mathcal{S}^\nu$.)

$$\begin{tikzcd}
0 \arrow[r] & (M/S)^\nu \arrow[r] \arrow[d, "\cong"'] & M^{\nu\nu} \arrow[r] \arrow[d, "\cong"] & S^\nu \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (M/S)^\nu \arrow[r] & M \arrow[r] & S^\nu \arrow[r] & 0
\end{tikzcd}$$

$\square$

From here we see that $(M/S)^\nu, S^\nu \in \mathcal{S}$. Since we showed above $\mathcal{S} = \mathcal{S}^{\nu\nu}$, our claim is proved.

Lemma 4.12. *If $\mathcal{S}$ is a Serre class of finitely generated (noetherian) modules, then $\mathcal{S}^\nu$ is a Serre class of artinian modules.*

Proof. Let $\mathcal{S}$ be a Serre class of noetherian modules. In a complete local noetherian ring, noetherian modules are reflexive. (We will prove this fact in chapter five.) Then by 4.11 we know that $\mathcal{S}^\nu$ is a Serre class. If $M \in \mathcal{S}$, M is finitely generated and M^ν is artinian by 4.10. So, letting $M^\nu \in \mathcal{S}^\nu$ gives by definition, $M \in \mathcal{S}$ which implies M is noetherian and M^ν is artinian. Given a short exact sequence of the form $0 \to S \to M^\nu \to M^\nu/S \to 0$ where $S \subset M^\nu$, we know S and M^ν/S are artinian because submodules and quotient modules of artinian modules are artinian.

Finally, if we begin with $M_1^\nu, M_3^\nu \in \mathcal{S}^\nu$ we must have that they are artinian. Given a short exact sequence $0 \to M_1^\nu \to M^\nu \to M_3^\nu \to 0$ yields $0 \to M_3 \to M \to M_1 \to 0$ short exact, giving $M \in \mathcal{S}$ and M^ν artinian. $\square$

In a complete local noetherian ring, artinian modules are also reflexive. We will prove this in the next chapter as well. Since M^ν is artinian as well as the dual of a reflexive, by 2.17 it will follow that artinian modules are reflexive.

Notice that there exists a bijective correspondence between Serre classes, $\mathcal{S}$ of noetherian modules and the Serre classes, $\mathcal{T}$ of artinian modules. We formalize this bijection in the following remark which is clear by the fact that both noetherian modules and artinian modules are reflexive. (This makes $\mathcal{S} = \mathcal{S}^{\nu\nu}$, for example.)

Remark 4.13. The correspondence $\mathcal{S} \leftrightarrow \mathcal{T}$ given by $\mathcal{S} \mapsto \mathcal{T}$ when $\mathcal{S}^\nu = \mathcal{T}$ (equivalently $\mathcal{T}^\nu = \mathcal{S}$) is a bijection.

We now characterize Serre classes of artinian modules by concretely relating an artinian Serre class to its dual. The bijection between Serre classes of noetherian modules and Serre classes of artinian modules is shown above. We have also shown that the dual of a Serre class of noetherian modules is artinian. All that remains to show is that the dual of an artinian Serre class of modules is a Serre class of noetherian modules.

Lemma 4.14. *If $\mathcal{T}$ is a Serre class of artinian modules then $\mathcal{T}^{\nu}$ is a Serre class of noetherian modules.*

Proof. Let $\mathcal{T}$ be a Serre class of artinian modules. By the bijection in 4.13, there exists a Serre class, $\mathcal{S}$ of noetherian modules such that $\mathcal{S}^{\nu} = \mathcal{T}$. Notice that $\mathcal{T}^{\nu} = \mathcal{S}^{\nu\nu} = \mathcal{S}$ which is a Serre class of noetherians. $\square$

Now, recall that in chapter four we described the characterization of all Serre classes of noetherian modules. These Serre classes, $\mathcal{S}$ are completely determined by the set of prime ideals $\mathfrak{p}$ of R so that $R/\mathfrak{p} \in \mathcal{S}$. We called this set $\mathcal{F}$ and also noted $\mathcal{F}$ is closed under specialization.

Based on 4.14, we see that any Serre class $\mathcal{T}$ of artinian modules is in bijective correspondence with a subset $\mathcal{G} \subset \operatorname{Spec}(R)$ where $\mathcal{G}$ is closed under specialization so that $\mathfrak{p} \in \mathcal{G}$ if and only if $(R/\mathfrak{p})^{\nu} \in \mathcal{T}$. Note that by the relationship given above, $R/\mathfrak{p} \in \mathcal{S}$. As we mentioned in chapter three, Manuel López achieved this result in his Wesleyan thesis only he used a different argument. In chapter six, we will turn our attention to characterize the Serre classes of reflexive modules over a complete local noetherian ring which will be an extension of the results in chapters three and four.

Chapter 5 Requiring R to be Complete

In the next chapter, the proof of the main result of this dissertation is completed. We will characterize all Serre classes of reflexive modules over a complete local noetherian ring R. We will need to understand the relationship between R^ν and $E = E(\kappa)$. To achieve the most beneficial relationship (ie: $E^\nu \cong R$) we will need R to be complete.

We will then show that there are many examples of reflexive modules for such an R. For example, if R is local, a direct sum of reflexives is reflexive. When R is also complete, R itself is reflexive. Additional examples of reflexive modules include any free module with a finite bases and any finitely generated $R-$module. Still other examples include noetherian modules and artinian modules.

Lemma 5.1. *In a local ring, R, $R^\nu \cong E$.*

Proof. Considering R as a module over itself, notice that $\varphi \in R^\nu$ is completely determined by $\varphi(1) = x$ where x is an element of E. In addition, each element $x \in E$ defines a homomorphism $\varphi \in R^\nu$. Hence $R^\nu \cong E$. $\square$

In the results that follow, suppose R is a local ring which has finite length as a module over itself. Note that this implies when R has finite length, E has finite length by 2.20. In fact, we can show that over a local ring, any module of finite length is reflexive.

Lemma 5.2. *Given M is a module of finite length, M is reflexive.*

Proof. We use induction. By 2.19 any module of length 1 is reflexive. Suppose any module of length $n-1$ is reflexive and let M have length n. Since $0 \to M_1 \hookrightarrow M \to M/M_1 \to 0$ is a short exact sequence and E is injective, $0 \to M_1^{\nu\nu} \hookrightarrow M^{\nu\nu} \to M/M_1^{\nu\nu} \to 0$ is short exact by 2.9. Because $\ell(M_1^{\nu\nu}) + \ell(M/M_1^{\nu\nu}) = \ell(M^{\nu\nu})$, 2.20 implies $\ell(M/M_1^{\nu\nu}) = n-1$. By assumption, $M/M_1^{\nu\nu}$ is reflexive and 2.10 implies M is reflexive. $\square$

Notice that if R has finite length as a module over itself, $R \cong R^{\nu\nu} \cong E^\nu$ by 5.1 making $E^\nu \cong R$. We will extend this result to rings that do not necessarily have finite length. This will take some work and relies on considering completeness with respect to the $\mathfrak{m}-$adic topology. For now we give an explicit isomorphism between E^ν and R when R has finite length as a module over itself.

Lemma 5.3. *$E^\nu \cong R$ where the isomorphism is described by $\dot{r} : R \to E^\nu$ where $\dot{r}(r) = (x \mapsto rx)$ for any $r \in R$.*

Proof. Consider the diagram below and notice that by showing φ is an isomorphism and that the diagram commutes we get $\dot{r}$ is an isomorphism.

$$\begin{tikzcd} R \arrow[r, "\cong"] \arrow[dr, "\dot{r}"'] & R^{\nu\nu} \arrow[d, "\varphi"] \\ & E^{\nu} \end{tikzcd}$$

Notice φ is an isomorphism as $R^{\nu\nu} \cong E^{\nu}$ by 5.1. This means mapping $r \in R$ right and down via the diagram results in $r \mapsto (\sigma \mapsto \sigma(r)) \mapsto (x \mapsto \sigma(r))$ where each σ can be described by a unique $x \in E$. Now, notice the homomorphism $(x \mapsto \sigma(r))$ maps x to $\sigma(r \cdot 1) = r(\sigma(1))$. As σ is defined by x, we have $\sigma(1) = x$ making $(x \mapsto \sigma(r)) = (x \mapsto rx)$. □

5.1 Relating R to a Module of Finite Length

We now develop a result which allows us to change from a local R to a local ring which also has finite length. After that we will use the results above in a more general case. We will find that $R/\mathfrak{m}^n$ has finite length and the following arguments will allow us to compare injective modules over a local ring R with injective modules over $R/\mathfrak{m}^n$.

The next few results are true for any commutative R, an injective left $R-$module E and $I \subset R$ a two-sided ideal of R. We will later take $I = \mathfrak{m}^n$ and require a local R.

Lemma 5.4. *With R, E and I as above, $0 \to \mathrm{Hom}_R(R/I, E) \to \mathrm{Hom}_R(R, E)$ is exact.*

Proof. We denote $\mathrm{Hom}_R(R/I, E)$ by $(R/I)^{\nu}$ and $\mathrm{Hom}_R(R, E)$ by R^{ν}. Notice that $R \xrightarrow{\sigma} R/I \to 0$ is exact where σ represents the projection map. Let $h \in (R/I)^{\nu}$ and recall $\sigma^{\nu}(h) = h \circ \sigma$ by definition. If $\sigma^{\nu}(h) = 0$, $h(\sigma(r)) = 0$ for all $r \in R$. Suppose $h \neq 0$. Then there exists some $s \in R$ where $s \notin I$ so that $h(s + I) \neq 0$. By the surjectivity and definition of σ, $\sigma(s) = s + I \neq 0$ which is a contradiction to $\sigma^{\nu}(h) = 0$ as $h(\sigma(s)) = 0$. □

We now show that $(R/I)^{\nu}$ can be identified with a submodule of E. This will be key in relating R to $R/\mathfrak{m}^n$.

Lemma 5.5. *Define $E' = \{x | x \in E, Ix = 0\}$ and notice $E' \subset E$. This E' is isomorphic to $(R/I)^{\nu}$.*

Proof. Consider the inclusion $(R/I)^{\nu} \hookrightarrow R^{\nu}$ and recall we have $R^{\nu} \cong E$. (Remember, what we do not have yet for a general local ring is $R \cong E^{\nu}$.) This means $(R/I)^{\nu}$ is isomorphic to some submodule of E, call it T. We claim that $T = E'$ and proceed by showing both inclusions.

Claim: $T \subset E'$. Let $\tau \in T$ and recall by 5.4 σ^{ν} is an injection. Consider $(R/I)^{\nu} \xrightarrow{\sigma^{\nu}} R^{\nu} \xrightarrow{\cong} E$. By the nature of the isomorphism between R^{ν} and E, we have that τ maps as follows: $\tau \mapsto \tau \circ \sigma \mapsto \tau(\sigma(1))$. Define $\tau(\sigma(1)) = x$ and notice if $r \in I$, $\tau(\sigma(r)) = rx$. However, by definition of σ, $\sigma(r) = 0$, so we have $rx = 0$. This shows that when τ is identified with an element $x \in E$, $Ix = 0$ implying $T \subset E'$.

Claim: $E' \subset T$. Let $x \in E'$ meaning $x \in E$ and $Ix = 0$. By $R^\nu \cong E$ we know x corresponds to the $\varphi \in R^\nu$ so that $\varphi(1) = x$. Since $\varphi : R \to E$ and because any $r \in I$, is such that $r \in \ker(\varphi)$, we have that φ defines a map $R/I \to E$. This shows $E' \subset T$. ∎

Now that we have $E' \cong (R/I)^\nu$, we can view E' as a left R/I module. The only tricky part is defining scalar multiplication. An easy check shows that $(r + I)x = rx$ where $r + I \in R/I$ and $x \in E'$ is well-defined. In fact, we can show that E' is an injective R/I module.

Lemma 5.6. *E' as defined above is an injective R/I module.*

Proof. By Baer's criterion we must show that if $\varphi : J/I \to E'$, where J/I is an ideal of R/I, then there exists an extension of φ which maps R/I to E'. The following claim will help us achieve such an extension.

Claim 5.7. A linear map $\sigma : R \to E$ is such that $\sigma(R) \subseteq E'$ if and only if $\sigma(1) = x$ where $Ix = 0$.

Proof. To prove the claim, let $\sigma \in R^\nu$ so that $\sigma(R) \subseteq E'$. If $\sigma(1) = x$ and $Ix \neq 0$ then $x \notin E'$. This implies $\sigma(1) \notin \sigma(R)$ which is impossible as $1 \in R$.

Now, if $\sigma(1) = x$ where $Ix = 0$, suppose there exists some $r \in R$ where $\sigma(r) \notin E'$. Then $\sigma(r) = y$ and $Iy \neq 0$. We also have $\sigma(r) = r\sigma(1) = rx$. This provides a contradiction as $0 \neq Iy = Irx = Ix = 0$.

∎

Now that our claim is proved, consider the commutative diagram below where each $\hookrightarrow$ represents an inclusion map and the homomorphism f exists by the injectivity of E.

$$
\begin{tikzcd}
J \arrow[r, hook] \arrow[d] & R \arrow[dd, "f"] \\
J/I \arrow[d, "\varphi"] & \\
E' \arrow[r, hook] & E
\end{tikzcd}
$$

Notice that our diagram is such that $If(1) = 0$ and by our claim this means $f(R) \subset E'$. Finally, notice $I \subset \ker(f)$ giving the induced map we desire. By Baer's criterion, E' is an injective R/I module. ∎

We can now define what sort of submodules of E' have E' as an injective envelope.

Lemma 5.8. *Let $M \subset E$ be a submodule of E so that E is an injective envelope of M. Define $M' = \{x | s \in M, Ix = 0\}$. As $R/I-$modules, $M' \subset E'$ is an injective envelope.*

Proof. Note that M' is an R/I module as $M' \subset E'$. We need only show that $M' \subset' E'$. Suppose $T \subset E'$ and $T \cap M' = 0$ but $T \neq 0$. By $M \subset' E$ and $T \neq 0$ we must have $T \cap M \neq 0$. Let $0 \neq t \in T \cap M$. Since $T \subset E'$, we have $t \in M$ which contradicts $T \cap M' = 0$. □

We now require R to be local with our usual $E = E(\kappa)$. We do not assume R to have finite length as a module over itself. In the results which follow, we see that $R \to E^{\nu}$ as defined in 5.3 might not always be injective. We need $R \cong E^{\nu}$, but this will require that R is complete with respect to the $\mathfrak{m}-$adic topology.

Note that the residue field of R which we are calling κ is the only simple module (up to isomorphism) over R. If V is a vector space over κ, then V can also be considered an $R-$module. If $\dim(V) = n < \infty$, then V is the direct sum of n copies of κ. This means that as an $R-$module, V has finite length.

Since R is noetherian, any ideal of R is finitely generated as an $R-$module. It can be checked that $I/\mathfrak{m}I$ is the direct sum of a finite number of copies of κ when $I \subset R$ is an ideal of R. Now, notice that $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n$. These two facts allow us to argue that $R/\mathfrak{m}^n$ has finite length.

Lemma 5.9. *For $n \geq 1$ $R/\mathfrak{m}^n$ has finite length.*

Proof. Note that $\ell(R/\mathfrak{m}) = 1$ and proceed by induction. Suppose $\ell(R/\mathfrak{m}^{n-1}) < \infty$. By the chain of inclusions above we have $R/\mathfrak{m}^n \xrightarrow{\varphi} R/\mathfrak{m}^{n-1} \to 0$ is exact. Additionally, if $r + \mathfrak{m}^{n-1} = 0$, $r \in \mathfrak{m}^{n-1}$ which implies $\ker(\varphi) = \{r + \mathfrak{m}^n | r \in \mathfrak{m}^{n-1}\} = \mathfrak{m}^{n-1}/\mathfrak{m}^n$. This implies $0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to R/\mathfrak{m}^n \xrightarrow{\varphi} R/\mathfrak{m}^{n-1} \to 0$ is short exact. By 2.27, $R/\mathfrak{m}^n$ has finite length.

□

In what follows, for $n \geq 1$, define $E_n = \{x | x \in E, \mathfrak{m}^n x = 0\}$ and $E_0 = 0$.

Lemma 5.10. *With E_n defined as above, $\cup_{n=0}^{\infty} E_n = E$.*

Proof. Clearly E_0 is a submodule of each E_n and $E_0 \subset E$. Let $x \in E_1$ and notice $x \in E$ and $\mathfrak{m}x = 0$. Notice $\mathfrak{m}^2 x = 0$, so $E_1 \subset E_2$, and we see that:

$$E_0 \subset E_1 \subset E_2 \subset \cdots \subset E.$$

This shows if $x \in \cup_{n=0}^{\infty} E_n$, $x \in E_i$ for some $i \geq 1$ which implies $x \in E$. Now, if $x \in E$, by 2.29 there is some $n \geq 1$ so that $\mathfrak{m}^n x = 0$ which implies $x \in E_n$. □

So far we know $R/\mathfrak{m}^n$ is an $R-$module with finite length, but we would like it to be local as well. Noticing that $\mathfrak{m}/\mathfrak{m}^n$ is the unique maximal ideal of $R/\mathfrak{m}^n$ gives just that. Now we define the residue field of $R/\mathfrak{m}^n$ as $\dfrac{R/\mathfrak{m}^n}{\mathfrak{m}/\mathfrak{m}^n} \cong R/\mathfrak{m}$.

Now, by 5.6, and 5.8 we have $R/\mathfrak{m} \subset' E_n$. This makes E_n an injective envelope of $R/\mathfrak{m}$. By our comments above, E_n is isomorphic to an injective envelope of the residue field of $R/\mathfrak{m}^n$.

Lemma 5.11. *When R is assumed local, $\dot{r} : R \to E^\nu$ which maps r to the linear map $x \mapsto rx$ is injective.*

Proof. Our approach will be to show $\ker(\dot{r}) \subset \mathfrak{m}^n$ for any n. This will give by 2.30 (Krull-Intersection) that $\ker(\dot{r}) = 0$.

By 5.9, $R/\mathfrak{m}^n$ has finite length, so we can use 5.3. After recalling that E_n is isomorphic to an injective envelope of the residue field of $R/\mathfrak{m}^n$, we see $R/\mathfrak{m}^n \cong \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = \mathrm{Hom}_R(E_n, E_n)$. Notice that if $\psi \in E^\nu$, $\psi(\mathfrak{m}^n x) = \psi(0) = 0$ for all $x \in E$ by 5.10. This implies $\mathfrak{m}^n \psi(x) = 0$ which puts $\psi(x) \in E_n$. This allows us to consider $\dot{r}$ as a map from R to $\mathrm{Hom}_R(E_n, E_n)$.

Considering the diagram below gives that $\ker(\dot{r}) \subset \mathfrak{m}^n$ for all n. This implies $\ker(\dot{r}) = 0$, making $\dot{r}$ injective.

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend left, "\dot{r}"] & R/\mathfrak{m}^n \cong \arrow[r] & \mathrm{Hom}_R(E_n, E_n) \end{tikzcd}$$

∎

Now, even though we have shown $\dot{r} : R \to E^\nu$ is injective, we must reduce back to the map $R \to \mathrm{Hom}_R(E_n, E_n)$ to get surjectivity.

Lemma 5.12. *With E_n as above, the map $R \to \mathrm{Hom}_R(E_n, E_n)$ defined by sending $r \in R$ to the map $x \mapsto rx$ is surjective, and its kernel is $\mathfrak{m}^n$.*

Proof. Let $\psi \in \mathrm{Hom}_R(E_n, E_n)$. By noticing $\mathrm{Hom}_R(E_n, E_n) \cong R/\mathfrak{m}^n$, ψ corresponds one-to-one with some unique $r + \mathfrak{m}^n$. The projection map from $R \to R/\mathfrak{m}^n$ is surjective, so our map from R to $\mathrm{Hom}_R(E_n, E_n)$ is surjective. Furthermore, the kernel of this map is the same as the kernel of the canonical projection, which is $\mathfrak{m}^n$. ∎

As mentioned in the beginning of this section, we will need a complete R to have $R \cong E^\nu$. Because we want $R \to E^\nu$ surjective, we really just need to answer the question: Given $\psi \in E^\nu$, when is there an $r \in R$ so that $\psi(x) = rx$ for all $x \in E$. We first form several equivalent conditions that give R complete, and then realize this answers our question.

Lemma 5.13. *R is complete if and only if whenever we have a sequence $r_0, r_1, \cdots$ of elements of R such that $r_n \cong r_{n+1} \mod \mathfrak{m}^n$ for each n there is an $r \in R$ so that $r \equiv r_n \mod \mathfrak{m}^n$ for each $n \geq 0$.*

Proof. Let R be complete and suppose $\{r_i\}_{i=0}^{\infty}$ is a sequence as above. Define $t_0 = r_0, t_1 = r_1 - r_0, t_2 = r_2 - r_1, \cdots, t_n = r_n - r_{n-1}, \cdots$.

Notice $r_n = \sum_{i=0}^{n} t_i$ and $\lim_{n\to\infty} t_n = 0$ since $r_n - r_{n+1} \in \cap_{n=0}^{\infty}\mathfrak{m}^n$ which by 2.30 we know to be zero. By the completeness of R, $\{r_i\}_{i=0}^{\infty}$ converges. In fact we see it converges to $r = \sum_{n=0}^{\infty} t_n$.

Now, let R have the property described above for sequences. Let $\{s_n\}_{n=0}^{\infty}$ be any Cauchy sequence in R. (Note that we mean Cauchy with respect to the linear topology associated with the subgroups $\mathfrak{m}^n$. This is the $\mathfrak{m}-$adic topology discussed by Lam ([10]).) Then for some $N \in \mathbb{N}$, when $m \geq N$, $|s_m - s_{m+1}| < \varepsilon$ for any given $\varepsilon > 0$. Now, because $\{s_n\}$ is a general Cauchy sequence, we may not have $s_n \cong s_{n+1}$ mod $\mathfrak{m}^n$, but a subsequence will have this property.

We use the notation as $\{s_m\}$, noting that we are now referring to the subsequence mentioned above. So, $s_m - s_{m+1} \in \mathfrak{m}^k$ and $s_m - s_{m+1} \notin \mathfrak{m}^{k+1}$ where $k > \ln(\frac{1}{\varepsilon} - 2)$. By choosing $\varepsilon = \frac{1}{e^m + 2}$, $s_m - s_{m+1} \in \mathfrak{m}^m$ for each m. By assumption, we have some $s \in R$ so that $s \equiv s_n \mod \mathfrak{m}^n$ for each $n \geq 0$. Letting $\varepsilon > 0$ and $N = \ln(\varepsilon - \frac{1}{2})$ we see that $s_n \to s$ as $n \to \infty$, thus proving R is complete. $\square$

Lemma 5.14. *If $\varphi_n \in \mathrm{Hom}_R(E_n, E_n)$ for each $n \geq 1$ and φ_{n+1} agrees with φ_n for each n, there exists a unique $\varphi \in \mathrm{Hom}_R(E, E)$ such that φ agrees with φ_n on E_n for each $n \geq 1$.*

Proof. If $x \in E$, $x \in E_n$ for some $n \geq 0$ by 5.10. Define $\varphi(x) = \varphi_n(x)$ where n is the smallest index where $x \in E_n$. The only tricky part of checking φ is well-defined is in checking $\varphi(x+y)$ for $x, y \in E$. Perhaps $x \in E_n$ and $y \in E_m$ where $x \notin E_i, y \notin E_j$ for $i < n$, $j < m$. Without loss of generality, suppose $m \geq n$. Then because $E_n \subset E_m$, $\varphi_m(x) = \varphi_n(x)$. So, $\varphi(x+y) = \varphi_m(x+y) = \varphi_n(x) + \varphi_m(y) = \varphi(x) + \varphi(y)$. This map is unique because if $\varphi_1 = \varphi_2$, there is some $x \in E$ so that $\varphi_1(x) \neq \varphi_2(x)$. But if both φ_1 and φ_2 satisfy our hypothesis, $\varphi_1(x) = \varphi_n(x) = \varphi_2(x)$, which is a contradiction. $\square$

With the arguments above, we can now answer the question of surjectivity for $R \to E^{\nu}$. For a local ring, this map is surjective if and only if R is complete.

Lemma 5.15. *When R is a local ring, R is complete if and only if for every $\varphi \in E^{\nu} = \mathrm{Hom}_R(E, E)$ there is an $r \in R$ so that $\varphi(x) = rx$ for all $x \in E$.*

Proof. Let R be complete and local and $\varphi \in E^{\nu}$. If $x \in E$, $x \in E_n$ as in 5.10. As $E_n \subset E_{n+1} \subset \cdots \subset E$, each map $\varphi_n, \varphi_{n+1}, \cdots$ mentioned in 5.12 is surjective. This implies we can find an r_n in R so that $r_n x = \varphi_n(x)$, and an $r_{n+1} \in R$ so that $\varphi_{n+1}(x) = r_{n+1}x = r_n x$, where $\varphi_i : E_i \to E_i$ is the map $x \mapsto r_i x$ for $i \geq n$. Continuing in this way, we form a sequence $\{r_i\}_{i=n}^{\infty}$ where $r_i x - r_{i+1} x = 0$. This means $r_i - r_{i+1}$ is in the kernel of the map mentioned in 5.12. So, $r_i - r_{i+1} \in \mathfrak{m}^n$ for all $i \geq n$. By 5.13 we realize our sequence converges to an $r \in R$ so that $\varphi(x) = rx$. This r is unique by 5.14.

Now, let R be local and have the property described above for every $\varphi \in E^\nu$. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence of elements of R so that $r_n \cong r_{n+1} \mod \mathfrak{m}^n$. By 5.13, we need only show there exists some $r \in R$ so that $r \equiv r_n \mod \mathfrak{m}^n$ for each $n \geq 0$. By our assumption, the φ defined by $\varphi_n(x)$ as above can be defined by one $r \in R$. This completes our proof. $\square$

By 5.11 and 5.15, we have the following theorem.

Theorem 5.16. *If R is a local ring, R is complete if and only if $R \cong E^\nu$ via the map defined by $r \mapsto (x \mapsto rx)$.*

5.2 Examples of Reflexive Modules

We finally come to our first example of a nontrivial reflexive module. When R is local and complete, R is reflexive.

Theorem 5.17. *A local ring R is reflexive if and only if it is complete.*

Proof. First, let R be complete and consider the canonical homomorphism ϕ as defined in 1.1. This map is injective because if $\phi(r_1) = \phi(r_2)$ for $r_1, r_2 \in R$, then $\varphi(r_1) = \varphi(r_2)$ for all $\varphi \in R^\nu$. Because we showed E is faithfully injective, $\varphi(r_1) - \varphi(r_2) = 0$ implies $r_1 - r_2 = 0$. This means ϕ is injective. To show that ϕ is surjective, let $\psi \in R^{\nu\nu}$. We have $R^{\nu\nu} \cong E^\nu$ by 5.1 and $E^\nu \cong R$ by 5.16 as R is complete. By the nature of the isomorphisims we have defined above for $R^{\nu\nu} \cong E^\nu \cong R$, it is easy to check that the $r \in R$ guaranteed by 5.16 is such that $\phi(r) = \psi$. We conclude ϕ is an isomorphism, making R reflexive.

Now, let R be reflexive and $\varphi \in E^\nu$. By 5.15, we need only show there exists some $r \in R$ so that $\varphi(x) = rx$ for all $x \in E$. By 5.1, $R^{\nu\nu} \cong E^\nu$, meaning φ corresponds bijectively with some $\psi \in R^{\nu\nu}$, which in turn corresponds to some $r \in R$ by the reflexivity assumed. Note that ψ acts on elements of R^ν, which are homomorphisms from R to E. The inverse of the canonical isomorphism from R to $R^{\nu\nu}$ preserves the desired property that $\psi(f) = f(r)$ when $f \in R^\nu$. We conclude $\varphi(x) = rx$ as desired. $\square$

So, in a local ring, R, we have a nontrivial reflexive module. It is helpful here to recall that the work of Matlis and Gabriel in [11] and [6] shows that under the Matlis dual, finitely generated (noetherian) and artinian modules are reflexive. We complete this result here by instead introducing a new type of submodule called the socle.

In [14] Walker and Walker showed that over a commutative noetherian ring, modules, M, with $\mathrm{soc}(M) \subset' M$ form a Serre class. We will show below that for an artinian module, its socle is an essential submodule, so this would give an alternate proof of 2.25. We will use the socle to prove that artinian modules are reflexive over a complete, local, noetherian ring.

Definition 5.18. *The* ***socle*** *of a module is defined as the intersection of all the essential submodules of M. In a local ring this can be described more explicitly as the set of all $x \in M$ so that $\mathfrak{m}x = 0$.*

The socle of a module can be made into a κ module by defining $(r + \mathfrak{m})x = rx$. This naturally makes $\operatorname{soc}(M)$ a vector space over κ. While we understand artinian modules are those which satisfy DCC, we will need a more useful characterization for proving the theorem at the end of this section. As it turns out, a module, M, is artinian if and only if its socle has finite dimension and is an essential submodule of M. We include here the direction which is relevant in this chapter. The second direction appears in chapter six.

Lemma 5.19. *If M is an artinian module, then $\operatorname{soc}(M)$ is a finite dimensional vector space over κ and $\operatorname{soc}(M) \subset' M$.*

Proof. First we will show if M is artinian, then its socle has finite dimension as a vector space over κ. It is clear that $\operatorname{soc}(M) \subset M$ which implies $\operatorname{soc}(M)$ is artinian. If $\operatorname{soc}(M)$ had infinite dimension, we could find a descending chain of submodules of $\operatorname{soc}(M)$ that did not stabilize by looking at submodules generated by the basis elements.

Now we show $\operatorname{soc}(M) \subset' M$. Let $0 \neq S \subset M$ and suppose $S \cap \operatorname{soc}(M) = \{0\}$. Because $S \subset M$, S is artinian which implies S contains a simple submodule, S', as any descending chain must stabilize. Because our ring is local, $S' \cong \kappa$. Let $s \in S'$ and notice $\mathfrak{m}s = 0$ which implies $s \in \operatorname{soc}(M)$. Since $s \in S \cap \operatorname{soc}(M)$, we have that $s = 0$ which shows $S' = 0$, contradicting that S' is simple and making $S = 0$. $\square$

Before we can show that noetherian and artinian modules are reflexive, we will show that when R is complete, direct sums of reflexives are reflexive and that evey free $R-$module with a finite base is reflexive.

Lemma 5.20. *Let M_1 and M_2 be $R-$modules. $M_1 \oplus M_2$ is reflexive if and only if M_1 and M_2 are reflexive.*

Proof. First, let $M_1 \oplus M_2$ be reflexive and consider the canonical homomorphism $M_1 \xrightarrow{\phi} M_1^{\nu\nu}$. Injectivity follows from 2.13. Let $\varphi \in M^{\nu\nu}$. Notice that this φ can be considered as a map on $(M_1 \oplus M_2)^{\nu\nu}$ by defining $\sigma(m_1 + m_2) = \sigma(m_1)$ where $m_1 \in M_1, m_2 \in M_2$ and $\sigma \in M_1^{\nu}$. By reflexivity of $M_1 \oplus M_2$, ϕ is surjective. The proof for M_2 is similar.

Now, let M_1, M_2 be reflexive and consider $M_1 \oplus M_2 \xrightarrow{\phi} (M_1 \oplus M_2)^{\nu\nu}$. Again, injectivity follows from 2.13. Let $\varphi \in (M_1 \oplus M_2)^{\nu\nu}$. For any $\sigma \in (M_1 \oplus M_2)^{\nu}$, the restrictions of σ to M_1 and M_2 respectively are surjective thereby making ϕ surjective. $\square$

Lemma 5.21. *If R is complete, then every free $R-$module with a finite base is reflexive.*

Proof. Note that a free module with a finite base is isomorphic to the direct sum of finitely many copies of R. By 5.17 and 5.20, R is reflexive. $\square$

Lemma 5.22. *If R is complete and M is a finitely generated $R-$module, then M is reflexive.*

Proof. Let M be finitely generated by $x_1, x_2, \ldots, x_n$ for some $n < \infty$. Consider the map $R^n \xrightarrow{\varphi} M$ which is defined as follows:

$$\begin{array}{ccc} (1,0,0,\ldots,0) & \mapsto & x_1 \\ (0,1,0,\ldots,0) & \mapsto & x_2 \\ & \ldots & \\ (0,\ldots,0,0,1) & \mapsto & x_n \end{array}$$

Notice φ is a homomorphism which makes $M \cong R^n/\ker(\varphi)$. By 5.21, R^n is reflexive. By 6.1 we notice quotient modules of reflexive modules are reflexive. Since M is isomorphic a reflexive module, M reflexive. $\square$

Lemma 5.23. *If M is artinian, then M is reflexive.*

Proof. For this proof, we will show that M is isomorphic to a submodule of a reflexive. By 5.19, $\mathrm{soc}(M)$ has finite dimension, as does κ^n making $\mathrm{soc}(M)$ and κ^n isomorphic. Consider the following commutative diagram:

$$\begin{tikzcd} \kappa^n \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{soc}(M) \arrow[r, hook] & M \arrow[dll, "\varphi"] \\ E^n & & \end{tikzcd}$$

where φ exists because E^n is injective. Now, $\varphi(M) \cong M/\ker\varphi$. By 5.19, $\mathrm{soc}(M) \subset' M$. Note that restricting φ to the socle of M gives an isomorphism and makes $\ker(\varphi) \cap \mathrm{soc}(M) = \{0\}$ thus implying $\ker(\varphi) = 0$. This means $M \cong \varphi(M) \subset E^n$.

We have shown R itself is reflexive, and by 2.17 R^ν is reflexive. In 5.1 we showed $R^\nu \cong E$, so E is reflexive. Finally, in 5.21 we showed E^n is reflexive. Now that M is isomorphic to a submodule of a reflexive, M is reflexive. $\square$

Now we have shown noetherian modules and artinian modules over a complete, local, noetherian ring are also reflexive. From our previous work, we know that finitely generated modules and artinian modules each form a Serre class. This shows that some subclasses of reflexive modules form a Serre class. We will continue to use the socle in the following section.

Recall that in chapter two we showed that noetherian and artinian left $R-$modules form a Serre class. Note that the following result shows that reflexive modules over a complete local noetherian ring form a Serre class.

Lemma 6.1. *When R is local, M is reflexive if and only if S and M/S are reflexive.*

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "i"] \arrow[d, "\phi_S"'] & M \arrow[r, "p"] \arrow[d, "\phi"'] & M/S \arrow[r] \arrow[d, "\phi_{M/S}"'] & 0 \\
0 \arrow[r] & S^{\nu\nu} \arrow[r, "i^{\nu\nu}"] & M^{\nu\nu} \arrow[r, "p^{\nu\nu}"] & (M/S)^{\nu\nu} \arrow[r] & 0
\end{tikzcd}$$

Proof. One direction is clear by 2.10. The fact that E is faithfully injective (2.15) implies that all vertical maps are injective (2.13). It remains to be shown that ϕ_S and $\phi_{M/S}$ are surjective. The latter is clear by the surjectivity of ϕ and $p^{\nu\nu}$. We see that ϕ_S is surjective by considering some nonzero $\psi \in S^{\nu\nu}$. By injectivity of $i^{\nu\nu}$, $i^{\nu\nu}(\psi) \neq 0$. The fact that ϕ is an isomorphism implies there exists a nonzero $m \in M$ so that $\phi(m) = i^{\nu\nu}(\psi)$. Because $\mathrm{Im}(i^{\nu\nu}) = \ker(p^{\nu\nu})$, we have $p^{\nu\nu}(i^{\nu\nu}(\psi)) = 0 = \phi_{M/S}(p(m))$. The injectivity of $\phi_{M/S}$ implies $p(m) = 0$ which places $m \in \ker(p)$. Since $\mathrm{Im}(i) = \ker(p)$, there exists some $s \in S$ so that $i^{\nu\nu}(\phi_S(s)) = \phi(i(s)) = i^{\nu\nu}(\psi)$. The injectivity of $i^{\nu\nu}$ allows us to conclude $\phi_S(s) = \psi$. □

6.1 A Useful Theorem

The very reason we can characterize all Serre classes of reflexive modules over a complete local noetherian ring is that reflexive modules can, in a sense, be built from noetherian and artinian modules. In the next section we show how this is achieved. In the process a bijection will be defined which will complete the characterization. First, however, we will need a useful theorem.

One key fact that is needed to build reflexive modules from noetherian modules and artinian modules was given as an exercise by Bourbaki in [3]. We will first need to prove several lemmas based on two exercises suggested by Bourbaki.

Lemma 6.2. *No infinite direct sum of nonzero modules is reflexive.*

Proof. One approach for proof might be to recall that in 2.1 we showed a vector space with an infinite base cannot be reflexive. We could relate this result to our new situation by considering cyclic ideals and quotient modules, then relate these to our κ. The alternate approach which follows uses a result from chapter 5.

For the sake of notation, assume the sum is countable and let $M = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$ where each M_i is nonzero. Note that we can think of M^ν as the product

$M^\nu = M_0^\nu \otimes M_1^\nu \otimes M_2^\nu \otimes \cdots$.

All but finitely many of the summands in the direct sum $\overline{M} = M_0^\nu \oplus M_1^\nu \oplus M_2^\nu \oplus \cdots$ are zero, so $\overline{M} \subsetneq M^\nu$. So, let $\varphi \in M^\nu$ where $\varphi \notin \overline{M}$. By 5.15, there exists a map $\psi \in M^{\nu\nu}$ so that $\psi(\varphi) \neq 0$ while $\psi(\overline{M}) = 0$.

Consider the canonical isomorphism, ϕ, between M and $M^{\nu\nu}$ and suppose there exists some $x \in M$ so that $\phi(x) = \psi$. This means that ψ is defined by evaluating homomorphisms from M^ν at x. Notice $x = (x_1, x_2, \ldots, x_n, 0, 0 \ldots)$ where only the first n elements may be nonzero and n is finite. Now, since $\varphi \in M^\nu$, $\varphi = (\sigma_1, \sigma_2, \ldots)$ where all maps may be nonzero. Consider $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, 0, 0, \ldots) \in \overline{M}$. By the definition of ψ, $\psi(\sigma(x)) = \sigma_1(x_1) + \sigma_2(x_2) + \cdots + \sigma_n(x_n) = 0$. However, $0 \neq \psi(\varphi(x)) = \sigma_1(x_1) + \sigma_2(x_2) + \cdots + \sigma_n(x_n) + 0 + 0 + \ldots$. This is a contradiction, showing M is not reflexive. $\square$

Definition 6.3. *A **subquotient** of M is a quotient module of a submodule of M. Such a subquotient has the form S/T where $T \subset S$ are submodules of M.*

Notice that by this definition, a subquotient of a subquotient of M is always isomorphic to a subquotient of M. Also notice that if M is reflexive, any submodule or quotient module of M is also reflexive, so subquotients of M are reflexive. This together with 6.2 gives the corollary below.

Corollary 6.4. If M is reflexive, no subquotient of M can be an infinite direct sum of nonzero modules.

Lemma 6.5 (Bourbaki Exercise 1)**.** *Let M be an $R-$module where R is complete, local and noetherian. M is artinian if and only if the dimension of* $\mathrm{soc}(M)$ *(as a vector space over κ) is finite and* $\mathrm{soc}(M) \subset' M$.

Proof. By 5.19 one direction is clear. Now, let $\mathrm{soc}(M) \subset' M$ and $\dim(\mathrm{soc}(M)) < \infty$. Since $\dim(\mathrm{soc}(M))$ is finite, we can think of $\mathrm{soc}(M)$ as a vector space over κ. So, $\mathrm{soc}(M) \cong \kappa \oplus \kappa \oplus \cdots \oplus \kappa \subset' M$. By the fact that $E \oplus E \oplus \cdots \oplus E$ is injective, we can complete the following commutative diagram with φ as shown.

$$\begin{tikzcd}
\mathrm{soc}(M) \arrow[r, "\cong"] \arrow[d, hook] & \kappa \oplus \kappa \oplus \cdots \oplus \kappa \arrow[r, hook] & M \arrow[dll, "\varphi"] \\
E \oplus E \oplus \cdots \oplus E & &
\end{tikzcd}$$

Because $\kappa \oplus \kappa \oplus \cdots \oplus \kappa$ is essential in M, we have $\ker(\varphi) = 0$. By 5.1 $E \cong R^\nu$. Because R is noetherian and has ACC, E has DCC by the results of chapter 5. This means $E \oplus E \oplus \cdots \oplus E$ satisfies DCC. Because $M \cong \varphi(M) \subset E \oplus E \oplus \cdots \oplus E$, M is isomorphic to a submodule of a module satisfying DCC which means M also satisfies DCC. $\square$

We will now prove a lemma for a more general situation than what we need. The notation in the corollary that follows will be more useful.

Lemma 6.6. *If M is not an artinian module, then there is a finitely generated submodule $U \subset M$ such that the map* $\operatorname{soc}(M) \to \operatorname{soc}(M/U)$ *induced by the map $M \to M/U$ is injective, but not surjective.*

Proof. By 6.5, if $\dim(\operatorname{soc}(M)) < \infty$ for a non-artinian M, then $\operatorname{soc}(M)$ is not essential in M. (We are choosing $\dim \operatorname{soc}(M) < \infty$ without loss of generality because when we use this lemma, this will be our situation.) This implies there exists some nonzero $T \subset M$ with $\operatorname{soc}(M) \cap T = \{0\}$. Without loss of generality, we can assume T is finitely generated. If not, choose a finitely generated submodule of T. Now, since T is finitely generated and nonzero, there exists a maximal submodule $U \subset T$ by Zorn's lemma, and T/U is simple.

Consider $f : M \to M/U$ and notice $\ker(f) = U$. This defines the induced map $g : \operatorname{soc}(M) \to M/U$ by restricting the domain of f to $\operatorname{soc}(M)$. Since $T \cap \operatorname{soc}(M) = 0$ and $\ker(g) = U \cap \operatorname{soc}(M)$, g is injective.

It is clear that g maps $\operatorname{soc}(M)$ into $\operatorname{soc}(M/U)$, but we show now that g is not surjective. First, we argue $T/U \subseteq \operatorname{soc}(M/U)$. Notice that $\operatorname{soc}(M/U) = \{x + U \in M/U | \mathfrak{m}x \in U\}$. Now, T/U is a simple submodule of M/U, so $T/U \cong \kappa = R/\mathfrak{m}$. Notice $\mathfrak{m}\,(T/U) \cong \mathfrak{m}\,(R/\mathfrak{m}) = 0$ which puts $T/U \subset \operatorname{soc}(M/U)$. Now, choose some nonzero $t+U \in T/U$. Suppose there exists a nonzero $x \in \operatorname{soc}(M)$ so that $g(x) = t+U$. This implies $x - t \in U \subset T$ so that $x \in V \cap \operatorname{soc}(M) = \{0\}$. This contradicts that x was nonzero thus proving $g(\operatorname{soc}(M)) \subsetneq \operatorname{soc}(M/U)$. ☐

The wording in following corollary to 6.6 will be more useful for what we need to prove.

Corollary 6.7. Let M be an $R-$module and suppose that for any finitely generated $S \subset M$ we have M/S is not artinian. Then if $T \subset M$ is finitely generated, there exists $T' \subset M$ that is finitely generated with $T \subset T'$ so that the map $\operatorname{soc}(M/T) \to \operatorname{soc}(M/T')$ (induced by the map $M/T \to M/T'$) is injective, but not surjective.

Theorem 6.8 (Bourbaki Exercise 2)**.** *If a module M is reflexive, then M has a finitely generated submodule $S \subset M$ so that M/S is artinian.*

Proof. Suppose every finitely generated $S \subset M$ is such that M/S is not artinian. Note that in the proof of 6.6 we assumed the dimension of the socle was finite. We have assumed here that M is reflexive, so its socle has finite dimension as a vector space over κ. This allows us to use (6.7) to find a finitely generated subquotient T/S where $T \subset S \subset M$ and $M/S \to \dfrac{M/S}{T/S} \cong M/T$ induces a map on the socles which is injective but not surjective. For notation, define $T^{(1)} = T/S$. Define $T^{(2)}$ by again taking the appropriate subquotients. Notice we have the following maps as well as their induced maps below:

$$M/S \to M/T^{(1)} \to M/T^{(2)} \to \cdots$$
$$\operatorname{soc}(M/S) \to \operatorname{soc}(M/T^{(1)}) \to \operatorname{soc}(M/T^{(2)}) \to \cdots$$

Each map along the bottom row is injective, but not surjective. This allows us to consider the map $M/T^{(n)} \to \dfrac{M}{\cup_{i=1}^{\infty} T^{(i)}}$ for any n. Note that the induced map on $\operatorname{soc}(M/T^{(n)}) \to \operatorname{soc}(\cup_{i=1}^{\infty} T^{(i)})$ is injective for each n.

Recall that for any module, its socle is a vector space over κ. As each successive map from $\operatorname{soc}(M/T^{(n)}) \to \operatorname{soc}(M/T^{(n+1)})$ is injective, but not surjective, each successive socle has dimension larger than the previous one. This implies $\dfrac{M}{\cup_{i=1}^{\infty} T^{(i)}}$ has a socle of infinite dimension.

Notice this means M has a subquotient that is an infinite direct sum of nonzero modules which contradicts that M is reflexive by 6.4. Hence we must have that for some finitely generated $S \subset M$ that M/S is artinian. $\square$

The following theorem is the useful one mentioned in the title of this section. Only part of this theorem was already known. The last piece will be particularly useful in the next section.

Theorem 6.9. *For a module M over a complete, local ring R the following are equivalent:*

(i) M is reflexive.

(ii) M has no subquotient that is an infinite direct sum of nonzero modules.

(iii) There is a finitely generated submodule $S \subset M$ so that M/S is artinian.

Proof. By 6.4, 6.8 our theorem is nearly complete. We need only show that if there is a finitely generated submodule $S \subset M$ so that M/S is artinian, then M is reflexive.

Suppose there exists such an $S \subset M$. Because noetherian modules and artinian modules in a complete, local ring are reflexive, we see that the canonical homomorphism between S and $S^{\nu\nu}$ and the canonical homomorphism between M/S and $(M/S)^{\nu\nu}$ are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, hook] \arrow[d, "\cong"'] & M \arrow[r] \arrow[d] & M/S \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & S^{\nu\nu} \arrow[r, hook] & M^{\nu\nu} \arrow[r] & (M/S)^{\nu\nu} \arrow[r] & 0
\end{tikzcd}$$

By 2.7, M is reflexive. $\square$

6.2 Building Reflexive Modules

At first we will begin with a Serre class of reflexive modules and show that we can develop a bijection between this Serre class and corresponding subclasses of noetherian and artinian modules. Next, we will begin with certain Serre classes of noetherian and artinian modules and show that we can build a Serre class of reflexive modules. It is the bijection we develop along the way that allows for the characterization of the Serre classes of reflexive modules.

For the following, let R be a complete, local ring with the usual κ. Also, let $\mathcal{U}$ be a Serre class of reflexive modules and let $\mathcal{S} \subset \mathcal{U}$, and $\mathcal{T} \subset \mathcal{U}$ consist of, respectively, all the noetherian (artinian) modules in $\mathcal{U}$. The fact that $\mathcal{S}$ and $\mathcal{T}$ are Serre classes in their own right follows easily from the fact that $\mathcal{S}, \mathcal{T} \subset \mathcal{U}$ and that $\mathcal{U}$ is assumed to be a Serre class.

Remark 6.10. If $\mathcal{U}$ is a Serre class of modules then $\mathcal{S}$ and $\mathcal{T}$ as defined above are Serre classes of modules.

With our definitions above, we can describe an important uniqueness result regarding the relationship between these Serre classes. To achieve this result, we must utilize the useful theorem from the previous section.

Lemma 6.11. *Let $\mathcal{U}$ and $\mathcal{U}'$ be Serre classes of reflexive modules. Let $\mathcal{S}, \mathcal{T}$ be defined as above. Let $\mathcal{S}', \mathcal{T}'$ be defined similarly as corresponding classes for $\mathcal{U}'$.*

Then $\mathcal{U} = \mathcal{U}'$ if and only if $\mathcal{S} = \mathcal{S}'$ and $\mathcal{T} = \mathcal{T}'$.

Proof. Letting $\mathcal{U} = \mathcal{U}'$, it is clear by definition that $\mathcal{S} = \mathcal{S}'$ and $\mathcal{T} = \mathcal{T}'$ because $\mathcal{U}$ and $\mathcal{U}'$ will have the exact same noetherian and artinian submodules.

Let $\mathcal{S} = \mathcal{S}'$ and $\mathcal{T} = \mathcal{T}'$. Also, suppose $\mathcal{U} \neq \mathcal{U}'$. Then without loss of generality, there exists some $M \in \mathcal{U}$ where $M \notin \mathcal{U}'$ so that M is reflexive, but not noetherian and not artinian. By 6.9 there exists some finitely generated $S \subset M$ where M/S is artinian.

Consider the short exact sequence $0 \to S \to M \to M/S \to 0$ for this particular S. Since S is noetherian and a submodule of M, $S \in \mathcal{S}$. Similarly $M/S \in \mathcal{T}$. By our equality, $S \in \mathcal{S}'$ and $T \in \mathcal{T}'$. Furthermore, $S, T \in \mathcal{U}'$. This means $M \in \mathcal{U}'$ by $\mathcal{U}'$ a Serre class. $\square$

This gives us a glimpse at a possible bijection between Serre classes of reflexives and these pairs of noetherians and artinians. It turns out that to get a true bijection, more is needed about the classes $\mathcal{S}$ and $\mathcal{T}$. For now, we will settle for a map whose injectivity follows directly from 6.11.

Corollary 6.12. The map defined by $\mathcal{U} \mapsto (\mathcal{S}, \mathcal{T})$ which maps a Serre class $\mathcal{U}$ to the pair $(\mathcal{S}, \mathcal{T})$ where $\mathcal{S}$ and $\mathcal{T}$ are the corresponding Serre classes of noetherian and artinian modules is injective.

To make the map above a bijection, we will first need to assume that neither $\mathcal{S}$ nor $\mathcal{T}$ are zero. If we assume that one is nonzero, we see below that both are nonzero.

Lemma 6.13. *With the notation above, $\mathcal{S} \neq 0$ if and only if $\mathcal{T} \neq 0$.*

Proof. First, let $\mathcal{S} \neq 0$. Then $\mathcal{S}$ contains a nonzero finitely generated module M with generator x. Then Rx is a cyclic submodule of M, putting $Rx \subset \mathcal{S}$. Note that $Rx \cong R/\operatorname{ann}(x) \in \mathcal{S}$. The short exact sequence $0 \to \mathfrak{m}/\operatorname{ann}(x) \to R/\operatorname{ann}(x) \to R/\mathfrak{m} \to 0$ shows $R/\mathfrak{m} \in \mathcal{S}$. Since $R/\mathfrak{m} \in \mathcal{S}$ and κ is simple, we know κ is artinian as well. Thus $R/\mathfrak{m} \in \mathcal{T}$ proving $\mathcal{T} \neq 0$.

Now, let $0 \neq N \in \mathcal{T}$. N contains a simple submodule by 1.6. This submodule must be isomorphic to κ because our ring is complete and local, which shows $R/\mathfrak{m} \in \mathcal{T}$. We must also have $R/\mathfrak{m} \in \mathcal{S}$ as it is a simple module, meaning $\mathcal{S} \neq 0$. $\square$

We get a corollary to 6.13 by again applying our useful theorem 6.9 from the previous section.

Corollary 6.14. If $\mathcal{S} = 0$, we must have that $\mathcal{U} = 0$.

Proof. Let $0 \neq M \in \mathcal{U}$ and use 6.9 on the resulting short exact sequence $0 \to S \to M \to M/S \to 0$ to get a contradiction. $\square$

Now we will build our reflexive modules by selecting the appropriate $\mathcal{S}$ and $\mathcal{T}$. In what follows, let $\mathcal{S}$ be a nonzero Serre class of finitely generated modules and let $\mathcal{T}$ be a nonzero Serre class of artinian modules. We show below that there exists a Serre class, $\mathcal{U}$ of reflexive modules so that $\mathcal{U} \mapsto (\mathcal{S}, \mathcal{T})$, making the map in 6.12 surjective. One important property that allows for this construction is that taking the intersection of any nonzero $\mathcal{S}$ and $\mathcal{T}$ results in precisely all $R-$modules which have finite length.

Lemma 6.15. *Let $\mathcal{S}$ and $\mathcal{T}$ be as described above and let M be any $R-$module. $M \in \mathcal{S} \cap \mathcal{T} \Leftrightarrow M$ is a module of finite length.*

Proof. Note this is just a way of rephrasing 2.26 and because κ is the unique simple module of R up to isomorphism, we also have $\kappa \in \mathcal{T}$. It is this property that concludes that in a local ring we have that $\mathcal{S} \cap \mathcal{T}$ consists of all modules of finite length. $\square$

So, we know that choosing a module from $\mathcal{S} \cap \mathcal{T}$ means we are choosing a module of finite length. Now, as our useful theorem 6.9 suggests, define $\mathcal{U}$ to consist of all modules M so that there exists a finitely generated submodule $S \subset M$ with M/S artinian and $S \in \mathcal{S}, M/S \in \mathcal{T}$. We will show that $\mathcal{U}$ is a Serre class of reflexive modules and that $\mathcal{S}$ and $\mathcal{T}$ consist, respectively, of all noetherian and artinian modules in $\mathcal{U}$. In doing this we will have that our map from 6.12 is bijective.

Lemma 6.16. *$\mathcal{U}$, as defined above, consists of reflexive modules.*

Proof. This is clear by 6.9 and the definition of U. □

Lemma 6.17 (Submodule Closure)**.** *If $M \in \mathcal{U}$ and $T \subset M$, then $T \in \mathcal{U}$.*

Proof. Since $M \in \mathcal{U}$, there exists is an $S \in \mathcal{S}$ so that $S \subset M$ and $M/S \in \mathcal{T}$. Consider the short exact sequence $0 \to S \to M \to M/S \to 0$ and the submodule $T \cap S \subset S \in \mathcal{S}$. We have $T \cap S$ is finitely generated and a submodule of T. We want to show $T/(T \cap S)$ is artinian and in $\mathcal{T}$.

Consider $T \to (T+S)/S$ defined by $t \mapsto t+S$. This map is surjective, so the elements in its kernel are exactly the elements of T that are also in S, so $T/(T \cap S) \cong (T+S)/S$. Furthermore, $(T+S)/S \subset M/S \in \mathcal{T}$ and since Serre classes preserve submodules and isomorphisms, $(T+S)/S \cong T/(T \cap S) \in \mathcal{T}$ and $T/(T \cap S)$ is artinian. □

Lemma 6.18 (Quotient Closure)**.** *If $M \in \mathcal{U}$ and $T \subset M$, then $M/T \in \mathcal{U}$.*

Proof. Let $T \subset M \in \mathcal{U}$ and $S \subset M$ where $S \in \mathcal{S}$ and $M/S \in \mathcal{T}$. Consider $(S+T)/T$ which is clearly a submodule of M/T.

Looking at short exact sequence $0 \to (S+T)/T \to M/T \to \dfrac{M/T}{(S+T)/T} \to 0$, we see we need to show $(S+T)/T \in \mathcal{S}$ and $\dfrac{M/T}{(S+T)/T} \cong M/S+T \in \mathcal{T}$.

By the second isomorphism theorem for modules, $(S+T)/T \cong S/(T \cap S)$. Since Serre classes preserve intersections, quotients, and isomorphisms, $(S+T)/T \in \mathcal{S}$.

By applying the first isomorphism theorem for modules twice,

$$\frac{M/T}{(S+T)/T} \cong M/(S+T) \cong \frac{M/S}{(S+T)/S}.$$

Since $M/S \in \mathcal{T}$ and $(S+T)/S \subset M/S$, we have $\dfrac{M/S}{(S+T)/S} \in \mathcal{S}$. This gives $\dfrac{M/T}{(S+T)/T} \cong M/S+T \in \mathcal{T}$ as desired. □

To show $\mathcal{U}$ is a Serre class, we still have to show a bit more than what has been established above. The following property gives motivation for one possible way to continue. While we will not use this property directly to finish our proof that the $\mathcal{U}$ built from a given $\mathcal{S}$ and $\mathcal{T}$ is a Serre class, it gives some insight on how to proceed. In addition, the property gives an alternate way to build $\mathcal{U}$.

Theorem 6.19. *If there exists $T \in \mathcal{T}$ so that $T \subset M$ and $M/T \in \mathcal{S}$, then there exists $S \in \mathcal{S}$ so that $S \subset M$ and $M/S \in \mathcal{T}$.*

Proof. Let $M/T \in \mathcal{S}$ and define S to consist of the elements in M which map onto the generators of M/T. This S is finitely generated and a submodule of M. In addition, S maps onto M/T by sending s to $s+T$. So, $\text{Im}(S) = (S+T)/T$ which implies $M = S+T$. Also notice that the kernel of the map $S \to M/T$ is $T \cap S \subset S$, which implies $T \cap S$ is finitely generated. In addition, $T \cap S \subset T \in \mathcal{T}$ which implies $S \cap T$ is a module of finite length. By 6.15, $T \cap S \in \mathcal{S}$.

Now, since the kernel of $S \to M/T$ is $S \cap T$, we have $S/(S \cap T) \cong M/T \in \mathcal{S}$ which implies $S/(S \cap T) \in \mathcal{S}$. Considering the short exact sequence $0 \to T \cap S \to S \to S/(T \cap S) \to 0$, we see that because both $(T \cap S)$ and $S/(T \cap S)$ are elements of $\mathcal{S}$ we get $S \in \mathcal{S}$ as desired.

Because $S+T = M$, we prove $M/S \in \mathcal{T}$ by showing $(S+T)/S \in \mathcal{T}$. The second isomorphism theorem gives $(S+T)/S \cong T/(S \cap T) \in \mathcal{T}$ since $T \in \mathcal{T}$. Thus $M/S \in \mathcal{T}$ as desired. $\square$

Theorem 6.20. *$\mathcal{U}$ is a Serre class.*

Proof. By 6.17 and 6.18 we know that if $M \in \mathcal{U}$, then $S, M/S \in \mathcal{U}$ for any submodule $S \subset M$. By 2.23 this shows half of what we need.

Suppose now that $M_1, M_2 \in \mathcal{U}$ and that $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence. We will show $M \in \mathcal{U}$. Notice since $M_1 \in \mathcal{U}$, there exists a finitely generated $T_1 \subset M_1$ so that $T_1 \in \mathcal{S}$ and $M_1/T_1 \in \mathcal{T}$. Also, there exists a finitely generated $T_2 \subset M_2$ so that $T_2 \in \mathcal{S}$ and $M_2/T_2 \in \mathcal{T}$.

Consider $T \subset M$ containing the elements of M which map to the generators of T_2. Then the map $T \to T_2$ is surjective. Notice that T may contain more elements than $\text{Im}(T_1)$, but as T maps onto T_2, T is still finitely generated. Since T_1 is a submodule of the kernel of the map $M_1 \to M_2$, we can form T_1' by adding any elements of M_1 which are missing from the kernel of $M_1 \to M_2$. (So, explicitly, T_1' is the set of all $x \in M_1$ so that x maps to an element in $\text{Im}(M_1 \cap T.)$ Now we have that $0 \to T_1' \to T \to T_2 \to 0$ is short exact.

Since T is finitely generated, $\text{Im}(M_1) \cap T$ is finitely generated. This implies T_1' is finitely generated. As suggested by 6.19, consider $(T_1' + T_1)/T_1 \subset M_1/T_1 \in \mathcal{T}$. By the second isomorphism theorem for modules, $T_1'/(T_1 \cap T_1') = T_1'/T_1 \in \mathcal{T}$. In addition, T_1' finitely genearated because it is isomorphic to a submodule of T which is finitely generated. This in turn makes T_1'/T_1 finitely generated. This means $T_1'/T_1 \in \mathcal{S} \cap \mathcal{T}$ by 6.15, which implies $T_1' \in \mathcal{S}$ by considering the short exact sequence $0 \to T_1 \to T_1' \to T_1'/T_1 \to 0$ where $T_1, T_1'/T_1 \in \mathcal{S}$. Finally, $0 \to T_1' \to T \to T_2 \to 0$ implies $T \in \mathcal{S}$.

We have $M_1/T_1 \in \mathcal{T}$ and $T_1'/T \in \mathcal{S} \cap \mathcal{T}$, so $T_1'/T \in \mathcal{T}$. Notice that $M_1/T_1' \cong \dfrac{M_1/T_1}{T_1'/T_1} \in \mathcal{T}$. We use the Nine Lemma to conclude, based on the following commutative diagram that $M/T \in \mathcal{T}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_1' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1/T_1' \arrow[r] \arrow[d] & M/T \arrow[r] \arrow[d] & M_2/T_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Now we know that we can build a Serre class of reflexives, $\mathcal{U}$ given a pair of non-empty Serre classes $(\mathcal{S}, \mathcal{T})$ as described. It remains to show that our map from 6.12 is surjective. For this, we need to show that $\mathcal{U}$ does not contain any finitely generated modules that are not in $\mathcal{S}$ and that $\mathcal{U}$ does not contain any artinian modules which are not in $\mathcal{T}$. We will also need to show that the map $\mathcal{U} \mapsto (\mathcal{S}, \mathcal{T})$ is well-defined. Our characterization of Serre classes of reflexive modules will follow as a consequence of this bijection.

Lemma 6.21. *If M is a finitely generated $R-$module and $M \in \mathcal{U}$, then $M \in \mathcal{S}$.*

Proof. Since $M \in \mathcal{U}$, our definition of $\mathcal{U}$ gives a finitely generated $S \subset M$ with $S \in \mathcal{S}$ and M/S artinian with $M/S \in \mathcal{T}$. Now, because M is finitely generated, M/S is finitely generated, being the quotient of a finitely generated module. Since M/S is artinian and noetherian, by 6.15 $M/S \in \mathcal{S} \cap \mathcal{T}$. The short exact sequence $0 \to S \to M \to M/S \to 0$ implies $M \in \mathcal{S}$ as both S and M/S are in $\mathcal{S}$. □

The proof of the following remark is similar to that above by again using 6.15 .

Remark 6.22. If M is an artinian $R-$module and $M \in \mathcal{U}$, then $M \in \mathcal{T}$.

Finally we can argue that our map $\mathcal{U} \mapsto (\mathcal{S}, \mathcal{T})$ is well-defined. Let $\mathcal{U} \mapsto (\mathcal{S}', \mathcal{T}')$. If we suppose $\mathcal{S}' \neq \mathcal{S}$, then there exists some $M \in \mathcal{U}$ so that $M \in \mathcal{S}$ and $M \notin \mathcal{S}'$. This contradicts the very definition of $\mathcal{S}$ and $\mathcal{S}'$, proving that $\mathcal{S} = \mathcal{S}'$. A similar argument shows that $\mathcal{T} = \mathcal{T}'$, giving that the map $\mathcal{U} \mapsto (\mathcal{S}, \mathcal{T})$ is a bijection.

6.3 Serre Classes of Reflexive Modules

We can now, given the bijection developed in the previous section, describe the characterization of all Serre classes of reflexive modules over a complete local noetherian ring. They are in one-to-one correspondence with the pairs $(\mathcal{F}, \mathcal{G})$ described in chapter five.

Theorem 6.23. *The Serre classes of reflexive modules over a complete local noetherian ring are in one-to-one correspondence with the pairs $(\mathcal{F}, \mathcal{G})$ where either both $\mathcal{F}$ and $\mathcal{G}$ are empty, or they are both nonempty and are as described in chapter five.*

Remark 6.24. The case where $\mathcal{F}$ and $\mathcal{G}$ are both empty corresponds to the Serre class consisting of the module 0. More precisely, this class consists of all modules with only one element.

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