2014

TIME DEPENDENT HOLOGRAPHY

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TIME DEPENDENT HOLOGRAPHY

DISSERTATION

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy in the
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By

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Lexington, Kentucky

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ABSTRACT OF DISSERTATION

TIME DEPENDENT HOLOGRAPHY

One of the most important results emerging from string theory is the gauge gravity duality (AdS/CFT correspondence) which tells us that certain problems in particular gravitational backgrounds can be exactly mapped to a particular dual gauge theory a quantum theory very similar to the one explaining the interactions between fundamental subatomic particles. The chief merit of the duality is that a difficult problem in one theory can be mapped to a simpler and solvable problem in the other theory. The duality can be used both ways.

Most of the current theoretical framework is suited to study equilibrium systems, or systems where time dependence is at most adiabatic. However in the real world, systems are almost always out of equilibrium. Generically these scenarios are described by quenches, where a parameter of the theory is made time dependent. In this dissertation I describe some of the work done in the context of studying quantum quench using the AdS/CFT correspondence. We recover certain universal scaling type of behavior as the quenching is done through a quantum critical point. Another question that has been explored in the dissertation is time dependence of the gravity theory. Present cosmological observations indicate that our universe is accelerating and is described by a spacetime called de-Sitter(dS). In 2011 there had been a speculation over a possible duality between de-Sitter gravity and a particular field theory (Euclidean SP(N) CFT). However a concrete realization of this proposition was still lacking. Here we explicitly derive the dS/CFT duality using well known methods in field theory. We discovered that the time dimension emerges naturally in the derivation. We also describe further applications and extensions of dS/CFT.

KEYWORDS: Holography, AdS/CFT correspondence, Quantum Quench, dS/CFT correspondence, Chaos

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Dedicated to **Maa** *and* **Baba**

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Chapter 1

Introduction

1.1 The problem of time dependence

Almost all systems in our real life are governed by dynamics which is time dependent. Two phenomena where this is strikingly obvious is the case of *quench* and *cosmology*. The problem of quantum quench is the response of a quantum system to a time dependent coupling[1]. Due to a plethora of experimental results[2] especially in cold atom physics this has attracted a lot of attention. When such quenches are carried out across critical points universal scaling laws for observables emerge. Suppose the coupling approaches the critical coupling linearly, i.e,

$$g - g_c \sim vt$$

Then *Kibble-Zurek*[3, 4] type of arguments show that the one point function of an operator with conformal dimension x at the critical point has the following scaling behavior:

$$\langle \mathcal{O}(t) \rangle \sim (v)^{\frac{x\nu}{z\nu+1}} F(tv^{\frac{z\nu}{z\nu+1}})$$

The arguments which lead to this scaling makes the assumption that as the coupling approaches critical value the quantum state of the system stays frozen. This is a very drastic assumption and hence the problem is conceptually unclear. In this dissertation we will use *Gauge-Gravity* duality to address this question.
The gauge-gravity duality or the AdS/CFT correspondence[5] is one of the recent developments of string theory. The statement of the correspondence is that certain d-dimensional quantum field theories are *exactly equivalent* to a $d+1$-dimensional theory of quantum gravity. This *duality* is extremely useful: when the field theory is strongly coupled the dual gravity theory is weakly coupled and classical, and indeed one can now use it to calculate field theory observables and critical properties which normally would have been utterly inaccessible. There are also regimes where the gravity problem is no longer classical and does not admit a direct analysis, but is mapped to a simpler problem in a weakly interacting gauge theory: the duality can thus be used both ways. As we shall see in more detail in the next two subsections, the duality relates the couplings of the field theory to the boundary conditions of the gravitational theory. Thus for quench in a strongly coupled field theory, the problem of time-dependent coupling translates to a problem of time-dependent boundary conditions in the gravity side. By analyzing the gravity equations we will be able to find hints of a mechanism that explains the emergence of the Kibble-Zurek type of scalings.

The theory of gravity in the present formulation of holography is gravity in asymptotically *Anti-de Sitter* spacetime. Pure AdS in Poincare patch and in $d+1$ dimensions is described by the following metric[6] :

$$ds^2 = \frac{L_{AdS}^2}{z^2}(-dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2)$$

where L_{AdS} is the associated lengthscale of AdS. It is however well known from cosmological observations that we exist in an expanding universe. The geometry of this spacetime is asymptotically *de Sitter*. Pure dS in Poincare patch and in $d+1$ dimensions is described by the following metric :

$$ds^2 = \frac{L_{dS}^2}{t^2}(-dt^2 + \sum_{i=1}^{d} dx_i^2)$$

The coordinate t is identified with 'time'. This time-dependent geometry which describes our universe is expanding at an accelerated rate. If this expansion persists, we will eventually head towards a cold and lonely world. The large structures in our universe will slowly dilute away, and after even longer time scales, all cosmic radiation will have stretched to sizes beyond the horizon[7]. Our whole observable world will be governed by thermal and quantum fluctuations at a Hawking temperature of $\sim 10^{-29}$K. What will be the relevant physics in the far future? From the perspective of an observer the basic theoretical problem that arises is the lack of a set of sharp observables, due to lack of any asymptotic accessible boundary[8]. A gauge-gravity duality for de Sitter will be a solution to this problem since it will identify a gauge theory which comes with a set of precise observables. Looking at the above two metrics it is clear that they share many symmetries and are also connected by analytic continuation. Hence it is natural to explore if AdS/CFT can be extended to dS/CFT, where the aim now is to find a precise correspondence which will help us to understand problems of quantum gravity in de Sitter space by mapping them to a field theory. In this dissertation we look at the dS/CFT proposition[9–12] in detail. We shall also see in subsection 1.3 that the holographic direction is to be identified with the energy scale of the field theory. In the field theory side, the renormalization group equations describe the evolutions of the couplings as we change energy. On the other hand if there exists a dS/CFT correspondence then the holographic coordinate is *time*. As the dual gravity theory is unitary, there is a well-defined time evolution of the bulk wavefunction. Thus one expects a connection between the β-functions of the field theory, and the time evolution in de Sitter via the duality which we explore in the dissertation.

An interesting regime of time-dependent dynamics is chaos. It is known that classical string dynamics in pure $AdS_5 \times S^5$ is integrable [13] and hence is non-chaotic. On the other hand there has been a huge concentration of efforts to construct particle physics models from string theory. One of the goals is to reproduce quantum chromodynamics or the theory of hadrons which exhibits confinement[14]. It turns out that this particular feature arises not in pure AdS but in a certain class of geometries which are asymptotically AdS and caps off in the interior. It is not known if classical string motion will continue to be integrable in such confining geometries, and this is another subject that we explore in this dissertation.

1.2 Holography in a nutshell

Holographic duality relates certain gauge theories to theories of gravity. When the rank of the gauge group N is taken large, the corresponding gravity theory is classical. Let us completely forget about gravity for a moment and think only about gauge theory with gauge group $SU(N)$, where we consider N to be large. Consider gauge invariant operators made from the gluon fields $\mathcal{O}_i$, where we normalize them so that they have a well-defined large-N limit. By a simple power counting[15] it is easy to show that the connected correlator of m of these satisfies,

$$\langle \mathcal{O}_1 ... \mathcal{O}_m \rangle_C \sim N^{2-2m}$$

In particular for the variance,

$$\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle = \langle \mathcal{O}\mathcal{O} \rangle_c \sim N^{-2}$$

Thus in the large N limit all the operators are peaked about their mean values, which is the very definition of what it means to be "classical". All observables are thus peaked about some "gauge field configuration" that dominates the functional integral as $N \to \infty$. Historically this solution is called the "master field"[16] and it should satisfy classical equations. But they should contain the information of infinitely many degrees of freedom per spacetime point, as $N = \infty$.

Keeping these thoughts in mind let us now turn to a question of gravity, how are the degrees of freedom encoded in spacetime? At the easiest level: if the gravitational coupling G_N is taken to be small, it is sensible to think of a theory of gravity and matter as an ordinary quantum field theory on a fixed background. If we consider a region of volume V and energy E, it is well known that the entropy scales like,

$$S \sim VF(\frac{E}{V})$$

Let us now consider turning G_N on. General relativity tells us that something very interesting happens when we make V small while holding E constant. Once the linear dimension characterizing V is smaller than the Schwarzchild radius $r_s(E)$,

$$r_s(E) = \frac{2G_N E}{c^4}$$

our system collapses into a black hole. Now one of the great results of semi-classical general relativity tells us that the counting of entropy has to be done differently, the answer is the Bekenstein-Hawking[17] entropy,

$$S = \frac{Ac^3}{4G_N \hbar}$$

where, A is the area (and *not* the volume) of the event horizon of the resulting black hole. This indicates that a theory of gravity behaves like it has one less dimension than expected.

Thus on one side, in d-dimensional large N gauge theory, we are looking for "classical" field configurations that somehow contain infinitely more degrees of freedom, and on the other hand in classical gravity in $d+1$-dimensions we see that somehow the counting of degrees of freedom enforces us to think gravity as a conventional (non-gravitational) theory in d dimensions. *Thus, the $N \to \infty$ limit of gauge theories is related to classical gravity in one higher dimension.* What happens if N is finite? One now expects that the fluctuations about the master field will also contribute. On the gravitational side, these fluctuations can be mapped to traditional quantum fluctuations about a bulk spacetime. Thus,

Certain finite N gauge theories are exactly equivalent to quantum gravity in one higher dimension.

There are many explicit examples of the duality arising from string theory. The most well-studied example is between maximally supersymmetric Yang-Mills theory with gauge group $SU(N)$ in four dimensions and Type IIB string theory on the product of a five dimensional Anti-de Sitter space with a 5 sphere, S^5. This example leads to a precise mapping of the parameters of the two theories:

$$\frac{\lambda}{N^2} = g_s \qquad (\frac{L_{AdS}}{l_s})^4 = 4\pi\lambda$$

where λ is the Yang-Mills t'Hooft coupling ($g^2_{YM}N$), L_{AdS} is the curvature radius of the bulk spacetime, l_s is the string length and g_s the string coupling. Notice when N is large the string coupling (which controls the bulk gravity effects) is small. Notice also that when λ is large the curvature is small. Thus, classical gravity is a good description for strongly coupled gauge theory.

1.3 Calculations in Holography

In this subsection we briefly review some of the basic aspects of gauge/gravity duality that will be required later. It is useful to keep in mind that the duality is a strong/weak correspondence : when the field theory side is strongly correlated, the gravitational description is weakly coupled. In the most well-studied examples of the correspondence gravity and matter fields propagating on a weakly curved Anti-de Sitter spacetime in $d+1$ dimensions is mapped to a strongly coupled conformally invariant quantum field theory that lives in d dimensions. In this limit the relevant gravity action in $d+1$-dimensions is the Einstein-Hilbert action:

$$S_{bulk}[g] = \frac{1}{16\pi G_N} \int d^{d+1}x\sqrt{g}\left(\mathcal{R} + \frac{d(d-1)}{L^2_{AdS}}\right)$$

Where $\mathcal{R}$ is the Ricci scalar built out of the bulk metric whose determinant is g. The AdS_{d+1} metric which has already been introduced is one of the solutions of the above action. This simplest solution represents the *vacuum* of the CFT. The vacuum is invariant under the conformal group in d-dimensions, which is precisely the group of isometry of AdS_{d+1} as well. One special isometry is *scaling* :

$$x^\mu \to \lambda x^\mu \qquad z \to \lambda z$$

We see that as we scale the energy we must scale the *holographic* coordinate z as well. It turns out that z represents the energy scale at which we consider the field theory with the UV at $z \to 0$ (boundary) and IR at $z \to \infty$. Thus the bulk *geometrizes the RG flow* of the field theory.
To answer most field theory questions it is sufficient to disturb the CFT vacuum with scalar operators $\mathcal{O}$:

$$\delta S_{CFT} = \int d^d x\, J(x)\mathcal{O}(x).$$

How do we study these excitations using gravity? There are two possible quantization schemes for the field theory. In the standard quantization, AdS/CFT gives us the following dictionary:

Field Theory	Gravity
Operator $\mathcal{O}$	Scalar field ϕ
Source J	$\phi_0 = \phi(z \to 0)$

Thus to study the perturbed strongly coupled CFT it is sufficient to consider a minimally coupled massive scalar field in the AdS_{d+1} background:

$$S_\phi = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left((\nabla\phi)^2 + m^2\phi^2\right)$$

From the equation of motion arising from the above action one can show that near the boundary the scalar has the expansion:

$$\phi(z \to 0, x^\mu) \sim A(x)z^{\Delta_-} + B(x)z^{\Delta_+}$$

where,

$$\Delta_\pm = \frac{d}{2} \pm \nu \quad \nu = \sqrt{\frac{d^2}{4} + m^2 L_{AdS}^2}$$

Δ_+ is the conformal dimension of the dual operator $\mathcal{O}$. The idea of AdS/CFT (referred to as the GKPW prescription[18, 19]) is that the generating functionals on both sides are equal. Schematically,

$$Z[J] = \left\langle e^{-\int d^d x\, \mathcal{O}(x)J(x)} \right\rangle = Z_{string}[\text{b.c depends on } J] \sim \exp\left(- S_{grav}\right)|_{A(x)=J(x)} \tag{1.3.1}$$

where the last approximation holds when gravity is classical, and the gravity path integral has been done by saddle point, i.e, S_{grav} is the on-shell action subject to the boundary condition $A(x) = J(x)$. Note however that since the equation of motion is second order we need another boundary condition to fully specify the solution. Generally one finds that by demanding that the solution be *regular* everywhere in the interior, this will fix the coefficient $B(x)$ in terms of $A(x)$. To use (1.3.1) to perform field theory computations are we note a key result : by taking functional derivatives of a regulated version of (1.3.1) one can show that the expectation value of $\mathcal{O}$ is

$$\langle \mathcal{O}(x) \rangle = 2\nu B(x)$$

For instance if one finds a regular solution with $A(x) = 0$ and $B(x) \neq 0$ this implies that even in the absence of the source the operator has spontaneously developed an expectation value. Note by studying the scaling properties of $B(x)$ one can verify that the conformal dimension of $\mathcal{O}$ is Δ_+. Similarly one can show that the two-point function $\langle \mathcal{O}\mathcal{O} \rangle$ is related to the ratio, $\frac{B}{A}$.
In our *quench* investigations we will repeatedly use the above results, where now a time-dependent coupling translates to time-dependent boundary conditions in the gravity side. For dS/CFT we will be interested to arrive at an analogous prescription like (1.3.1).

1.4 Contents of the dissertation

In chapter 2 we study quantum quench in a holographic model of a zero temperature insulator-superfluid transition. The model is a modification of that of [20] and involves a self-coupled complex scalar field, Einstein gravity with a negative cosmological constant, and Maxwell field with one of the spatial directions compact. In a suitable regime of parameters, the scalar field can be treated as a probe field whose backreaction to both the metric and the gauge field can be ignored. We show that when the chemical potential of the dual field theory lies between two critical values, the equilibrium background geometry is a AdS soliton with a constant gauge field, while the complex scalar condenses leading to broken symmetry. We then turn on a time dependent source for the order parameter which interpolates between constant values and crosses the order-disorder critical point. In the critical region adiabaticity breaks down, but for a small rate of change of the source v there is a new small-v expansion in fractional powers of v. The resulting critical dynamics is dominated by a zero mode of the bulk field. To lowest order in this small-v expansion, the order parameter satisfies a time dependent Landau-Ginsburg equation which has $z = 2$, but non-dissipative. These predictions are verified by explicit numerical solutions of the bulk equations of motion.

We consider quantum quench by a time dependent double trace coupling in a strongly coupled large N field theory which has a gravity dual via the AdS/CFT correspondence in chapter 3. The bulk theory contains a self coupled neutral scalar field coupled to gravity with negative cosmological constant. We study the scalar dynamics in the probe approximation in two backgrounds: AdS soliton and AdS black brane. In either case we find that in equilibrium there is a critical phase transition at a *negative* value of the double trace coupling κ below which the scalar condenses. For a slowly varying homogeneous time dependent coupling crossing the critical point, we show that the dynamics in the critical region is dominated by a single mode of the bulk field. This mode satisfies a Landau-Ginsburg equation with a time dependent mass, and leads to Kibble Zurek type scaling behavior. For the AdS soliton the system is non-dissipative and has $z = 1$, while for the black brane one has dissipative $z = 2$ dynamics. We also discuss the features of a holographic model which would describe the non-equilibrium dynamics around quantum critical points with arbitrary dynamical critical exponent z and correlation length exponent ν. These analytical results are supported by direct numerical solutions.

In chapter 4 we derive a collective field theory of the singlet sector of the $Sp(2N)$ sigma model. Interestingly the hamiltonian for the bilocal collective field is the same as that of the $O(N)$ model. However, the large-N saddle points of the two models differ by a sign. This leads to a fluctuation hamiltonian with a negative quadratic term and alternating signs in the nonlinear terms which correctly reproduces the correlation functions of the singlet sector. Assuming the validity of the connection between $O(N)$ collective fields and higher spin fields in AdS, we argue that a natural interpretation of this theory is by a double analytic continuation, leading to the dS/CFT correspondence proposed by Anninos, Hartman and Strominger. The bilocal construction gives a map into the bulk of de Sitter space-time. Its geometric pseudospin-representation provides a framework for quantization and definition of the Hilbert space. We argue that this is consistent with finite N grassmanian constraints, establishing the bi-local representation as a nonperturbative framework for quantization of Higher Spin Gravity in de Sitter space.

If there is a dS/CFT correspondence, time evolution in the bulk should translate to RG flows in the dual euclidean field theory. Consequently, although the dual field is expected to be non-unitary, its RG flows will carry an imprint of the unitary time evolution in the bulk. In chapter 5 we examine the prediction of holographic RG in de Sitter space for the flow of double and triple trace couplings in any proposed dual. We show quite generally that the correct form of the field theory beta functions for the double trace couplings is obtained from holography, provided one identifies the scale of the field theory with $(i|T|)$ where T is the 'time' in conformal coordinates. For dS_4, we find that with an appropriate choice of operator normalization, it is possible to have real n-point correlation functions as well as beta functions with real coefficients. This choice leads to an RG flow with an IR fixed point at negative coupling unlike in a unitary theory where the IR fixed point is at positive coupling. The proposed correspondence of $Sp(2N)$ vector models with de Sitter Vasiliev gravity provides a specific example of such a phenomenon. For dS_{d+1} with even d, however, we find that no choice of operator normalization exists which ensures reality of coefficients of the beta-functions as well as absence of n-dependent phases for various n-point functions, as long as one assumes real coupling constants in the bulk Lagrangian.

In chapter 6 we describe a class of spacetimes that are asymptotically de Sitter in the Poincare slicing. Assuming that a dS/CFT correspondence exists, we argue that these are gravity duals to a CFT on a circle leading to uniform energy-momentum density, and are equivalent to an analytic continuation of the Euclidean AdS black brane. These are solutions with a complex parameter which then gives a real energy-momentum density. We also discuss a related solution with the parameter continued to a real number, which we refer to as a de Sitter "bluewall". This spacetime has two asymptotic de Sitter universes and Cauchy horizons cloaking timelike singularities. We argue that the Cauchy horizons give rise to a blue-shift instability.

In chapter 7 we investigate similar classical integrability for a more realistic confining background and provide a negative answer. The dynamics of a class of simple string configurations in AdS soliton background can be mapped to the dynamics of a set of non-linearly coupled oscillators. In a suitable limit of small fluctuations we discuss a quasi-periodic analytic solution of the system. However numerics indicates chaotic behavior as the fluctuations are not small. Integrability implies the existence of a regular foliation of the phase space by invariant manifolds. Our numerics shows how this nice foliation structure is eventually lost due to chaotic motion. We also verify a positive Lyapunov index for chaotic orbits. Our dynamics is roughly similar to other known non-integrable coupled oscillators systems like Henon-Heiles equations.

Using methods of Hamiltonian dynamical systems, we show analytically in chapter 8 that a dynamical system connected to the classical spinning string solution holographically dual to the principal Regge trajectory is non-integrable. The Regge trajectories themselves form an integrable island in the total phase space of the dynamical system. Our argument applies to any gravity background dual to confining field theories and we verify it explicitly in various supergravity backgrounds: Klebanov-Strassler, Maldacena-Nunez, Witten QCD and the AdS soliton. Having established non-integrability for this general class of supergravity backgrounds, we show explicitly by direct computation of the Poincare sections and the largest Lyapunov exponent, that such strings have chaotic motion.

Chapter 2

Quantum Quench Across a Zero Temperature Holographic Superfluid Transition

2.1 Introduction and summary

Recently there has been several efforts to understand the problem of quantum or thermal quench [1, 21, 2, 22–26] in strongly coupled field theories using the AdS/CFT correspondence [5, 27, 28, 15]. This approach has been used to explore two interesting issues. The first relates to the question of thermalization. In this problem one typically considers a coupling in the hamiltonian which varies appreciably with time over some finite time interval. Starting with a nice initial state (e.g. the vacuum) the question is whether the system evolves into some steady state and whether this steady state resembles a thermal state in a suitably defined sense. In the bulk description a time dependent coupling of the boundary field theory is a time dependent boundary condition. For example, with an initial AdS this leads to black hole formation under suitable conditions. This is a holographic description of thermalization, which has been widely studied over the past several years [29–44] with other initial conditions as well.

Many interesting applications of AdS/CFT duality involve a subset of bulk fields whose backreaction to gravity can be ignored, so that they can be treated in a *probe approximation*. One set of examples concern probe branes in AdS which lead to hypermultiplet fields in the original dual field theory. Even though the background does not change in the leading order, it turns out that thermalization of the hypermultiplet sector is still visible - this manifests itself in the formation of apparent horizons on the worldvolume [45–51].

The second issue relates to quench across critical points [1, 21, 2, 22–26]. Consider for example starting in a gapped phase, with a parameter in the Hamiltonian varying slowly compared to the initial gap, bringing the system close to a value of the parameter where there would be an equilibrium critical point. As one comes close to this critical point, adiabaticity is inevitably broken. Kibble and Zurek [3, 4, 1, 52, 53] argued that in the critical region the dynamics reflects universal features leading to scaling of various quantities. These arguments are based on rather drastic approximations, and for strongly coupled systems there is no theoretical framework analogous to renormalization group which leads to such scaling. For two-dimensional theories which are *suddenly* quenched to a critical point, powerful techniques of boundary conformal field theory have been used in [24–26] to show that ratios of relaxation times of one point functions, as well as the length/time scales associated with the behavior of two point functions of different operators, are given in terms of ratios of their conformal dimensions at the critical point, and hence universal.

In [20] quench dynamics in the critical region of a finite chemical potential holographic critical point was studied in a probe approximation. The "phenomenological" model used was that of [54] which involves a neutral scalar field with quartic self-coupling with a mass-squared lying in the range $-9/4 < m^2 < -3/2$ in the background of a *charged* AdS_4 black brane. The self coupling is large so that the backreaction of the scalar dynamics on the background geometry can be ignored. The background Maxwell field gives rise to a nonzero chemical potential in the boundary field theory. In [54] it was shown that for low enough temperatures, this system undergoes a critical phase transition at a mass m_c^2. For $m^2 < m_c^2$ the scalar field condenses, in a manner similar to holographic superfluids [55–62]. The critical point at $m^2 = m_c^2$ is a standard mean field transition at any non-zero temperature, and becomes a Berezinski-Kosterlitz-Thouless transition at zero temperature, as in several other examples of quantum critical transitions. In [20] the critical point was probed by turning on a time dependent source for the dual operator, with the mass kept exactly at the critical value, i.e. a time dependent boundary value of one of the modes of the bulk scalar. The source asymptotes to constant values at early and late times, and crosses the critical point at zero source at some intermediate time. The rate of time variation v is slow compared to the initial gap. As expected, adiabaticity fails as the equilibrium critical point at vanishing source is approached. However, it was shown that for any non-zero temperature and small enough v, the bulk solution in the critical region can be expanded in *fractional* powers of v. To lowest order in this expansion, the dynamics is dominated by a single mode - the zero mode of the linearized bulk equation, which appears exactly at $m^2 = m_c^2$. The resulting dynamics of this zero mode is in fact a *dissipative* Landau-Ginsburg dynamics with a dynamical critical exponent $z = 2$, and the order parameter was shown to obey Kibble-Zurek type scaling.

The work of [20] is at finite temperature - the dissipation in this model is of course due to the presence of a black hole horizon and is expected at any finite temperature. It is interesting to ask what happens at zero temperatures. It turns out that the model of [54] used in [20] becomes subtle at zero temperature. In this case, there is no conventional adiabatic expansion even away from the critical point (though there is a different low energy expansion, as in [63]). Furthermore, the susceptibility is finite at the transition, indicating there is no zero mode. While it should be possible to examine quantum quench in this model by numerical methods, we have not been able to get much analytic insight.

In this paper we study a different model of a quantum critical point, which is a variation of the model of insulator-superconductor transition of [64]. The model of [64] involves a *charged* scalar field minimally coupled to gravity with a negative cosmological constant and a Maxwell field. One of the spatial directions is compact with some radius R, and in addition one can have a non-zero temperature T and a non-zero chemical potential μ corresponding to the boundary value of the Maxwell field. In the absence of the scalar field this model has a line of Hawking-Page type first order phase transitions in the T-μ plane which separates an (hot) AdS soliton and a (charged) black brane. Exactly on the $T = 0$ line, the two phases correspond to the AdS soliton with a constant Maxwell scalar potential, and an extremal black hole. In [64] it was shown that in the presence of a minimally coupled charged scalar, the phase diagram changes. When the charge is large the scalar and the gauge fields can be regarded as probe fields which do not affect the geometry. Now there is a phase with a trivial scalar and a phase with a scalar condensate. In the boundary theory the latter is a superfluid phase. This phase transition persists at zero temperature, where it separates an unbroken phase at low chemical potential and a broken phase - in both cases the background geometry is the AdS soliton, while the gauge field is non-trivial in the superfluid phase. The phase diagram is given in Figure 9 of [64].

The idea now is to probe the dynamics of this insulator-superfluid transition at zero temperature by turning on a time dependent source for the operator dual to the charged field. So long as the scalar is minimally coupled and the charge q is large, this would involve analyzing a coupled set of equations of the scalar field and the gauge field.

However, it turns out that a slight modification of the model allows us to ignore the backreaction of the scalar to the gauge field as well. This involves the introduction of a quartic self coupling of the scalar λ. Then in the regime $\lambda \gg q^2$ and $\lambda \gg \kappa^2$ (where κ is the gravitational coupling), we can consider the dynamics of the charged scalar in isolation.

In this work we first show that in this regime of the parameters the insulator-superfluid transition persists. Concretely, for a sufficiently small negative m^2, there is a critical value of the background chemical potential beyond which a nontrivial static solution for the scalar becomes thermodynamically favored. Note that unlike other models of holographic superconductors the trivial solution does not become dynamically unstable. Rather the non-trivial solution has lower energy. The transition is a standard mean field critical transition. The background geometry remains an AdS soliton and the background gauge potential remains a constant, which is the chemical potential μ. At the transition, the linearized equation has a zero mode solution which is regular both at the boundary and at the tip.

We then turn on a time dependent boundary condition and find that the breakdown of adiabaticity for a small rate v is characterized by exponents which are appropriate for a dynamical critical exponent $z = 2$. In a way quite similar to [20] we find that in the critical region there is a new small v expansion in fractional powers of v, and the dynamics is once again dominated by a zero mode. The real and imaginary parts of the zero mode now satisfy a coupled set of Landau-Ginsburg type equation with first order time derivatives. However the resulting system is oscillatory rather than dissipative - this is expected since the background geometry has no horizon so that we have is a *closed* system. The order parameter is shown to obey a Kibble-Zurek type scaling. Finally we solve the bulk equations numerically and verify the scaling property obtained from the above small-v expansion.

Thermal quench in holographic superfluids with backreaction has been recently studied in [65, 66]. This work addresses a different issue - here the quench is applied to the system in the ordered phase *away from the critical point* and the resulting late time relaxation of the order parameter is studied. Our emphasis is on probing a possible Kibble-Zurek scaling when the quench crosses the critical point.

In Section 2 we define the model and discuss its equilibrium phases. In Section 3 we study quantum quench in this model by turning on a time dependent source, discuss the breakdown of adiabaticity and show that the critical region dynamics is dominated by the zero mode, leading to scaling behavior. In Section 4 we present the results of a numerical solution of the equations, verifying the scaling behavior. In an appendix we discuss a Landau-Ginsburg model similar to the critical dynamics of our holographic model.

2.2 The model and equilibrium phases

The "phenomenological" holographic model we consider is a slight variation of the model of [64]. The bulk action in $(d+2)$-dimensions is

$$S = \int d^{d+2}x\sqrt{g}\left[\frac{1}{2\kappa^2}\left(R+\frac{d(d+1)}{L^2}\right)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{\lambda}\left(|\nabla_\mu\Phi-iqA_\mu\Phi|^2-m^2|\Phi|^2-\frac{1}{2}|\Phi|^4\right)\right] , \tag{2.2.1}$$

where Φ is a complex scalar field and A_μ is an abelian gauge field, and the other notations are standard. Henceforth we will use $L=1$ units.

One of the spatial directions, which we will denote by θ will be considered to be compact. We will consider the regime

$$\lambda \gg q^2 \ , \qquad \lambda \gg \kappa^2 \ . \tag{2.2.2}$$

In this regime the scalar field is a probe field, and its backreaction to both the metric and the gauge field can be ignored.

2.2.1 The background

The background metric and the gauge field can be then obtained by solving the Einstein-Maxwell equations with the appropriate periodicity condition on θ. It is well known that there are two possible solutions. The first is the AdS_{d+2} soliton,

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2 f_{sl}(r)} + r^2\left(-dt^2+\sum_{i=1}^{d-1}dx_i^2\right)+r^2 f_{sl}(r)d\theta^2 \ , \\ f_{sl}(r) &= 1-\left(\frac{r_0}{r}\right)^{d+1} \ , \\ A_t &= \mu \ , \end{aligned} \tag{2.2.3}$$

with constant parameters μ and r_0. The periodicity of θ in this solution is

$$\theta \sim \theta + \frac{4\pi}{(d+1)r_0} \ , \tag{2.2.4}$$

while the temperature can be arbitrary. The second solution is a AdS_{d+2} charged black hole

$$ds^2 = -r^2 f_{bh}(r)dt^2 + \frac{dr^2}{r^2 f_{bh}(r)} + r^2\left(\sum_{i=1}^{d-1}dx_i^2 + d\theta^2\right) \ ,$$

$$\begin{aligned} f_{bh}(r) &= 1 - \left[1 + \frac{d-1}{2d}\left(\frac{\mu}{r_+}\right)^2\right]\left(\frac{r_+}{r}\right)^{d+1} + \frac{d-1}{2d}\left(\frac{\mu}{r_+}\right)^2\left(\frac{r_+}{r}\right)^{2d} , \\ A_t &= \mu\left[1 - \left(\frac{r_+}{r}\right)^{d-1}\right] . \end{aligned} \tag{2.2.5}$$

The temperature of this black brane is

$$T = \frac{r_+}{4\pi}\left[d + 1 - \frac{(d-1)^2}{2d}\left(\frac{\mu}{r_+}\right)^2\right] , \tag{2.2.6}$$

while the period of θ is arbitrary. As shown in [64], this system undergoes a phase transition between these two solutions when

$$r_0^{d+1} = r_+^{d+1}\left[1 + \frac{d-1}{2d}\left(\frac{\mu}{r_+}\right)^2\right] . \tag{2.2.7}$$

The AdS soliton is stable when the temperature and the chemical potential are small. At $T = 0$ the transition happens at a critical chemical potential μ_{c2} given by

$$\mu_{c2} = \frac{r_0(d+1)(2d)^{\frac{d-1}{2(d+1)}}}{(d-1)^{\frac{d}{d+1}}(d+1)^{1/2}} . \tag{2.2.8}$$

2.2.2 Scalar condensate

Consider now the scalar wave equation in the AdS soliton background (2.2.3). We first rescale

$$r \to \frac{r}{r_0} , \quad t \to t r_0 , \quad \mu \to \frac{\mu}{r_0} . \tag{2.2.9}$$

In the rest of the paper we will use these rescaled coordinates (i.e., $r_0 = 1$) and chemical potentials.

For fields which depend only on t and r, the equation of motion is given by

$$\left[-\frac{1}{r^2}(\partial_t - i\mu)^2 + \frac{1}{r^d}\partial_r\left(r^{d+2} f_{sl}(r)\partial_r\right)\right]\Phi - m^2\Phi - \Phi|\Phi|^2 = 0 . \tag{2.2.10}$$

In this paper we will consider $-\frac{(d+1)^2}{4} < m^2 < -\frac{d(d-1)}{4}$. The asymptotic behavior of the solution at the AdS boundary $r \to \infty$ is of the standard form

$$\Phi(r,t) = J(t)\, r^{-\Delta_-}[1 + O(1/r^2)] + A(t) r^{-\Delta_+}[1 + O(1/r^2)] + \cdots , \tag{2.2.11}$$

where

$$\Delta_\pm = \frac{d+1}{2} \pm \sqrt{m^2 + \frac{(d+1)^2}{4}} . \tag{2.2.12}$$

In "standard quantization" $J(t)$ is the source, while the expectation value of the dual operator is given by

$$\langle \mathcal{O} \rangle = A(t) . \tag{2.2.13}$$

In "alternative quantization" the role of $J(t)$ and $A(t)$ are interchanged. In this mass range both $\Delta_\pm$ are positive and both the solutions of the linear equation vanish at the boundary. Thus the nonlinear terms in the equation (2.2.10) are subdominant - which is why the leading solution near the boundary is the same as those of the linear equation, as written above.

We need to find time independent solutions of the equation (2.2.11). Because of gauge invariance, we need to specify a gauge to qualify what we mean by time independence. For the equilibrium solution we require the solution to be *real* - this fixes the gauge. Note that the tip of the soliton is locally two-dimensional flat space. Therefore we need to require the solution to be regular at the tip $r = 1$. This leads to the following boundary condition at $r = 1$

$$\Phi(r) = \Phi_h + \Phi_h'(r-1) + \cdots \ , \tag{2.2.14}$$

where regularity requires

$$\Phi_h' = \frac{1}{(d+1)}\,\Phi_h(\Phi_h^2 + m^2) + \frac{1}{(d+1)}\,\Phi_h\mu^2 \ . \tag{2.2.15}$$

To examine the phase structure we need to find time independent solutions with a vanishing source.

Clearly $\Phi = 0$ is always a solution. We have solved the equations numerically and found that there is a critical value of the chemical potential μ_{c1} beyond which there is another solution with a non-trivial r dependence which is thermodynamically preferred. This means that for $\mu > \mu_{c1}$, the operator dual to the bulk scalar has a vacuum expectation value, i.e., the global $U(1)$ symmetry of the boundary theory is spontaneously broken. Although this could happen both in the standard and alternative quantizations, we need to check the critical value is less that that of the phase transition between the AdS soliton and AdS black hole: $\mu_{c1} < \mu_{c2}$. Otherwise, the scalar condensate phase is not available on the AdS soliton.

Figure 2.1 shows the behavior of the expectation value $\langle\mathcal{O}\rangle$ for $m^2 = -15/4$ for standard and quantization. We are plotting the condensation with respect to μq and the phase transition happens at $\mu_{c1} q \sim 1.89$, which means the critical chemical potential is very small of order $O(1/q)$ in the probe limit. It follows from (2.2.8) that μ_{c1} is always much smaller than $\mu_{c2} \sim 1.86$ and there exists a scalar condensate phase on the AdS soliton. Similarly for any given m^2, $\mu = \mu_{c1}$ is a critical point by letting q be large enough.

This transition was first found in [64] for a *minimally coupled* complex scalar - in this case the backreaction to the gauge field cannot be ignored, and the result follows from an analysis of the coupled set of equations for the scalar and the gauge field. In this case the gauge field introduces non-linearity in the problem which is necessary for condensation of the scalar. What we found is that a self-coupling does the same same job.

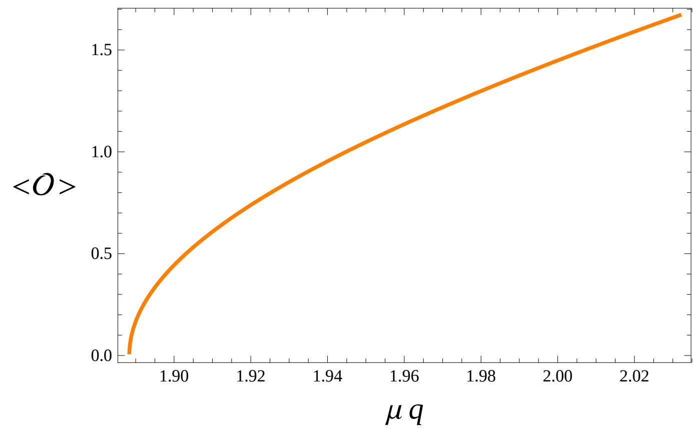


Figure 2.1: The condensations of the scalar operators.

2.2.3 The zero mode at the critical point

To get some insight into this transition it is useful to write the equation (2.2.10) as a Schrödinger problem. First define a new coordinate

$$\rho(r) = \int_r^\infty \frac{ds}{s^2 f_{sl}^{1/2}(s)} \ , \tag{2.2.16}$$

which is the "tortoise coordinate" for the AdS soliton. $\rho(r)$ is a monotonic function of r with the behavior

$$\begin{aligned} \rho &\sim 1/r \ , && r \to \infty \ , \\ \rho &\to \rho_\star + \frac{2\sqrt{r-1}}{\sqrt{d+1}} \ , && r \to 1 \ . \end{aligned} \tag{2.2.17}$$

For example, for $d = 3$ (asymptotically AdS_5 spacetime) soliton $\rho_\star = 1.311$. Let us now redefine the field by

$$\Phi(r,t) = \frac{1}{[r(\rho)]^{\frac{(d-2)}{2}}} \left(\frac{d\rho}{dr}\right)^{1/2} \Psi(\rho, t) \ . \tag{2.2.18}$$

Then $\Psi(\rho, t)$ satisfies the equation

$$\left[-\partial_t^2 + 2i\mu\partial_t\right] \Psi = \mathcal{P}\Psi - \mu^2\Psi + \frac{r^{2-d}}{\sqrt{f_{sl}(r)}} |\Psi|^2\Psi \ . \tag{2.2.19}$$

The operator $\mathcal{P}$ is

$$\begin{aligned} \mathcal{P} &= -\partial_\rho^2 + V_0(\rho) \ , \\ V_0(\rho) &= m^2 r^2 + \frac{4d(d+2)r^{2d+2} - 4d(d+3)r^{d+1} - (d-1)^2}{16 r^{d-1}(r^{d+1} - 1)} \ , \end{aligned} \tag{2.2.20}$$

where r has to be expressed as a function of ρ using (2.2.16).

The potential $V_0(\rho)$ has the following behavior near the boundary and the tip

$$\begin{aligned} V_0(\rho) &= \frac{m^2 + \frac{d(d+2)}{4}}{\rho^2} + O(\rho^2) \ , && \rho \to 0 \ , \\ V_0(\rho) &= -\frac{1}{4(\rho_\star - \rho)^2} + O(1) \ , && \rho \to \rho_\star \ . \end{aligned} \tag{2.2.21}$$

The behavior at the boundary $\rho = 0$ is of course the same as in pure AdS_{d+2}. The behavior near the tip $\rho = \rho_\star$ is in fact the correct behavior expected from a flat two dimensional space. Near the tip of the soliton the space becomes $\mathbb{R}^2 \times \mathbb{R}^{d-1}$ with $y \equiv (\rho_\star - \rho)$ playing the role of a radial variable and θ playing the role of the polar angle. Indeed with the redefined field

$$\tilde{\Psi}(y) = \frac{\Psi(\rho)}{\sqrt{\rho_\star - \rho}} \ , \tag{2.2.22}$$

the operator $\mathcal{P}$ becomes, near $y = 0$, the zero angular momentum Laplacian in two dimensions

$$\mathcal{P} \underset{y \to 0}{\to} -\frac{1}{y}\partial_y(y\partial_y) = -(\nabla_2)^2|_0 + \text{constant} \ . \tag{2.2.23}$$

In fact the eigenvalues of the operator $\mathcal{P}$ which acts on $\tilde{\Psi}$

$$\mathcal{P} = -(\nabla_2)^2|_0 + V_1(y) \ , \qquad \left(V_1(y) \equiv V_0(y) + \frac{1}{4y^2}\right) \ , \tag{2.2.24}$$

are all positive. For $d = 3$ the proof is the following. Let us rewrite the potential $V_1(y)$ as follows

$$V_1(y) = (m^2 + \frac{15}{4})r^2 + V_2(y) \tag{2.2.25}$$

where

$$V_2(y) = \frac{1}{4}\left[\frac{1+3r^4}{r^2 - r^6} + \frac{1}{y(r)^2}\right] \tag{2.2.26}$$

The term $V_2(y)$ is explicitly positive for all r. This may be seen as follows. The condition for positivity of $V_2(y)$ is

$$\sqrt{\frac{r^6 - r^2}{1 + 3r^4}} - y(r) \geq 0 \tag{2.2.27}$$

The inequality is saturated for $y = 0$ ($r = 1$). Furthermore the first derivative of the left hand side becomes

$$\frac{1}{\sqrt{r^4 - 1}}\left[\frac{3r^8 + 6r^4 - 1}{(1 + 3r^4)^{3/2}} - 1\right] \tag{2.2.28}$$

This can be explicitly checked to be positive for all $r > 1$ (e.g. by squaring the expression). Therefore $V_2(y) \geq 0$ for all $r > 1$. The first term in $V_1(y)$ in (2.2.25) is the asymptotic potential in AdS_5 - when $m^2 + \frac{15}{4} > -\frac{1}{4}$ (which is the BF bound), this potential does not have any bound state. Since $V_2(y)$ differs from this asymptotic potential by a positive function, the full potential $V_1(y)$ does not have any bound state.

To look for a condensate in standard quantization, we need to find time independent solutions of the equation (2.2.19) which satisfy the boundary condition $J = 0$ at $\rho = 0$ and is regular at the tip $\rho = \rho_\star$. With these boundary conditions the operator $\mathcal{P}$ has a *discrete* and *positive* spectrum. This means that for sufficiently large μ the operator

$$\mathcal{D} \equiv \mathcal{P} - \mu^2 \ , \tag{2.2.29}$$

will have a negative eigenvalue. This is what we found numerically.

At the critical value $\mu = \mu_{c1}$ the operator $\mathcal{D}$ has a zero eigenvalue, i.e. a zero mode which satisfies the appropriate boundary conditions both at the tip and at the boundary. This zero mode will play a key role in the following.

Note that even though the operator $\mathcal{D}$ has negative eigenvalues in the condensed phase, the trivial solution does not become unstable. This is clear from (2.2.19) and from the fact the spectrum of $\mathcal{P}$ is positive, which shows that the frequencies of the solutions to the linearized equation are all real.

Following the arguments of [54] it can be easily checked that the transition is standard mean field. This means that

$$\begin{aligned} \langle \mathcal{O} \rangle_{J=0} &\sim \sqrt{|\mu_{c1} - \mu|} \ , \\ \langle \mathcal{O} \rangle_{\mu=\mu_{c1}} &\sim |J|^{1/3} \ . \end{aligned} \tag{2.2.30}$$

We expect that this transition extends to non-zero temperature, though we have not checked this explicitly.

2.3 Quantum quench with a time dependent source

We will now probe the quantum critical point by quantum quench with a time dependent homogeneous source $J(t)$ for the dual operator $\mathcal{O}$, with the chemical potential tuned to $\mu = \mu_{c1}$. The function $J(t)$ will be chosen to asymptote to constants at early and late times, e.g.

$$J(t) = J_0 \tanh(vt) \ . \tag{2.3.1}$$

Note that we are using units with $r_0 = 1$. The system then crosses the equilibrium critical point at time $t = 0$. The idea is to start at some early time with initial conditions provided by the *instantaneous solution* and calculate the one point function $\langle \mathcal{O}(t) \rangle$. In standard quantization this means that we impose a time dependent boundary condition as in (2.2.11) and calculate $A(t)$. In alternative quantization the source should equal $A(t)$. In this paper we discuss the problem in standard quantization : the treatment in alternative quantization is similar.

2.3.1 Breakdown of adiabaticity

With a $J(t)$ of the form described above (e.g. (2.3.1)), one would expect that the initial time evolution is adiabatic for small v so long as J_0 is not too small. As one approaches $t = 0$ adiabaticity inevitably breaks down and the system gets excited. In this subsection we determine the manner in which this happens.

An adiabatic solution of (2.2.19) is of the form

$$\Psi(\rho, t) = \Psi^{(0)}(\rho, J(t)) + \epsilon \Psi^{(1)}(\rho, t) + \epsilon^2 \Psi^{(2)} + \cdots \ , \tag{2.3.2}$$

where ϵ is the adiabaticity parameter. The leading term is the instantaneous solution of (2.2.19), which is (using the definition (2.2.29))

$$\mathcal{D}\Psi^{(0)} + G(\rho)|\Psi^{(0)}|^2 \Psi^{(0)} = 0 \ , \tag{2.3.3}$$

satisfying the required boundary condition. Here we have defined

$$G(\rho) \equiv \frac{r^{2-d}}{\sqrt{f_{sl}(r)}} \ . \tag{2.3.4}$$

From (2.2.30) we know that for a real $J(t)$, this is real and has a form

$$\Psi^{(0)} \sim \rho^{\alpha} J(t) \left[1 + O(\rho^2)\right] + \rho^{1-\alpha} \left[J(t)\right]^{1/3} \left[1 + O(\rho^2)\right] \ , \tag{2.3.5}$$

where

$$\alpha \equiv \Delta_- - d/2 \ . \tag{2.3.6}$$

This follows from the equations (2.2.17), (2.2.18) and (3.1.10). The adiabatic expansion now proceeds by replacing $\partial_t \to \epsilon \partial_t$ in (2.2.19) substituting (2.3.2) and equating terms order by order in ϵ. The n-th order contribution $\Psi^{(n)}$ satisfies a *linear, inhomogeneous* ordinary differential equation with a source term which depends on the previous order solution $\Psi^{(n-1)}$. To lowest order we have the following equations for the real and imaginary parts of $\Psi^{(1)}$

$$\begin{aligned} \left[\mathcal{D} + 3G(\rho)(\Psi^{(0)})^2\right] (\mathrm{Re}\,\Psi^{(1)}) &= 0 \ , \\ \left[\mathcal{D} + G(\rho)(\Psi^{(0)})^2\right] (\mathrm{Im}\,\Psi^{(1)}) &= 2\mu\, \partial_t \Psi^{(0)} \ . \end{aligned} \tag{2.3.7}$$

Note that in these equations the time dependence of $J(t)$ should be ignored. The full function Ψ must satisfy the boundary condition $\lim_{\rho \to 0} [\rho^{-\alpha}\Psi(\rho, t)] = J(t)$. This means that the adiabatic corrections must start with the subleading terms, $\Psi^{(1)} \sim \rho^{1-\alpha}$ as $\rho \to 0$ and has to be regular as $\rho \to \rho_\star$. These provide the boundary conditions for solving the equations (2.3.7). Consider first the equation for $\mathrm{Im}\,\Psi^{(1)}$. Since the time dependence of $\Psi^{(0)}$ is entirely through $J(t)$ the solution may be written as

$$\mathrm{Im}\,\Psi^{(1)}(\rho, t) = 2\mu\, \dot{J}(t) \int_0^{\rho_\star} d\rho'\, \mathcal{G}(\rho, \rho')\, \frac{\partial \Psi^{(0)}}{\partial J(t)}(\rho', J(t)) \ , \tag{2.3.8}$$

where $\mathcal{G}(\rho, \rho')$ is the Green's function for the operator $\mathcal{D} + G(\rho)(\Psi^{(0)})^2$,

$$\mathcal{G}(\rho, \rho') = \frac{1}{W(\psi_1, \psi_2)}\, \psi_1(\rho')\psi_2(\rho) \ , \quad \rho < \rho' \ , \tag{2.3.9}$$

$$= \frac{1}{W(\psi_1, \psi_2)} \psi_2(\rho')\psi_1(\rho) \ , \quad \rho > \rho', \tag{2.3.10}$$

where ψ_1 and ψ_2 are solutions of the homogeneous equation $\left[\mathcal{D} + G(\rho)(\Psi^{(0)})^2\right] \psi_{1,2} = 0$ which satisfy the appropriate boundary conditions at the tip $\rho = \rho_\star$ and at the boundary $\rho = 0$ respectively. The Wronskian $W(\psi_1, \psi_2)$ for this operator is clearly constant and is conveniently evaluated near the tip. Near $\rho = \rho_\star$ these solutions behave as

$$\psi_1 \sim C\sqrt{\rho_\star - \rho} \ , \qquad \psi_2 \sim A\sqrt{\rho_\star - \rho} + B\sqrt{\rho_\star - \rho} \ \log(\rho_\star - \rho) \ , \tag{2.3.11}$$

where A, B, C are constants which depends on $J(t)$ [1]. Thus the Wronskian is

$$W(\psi_1, \psi_2) = -BC \ . \tag{2.3.12}$$

As noted in the previous section, the operator $\mathcal{D}$ has a zero mode, i.e. $\left[\mathcal{D} + G(\rho)(\Psi^{(0)})^2\right]$ has a zero mode when $\Psi^{(0)} = 0$, i.e. exactly at the equilibrium critical point. Thus, at this point we must have $B = 0$. This is why the first adiabatic correction $\mathrm{Im}\,\Psi^{(1)}(\rho, t)$ diverges at this point. For small $J(t)$ we can use perturbation theory to estimate the value of B. For small J the zeroth order solution $\Psi^{(0)}$ behaves as $J^{1/3}$ (the first term in (2.3.5) is subdominant). This is explicit to all orders in the expansion of the solution around the boundary. However, this is also justified by the results of the next section where we show that in the critical region the dynamics is dominated by a zero mode. The coefficient of the zero mode can be seen to be proportional to $J^{1/3}$ using a regularity argument similar to that in [54] so that the additional term in the operator behaves as $G(\rho)(\Psi^{(0)})^2 \sim [J(t)]^{2/3}$. This yields $B \sim J^{2/3}$ as well. Thus the Green's function which appears in (2.3.8) behaves as $J^{-2/3}$ so that the correction behaves as

$$\mathrm{Im}\,\Psi^{(1)} \sim \frac{\dot{J}(t)}{J^{2/3}} \frac{\partial \Psi^{(0)}}{\partial J(t)} \sim \frac{\dot{J}(t)}{J^{4/3}} \ . \tag{2.3.13}$$

The same argument shows that $\mathrm{Re}\,\Psi^{(1)} = 0$, so that $|\Psi^{(1)}| \sim \frac{\dot{J}}{J^{4/3}}$ as well. Therefore adiabaticity breaks when

$$|\Psi^{(1)}| \sim |\Psi^{(0)}| \implies \dot{J}(t) \sim J^{5/3} \ . \tag{2.3.14}$$

For sources which vanish linearly at $t = 0$, i.e. $J(t) \sim vt$ (e.g. of the form (2.3.1)) this means that if the source is turned on at some early time, adiabaticity breaks at a time

$$t_{adia} \sim v^{-2/5} \ . \tag{2.3.15}$$

while at this time the value of the order parameter $\langle \mathcal{O} \rangle$ is

$$\langle \mathcal{O}(t_{adia}) \rangle \sim [J(t_{adia})]^{1/3} = [v t_{adia}]^{1/3} \sim v^{1/5} \ . \tag{2.3.16}$$

With the usual adiabatic-diabatic assumption, these exponents lead to Kibble-Zurek scaling for a dynamical critical exponent $z = 2$, even though the underlying dynamics is relativistic and non-dissipative. From the above analysis it is clear that this happened because the leading adiabatic correction is provided by the chemical potential term, which multiplies a first order time derivative of the bulk field.

[1] Note that in the equations (2.3.7) the time is simply a parameter.

2.3.2 Critical dynamics of the order parameter

The breakdown of adiabaticity means that an expansion in time derivatives fail. In this subsection we show, following closely the treatment of [20], that we now have a *different* small v expansion in *fractional* powers of v during the period when the sources passes through zero. This will lead to a scaling form of the order parameter in the critical region. In the following we will demonstrate this for the case where $J(t) \sim vt$ near $t \approx 0$. However the treatment can be easily generalized to a $J(t) \sim (vt)^n$ for any integer n.

To establish this, it is convenient to separate out the source term in the field $\Psi(\rho, t)$,

$$\Psi(\rho, t) = \rho^\alpha J(t) + \Psi_s(\rho, t) \ , \qquad \alpha = \Delta_- - d/2 \ , \tag{2.3.17}$$

where we have used the relation (2.2.18) and the fact that near the boundary $\rho \sim 1/r$. The equation of motion (2.2.19) then becomes

$$\begin{aligned} -\partial_t^2 \Psi_s + 2i\mu\, \partial_t \Psi_s \;\; = \;\; & (\mathcal{D}\rho^\alpha) J(t) + \mathcal{D}\Psi_s + G(\rho) \left[\rho^{3\alpha}[J(t)]^3 + \rho^{2\alpha}[J(t)]^2 (2\Psi_s + \Psi_s^\star)\right] \\ & + G(\rho) \left[\rho^\alpha J(t)(2|\Psi_s|^2 + \Psi_s^2) + |\Psi_s|^2 \Psi_s\right] \\ & + \rho^\alpha [\partial_t^2 J - 2i\mu \partial_t J] \ . \end{aligned} \tag{2.3.18}$$

This separation is useful because we know that in the presence of a constant source $J(t) = \bar{J}$, the static solution has the asymptotic form

$$\Psi_s \sim \rho^{1-\alpha} \left[|\bar{J}|^{1/3} + O(\rho^2)\right] + \bar{J} \rho^{\alpha+2} \left[1 + O(\rho^2)\right] \ , \tag{2.3.19}$$

which follows from (2.2.30).

The scaling relations (2.3.15) and (3.2.25) suggest that we perform the following rescaling of the time and the field

$$t = v^{-2/5} \eta \ , \qquad\qquad \Psi_s = v^{1/5} \chi \ . \tag{2.3.20}$$

In the critical region we can now use $J(t) = vt = v^{3/5} \eta$ and rewrite (2.3.18) as an expansion in powers of $v^{2/5}$,

$$\mathcal{D}\chi = v^{2/5} \left[2i\mu \partial_\eta \chi - G(\rho)|\chi|^2 \chi - \eta(\mathcal{D}\rho^\alpha)\right] + O(v^{4/5}) \ . \tag{2.3.21}$$

As noted above, because of the boundary condition at $\rho = 0$ and the regularity condition at $\rho = \rho_\star$ the spectrum of $\mathcal{D}$ is discrete. Let φ_n be the orthonormal set of eigenfunctions of the operator $\mathcal{D}$

$$\mathcal{D}\varphi_n(\rho) = \lambda_n \varphi_n(\rho) \ , \qquad n = 0, 1, \cdots \ , \tag{2.3.22}$$

with $\lambda_0 = 0$. $\varphi_0(\rho)$ is the zero mode which we discussed earlier. Since μ has been tuned to be equal to μ_{c1}, all the higher eigenvalues are positive.

We now expand

$$\chi(\rho, \eta) = \sum_n \chi_n(\eta) \varphi_n(\rho) \ , \tag{2.3.23}$$

and rewrite the equation (2.3.21) in terms of the modes $\chi_n(\eta)$

$$\lambda_n\chi_n = v^{2/5}\left[2i\mu(\partial_\eta\chi_n) - \sum_{n_1n_2n_3}\mathcal{C}^n_{n_1n_2n_3}\chi^\star_{n_3}\chi_{n_2}\chi_{n_1} + \mathcal{J}_n\eta\right] + O(v^{4/5}) \ , \qquad (2.3.24)$$

where we have defined

$$\begin{aligned}\mathcal{J}_n &= \int d\rho\varphi^\star_n(\rho)(\mathcal{D}\rho^\alpha) \ , \\ \mathcal{C}^n_{n_1n_2n_3} &= \int d\rho\varphi^\star_n(\rho)\varphi^\star_{n_3}(\rho)\varphi_{n_2}(\rho)\varphi_{n_1}(\rho)G(\rho) \ .\end{aligned} \qquad (2.3.25)$$

It is clear from (2.3.24) that the zero mode part of the bulk field dominates the dynamics in the critical region. In fact for small v a solution is of the form

$$\chi_n(\eta) = \delta_{n0}\xi_0(\eta) + v^{2/5}\xi_n + O(v^{4/5}) \ . \qquad (2.3.26)$$

The zero mode satisfies a $z = 2$ Landau-Ginsburg equation

$$-2i\mu\partial_\eta\xi_0 + \mathcal{C}^0_{000}|\xi_0|^2\xi_0 + \mathcal{J}_0\eta = 0 \ . \qquad (2.3.27)$$

Reverting back to the original variables we therefore have

$$\Psi_s(\rho,t,v) = v^{1/5}\Psi_s(\rho, tv^{2/5}, 1) \ , \qquad (2.3.28)$$

which implies a Kibble-Zurek scaling for the order parameter with $z = 2$

$$\langle\mathcal{O}(t,v)\rangle = v^{1/5}\langle\mathcal{O}(v^{2/5}t, 1)\rangle \ . \qquad (2.3.29)$$

Note that the effective Landau-Ginsburg equation (2.3.27) is not dissipative because the first order time derivative is multiplied by a purely imaginary constant. In fact, in the absence of a source term the quantity $\frac{1}{2}(|\xi_0|^2)^2$ is independent of time.

Beyond the critical region, we cannot use the approximation $J(t) \sim vt$ and there is no useful simplification in terms of the zero mode. However, the boundary conditions at the tip are perfectly reflecting boundary conditions (as appropriate for the origin of polar coordinates in two dimensions) so that there is a conserved energy in the problem. This is in contrast to a black hole background where there is a net ingoing flux at the horizon causing the system to be dissipative. Indeed in the quench problem considered in [20] arguments similar to those used in this section also led to an effective Landau-Ginsburg dynamics with $z = 2$, but which is dissipative.

In the appendix we analyze a Landau-Ginsburg toy model motivated by the results of this section.

2.4 Numerical results

In this section we summarize our numerical results. We have solved the bulk equation of motion numerically for $d = 3$. The results for different values of m^2 are similar.

We present detailed results for $m^2 = -15/4$. In this case the critical value of the chemical potential is $\mu_{c1} q \approx 1.88$.

We discretize the partial differential equations (PDEs) (2.2.19) (written in the y-coordinate) in a radial Chebyshev grid to study the numerical problem. Once discretized in radial direction, the PDEs become a series of ordinary differential equations (ODEs) in the temporal variable. The resulting ODEs are solved with a standard ODE solver (e.g. CVODE). The time dependence is chosen to be of the form dependent source as in (2.3.1). In principle one may study with any kind of time dependent source.

We will consider the problem in two regimes. The first is the "slow" regime where we expect our analytic arguments to be accurate, the other is a "fast" regime where there is no adiabatic region whatsoever. In the slow regime we will try to zoom on the scaling region around the phase transition. In the fast regime we will find a large deviation from the adiabatic behavior and possible chaotic behavior.

2.4.1 Slow regime

Since our main interest is quench through the critical point, we concentrate mainly near the phase transition. We choose $\mu q = \mu_{c1} q$ (≈ 1.88), so that the system is critical in the absence of any source. In the presence of a time dependent source of the form (2.3.1) we calculate the bulk field $\tilde{\Psi}(t)$ and extract from this the value of $\langle \mathcal{O}(t) \rangle$ of the dual field theory. A typical plot of the real part of $\langle \mathcal{O}(t) \rangle$ for slow quench through the phase transition is presented in Figure 2.2.

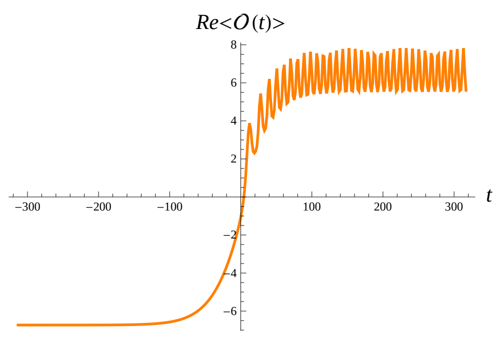


Figure 2.2: The plot of $\mathrm{Re}\,\langle \mathcal{O}(t) \rangle$ with $v = 0.02$.

Clearly the late time behavior is oscillatory, reflecting the fact that we are dealing with a closed and non-dissipative system.

We then zoom on the critical region near $t = 0$ for various value of v to look for any scaling behavior. One way to look for this is to consider the behavior of $\langle \mathcal{O} \rangle$ at $t = 0$. Equation (2.3.29) then predicts a scaling behavior $\langle \mathcal{O}(0) \rangle \sim v^{1/5}$.

Figure 2.3 shows a plot of $\log(\mathrm{Re}\,\langle \mathcal{O}(0) \rangle)$ for different v. We fit the data points with a function $f(x) = A + Bx + C/x$, where x is $\log(v)$. Here we kept a sublinear ($O(1/x)$) term to understand how the fit function approaches a linear regime. From our analytic argument we expect $B = 1/5$. A fit of the numerical results yields

$f(x) = 0.794 - 0.490/x + 0.206x$. Changing the number of fit points and range changes the values of fit parameters a bit, however we always get a value of B which is close to $1/5$ with only a few percentage deviation. The imaginary part ($\text{Im}\,\langle\mathcal{O}(0)\rangle$) also satisfies the same scaling.

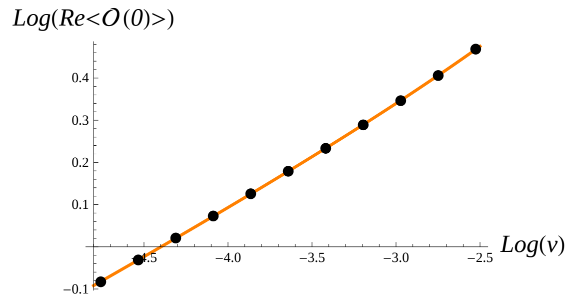


Figure 2.3: The plot of $\log(\text{Re}\,\langle\mathcal{O}(0)\rangle)$ vs $\log(v)$. We also plotted the closest fit (see text).

2.4.2 Fast regime

In the fast regime we see a large deviation from the adiabatic behavior, as shown in Figure 2.4.

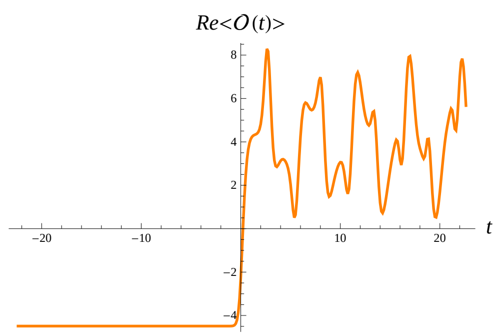


Figure 2.4: Plot of $\text{Re}\,\langle\mathcal{O}(t)\rangle$ showing chaotic behavior with a large value of $v = 2.0$ and $\mu = 1$.

The motion in this regime becomes possibly chaotic. Here we have a system with a conserved energy. Once we put some energy in the system, the non-linearity possibly takes the system over the whole phase space (Arnold diffusion). It is expected that if we wait long enough the probe approximation actually breaks down [67, 68] and we have to consider the fully backreacted problem. We plan to attack this problem in the near future.

Chapter 3

Quantum Quench and Double Trace Couplings

3.1 Introduction and Summary

There has been a lot of interest in understanding the problem of thermal or quantum quench [1, 21, 2, 22–26] using gauge-gravity duality [5, 27, 28, 15]. One set of works concentrate on the question of thermalization by horizon formation [29–51] and possible resolutions of spacelike singularities [69–71]. Recently there have been several studies of holographic quench which involve critical points. In [20] two of us initiated the study of holographic quench across finite temperature and finite chemical potential critical points, and found hints of a mechanism which gives rise to Kibble-Zurek scaling in critical dynamics [3, 4, 1, 21, 2]. This mechanism was confirmed for a zero temperature but nonzero chemical potential quantum critical point in [72]. In slightly different directions [65, 66] studied relaxation dynamics following a thermal quench from a broken symmetry phase and [73–75] studied scaling behavior of final values of observables due to a thermal quench. Quantum quench in solvable large-N field theories without the use of gauge-gravity duality has been studied in [76, 77, 52, 53, 78].

In [20] and [72] the quench was due to a homogeneous time dependent source for a scalar order parameter which translates to a time dependent Dirichlet boundary condition on the strongly self-coupled bulk scalar field. The other parameters in the theory were tuned such that in the absence of a source the theory is critical. The dynamics was then studied in the probe approximation by considering a source which is slowly varying at early and late times and which crosses zero (i.e. the location of the critical point) at some intermediate time. In this setup scaling behavior appears due to a few key facts

- At the equilibrium critical point the linearized bulk equation of motion for the scalar has a zero mode. This results in a breakdown of adiabaticity when the source becomes small characterized by a power law in the rate of change of the source v.
- In the critical region, and only in this region, there a new expansion for small v. This is an expansion in fractional powers of v, with exponents determined by the equilibrium critical exponents.
- In the lowest order of this expansion in fractional powers of v, the bulk dynamics is dominated by the zero mode. This zero mode then satisfies an ordinary differential equation which is basically the dynamics of the order parameter. In this equation, the boundary condition appears as a source term. This equation has a scaling solution displaying Kibble-Zurek scaling.

The setup in [20] and [72] involved a nonzero chemical potential and/or nonzero temperature. The background in [20] is a charged black brane with a neutral self coupled scalar [54], while that in [72] is an AdS soliton with a constant gauge field and a self coupled scalar - a variation of the setup of [64, 79]. In both cases the resulting dynamics of the order parameter is non-relativistic with dynamical critical exponent $z = 2$, even though the underlying bulk dynamics is relativistic. It is possible that the zero temperature limit of the setup of [20] may lead to a $z = 1$ dynamics. However the zero temperature limit the phase transition found in [54] and probed in [20] becomes a Berezinski-Kosterlitz-Thouless transition and we were not able to get any analytic handle on the dynamics.

So far all studies of quantum or thermal quench using holographic methods have dealt with time dependent external sources. A useful example to keep in mind is a magnet in the presence of a time dependent magnetic field. Critical dynamics can be then studied by tuning the temperature to the critical value. In a Landau-Ginsburg language this corresponds to a time dependent inhomogeneous term in the LG equation. In many situations, this is not a natural thing to do. For example in a superconductor an external source for the order parameter is not very natural, though it can be achieved by considering junctions. On the other hand, the standard tuning parameter in a critical transition is the term in a LG hamiltonian which is quadratic in the order parameter: we will call this a LG mass term.

In this paper we initiate the study of quench by such a time dependent LG mass using holographic techniques. While studying holographic quench with time dependent external source is straightforward because it maps to a time dependent boundary condition for the dual field, a time dependent LG mass quench would involve addition of a *double trace deformation* with a time dependent coefficient, $\kappa(t)$. As is well known this implies a modified boundary condition for the bulk scalar [80–86]. In equilibrium [87] found that for a class of scalar potentials, there is a critical phase transition at $\kappa = \kappa_c$ where $\kappa_c < 0$. Naively, from the field theory viewpoint, a deformation with negative κ appears to lead to an instability. However it has been shown in [87] and [88] this is not necessarily correct - typically there is a stable ground state with scalar hair for $\kappa < \kappa_c$. For vanishing temperature and vanishing chemical potential $\kappa_c = 0$, while for a nonzero temperature (i.e a black hole background) T one has $\kappa_c \propto T$. In the following we will show, not surprisingly, that there is a similar transition when the background is a AdS soliton.

We consider the simplest situation where such a transition occurs. The bulk action is given in $L_{AdS} = 1$ units

$$S = \int d^{d+2}x \, \sqrt{g} \left[\frac{1}{8\pi G_N} \left(R + d(d+1) \right) - \frac{1}{\lambda} \left((\nabla\phi)^2 + m^2\phi^2 + V(\phi) \right) \right] \tag{3.1.1}$$

where ϕ is a neutral bulk scalar. We will consider the limit $\lambda \gg G_N$ so that the scalar can be treated as a probe field whose dynamics does not affect the gravity background. We will consider potentials $V(\phi)$ which have a power series expansion in ϕ. As will become clear soon, the critical behavior is determined by the leading nonlinearity in $V(\phi)$, so it would be sufficient to consider monomials.

First we study the equilibrium transition in three such backgrounds. The first is pure AdS_{d+2}, which is the relevant geometry when all the spatial directions are noncompact,

$$ds^2 = r^2(-dt^2 + d\vec{x}^2 + dw^2) + \frac{dr^2}{r^2} \tag{3.1.2}$$

The second is a AdS_{d+2} soliton which is the relevant geometry when one of the spatial directions, w is compact with some radius R_0,

$$\begin{aligned} ds^2 &= r^2(-dt^2 + d\vec{x}^2 + f_s(r)du^2) + \frac{dr^2}{r^2 f_s(r)} \\ f_s(r) &= 1 - \left(\frac{r_0}{r}\right)^{d+1}, \qquad r_0 = \frac{4\pi}{(d+1)R_0} \end{aligned} \tag{3.1.3}$$

The third is a AdS_{d+2} black brane with all boundary directions non-compact. The metric is

$$\begin{aligned} ds^2 &= -r^2 f_b(r)dt^2 + r^2(d\vec{x}^2 + dw^2) + \frac{dr^2}{r^2 f_b(r)} \\ f_b(r) &= 1 - \left(\frac{\bar{r}_0}{r}\right)^{d+1}, \qquad \bar{r}_0 = \frac{4\pi T}{(d+1)} \end{aligned} \tag{3.1.4}$$

In all these cases the asymptotic form of the solution for the scalar has the form

$$\phi(r,\vec{x},t,u) \to r^{-\Delta_-}\mathcal{A}(\vec{x},u,t)\left(1 + O(1/r^2)\right) + r^{-\Delta_+}\mathcal{B}(\vec{x},u,t)\left(1 + O(1/r^2)\right) \tag{3.1.5}$$

provided the solution becomes small near the boundary. In (3.1.5)

$$\Delta_\pm = (d+1)/2 \pm \sqrt{(d+1)^2/4 + m^2} \tag{3.1.6}$$

We will work in the mass range $-(d+1)^2/4 \le m^2 \le -(d+1)^2/4 + 1$ so that we have two possible quantizations [89]: the standard quantization with $\mathcal{A}$ as the source and the alternative quantization with $\mathcal{B}$ as the source. The dimensions of the dual operator $\mathcal{O}$ in these two quantizations are Δ_+ and Δ_- respectively.

It is also possible to impose boundary conditions

$$\mathcal{B}(\vec{x},u,t) = \kappa(\vec{x},u,t)\mathcal{A}(\vec{x},u,t) \tag{3.1.7}$$

As is well known, this corresponds to addition of a term [80]

$$\int d^{d+1}x \ \kappa\mathcal{O}^2 \tag{3.1.8}$$

to the field theory action.

We will first show explicitly for suitable potentials that for a constant κ all these backgrounds admit critical points. For AdS_{d+2} the critical value is $\kappa = 0$: for $\kappa < 0$ there is a nontrivial solution of the equations of motion which is regular everywhere and satisfies the specified boundary conditions. This means that for the dual operator $\langle \mathcal{O} \rangle \neq 0$. Near the critical point we verify that

$$\langle \mathcal{O} \rangle \sim (-\kappa)^{\Delta_-/(\Delta_+ - \Delta_-)} \tag{3.1.9}$$

as required by scale symmetry. The scaling behavior is independent of the nature of the potential whose properties enter only in the overall coefficient.

For the AdS_{d+2} soliton as well as the AdS_{d+2} black brane, the critical value of κ is at some finite value $\kappa_c < 0$ and the condensate appears for $\kappa < \kappa_c$. This is shown by a direct numerical solution. It turns out that the value of κ_c can be determined analytically in closed form, following the treatment of [87] and is independent of the nature of the non-linearity. As is usual in such situations, there is a zero mode of the linearized equation at $\kappa = \kappa_c$: here the zero mode has a closed form in terms of hypergeometric functions. We verify that our numerical solution for ϕ^4 and ϕ^6 potentials agrees with this. The critical exponent can be also determined analytically. When the leading nonlinearity is ϕ^{n+1} , one gets

$$\langle \mathcal{O} \rangle \sim (\kappa_c - \kappa)^{1/(n-1)} \tag{3.1.10}$$

Note that in standard notation the critical exponent β is given by

$$\beta = 1/(n-1) \tag{3.1.11}$$

Our numerical results are consistent with the behavior (3.1.10). We also verify numerically that the critical behavior is indeed determined by the leading non-linearity.

We then consider the response of the system to a time dependent but homogeneous $\kappa(t)$ for the AdS_{d+2} soliton and AdS_{d+2} black brane backgrounds, staying in the probe approximation. For these backgrounds, the radius of the compact dimension (for the soliton) or the temperature (for the black brane) provides a scale, so that we can meaningfully talk about slow and fast quenches. We concentrate on slow quench starting deep in the ordered phase, crossing the critical point κ_c at some intermediate point and asymptoting to some other constant value at late times. Following the lines of [20, 72] we study the breakdown of adiabaticity and show that in a way similar to these works the critical region is characterized by an expansion in fractional powers of the rate and by the dominance of the zero mode. For a fast quench we expect a chaotic behavior to set in [90]. Unlike these previous works, the function $\kappa(t)$ now appears as a time dependent mass term in the effective LG dynamics of the zero mode and hence the order parameter. This is consistent with the fact that in the field theory, $\kappa(t)$ is indeed the coefficient of $\mathcal{O}^2$.

The ensuing critical dynamics for the soliton and the black brane are different. For the soliton, the dynamics is relativistic and non-dissipative. This is expected since in the field theory is at zero temperature and there is no chemical potential. The dynamics in the black brane background has $z = 2$ and is dissipative, as would be expected for a finite temperature situation.

Finally we solve the time evolution numerically and provide evidence for the scaling behavior discussed above.

In Section 2 we set up the equilibrium problems, show the existence of the critical point for negative constant κ and derive the critical exponents. Sections 3 and 4 deal with quantum quench due to a time dependent $\kappa(t)$ for the soliton and black brane backgrounds respectively. In section 5 we present our numerical results. In section 6 we discuss arbitrary critical exponents z and ν and the relationship of our scaling solutions with standard Kibble-Zurek scaling. Section 7 contains brief remarks and the appendix B discusses the solution of a toy model which justifies some key ingredients in our discussion of section 4.

3.2 The equilibrium critical point

In the probe approximation the only relevant equation we need to solve is the scalar field equation. For the backgrounds (3.1.2) or (3.1.3) and fields which depend only on t and r this equation is

$$-\frac{1}{h(r)}\partial_t^2\phi + \frac{1}{r^{d-2}}\partial_r(r^d g(r)\partial_r\phi) - m^2r^2\phi - r^2V'(\phi) = 0 \tag{3.2.1}$$

where

$$g(r) = \begin{cases} r^2, & \text{for } AdS_{d+2} \\ r^2 f_s(r) & \text{for } AdS_{d+2} \text{ soliton} \\ r^2 f_b(r) & \text{for } AdS_{d+2}\text{black brane} \end{cases} \tag{3.2.2}$$

and

$$h(r) = \begin{cases} 1 & \text{for } AdS_{d+2}\text{and } AdS_{d+2} \text{ soliton} \\ f_b(r) & \text{for } AdS_{d+2} \text{ black brane} \end{cases} \tag{3.2.3}$$

We first need to find static solutions of (3.2.1) which are regular in the interior and which satisfy the boundary condition (3.1.7) at the boundary (with constant $\mathcal{A}, \mathcal{B}, \kappa$).

3.2.1 Pure AdS_{d+2}

For pure AdS_{d+2} and a ϕ^4 potential regularity means that the value of the field at $r = 0$ is fixed to the attractor value

$$\phi(r=0)_{AdS} = \sqrt{-m^2} \tag{3.2.4}$$

To find a solution to the non-linear equation consider integrating the equation by imposing the condition at small ϵ

$$\phi(\epsilon) = \phi(r=0)_{AdS} - c\epsilon^{\Delta_v} \quad c > 0 \quad \Delta_v = \sqrt{(d+1)^2/4 - 2m^2} - (d+1)/4 \tag{3.2.5}$$

This form is dictated by the solution near $r = 0$ where the departure from the attractor value is small so that the equation can be linearized. The solution to the full nonlinear equation may be therefore written as $\phi(r, c)$, which gives us a one parameter class of solutions. However the equation has a scaling symmetry under

$$r \to \lambda r \qquad\qquad \phi \to \phi \tag{3.2.6}$$

which immediately implies that the solution satisfies

$$\phi_{AdS}(r,c) = \phi_{AdS}(rc^{-1/\Delta_v}, 1) \tag{3.2.7}$$

The solution near the boundary $r = \infty$ is of the form (3.1.5) with constant $\mathcal{A}$ and $\mathcal{B}$ - the scaling symmetry then implies

$$|\mathcal{A}| \sim c^{\frac{\Delta_-}{\Delta_v}} \qquad |\mathcal{B}| \sim c^{\frac{\Delta_+}{\Delta_v}} \qquad \Longrightarrow \qquad |\kappa| \sim c^{\frac{\Delta_+ - \Delta_-}{\Delta_v}} \tag{3.2.8}$$

In alternative quantization, $\mathcal{A}$ is the expectation value of the dual operator and the above relations immediately implies (3.1.9).

The solution $\phi(r,c)$ can be found easily by numerically solving the nonlinear equation. We find that there is a nonsingular solution for any negative κ which satisfies the above scaling behavior.

Note that the scaling argument given above does not depend on the potential being ϕ^4, and is valid for any potential $V(\phi)$ with $n \neq 1$. The value of the attractor is generally given by

$$m^2\phi + V'(\phi) = 0 \tag{3.2.9}$$

and the behavior for small r becomes a bit complicated, though still determined by a linear equation. Nevertheless the same scaling behavior (3.1.9) would follow. The numerical coefficient will of course depend on the details of the potential.

3.2.2 AdS_{d+2} soliton

For the AdS_{d+2} soliton (3.1.3) regularity at the tip $r = r_0 = 1$ implies that the field can attain any value ϕ_0 at $r = r_0$ while the derivative is given by

$$\frac{d\phi}{dr}(r = r_0) = \frac{1}{d+1}\left[m^2\phi_0 + V'(\phi_0)\right] \tag{3.2.10}$$

The static solution may be now obtained by starting at some ϕ_0 and integrating out to $r = \infty$. As we will see below, straightforward numerical integration then shows that a non-trivial regular solution exists only when $\kappa < \kappa_c$ where the critical value κ_c is a negative number to be determined shortly.

In the rest of the paper we will use $r_0 = 1$ units

As is usual in such cases (e.g. for holographic superconductors [56, 57]) the trivial solution with $\phi = 0$ is in fact unstable for $\kappa < \kappa_c$. To see this, let us write the linearized equation of motion as

$$\begin{aligned} -\partial_t^2\phi &= \tilde{\mathcal{D}}\phi \\ \tilde{\mathcal{D}} &\equiv -r^{2-d}\frac{\partial}{\partial r}\left(r^{d+2}f_s(r)\frac{\partial}{\partial r}\right) + m^2r^2 \end{aligned} \tag{3.2.11}$$

This equation can be cast into a Schrodinger form by changing coordinates to ρ

$$\rho(r) = \int_r^\infty \frac{ds}{s^2 f_s^{1/2}(s)} \tag{3.2.12}$$

and redefining the field to $\psi(\rho, t)$

$$\phi(r,t) = \frac{1}{[r(\rho)]^{\frac{d}{2}-1}} \left(\frac{d\rho}{dr}\right)^{1/2} \psi(\rho, t) \tag{3.2.13}$$

Note that

$$\begin{aligned} \rho &\sim 1/r & r \to \infty \\ \rho &\sim \rho_\star + \frac{2\sqrt{r-1}}{\sqrt{d+1}} & r \to 1 \end{aligned} \tag{3.2.14}$$

where $\rho_\star$ is finite. For $d = 3$ we get $\rho_\star = 1.311$. Using the explicit form of $f_s(r)$ the equation (3.2.11) becomes

$$-\partial_t^2 \psi = \mathcal{D}\psi \tag{3.2.15}$$

where

$$\begin{aligned} \mathcal{D} &= -\partial_\rho^2 + V_0(\rho) \\ V_0(\rho) &= m^2 r^2 + \frac{4d[(d+2)r^{2d+2} - (d+3)r^{d+1}] - (d-1)^2}{16 r^{d-1}(r^{d+1}-1)} \end{aligned} \tag{3.2.16}$$

This operator appeared in [72] where it was shown that with boundary conditions corresponding to either standard or alternative quantization this has a positive spectrum. However with the modified boundary condition $\mathcal{B} = \kappa\mathcal{A}$ with $\kappa < 0$ this is no longer true. In fact there is a specific value of κ where the operator $\mathcal{D}$ has a zero mode. The equation $\tilde{\mathcal{D}}\phi = 0$ (which is equivalent to $\mathcal{D}\psi = 0$) is in fact the same as equation (B.1) in [87] and we can borrow the results. The solution which is regular at $r = 1$ is given by

$$\phi_0(r) = A\left(r^{-\Delta_-} F_1^2\left[\frac{\Delta_-}{d+1}, \frac{\Delta_-}{d+1}, \frac{2\Delta_-}{d+1}, r^{-(d+1)}\right] + B r^{-\Delta_+} F_1^2\left[\frac{\Delta_+}{d+1}, \frac{\Delta_+}{d+1}, \frac{2\Delta_+}{d+1}, r^{-(d+1)}\right]\right) \tag{3.2.17}$$

where

$$B = -\frac{\Gamma(\frac{2\Delta_-}{d+1})[\Gamma(1-\frac{\Delta_-}{d+1}]^2}{\Gamma(2-\frac{2\Delta_-}{d+1})[\Gamma(\frac{\Delta_-}{d+1})]^2}. \tag{3.2.18}$$

The asymptotic expansion of this solution at $r = \infty$ can be read off trivially. Clearly the κ for this solution is $\kappa = B$. This must be the critical value, κ_c which is thus determined to be

$$\kappa_c = -B\ (r_0)^{d+1-2\Delta_-} \tag{3.2.19}$$

where we have restored factors of r_0. The zero mode ϕ_0 will play a key role in what follows.

For $\kappa < \kappa_c$ the operator has negative eigenvalues which implies that the trivial solution is unstable.

In the following we will also need the behavior of the lowest eigenvalue λ_0 of $\mathcal{D}$ for $\kappa < \kappa_c$. Generically one would expect that this would vanish linearly,

$$\lambda_0(\kappa) = -c_0(\kappa_c - \kappa). \tag{3.2.20}$$

We have checked this numerically for $d = 3$ and $m^2 = -15/4$ and obtained $c_0 = 0.762589$. We also checked that $\kappa_c = -0.495$ which is consistent with (3.2.19) and 3.2.18). This behavior will be important in the dynamics.

3.2.2.1 Effect of non-linearity

We now consider the effect of non-linearity in the static solution. Consider a Z_2 invariant potential of the form [1]

$$V(\phi) = \sum_{q=2}^{\infty} \lambda_q \phi^{2q}. \tag{3.2.21}$$

For simplicity we will assume all the λ_q's to be positive. We want to find solutions of the full nonlinear equation with specified boundary conditions at $r = \infty$ and regular in the interior. Such solutions can be constructed by numerical integration starting with a given value of ϕ_0 and obtaining the solution $\phi(r; \phi_0)$ from which the leading and subleading terms in the asymptotic expansion, $\mathcal{A}$ and $\mathcal{B}$ can be calculated, thus determining $\kappa(\phi_0)$. In all the cases we have studied, the solution is trivial for any $\kappa > \kappa_c$ while for $\kappa < \kappa_c$ there is a nontrivial solution, leading to a nonzero order parameter in the boundary theory. It is expected (and we can also verify numerically) that κ_c is only a function of m^2 and it is independent of λ_q. Furthermore, as is usual in mean field theory, the critical exponent is determined by the leading nonlinearity. For example, if lowest nonvanishing term is $V(\phi) = \frac{1}{4}\phi^4$ then one expects

$$\langle \mathcal{O} \rangle_{soliton} \sim (\kappa_c - \kappa)^{1/2}. \tag{3.2.22}$$

This is standard mean field behavior. The exponent should not be affected by the presence of nonvanishing λ_q with $q > 2$.

Figure (3.1) shows the result of a numerical solution of the static equations of motion for $d = 3, m^2 = -15/4$ for two potentials : (i) $\lambda_2 = 1$ with all the other λ_q vanishing and (ii) $\lambda_2 = 1, \lambda_3 = 20$ with the other λ_q vanishing

The critical coupling κ_c is found to be $\kappa_c = -0.495129$ which is the same for both potentials and consistent with (3.2.18) and (3.2.19) for this value of d, m^2. Clearly the behavior of the order parameter near the critical point is the same for both potentials while the behavior differs far away from the critical point. Figure (3.2) shows the determination of the critical exponent for bth potentials.

If the leading non-vanishing non-linear term is of $O(\phi^{n+1})$, i.e. $V(\phi) = \frac{1}{n+1}\phi^{n+1} + \cdots$, then we get,

$$\langle \mathcal{O} \rangle_{soliton} \sim (\kappa_c - \kappa)^{\beta} \tag{3.2.23}$$

where $\beta = \frac{1}{(n-1)}$. This is standard mean field multicritical behavior (for a numerical verification see Fig 3.4).

The critical exponent in fact follows from the equation itself. In terms of the redefined field $\psi(\rho, t)$ (see Eqs. (3.2.11) - (3.2.16)), the static equation of motion is

$$\mathcal{D}\psi + G(\rho)\psi^n = 0 \tag{3.2.24}$$

[1] See [91] for a discussion of scalar effective potential in AdS background.

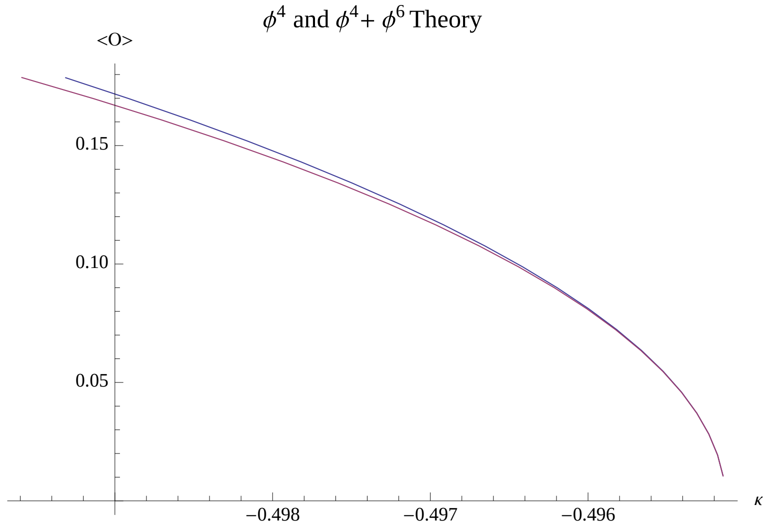


Figure 3.1: The order parameter as a function of κ for ϕ^4 (blue) and $\phi^4 + \phi^6$ (red) theory. The critical value is around -0.495129

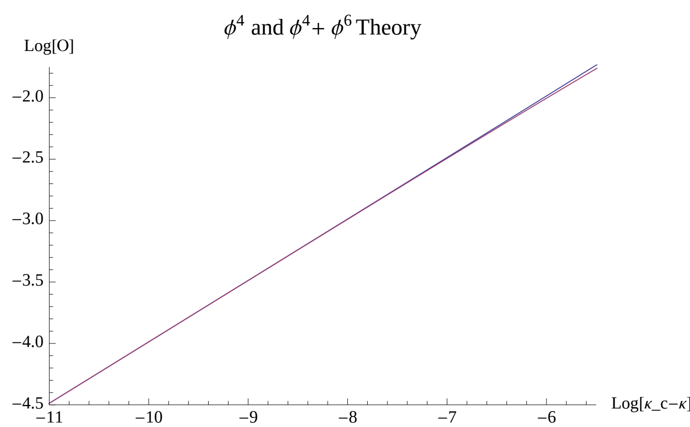


Figure 3.2: Plot of $\log\langle\mathcal{O}\rangle$ versus $\log(\kappa_c - \kappa)$ for ϕ^4 (blue) and $\phi^4 + \phi^6$ (red) theory. The fit for blue line is, $\log\langle\mathcal{O}\rangle = 1.01802 + 0.500572\log(\kappa_c - \kappa)$ and that for the red line is $\log\langle\mathcal{O}\rangle = 0.96698 + 0.495077\log(\kappa_c - \kappa)$

where

$$G(\rho) \equiv \frac{r^{2-d}}{f_s(r)^{1/2}} \tag{3.2.25}$$

Near $\kappa = \kappa_c$ the solution itself is small and may be expanded as

$$\psi(\rho;\kappa) = \epsilon^\beta \left(\psi^{(0)}(\rho) + \epsilon\psi^{(1)}(\rho) + \epsilon^2\psi^{(2)}(\rho) + \cdots\right) \tag{3.2.26}$$

where

$$\epsilon \equiv (\kappa_c - \kappa) \tag{3.2.27}$$

where the number β has to be determined by substituting the expansion in (3.2.25) and equating terms order by order in ϵ. This may be easily seen to determine $\beta = \frac{1}{(n-1)}$.

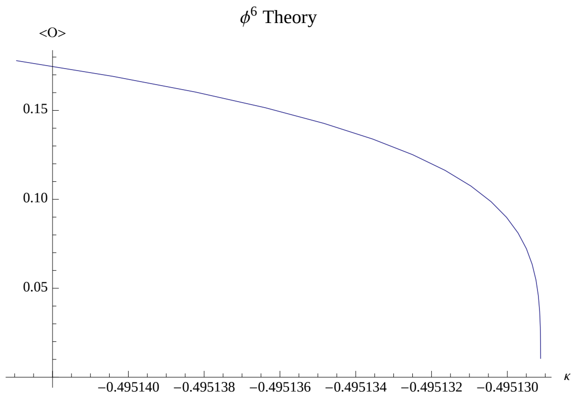


Figure 3.3: The order parameter as a function of κ for ϕ^6 theory. The critical value is around -0.495129.

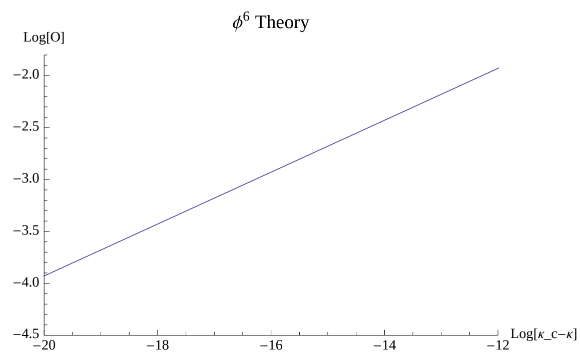


Figure 3.4: Plot of $\log\langle\mathcal{O}\rangle$ versus $\log(\kappa_c - \kappa)$ for ϕ^6 theory. The fit is $\log\langle\mathcal{O}\rangle = 1.07161 + 0.250041\log(\kappa_c - \kappa)$

3.2.3 AdS_{d+2} Black Brane

The equilibrium solutions for the AdS_{d+2} black brane are identical to those for the AdS_{d+2} soliton with the replacement $r_0 \to \bar{r}_0$. This is clear from the full equation (3.2.1) and the form of the functions $f_b(r)$ and $f_s(r)$ in (3.1.3) and (3.1.4).

However, the passage to the Schrodinger form of the equations is different, which leads to a different dynamics. As explained below, it is useful to use Eddington-Finkelstein coordinates which are regular at the horizon,

$$u = \bar{\rho} - t \qquad d\bar{\rho} = -\frac{dr}{r^2 f_b(r)} \tag{3.2.28}$$

so that the metric becomes

$$ds^2 = r^2(d\vec{x}^2 + dw^2) - 2dudr - r^2 f_b(r)du^2 \tag{3.2.29}$$

In terms of fields

$$\chi(u,\bar{\rho}) = r^{d/2}\phi(r,t) \tag{3.2.30}$$

The full equation of motion becomes

$$-2\partial_u\partial_{\bar\rho}\chi = \mathcal{P}\chi + \bar G(\bar\rho)\chi^3 \tag{3.2.31}$$

where

$$\begin{aligned}
\mathcal{P} &\equiv -\partial_{\bar\rho}^2 + \bar V_0(\bar\rho) \\
\bar V_0(\bar\rho) &\equiv r^2 f_b(r)\left[\frac{d}{2} r\partial_r f_b(r) + \frac{d(d+2)}{4} f_b(r) + m^2\right] \\
\bar G(\bar\rho) &\equiv \frac{f_b(r)}{r^{d-2}}
\end{aligned} \tag{3.2.32}$$

In the following we will use this form of the equations of motion to examine the dynamics.

The discussion of multicritical points is exactly the same as that in the AdS soliton background in the previous section and will not be repeated here.

3.3 Slow Quench with a time dependent κ in AdS_{d+2} soliton background

We now study the response of the system in the AdS_{d+2} soliton background to a time dependent κ which starts off *slowly* at early times in the ordered phase $\kappa_i < \kappa_c$, crosses κ_c and asymptotes at late times for some other value $\kappa_f > \kappa_c$. The details of the protocol are not important - however the manner in which the coupling crosses the critical value is relevant. We consider a quench which is linear near $\kappa \sim \kappa_c$, though all the considerations can be trivially extended to nonlinear quenches. For concreteness we consider the protocol

$$\kappa(t) = \kappa_c + a\tanh(vt) \tag{3.3.1}$$

with $v \ll 1$. Note that we are using units $r_0 = 1$ so what we really mean is that $v \ll r_0$.

3.3.1 Breakdown of Adiabaticity

At early times, the response of the system is adiabatic. The solution to the equation of motion (3.2.24) can be then obtained in an adiabatic expansion

$$\psi(\rho, t; \kappa) = \psi_0(\rho; \kappa(t)) + \epsilon\psi_1(\rho, t; \kappa) + \epsilon^2\psi_2(\rho, t; \kappa) + \cdots, \tag{3.3.2}$$

where the static solution is denoted by $\phi_0(r; \kappa)$ and ϵ is the adiabaticity parameter. In the left hand side of equation (3.2.24) we now need to replace $\partial_t \to \epsilon\partial_t$ and equate terms with the same power of ϵ. The n-th order contribution to the solution, ϕ_n satisfies a *linear, inhomogeneous* equation with the source being determined by the ϕ_m with $m < n$.

For the ϕ^4 theory the two lowest order corrections satisfy

$$\left[\mathcal{D} + 3G(\rho)\psi_0^2\right]\psi_1 = 0$$

$$\left[\mathcal{D} + 3G(\rho)\psi_0^2\right]\psi_2 \;\;=\;\; -\partial_t^2\psi_0 - 3G(\rho)\psi_1^2\psi_0 \tag{3.3.3}$$

Note that ψ_n for $n > 0$ satisfy vanishing boundary conditions at infinity and regularity conditions in the interior. Since ψ_1 satisfies a homogeneous equation there is no nontrivial solution. The lowest order correction to the adiabatic solution is therefore ψ_2

$$\psi_2(\rho, t; \kappa) = -\int_0^{\rho_\star} d\rho' \mathcal{G}(\rho, \rho')\partial_t^2\psi_0(\rho', \kappa(t)) \tag{3.3.4}$$

where $\mathcal{G}(\rho, \rho')$ is the Green's function of the operator $[\mathcal{D} + 3G(\rho)\psi_0^2]$.

Exactly at $\kappa = \kappa_c$ the operator $\mathcal{D}$ has a zero mode. At this point the Green's function diverges and adiabaticity fails. As found in the previous section, the smallest eigenvalue for a κ close to κ_c is proportional to $(\kappa - \kappa_c)$. Furthermore we also found that $\psi_0 \sim (\kappa_c - \kappa)^{1/2}$. Thus the lowest eigenvalue of the entire operator $[\mathcal{D} + 3G(\rho)\psi_0^2]$ is proportional to $(\kappa_c - \kappa)$. This gives an estimate of ψ_2 as we approach the critical point,

$$\psi_2 \sim \frac{1}{\kappa_c - \kappa(t)}\partial_t^2\sqrt{\kappa_c - \kappa(t)} = \frac{1}{2(\kappa_c - \kappa)^{3/2}}\left[\partial_t^2\kappa(t) + \frac{(\partial_t\kappa)^2}{\kappa_c - \kappa(t)}\right] \tag{3.3.5}$$

The adiabatic expansion breaks down once $\psi_2 \sim \psi_0$ which leads to the condition

$$\left[\partial_t^2\kappa(t) + \frac{(\partial_t\kappa)^2}{\kappa_c - \kappa(t)}\right] \sim (\kappa_c - \kappa)^2 \tag{3.3.6}$$

For the protocol like (3.3.1), or any other protocol which is linear in time as it crosses κ_c this leads to the estimate for the time when adiabaticity fails, t_{ad}

$$t_{ad} \sim v^{-1/3} \tag{3.3.7}$$

At this time the value of the order parameter is then

$$\langle\mathcal{O}\rangle \sim (v t_{ad})^{1/2} \sim v^{1/3} \tag{3.3.8}$$

This analysis can be easily repeated for multicritical points - this will be discussed in detail in a separate section.

3.3.2 Dynamics in the critical region

Once adiabaticity is broken there is no power series expansion in v. We will now show that there is nevertheless an expansion for small v, but in fractional powers of v. To see this let us rescale

$$\psi(\rho, t) = v^{1/3}\varphi(\rho, t) \qquad\qquad t = v^{-1/3}\eta \tag{3.3.9}$$

The equation of motion (3.2.15) now becomes

$$\mathcal{D}\varphi = v^{2/3}\left(-\partial_\eta^2\varphi - G(\rho)\varphi^3\right) \tag{3.3.10}$$

Now decompose the field in terms of eigenfunctions of $\mathcal{D}$

$$\begin{aligned}\varphi(\rho,\eta) &= \sum_n \chi_n(\rho)\xi_n(\eta) \\ \mathcal{D}\chi_n &= \lambda_n(\kappa)\chi_n \end{aligned} \tag{3.3.11}$$

The eigenvalues of course depend on the boundary conditions. We have expressed this explicitly by denoting them by $\lambda_n(\kappa)$. The equation (3.3.10) becomes

$$\lambda_n(\kappa)\xi_n(\eta) = v^{2/3}\left(-\partial_\eta^2\xi_n - C^n_{m_1,m_2,m_3}\xi_{m_1}\xi_{m_2}\xi_{m_3}\right) \tag{3.3.12}$$

where

$$C^n_{m_1,m_2,m_3} \equiv \int_0^{\rho_\star} d\rho G(\rho)\varphi_n^\star(\rho)\varphi_{m_1}(\rho)\varphi_{m_2}(\rho)\varphi_{m_3}(\rho) \tag{3.3.13}$$

In the previous section we showed explicitly that the lowest eigenvalue of $\mathcal{D}$ is of order $(\kappa_c - \kappa)$. In fact generically for theories with $\nu = 1/2$

$$\lambda_n(\kappa) = \lambda_n(\kappa_c) - c_n(\kappa_c - \kappa) + O[(\kappa_c - \kappa)^2] \qquad c_n > 0 \tag{3.3.14}$$

Using the fact that

$$\kappa_c - \kappa(t) \sim -a(vt) = -av^{2/3}\eta \tag{3.3.15}$$

in the critical region, the equation (3-12) becomes

$$\lambda_n(\kappa_c)\xi_n(\eta) = v^{2/3}\left(-\partial_\eta^2\xi_n - ac_n\eta\xi_n - C^n_{m_1,m_2,m_3}\xi_{m_1}\xi_{m_2}\xi_{m_3}\right) \tag{3.3.16}$$

The boundary condition gives rise to a time dependent mass term in the equation for the mode functions. Recall that $\lambda_0(\kappa_c) = 0$. The dominance of this zero mode for small v is manifest in this equation. All the other modes are at least $O(v^{2/3})$. The zero mode satisfies an effective Landau-Ginsburg dynamics,

$$\partial_\eta^2\xi_0 + c_0\eta + C^0_{000}\xi_0^3 = 0 \tag{3.3.17}$$

The order parameter, which is given in terms of the asymptotic behavior of the field, also satisfies this equation to lowest order. Reverting to the original variables we see that the order parameter as a function of time has the scaling behavior

$$\langle\mathcal{O}\rangle(t;v) = v^{1/3}\langle\mathcal{O}\rangle(tv^{1/3};1) \tag{3.3.18}$$

The dynamics is relativistic and, as will be discussed in a later section, consistent with $z = 1$ Kibble Zurek scaling.

Once again the scaling solution for multicritical points follow along similar lines, as discussed below.

3.4 Slow quench with a time dependent κ: AdS_{d+2} black brane

The analysis for the response to a slow quench in a black brane background is quite similar to above, but the results are rather different. We will not detail the analysis, but give the essential equations, highlighting the results. The key difference arises from the presence of a horizon in this geometry. We need to impose ingoing boundary conditions at the horizon. Equivalently, in the ingoing Eddington-Finkelstein coordinates we are using we need to impose a regularity at the horizon $r = 1$ (in $\bar{r}_0 = 1$ units) [92, 93].

The time coordinate is now u, so that the protocol is

$$\kappa(u) = \kappa_c + a \tanh(vu) \tag{3.4.1}$$

Note that on the boundary u becomes the same as the usual time t and in fact for any r we have $\partial_t|_r = \partial_u|r$, so that on the boundary this represents a time dependence identical to (3.3.1).

3.4.1 Breakdown of Adiabaticity

Let us first discuss usual critical points (ϕ^4 potential). Since the equation of motion (3.2.31) is first order in u derivatives the first order correction to the adiabatic result is non-vanishing. In the expansion

$$\chi(\bar{\rho}, u; \kappa) = \chi_0(\bar{\rho}, \kappa(u)) + \epsilon\chi_1(\bar{\rho}, u) + \cdots \tag{3.4.2}$$

the first order correction χ_1 satisfies

$$\left[\mathcal{P} + 3\bar{G}(\bar{\rho})\chi_0^2\right]\chi_1 = -2\partial_u\partial_{\bar{\rho}}\chi_0 \tag{3.4.3}$$

An analysis identical to the one between equations (3.3.3) and (3.3.5) then leads to

$$\chi_1 \sim \frac{1}{\kappa_c - \kappa(u)}\partial_u\sqrt{\kappa_c - \kappa(u)} \tag{3.4.4}$$

The condition $\chi_0 \sim \chi_1$ then leads to the adiabaticity breaking time

$$u_{ad} \sim v^{-1/2} \tag{3.4.5}$$

while the expectation value of the operator at this time is

$$\langle\mathcal{O}\rangle \sim v^{1/4} \tag{3.4.6}$$

The extension of these results to multicritical points with the leading term in the potential being ϕ^{n+1} is straightforward, leading to

$$u_{ad} \sim v^{-1/2} \qquad \langle\mathcal{O}\rangle \sim v^{\frac{1}{2(n-1)}} = v^{\beta/2} \tag{3.4.7}$$

We will show below that these results are consistent with the general Kibble-Zurek relations.

3.4.2 Dynamics in The Critical Region

For the ϕ^4 theory we first rescale

$$\chi(\bar{\rho}, u) = v^{1/4}\bar{\chi}(\bar{\rho}, \eta) \qquad u = v^{-1/2}\eta \tag{3.4.8}$$

so that the equation (3.2.31) becomes

$$\mathcal{P}\bar{\chi} = -v^{1/2}\left[2\partial_{\bar{\rho}}\partial_\eta\bar{\chi} + \bar{G}(\bar{\rho})\bar{\chi}^3\right] \tag{3.4.9}$$

Unlike the soliton, the spectrum of $\mathcal{P}$ is now continuous. Therefore the mode decomposition (3.3.11) is replaced by

$$\bar{\chi}(\bar{\rho}, \eta) = \int dk\ \bar{\chi}_k(\bar{\rho}; \kappa)\bar{\xi}^k(\eta) \tag{3.4.10}$$

where

$$\mathcal{P}\bar{\chi}_k(\bar{\rho}; \kappa) = \bar{\lambda}(k; \kappa)\bar{\chi}_k(\bar{\rho}; \kappa) \tag{3.4.11}$$

so that instead of (3.3.12) we get

$$\bar{\lambda}(k; \kappa)\bar{\xi}_k = -v^{1/2}\left[\int dk'\bar{B}_{kk'}\partial_\eta\bar{\xi}^{k'}(\eta) + \int dk_1 dk_2 dk_3\ \bar{C}^k_{k_1k_2k_3}\bar{\xi}^{k_1}\bar{\xi}^{k_2}\bar{\xi}^{k_3}\right] \tag{3.4.12}$$

where

$$\begin{aligned}\bar{B}_{kk'} &= \int d\bar{\rho}\ \bar{\chi}_k(\bar{\rho})\partial_{\bar{\rho}}\bar{\chi}_{k'}(\bar{\rho}) \\ \bar{C}^k_{k_1k_2k_3} &= \int d\bar{\rho}\ \bar{G}(\bar{\rho})\bar{\chi}_k(\bar{\rho})\bar{\chi}_{k_1}(\bar{\rho})\bar{\chi}_{k_2}(\bar{\rho})\bar{\chi}_{k_3}(\bar{\rho})\end{aligned} \tag{3.4.13}$$

Since the operator $\mathcal{P}$ is related to the operator $\mathcal{D}$ in (3.2.11) by a similarity transformation (with the replacement $\bar{r}_0 \to r_0$) the behavior of the eigenvalues $\bar{\lambda}(k; \kappa)$ near $\kappa = \kappa_c$ is the same as that of λ_n in (3.3.14)

$$\bar{\lambda}(k; \kappa) = \bar{\lambda}(k; \kappa_c) - \bar{c}(k)(\kappa_c - \kappa) + O((\kappa_c - \kappa)^2) \tag{3.4.14}$$

and using the time dependence of $\kappa(u)$ near κ_c we get

$$\bar{\lambda}(k; \kappa_c)\bar{\xi}_k = -v^{1/2}\left[a\bar{c}(k)\eta\bar{\xi}^k + \int dk'\bar{B}_{kk'}\partial_\eta\bar{\xi}^{k'}(\eta) + \int dk_1 dk_2 dk_3\ \bar{C}^k_{k_1k_2k_3}\bar{\xi}^{k_1}\bar{\xi}^{k_2}\bar{\xi}^{k_3}\right] \tag{3.4.15}$$

Recall that there is a zero mode at $\kappa = \kappa_c$ where the left hand side of (3.4.15) vanishes. If the spectrum of $\bar{\lambda}(k; \kappa_c)$ were discrete it is clear from (3.4.15) that the zero mode dominates the dynamics. This is what happens for the AdS soliton in the previous section. However the operator $\mathcal{P}$ with $\kappa = \kappa_c$ has a continuous spectrum and one has to careful. This analysis is, however, identical to that of [**?**].

The equation (3.4.15) suggests a solution which is an expansion in $v^{1/2}$ as follows

$$\bar{\xi}^k(\eta) = \delta(k)\tilde{\xi}^0(\eta) + v^{1/2}\tilde{\xi}^k(\eta) + O(v) \tag{3.4.16}$$

where again to lowest order in small v

$$\begin{aligned} 0 &= \mathcal{B}_{00}\partial_\eta\tilde{\xi}^0(\eta) + a\bar{c}_0\eta\tilde{\xi}^0(\eta) + \bar{C}^0_{000}(\tilde{\xi}^0)^3 \\ \tilde{\xi}^k &= -\frac{1}{\bar{\lambda}(k;\kappa_c)}\left[\mathcal{B}_{k0}\partial_\eta\tilde{\xi}^0(\eta) + a\bar{c}_k\eta\tilde{\xi}^0(\eta) + \bar{C}^k_{000}(\tilde{\xi}^0)^3\right] \end{aligned} \tag{3.4.17}$$

Combining the two equations in (3.4.17) one has

$$\tilde{\xi}^k = -\frac{1}{\bar{\lambda}(k;\kappa_c)}\left[(\mathcal{B}_{k0}-\mathcal{B}_{00})\partial_\eta\tilde{\xi}^0(\eta) + a(\bar{c}_k-\bar{c}_0)\eta\tilde{\xi}^0(\eta) + (\bar{C}^k_{000}-\bar{C}^0_{000})(\tilde{\xi}^0)^3\right] \tag{3.4.18}$$

We know that all the eigenvalues $\bar{\lambda}(k;\kappa_c)$ are positive except the one which is zero. Since these positive eigenvalues form a continuum we can, without loss of generality, write $\lambda(k;\kappa_c) = k^2$. This means that our expansion (3.4.16) is valid only if the quantities $(\mathcal{B}_{k0}-\mathcal{B}_{00}), (\bar{c}_k-\bar{c}_0), (\bar{C}^k_{000}-\bar{C}^0_{000})$ go to zero *at least as fast as* k^2. In a way quite similar to [20] it turns out that this is indeed true - precisely when $\kappa = \kappa_c$. This is shown in detail for a toy model which is quite similar to our case in the appendix B.

We therefore conclude that the dynamics in the critical region is again dominated by the zero mode which now satisfies a Landau-Ginsburg equation with a first order time derivative - the first equation in (3.4.17). This clearly yields a dissipative time evolution. The dissipation is of course due to a finite temperature and is caused by inflow into the horizon. Reverting to the original variables and noting that on the boundary $u = t$, the time of the field theory, we get a scaling solution

$$\langle\mathcal{O}\rangle(t;v) = v^{1/4}\langle\mathcal{O}(tv^{1/2};1)\rangle \tag{3.4.19}$$

This will be shown to be consistent with Kibble-Zurek scaling with $z = 2, \nu = 1/2$.

3.5 Numerical Results

3.5.1 Soliton background

After suitable changes of variables and field redefinitions for simplification, we solved the resulting equation of motion on a Chebyshev grid using pseudo-spectral derivative method. The k-th lattice point on a Chebyshev grid is defined in the following way,

$$\rho_k = \rho_\star\left(1-\cos\frac{k\pi}{N}\right) \tag{3.5.1}$$

where, N denotes the total number of points on the grid. At the center of the soliton we put a regularity condition on the field ϕ.

We dealt with a specific case of the AdS_{d+2} soliton, taking $d = 3$ and $m^2 = -15/4$ on a grid with total number of points, $N = 61$. Setting the mass parameter at the conformal value simplifies the numerics. The numerical calculation of the critical exponent involves following steps :

- First we calculated κ_c using the linear static equation and obtained $\kappa_c \approx -0.495129$.
- Next, we solved the non-linear static equation on the Chebyshev grid iteratively using a $\kappa = \kappa_c - a$ in the boundary condition. a is an arbitrary constant chosen to be $a = 0.1$.
- The above field configuration was used to specify the initial conditions at some early time $t = -t_{max}$ in the full dynamic equation, which was solved using a time dependent κ-profile of the form $\kappa(t) = \kappa_c + a \tanh(vt)$. Near the phase transition point (i.e. $t = 0$) κ behaves linearly like $\kappa \approx \kappa_c + a\, vt$.
- This was done for various values of v. Using small numerical values of v we expect to find the system in a scaling regime. At time $t = 0$ the value of the order parameter, $\langle \mathcal{O} \rangle$ was numerically calculated from the solution and then the suitable plot [see Figure(3.5)] was made to check the scaling.

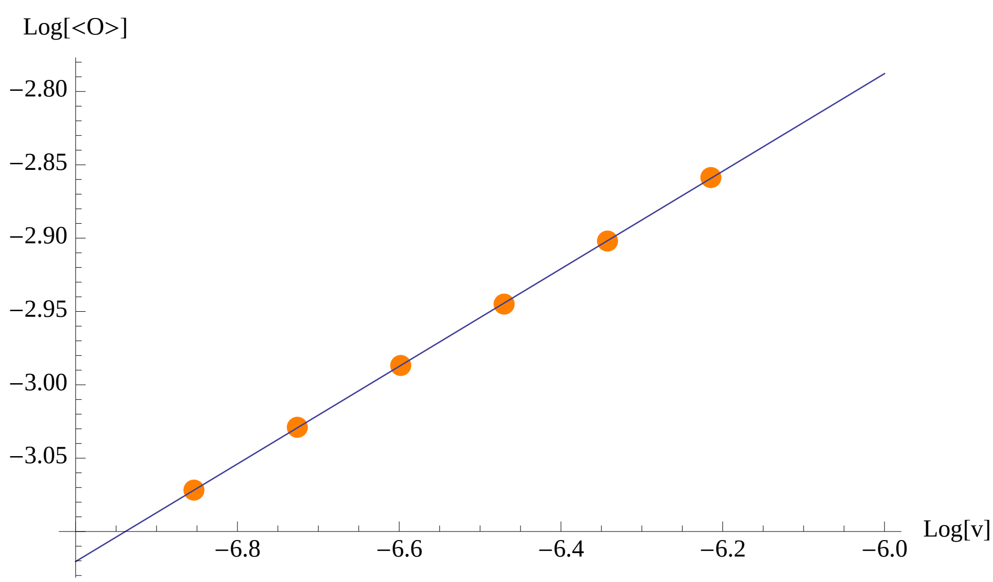


Figure 3.5: The scaling behavior of the order parameter $\mathcal{O}$ as a function of v in in a ϕ^4 theory in AdS soliton geometry. The fit gives, $\ln\langle \mathcal{O} \rangle = -0.791971 + 0.332643 \ln v$.

The above fit clearly confirms our analytical expectation, viz.,

$$\langle \mathcal{O}(0) \rangle \sim v^{1/3} \tag{3.5.2}$$

We also checked that changing $d\kappa$ and N does not significantly change the exponent. To understand the full time dependence and the scaling of time (Eq. 3.3.18) one can plot the scaled response (Fig 3.6).

3.5.2 Black Brane background

Here we solve the PDE's by slightly different method by calculating finite difference derivative on a lattice. We choose lattice size to be $npoints = 500$. The resulting

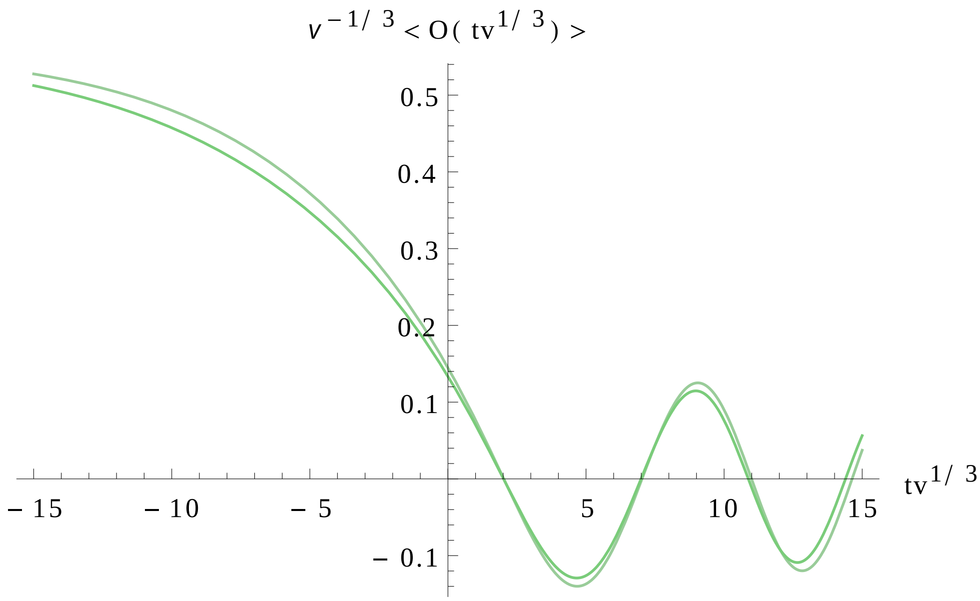


Figure 3.6: The scaling behavior of the order parameter $\mathcal{O}$ as a function of t in in a ϕ^4 theory in AdS soliton geometry. Plots from the top are for $v = 0.03, 0.024$. These plots show scaling consistent with Eq. 3.3.18.

discretized equations are again solved by method of lines. Near the black hole horizon we impose an ingoing boundary condition. The main steps of the numerics, including the value of κ_c and the time dependent profile $k(t)$, are identical to the soliton case : we do not repeat the details. The best fit here [see Figure(3.7)] is given by, $\ln\langle\mathcal{O}\rangle = 0.253967\ln(v) - 0.195079$ which conforms with our analytic result,

$$\langle\mathcal{O}(0)\rangle \sim v^{1/4}. \tag{3.5.3}$$

Like the soliton case, we have also checked that the temporal scaling matches with Eq. 3.4.19.

In the probe approximation the late time behavior of the scalar field in black hole and soliton backgrounds are very different due to presence of the horizon in a black hole background. Any excess energy in bulk is gradually engulfed by the black hole and at very late time we have a almost static scalar profile. The late time decay of excitations of the scalar is determined by the quasi-normal modes. In a soliton background, the excess energy does not dissipate once the quenching is stopped and the scalar field shows temporal oscillation at late time. Our numerics confirm these assertions.

3.6 Arbitrary exponents and Kibble-Zurek Scaling

In this section we discuss the connection of the holographic *derivation* for scaling behavior in critical dynamics with the standard *arguments* leading to Kibble Zurek scaling.

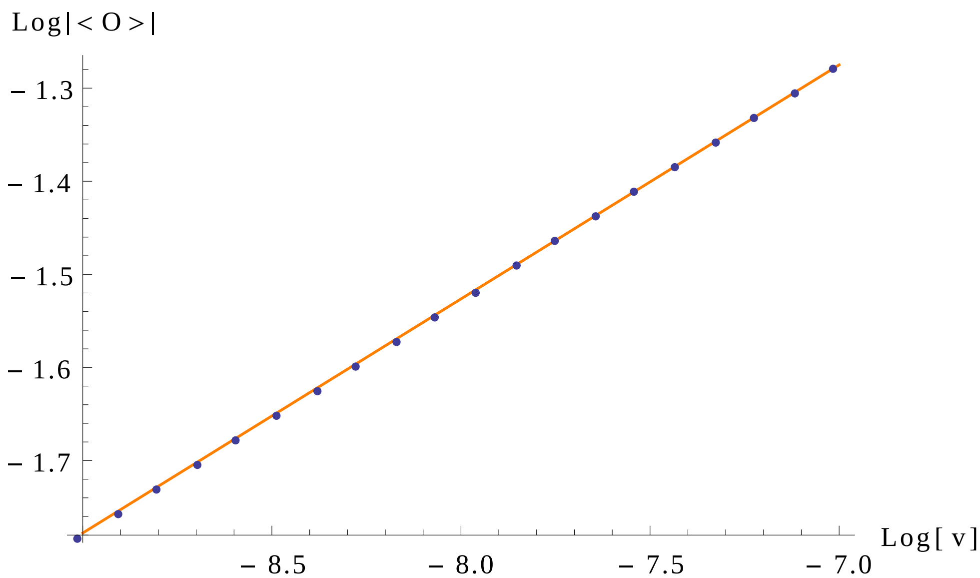


Figure 3.7: The scaling behavior of the order parameter $\mathcal{O}$ as a function of v for a ϕ^4 theory in the AdS blackhole background. The fit gives, $\ln\langle\mathcal{O}\rangle = 0.253967\ln(v) - 0.195079$.

The standard argument for Kibble-Zurek scaling for a quantum critical point proceeds as follows [3, 4, 1, 21, 2]. In the vicinity of such a transition the energy gap Δ depends on the control parameter λ (with the critical value of λ chosen to zero without loss of generality) as $\Delta \sim \lambda^{z\nu}$, where z is the dynamical critical exponent and the ν the correlation length exponents. Consider quenching this control parameter across this transition so that near the critical point $\lambda \sim (vt)^\alpha$. Then the instantaneous value of the energy gap in this region is given by $\Delta_{inst} \sim (vt)^{\alpha z\nu}$. The criteria for the breakdown of adaibaticity during such a quench is [1] $d\Delta/dt \sim \Delta^2$. Substituting the expression for Δ and noting that the critical point is reached at $t = 0$, one finds that the time spent by the system in the non-adiabatic regime is

$$T \sim v^{-\alpha z\nu/(\alpha z\nu+1)} \tag{3.6.1}$$

Next one makes the important assumption that the time evolution after the breakdown of adiabaticity is *diabatic*. This means that one can then argue that the order parameter is determined by the instantaneous value at time T. Furthermore, for slow quenches, the breakdown of adiabaticity occurs in the critical region sufficiently close to the critical point so that one can assume standard critical scaling holds. Since in this region the order parameter $\mathcal{O}$ varies with the control parameter λ as $\mathcal{O} \sim \lambda^\beta$ we get

$$\langle\mathcal{O} >\sim (vT)^{\alpha\beta} \sim v^{\alpha\beta/(\alpha z\nu+1)} \tag{3.6.2}$$

The adiabatic-diabatic assumption is rather drastic. In contrast, the holographic treatment of the present paper as well as that in [20] and [72] we *derived* a set of scaling relations from the properties of the solutions to the bulk equations of motion without any assumption about the nature of time evolution after breakdown of adiabaticity. The physics of the bulk is essential in this derivation, which is not at all transparent in the boundary field theory description. We will now show that the scaling relations we obtained are, nevertheless, consistent with the standard Kibble Zurek results described above.

There are several critical exponents involved in these relations. First, the static exponent β follows from the leading nonlinearity of the bulk potential, as argued in section 2. If the leading term in the potential is ϕ^{n+1} the value of β is given by equation (3.2.23), $\beta = 1/(n-1)$. To find the values of z and ν we need to look at the dispersion relation of small fluctuations around the static solution. The linearized fluctuations would satisfy an equation of the form

$$\partial_t^m \delta\psi = \left[\mathcal{Q} + nF(\rho)\psi_0^{n-1}\right]\delta\psi \tag{3.6.3}$$

where ψ_0 is the static solution, and in the examples described in this paper we have $m = 2, \mathcal{Q} = \mathcal{D}, F(\rho) = G(\rho)$ for the soliton and $m = 1, \mathcal{Q} = \mathcal{P}, F(\rho) = \bar{G}(\bar{\rho})$ for the black hole. The control parameter is $\lambda = (\kappa_c - \kappa)$. The second term on the right hand side is therefore always of the order $O(\lambda)$. The dependence of the first term on λ is determined by the nature of the background. Suppose the smallest eigenvalue of $\mathcal{Q}$ is $O(\lambda^{1/p})$. In both the soliton and the black hole we had $p = 1$: here we have allowed for the possibility of other backgrounds with arbitrary p. Then the energy of excitations E is given by

$$\begin{aligned} E^m &\sim \lambda^{1/p} \qquad p > 1 \\ E^m &\sim \lambda \qquad p < 1 \end{aligned} \tag{3.6.4}$$

From the definition of the standard exponents we therefore have

$$\begin{aligned} z\nu &= \frac{1}{pm} \qquad p > 1 \\ z\nu &= \frac{1}{m} \qquad p < 1 \end{aligned} \tag{3.6.5}$$

Now consider the condition for breakdown of adiabaticity. This again involves a solution of an equation of the form

$$\partial_t^m \psi_0 = \left[\mathcal{Q} + nF(\rho)\psi_0^{n-1}\right]\psi' \tag{3.6.6}$$

where ψ' denotes the leading correction to the adiabatic result. It is then clear that the condition $\psi' \sim \psi_0$ leads to an adiabaticity breaking time t_{ad}

$$\begin{aligned} t_{ad} &\sim (v)^{-\frac{\alpha}{\alpha+pm}} \qquad p > 1 \\ t_{ad} &\sim (v)^{-\frac{\alpha}{\alpha+m}} \qquad p < 1 \end{aligned} \tag{3.6.7}$$

With the value of $(z\nu)$ obtained in (3.6.5) this reproduces the relation (3.6.1). The instantaneous value of ψ, and therefore the order parameter at this time is then clearly seen to be

$$\langle \mathcal{O} \rangle (t_{ad}) \sim (v)^{\frac{m\alpha}{(n-1)(\alpha+m)}} = \mathcal{O}_{ad} \qquad p > 1$$

$$\langle \mathcal{O} \rangle(t_{ad}) \sim (v)^{\frac{mp\alpha}{(n-1)(\alpha+mp)}} = \mathcal{O}_{ad} \quad p < 1 \tag{3.6.8}$$

The value of z requires knowledge of the way space derivatives appear in the equation of motion. In the examples we have discussed in this paper (as well as in [20, 72]) the bulk equation of motion contains two space derivatives. Thus with m time derivatives we have $z = 2/m$. It is then clear that (3.6.8) reproduces (3.6.2) with $\beta = 1/(n-1)$ as derived above.

Once the scalings of t_{ad} and $\mathcal{O}$ are known the rescaling of fields and time required to expose the dynamics in the critical region after breakdown of adiabaticity is clear - we need to perform

$$t \to \eta = \frac{t}{t_{ad}} \qquad \psi \to \chi = \frac{\psi}{\mathcal{O}_{ad}} \tag{3.6.9}$$

The analysis of sections (3.1) and (4.2) can be carried out in a straightforward fashion leading to a scaling solution

$$\langle \mathcal{O} \rangle(t, v) = (v)^{\frac{mp\alpha}{(n-1)(\alpha+mp)}} \langle \mathcal{O} \rangle(t/t_{ad}, 1) \tag{3.6.10}$$

which agrees with the Kibble-Zurek solution obtained earlier.

In the above discussion we have indicated what should be the nature of the bulk theory which leads to nontrivial values of z and ν. In a relativistic bulk theory, we always start with two time derivatives in the equations of motion. However the presence of a gauge field and/or a black hole horizon effectively leads to $m = 1$. Values of $m \geq 3$ would be rather pathological in a bulk theory.

3.7 Remarks

As in [20] and [72] we have demonstrated the emergence of a scaling solution in the critical region in the probe approximation for a quench which is more natural from the boundary field theory point of view. The next obvious step is to study this issue with gravitational backreaction, particularly for the soliton background. Pretty much like global AdS we expect that for a *slow* quench which does not come close to a critical point, a black hole is not formed immediately [33]. A black hole may, however, form a late times [67, 68, 94]. However near the critical point we expect that a breakdown of adiabatic evolution leads to a black hole formation at early times, and it would be interesting to look for critical behavior in this collapse. This would involve coupled partial differential equations - nevertheless we expect that the zero mode will continue to play a key role and dominate critical dynamics. In the full problem, however, it is important to consider potentials which follow from a superpotential [87, 88] so that the static solution is stable. However once again we expect that near the critical point the leading non-linearity determines the dynamics. These issues are being explored at the moment.

Chapter 4

Bi-local Construction of $Sp(2N)$/dS Higher Spin Correspondence

4.1 Introduction and summary

The proposed duality [95] of the singlet sector of the $O(N)$ vector model in three space-time dimensions and Vasiliev's higher spin gauge theory in AdS_4 [96] has received a definite verification[97, 98] and has also thrown valuable light on the origins of holography. Since the field theory is solvable in the large-N limit, one might hope that there is an explicit derivation of the higher spin gauge theory from the vector model, thus providing an explicit understanding of the emergence of the holographic direction. Indeed, the singlet sector of the $O(N)$ model can be expressed in terms of a Hamiltonian for the bi-local collective field, $\sigma(\vec{x},\vec{y}) = \phi^i(\vec{x})\phi^i(\vec{y})$ where $\phi^i(\vec{x}), i = 1 \cdots N$ is the $O(N)$ vector field . In [99] it was proposed that Vasiliev's fields are in fact components of $\sigma(\vec{x},\vec{y})$. The precise connection between the bi-local and HS bulk fields was written explicitly in the light cone frame[100–102] : the correspondence in general involves a nonlocal transformation corresponding to a canonical transformation in phase space. This provides a direct understanding of the emergence of a holographic direction from the large-N degrees of freedom, in a way similar to the well known example of the $c = 1$ Matrix model [103]. In both these models, the large-N degrees of freedom gave rise to an additional dimension which had to be interpreted as a *spatial* dimension [1].

In contrast to AdS/CFT correspondence, any dS/CFT correspondence [9, 10] involves an emergent holographic direction which is *timelike*. It is then of interest to understand how a *timelike* dimension is generated from large-N degrees of freedom. Recently, Anninos, Hartman and Strominger [107] put forward a conjecture that the *euclidean* $Sp(2N)$ vector model in three dimensions is dual to Vasiliev higher spin theory in four dimensional de Sitter space.

In this work we construct a collective field theory of the *Lorentzian* $Sp(2N)$ model which captures the singlet state dynamics of the $Sp(2N)$ vector model. Using the results of [99] and [100] we then argue that a natural interpretation of the resulting action is by double analytic continuation which makes the emergent direction time-like, relating this to higher spin theory in dS_4, in a way reminiscent of the way the Louiville mode in worldsheet string theory has to be interpreted as a time beyond critical dimensions [108]. Our map establishes the bi-local theory as the bulk space-time representation of de Sitter higher spin gravity.

The bilocal collective field is a composite of two Grassmann variables and therefore might not appear to be a genuine bosonic field. In particular for finite N a sufficiently

[1] Other instances of emergence of dimensions from large-N degrees of freedom, e.g. Eguchi-Kawai models [104], Matrix Theory [105, 106] also lead to spatial directions in Lorentzian signature or Euclidean theories.

large power of the field operator vanishes, reflecting its Grassmannian origin [2]. This is further reflected on the size of its Hilbert space. The bulk theory cannot be a usual bosonic theory defined on dS space, though it may be regarded as such in a perturbative $1/N$ expansion.

The implementation of the Grassmann origin of the Hilbert space will be given a central attention in the present work. For this we will describe a geometric (pseudo-spin) version of the collective theory which will be seen to incorporate these effects. For dS/CFT, this implies that the true number of degrees of freedom in the dual higher spin theory in dS is in this framework reduced from what is seen perturbatively (with $G = R_{dS}^2/N$ being the coupling constant squared). The issue of the size of the Hilbert space is of central relevance for possible accounting of entropy of de Sitter space. For pure Gravity in de Sitter space, it has argued that the Entropy being $S = A/4G$ with a finite area of the horizon requires a finite dimensional Hilbert space [110–112]. Interesting quantum mechanical models have been proposed [111, 113–115] to account for this. But apparent conflicts between a finite entropy of de Sitter space with the usual formulations of dS/CFT have been discussed for example in in [116, 117]. In the present case of dS/CFT we are dealing with N-component quantum field theory with d=3 dimensional space so clearly the number of degrees of freedom must be infinite. Consequently the question of Entropy remains open and is an interesting topic for further investigations.

4.2 The $Sp(2N)$ vector model

The $Sp(2N)$ vector model in d spacetime dimensions is defined by the action

$$S = i \int dt d^{d-1}x \; \left[\{ \partial_t \phi_1^i \partial_t \phi_2^i - \nabla \phi_1^i \nabla \phi_2^i \} - V(i\phi_1^i \phi_2^i) \right] \tag{4.2.1}$$

where ϕ_1^i, ϕ_2^i with $i = 1 \cdots N$ are N pairs of Grassmann fields. This is of course a model of ghosts.

In this section we will quantize this model following [118, 119] and [120]. In this quantization, the fields ϕ_1^i and ϕ_2^i are hermitian operators, while the canonically conjugate momenta

$$P_1^i = i\partial_t \phi_2^i \;\; , \qquad\qquad P_2^i = -i\partial_t \phi_1^i \tag{4.2.2}$$

are anti-hermitian. The Hamiltonian H is hermitian

$$H = i \int d^{d-1}x \left[P_2^i P_1^i + \nabla \phi_1^i \nabla \phi_2^i + V(i\phi_1^i \phi_2^i) \right] \tag{4.2.3}$$

The (equal time) canonical anticommutation relations are

$$\begin{aligned} \{\phi_i^a(\vec{x}), P_j^b(\vec{y})\} &= -i\delta_{ij}\delta^{ab}\delta^{d-1}(\vec{x} - \vec{x}') \\ \{\phi_a^i(\vec{x}), \phi_b^j(\vec{y})\} &= \{P_a^i(\vec{x}), P_b^j(\vec{y})\} = 0 \;\; , \qquad\qquad (a, b = 1, 2) \end{aligned} \tag{4.2.4}$$

[2]This property of higher spin currents has been already recognized in [109]

with all other anticommutators vanishing. With these anticommutators the equations of motion for the corresponding Heisenberg picture operators

$$\partial_t^2 \phi_a^i - \nabla^2 \phi_a^i + V' = 0 \tag{4.2.5}$$

follow. The operator relations (4.2.4) allow a representation of the operators as follows

$$\phi_i^a(\vec{x}) \to \phi_i^a(\vec{x}) \quad , \qquad P_i^a \to -i\frac{\delta}{\delta\phi_i^a(\vec{x})} \tag{4.2.6}$$

where ϕ_a^i are now Grassmann fields.

For the free theory, the solution to the equation of motion is

$$\phi_a^i(\vec{x},t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}\sqrt{2|k|}} \left[\alpha_a^i(\vec{k})e^{-i(|k|t-\vec{k}\cdot\vec{x})} + \alpha_a^{i\dagger}(\vec{k})e^{i(|k|t-\vec{k}\cdot\vec{x})}\right] \tag{4.2.7}$$

and the operators α_a^i satisfy

$$\{\alpha_1^i(\vec{k}), \alpha_2^{\dagger j}(\vec{k}')\} = i\delta^{ij}\delta(\vec{k}-\vec{k}') \quad , \qquad \{\alpha_1^{\dagger i}(\vec{k}), \alpha_2^j(\vec{k}')\} = -i\delta^{ij}\delta(\vec{k}-\vec{k}') \tag{4.2.8}$$

with all the other anticommutators vanishing. The Hamiltonian is given by

$$H = i\int [d\vec{k}]\ |\vec{k}| \left[\alpha_1(\vec{k})^\dagger \alpha_2(\vec{k}) - \alpha_2(\vec{k})^\dagger \alpha_1(\vec{k})\right] \tag{4.2.9}$$

The basic commutators lead to

$$[H, \alpha_a^i(k)] = -k\alpha_a^i(k) \quad , \qquad [H, \alpha_a^{i\dagger}] = k\alpha_a^{i\dagger}(k) \tag{4.2.10}$$

To discuss the quantization of the free theory it is useful to review the quantization of the $Sp(2N)$ oscillator, following [120] [3]. The Hamiltonian is

$$H = i(-\frac{\partial^2}{\partial\phi_2^i\partial\phi_1^i} + k^2\phi_1^i\phi_2^i) \tag{4.2.11}$$

where ϕ_1^i, ϕ_2^i are N pairs of Grassmann numbers. Because of the Grassmann nature of the variables the spectrum of the theory is bounded both from below and from above. The oscillators are defined by (in the Schrodinger picture)

$$\phi_a^i = \frac{1}{\sqrt{2k}}[\alpha_a^i + \alpha_a^{i\dagger}] \tag{4.2.12}$$

while the momenta are

$$P_a^i = \epsilon_{ab}\sqrt{\frac{k}{2}}(\alpha_b^i - \alpha_b^{i\dagger}) \tag{4.2.13}$$

The ground state $|0\rangle$ and the highest state $|2N\rangle$ are then given by the conditions

$$\alpha_a^i|0\rangle = 0 \quad , \qquad \alpha_a^{i\dagger}|2N\rangle = 0 \tag{4.2.14}$$

[3] Note that our notation is different from that of [120]

with the wavefunctions

$$\Psi_0 = \exp[-ik\phi_1^i\phi_2^i] \quad , \qquad\qquad \Psi_{2N} = \exp[ik\phi_1^i\phi_2^i] \tag{4.2.15}$$

and the energy spectrum is given by

$$E_n = k[n-N] \quad , \qquad\qquad n = 0, 1, \cdots, 2N \tag{4.2.16}$$

Finally, the Feynman correlator of the Grassmann coordinates may be easily seen to be

$$\langle 0|T[\phi_1^i(t)\phi_2^j(t')]|0\rangle = \frac{i\delta^{ij}}{2k}\ e^{-ik|t-t'|} \tag{4.2.17}$$

Extension of these results to the free field theory is straight forward: for each momentum $\vec{k}$, we have a fock space with a finite number of states.

4.3 Collective Field Theory for the $Sp(2N)$ model

In the representation (4.2.6) a general wavefunctional is given by $\Psi[\phi_a^i(\vec{x}), t]$. Our aim is to obtain a description of the singlet sector of the theory, i.e. wavefunctionals which are invariant under the $Sp(2N)$ rotations of the fields $\phi_a^i(\vec{x})$. All the invariants in field space are functions of the bilocal collective fields

$$\rho(\vec{x}, \vec{y}) \equiv i\epsilon^{ab}\phi_a^i(\vec{x})\phi_b^i(\vec{y}) \tag{4.3.1}$$

We have defined this collective field to be hermitian (which is why there is a i in the definition). Clearly $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$. The aim now is to rewrite the theory in terms of a Hamiltonian which is a functional of $\rho(\vec{x}, \vec{y})$ and its canonical conjugate $-i\frac{\delta}{\delta\rho(\vec{x},\vec{y})}$ which acts on wavefunctionals which are functionals of $\rho(\vec{x}, \vec{y})$.

It is important to remember that $\rho(\vec{x}, \vec{y})$ is not a genuine bosonic field. This will have important consequences at finite N. In a perturbative expansion in $1/N$, however, there is no problem [121] in treating $\rho(\vec{x}, \vec{y})$ as a bosonic field.

Before dealing with the $Sp(2N)$ field theory, it is useful to review some aspects of the collective theory for the usual $O(N)$ model, starting with the $O(N)$ oscillator.

4.3.1 Collective fields for the $O(N)$ theory

In this section we review the bi-local collective field theory construction for the $O(N)$ field theory, starting with the $O(N)$ oscillator. This has a Hamiltonian

$$H = \frac{1}{2}[P^iP^i + k^2X^iX^i] \tag{4.3.2}$$

The collective variable is the square of the radial coordinate $\sigma = X^iX^i$ and the Jacobian for transformation from X^i to σ and the angles is

$$J(\sigma) = \frac{1}{2}t\sigma^{(N-2)/2}\Omega_{N-1} \tag{4.3.3}$$

where Ω_{N-1} is the volume of unit S^{N-1}. The idea is to find the Hamiltonain $H(\sigma, \frac{\partial}{\partial\sigma})$ which acts on wavefunctions $[J(\sigma)]^{1/2}\Psi(\sigma)$. The key observation of [122] is that this can also be obtained by requiring that $H(\sigma, \frac{\partial}{\partial\sigma})$ acting on wavefunctions $[J(\sigma)]^{1/2}\Psi(\sigma)$ is hermitian with the trivial measure $d\sigma$. This determines both the Jacobian and the Hamiltonian and the technique generalizes to higher dimensional field theory. The final result is well known,

$$H_{coll} = -2\frac{\partial}{\partial\sigma}\sigma\frac{\partial}{\partial\sigma} + \frac{(N-2)^2}{8\sigma} + \frac{1}{2}k^2\sigma \tag{4.3.4}$$

The large-N expansion then proceeds as usual by expanding around the saddle point solution σ_0 which minimizes the potential [4],

$$\sigma_0^2 = \frac{N^2}{4k^2} \tag{4.3.5}$$

Clearly, we have to choose the positive sign since in this case σ is a *positive* real quantity,

$$\sigma_0 = \frac{N}{2k} \tag{4.3.6}$$

which reproduces the coincident time two point function $\langle 0|X^i(t)X^i(t)|0\rangle$ and the correct ground state energy, $E_0 = \frac{N}{2}k$. The subleading contributions are then obtained by expanding around the saddle point,

$$\sigma = \sigma_0 + \sqrt{\frac{2N}{k}}\eta \quad , \qquad \Pi_\sigma = \sqrt{\frac{k}{2N}}\pi_\eta \tag{4.3.7}$$

The quadratic part of the Hamiltonian becomes

$$H^{(2)} = \frac{1}{2}\left[\pi_\eta^2 + 4k^2\eta^2\right] \tag{4.3.8}$$

This leads to the excitation spectrum to $O(1)$, $E_n = 2nk$ with $n = 0, 1, \cdots, \infty$. The Hamiltonian of course contains all powers of η. *Terms with even number of the fluctuations* (π_η, η) *come with odd factors of* σ_0. This fact will play a key role in the following.

In the following it will be necessary to consider wavefunctions. It follows directly from (4.3.2) that the ground state wavefunction is given by (up to a normalization which is not important for our purposes)

$$\Psi_0(X^i) = \exp[-\frac{k}{2}\sigma] \sim \exp[-\sqrt{\frac{Nk}{2}}\eta] \tag{4.3.9}$$

where we have expanded σ as in (4.3.7), used (4.3.6) and ignored an overall constant. We should get the same result from the collective theory. Recalling that the collective

[4] To see why the saddle point approximation is valid, rescale $\sigma \to N\sigma$ and $\Pi_\sigma \to \frac{1}{N}\Pi_\sigma$ so that there is an overall factor of N in front of the potential energy term. We will, however, stick to the unrescaled fields.

wavefunction is related to the original wavefunction by a Jacobian factor, the ground state wavefunction follows from (4.3.8)

$$\Psi_0'(\eta) = [J(\sigma)]^{-\frac{1}{2}} \exp[-k\eta^2] \tag{4.3.10}$$

The presence of the Jacobian is crucial in obtaining agreement with (4.3.9) [123]. Expanding the argument in the Jacobian in powers of η according to (4.3.7) it is easy to see that the quadratic term in η coming from the Jacobian exactly cancels the explicit quadratic term in (4.3.10) and the linear term in η is in exact agreement with (4.3.9). The expression (4.3.10) of course contain all powers of η once exponentiated - these should also cancel once one takes into account the cubic and higher terms in the collective Hamiltonian as well as finite N corrections which we have ignored to begin with. The above formalism can be easily generalized to an additional invariant potential, since the latter would be a function of σ.

The collective theory for $O(N)$ field theory can be constructed along identical lines. We reproduce the relevant formulae from [122] which are direct generalizations of the formulae for the oscillator. The $O(N)$ model has the Hamiltonian

$$H = \frac{1}{2}\int d^{d-1}x \left[-\frac{\delta^2}{\delta\phi^i(\vec{x})\delta\phi^i(\vec{x})} + \nabla\phi^i(\vec{x})\nabla\phi^i(\vec{x}) + U[\phi^i(\vec{x})\phi^i(\vec{x})]\right] \tag{4.3.11}$$

The singlet sector Hamiltonian in terms of the bi-local collective field $\sigma(\vec{x},\vec{y}) = \phi^i(\vec{x})\phi^i(\vec{y})$ and its canonically conjugate momentum $\Pi_\sigma(\vec{x},\vec{y})$ is, to leading order in $1/N$ [5]

$$H_{coll}^{O(N)} = 2\mathrm{Tr}\left[(\Pi_\sigma\sigma\Pi_\sigma) + \frac{N^2}{16}\sigma^{-1}\right] - \frac{1}{2}\int d\vec{x}\nabla_x^2\sigma(\vec{x},\vec{y})|_{\vec{y}=\vec{x}} + U(\sigma(\vec{x},\vec{x})) \tag{4.3.12}$$

where the spatial coordinates are treated as matrix indices.

So far our considerations are valid for an arbitrary interaction potential U. Let us now restrict ourselves to the free theory, $U = 0$ to discuss the large-N solution explicitly. In momentum space the saddle point solution is

$$\sigma(\vec{k}_1,\vec{k}_2) = \frac{N}{2|\vec{k}_1|}\delta(\vec{k}_1 - \vec{k}_2) \tag{4.3.13}$$

Once again we have chosen the positive sign in the solution of the saddle point equation, and the saddle point value of the collective field agrees with the two point correlation function of the basic vector field, which should be positive. The $1/N$ expansion is generated in a fashion identical to the single oscillator,

$$\sigma(\vec{k}_1,\vec{k}_2) = \sigma_0(\vec{k}_1,\vec{k}_2) + \left(\frac{|\vec{k}_1||\vec{k}_2|}{N(|\vec{k}_1|+|\vec{k}_2|)}\right)^{-\frac{1}{2}}\eta(\vec{k}_1,\vec{k}_2) \ , \ \Pi_\sigma = \left(\frac{|\vec{k}_1||\vec{k}_2|}{N(|\vec{k}_1|+|\vec{k}_2|)}\right)^{\frac{1}{2}}\pi_\eta(\vec{k}_1,\vec{k}_2) \tag{4.3.14}$$

[5] To subleading order there are singular terms which are crucial for reproducing the correct $1/N$ contributions.

the quadratic piece becomes

$$H^{(2)} = \frac{1}{2}\int d\vec{k}_1 d\vec{k}_2 \left[\pi_\eta(\vec{k}_1,\vec{k}_2)\pi_\eta(\vec{k}_1,\vec{k}_2) + (|\vec{k}_1| + |\vec{k}_2|)^2\eta(\vec{k}_1,\vec{k}_2)\eta(\vec{k}_1,\vec{k}_2)\right] \quad (4.3.15)$$

so that the energy spectrum is given by

$$E(\vec{k}_1,\vec{k}_2) = |\vec{k}_1| + |\vec{k}_2| \quad (4.3.16)$$

as it should be. It is easy to check that the unequal time two point function of the fluctuations reproduces the connected part of the two point function of the full collective field as calculated from the free field theory. A nontrivial U can be reinstated easily (see e.g. the treatment of the $(\vec{\phi}^2)^2$ model in [99], which discusses the RG flow to the nontrivial IR fixed point).

4.3.2 Collective theory for the $Sp(2N)$ oscillator

Since there is a representation of the field operator and the conjugate momentum operator of the $Sp(2N)$ theory in terms of Grassmann fields, (4.2.6), it is clear that the derivation of the collective field theory of the $Sp(2N)$ model closely parallels that of the $O(N)$ theory. In this subsection we consider the $Sp(2N)$ oscillator. The Hamiltonian is given by (4.2.11. The collective variable is

$$\rho = i\epsilon^{ab}\phi_a^i\phi_b^i \quad (4.3.17)$$

The fully connected correlators of this collective variable have a simple relationship with those of the $O(2N)$ harmonic oscillator,

$$\langle\rho(t_1)\rho(t_2)\cdots\rho(t_n)\rangle^{conn}_{Sp(2N)} = -\langle\sigma(t_1)\sigma(t_2)\cdots\sigma(t_n)\rangle^{conn}_{SO(2N)} \quad (4.3.18)$$

This result follows from (4.2.17) and the application of Wick's theorem for Grassmann variables.

The collective variable ρ is a Grassmann even variable - it is not an usual bosonic variable. This key fact is intimately related to the finite number of states of the $Sp(2N)$ oscillator. In this section we will show that in a $1/N$ expansion we can nevertheless proceed, defering a proper discussion of this point to a later section.

The Hamiltonian for the collective theory is obtained by the same method used to obtain the collective theory in the bosonic case, with various negative sign coming from the Grassmann nature of the variables. Using the chain rule and taking care of negative signs coming because of Grassmann numbers, one gets the Jacobian $J'(\rho)$ (determined by requiring the hermicity of $J^{-1/2}HJ^{1/2}$)

$$J'(\rho) = A'\ \rho^{-(N+1)} \quad (4.3.19)$$

where A' is a constant. The negative power of ρ of course reflects the Grassmann nature of the variables [6] Despite this difference, the final collective Hamiltonian is in

[6] This ρ dependence of the Jacobian follows from a direct calculation $J'(\rho) = \int d\phi_1^i d\phi_2^i \delta(\rho - i\phi_1^i\phi_2^i) = \int d\lambda e^{i\lambda\rho}\int d\phi_1^i d\phi_2^i\ e^{-i\lambda\phi_1^i\phi_2^i} \sim \rho^{-(N+1)}$

fact *identical* to the $O(2N)$ oscillator collective Hamiltonian

$$H_{coll}^{Sp(2N)} = -2\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{N^2}{2\rho} + \frac{1}{2}k^2\rho \tag{4.3.20}$$

This leads to the same saddle point equation, and the solutions satisfy the same equation as (4.3.5) with $N \to 2N$.

In the $O(2N)$ oscillator, we had to choose the positive sign, since σ is by definition a real *positive* variable. In this case, there is no reason for ρ to be positive. In fact we need to choose the negative sign, since (4.3.18) requires that the one point function of ρ must be the negative of the one point function of σ.

$$\rho_0 = -\frac{N}{k} \tag{4.3.21}$$

It is interesting that the singlet sectors of the $O(2N)$ and $Sp(2N)$ models are described by two different solutions of the *same* collective theory.

The leading order ground state energy is the Hamiltonian evaluated on the saddle point,

$$E_{gs} = -Nk \tag{4.3.22}$$

in agreement with (4.2.16). The fluctuation Hamiltonian is obtained as usual by expanding

$$\rho = \rho_0 + \sqrt{\frac{4N}{k}}\xi \quad , \qquad \Pi_\rho = \sqrt{\frac{k}{4N}}\pi_\xi \tag{4.3.23}$$

The quadratic Hamiltonian is now *negative*, essentially because of the negative sign in the saddle point,

$$H_\xi^{(2)} = -\frac{1}{2}\left[\pi_\xi^2 + 4k^2\xi^2\right] \tag{4.3.24}$$

A standard quantization of this theory leads to a spectrum which is unbounded from below. We will now argue that we need to quantize this theory rather differently, in a way similar to the treatment of [124]. This involves defining annihilation and creation operators $a_\xi, a_\xi^\dagger$

$$\xi = \frac{1}{\sqrt{4k}}[a_\xi + a_\xi^\dagger] \quad , \qquad \pi_\xi = i\sqrt{k}[a_\xi - a_\xi^\dagger] \tag{4.3.25}$$

which now satisfy

$$[a_\xi, a_\xi^\dagger] = -1 \quad , \quad [H, a_\xi] = -2k a_\xi \quad , \quad [H, a_\xi^\dagger] = 2k a_\xi^\dagger \tag{4.3.26}$$

Because of the negative sign of the first commutator in (4.3.26) a standard quantization will lead to a *highest energy* state annihilated by $a_\xi^\dagger$, and then the action of powers of a_ξ leads to an infinite tower of states with lower and lower energies. The highest state has a normalizable wavefunction of the standard form $e^{-k\xi^2}$ (Note that the expression for π_ξ has a negative sign compared to the usual harmonic oscillator). It is easy to see that this standard quantization does not reproduce the correct two-point function of the $Sp(2N)$ theory, does not lead to the correct spectrum (4.2.16) and, as shown below, does not lead to the correct wavefunction.

All this happens because ρ and hence ξ is not really a bosonic variable, and this allows other possibilities. Consider now a state $|0\rangle_\xi$ which is annihilated by the annihilation operator a_ξ. This leads to a wavefunction $\exp[k\xi^2]$, which is inadmissible if ξ is really a bosonic variable since it would be non-normalizable. However the true integration is over the Grassmann partons of these collective fields, and in terms of Grassmann integration this wavefunction is perfectly fine. This is in fact the state which has to be identified with the ground state of the $Sp(2N)$ oscillator. Including the factor of the Jacobian, the full wavefunction is (at large N)

$$\Psi'_{0\xi}[\xi] = [J'(\rho)]^{-1/2}\exp[k\xi^2] = [-\frac{N}{k} + 2\sqrt{\frac{N}{k}}\xi]^{N/2}\exp[k\xi^2] \tag{4.3.27}$$

Expanding the Jacobian factor in powers of ξ one now sees that the term which is quadratic in ξ cancels exactly, leaving with

$$\Psi'_{0\xi}[\xi] = \exp[-\sqrt{Nk}\xi + O(\xi^3)] \tag{4.3.28}$$

This is easily seen to exactly agree with Ψ_0 in (4.2.15)

$$\Psi_0 \sim \exp[-\frac{1}{2}k\rho] \sim \exp[-\sqrt{Nk}\xi] \tag{4.3.29}$$

up to a constant. Once again we need to take into account the interaction terms in the collective Hamiltonian to check that the $O(\xi^3)$ terms cancel. It can be easily verified that the propagator of fluctuations ξ will now be *negative* of the usual harmonic oscillator propagator. Furthermore the action of $a_\xi^\dagger$ now generates a tower of states with the energies (4.2.16) - except that the integer n is not bounded by N.

The fact that we get an unbounded (from above) spectrum from the collective theory is not a surprise. This is an expansion around $N = \infty$ and at $N = \infty$ the spectrum of $Sp(2N)$ is also unbounded. At finite N a change of variables to ρ is not useful because of the constraints coming from the Grassmann origin of ρ. Nevertheless, even in the $1/N$ expansion, the Grassmann origin allows us to consider wavefunctions which would be otherwise considered inadmissible.

The negative propagator ensures that the relationship (4.3.18) is satisfied for the 2 point functions. Once this choice is made, the relationship (4.3.18) holds for all m-point functions to the leading order in the large-N limit. As commented earlier, a term with even number of π_ξ or ξ would have an odd number of factors of ρ_0. Therefore a n-point vertex in the theory will differ from the corresponding n-point vertex of the $O(N)$ theory by a factor of $(-1)^{n+1}$. The connected correlator which appears in (4.3.18) is the sum of all connected tree diagrams with n external legs. The collective theory gives us the following Feynman rules

1 Every propagator contributes to a negative sign.

2 A p point vertex has a factor of $(-1)^{p+1}$

We now argue that these rules ensure the validity of the basic relation (4.3.17). We do it by the following simple diagrammatic method:

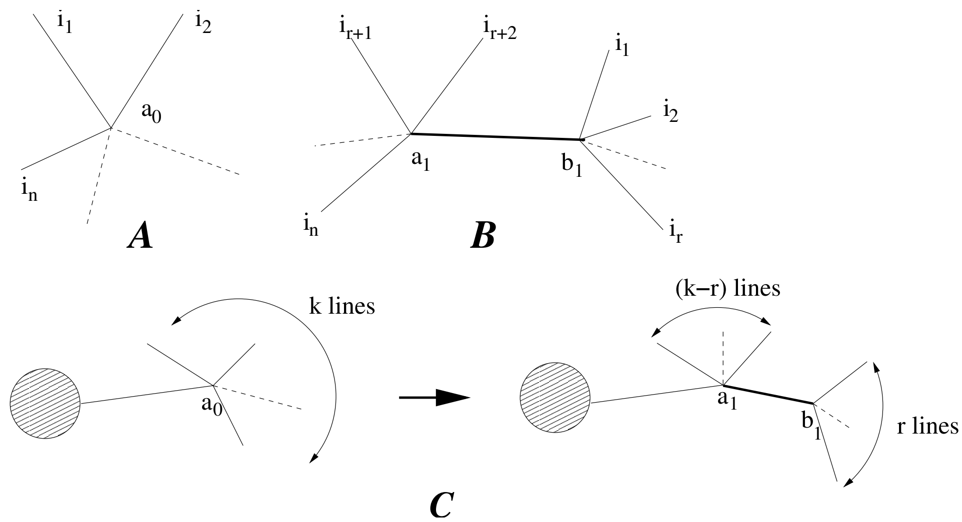


Figure 4.1: Connected tree level correlators of the collective theory

Consider first the simplest diagram for a n-point function, figure A, which is a star graph. The net sign of the diagram is $(-1)^{n+1} \times (-1)^n = -1$, where the first factor is from the vertex a_0 and the second one from the number of lines. Now we proceed to construct all other tree level diagrams from A, by pulling 'r' lines resulting in figure B, which now has vertices, a_1 and b_1 joined by a new line. It is easy to see, that the sign of figure A is not changed by this operation. The net sign of figure B is $(-1)^{(n-r+1)+1} \times (-1)^{(r+1)+1} \times (-1)^{(n+1)} = -1$, where the 3 factors are from a_1, b_1 and the number of lines respectively. In figure C we repeat this method for the substar diagrams until we exhaust all possibilities. It is easy to see that the sign stays invariant. Assigning a sign α to the blob, we first find the net sign of the left diagram in figure C. It turns out to be, $\alpha \times (-1)^{(k+1)+1} \times (-1)^{k+1} = -\alpha$. After the "pulling" operation we get $\alpha \times (-1)^{(k-r+2)+1} \times (-1)^{(r+1)+1} \times (-1)^{k+1+1} = -\alpha$. Thus it is proved that in every move the sign is preserved. This proves the relationship (4.3.18) for all correlation functions.

4.3.3 $Sp(2N)$ Correlators

Our discussion of the bosonic $O(N)$ collective field theory shows that the $Sp(2N)$ collective field theory in momentum space is a straightforward generalization. In this subsection we discuss the relevant features of the collective theory for the free $Sp(2N)$ model.

The collective Hamiltonian is again exactly the same as in the $O(N)$ theory, given by (4.3.12) with $\sigma \to \rho$. Since the connected correlators of the collective fields satisfy

$$\langle \rho(\vec{k}_1, \vec{k}'_1, t_1)\rho(\vec{k}_2, \vec{k}'_2, t_2) \cdots \rho(\vec{k}_n, \vec{k}'_n, t_n)\rangle^{conn}_{Sp(2N)} = -\langle \sigma(\vec{k}_1, \vec{k}'_1, t_1)\sigma(\vec{k}_2, \vec{k}'_2, t_2) \cdots \sigma(\vec{k}_n, \vec{k}'_n, t_n)\rangle^{conn}_{SO(2N)} \tag{4.3.30}$$

we now need to choose the negative saddle point,

$$\rho_0(\vec{k}, \vec{k}', t) = -\frac{N}{|\vec{k}|}\delta(\vec{k} - \vec{k}') \tag{4.3.31}$$

The fluctuation Hamiltonian once again has a factor of $(-1)^{n+1}$ for the n-point vertex. In particular, the propagator of the collective field is negative of that of the $O(N)$ collective field - the quadratic Hamiltonian has an overall negative sign! This is required - the diagramatic argument for the $Sp(2N)$ oscillator generalizes in a straightforward fashion, ensuring that (4.3.30) holds.

4.4 Bulk Dual of the $Sp(2N)$ model

In [99], it was proposed that the collective field theory for the d dimensional free $O(N)$ theory is in fact Vasiliev's higher spin theory in AdS$_{d+1}$. It is easy to see that the collective field has the right collection of fields. Consider for example $d = 3$. The field depends on four spatial variables, which may be reorganized as three spatial coordinates one of which is restricted to be positive and an angle. A fourier series in the angle then gives rise to a set of fields $\chi_{\pm n}$ which depend on three spatial variables, with the integer n denoting the conjugate to the angle. Symmetry under interchange of the arguments of the collective field then requires n to be even integers. But this is precisely the content of a theory of massless even spin fields in four space-time dimensions, with n labelling the spin and the two signs corresponding to the two helicities. (Recall that in four space-time dimensions massless fields with any spin have just two helicity states).

The precise relationship between collective fields and higher spin fields in AdS was found in [100] which we now summarize for $d = 3$. The correspondence is formulated in the light front quantization. Denote the usual Minkowski coordinates on the space-time on which the $O(N)$ fields live by t, y, x and define light cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(t \pm y) \tag{4.4.1}$$

The conjugate momenta to x^+, x^- are denoted by p^-, p^+. Then in light front quantization where x^+ is treated as time, the Schrodinger picture fields are $\phi^i(x^-, x)$ while the momentum space fields are given by $\phi^i(p^+, p)$. The corresponding collective field is then defined as

$$\sigma(p_1^+, p_1; p_2^+, p_2) = \phi^i(p_1^+, p_1)\phi^i(p_2^+, p_2) \tag{4.4.2}$$

The fluctuation of this field around the saddle point is denoted by $\Psi(p_1^+, p_1; p_2^+, p_2)$. Now define the following bilocal field

$$\Phi(p^+, p^x, z, \theta) = \int dp^z dp_1^+ dp_2^+ dp_1 dp_2 \; K(p^+, p^x, z, \theta; p_1^+, p_1, p_2^+, p_2)\Psi(p_1^+, p_1; p_2^+, p_2) \tag{4.4.3}$$

where the kernel is given by

$$\begin{aligned} K(p^+, p^x, z, \theta; p_1^+, p_1, p_2^+, p_2) \;\; &= \;\; z\, e^{izp_z}\; \delta(p_1^+ + p_2^+ - p^+)\; \delta(p_1 + p_2 - p) \\ &\quad \delta(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z)\; \delta(2\tan^{-1}\sqrt{\frac{p_2^+}{p_1^+}} - \theta) \end{aligned}$$

In [100] it was shown that the Fourier transforms of the field $\Phi(p^+, p^x, z, \theta)$ with respect to θ satisfy the same linearized equation of motion as the physical helicity modes of higher spin gauge fields in AdS$_4$ in light cone gauge. The metric of this AdS$_4$ is given by the standard Poincare form

$$ds^2 = \frac{1}{z^2}[-2dx^+dx^- + dx^2 + dz^2] = \frac{1}{z^2}[-dt^2 + dy^2 + dx^2 + dz^2] \tag{4.4.4}$$

The momenta p^+, p are conjugate to x^-, x. The additional dimension generated from the large-N degrees of freedom is z, which is canonically conjugate to p^z and is given in terms of the phase space coordinate of the bi-locals by

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} \tag{4.4.5}$$

In particular, the linearized equation for the spin zero field, $\varphi(x^-, x, z)$, follows from the quadratic action

$$S = \frac{1}{2}\int dx^+dx^-dzdx \left[\frac{1}{z^2}\left(-2\partial_+\varphi\partial_-\varphi - (\partial_x\varphi)^2 - (\partial_z\varphi)^2\right) + \frac{2}{z^4}\varphi^2\right] \tag{4.4.6}$$

which is of course the action of a conformally coupled scalar in the AdS$_4$ with coordinates given by (4.4.5). The actions for the spin-$2s$ fields can be similarly written down. Even though these actions are derived using light cone coordinates, they can be covariantized easily since these are free actions. In terms of the coordinates t, y, x, z the scalar action is given by

$$S = \frac{1}{2}\int dtdzdxdy \left[\frac{1}{z^2}\left((\partial_t\varphi)^2 - (\partial_y\varphi)^2 - (\partial_x\varphi)^2 - (\partial_z\varphi)^2\right) + \frac{2}{z^4}\varphi^2\right] \tag{4.4.7}$$

Let us now turn to the $Sp(2N)$ collective theory. One can define once again the fields as in (4.4.3) and (4.4.4). The coordinates (x^+, x^-, x, z) will continue to transform appropriately under AdS isometries. However, we saw earlier that the quadratic part of the Hamiltonian, and therefore the quadratic part of the action will have an overall *negative* sign.

A negative kinetic term signifies a pathology. Indeed we derived this theory with the Lorentzian signature $Sp(2N)$ model, which has negative norm states. The negative kinetic term of the collective theory is possibly intimately related to this lack of unitarity.

However, the form of the action (4.4.7) cries out for a *analytic continuation*

$$z = i\tau \quad , \qquad t = -iw \tag{4.4.8}$$

Under this continuation the action, S becomes

$$S' = \frac{1}{2}\int d\tau dwdxdy \left[\frac{1}{\tau^2}\left((\partial_\tau\varphi)^2 - (\partial_y\varphi)^2 - (\partial_x\varphi)^2 - (\partial_w\varphi)^2\right) - \frac{2}{\tau^4}\varphi^2\right] \tag{4.4.9}$$

The sign of the mass term *has not changed* in this analytic continuation, and this action has become the action of a conformally coupled scalar field in de Sitter space with the metric

$$ds^2 = \frac{1}{\tau^2}[-d\tau^2 + dx^2 + dy^2 + dw^2] \qquad (4.4.10)$$

This mechanism works for all *even* higher spin fields at the quadratic level.

To summarize, the collective field theory of the three dimensional Lorentzian $Sp(2N)$ model can be written as a theory of massless even spin fields in AdS_4, but with negative kinetic terms. Under a double analytic continuation this becomes the action in dS_4 with positive kinetic terms. This is consistent with the conjecture of [107] that the *euclidean* $Sp(N)$ model is dual to Vasiliev theory in dS_4. It is interesting to note that the way an emergent holographic direction is similar to the way the Liouville mode has to be interpeted as a time dimension in worldsheet supercritical string theory [108]. In this latter case, the sign of the kinetic term for the Liouville mode is negative for $d > d_{cr}$.

Even for the $O(N)$ model, the collective field is an represents seemingly an over-complete description, since for a finite number of points in space K, one replaces at most NK variables by K^2 variables, which is much larger in the thermodynamic and continuum limit. However, in the perturbative $1/N$ expansion this is not an issue and the collective theory is known to reproduce the standard results of the $O(N)$ model. The issue becomes of significance at finite N level.The relevance of incorporating for such features has been noted in [109, 125].

For the fermionic $Sp(2N)$ model, there appears potentially an even more important redundancy related to the Grassmannian origin of the construction. Consequently the fields are to obey nontrivial constraint relationships and the Hilbert space is subject to a cutoff of highly excited states. This 'exclusion principle' was noted already in the AdS correspondence involving S_N orbifolds[126–128].

In an expansion around $N = \infty$ most effects of this are invisible and our discussion shows that this can be regarded as a theory of higher spin fields in dS is insensitive to these effects. However, as we saw above, the Grassmann origin was already of importance in choosing the correct saddle point and the correct quanization of the quadratic hamiltonian. In the next section we will address the question of finite N and the Hilbert space of the bi-local theory. In the framework of geometric (pseudospin) representation we will give evidence that the bi-local theory is non-perturbatively satisfactory at the finite N level.

4.5 Geometric Representation and The Hilbert Space

The bi-local collective field representation is seen to give a bulk description dS space and the Higher Spin fields. It provides an interacting theory with vertices governed by $G = 1/N$ as the coupling constant. We would now to show that the collective theory has an equivalent geometric (Pseudo) Spin variable description appropriate for nonperturbative considerations. The essence of this (geometric) description is in reinterpreting the bi-local collective fields (and their canonical conjugates) as matrix variables (of infinite dimensionality) endowed with a Kahler structure.This geometric description will provide a tractable framework for quantization and non-perturbative definition of the bi-local and HS de Sitter theory. It will be seen capable to incorporate non-perturbative features related to the Grassmannian origin of bi-local fields and its Hilbert space. Pseudo-spin collective variables represent all $Sp(2N)$ invariant variables of the theory (both commuting and non-commuting). These close a compact algebra and at large N are constrained by the corresponding Casimir operator. One therefore has an algebraic pseudo-spin system whose nonlinearity is governed by the coupling constant $G = 1/N$. As such they have been employed earlier for developing a large N expansion [129] and as a model for quantization [130]. This version of the theory is in its perturbative $(1/N)$ expansion identical to the bi-local collective representation. It therefore has the same map to and correspondence with Higher Spin dS_4 at perturbative level. We will see however that the geometric representation becomes of use for defining (and evaluating) the Hilbert space and its quantization.

To describe the pseudo-spin description of the $Sp(2N)$ theory we will follow the quantization procedure of [131]. In this approach one starts from the action:

$$S = \int d^d x \; dt (\partial^\mu \eta_1^i \partial_\mu \eta_2^i) \tag{4.5.1}$$

and deduces the canonical anti-commutation relations

$$\{\eta_1^i(x,t)\partial_t \eta_2^j(x',t)\} = -\{\eta_2^i(x,t)\partial_t \eta_1^j(x',t)\} = i\delta^d(x-x')\delta^{ij} \tag{4.5.2}$$

The quantization based on the mode expansion

$$\begin{aligned} \eta_1^i(x) &= \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (a_{k+}^{i\dagger} e^{-ikx} + a_{k-}^{i} e^{ikx}) \\ \eta_2^i(x) &= \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (-a_{k-}^{i\dagger} e^{-ikx} + a_{k+}^{i} e^{ikx}) \end{aligned} \tag{4.5.3}$$

with

$$\{a_{k-}^i, a_{k'-}^{j\dagger}\} = \{a_{k+}^i, a_{k'+}^{j\dagger}\} = \delta^d(k-k')\delta^{ij} \tag{4.5.4}$$

Note that in this approach the operators η_a^i are not hermitian, but pseudo-hermitian in the sense of [132].

Pseudo-spin bi-local variables will be introduced based on $Sp(2N)$ invariance, we have the vectors:

$$\begin{aligned} \eta &= (\eta_1^1, \eta_2^1, \eta_1^2, \eta_2^2, \cdots, \eta_1^N, \eta_2^N) \\ a(k) &= (a_{k-}^1, a_{k+}^1, a_{k-}^2, a_{k+}^2, \cdots, a_{k-}^N, a_{k+}^N) \\ \tilde{a}(k) &= (a_{k+}^{1\dagger}, -a_{k-}^{1\dagger}, a_{k+}^{2\dagger}, -a_{k-}^{2\dagger}, \cdots, a_{k\dagger}^{N+}, -a_{k-}^{N\dagger}) \end{aligned} \tag{4.5.5}$$

and the notation:

$$\eta(x) = \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (\tilde{a}(k) e^{-ikx} + a(k) e^{ikx}) \tag{4.5.6}$$

so that a complete set of $Sp(2N)$ invariant operators now follows:

$$\begin{aligned} S(p_1,p_2) &= \frac{-i}{2\sqrt{N}} a^T(p_1)\epsilon_N a(p_2) = \frac{i}{2\sqrt{N}}\sum_{i=1}^{N}(a^i_{p_1+}a^i_{p_2-} + a^i_{p_2+}a^i_{p_1-}) \\ S^\dagger(p_1,p_2) &= \frac{-i}{2\sqrt{N}} \tilde{a}^T(p_1)\epsilon_N \tilde{a}(p_2) = \frac{i}{2\sqrt{N}}\sum_{i=1}^{N}(a^{i\dagger}_{p_1+}a^{i\dagger}_{p_2-} + a^{i\dagger}_{p_2+}a^{i\dagger}_{p_1-}) \\ B(p_1,p_2) &= \tilde{a}^T(p_1)\epsilon_N a(p_2) = \sum_{i=1}^{N} a^{i\dagger}_{p_1+}a^i_{p_2+} + a^{i\dagger}_{p_1-}a^i_{p_2-} \end{aligned} \tag{4.5.7}$$

and $\epsilon_N = \epsilon\otimes\mathbb{I}_N$, $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

These invariant operators close an invariant algebra. The commutation relations are found to equal:

$$\begin{aligned} \left[S(\vec{p}_1,\vec{p}_2), S^\dagger(\vec{p}_3,\vec{p}_4)\right] &= \frac{1}{2}\left(\delta_{\vec{p}_2,\vec{p}_3}\delta_{\vec{p}_4,\vec{p}_1} + \delta_{\vec{p}_2,\vec{p}_4}\delta_{\vec{p}_3,\vec{p}_1}\right) - \frac{1}{4N}\left[\delta_{\vec{p}_2,\vec{p}_3}B(\vec{p}_4,\vec{p}_1) + \delta_{\vec{p}_2,\vec{p}_4}B(\vec{p}_3,\vec{p}_1)\right. \\ &\quad \left. + \delta_{\vec{p}_1,\vec{p}_3}B(\vec{p}_4,\vec{p}_2) + \delta_{\vec{p}_1,\vec{p}_4}B(\vec{p}_3,\vec{p}_2)\right] \\ \left[B(\vec{p}_1,\vec{p}_2), S^\dagger(\vec{p}_3,\vec{p}_4)\right] &= \delta_{\vec{p}_2,\vec{p}_3}S^\dagger(\vec{p}_1,\vec{p}_4) + \delta_{\vec{p}_2,\vec{p}_4}S^\dagger(\vec{p}_1,\vec{p}_3) \\ \left[B(\vec{p}_1,\vec{p}_2), S(\vec{p}_3,\vec{p}_4)\right] &= -\delta_{\vec{p}_1,\vec{p}_3}S(\vec{p}_2,\vec{p}_4) - \delta_{\vec{p}_1,\vec{p}_4}S(\vec{p}_2,\vec{p}_3) \end{aligned} \tag{4.5.8}$$

The singlet sector of the original $Sp(2N)$ theory is characterized by a further constraint. This constraint is is associated with the Casimir operator of of the algebra and can be shown to take the form:

$$\frac{4}{N}S^\dagger \star S + (1-\frac{1}{N}B)\star(1-\frac{1}{N}B) = \mathbb{I} \tag{4.5.9}$$

Here we have used the matrix star product notation: $\star$ product as: with $A\star B = \int d\vec{p}_2 A(\vec{p}_1\vec{p}_2)B(\vec{p}_2\vec{p}_3)$.

The form of the Casimir, which commutes with the above pseudo-spin fields points to the compact nature of the bi-local pseudo-spin algebra associated with the $Sp(2N)$ theory. This will have major consequences which we will highlight later.

Indeed it is interesting to compare the algebra with the bosonic case, where we have:

$$\begin{aligned} S(p_1,p_2) &= \frac{1}{2\sqrt{N}}\sum_{i=1}^{2N} a_i(p_1)a_i(p_2) \\ S^\dagger(p_1,p_2) &= \frac{1}{2\sqrt{N}}\sum_{i=1}^{2N} a_i^\dagger(p_1)a_i^\dagger(p_2) \\ B(p_1,p_2) &= \sum_{i=1}^{2N} a_i^\dagger(p_1)a_i(p_2) \end{aligned} \tag{4.5.10}$$

with the commutation relations:

$$\left[S(\vec{p}_1,\vec{p}_2), S^\dagger(\vec{p}_3,\vec{p}_4)\right] = \frac{1}{2}\left(\delta_{\vec{p}_2,\vec{p}_3}\delta_{\vec{p}_4,\vec{p}_1} + \delta_{\vec{p}_2,\vec{p}_4}\delta_{\vec{p}_3,\vec{p}_1}\right) + \frac{1}{4N}\left[\delta_{\vec{p}_2,\vec{p}_3}B(\vec{p}_4,\vec{p}_1) + \delta_{\vec{p}_2,\vec{p}_4}B(\vec{p}_3,\vec{p}_1)\right.$$

$$
\begin{aligned}
& \qquad + \delta_{\vec{p}_1,\vec{p}_3} B(\vec{p}_4,\vec{p}_2) + \delta_{\vec{p}_1,\vec{p}_4} B(\vec{p}_3,\vec{p}_2)] \\
\left[B(\vec{p}_1,\vec{p}_2), S^\dagger(\vec{p}_3,\vec{p}_4)\right] &= \delta_{\vec{p}_2,\vec{p}_3} S^\dagger(\vec{p}_1,\vec{p}_4) + \delta_{\vec{p}_2,\vec{p}_4} S^\dagger(\vec{p}_1,\vec{p}_3) \\
\left[B(\vec{p}_1,\vec{p}_2), S(\vec{p}_3,\vec{p}_4)\right] &= -\delta_{\vec{p}_1,\vec{p}_3} S(\vec{p}_2,\vec{p}_4) - \delta_{\vec{p}_1,\vec{p}_4} S(\vec{p}_2,\vec{p}_3)
\end{aligned}
\tag{4.5.11}
$$

In this case the Casimir constraint is found to equal:

$$
-\frac{4}{N} S^\dagger \star S + (1 + \frac{1}{N} B) \star (1 + \frac{1}{N} B) = \mathbb{I} \tag{4.5.12}
$$

featuring the non-compact nature of the bosonic problem.

We can see therefore that the singlet sectors of the fermionic $Sp(2N)$ theory and the bosonic $O(2N)$ theory can be described in analogous a bi-local pseudo-spin algebraic formulations with a quadratic Casimir taking the form:

$$
4\gamma S^\dagger \star S + (1 - \gamma B) \star (1 - \gamma B) = \mathbb{I} \tag{4.5.13}
$$

the difference being that with $\gamma = \frac{1}{N}(-\frac{1}{N})$ for the fermionic (bosonic) case respectively. This signifies the compact versus the non-compact nature of the algebra, but also exhibits the relationship obtained through the $N \leftrightarrow -N$ switch that was central in the argument for de Sitter correspondence in [107].

From this algebraic bi-local formulation one can easily see the the Collective field representation(s) that we have discussed in sections 2 and 3. Very simply, the Casimir constraints can be solved, and the algebra implemented in terms of a canonical pair of bi-local fields:

$$
\begin{aligned}
S(p_1 p_2) &= \frac{\sqrt{-\gamma}}{2} \int dy_1 dy_2 e^{-i(p_1 y_2 + p_2 y_2)} \{ -\frac{2}{\kappa_{p_1}\kappa_{p_2}} \Pi \star \Psi \star \Pi(y_1 y_2) - \frac{1}{2\gamma^2 \kappa_{p_1}\kappa_{p_2}} \frac{1}{\Psi}(y_1 y_2) \\
& + \frac{\kappa_{p_1}\kappa_{p_2}}{2} \Psi(y_1 y_2) - i \frac{\kappa_{p_1}}{\kappa_{p_2}} \Psi \star \Pi(y_1 y_2) - i \frac{\kappa_{p_2}}{\kappa_{p_1}} \Pi \star \Psi(y_1 y_2) \} \\
S^\dagger(p_1 p_2) &= \frac{\sqrt{-\gamma}}{2} \int dy_1 dy_2 e^{-i(p_1 y_2 + p_2 y_2)} \{ -\frac{2}{\kappa_{p_1}\kappa_{p_2}} \Pi \star \Psi \star \Pi(y_1 y_2) - \frac{1}{2\gamma^2 \kappa_{p_1}\kappa_{p_2}} \frac{1}{\Psi}(y_1 y_2) \\
& + \frac{\kappa_{p_1}\kappa_{p_2}}{2} \Psi(y_1 y_2) + i \frac{\kappa_{p_1}}{\kappa_{p_2}} \Psi \star \Pi(y_1 y_2) + i \frac{\kappa_{p_2}}{\kappa_{p_1}} \Pi \star \Psi(y_1 y_2) \} \\
B(p_1 p_2) &= \frac{1}{\gamma} + \int dy_1 dy_2 e^{-i(p_1 y_2 + p_2 y_2)} \{ \frac{2}{\kappa_{p_1}\kappa_{p_2}} \Pi \star \Psi \star \Pi(y_1 y_2) + \frac{1}{2\gamma^2 \kappa_{p_1}\kappa_{p_2}} \frac{1}{\Psi}(y_1 y_2) \\
& + \frac{\kappa_{p_1}\kappa_{p_2}}{2} \Psi(y_1 y_2) - i \frac{\kappa_{p_1}}{\kappa_{p_2}} \Psi \star \Pi(y_1 y_2) + i \frac{\kappa_{p_2}}{\kappa_{p_1}} \Pi \star \Psi(y_1 y_2) \}
\end{aligned}
\tag{4.5.14}
$$

where $\kappa_p = \sqrt{\omega_p}$.

Recalling that the Hamiltonian is given in terms of B we now see that its bi-local form is the same in the fermionic and the bosonic case. This explains the feature that we have established by direct construction in Sec. 2,3. While the bi-local field representation of B is the same in the fermionic and bosonic cases, the difference is seen in the representations of operators S and $S^\dagger$. These operators create singlet states in the Hilbert space and the difference contained in the sign of gamma implies the opposite shifts for the background fields that we have identified in Sec. 2,3. The algebraic pseudo spin reformulation is therefore seen to account for all the perturbative $(1/N)$ features of the the bi-local theory that we have identified in Sec. 2,3. However, in addition and we would like to emphasize that, the algebraic formulation provides a proper framework for defining the bi-local Hilbert space.

4.5.1 Quantization and the Hilbert Space

The bi-local pseudo-spin algebra has several equivalent representations that turn out to be useful. Beside that collective representation that we have explained above, one has the simple oscillator representation:

$$\begin{aligned}
S(p_1,p_2) &= \alpha\star(1-\frac{1}{N}\alpha^\dagger\star\alpha)^{\frac{1}{2}}(p_1,p_2)\\
S^\dagger(p_1,p_2) &= (1-\frac{1}{N}\alpha^\dagger\star\alpha)^{\frac{1}{2}}\star\alpha^\dagger(p_1,p_2)\\
B(p_1,p_2) &= 2\,\alpha^\dagger\star\alpha(p_1,p_2)
\end{aligned}\tag{4.5.15}$$

with standard canonical canonical commutators (or Poisson brackets).

A more relevant geometric representation is obtained through a change:

$$\begin{aligned}
\alpha &= Z(1+\frac{1}{N}\bar{Z}Z)^{-\frac{1}{2}}\\
\alpha^\dagger &= (1+\frac{1}{N}\bar{Z}Z)^{-\frac{1}{2}}\bar{Z}
\end{aligned}\tag{4.5.16}$$

The pseudo-spins in the Z representation are given by:

$$\begin{aligned}
S(p_1,p_2) &= Z\star(1+\frac{1}{N}\bar{Z}\star Z)^{-1}(p_1,p_2)\\
S^\dagger(p_1,p_2) &= (1+\frac{1}{N}\bar{Z}\star Z)^{-1}\star\bar{Z}(p_1,p_2)\\
B(p_1,p_2) &= 2\,Z\star(1+\frac{1}{N}\bar{Z}\star Z)^{-1}\star\bar{Z}(p_1,p_2)
\end{aligned}\tag{4.5.17}$$

It's easy to see that this satisfy the Casimir constraint: $\frac{4}{N}S^\dagger\star S+(1-\frac{1}{N}B)^2=1$
One can write the Lagrangian in this Z representation as:

$$\mathcal{L}=i\int dt\,\mathrm{tr}[Z(1+\frac{1}{N}\bar{Z}Z)^{-1}\dot{\bar{Z}}-\dot{Z}(1+\frac{1}{N}\bar{Z}Z)^{-1}\bar{Z}]-\mathcal{H}\tag{4.5.18}$$

For regularization purposes, it is useful to consider putting $\vec{x}$ in a box and limiting the momenta by a cutoff Λ: this makes the bi-local fields into finite dimensional matrices (which we will take to be a size K). For $Sp(2N)$ one deals with a $K\times K$ dimensional complex matrix Z and we have obtained in the above a compact symmetric (Kahler) space :

$$ds^2=\mathrm{tr}[dZ(1-\bar{Z}Z)^{-1}d\bar{Z}(1-Z\bar{Z})^{-1}]\tag{4.5.19}$$

According to the classification of [133], this would correspond to manifold $M_I(K,K)$. We note that the standard fermionic problem which was considered in detail in [130] corresponds to manifold $M_{III}(K,K)$ of complex antisymmetric matrices.

Quantization on Kahler manifolds in general has been formulated in detail by Berezin [130]. We also note that the usefullnes of Kahler quantization for discretizing de Sitter space was pointed out by A. Volovich in a quantum mechanical scenario[113]. In the present Quantization we are dealing with a field theory with infinitely many degrees of freedom and infinite Khaler matrix variables. We will now summarize some of the results of quantization which are directly relevant to the $Sp(2N)$ bi-local collective fields theory. Commutation relations of this system follow from the Poisson Brackets associated with the Lagrangian $\mathcal{L}(\bar{Z}, Z)$. States in the Hilbert space are represented by (holomorphic) functions (functionals) of the bi-locals $Z(k,l)$. A Kahler scalar product defining the bi-local Hilbert space reads:

$$(F_1, F_2) = C(N,K) \int d\mu(\bar{Z}, Z) F_1(Z) F_2(\bar{Z}) \det[1 + \bar{Z}Z]^{-N} \tag{4.5.20}$$

with the (Kahler) integration measure:

$$d\mu = \det[1 + \bar{Z}Z]^{-2K} d\bar{Z} dZ \tag{4.5.21}$$

The normalization constant is found from requiring $(F_1, F_1) = 1$ for $F = 1$. Let:

$$a(N,K) = \frac{1}{C(N,K)} = \int d\mu(\bar{Z}, Z) \det[1 + \bar{Z}Z]^{-N} \tag{4.5.22}$$

This leads to the matrix integral (complex Penner Model)

$$a(N,K) = \frac{1}{C(N,K)} = \int \prod_{k,l=1}^{K} d\bar{Z}(k,l) dZ(k,l) \det[1 + \bar{Z}Z]^{-2K-N} \tag{4.5.23}$$

which determines $C(N,K)$.

The following results on quantization of this type of Kahler system are of note: First, the parameter N: much like for ordinary spin, one can show that N (and therefore G in Higher Spin Theory) can only take integer values, i.e. $N = 0, 1, 2, 3, \cdots$. Next, one has question about the total number of states in the above Hilbert space. Naively, bi-local theory would seem to grossly overcount the number of states of the original fermionic theory. Originally one essentially had $2NK$ fermionic degrees of freedom with a finite Hilbert space. The bi-local description is based on (complex) bosonic variables of dimensions K^2 and the corresponding Hilbert space would appear to be much larger. But due to the compact nature of the phase space, the number of states much smaller.

We will now evaluate this number (at finite N and K) for the present case of $Sp(2N)$ (in [130] ordinary fermions were studied) and show that the exact dimension of the bi-local Hilbert space in geometric (Kahler) quantization agrees with the dimension of the singlet Hilbert space of the $Sp(2N)$ fermionic theory.

The dimension of quantized Hilbert space is found as follows: Considering the operator $\hat{O} = I$ one has that:

$$\mathrm{Tr}(I) = C(N,K) \int \prod_{k,l=1}^{K} d\bar{Z}(k,l) dZ(k,l) \det[1 + \bar{Z}Z]^{-2K} \tag{4.5.24}$$

Consequently the dimension of the bi-local Hilbert space is given by:

$$\mathrm{Dim}\ \mathcal{H}_B = \frac{C(N,K)}{C(0,K)} = \frac{a(0,K)}{a(N,K)} \tag{4.5.25}$$

The evaluation of the matrix (Penner) integral therefore also determines the dimension of the bi-local Hilbert space. Since this evaluation is a little bit involved, we present it in the following. Evaluation of matrix integrals (for real matrices) is given in [134] the extension to the complex case was considered in[135].

We will use results of [133], whereby every (complex) matrix can be reduced through (symmetry) transformations to a diagonal form:

$$Z(k,l) \to \begin{bmatrix} \omega_1 & & & & \\ & \omega_2 & & 0 & \\ & & \omega_3 & & \\ & 0 & & \ddots & \\ & & & & \omega_K \end{bmatrix} \tag{4.5.26}$$

and the matrix integration measure becomes:

$$[d\bar{Z}dZ] = |\Delta(\omega)|^2 \prod_{l=1}^{K} d\omega_l d\Omega \tag{4.5.27}$$

where $d\Omega$ denotes "angular" parts of the integration and $\Delta(x_1, \cdots, x_K) = \prod_{k<l}(x_k - x_l)$ is a Vandermonde determinant, with $x_i = \omega_i^2$. Consequently the matrix integral for $a(N,K)$ (and $C(N,K)$) becomes:

$$a(N,K) = \frac{\text{Vol } \Omega}{K!} \int \Delta(x_1, \cdots, x_K)^2 \prod_l (1+\omega_l^2)^{-2K-N} \prod_l d\omega_l \tag{4.5.28}$$

changing variables: $x_i = -\frac{y_i}{1-y_i}$, we get:

$$a(N,K) = \frac{\text{Vol } \Omega}{2^K K!} \int_0^\Lambda \prod_i^K dy_i \Delta(y_1, \cdots, y_K)^2 \prod_i (1-y_i)^N \tag{4.5.29}$$

This integral can be evaluated exactly. It belongs to a class of integrals evaluated by Selberg in 1944 [136]:

$$\begin{aligned} I(\alpha,\beta,\gamma,n) &= \int_0^1 dx_1 \cdots \int_0^1 dx_n |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^{\alpha-1}(1-x_j)^{\beta-1} \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)} \end{aligned} \tag{4.5.30}$$

we have the case with $\alpha = 1, \ \ \beta = N+1, \ \ \gamma = 1, \ \ n = K$ and

$$I(1, N+1, 1, K) = \prod_{j=0}^{K-1} \frac{\Gamma(2+j)\Gamma(1+j)\Gamma(N+1+j)}{\Gamma(2)\Gamma(N+K+j+1)} \tag{4.5.31}$$

We therefore obtain the following formula for the number of states in our Bi-local $Sp(2N)$ Hilbert space:

$$\text{Dim } \mathcal{H}_B = \prod_{j=0}^{K-1} \frac{\Gamma(j+1)\Gamma(N+K+j+1)}{\Gamma(K+j+1)\Gamma(N+j+1)} \tag{4.5.32}$$

We have compared this number with explicit enumeration of $Sp(2N)$ invariant states in the fermionic Hilbert space (for low values of N and K) and found complete agreement. It is probably not that difficult to prove agreement for all N, K. This settles however the potential problem of overcompletness of the bi-local representation. Since the $Sp(2N)$ counting uses the fermionic nature of creation operators and features exclusion when occupation numbers grow above certain limit it is seen that bi-local geometric quantization elegantly incorporates these effects. The compact nature of the associated infinite dimensional Kahler manifold secures the correct dimensionality of the the singlet Hilbert space. By using Stirling's approximation for the number of states in the bi-local Hilbert space (4.5.32), we see the dimension growing linearly in N (with $K \gg N$):

$$\ln(\text{Dim } \mathcal{H}_B) \sim 2NK \ \ln 2 \qquad \text{at the leading order} \tag{4.5.33}$$

This is a clear demonstration of the presence of an N-dependent cutoff in agreement with the fermionic nature of the original $Sp(2N)$ Hilbert space. So in the nonlinear bi-local theory with $G = 1/N$ as coupling constant, we have the desired effect that the Hilbert space is cutoff through $1/G$ effects. Consequently we conclude that the geometric bi-local representation with infinite dimensional matrices $Z(k, l)$ provides a complete framework for quantization of the bi-local theory and of de Sitter HS Gravity.

4.6 Comments

We have motivated the use of double analytic continuation and hence the connection between the $Sp(2N)$ model and de Sitter higher field theory for the quadratic action for the collective field. To establish this connection one of course needs to establish this for the interaction terms. This is of course highly nontrivial, and in fact the connection between the collective theory for the $O(N)$ model and the AdS higher spin theory is only beginning to be understood. We believe that once this is understood well enough one can address the question for the $Sp(2N)$-dS connection.

In this paper we have dealt mostly with the *free* $Sp(2N)$ vector model. As the parallel $O(N)$/AdS case this theory is characterized with an infinite sequence of conserved higher spin currents and associated conserved charges. The question regarding the implementation of the Coleman-Mandula theorem then arises, this question was discussed recently in [137–139]. One can expected that identical conclusions hold for the present $Sp(2N)$ case. The bi-local collective field theory technqiue is trivially extendible to the linear sigma model based on $Sp(2N)$, as commented in section (4.2). Of particular interest is the IR behavior of the theory which presumably takes the theory from the Gaussian fixed point to a nontrivial fixed point.

It is well known that dS/CFT correspondence is quite different from AdS/CFT correspondence, particularly in the interpretation of bulk correlation functions [9–12]. We have not addressed these issues in this paper. Recently it has been proposed that the $Sp(2N)$/dS connection can be used to understand subtle points about dS/CFT [109]. We hope that an explicit construction as described in this paper will be valuable for a deeper understanding of these issues.

The bi-local formulation that we have presented was cast in a geometric, pseudo-spin framework. We have suggested that this representation offers the best framework for quantization of the bi-local theory and consequently the Hilbert space in dS/CFT. We have demonstrated through counting of the size of the Hilbert space that it incorporates finite N effects through a cutoff which depends on the coupling constant of the theory: $G = 1/N$. Most importantly it incorporates the finite N exclusion principle and provides an explanation on the quantization of $G = 1/N$ from the bulk point of view. These features are obviously of definite relevance for understanding quantization of Gravity in de Sitter space-time. Nevertheless the question of understanding de Sitter Entropy from this 3 dimensional CFT remains an interesting and challenging problem.

It would be interesting to consider the analogues of $Sp(2N)$/dS correspondence in the CFT_2/Chern-Simons version[140–142], as well as to three dimensional conformal theories which have a line of fixed points, as in [143]. Finally higher spin theories arise as limits of string theory in several contexts, e.g. [144] and [143]. It would be interesting to see if these models can be modified to realize a dS/CFT correspondence in string theory.

Chapter 5

Double Trace Flows and Holographic RG in dS/CFT correspondence

5.1 Introduction

The dS/CFT correspondence [110, 9, 11] proposes that quantum gravity in asymptotically de Sitter space is dual to a *Euclidean* conformal field theory which lives on $\mathscr{I}^+$ or $\mathscr{I}^-$. Specifically, it has been proposed that the partition function of the CFT deformed by single trace operators (which equals the generating functional for correlators of the CFT) is the Bunch-Davies wavefunctional obtained by performing the bulk path integral with Dirichlet boundary conditions on $\mathscr{I}^+$ and Bunch-Davies condition in the infinite past. Unlike in AdS/CFT [5]-[15], the meaning of this correspondence is not completely clear, particularly because of the difficulty in defining observables in de Sitter space [110]. While these issues are obviously important, one can nevertheless perform computation in the dS bulk where gravity is treated semi-classically [11]. Keeping this in view, in this note we will address the question: if a dS/CFT correspondence does exist, what does it say about the dual field theory?

To begin with, the dual field theory cannot be unitary in the usual sense [11, 107]. The symmetry group of the putative d-dimensional Euclidean CFT, $SO(d+1,1)$, is the isometry group of both dS_{d+1} and Euclidean AdS_{d+1}. If the CFT is unitary, one would expect that the dual is a bulk theory living in Euclidean AdS_{d+1}. Thus, the CFT dual to dS_{d+1} is non-unitary. On the other hand, there is a unitary time evolution in the dS_{d+1} bulk (examples of which we will consider explicitly below); if the holographic correspondence is true, this will clearly imply some constraints on the dual field theory. In this note, we will explore these constraints on the RG flow of double and triple trace deformations in the dual field theory. For double trace couplings, the story for AdS is well known [80, 83, 85]: for a relevant deformation with positive coupling, the theory flows into a IR fixed point, in complete agreement with the prediction of the dual large-N field theory.

We will calculate the beta function for the double and triple trace couplings of a proposed CFT dual to de Sitter space using the holographic renormalization group techniques of [145] and [146] (for previous work on the subject, see [147]-[148]). We will show that the beta function has the same structure as that expected from general field theory considerations, along with holographically determined coefficients. In particular the coefficient of the quadratic term of the double trace beta function equals the normalization of the two point function; similar statements are true for the triple trace beta function. For dS_4, we find that the specific choice of operator normalization which leads to real n-point correlation functions [107] also leads to beta functions with real coefficients. This leads to a beta function whose quadratic term differs in sign from that in Euclidean AdS_4, so that the IR fixed point now appears at negative rather than positive coupling. The recent proposal of a duality between $Sp(N)$ vector models in three Euclidean dimensions and Vasiliev theory in

dS_4 [107][149] provides a specific realization of the above result.

For dS_{d+1} with even d, however, we find, first of all, that no choice of operator normalization exists which ensures absolute reality of the n-point functions; furthermore, any choice of operator normalization which ensures reality of coefficients of the beta-functions forces us to have n-point functions with very specific n-dependent complex phases, $\langle O_1 \cdots O_n \rangle \sim i^{(n-2)(1-d)/2}$ as explained in Section 5.7. These assertions are proved in Section 5.7 under the general condition of real coupling constants in the bulk Lagrangian. It is important to note that the reality of the coefficients of the bulk Lagrangian, which is tied to the unitarity of the bulk field theory, plays a crucial role here.

5.2 The main result

In this section we first derive the field theory beta function at leading order of $1/N$. We then summarize our findings for the holographic beta function.

5.2.1 Field theory: 2-pt function vs. double trace beta-function

Consider the two-point function of an operator $\mathcal{O}(x)$ in a d-dimensional *Euclidean* CFT:

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\rangle_0 = G_0(k)(2\pi)^d \delta(k_1+k_2),\ G_0(k) = bk^{-2\nu},\ 2\nu \equiv d - 2\Delta \qquad (5.2.1)$$

where $\mathcal{O}$ is a scalar operator of dimension Δ [1]. The exponent of k follows from dimensional analysis; the subscript 0 implies that the correlator is computed in the unperturbed CFT. The constant b denotes the normalization of the operator $\mathcal{O}$.

In the following we will assume that, for large central charge c of the CFT, the leading contribution to the 2n-point function of $\mathcal{O}$ has a factorized form (similar to Wick's theorem):

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)....\mathcal{O}(k_{2n}\rangle = \left[\sum_{\text{permutations}} \langle \mathcal{O}(k_{i_1})\mathcal{O}(k_{i_2})\rangle ... \langle \mathcal{O}(k_{i_{2N-1}})\mathcal{O}(k_{i_{2N}})\rangle\right] + ... \qquad (5.2.2)$$

where the ... terms at the end denote $O(1/c)$ corrections. Well-known CFT's with such properties are conformal large N gauge theories with $\mathcal{O}$ a single trace operator (or conformal large N vector theories with $\mathcal{O}$ some appropriate bilinear of vectors)[2].

This has the following consequences:

1. The dimension of the "double trace" operator $\mathcal{O}^2$ is 2Δ. [3]

[1] In the context of (A)dS/CFT, we will consider alternative quantization, where $\mathcal{O}$ will be identified with $\mathcal{O}_-$, as in (5.3.15). In that case, $\Delta = \Delta_-$ (see (5.3.6)), and the value of ν follows the usual definition. Among other things, the choice of alternative quantization ensures that the double trace flow is relevant.

[2] For more general examples, see, e.g. [150].

[3] We will call $\mathcal{O}$ and $\mathcal{O}^2$ "single trace" and "double trace" operators, respectively, by analogy with large N gauge theories; however, at least for the purposes of this section, this only implies the factorization property (5.2.2).

2. Under a double trace deformation (f_0 is a bare coupling)

$$S = S_0 + \frac{f_0}{2}\int d^d x \mathcal{O}(x)^2 \qquad (5.2.3)$$

the Green's function (5.2.1) changes to [4]

$$G_f(k) = G_0(k) - f_0 G_0(k)^2 + ... = \frac{G_0(k)}{1 + f_0 G_0(k)} \qquad (5.2.4)$$

We will derive the same equation in (5.4.4) from a dS bulk dual.

The above Green's function implies the following 'running coupling constant' [5]

$$f(k) = \frac{f_0}{1 + f_0 G_0(k)} \qquad (5.2.5)$$

Let us define a dimensionless renormalized coupling $\lambda(\mu)$ by the relation [6]

$$f(\mu) = \lambda(\mu)\mu^{2\nu} \qquad (5.2.6)$$

By using the above equations, we get

$$\lambda(\mu) = \frac{f_0}{\mu^{2\nu} + f_0 b}$$

Since f_0 is a bare coupling, it should not depend on μ. By differentiating the above with respect to μ, we get

$$\mu\frac{d\lambda(\mu)}{d\mu} = -2\nu\lambda + 2\nu b\lambda^2 \qquad (5.2.7)$$

At this stage the constant b is arbitrary and is not necessarily real; holography allows us to determine the value of b, as in (5.3.18) (for a dS/CFT) where b is complex and (5.3.21) (for AdS/CFT) where b is real and positive. For unitary theories, on general grounds, b must be real and positive and we have the well-known result that the theory flows to a IR fixed point at positive coupling.

Note that we have arrived at (5.2.7) with minimal assumptions about the CFT and about the operator O (essentially its scaling and factorization).

5.2.2 Bulk dual

Let us now assume that our CFT has a bulk dual. The $SO(d+1,1)$ conformal symmetry implies that the bulk must be either AdS_{d+1} or dS_{d+1}. A double trace deformation then translates to modified boundary conditions for the dual bulk field

[4] This is easy to derive by expanding $\exp[-S] = \exp[-S_0](1 - S_{int} + \frac{1}{2}S_{int}^2 - ...)$, and using (5.2.2).

[5] We define the running coupling $f(k)$ by $G_f(k) =: G_0(k) - f(k)G_0(k)^2$ (thus $f(k)$ represents the Dyson Schwinger sum of an infinite number of Feynman diagrams in the middle expression of (5.2.4)).

[6] Note that $f(k)$ is of dimension $2\nu \equiv d - 2\Delta$ since $\mathcal{O}^2$ is of dimension 2Δ.

[80]. For $\nu > 0$ in (5.2.1) the deformation has to be around alternative quantization. Following the procedure of integrating out geometry devised in [145] and [146] we will derive the beta-function of the field theory from bulk Schrodinger equations. For AdS the time in the Schrodinger equation is euclidean and identified with the radial coordinate, which is identified with the RG scale of the field theory: this derivation is already contained in [145, 146]. For dS, bulk evolution is in real time, and the precise relationship of time with the field theory scale is less clear. If T denotes the bulk time in inflationary coordinates (which in our convention is negative), we will find that the beta function (5.2.7) is again reproduced, provided we identify $(-iT)$ with the RG scale of the dual theory.

We will find below that, the equation (5.2.4) is reproduced holographically both in the case of AdS and dS (see (5.4.3) and (5.4.4)). Further, with the above holographic identification of the field theory cut-off, the beta-function (5.2.7) is reproduced exactly in both cases. For dS, unlike in AdS we cannot demand that $b > 0$ or even real in the field theory. However, for dS_4 it was argued in [107] that the only way to ensure real n point functions is to have b real and negative. This is the normalization used in [11] as well. This leads to the conclusion that the IR fixed point of the dual theory is at *negative* coupling. This is consistent with the conjecture of [107]: indeed a calculation of the beta function of $Sp(N)$ field theory leads to the same beta function (this has been calculated to one loop in [131]).

However, for dS_{d+1} with even d, as explained at the end of the Introduction, reality of b is only possible if one allows for specific n-dependent complex phases of the n-point correlation functions (see Section 5.7 for details).

5.3 Holographic dictionaries

5.3.1 dS/CFT dictionary

We will consider the inflationary patch of dS_{d+1} with a metric

$$ds^2 = \frac{L_{dS}^2}{T^2}\left[-dT^2 + d\vec{x}^2\right] \tag{5.3.1}$$

with $-\infty \leq T \leq 0$ We will consider a massive minimally coupled scalar in this geometry with the action

$$S_\epsilon = S_{gr} + \frac{1}{2G_N}\int_{-\infty}^{\epsilon} dT \int d^d x \left(\frac{L_{dS}}{-T}\right)^{d+1}\left[\left(\frac{-T}{L_{dS}}\right)^2 [(\partial_T\phi)^2 - (\nabla\phi)^2] - m^2\phi^2\right] \tag{5.3.2}$$

where S_{gr} is the gravity action and ϵ is a cutoff. In the following we will consider the dynamics of the scalar - we will therefore drop the gravity part. We will work in a probe approximation and ignore the backreaction on gravity. A bulk wavefunction can be now defined by the path integral

$$\Psi[\phi_0(\vec{x}), \epsilon] = \int_{\phi(\epsilon,\vec{x})=\phi_0(\vec{x})} \mathcal{D}\phi(T,\vec{x})\exp\left(iS_\epsilon\right) \tag{5.3.3}$$

where the field satisfies Bunch-Davies conditions at $T = -\infty$. S_ϵ is the action obtained by integrating from $T = -\infty$ to $T = \epsilon$, and $\epsilon < 0$.

In the following we will use a notation

$$\rho \equiv \sqrt{\frac{L_{dS}^{d-1}}{G_N}} \tag{5.3.4}$$

The dS/CFT correspondence as interpreted in [11, 12, 107, 149] then claims that this wavefunctional is related to the partition function of a dual CFT in the presence of a source. More precisely, in the standard quantization of the CFT

$$\langle \exp\left[\int d^d x\, \phi_0(\vec{x})\, \mathcal{Z}(\epsilon)\, \mathcal{O}_+(\vec{x})\right]\rangle_{st} = \Psi[\phi_0(\vec{x}), \epsilon], \quad \mathcal{Z}(\epsilon) = \frac{\rho}{\sqrt{\gamma}}(-i\epsilon)^{-\Delta_-} \tag{5.3.5}$$

where

$$\Delta_\pm = d/2 \pm \nu,\ \nu \equiv \sqrt{d^2/4 - m^2 L_{dS}^2} \tag{5.3.6}$$

Here $\mathcal{Z}(\epsilon)$ is a normalization factor used to define the GKPW relation (5.3.5). The important part of this factor is the numerical coefficient γ which we treat *a priori* to be complex. This constant is taken to be $\gamma = 1$ in [12, 107, 149]. We will come back to a detailed discussion of this coefficient later. Note that the factor $(-i\epsilon)$ is naturally identified with the field theory UV cutoff [107].

We will be concerned with the semiclassical limit where the functional integral on the right hand side of (5.3.3) can be evaluated by saddle point. The classical solution which satisfies the Bunch-Davies condition at $T = -\infty$ and the specified boundary condition at $T = \epsilon$ is given, in momentum space, by

$$\phi(T, k) = \left(\frac{T}{\epsilon}\right)^{d/2} \frac{H_\nu^{(2)}(-kT)}{H_\nu^{(2)}(-k\epsilon)} \phi_0(\vec{k}) \tag{5.3.7}$$

where ν is given by (5.3.6). This leads to the following on-shell action

$$iS_{on} = -\frac{i}{2G_N} \int [dk]\ L_{dS}^{d-1} \left(\frac{\Delta_-}{(-\epsilon)^d} - \frac{k\epsilon H_{\nu-1}^{(2)}(-k\epsilon)}{(-\epsilon)^d H_\nu^{(2)}(-k\epsilon)} \right) \phi_0(\vec{k}) \phi_0(-\vec{k}) \tag{5.3.8}$$

At late times $k|\epsilon| \ll 1$

$$\begin{aligned} iS_{on} &= -i\frac{\rho^2}{2} \int [dk] \left(\frac{\Delta_-}{(-\epsilon)^d} - \frac{\Gamma(1-\nu)}{\Gamma(2-\nu)} \frac{k^2}{2(-\epsilon)^{d-2}} \right) \phi_0(\vec{k}) \phi_0(-\vec{k}) \\ &+ \frac{\rho^2}{2} \int [dk] \phi_0(\vec{k}) \phi_0(-\vec{k}) (-i\epsilon)^{-2\Delta_-}\ H(k) \end{aligned} \tag{5.3.9}$$

where

$$H(k) = (i)^{d-1} C_1(\nu) k^{2\nu}, \quad C_1(\nu) \equiv -2\nu \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} 2^{-2\nu} \tag{5.3.10}$$

In the semiclassical limit the wavefunction is then

$$\Psi[\phi_0(\vec{x}), \epsilon] \sim \exp[iS_{on}] \tag{5.3.11}$$

It may be easily checked that at early times $k|\epsilon| \gg 1$ this reproduces the ground state of a bunch of harmonic oscillators with "coordinates" $\chi_\epsilon(k) = (-\epsilon)^{\frac{1-d}{2}}\phi(k)$. At late times $k|\epsilon| \ll 1$ we need to remove the divergent piece by holographic renormalization and define the wavefunction by

$$\Psi[\phi_0(\vec{x}), \epsilon] \sim \exp[iS'_{on}] \tag{5.3.12}$$

where

$$iS'_{on} = \frac{L_{dS}^{d-1}}{2G_N}\int [dk]\phi_0(\vec{k})\phi_0(-\vec{k})(-i\epsilon)^{-2\Delta_-}\ H(k) \tag{5.3.13}$$

is the finite part of the on-shell action. The divergent first term in (5.3.9) has to be removed by addition of a counterterm to the action. Using (5.3.5) the two point correlator of the dual operator $\mathcal{O}_+$, is given by

$$\langle \mathcal{O}_+(k)\mathcal{O}_+(-k)\rangle_{st} = G_{st}(k) = \gamma H(k) = \gamma\, i^{d-1}C_1(\nu)\, k^{2\nu} \tag{5.3.14}$$

We will be interested in *alternative* quantization. The generating functional for correlators in the appropriate CFT in this case is obtained by extending the corresponding prescription in AdS [89],

$$\langle \exp\left[\int d^dx J(\vec{x})\mathcal{O}_-(\vec{x})\right]\rangle_{alt} = \int \mathcal{D}\phi_0(\vec{x})\langle \exp\left[\int d^dx\phi_0(\vec{x})\,\mathcal{Z}(\epsilon)\,\mathcal{O}_+(\vec{x})\right]\rangle_{st} \exp\left[\mathcal{Z}(\epsilon)\int d^dx\frac{J(\vec{x})}{2\nu}\phi_0(\vec{x})\right] \tag{5.3.15}$$

In the semiclassical approximation we may replace the generating functional of standard quantization by the wavefunction (5.3.12). Performing the ϕ_0 integral leads to a two point correlator in alternative quantization

$$G_{alt}(k) = \frac{\delta^2}{\delta J(k)\delta J(-k)}\langle e^{\int d^dx J(\vec{x})\mathcal{O}_-(\vec{x})}\rangle_{alt} = -\frac{1}{(2\nu)^2 G_{st}(k)} \tag{5.3.16}$$

This inverse relation between the Green's function is exactly the same as in AdS/CFT [89]. Combining (5.3.16),(5.3.14) and (5.3.10) we get

$$\langle \mathcal{O}_-(k)\mathcal{O}_-(-k)\rangle_{alt} = G_{alt}(k) = \frac{i^{1-d}}{\gamma}\ C(\nu)k^{-2\nu},\ C(\nu) \equiv \frac{2^{2\nu}}{(2\nu)^3}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \tag{5.3.17}$$

Comparing with (5.2.1), we get the following holographically determined value of b:

$$b_{dS} = \frac{i^{1-d}}{\gamma}\ C(\nu) \tag{5.3.18}$$

In case of dS_4, Ref. [107] chose $\gamma = 1$ in keeping with the reality of the n-point functions, which was also reproduced by a CFT calculation using $SP(N)$. However, in this paper we are dealing with dS_{d+1} for arbitrary d and will keep γ arbitrary and in principle complex. We will come back to the important issue of the phase of γ (equivalently of $\mathcal{Z}$) and its relation to the phases of the n-point functions and beta-function coefficients in detail in Section 5.7.

The relationship (5.3.15) can be inverted to rewrite the Bunch-Davies wavefunction in terms of the generating functional in alternative quantization,

$$\Psi[\phi_0(\vec{x}), \epsilon] = \int \mathcal{D}J(\vec{x})\ \exp\left[-\mathcal{Z}(\epsilon)\int d^dx\frac{J(\vec{x})}{2\nu}\phi_0(\vec{x})\right]\langle \exp\left[\int d^dx J(\vec{x})\mathcal{O}_-(\vec{x})\right]\rangle_{alt} \tag{5.3.19}$$

5.3.2 The formulae for AdS

It will be useful to record the corresponding well known formulae in euclidean AdS space. The GKPW prescription for the generating functional for correlators in standard quantization reads

$$\langle \exp\left[\int d^d x (\epsilon)^{-\Delta_-}\phi_0(\vec{x})\tilde{\mathcal{Z}}(\epsilon)\mathcal{O}_+(\vec{x})\right]\rangle_{st} = Z[\phi_0(\vec{x}),\epsilon] \quad \tilde{\mathcal{Z}}(\epsilon) \equiv \frac{\rho}{\sqrt{\gamma}}(\epsilon)^{-\Delta_-} \tag{5.3.20}$$

where we of course need to replace $L_{dS} \to L_{AdS}$. There are no factors of i in the formulae, the rescaling factor γ has to be real, the Hankel functions are replaced by Modified Bessel functions and the quantity in square brackets in (5.3.10) is the boundary Green's function in AdS_{d+1} leading to the proportionality constant

$$b_{AdS} = \frac{1}{\gamma}\, C(\nu) \tag{5.3.21}$$

where $C(\nu)$ is defined in (5.3.17). Since everything needs to be real, (5.3.20) requires γ to be real and positive, leading to a real positive b_{AdS}. Finally, the analog of (5.3.19) for AdS may be obtained by replacing $(-i\epsilon) \to \epsilon$.

5.4 Double Trace deformations

In the following we will be interested in the deformation of the CFT dual to alternative quantization in dS_{d+1} by a double trace operator. The Euclidean field theory action is given by (5.2.3). As argued in Sec 5.2.1, to leading order in large N, the dimension of $\mathcal{O}^2$ is then 2Δ. We require the perturbation to be relevant, which means that the CFT action S_0 must correspond to alternative quantization (see also footnote 1), ensuring that $2\Delta = 2\Delta_- < d$ (see (5.3.6)). The generating function for correlators in the presence of the deformation may be now written using a Hubbard-Stratanovich transformation,

$$\langle \exp\left[\int d^d x J(\vec{x})\mathcal{O}(\vec{x})\right]\rangle_{alt}^{f_0} = \int \mathcal{D}\sigma \exp\left[\frac{1}{2f_0}\int d^d x\, \sigma(\vec{x})^2\right]\langle \exp\left[\int d^d x (J(\vec{x})+\sigma(\vec{x}))\mathcal{O}(\vec{x})\right]\rangle_{alt} \tag{5.4.1}$$

where the notation $\langle ...\rangle_{alt}^{f_0}$ denotes correlations in presence of the double trace deformation (5.2.3). Using (5.3.5),(5.3.12) and (5.3.15) we get

$$\langle \exp\left[\int d^d x J(\vec{x})\mathcal{O}(\vec{x})\right]\rangle_{alt}^{f_0} = \int \mathcal{D}\phi_0 \, \exp[iI_{f_0}(\phi_0)] \tag{5.4.2}$$

where

$$iI_{f_0}(\phi_0) = iS'_{on}(\phi_0) + \int d^d x \left[\mathcal{Z}(\epsilon)\frac{J(\vec{x})}{2\nu}\phi_0(\vec{x}) - \mathcal{Z}(\epsilon)^2\frac{f_0}{2}\left(\frac{\phi_0(\vec{x})}{2\nu}\right)^2\right] \tag{5.4.3}$$

Using (5.3.13) and performing the integral over ϕ_0 this leads to the prediction that the deformed CFT has a Green's function

$$G_f(k) = \frac{G_{alt}(k)}{1+f_0 G_{alt}(k)} \tag{5.4.4}$$

This relation can be of course obtained directly from the large-N field theory (5.2.3) (see Eq. (5.2.4)). The holographic derivation of this formula is a consistency check on the above dS/CFT prescription.

5.5 Holographic RG

We now adapt the holographic renormalization group procedure developed in [145, 146] to de Sitter space. we rewrite the right hand side of (5.3.3) by introducing a floating cutoff at $T = l$,

$$\Psi[\phi_0(\vec{x}), \epsilon] = \int \mathcal{D}\tilde{\phi}(\vec{x}) \Psi_{IR}[\tilde{\phi}, l] \Psi_{UV}[\tilde{\phi}, \phi_0] \tag{5.5.1}$$

where

$$\Psi_{IR}[\tilde{\phi}] = \Psi[\tilde{\phi}(\vec{x}), l] \tag{5.5.2}$$

and

$$\Psi_{UV}[\tilde{\phi}, \phi_0] = \int_{\phi(l,\vec{x})=\tilde{\phi}(\vec{x})}^{\phi(\epsilon,\vec{x})=\phi_0(\vec{x})} \mathcal{D}\phi(T, \vec{x}) \exp\left(i \int_l^\epsilon dT\ L\right) \tag{5.5.3}$$

where L is the Lagrangian.

The idea is now to obtain an effective action of the dual theory at a finite cutoff l by extending the dS/CFT relationship (5.3.19) for $\Psi_{IR}[\tilde{\phi}, l]$,

$$\langle e^{-S_{eff}(l)} \rangle_{alt} = \int \mathcal{D}\tilde{\phi}(\vec{x}) \int \mathcal{D}J(\vec{x})\, \Psi_{UV}[\tilde{\phi}, \phi_0] \exp\left[-\mathcal{Z}(l) \int d^d x \frac{J(\vec{x})}{2\nu} \tilde{\phi}(\vec{x})\right] \langle \exp\left[\int d^d x J(\vec{x}) \mathcal{O}_-(\vec{x})\right] \rangle_{alt}$$

where $\mathcal{Z}(l)$ is defined as in (5.3.5), with ϵ replaced by l. This relates the parameters in Ψ_{UV} to couplings in the effective action. The expression for $\langle e^{\int d^d x J(\vec{x}) \mathcal{O}_-(\vec{x})} \rangle_{alt}$ in terms of bulk quantities in (5.3.12) and (5.3.15) are valid in the $\epsilon \to 0$ limit. When we use these expressions for finite l, there is a freedom of choosing counterterms [151, 152]. We will stick to the counterterm implied in (5.3.12), and comment on the implications of this freedom later.

From the definition (5.5.3), Ψ_{UV} satisfies a Schrodinger equation with the Hamiltonian derived from the Lagrangian,

$$i G_N \frac{\partial}{\partial(-l)} \Psi_{UV}(\tilde{\phi}, l) = H(l) \Psi_{UV}(\tilde{\phi}, l) \tag{5.5.4}$$

which give flow equations for the parameters in Ψ_{UV} and hence couplings in the effective action. The negative sign in the left hand side of (5.5.4) comes because time evolution corresponds to decreasing l, which appears as the lower limit of integration in (5.5.3).

For the free scalar field we are considering the hamiltonian at some time slice T is given by

$$H(T) = \frac{1}{2} \int d^d x \left[-G_N^2 \left(\frac{-T}{L_{dS}}\right)^{d-1} \frac{\delta^2}{\delta\phi^2} + \left(\frac{L_{dS}}{-T}\right)^{d-1} (\nabla\phi)^2 + \left(\frac{L_{dS}}{-T}\right)^{d+1} m^2 \phi^2 \right] \tag{5.5.5}$$

In the semiclassical limit $G_N \ll L_{dS}^{d-1}$ the Schrodinger equation reduces to a Hamilton-Jacobi equation. For a wavefunction

$$\Psi_{UV} = \exp[iK] \tag{5.5.6}$$

the Hamilton-Jacobi equation is given by

$$\frac{1}{2}\left[G_N^2\left(\frac{-l}{L_{dS}}\right)^{d-1}\left(\frac{\delta K}{\delta\phi}\right)^2 + \left(\frac{L_{dS}}{-l}\right)^{d-1}(\nabla\phi)^2 + \left(\frac{L_{dS}}{-l}\right)^{d+1} m^2\phi^2\right] + G_N\frac{\partial K}{\partial(-l)} = 0 \tag{5.5.7}$$

Consider now a general quadratic form for K

$$K = \rho^2\,(-l)^{-d}\int d^dx\left[-\frac{1}{2}g(l)\tilde{\phi}^2 + h(l)\tilde{\phi} + c(l)\right] \tag{5.5.8}$$

Note that the parameters in (5.5.8) depend on the cutoff l. The flow equations for these parameters follow from substituting (5.5.8) in (5.5.7). For consistency we really need to replace these parameters by space-dependent parameters (e.g. $g(x)$). However as shown in [145] and [153] the flow equations for the zero momentum modes of these couplings decouple from the non-zero momentum modes. With this understanding,

$$\begin{aligned}\beta_g &= -(-il)\frac{\partial g}{\partial(-il)} = -g^2 - dg - m^2L_{dS}^2\\ \beta_h &= -(-il)\frac{\partial h}{\partial(-il)} = -h(g+d)\end{aligned} \tag{5.5.9}$$

As is clear from the discussion of [151] and [152], the freedom of choosing different counterterms at finite l modifies the last term in the first equation of (5.5.9). We have written the equations (5.5.9) using $(-il)$ as a cutoff scale. This is a natural choice (as will be discussed further below).

The zeroes of β_g are at $g_\pm = -\Delta_\pm$ and alternative quantization means we have to expand the coupling as

$$g = g_- + \delta g \tag{5.5.10}$$

The beta function for δg is given by

$$\beta_{\delta g} = -(-il)\frac{\partial\delta g}{\partial(-il)} = -2\nu(\delta g) - (\delta g)^2 \tag{5.5.11}$$

To relate this flow equations to beta functions of the dual field theory we need to establish a relationship between g, f and the couplings of the field theory. This may be done by substituting (5.5.8) in (5.5.4) and performing the integrals over $J(\vec{x})$ and $\tilde{\phi}(\vec{x})$ by saddle point method. This leads to a field theory effective action

$$S_{eff} = \frac{f}{2}\int d^dx\;\mathcal{O}_-^2 + j\int d^dx\;\mathcal{O}_- + c \tag{5.5.12}$$

where

$$j = -2\nu\rho\sqrt{\gamma}(i)^{d+1}(-il)^{-\Delta_+}h(l) \tag{5.5.13}$$

$$f = (i)^{d+1}(-il)^{-2\nu}(2\nu)^2 g\,\gamma = -(2\nu)^2 C(\nu)\frac{1}{b_{dS}}\,(-il)^{-2\nu}\,g \qquad (5.5.14)$$

and c is a constant independent of the operator $\mathcal{O}$. In the above we have used the expression for b_{dS} in (5.3.18).

The fixed point values of the parameter g simply corresponds to the minimal counterterm in the bulk action . The field theory couplings, which are defined as departures from a CFT have to be related to the departure from the fixed point.

The couplings f, j and hence δf and δj have the appropriate dimensions 2ν and Δ_+ respectively, as is clear from the powers of l which appear in (5.5.14). The beta functions of the field theory are, however, those of *dimensionless* couplings. In the field theory this is done by multiplying by an appropriate power of the cutoff or renormalization scale, as in (5.2.6). In the holographic setup this requires specifying a relationship between the cutoff in the bulk with a UV cutoff on the boundary. As is quite clear from all the formulae above, it is natural to identify $(-il)$ as the renormalization scale μ of the field theory. Let us identify the field theory renormalization scale μ to be $a^{1/(2\nu)}$ times the holographic cut-off scale $1/(-il)$, for some positive constant a. With this choice, we have the following identification of the dimensionless coupling of the field theory λ with the departure from the fixed point,

$$\lambda\, a\left((-il)^{-1}\right)^{2\nu} \equiv \delta f = -(2\nu)^2 C(\nu)\frac{1}{b_{dS}}\,(-il)^{-2\nu}\,\delta g \qquad (5.5.15)$$

where we have used (5.2.6). Making the convenient choice $a = (2\nu)^3 C(\nu)$ (which gives a specific choice of the field theory renormalization scale), we get

$$\delta g = -2\nu b_{dS}\,\lambda \qquad (5.5.16)$$

Substituting this in (5.5.11) finally leads to a beta function for λ

$$\beta_\lambda = -2\nu\lambda + 2\nu\, b_{dS}\lambda^2 = -2\nu\lambda + 2\nu\,\frac{i^{1-d}}{\gamma}C(\nu)\lambda^2 \qquad (5.5.17)$$

This is the same as the general field theory answer, (5.2.7).

As we have remarked above and will discuss in detail in Section 5.7, the requirement that there are no relative phases between various n-point functions of the dual field theory [7] implies that $b_{dS} \sim i^{d-1}$. This implies, in turn, that for even d we have purely imaginary b_{dS} and hence a complex beta function.

5.5.1 Results in AdS

For comparison let us recall the results of the above analysis in euclidean AdS. In this case the range of the radial coordinate is $0 \leq z \leq \infty$. The radial evolution equation satisfied by $\Psi_{UV.AdS}$ is

$$G_N \frac{\partial}{\partial(l)}\Psi_{UV,AdS}(\tilde{\phi}, l) = -H_{AdS}(l)\Psi_{UV,AdS}(\tilde{\phi}, l) \qquad (5.5.18)$$

[7] It is clear from our discussion in Section 5.7 that under no circumstance can the n-point functions be all real.

where

$$H(l) = \frac{1}{2}\int d^d x \left[-G_N^2 \left(\frac{l}{L_{AdS}}\right)^{d-1} \frac{\delta^2}{\delta\phi^2} + \left(\frac{L_{AdS}}{l}\right)^{d-1} (\nabla\phi)^2 + \left(\frac{L_{AdS}}{l}\right)^{d+1} m^2\phi^2\right] \tag{5.5.19}$$

With the form

$$\Psi_{UV,AdS} = \exp\left[\left(\frac{L_{AdS}^{d-1}}{G_N}\right) l^{-d} \int d^d x \left[-\frac{1}{2}g'(l)\tilde{\phi}^2 + h'(l)\tilde{\phi} + c'(l)\right]\right] \tag{5.5.20}$$

which leads to the flow equation

$$l\frac{\partial g'}{\partial l} = -(g')^2 - dg' + m^2 L_{AdS}^2 \tag{5.5.21}$$

The expressions for the fixed points are changed appropriately, but the flow equation for the departure from the fixed point $\delta g'$ is, instead of (5.5.11)

$$\beta_{\delta g'} = -l\frac{\partial \delta g'}{\partial l} = -2\nu(\delta g') + (\delta g')^2 \tag{5.5.22}$$

Finally the relationship between the field theory dimensionless coupling and $\delta g'$ is

$$\lambda = \gamma(2\nu)^2\ \delta g' = (2\nu)2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{1}{b_{AdS}}\delta g' \tag{5.5.23}$$

which leads once again to a beta function of the expected form (5.2.7)

5.6 Beta function of Triple and Higher trace couplings

In this section we will discuss a generalization of the above methods to derive the holographic beta-function of triple and higher trace couplings (in perturbation theory). We will be brief, emphasizing mainly the new features.

For concreteness, we will focus on triple trace couplings, of the form $\mathcal{O}_-^3$; however, the generalization to higher trace operators is straightforward. Triple trace operators are induced in a holographic RG, as we will see, when the dual scalar field theory has a cubic coupling

$$\Delta S_\epsilon = \frac{1}{G_N}\int_{-\infty}^{\epsilon} dT \int d^d x \left(\frac{L_{dS}}{-T}\right)^{d+1} \left[-\frac{r}{3}\phi^3\right] \tag{5.6.1}$$

in additional to the quadratic action (5.3.2). The Hamilton-Jacobi equation (5.5.7) gets modified by the addition of a cubic term

$$\left(\frac{L_{dS}}{-l}\right)^{d+1} \frac{2r}{3}\phi^3$$

to the term inside the square bracket. It is easy to see that a quadratic ansatz for the kernel K such as (5.5.8) will not satisfy such a Hamilton-Jacobi equation. Let us, therefore, take K to be cubic, *viz.* of the form

$$K = \rho^2 \epsilon^{-d} \int d^d x \Big(-\frac{g}{2}\tilde{\phi}^2 + h\tilde{\phi} + c + A\frac{\tilde{\phi}^3}{3} \Big) \tag{5.6.2}$$

By repeating the steps leading to (5.5.9), and equating the coefficients of $\tilde{\phi}, \tilde{\phi}^2$ and $\tilde{\phi}^3$ in the Hamilton-Jacobi equation,[8] we now get the following cut-off dependence of the couplings in (5.6.2)

$$\begin{aligned} &\beta_g = -g^2 - d\ g - \bar{m}^2 - 2hA, \quad \bar{m} = mL_{dS}, \\ &\beta_A = (-3g - d)A + \bar{r}, \quad \bar{r} = rL_{dS}^2, \\ &\beta_h = (-g - d)h \end{aligned} \tag{5.6.3}$$

Note that this generalizes (5.5.9), and reduces to it for $A = 0$. It is easy to find the following UV fixed point (near which β_g is negative):

$$h_c = 0,\ g_c = -\Delta_-,\ A_c = \bar{r}/(d - 3\Delta_-) \tag{5.6.4}$$

The linearized beta-functions for the deformations $\delta h, \delta g$ and δA (measured from this fixed point) are

$$\beta_{\delta g} = -2\nu\delta g, \quad \beta_{\delta A} = (3\Delta_- - d)\delta A, \quad \beta_{\delta h} = (\Delta_- - d)\delta h \tag{5.6.5}$$

How does one read off the field theory beta-functions from these? We can, once again, use (5.5.4), and show that it leads to a field theory with the following effective action

$$S_{eff} = \int d^d x \left(\frac{f}{2}\mathcal{O}_-^2 + j\mathcal{O}_- + \frac{B}{3}\mathcal{O}_-^3 + c \right) \tag{5.6.6}$$

where

$$\begin{aligned} j &= -(i)^{d+1}(-il)^{-\Delta_+} h(l) 2\nu\sqrt{\gamma}\rho \\ f &= (i)^{d+1}(-il)^{-2\nu}(2\nu)^2 g(l)\ \gamma \\ B &= -(i)^{d+1}(-il)^{3\Delta_- - d}(2\nu)^3 A(l)\frac{\gamma^{3/2}}{\rho} \end{aligned} \tag{5.6.7}$$

which generalizes the equation (5.5.14) encountered for double trace couplings. The beta-function for the field theory couplings f, j, B can easily be read off from the above identifications (5.6.7) with the bulk couplings g, h, A and their beta-functions

[8] Our approach here is perturbative; the Hamilton-Jacobi analysis generates $\tilde{\phi}^4$ terms. We imagine them to be taken care of by higher couplings, and focus here on couplings up to cubic order. It is straightforward, although cumbersome, to write more general beta-functions involving arbitrary Wilsonian couplings.

(5.6.3) or (5.6.5). The beta function for the dimensionless cubic trace coupling ($\delta\bar{B}$), which measures the deviation from the fixed point, turns out to be,

$$\beta_{\delta\bar{B}} = -3\Delta_-\delta\bar{B} + 3\frac{i^{1-d}}{(2\nu)^2\gamma}\delta\bar{f}\delta\bar{B} = -3\Delta_-\delta\bar{B} + 3\ b_{dS}\ \delta\bar{f}\,\delta\bar{B}\ \frac{\Gamma(1-\nu)}{2^{2\nu}(2\nu)\Gamma(1+\nu)} \quad (5.6.8)$$

where, $\delta\bar{f}$ is the deviation of the dimensionless double trace coupling from the fixed point. One can easily check that the field theory beta-functions have the correct form. E.g., $\beta_{\delta\bar{B}}$ includes a term $\propto \delta\bar{f}\delta\bar{B}$; to see this from a field theory reasoning, one needs to simply note that the three-point function $\langle\mathcal{O}_-(x)\mathcal{O}_-(y)\mathcal{O}_-(z)\rangle$ has a perturbative expansion of the schematic form $\delta\bar{B}\int d^dw G_0(x-w)G_0(y-w)G_0(z-w) + \delta\bar{f}\delta\bar{B}\int d^dw\ d^dw' G_0(x-w)G_0(y-w)G_0(z-w')G_0(w-w')$ (where we have shown only the first two terms). Using large N methods, one can organize such perturbation expansions [80, 83, 85].

Significantly, the beta function for A does not have an A^2 term (in field theory terms, $\beta_{\delta\bar{B}}$ does not have a $\delta\bar{B}^2$ term), and is in fact the same as in AdS. For the special case where $d = 3, \Delta_- = 1$ (which implies $\bar{m}^2 = 2, \Delta_+ = 2, \nu = 1/2$) the linearized beta-function indicates correctly the fact that the cubic coupling is marginal [9]. This is consistent with the known field theory result for vector models that a $[(\vec{\phi})^2]^3$ coupling acquires a nontrivial beta function only due to $1/N$ corrections. Our holographic result shows that this is a general result in large-N field theories.

5.7 Complex Phases

Here we focus on the structure of complex phases of the n-point correlation functions of the field theory. As seen in [107] even with interactions present in the bulk, the overall factor in $iI_{on-shell}$ is i^{d-1}. This implies the following schematic relations for leading order contributions to the first few n-point correlation functions,

$$\begin{aligned} i^{d-1} &= \mathcal{Z}^2\langle OO\rangle = \gamma^{-1}b_{dS} \\ r_3 i^{d-1} &= \mathcal{Z}^3\langle OOO\rangle \\ r_4 i^{d-1} &= \mathcal{Z}^4\langle OOOO\rangle \end{aligned} \quad (5.7.1)$$

In these equations, we display only those quantities which possibly contain complex phases. The quantity $\mathcal{Z}$ is defined in (5.3.5); since $-i\epsilon$ has been identified with a real cut-off of the field theory, i.e. $-i\epsilon \propto 1/\Lambda_{UV}$, $\mathcal{Z}$ is essentially equal to $1/\sqrt{\gamma}$ so far as keeping track of complex phases is concerned. Similarly, we have written $\langle OO\rangle \propto b_{dS}$. The left hand sides of the above set of equations are obtained from the bulk; e.g. the LHS of the top equation displays the complex phase of (5.3.13). The couplings r_3, r_4 represent cubic, quartic, etc. couplings of the scalar Lagrangian (e.g. r_3 is the same as r in (5.6.1)). In keeping with unitarity of the bulk field theory, we will assume that

[9]The fixed point value of A is infinite for these values, as can be seen from (5.6.4). However, as remarked earlier, the fixed point value of holographic couplings is non-universal as they are affected by the choice of holographic counterterms. The linearized beta-functions (5.6.5) are free of such non-universalities.

these coefficients are all real. The right hand sides of equations (5.7.1) are obtained by the GKPW prescription, i.e. by expanding $\Psi[\phi_0(x), \epsilon]$ in (5.3.5) in powers of $\phi_0(x)$. Now, if we require that there is no relative phase between the correlation functions, i.e, the phase of $\langle O_1 \cdots O_n \rangle$ = the phase of $\langle O_1 \cdots O_{n+1} \rangle$, then we must have $\mathcal{Z}$ real. Recalling that $\mathcal{Z}(\epsilon) \propto 1/\sqrt{\gamma}$ (where the proportionality constant is positive), the reality of $\mathcal{Z}$ implies that γ is real. Thus, so far as keeping track of complex phases is concerned, it can be taken to be 1. It then follows immediately that b_{dS} is complex for even d leading to complex beta functions.

Alternatively if we want to require the beta function to be always real, i.e, b_{dS} to be real, then we must choose the phase of $\mathcal{Z}$ to be $i^{(d-1)/2}$. However since the phases of the left hand sides of (5.7.1) are all equal, this will now imply the following relative complex phase,

$$\langle O_1 \cdots O_{n+1} \rangle = i^{(1-d)/2} \langle O_1 \cdots O_n \rangle \tag{5.7.2}$$

In particular, since in this choice $\langle OO \rangle$ is real, we get that the phase of $\langle O_1 \cdots O_n \rangle$ is $i^{(n-2)(1-d)/2}$. This clearly shows that we cannot have both the beta function as real and the absence of n-dependent phases.

Chapter 6

dS/CFT at uniform energy density and a de Sitter "bluewall"

6.1 Introduction and summary

One version of the dS/CFT correspondence [9, 110, 11] states that quantum gravity in de Sitter space is dual to a Euclidean CFT living on the boundary $\mathcal{I}^+$ or $\mathcal{I}^-$. More specifically, the partition function of the CFT with specified sources $\phi_{i0}(\vec{x})$ coupled to operators $\mathcal{O}_i$ is identified with the wavefunctional of the bulk theory as a functional of the boundary values of the fields dual to $\mathcal{O}_i$ given by $\phi_{i0}(\vec{x})$. In the semiclassical regime this becomes

$$\Psi[\phi_{i0}(\vec{x})] = \exp\left[iI_{cl}(\phi_{i0})\right] \tag{6.1.1}$$

where we need to impose regularity conditions on the cosmological horizon. This has been developed further in [12]. Unlike AdS/CFT, there are few concrete realizations of dS/CFT (for a recent proposal see [107] and *e.g.* [109, 154, 155, 149, 156–159] for related work). Nevertheless, it is interesting to explore the consequences of such a correspondence, assuming it exists.

In this note we address the question of what the bulk dual is of a euclidean CFT with *constant* spatially uniform energy-momentum density. One way to achieve this is to put the CFT on a circle. It is well known that the dual CFT to de Sitter space cannot be a usual unitary (more precisely reflection-positive) quantum field theory. The bulk dual of such a theory would be euclidean AdS. Such a CFT on a circle has a uniform energy-momentum density, describing the corresponding Lorentzian theory in a thermal state. The dual of this is a Euclidean AdS black brane, not a Lorentzian geometry.

In this context, consider a class of asymptotically de Sitter spacetimes

$$ds^2 = -\frac{R_{dS}^2 d\tau^2}{\tau^2(1+\frac{C}{\tau^d})} + \frac{\tau^2}{R_{dS}^2}\Big(1+\frac{C}{\tau^d}\Big)dw^2 + \frac{\tau^2}{R_{dS}^2}dx_i^2\ , \qquad C \propto \tau_0^d\ , \tag{6.1.2}$$

with C a general complex parameter and τ_0 is real. This metric should be regared as a (generally complex) saddle point in a functional integral. Motivated by this, we impose a requirement that the euclidean metric obtained by Wick rotation of the time coordinate τ is real and regular – this fixes the parameter $C = -i^d\tau_0^d$, and requires w to be periodic. However, as will be clear in the following, the Lorentzian metric can become singular for even d. Our solutions are similar to those in [149] who considered solutions with $S^{d-1}\times S^1$ boundaries. In fact (6.1.2) can be obtained as a limit of the solution [149] when the radius of the S^1 is much smaller than the radius of S^{d-1} (or equivalently, as S^{d-1} decompactifies).

The resulting spacetime can equivalently be obtained from the Euclidean AdS black brane by the analytic continuation from AdS to dS familiar in dS/CFT [9, 11].

Clearly this leads to a *complex* solution for odd d. This is in fact quite common in the dS/CFT correspondence [11, 12]. Indeed we will show that the energy-momentum tensor $T_{ij} \sim \frac{\delta\Psi}{\delta h^{ij}}$ in the CFT which follows from (6.1.1) is real for odd d. This is consistent with known results for correlators in pure dS [1]. For example in $d = 3$ we get $\langle T_{ij}\rangle \propto \frac{\tau_0^3}{G_4 R_{dS}^4}$, which is exactly what we need. For even d, the solution (6.1.2) is real and the boundary energy-momentum tensor is purely imaginary.

While real energy momentum tensors are thus obtained only for complex solutions, it is interesting to consider the geometry of the solutions with real parameters. The geometry is bounded by asymptotically de Sitter spacelike $\mathcal{I}^{\pm}$ and time-like singularities at the two ends of space. The null lines $\tau = \tau_0$ are Cauchy horizons. In fact the geometry bears some resemblance to the interior of the Reissner-Nordstrom black hole [160, 6]. The physics of physical observers is also quite similar. Timelike geodesics originating from $\mathcal{I}^-$ are repelled by the singularities. As an observer approaches the horizon, light coming from $\mathcal{I}^-$ is infinitely blueshifted, just as in the RN interior. It is natural to expect that this blueshift signals an instability, preserving cosmic censorship and distinguishing these from naked singularities. It is intriguing to note that from a dS/CFT perspective, the energy-momentum tensor is purely imaginary. It is tempting to think of this imaginary T_{ij} as a possible dual signature of the Cauchy horizon blue-shift instability that we have seen. It would be interesting to explore this and more generally cosmic censorship in dS/CFT. We dub these solutions "bluewalls".

6.2 dS/CFT at uniform energy-momentum density

The CFT correlation functions in dS/CFT correspondence follow from analytic continuation from euclidean AdS (or double analytic continuation from lorentzian AdS), with the interpretation that the wavefunctional is the generating functional of correlators. One half of dS_{d+1}, e.g. the upper patch being $\mathcal{I}^+$ at $\tau = \infty$ with a coordinate horizon at $\tau = 0$ is described in the planar coordinate foliation by the metric

$$ds^2 = -R_{dS}^2 \frac{d\tau^2}{\tau^2} + \frac{\tau^2}{R_{dS}^2}\delta_{ij}dx^i dx^j \ . \tag{6.2.1}$$

This may be obtained by analytic continuation of a Poincare slicing of $EAdS$,

$$r \to -i\tau \ , \qquad R_{AdS} \to -iR_{dS} \ . \tag{6.2.2}$$

In fact the analytic continuation of the smooth euclidean solutions lead to Bunch-Davies initial conditions on the cosmological horizon.

Consider the asymptotically de Sitter spacetime

$$ds^2 = -\frac{R_{dS}^2 d\tau^2}{\tau^2(1+\frac{C}{\tau^d})} + \frac{\tau^2}{R_{dS}^2}\Big(1+\frac{C}{\tau^d}\Big)dw^2 + \frac{\tau^2}{R_{dS}^2}dx_i^2 \ , \tag{6.2.3}$$

[1] One may wonder if there could be an additional factor of i in this relation. However, the requirement that the n-point correlator does not have a n-dependent phase rules this out [107].

with C a general complex parameter. This is a *complex* metric which satisfies Einstein's equation with a positive cosmological constant

$$R_{MN} = \frac{d}{R_{dS}^2} g_{MN} \ , \qquad \Lambda = \frac{d(d-1)}{2R_{dS}^2} \ . \tag{6.2.4}$$

With a view to requiring an analog of regularity in the interior for an asymptotically AdS solution, consider a Wick rotation of the time coordinate τ above. Then (6.2.3) becomes

$$\tau = il \quad \Rightarrow \quad ds_E^2 = -\frac{R_{dS}^2 dl^2}{l^2(1+\frac{C}{i^d l^d})} - \frac{l^2}{R_{dS}^2}\Big(1+\frac{C}{i^d l^d}\Big) dw^2 - \frac{l^2}{R_{dS}^2} dx_i^2 \ . \tag{6.2.5}$$

With a further continuation $R_{dS} \to iR'$, this is in general a complex euclidean metric. We require that this euclidean spacetime is real and regular in the interior, by which we demand that the spacetime in the interior approaches flat Euclidean space in the (l, w)-plane with no conical singularity. This is true if

$$C = -i^d \tau_0^d \ , \qquad l \geq \tau_0 \ , \qquad w \simeq w + \frac{4\pi}{(d-1)\tau_0} \ , \tag{6.2.6}$$

where τ_0 is some real parameter of dimension length, and the w-coordinate is compactified with the periodicity fixed by demanding that there is no conical singularity.

This requirement of regularity is similar to the one we use in an asymptotically AdS spacetime, where *e.g.* Wick rotating the time coordinate renders the resulting Euclidean space regular if the time coordinate is regarded as compact with a periodicity that removes any conical singularity (thus rendering it sensible for a Euclidean path integral). A sharp difference in the asymptotically de Sitter case is that we Wick rotate the asymptotic bulk time coordinate but the absence of a conical singularity fixes the w-coordinate to be compact with appropriate periodicity. This, however, is at odds with the regularity of the real time metric with this value of C when d is even. In that case, a periodic w leads to a conical type singularity at $\tau = \tau_0$ pretty much like the Milne universe with a compact spatial direction. In this regard, it is interesting to consider the asymptotically dS_5 solution above: then the above Wick rotation procedure fixes $C = -\tau_0^4$ and the periodicity of the w-coordinate and the solution is

$$ds^2 = -\frac{R_{dS}^2 d\tau^2}{\tau^2(1-\frac{\tau_0^4}{\tau^4})} + \frac{\tau^2}{R_{dS}^2}\Big(1-\frac{\tau_0^4}{\tau^d}\Big) dw^2 + \frac{\tau^2}{R_{dS}^2} dx_i^2 \ . \tag{6.2.7}$$

The metric in the vicinity of $l = \tau_0$ is $ds^2 \sim -dT^2 + T^2 dw^2 + \tau_0^2 dx_i^2$, where $T \sim l - \tau_0$. This is Milne space in the (T, w)-plane, with w compact (and thus a resulting singularity). We note that Wick rotating the coordinate T does not give a Euclidean space and is not equivalent to the above procedure of Wick rotating the asymptotic time coordinate τ.

As expected, this entire procedure is equivalent to analytically continuing from the Euclidean AdS black brane

$$ds^2 = R_{AdS}^2 \frac{dr^2}{r^2(1-\frac{r_0^d}{r^d})} + \frac{r^2}{R_{AdS}^2}\Big(1-\frac{r_0^d}{r^d}\Big) d\theta^2 + + \frac{r^2}{R_{AdS}^2} \sum_{i=1}^{d-1} dx^i dx^i \ , \tag{6.2.8}$$

where $\theta \sim \theta + \frac{4\pi}{(d-1)r_0}$, to the asymptotically de Sitter spacetime (6.2.3) using (6.2.2) and we identify $r_0 \equiv \tau_0$. The phase obtained by this analytic continuation is $\frac{-1}{(-i)^d}$ which can be seen as identical to $-i^d$ in (6.2.6). The regularity criterion (6.2.6) itself is then seen to simply be the analog of regularity of the $EAdS$ black brane. The condition $l \geq \tau_0$ is equivalent to the radial coordinate having the range $r \geq r_0$. In the Lorentzian signature spacetime (6.2.3), the time τ-coordinate extends to $\tau \to 0$. The curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges as $\tau \to 0$.

Near $\mathcal{I}^+$, *i.e.* $\tau \to \infty$, the metric (6.2.3) approaches that of de Sitter space with a Fefferman-Graham expansion

$$ds^2 = -\frac{R_{dS}^2}{\tau^2}d\tau^2 + h_{ij}dy^i dy^j = -\frac{R_{dS}^2}{\tau^2}d\tau^2 + \frac{\tau^2}{R_{dS}^2}\Big[g_{ij}^{(0)}(y^i) + \frac{R_{dS}^2}{\tau^2}g_{ij}^{(2)}(y^i) + \ldots\Big]dy^i dy^j \ . \tag{6.2.9}$$

It is clear from (6.2.9) that "normalizable" metric pieces are turned on in (6.2.3) [2]. We then expect a nonzero expectation value for the energy-momentum tensor here, as in the AdS context [161–163, 148]. For concreteness, let us consider the asymptotically dS_4 solution (6.2.3) with the regularity conditions (6.2.6),

$$ds^2 = -\frac{R_{dS}^2 d\tau^2}{\tau^2(1+\frac{i\tau_0^3}{\tau^3})} + \frac{\tau^2}{R_{dS}^2}\Big(1+\frac{i\tau_0^3}{\tau^3}\Big)dw^2 + \frac{\tau^2}{R_{dS}^2}dx_i^2 \ . \tag{6.2.10}$$

The calculation of the energy momentum tensor proceeds in a way entirely analogous to that in AdS. The total action, obtained by adding suitable Gibbons-Hawking surface terms and counterterms to the bulk action is

$$I = \frac{1}{16\pi G_4}\int_{\mathcal{M}} d\tau d^3x\sqrt{-g}\ (R-2\Lambda) + \frac{1}{8\pi G_4}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\ \big(K + \frac{2}{R_{dS}}\big) \tag{6.2.11}$$

The counterterms have been engineered to remove divergences in the bulk action coming from the boundary at $\tau \to \infty$. Here h_{ij} is the boundary metric and K is the trace of the extrinsic curvature. This renormalized action appears in (6.1.1). This leads to the energy momentum tensor[3]

$$T_{ij} = \lim_{\tau\to\infty}\frac{\tau}{R_{dS}}\,\frac{2}{\sqrt{h}}\frac{\delta\Psi}{\delta h^{ij}} \sim \lim_{\tau\to\infty}\frac{\tau}{R_{dS}}\frac{i}{G_4}\Big(K_{ij} - Kh_{ij} - \frac{2}{R_{dS}}h_{ij}\Big) \ , \tag{6.2.12}$$

We have used the standard relationship $\sqrt{h^B}h^B_{\mu\nu}T^{\nu\rho} = \sqrt{h}h_{\mu\nu}\tau^{\nu\rho}$ between the energy momentum tensor of the boundary theory and the quasi-local stress tensor $\tau^{\mu\nu}$, with $h^B_{\mu\nu} = \lim_{\tau\to\infty}\frac{R_{dS}^2}{\tau^2}h_{\mu\nu}$ the boundary metric. The above energy-momentum tensor vanishes for pure dS_4 as expected. For the spacetime (6.2.10), we obtain

$$T_{ww} = -2T_{ii} \sim \frac{i}{G_4}\Big(\frac{i\tau_0^3}{R_{dS}^4}\Big) = -\frac{\tau_0^3}{G_4 R_{dS}^4} \ , \tag{6.2.13}$$

[2]Scalar modes in dS_{d+1} near the boundary are $\phi \sim \tau^\Delta$, with $\Delta(\Delta-d) = -m^2R^2$. For $m^2 = 0$, we have $\Delta = d$ as analogous to a "normalizable" mode (in AdS): this is the mode turned on in (6.2.10).

[3]Note that our definition is consistent with [11] (also [107]), but differs from *e.g.* [9, 164] which use a derivative of the action rather than the wavefunction.

which is a real and spatially uniform energy-momentum density. Since (6.2.10) is a complex solution, its conjugate is also a solution (obtained by analytically continuing the opposite way), giving T_{ij} of the opposite sign as above. In the AdS case, spacetimes of this sort which are solutions in pure gravity have $I_{bulk} = \frac{1}{16\pi G_4}\int_{\mathcal{M}} drd^3x\sqrt{-g}\,(R-2\Lambda) = \frac{1}{16\pi G_4}\int drd^3x\frac{R^4}{r^4}(\frac{-6}{R^2})$. Under the analytic continuation (6.2.2), we have

$$I_{EAdS} \;\rightarrow\; \frac{1}{16\pi G_4}\int(-id\tau)d^3x\frac{R_{dS}^4}{\tau^4}\left(\frac{-6}{-R_{dS}^2}\right) = -iI_{dS}\ , \tag{6.2.14}$$

where I_{dS} is the action for the asymptotically de Sitter solution[4]. Thus the energy-momentum tensor is continued as $\frac{2}{\sqrt{h}}\frac{\delta(-I_{EAdS})}{\delta h^{\mu\nu}} \rightarrow \frac{2}{\sqrt{h}}\frac{\delta(iI_{dS})}{\delta h^{ij}}$. Note that T_{ij} in (6.2.13) is traceless ($T_{ww}+2T_{ii}=0$) as expected for a CFT.

Thus this asymptotically dS_4 complex solution is dual to a euclidean CFT with spatially uniform energy-momentum density, *i.e.* uniform T_{ij} expectation value (6.2.13) in the Euclidean CFT dual to dS_4 (in Poincare slicing). Loosely speaking, this Euclidean partition corresponds to a thermal state of the would-be corresponding Lorentzian theory (on $R\times R^2$), analogous to the AdS_4 Schwarzschild black brane dual to a thermal state in the SYM CFT with uniform $T_{\mu\nu}$.

Similar arguments apply in other dimensions but with different results. Using the Fefferman-Graham expansion (6.2.9) for an asymptotically dS_{d+1} spacetime, we see that "normalizable" metric modes $g_{\mu\nu}^{(d)}$ turned on give rise to a nonzero expectation value for the holographic energy-momentum tensor

$$T_{ij} = \lim_{\tau\to\infty}\frac{\tau^{d-2}}{R_{dS}^{d-2}}\frac{2}{\sqrt{h}}\frac{\delta\Psi}{\delta h^{ij}} \sim \lim_{\tau\to\infty}\frac{\tau^{d-2}}{R_{dS}^{d-2}}\frac{i}{G_{d+1}}\Big(K_{ij}-Kh_{ij}-\frac{d-1}{R_{dS}}h_{ij}\Big) \propto \frac{i}{G_{d+1}R_{dS}}g_{ij}^{(d)}\ , \tag{6.2.15}$$

where the form of (6.2.9) shows $g_{ij}^{(d)}$ to be the dimensionless coefficient of the normalizable $\frac{1}{\tau^{d-2}}$ term, and the i arises from Ψ, the wavefunction of the universe. In effect, this dS/CFT energy-momentum tensor can be thought of as the analytic continuation of the $EAdS$ one, with the i arising from $R_{AdS}\rightarrow -iR_{dS}$, and the metric modes also continuing correspondingly. The spacetime (6.2.3) with the parameter $C=-i^d\tau_0^d$ in (6.2.6) gives

$$g_{ww}^{(d)} \sim -\frac{i^d\tau_0^d}{R_{dS}^d} \qquad \Rightarrow \qquad T_{ww} \sim -\frac{i^{d+1}\tau_0^d}{G_{d+1}R_{dS}^{d+1}} = \frac{i^{d-1}\tau_0^d}{G_{d+1}R_{dS}^{d+1}}\ , \tag{6.2.16}$$

with $T_{ww}+(d-1)T_{ii}=0$. The phase i^{d-1} is equivalent to that in general dS/CFT correlation functions arising from the analytic continuation (6.2.2) from $EAdS$ correlators [157], following the arguments of [107]. For even d the energy momentum tensor is imaginary – this is also the case when the Lorentzian signature metric is singular at $\tau=\tau_0$.

[4] As we saw, the divergent terms in I_{dS} cancel: this gives a single new term from the τ-location where h_{ij} departs from the dS_4 value. The on-shell wavefunction for these solutions becomes $\Psi \sim \Psi_{dS}\ exp[\frac{V_3\tau_0^3}{8\pi G_4 R_{dS}^4}]$.

We thus see that a real energy-momentum density must arise from a metric mode $g^{(d)}$ that is pure imaginary: in other words, the spacetime (6.2.3) with a pure imaginary parameter C is dual to a CFT with real spatially uniform energy-momentum density. For the dS_5 case, we see that the regularity criterion for the Euclidean solution (or equivalently the analytic continuation from the $EAdS$ black brane) gives the spacetime (6.2.7) which is real: this metric which is singular gives an imaginary T_{ij} above.

In summary, we have described asymptotically de Sitter spacetimes (6.2.3) which under a Wick rotation are regular in the interior for certain values of the general complex parameter (6.2.6). The resulting spacetime can then be equivalently obtained by analytic continuation (6.2.2) from the Euclidean AdS black brane (6.2.8). These spacetimes give rise to a spatially uniform holographic energy-momentum density (6.2.15), which is real if the spacetime is complex (for odd d). Conversely, given a T_{ij} expectation value in dS/CFT, we could ask what the gravity dual is. An asymptotically de Sitter spacetime with the Fefferman-Graham series expansion (6.2.9) and thus corresponding T_{ij} then in fact sums to the closed form expression (6.2.3).

6.3 Real parameter C: a de Sitter "bluewall"

Even though the metric needs to be complex to yield a real energy momentum tensor, it is interesting to explore the properties of metrics of the form (6.2.3), (6.2.6), but with the parameter τ_0^d also continued to be real, *i.e.*

$$ds^2 = -\frac{d\tau^2}{f(\tau)} + f(\tau)dw^2 + \tau^2 dx_i^2 \,, \qquad f(\tau) = \tau^2\Big(1 - \frac{\tau_0^d}{\tau^d}\Big) \,, \tag{6.3.1}$$

with a nonzero parameter τ_0, and x_i are $d-1$ of the d spatial dimensions. The w-coordinate here has the range $-\infty \leq w \leq \infty$. This can be recast in FRW form as an asymptotically deSitter cosmology with anisotropy in the w-direction. The metric (6.3.1) is simply the analytic continuation of AdS-Schwarzschild with a further continuation of the mass parameter. We do not speculate about the significance of this real solution for dS/CFT for odd d.

The lines $\tau = \tau_0$ are coordinate singularities whose nature will be explored below. For concreteness, we focus on $d = 3$. The maximally extended geometry in Kruskal type coordinates (Appendix A, eq.(C.0.2)) is

$$ds^2 = \tau^2 \left[-\frac{4}{9}\Big(1 + \frac{\tau_0}{\tau} + \frac{\tau_0^2}{\tau^2}\Big)^{3/2} e^{-\sqrt{3}\tan^{-1}(\frac{2\frac{\tau}{\tau_0}+1}{\sqrt{3}})} d\tilde{u}d\tilde{v} + dx_i^2 \right] . \tag{6.3.2}$$

The Penrose diagram[5] Figure 6.1 shows the following key features of the geometry.
Two asymptotic dS***-regions:*** $v^2 - u^2 = \tilde{u}\tilde{v} > 0$ both map to $\tau \gg \tau_0$, using (C.0.2).
Cauchy horizons: $\quad \tau = \tau_0 \quad \Rightarrow \quad \tilde{u}\tilde{v} = 0 \,, \quad i.e. \quad u = \pm v.$
Using (C.0.2), we see that $\tanh \frac{3w\tau_0}{2} = \frac{\tilde{u}-\tilde{v}}{\tilde{u}+\tilde{v}}$ so that the two horizons are $\tilde{u} = 0 \Rightarrow \tau =$

[5]The Penrose diagram Figure 6.1 also appears in [165] but corresponds to a distinct spacetime (with an inhomogenuous energy-momentum tensor).

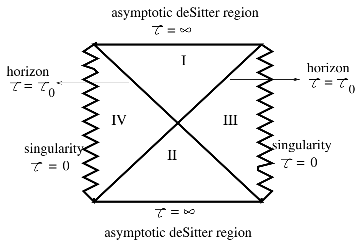


Figure 6.1: de Sitter "bluewall" Penrose diagram. This resembles the Penrose diagram of the AdS Schwarzschild black brane rotated by $\frac{\pi}{2}$.

$\tau_0,\ w = -\infty$, and $\tilde{v} = 0 \Rightarrow \tau = \tau_0,\ w = +\infty$. These are Cauchy horizons, as we discuss later. We refer to the intersection of the horizons $\tilde{u} = 0 = \tilde{v}$, *i.e.* $u = 0 = v$ or $\tau = \tau_0$ as the **bifurcation region**: the w-coordinate can take any value here.
Timelike singularities: $\quad \tau = 0 \quad \Rightarrow \quad \tilde{u}\tilde{v} = -e^{\frac{\pi}{2\sqrt{3}}} \ \sim\ v^2 - u^2$
In the Kruskal diagram these are hyperbolae with $u^2 > v^2$. There are two singularity loci $\tilde{v} = -\frac{c}{\tilde{u}}$ with $\tilde{u} > 0$ and $\tilde{u} < 0$. The curvature invariants for (6.3.1) are

$$R = d(d+1)\ , \quad R_{\mu\nu}R^{\mu\nu} = d^2(d+1)\ , \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 2d\Big(d+1+\tfrac{(d-2)(d-1)^2}{2}\tfrac{\tau_0^{2d}}{\tau^{2d}}\Big).$$

The divergence in $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ implies a curvature singularity as $\tau \to 0$: this is what the Schwarzschild interior singularity becomes after analytic continuation. Interestingly, the dS_3 solution is singularity-free: the metric in this case is isomorphic to dS_3 in static coordinates.

Near $\tau \to 0$, the metric approaches $ds^2 \sim -\frac{dw^2}{\tau^{d-2}} + \tau^{d-2}d\tau^2 + \frac{dx_i^2}{\tau^2} \ \sim\ \frac{1}{\tau}d\tilde{u}d\tilde{v} + \tau^2 dx_i^2$. For $\tau < \tau_0$, the τ-coordinate is spacelike while w is timelike. Then the singularity which occurs on a constant-τ slice is timelike (metric approaching $ds^2 \sim -\frac{dw^2}{\tau^{d-2}}$).

Note that these features (Cauchy horizons, timelike singularities) resemble the interior of the Reissner-Nordstrom black hole or "wormhole" (discussed in *e.g.* [6]). Recall that the latter geometry is of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2\ , \qquad f(r) = (r - r_+)(r - r_-)\ . \tag{6.3.3}$$

Near the inner horizon r_- this can be approximated as $ds^2 \sim -\frac{dr^2}{k(r-r_-)} + k(r - r_-)dt^2 + r^2 d\Omega^2$, where $k = r_+ - r_-$. In the region $r_- < r < r_+$, the radial coordinate r is timelike. Thus we see that the geometry near the inner horizon r_- in fact resembles the geometry $ds^2 \sim -\frac{d\tau^2}{\tau-\tau_0} + (\tau - \tau_0)dw^2 + \tau^2 dx_i^2$ near the horizon τ_0 in the present dS-case. Thus it is not surprising that the Penrose diagram and associated physics are similar in both cases.

For general timelike geodesic trajectories the momenta satisfy $p_\mu p^\mu = -m^2$ and the action is $S = \int d\tau \frac{m}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, with λ the affine parameter. Sinc ∂_w is a Killing vector the associated momentum p_w is conserved

$$\frac{p_w}{m} = \Big(\tau^2 - \frac{\tau_0^3}{\tau}\Big)\frac{dw}{d\lambda}\ , \quad \frac{\dot{\tau}^2 - \ p_w^2/m^2}{\tau^2 - \frac{\tau_0^3}{\tau}} = 1, \quad \frac{dw}{d\tau} = \frac{\pm\frac{p_w}{m}}{\tau^2(1-\frac{\tau_0^3}{\tau^3})\sqrt{\frac{p_w^2}{m^2} + \tau^2(1-\frac{\tau_0^3}{\tau^3})}}\ . \tag{6.3.4}$$

In these coordinates static observers are timelike geodesics at const-w, x_i with $p_w = 0,\ g_{\tau\tau}(\frac{d\tau}{d\lambda})^2 = -1$. Using (C.0.2), these are $\frac{u}{v} = -\tanh\frac{3w\tau_0}{2} = const$ *i.e.* straight lines crossing from the past universe II to the future one I through the bifurcation region. Generic observers have $p_w \neq 0$: as $\tau \to \tau_0$, they approach the horizon with increasing coordinate speed $|\frac{dw}{d\tau}| \to \infty$ and fall through the horizon. They do not hit the singularity however: the singularity appears to be repulsive. This can be seen from (6.3.4) by noting that $\dot{\tau}^2 = \frac{p_w^2}{m^2} + \tau^2 - \frac{\tau_0^3}{\tau} > 0$ implies a turning point $\tau_{min} = \frac{\tau_0^3}{p_w^2}$, and likewise $(\frac{dw}{d\tau})^2 > 0 \quad \Rightarrow \quad \tau \geq \tau_{min}$, so that timelike geodesic trajectories never reach the singularity. We see that in this "deSitter bluewall" solution, particles can apparently pass from the past universe through the horizons avoiding the timelike singularities behind the horizons and emerge in the future universe. Whether such trajectories can actually go across is unclear due to a blue-shift instability stemming from the Cauchy horizons, as we discuss now.

Bluewalls: We now discuss the role of the Cauchy horizons and the possibility of traversing from the past universe to the future one. First recall that the horizons are $\tilde{u} = 0$ at $\tau = \tau_0, w = -\infty$ and $\tilde{v} = 0$ at $\tau = \tau_0, w = +\infty$. Thus the horizons are actually infinitely far away in the w-direction. As we have seen, trajectories from the past universe (beginning at some point on $\mathcal{I}^-$) can pass through the horizons into the interior regions: however there are timelike and null geodesics which begin in the interior regions alone and thus cannot be obtained by time development of any Cauchy data on $\mathcal{I}^-$. Thus the past horizons are future Cauchy horizons for Cauchy data on $\mathcal{I}^-$. Likewise the future horizons are causal boundaries for the future universe, so that these are past Cauchy horizons for data on $\mathcal{I}^+$.

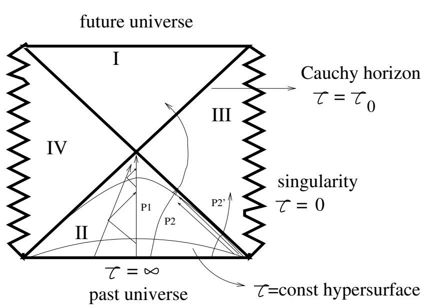


Figure 6.2: Trajectories in the de Sitter bluewall and the Cauchy horizon. Observers P_1 are static while P_2 has w-momentum p_w, crosses the horizon, turns around inside and appears to re-emerge in the future universe. Also shown are incoming lightrays from infinity which "crowd near" the Cauchy horizon.

Two static observers at different w-locations communicating by lightray signals are always in contact with each other. Consider observers P_1, P_2, with P_1 a static observer while P_2 is a geodesic infalling observer with some w-momentum. P_2 falls freely through the horizon, turns around somewhere in the interior and then appears to re-emerge in the future universe. From the point of view of P_1, the observer P_2 appears to be going to $|w| \to \infty$. Eventually P_2 sends, from $|w| \to \infty$, a "final" lightray which is the generator of the corresponding horizon. Similarly one can consider signals received by infalling observers P_2 at late times, sent by infalling observers $P_{2'}$ at early times. Such observers $P_2, P_{2'}$ have $\dot{\tau}^2 \equiv (\frac{d\tau}{d\lambda})^2 = \frac{p_w^2}{m^2} + \tau^2 - \frac{\tau_0^3}{\tau}$, using (6.3.4), so that we have the proper time intervals

$$\Delta\lambda_{P_2} \sim \frac{1}{p_w}\Delta\tau \qquad (\text{near } \tau_0) \ , \qquad \Delta\lambda_{P_{2'}} \sim \frac{\Delta\tau}{\tau} \qquad (\text{early times}) \ . \qquad (6.3.5)$$

Ingoing lightray congruences of the form $U = w - \tau_* = c$ have a cross-sectional vector $v = w + \tau_*$. To analyse these transmitting-receiving events further, it is convenient to use Eddington-Finkelstein-type coordinates here: defining the ingoing coordinate $v = w + \tau_*$, the metric (6.3.1) becomes

$$ds^2 = f(\tau)dv^2 - 2dvd\tau + \tau^2 dx_i^2 \ , \tag{6.3.6}$$

and infalling geodesic observers at const-x_i (*i.e.* τ, w decreasing with proper time λ) are

$$f\dot{v}^2 - 2\dot{v}\dot{\tau} = -1 \ , \qquad -f\dot{v} + \dot{\tau} = -f\dot{w} = p_v > 0 \qquad (i.e.\ \dot{w} < 0) \ , \tag{6.3.7}$$

$$\Rightarrow \ \frac{d\tau}{d\lambda} = -\sqrt{f(\tau) + p_v^2} < 0 \ , \quad \frac{dv}{d\lambda} = -\frac{p_v + \sqrt{f(\tau)+p_v^2}}{f(\tau)} \ , \quad \frac{dv}{d\tau} = \frac{p_v + \sqrt{f(\tau)+p_v^2}}{f(\tau)\sqrt{f(\tau)+p_v^2}} \ . \tag{6.3.8}$$

Figure 6.2 shows infalling observers P_2 approaching the horizon, receiving at late times ($\tau \sim \tau_0$) light signals that emanate from early times ($\tau \sim \infty$): the latter can be thought of as signals transmitted by infalling observers $P_{2'}$ at early times. It can be seen from Figure 6.2 that such events (transmission-reception of such light signals) are consistent with the causal (lightcone) structure of the spacetime. The light rays in question have constant v which is very large and negative.

Let us denote the conserved momenta for the two geodesics P_2 and $P_{2'}$ by p_v and p_v' respectively. Suppose $P_{2'}$ sends out successive light signals along constant v and constant $v + dv$, at coordinate times τ' and $\tau' + d\tau'$, and these are received by P_2 at coordinate times τ and $\tau + d\tau$. The proper time between emission these signals is $d\lambda_{P_2'}$ while the proper time between reception of the same two signals by P_2 is $d\lambda_{P_2}$. Then equations (6.3.8) yield

$$\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} = \frac{f(\tau)}{f(\tau')} \frac{p_v' + \sqrt{f(\tau') + (p_v')^2}}{p_v + \sqrt{f(\tau) + p_v^2}} \tag{6.3.9}$$

When both observers are at rest, $p_v = p_v' = 0$ this leads to the standard formula for the gravitaional redshift/blueshift. In our setup $\tau \sim \tau_0$ while $\tau' \to \infty$, so that

$$f(\tau) \sim 3\tau_0(\tau - \tau_0) \ , \qquad f(\tau') \sim (\tau')^2 \ , \tag{6.3.10}$$

which leads to

$$\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} \sim \frac{3\tau_0(\tau - \tau_0)}{2p_v\tau'} \ . \tag{6.3.11}$$

We now need to express the ratio in (6.3.11) in terms of v. It is clear from the last equation in (6.3.8) that at early times the geodesic $P_{2'}$ is described by

$$v(\tau') = -\frac{1}{\tau'} + c' \ , \tag{6.3.12}$$

where c' is a constant. Note that we are considering light rays which have $v \sim -\infty$, so that the integration constant c' must be large and negative (since $(v - c') = -\frac{1}{\tau'}$ is small as $\tau' \to \infty$). The trajectory P_2 is described in the vicinity of $\tau = \tau_0$ by

$$\tau - \tau_0 \sim A \exp\left[\frac{3\tau_0}{2} v\right] \ , \tag{6.3.13}$$

where A is a finite constant of integration. Substituting (6.3.12) and (6.3.13) in (6.3.11) we get

$$\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} \sim -\frac{3A\tau_0}{2p_v}\,(v-c')\exp\left[\frac{3\tau_0}{2}v\right] . \tag{6.3.14}$$

Thus for a fixed proper time between the signals during emission, the proper time interval for reception becomes *exponentially small* as $v \to -\infty$.

It is interesting to compare the situation with pure de Sitter. Here $f(\tau) = \tau^2$ for all τ and the cosmological horizon is at $\tau = 0$. In this case the ratio of the proper time interval (6.3.9) for the observers P_2 and $P_{2'}$ becomes, instead of (6.3.11),

$$\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} \sim \frac{\tau^2}{2p_v\tau'} . \tag{6.3.15}$$

Near $\tau = 0$, the trajectory P_2 can be obtained by solving the second equation in (6.3.8) with $f(\tau) = \tau^2 \sim 0$. Since u is finite it is easy to see that one needs $p_v \neq 0$ and one gets

$$v(\tau) = -\frac{2}{\tau} + a , \tag{6.3.16}$$

where the constant of integration a is finite. The trajectory $P_{2'}$ is exactly the same as (6.3.12). Using this, the equation (6.3.15) becomes

$$\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} \sim -\frac{2(v-c')}{(v-a)^2p_v} \sim -\frac{2(v-c')}{v^2p_v} , \tag{6.3.17}$$

where in the second equation above we have used finiteness of a. Once again $\frac{d\lambda_{P_2}}{d\lambda_{P_2'}} \to 0$, however in a power law fashion. This is a much milder blueshift than what is experienced for our bluewall solution.

This exponentially vanishing blueshift is a reflection of the "crowding" of lightrays near the horizon. The energy flux that the infalling observer measures is $T_{\mu\nu}v^\mu v^\nu \sim T_{vv}\dot{v}^2$. From above, we see that the infalling observer thus crosses a diverging flux of incoming lightrays in finite proper time as he approaches the horizon[6], suggesting an instability. This is somewhat akin to the Reissner-Nordstrom black hole inner horizon (see *e.g.* [167]) where an infalling observer receives signals from the exterior region in vanishingly small proper time ("seeing entire histories in a flash"). However note that here, this occurs for the late time infalling observer only as he approaches the horizon and only from signals emanating at early times from "infinity" ($|w| \to \infty$). Now applying the energy-momentum calculation earlier gives an imaginary energy density $\langle T_{ij}\rangle$: it is interesting to ask if this is the dual CFT signature of the blue-shift instability. It would be interesting to explore these further, perhaps keeping in mind black holes, firewalls and entanglement [168–170].

[6]It is a reasonable assumption that T_{vv} for the lightrays follows a power law in v, akin to [166]. From (6.3.8), (6.3.13), we have $\dot{v} \sim e^{-(3\tau_0/2)v}|_{v\to-\infty} \to \infty$. Thus $T_{vv}\dot{v}^2$ diverges as $\tau \to \tau_0$.

Chapter 7

Integrability Lost

7.1 Introduction

AdS/CFT duality [5] was a major step towards the goal of recasting large-N QCD as a string theory. One of the interesting aspects of AdS/CFT is integrability (see the review [171]). Integrability has allowed us to obtain many classical solutions of the theory that would otherwise have been impossible to find [27, 172]. One may approach integrability from two sides corresponding to the two extreme values of the 't Hooft coupling. On the supergravity side (which is a good description as $\lambda \to \infty$), integrability of the classical sigma model on $AdS_5 \times S^5$ was established for bosonic sector in [13] and fully completed with the inclusion of fermions in [173]. It has been shown that classical string motion in $AdS_5 \times S^5$ has an infinite number of conserved charges.[1] This is the closest to solvability that we can get currently. It is also to be noted that the study of integrability in non-linear sigma models has a long history [174, 175]. The other approach to integrability is perturbative in the weakly coupled gauge theory [171]. We will restrict ourselves to the former approach.

An important open question is whether integrability can be extended to more QCD-like theories.[2] The original form of AdS/CFT duality was for conformally invariant $\mathcal{N} = 4$ SYM theory but this can be deformed in various ways to produce string duals to confining gauge theories with less or no supersymmetry. The construction of [13, 173] does not readily generalize to these less symmetric backgrounds. One prime example of a confining background is the AdS soliton [176, 28]. Similar geometries have been used extensively to model various aspects of QCD in the context of holography [177]. Here we will look at the question of integrability of bosonic strings on an AdS soliton background. Although it is much more interesting to explore full quantum integrability, to begin with we may ask whether we can find enough conserved charges even at a purely classical level. The answer turns out to be negative. By choosing a class of simple classical string configurations, we show that the Lagrangian reduces to a set of coupled harmonic and anharmonic oscillators that correspond to the size fluctuation and the center of mass of motion of the string. The oscillators decouple in the low energy limit. With increasing energy the oscillators become nonlinearly coupled. Many such systems are well known to be chaotic and nonintegrable [178, 179]. It is no surprise that our system also shows a similar behaviour. Possibly chaotic behaviour of a test string has been argued previously in black hole backgrounds [180, 181]. However our problem is somewhat different as we are looking at a zero temperature geometry without a horizon. In a companion paper [182] non-integrability of string theory in $AdS_5 \times T^{1,1}$ is discussed.

[1] However the commutator algebra of such charges is not fully understood.

[2] This is one of the motivations discussed in the introduction of [173].

We start by discussing the AdS soliton background and our test string ansatz§7.2. We discuss some quasi-periodic solutions for small oscillation regime. Our argument for the non-integrability is via numerical solution of the EOM's§7.3. In a certain regime of parameter space the system shows a zigzag aperiodic motion characteristic of a chaotic system. We then look at the phase space. Integrability implies the existence of a regular foliation of the phase space by invariant manifolds, known as KAM (Kolmogorov-Arnold-Moser) tori, such that the Hamiltonian vector fields associated with the invariants of the foliation span the tangent distribution. Our numerics shows how this nice foliation structure is gradually lost as we increase the energy of the system§7.3.1. To be complete we also calculate Lyapunov indices for various parameter ranges and find large positive values in chaotic regimes§8.4.2. We discuss open questions and possible extensions in the conclusion§7.4.

7.2 Setup

The AdS soliton ($\mathcal{M}$) metric for an asymptotically AdS_{d+2} background is given by [176],

$$\begin{aligned} ds^2 &= L^2\alpha'\left(e^{2u}(-dt^2+T_{2\pi}(u)d\theta^2+dw_i^2)+\frac{1}{T_{2\pi}(u)}du^2\right)\,, \\ \text{where}\ \ T_{2\pi}(u) &= 1-\left(\frac{d+1}{2}e^u\right)^{-(d+1)}\,. \end{aligned} \tag{7.2.1}$$

At large u, $T_{2\pi}(u)\approx 1$ and (7.2.1) reduces to AdS_{d+2} in Poincaré coordinates. However one of the spatial boundary coordinates θ is compactified on a circle. The remaining boundary coordinates w_i and t remain non-compact. The dual boundary theory may be thought of as a Scherk-Schwarz compactification on the θ cycle. The θ cycle shrinks to zero at a finite value of u, smoothly cutting off the IR region of AdS. This cutoff dynamically generates a mass scale in the theory, very much like in real QCD. The resulting theory is confining and has a mass gap.

Here we will work with[3] $d=4$ and make a coordinate transformation $u=u_0+ax^2$ with $u_0=\log(2/5)$ and $a=5/4$, such that $T_{2\pi}(u_0)=0$ and for small x the x-θ part of the metric looks flat, $ds^2\approx dx^2+x^2\,d\theta^2$. In these coordinates the metric is

$$\begin{aligned} ds^2 &= e^{2u_0+2ax^2}(-dt^2+T(x)d\theta^2+dw_i^2)+\frac{4a^2x^2}{T(x)}dx^2\,, \\ \text{where}\ \ T(x) &= 1-e^{-5ax^2}\,. \end{aligned} \tag{7.2.2}$$

7.2.1 Classical string in AdS-soliton

We start with the Polyakov action:

$$S_P=-\frac{1}{2\pi\alpha''}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{ab}G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu \tag{7.2.3}$$

[3] The analysis for any other $d\geq 3$ proceeds along the same lines and almost identical results can be obtained.

where X^μ are the coordinates of the string, $G_{\mu\nu}$ is the spacetime metric of the fixed background, γ_{ab} is the worldsheet metric, the indices a, b represent the coordinates on the worldsheet of the string which we denote as (τ, σ). We work in the conformal gauge $\gamma_{ab} = \eta_{ab}$ and use the following embedding for a closed string (partially motivated by [180]):

$$\begin{aligned} &t = t(\tau),\ \theta = \theta(\tau),\ x = x(\tau),\\ &w_1 = R(\tau)\cos\left(\phi(\sigma)\right),\ w_2 = R(\tau)\sin\left(\phi(\sigma)\right)\ \text{with}\ \phi(\sigma) = \alpha\sigma\,. \end{aligned} \tag{7.2.4}$$

The string is at located at a certain value of u and is wrapped around a pair of w-directions as a circle of radius R. It is allowed to move along the potential in u direction and change its radius R. Here $\alpha \in \mathbb{Z}$ is the winding number of the string. The test string Lagrangian takes the form:

$$\mathcal{L} \propto \frac{2}{5}e^{2ax^2}\left\{-\dot{t}^2 + T(x)\dot{\theta}^2 + \dot{w}_i^2 - w_i'^2\right\} + \frac{2a^2x^2}{T(x)}\dot{x}^2 \tag{7.2.5}$$

$$= \frac{2}{5}e^{2ax^2}\left\{-\dot{t}^2 + T(x)\dot{\theta}^2 + \dot{R}^2 - R^2\alpha^2\right\} + \frac{2a^2x^2}{T(x)}\dot{x}^2\,, \tag{7.2.6}$$

where dot and prime denote derivatives w.r.t τ and σ respectively. The coordinates t and θ are *ignorable* and the corresponding momenta are constants of motion. The test string Lagrangian differs from a test particle Lagrangian because of the potential term in $R(\tau)$. The coordinate R would be ignorable without a potential term. In general it can be easily argued that for a generic motion of a test particle in an AdS soliton background, all the coordinates other than x are ignorable and the equations of motion can be reduced to a Lagrangian dynamics in one variable x. This implies integrability.

Here the conserved momenta conjugate to t and θ are,

$$\begin{aligned} p_t &= -\frac{4}{5}e^{2ax^2}\dot{t} \equiv -E\\ p_\theta &= \frac{4}{5}e^{2ax^2}T(x)\dot{\theta} \equiv k. \end{aligned} \tag{7.2.7}$$

The conjugate momenta corresponding to the other coordinates are:

$$\begin{aligned} p_R &= \frac{4}{5}e^{2ax^2}\dot{R}\\ p_x &= \frac{4a^2x^2}{T(x)}\dot{x}\,. \end{aligned} \tag{7.2.8}$$

With these we can construct the Hamiltonian density:

$$\mathcal{H} = \frac{5}{8}\left\{\left(-E^2 + \frac{k^2}{T(x)} + p_R^2\right)e^{-2ax^2} + \frac{T(x)p_x^2}{5a^2x^2} + \frac{16}{25}R^2\alpha^2e^{2ax^2}\right\} \tag{7.2.9}$$

Hamilton's equations of motion give:

$$\dot{R} = \frac{5}{4}p_Re^{-2ax^2} \tag{7.2.10}$$

$$\dot{p_R} = -\frac{4}{5}R\alpha^2 e^{2ax^2} \tag{7.2.11}$$

$$\dot{x} = \frac{T(x)p_x}{4a^2x^2} \tag{7.2.12}$$

$$\dot{p_x} = -\frac{5}{8}\left\{4ax\left[\left(E^2 - \frac{k^2}{T(x)} - p_R^2\right)e^{-2ax^2} + \frac{16}{25}R^2\alpha^2 e^{2ax^2}\right] - \frac{2T(x)p_x^2}{5a^2x^3} + \left[\frac{p_x^2}{5a^2x^2} - \frac{k^2e^{-2ax^2}}{T(x)^2}\right]\partial_x T(x)\right\} \tag{7.2.13}$$

We also have the constraint equations:

$$G_{\mu\nu}\left(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu\right) = 0\,, \tag{7.2.14}$$

$$G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0\,. \tag{7.2.15}$$

The first equation takes the form $\mathcal{H} = 0$ [4] and the second equation is automatically satisfied for our embedding.

7.3 Dynamics of the system

At $k = 0$, an exact solution to the EOM's is a fluctuating string at the tip of the geometry, given by

$$x(\tau) = 0 \tag{7.3.1}$$

$$R(\tau) = A\sin(\tau + \phi). \tag{7.3.2}$$

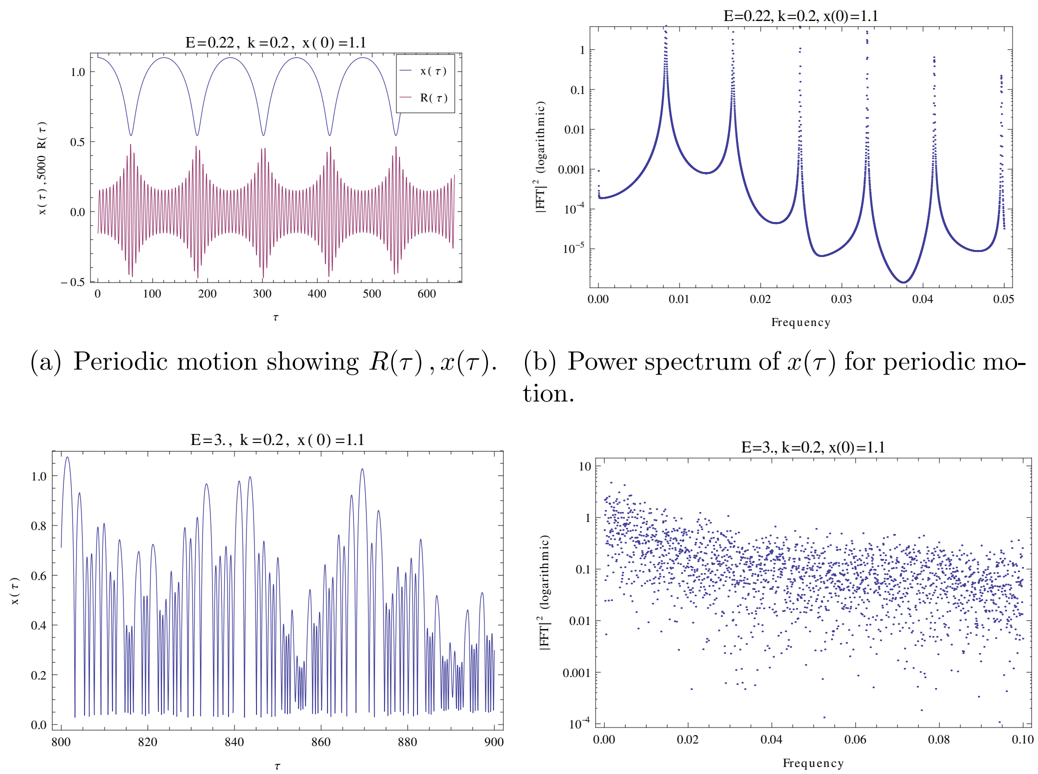
where A, ϕ are integration constants. No such solution with constant $x(\tau)$ exists for $k \neq 0$. However one may construct approximate quasi-periodic solutions for small $R(\tau), p_R(\tau)$. It should be noted that with $R, p_R = 0$ the zero energy condition Eqn.(7.2.14) becomes similar to the condition for a massless particle and the string escapes from AdS following a null geodesic. For small nonzero values of R_0, p_R, the motion in the x-direction will have a long time period. However the fluctuations in the radius will have a frequency proportional to the winding number which is of $\mathcal{O}(1)$. This is a perfect setup to do a two scale analysis. In the equation for $\dot{p_R}$ we may replace $R(\tau)^2$ by a time average value. With this approximation, motion in the x-direction becomes an anharmonic problem in one variable which is solvable in principle. The motion is also periodic [Fig.7.1(a)]. On the other hand to solve for $R(\tau)$ we treat $x(\tau)$ as a slowly varying field. In this approximation the solution for $R(\tau)$ is given by

$$R(\tau) \approx \exp(-a\,x(\tau)^2)A\sin(\tau + \phi). \tag{7.3.3}$$

[4] The Hamiltonian constraint could be tuned to a nonzero value by adding a momentum in a decoupled compact direction. For example if the space is $\mathcal{M} \times S^5$ then giving a non-zero angular momentum in an S^5 direction would do the job. However we choose to confine the motion within $\mathcal{M}$ here.

Hence $R(\tau)$ is quasi-periodic [Fig.7.1(a)]. We have verified that in the small R regime, the semi-analytic solution matches quite well with our numerics.

Once we start moving away from the small R limit the the above two scale analysis breaks down and the nonlinear coupling between two oscillators gradually becomes important. In short the coupling between oscillators tends to increase as we increase the energy of the string. Due to the nonlinearity, the fluctuations in the x- and R-coordinates influence each other and the motions in both coordinates become aperiodic. Eventually the system becomes completely chaotic [Fig.7.1(c)]. The power spectrum changes from peaked to noisy as chaos sets in [Fig.7.1]. As we discuss in the next subsection, the pattern follows general expectations from the KAM theorem.



(a) Periodic motion showing $R(\tau)\,, x(\tau)$.

(b) Power spectrum of $x(\tau)$ for periodic motion.

(c) Chaotic motion of string showing $x(\tau)$.

(d) Power spectrum of $x(\tau)$ for chaotic motion.

Figure 7.1: Numerical simulation of the motion of the string and the corresponding power spectra for small and large values of E. The initial momenta $p_x(0), p_R(0)$ have been set to zero. For a small value of $E = 0.22$, we see a (quasi-)periodicity in the oscillations. The power spectrum shows peaks at discrete harmonic frequencies. However for a larger value of $E = 3.0$, the motion is no longer periodic. We only show $x(\tau)$ but $R(\tau)$ is similar. The power spectrum is white.

7.3.1 Poincaré sections and the KAM theorem

An integrable system has the same number of conserved quantities as degrees of freedom. A convenient way to understand these conserved charges is by looking at the phase space. Let us assume that we have a system with N position variables q_i with conjugate momenta p_i. The phase space is $2N$-dimensional. Integrability means that there are N conserved charges $Q_i = f_i(p, q)$ which are constants of motion. One of them is the energy. These charges define a N-dimensional surface in the phase space which is a topological torus (KAM torus). The $2N$-dimensional phase space is nicely foliated by these N-dimensional tori. In terms of action-angle variables (I_i, θ_i) these tori just become surfaces of constant action. With each torus there are N associated frequencies $\omega_i(I_i)$, which are the frequencies of motion in each of the action-angle directions.

It is interesting to study what happens to these tori when an integrable Hamiltonian is perturbed by a small nonintegrable piece. The KAM theorem states that most tori survive, but suffer a small deformation [178, 179]. However the *resonant* tori which have rational ratios of frequencies, i.e. $m_i\omega_i = 0$ with $m \in \mathcal{Q}$, get destroyed and motion on them become chaotic. For small values of the nonintegrable perturbations, these chaotic regions span a very small portion of the phase space and are not readily noticeable in a numerical study. As the strength of the nonintegrable interaction increases, more tori gradually get destroyed. A nicely foliated picture of the phase space is no longer applicable and the trajectories freely explore the entire phase space with energy as the only constraint. In such cases the motion is completely chaotic.

To numerically investigate this gradual disappearance of foliation we look at the Poincaré sections. For our system, the phase space has four variables x, R, p_x, p_R. If we fix the energy we are in a three dimensional subspace. Now if we start with some initial condition and time-evolve, the motion is confined to a two dimensional torus for the integrable case. This 2d torus intersects the $R = 0$ hyperplane at a circle. Taking repeated snapshots of the system as it crosses $R = 0$ and plotting the value of (x, p_x), we can reconstruct this circle. Furthermore varying the initial conditions (in particular we set $R(0) = 0$, $p_x(0) = 0$, vary $x(0)$ and determine $p_R(0)$ from the energy constraint), we can expect to get the foliation structure typical of an integrable system.

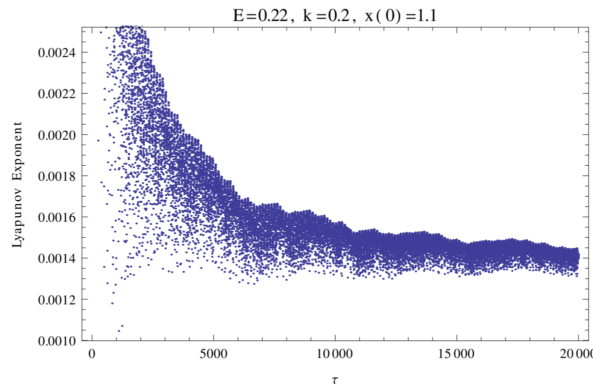
Indeed we see that for smaller value of energies, a distinct foliation structure exists in the phase space [Fig.7.3(a)]. However as we increase the energy some tori get gradually dissolved [Figs.7.3(b)-7.3(f)]. The tori which are destroyed sometimes get broken down into smaller tori [Figs.7.3(c)-7.3(d)]. Eventually the tori disappear and become a collection of scattered points known as cantori. However the breadths of these cantori are restricted by the undissolved tori and other dynamical elements. Usually they do not span the whole phase space [Figs.7.3(c)-7.3(f)]. For sufficiently large values of energy there are no well defined tori. In this case phase space trajectories are all jumbled up and trajectories with very different initial conditions come arbitrary close to each other [Fig.7.3(h)]. The mechanism is very similar to what happens in well known nonintegrable systems like Hénon-Heiles models [178, 179].

7.3.2 Lyapunov exponent

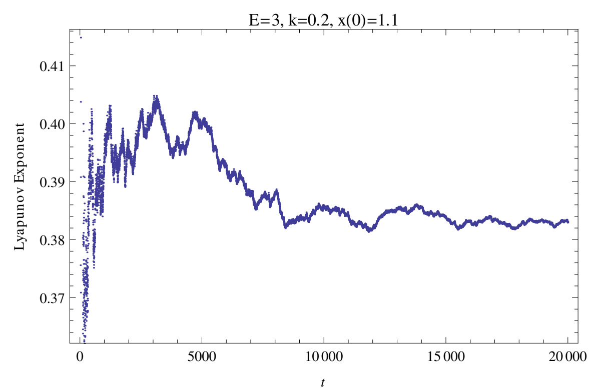
One of the trademark signatures of chaos is the sensitive dependence on initial conditions, which means that for any point X in the phase space, there is (at least) one point arbitrarily close to X that diverges from X. The separation between the two is also a function of the initial location and has the form $\Delta X(X_0, \tau)$. The Lyapunov exponent is a quantity that characterizes the rate of separation of such infinitesimally close trajectories. Formally it is defined as,

$$\lambda = \lim_{\tau \to \infty} \lim_{\Delta X_0 \to 0} \frac{1}{\tau} \ln \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)} \tag{7.3.4}$$

In practice we use an algorithm by Sprott [183], which calculates λ over short intervals and then takes a time average. We should expect to observe that, as time τ is increased, λ settles down to oscillate around a given value. For trajectories belonging to the KAM tori, λ is zero, whereas it is expected to be non-zero for a chaotic orbit. We verify such expectations for our case. We calculate λ with various initial conditions and parameters. For apparently chaotic orbits we observe a nicely convergent positive λ [Fig.7.2].

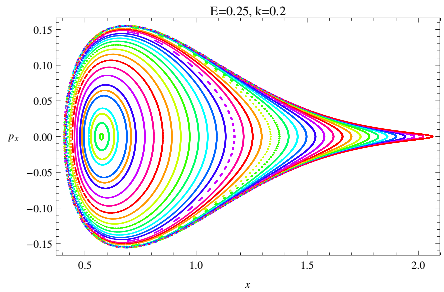


(a) Lyapunov index for periodic motion.

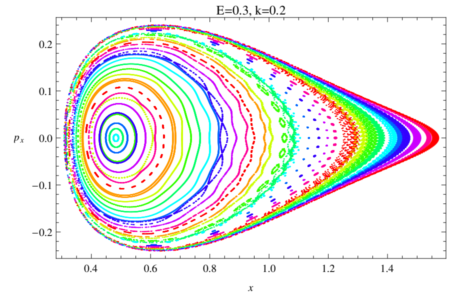


(b) Lyapunov index for chaotic motion.

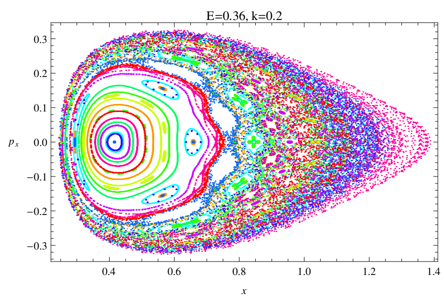
Figure 7.2: Lyapunov indices for the same values of parameters as in Fig.(7.1). For $E = 0.22$, the Lyapunov exponent falls off to zero. For $E = 3.0$, the Lyapunov exponent converges to a positive value of about 0.38.



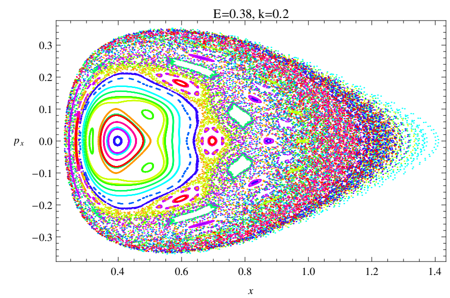
(a) $E = 0.25$



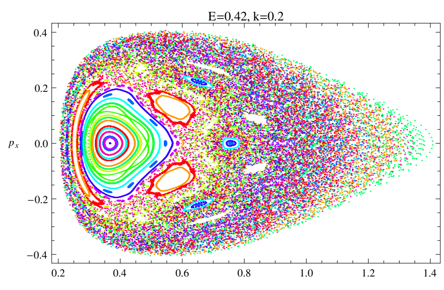
(b) $E = 0.30$



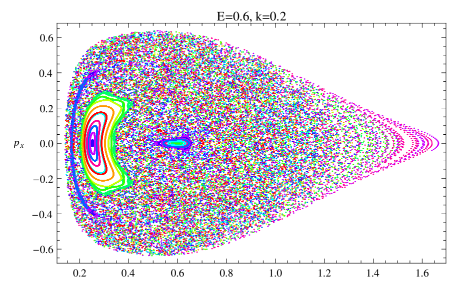
(c) $E = 0.36$



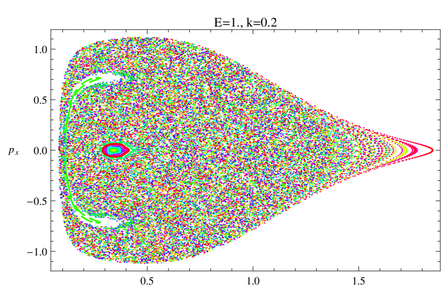
(d) $E = 0.38$



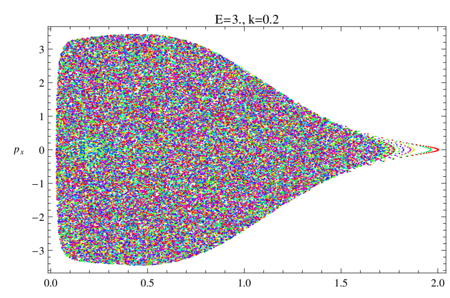
(e) $E = 0.42$



(f) $E = 0.60$



(g) $E = 1.0$



(h) $E = 3.0$

Figure 7.3: Poincaré sections demonstrate breaking of the KAM tori en route to chaos. Each colour represents a different initial condition. For smaller values of E the sections of the KAM tori are intact curves, except for the resonant ones. The tori near the resonant ones start breaking as E is increased. For very large values of E all the colours get mixed – this indicates that all the tori get broken and they fill the entire phase space.

7.4 Conclusion

In this work, we argue using numerical techniques that classical string motion in the AdS soliton background is nonintegrable. This certainly restricts the solvability of such theories. Also our results give a perspective on how much of classical integrability may be extended to various holographic backgrounds, especially those with less symmetry than AdS_5. Non-integrability is possibly quite generic and might be demonstrated by studying time evolution of simple string configurations. In particular, the basic construction of our work seems to be extendible to other confining backgrounds [184]. It would be nice to explore these directions. Another interesting extension would be to include world sheet fermions. It is also to be kept in mind that the AdS soliton is not an exact string background and possibly has α' corrections. However these effects are unlikely to change the main result of the current work.

One big question is the implication of our result for the full quantum spectrum. This is in turn connected with the glueball spectrum of the dual theory. The low-lying string modes will possibly be decoupled from the center of mass motion and will be more like the flat space counterpart. However the higher modes will be affected by the nonlinearity. Many exact results are known for the quantum spectrum of a chaotic theory. It would be interesting to explore how these results apply in a mini-superspace quantization of our system.

It would also be interesting understand more on the gauge theory side [185]. The dual gauge theory is the $\mathcal{N} = 4$ SYM theory with one compact direction with aperiodic boundary condition for fermions. This breaks supersymmetry and the low energy dynamics of the theory is confining. Here, a simple change in boundary condition is changing the integrability of the theory. It is also to be noted that the full SYM theory with $\frac{1}{N}$ corrections is almost surely nonintegrable. Any apparent integrability would then be a property of the large-N saddle points.

Chapter 8

Chaos around Holographic Regge trajectories

8.1 Introduction

The fact that the quantum numbers of certain operators or states in field theory can be well described by the corresponding classical string is the idea at the heart of Regge trajectories where the hadronic relationship $J \sim M^2$ is realized by a spinning string. This fact has a long history dating back to the Chew-Frautschi plots [186]. In the context of the AdS/CFT correspondence a better understanding of the role played by classical trajectories has been at the center of a substantial part of recent developments. More generally, the AdS/CFT correspondence provides a dictionary that identifies states in string theory with operators in field theory. One of the most prominent examples is provided by the Berenstein-Maldacena-Nastase (BMN) operators. The BMN operators [187] can be described as a string moving at the speed of light in the large circle of S^5, the operator corresponding to the ground states is given by $\mathcal{O}^J = (1/\sqrt{JN^J})\mathrm{Tr}\, Z^J$. Another interesting class of operators which are nicely described as semiclassical strings in the $AdS_5 \times S^5$ background are the Gubser-Klebanov-Polyakov (GKP) operators discussed in [27]. They are natural generalizations of twist-two operators in QCD and in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills they look like $\mathrm{Tr}\Phi^I \nabla_{(a_1} \ldots \nabla_{a_n)} \Phi^I$. A very important property of these operators is that their anomalous dimension can be computed using a simple classical calculation and yields a prediction for the result at strong coupling $\Delta - S = (\sqrt{\lambda}/\pi) \ln S$. This expression is similar to the QCD relation obtained originally by Gross and Wilczek [188].

Right after the original formulation of the AdS/CFT correspondence [5, 19, 18, 15] an important direction emerged surrounding the question of how to approach more realistic theories using the methods of the gauge/gravity correspondence. There is by now a well established body of results in this direction. In particular, general conditions on the supergravity backgrounds have been found that correspond to the existence of the area law for the Wilson loop in the field theory [189, 190]. Similarly, the classical string configuration corresponding to the Regge trajectories have been extensively studied.

In this paper we study properties of a configuration of classical strings in supergravity backgrounds dual to confining field theories. Our study goes beyond particular trajectories and explores the phase space. We show that a class of strings that naturally generalizes those corresponding to Regge trajectories is non-integrable. Further, we show explicitly that the motion of such strings is chaotic with the Regge trajectories being an integrable island in the phase space. It turns out that technically the problem is similar to the study of the spectrum of quadratic fluctuations. The study of quantum corrections to Regge trajectories in the context of the AdS/CFT correspondence was initiated in [191] and was extended to other backgrounds [192].

Other recent studies of chaotic behavior of classical strings in the context of the gauge/gravity correspondence include [193, 194, 182, 195]. We will in particular draw on modern Hamiltonian methods used in [195] and the concrete discussion of the AdS soliton background presented in [194].

One of the questions driving our program is how to interpret chaos in AdS/CFT, that is, what is the field theory dual of chaotic quantities? We ask whether we can test some of the ideas in the context of confinement. Are there any universal features of various confining theories? We come up with a unified approach to study integrability in a class of confining backgrounds that include many of the commonly-cited examples of confining geometries like Klebanov-Strassler, Maldacena-Núñez, Witten QCD and AdS-soliton. QCD, in the asymptotic free regime, has been argued to be possibly integrable. One particularly important lead in this direction comes from the integrable Regge trajectories. However our results show that the Regge trajectories are just integrable islands in a wider sea of nonintegrability. One is naturally led to ask the question whether there are more similar subdomains of integrability. In this work, we answer some of the questions above, while some of them still remain open.

The rest of the paper is organized as follows. In section 8.2 we consider two classes of closed spinning strings and discuss some of their properties in supergravity backgrounds dual to confining field theories. In section 8.3, for the sake of the readers, we review the main results of the literature of analytic non-integrability of Hamiltonian systems. In that section we also show that the motion of the string in supergravity backgrounds dual to confining field theories is non-integrable using analytic methods. Since analytic non-integrability is not a sufficient condition for chaotic behavior, we study numerically a particular background and show strong evidence of chaotic behavior in section 8.4. We conclude in section 8.5. In appendix D we present the main equation in the non-integrability paradigm of various supergravity backgrounds dual to known confining field theories.

8.2 Closed spinning strings in supergravity backgrounds

The Polyakov action and the Virasoro constraints characterizing the classical motion of the fundamental string are:

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{-g}g^{ab}G_{MN}\partial_a X^M \partial_b X^N, \tag{8.2.1}$$

where G_{MN} is the spacetime metric of the fixed background, X^μ are the coordinates of the string, g_{ab} is the worldsheet metric, the indices a, b represent the coordinates on the worldsheet of the string which we denote as (τ, σ). We will use to work in the conformal gauge in which case the Virasoro constraints are

$$\begin{aligned} 0 &= G_{MN}\dot{X}^M X'^N, \\ 0 &= G_{MN}\left(\dot{X}^M \dot{X}^N + X'^M X'^N\right), \end{aligned} \tag{8.2.2}$$

where dot and prime denote derivatives with respect to τ and σ respectively.

We are interested in the classical motion of the strings in background metrics G_{MN} that preserve Poincaré invariance in the coordinates (X^0, X^i) where the dual field theory lives:

$$ds^2 = a^2(r) dx_\mu dx^\mu + b^2(r) dr^2 + c^2(r) d\Omega_d^2. \tag{8.2.3}$$

Here $x^\mu = (t, x_1, x_2, x_3)$ and $d\Omega_d^2$ represents the metric on a d-dimensional sub-space that, can also have r-dependent coefficients. In the case of supergravity backgrounds in IIB, we have $d = 5$ but we leave it arbitrary to also accommodate backgrounds in 11-d supergravity in which case $d = 6$.

The relevant classical equations of motion for the string sigma model in this background are

$$\begin{aligned}\partial_a(a^2(r)\eta^{ab}\partial_b x^\mu) &= 0,\\ \partial_a(b^2(r)\eta^{ab}\partial_b r) &= \frac{1}{2}\partial_r(a^2(r))\eta^{ab}\partial_a x_\mu \partial_b x^\mu + \frac{1}{2}\partial_r(b^2(r))\eta^{ab}\partial_a r \partial_b r.\end{aligned} \tag{8.2.4}$$

They are supplemented by the Virasoro constraints. We will construct spinning strings by starting with the following Ansatz (**Ansatz I**):

$$\begin{aligned}x^0 &= e\,\tau,\\ x^1 &= f_1(\tau)\,g_1(\sigma), \qquad x^2 = f_2(\tau)\,g_2(\sigma),\\ x^3 &= \text{constant}, \qquad \mathrm{r} = \mathrm{r}(\sigma).\end{aligned} \tag{8.2.5}$$

We will also consider a slight modification of the above Ansatz as follows (**Ansatz II**):

$$\begin{aligned}x^0 &= e\,\tau,\\ x^1 &= f_1(\tau)\,g_1(\sigma), \qquad x^2 = f_2(\tau)\,g_2(\sigma),\\ x^3 &= \text{constant}, \qquad \mathrm{r} = \mathrm{r}(\tau).\end{aligned} \tag{8.2.6}$$

The main modification is that the radial coordinate is now a function of the worldsheet time $r = r(\tau)$.

With Ansatz I (8.2.5) the equation of motion for x^0 is trivially satisfied. Let us first show that the form of the functions f_i is fairly universal for this Ansatz. The equation of motion for x^i is

$$-\,a^2\,g_i \ddot{f}_i + f_i \partial_\sigma(a^2 g_i') = 0, \tag{8.2.7}$$

where a dot denotes a derivative with respect to τ and a prime denotes a derivative with respect to σ. Enforcing a natural separation of variables we see that

$$\ddot{f}_i + (e\,\omega)^2 f_i = 0, \qquad \partial_\sigma(a^2\,g_i') + (e\,\omega)^2 a^2 g_i = 0. \tag{8.2.8}$$

The radial equation of motion is

$$(b^2 r')' = \frac{1}{2}\partial_r(a^2)\big[e^2 - g_i^2 \dot{f}_i^2 + f_i^2 g'^2_i\big] + \frac{1}{2}\partial_r(b^2) r'^2. \tag{8.2.9}$$

Finally the nontrivial Virasoro constraint becomes

$$b^2 r'^2 + a^2\big[-e^2 + g_i^2 \dot{f}_i^2 + f_i^2 g_i'^2\big] = 0. \tag{8.2.10}$$

We are particularly interested in the integrals of motion describing the energy and the angular momentum

$$E = \frac{e}{2\pi\alpha'} \int a^2 d\sigma, \tag{8.2.11}$$

$$J = \frac{1}{2\pi\alpha'} \int a^2 \big[x_1 \partial_\tau x_2 - x_2 \partial_\tau x_1\big] d\sigma = \frac{1}{2\pi\alpha'} \int a^2 g_1 g_2 \big[f_1 \partial_\tau f_2 - f_2 \partial_\tau f_1\big] d\sigma \tag{8.2.12}$$

The above system can be greatly simplified by further taking the following particular solution:

$$f_1 = \cos e\omega\,\tau, \quad f_2 = \sin e\omega\,\tau, \quad \text{and} \quad \mathrm{g}_1 = \mathrm{g}_2 = \mathrm{g}. \tag{8.2.13}$$

Under these assumptions the equation of motion for r and the Virasoro constraint become

$$(b^2 r')' - \frac{1}{2}\partial_r(a^2)\big[e^2 - (e\omega)^2\, g^2 + g'^2\big] - \frac{1}{2}\partial_r(b^2) r'^2 = 0, \tag{8.2.14}$$

$$b^2 r'^2 + a^2\big[-e^2 + (e\omega)^2\, g^2 + g'^2\big] = 0. \tag{8.2.15}$$

The angular momentum is then

$$J = \frac{e\omega}{2\pi\alpha'} \int a^2 g^2 d\sigma. \tag{8.2.16}$$

Since we are working in Poincaré coordinates the quantity canonically conjugate to time is the energy of the corresponding state in the four dimensional theory. The angular momentum of the string describes the spin of the corresponding state. Thus a spinning string in the Poincaré coordinates is dual to a state of energy E and spin J. In order for our semiclassical approximation to be valid we need the value of the action to be large, this imply that we are considering gauge theory states in the IR region of the gauge theory with large spin and large energy. In the cases we study, expressions (8.2.11) and (8.2.16) yield a dispersion relation that can be identified with Regge trajectories.

8.2.1 Regge trajectories from closed spinning strings in confining backgrounds

Let us show that there exists a simple solution of the equations of motion (8.2.14) for any gravity background dual to a confining gauge theory. The conditions for a SUGRA background to be dual to a confining theory have been exhaustively explored [189, 190] using the fact that the corresponding Wilson loop in field theory should exhibit area law behavior. The main idea is to translate the condition for the vev of the rectangular Wilson loop to display an area law into properties that the metric of the supergravity background must satisfy through the identification of the vacuum expectation value of the Wilson loop with the value of the action of the corresponding classical string. It has been established that one set of necessary conditions is for g_{00} to have a nonzero minimum at some point r_0 usually known as the end of the space wall [189, 190]. Note that precisely these two conditions ensure the existence of a solution of (8.2.14). Namely, since $g_{00} = a^2$ we see that for a point $r = r_0 =$ constant is a solution if

$$\partial_r(g_{00})|_{r=r_0} = 0, \qquad g_{00}|_{r=r_0} \neq 0. \tag{8.2.17}$$

The first condition solves the first equation in (8.2.14) and the second condition makes the second equation nontrivial. Interestingly, the second condition can be interpreted as enforcing that the quark-antiquark string tension be nonvanishing as it determines the value of the string action. It is worth mentioning that due to the UV/IR correspondence in the gauge/gravity duality the radial direction is identified with the energy scale. In particular, $r \approx r_0$ is the gravity dual of the IR in the gauge theory. Thus, the string we are considering spins in the region dual to the IR of the gauge theory. Therefore we can conclude that it is dual to states in the field theory that are characteristic of the IR.

Let us now explicitly display the Regge trajectories. The classical solution is given by (8.2.5) with $g(\sigma)$ solving the second equation from (8.2.14), that is, $g(\sigma) = (1/\omega)\sin(e\omega\sigma)$. Imposing the periodicity $\sigma \to \sigma + 2\pi$ implies that $e\omega = 1$ and hence

$$x^0 = e\,\tau, \qquad x^1 = e\cos\tau\,\sin\sigma, \qquad x^2 = e\sin\tau\,\sin\sigma. \tag{8.2.18}$$

The expressions for the energy and angular momentum of the string states are:

$$E = 4\frac{e\,g_{00}(r_0)}{2\pi\alpha'}\int d\sigma = 2\pi g_{00}(r_0)T_s e, \qquad J = 4\frac{g_{00}(r_0)e^2}{2\pi\alpha'}\int \sin^2\sigma d\sigma = \pi g_{00}(r_0)T_s e^2. \tag{8.2.19}$$

Defining the effective string tension as $T_{s,\ eff} = g_{00}(r_0)/(2\pi\alpha')$ and $\alpha'_{eff} = \alpha'/g_{00}$ we find that the Regge trajectories take the form

$$J = \frac{1}{4\pi T_{s,\ eff}}E^2 \equiv \frac{1}{2}\alpha'_{eff}\ t. \tag{8.2.20}$$

Notice that the main difference with respect to the result in flat space dating back to the hadronic models of the sixties is that the slope is modified to $\alpha'_{eff} = \alpha'/g_{00}$. It is expected that a confining background will have states that align themselves in Regge trajectories.

8.2.2 Ansatz II

In this subsection we consider the Ansatz given in equation (8.2.6). Note that the analysis given in the previous sections can be applied *mutatis mutandis* to this Ansatz. In particular, the separation of variables described in equation (8.2.8) can be performed in a symmetric way and one obtains:

$$g_i'' + \alpha^2 g_i = 0, \qquad \partial_\tau(a^2\,\partial_\tau f_i) + \alpha^2 a^2\, f_i = 0. \tag{8.2.21}$$

The Ansatz given in (8.2.6, 8.2.13) becomes

$$\begin{aligned} t &= t(\tau), \qquad r = r(\tau), \\ x_1 &= R(\tau)\sin\alpha\sigma, \qquad x_2 = R(\tau)\cos\alpha\sigma. \end{aligned} \tag{8.2.22}$$

The Polyakov action is:

$$\mathcal{L} \propto a^2(r)\big[-\dot{t}^2 + \dot{R}^2 - \alpha^2 R^2\big] + b^2(r)\dot{r}^2. \tag{8.2.23}$$

The above Ansatz satisfies the first constraint automatically and the second constraint leads to a Hamiltonian constraint:

$$a^2(r)[\dot{t}^2 + \dot{R}^2 + \alpha^2 R^2] + b^2(r)\dot{r}^2 = 0\,. \tag{8.2.24}$$

We also have that

$$\dot{t} = E/a^2(r), \tag{8.2.25}$$

where E is an integration constant. This gives

$$\mathcal{L} \propto -\frac{E^2}{a^2(r)} + a^2(r)\big[\dot{R}^2 - \alpha^2 R^2\big] + b^2(r)\dot{r}^2\,. \tag{8.2.26}$$

From the above Lagrangian density the equations of motion for $r(\tau)$ and $R(\tau)$ are

$$\begin{aligned} \frac{d}{d\tau}\left(b^2(r)\frac{d}{d\tau}r(\tau)\right) &= \frac{E^2}{a^3(r)}\frac{d}{dr}a(r) + a(r)\frac{d}{dr}a(r)\big[\dot{R}^2 - \alpha^2 R^2\big] + b(r)\frac{d}{dr}b(r)(\frac{d}{d\tau}r)^2, \\ \frac{d}{d\tau}\left(a^2(r)\frac{d}{d\tau}R(\tau)\right) &= -\alpha^2 a^2(r)R(\tau). \end{aligned} \tag{8.2.27}$$

We can once again check the claim that for confining backgrounds there is always a confining wall which defines a straight line solution. Since one can always argue for confining backgrounds,

$$a(r) \approx a_0 - a_2(r - r_0)^2. \tag{8.2.28}$$

It is easily seen that in this region both equations above can be satisfied. The equation for $r(\tau)$ is satisfied by $r = r_0$ and $dr/d\tau = 0$. The solution for $R(\tau)$ is simply

$$\frac{d^2}{d\tau^2}R(\tau) + \alpha^2 R(\tau) = 0, \longrightarrow R(\tau) = A\sin(\alpha\tau + \phi_0). \tag{8.2.29}$$

This is precisely the solution discussed in the previous section that corresponds to the Regge trajectories in the dual field theory.

8.3 Analytic Non-integrability: From Ziglin to Galois Theory

Let us review, for the benefit of the reader, the main statements of the area of analytic non-integrability [196–198]. First, the term analytic is identified with meromorphic. A meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all D except a set of isolated points, which are poles of the function. The central place in the study of integrability and non-integrability of dynamical systems is occupied by ideas developed in the context of the KAM theory. The KAM theorem describes how an integrable system reacts to small deformations. The loss of integrability is readily characterized by the resonant properties of the corresponding phase space tori, describing integrals of motion in the action-angle variables. These ideas were already present in Kovalevskaya's work but were made precise in the context of KAM theory.

Consider a general system of differential equations $\dot{\vec{x}} = \vec{f}(\vec{x})$. The general basis for proving nonintegrability of such a system is the analysis of the variational equation around a particular solution $\bar{x} = \bar{x}(t)$ which is called the straight line solution. The variational equation around $\bar{x}(t)$ is a linear system obtained by linearizing the vector field around $\bar{x}(t)$. If the nonlinear system admits some first integrals so does the variational equation. Thus, proving that the variational equation does not admit any first integral within a given class of functions implies that the original nonlinear system is nonintegrable. In particular when one works in the analytic setting where inverting the straight line solution $\bar{x}(t)$, one obtains a (noncompact) Riemann surface Γ given by integrating $dt = dw/\dot{\bar{x}}(w)$ with the appropriate limits. Linearizing the system of differential equations around the straight line solution yields the *Normal Variational Equation* (NVE), which is the component of the linearized system which describes the variational normal to the surface Γ.

The methods described here are useful for Hamiltonian systems, luckily for us, the Virasoro constraints in string theory provide a Hamiltonian for the systems we consider. This is particularly interesting as the origin of this constraint is strictly stringy but allows a very intuitive interpretation from the dynamical system perspective. One important result at the heart of a analytic non-integrability are Ziglin's theorems. Given a Hamiltonian system, the main statement of Ziglin's theorems is to relate the existence of a first integral of motion with the monodromy matrices around the straight line solution [199, 200]. The simplest way to compute such monodromies is by changing coordinates to bring the normal variational equation into a known form (hypergeometric, Lamé, Bessel, Heun, etc). Basically one needs to compute the monodromies around the regular singular points. For example, in the case where the NVE is a Gauss hypergeometric equation $z(1-z)\xi'' + (3/4)(1+z)\xi' + (a/8)\xi = 0$, the monodromy matrices can be expressed in terms of the product of monodromy matrices obtained by taking closed paths around $z = 0$ and $z = 1$. In general the answer depends on the parameters of the equation, that is, on a above. Thus, integrability is reduced to understanding the possible ranges of the parameter a.

Morales-Ruiz and Ramis proposed a major improvement on Ziglin's theory by introducing techniques of differential Galois theory [201–204]. The key observation is to change the formulation of integrability from a question of monodromy to a question of the nature of the Galois group of the NVE. In more classical terms, going back to Kovalevskaya's formulation, we are interested in understanding whether the KAM tori are resonant or not. In simpler terms, if their characteristic frequencies are rational or irrational (see the pedagogical introductions provided in [197, 205]). This statement turns out to be dealt with most efficiently in terms of the Galois group of the NVE. The key result is now stated as: If the differential Galois group of the NVE is non-virtually Abelian, that is, the identity connected component is a non-Abelian group, then the Hamiltonian system is non-integrable. The calculation of the Galois group is rather intricate, as was the calculation of the monodromies, but the key simplification comes through the application of Kovacic's algorithm [206]. Kovacic's algorithm is an algorithmic implementation of Picard-Vessiot theory (Galois theory applied to linear differential equations) for second order homogeneous linear differential equations with polynomial coefficients and gives a constructive answer to the existence of integrability by quadratures. Kovacic's algorithm is implemented in most computer algebra software including Maple and Mathematica. It is a little tedious but straightforward to go through the steps of the algorithm manually. So, once we write down our NVE in a suitable linear form it becomes a simple task to check their solvability in quadratures. An important property of the Kovacic's algorithm is that it works if and only if the system is integrable, thus a failure of completing the algorithm equates to a proof of non-integrability. This route of declaring systems non-integrable has been successfully applied to various situations, some interesting examples include: [207–210]. See also [211] for nonintegrability of generalizations of the Hénon-Heiles system [205]. A nice compilation of examples can be found in [197]. In the context of string theory it was first applied in [195].

8.3.1 Analytic Nonintegrability in Confining Backgrounds

8.3.1.1 Ansatz II

For confining backgrounds we have that the conditions on g_{00} described in (8.2.17) imply that:

$$a(r) \approx a_0 - a_2(r - r_0)^2, \tag{8.3.1}$$

where a_0 is the nonzero minimal value of $g_{00}(r_0)$ and the absence of a linear terms indicates that the first derivative at r_0 vanishes.

In this region is easy to show that both equations in (8.2.27) can be satisfied. The equation for $r(\tau)$ is satisfied by $r = r_0$ and $dr/d\tau = 0$. The straight line equation for $R(\tau)$ is simply

$$\frac{d^2}{d\tau^2}R(\tau) + \alpha^2 R(\tau) = 0, \longrightarrow R(\tau) = A\sin(\alpha\tau + \phi_0). \tag{8.3.2}$$

We can now write down the NVE equation by considering an expansion around the *straight line* solution, that is,

$$r = r_0 + \eta(\tau). \tag{8.3.3}$$

We obtain

$$\ddot{\eta} + \frac{a_2 E^2}{2b_0^2 a_0^3}\left[1 + \frac{2\alpha^2 A^2 a_0^4}{E^2}\cos 2\alpha\tau\right]\eta = 0. \tag{8.3.4}$$

The question of integrability of the system (8.2.27) has now turned into whether or not the NVE above can be solved in quadratures. The above equation can be easily recognized as the Mathieu equation. The analysis above has naturally appeared in the context of quantization of Regge trajectories and other classical string configurations. For example, [191, 192] derived precisely such equation in the study of quantum corrections to the Regge trajectories, those work went on to compute one-loop corrections in both, fermionic and bosonic sectors. Our goal here is different, for us the significance of (8.3.4) is as the Normal Variational Equation around the dynamical system (8.2.27) whose study will inform us about the integrability of the system.

The solution to the above equation (8.3.4) in terms of Mathieu functions is

$$\eta(\tau) = c_1\; C(\frac{\theta}{\alpha^2}, \frac{\theta\beta}{2\alpha^2}, \alpha\,\tau) + c_2\; S(\frac{\theta}{\alpha^2}, \frac{\theta\beta}{2\alpha^2}, \alpha\,\tau), \tag{8.3.5}$$

where c_1 and c_2 are constants and

$$\theta = \frac{a_2 E^2}{2 b_0^2 a_0^3}, \qquad \beta = \frac{2\alpha^2 A^2 a_0^4}{E^2}. \tag{8.3.6}$$

A beautiful description of a similar situation is presented in [212] where non-integrability of some Hamiltonians with rational potentials is discussed. In particular, the extended Mathieu equation is considered as an NVE equation

$$\ddot{y} = (a + b\sin t + c\cos t)y. \tag{8.3.7}$$

Our equation 8.3.4 is of this form with $2\alpha\tau \to t$ and $b = 0$. To aid the mathematically minded reader, and to make connection with our introduction to non-integrability in the beginning of section 8.3, we show that the extended Mathieu equation can be brought to an algebraic form using $x = e^{it}$ which leads to:

$$y'' + \frac{1}{x}y' + \frac{(b+c)x^2 + 2ax + c - b}{2x^3}y = 0. \tag{8.3.8}$$

The above equation is perfectly ameanable to the application of Kovacic's algorithm. It was shown explicitly in [212] that our case ($b \neq -c$ above) corresponds to a non-integrable equation. More precisely, the Galois group is the connected component of $SL(2,\mathbb{C})$ and the identity component of the Galois group for (8.3.7) is exactly $SL(2,\mathbb{C})$, which is a non-Abelian group.

8.4 Explicit Chaotic Behavior

Analytic non-integrability does not, by itself, imply the presence of chaotic behavior. To logically close the circle we should also show chaotic behavior explicitly by computing chaos indicators such as Poincaré sections and the largest Lyapunov exponent. Conveniently, the work of some of the authors showed precisely just that. Namely, in [194] it was shown that the spinning string in the AdS soliton supergravity background, which is a background in the class of confining backgrounds we are interested in, is chaotic . Since a separate an exhaustive publication was devoted to strings in the AdS soliton background here we focus in the Maldacena-Núñez background and show explicitly that non-integrability is accompanied by positive indicators of chaos. We find a rather unifying pictures as both systems behave analogously. Our explicit work provides strong evidence that, indeed, the dynamical system of the classical string which include the Regge trajectory as a particular point in phase space is chaotic.

The expression for the functions a and b in the main dynamical system (8.2.27) can be read directly from the MN background (D.0.10)(see appendix for details of the background)

$$a(r)^2 = e^{-\phi_0}\frac{\sqrt{\sinh(2r)/2}}{(r\coth 2r - \frac{r^2}{\sinh^2(2r)} - \frac{1}{4})^{1/4}}, \qquad b(r)^2 = \alpha'\, g_s N a(r)^2. \tag{8.4.1}$$

It is crucial that

$$\lim_{r\to 0} a(r)^2 \to e^{-\phi_0}, \tag{8.4.2}$$

which is a nonzero constant that determines the tension of the confining string.

8.4.1 Poincaré sections

An integrable system has the same number of conserved quantities as degrees of freedom. A convenient way to understand these conserved charges is by looking at the phase space using action-angle variables. Let us assume that we have a system with N position variables q_i with conjugate momenta p_i. The phase space is $2N$-dimensional. Integrability means that there are N conserved charges $Q_i = f_i(p,q)$ which are constants of motion. One of them is the energy. These charges define a N-dimensional surface in phase space which is a topological torus (KAM torus). The $2N$-dimensional phase space is nicely foliated by these N-dimensional tori. In terms of action-angle variables (I_i, θ_i) these tori just become surfaces of constant action.

It is interesting to study what happens to these tori when an integrable Hamiltonian is perturbed by a small nonintegrable piece. The KAM theorem states that most tori survive, but suffer a small deformation [178, 179]. However the *resonant* tori which have rational ratios of frequencies, i.e. $m_i\omega_i = 0$ with $m \in \mathcal{Q}$, get destroyed and motion on them become chaotic. For small values of the nonintegrable perturbations, these chaotic regions span a very small portion of the phase space and are not readily noticeable in a numerical study. As the strength of the nonintegrable interaction increases, more tori gradually get destroyed. A nicely foliated picture of the phase space is no longer applicable and the trajectories freely explore the entire phase space with energy as the only constraint. In such cases the motion is completely chaotic.

To numerically investigate this gradual disappearance of foliation we look at the Poincaré sections. For our system, the phase space has four variables r, R, p_r, p_R. If we fix the energy we are in a three dimensional subspace. Now if we start with some initial condition and time-evolve, the motion is confined to a two dimensional torus for the integrable case. This two-dimensional torus intersects the $R = 0$ hyperplane at a circle. Taking repeated snapshots of the system as it crosses $R = 0$ and plotting the value of (r, p_r), we can reconstruct this circle. Furthermore varying the initial conditions (in particular we set $R(0) = 0$, $p_r(0) = 0$, vary $r(0)$ and determine $p_R(0)$ from the Virasoro constraint), we can expect to get the foliation structure typical of an integrable system. In the figures below different colors correspond to different values of $r(0)$. Note that for the MN background the confining wall is located at $r_0 = 0$ and precisely around that point we see islands of integrability.

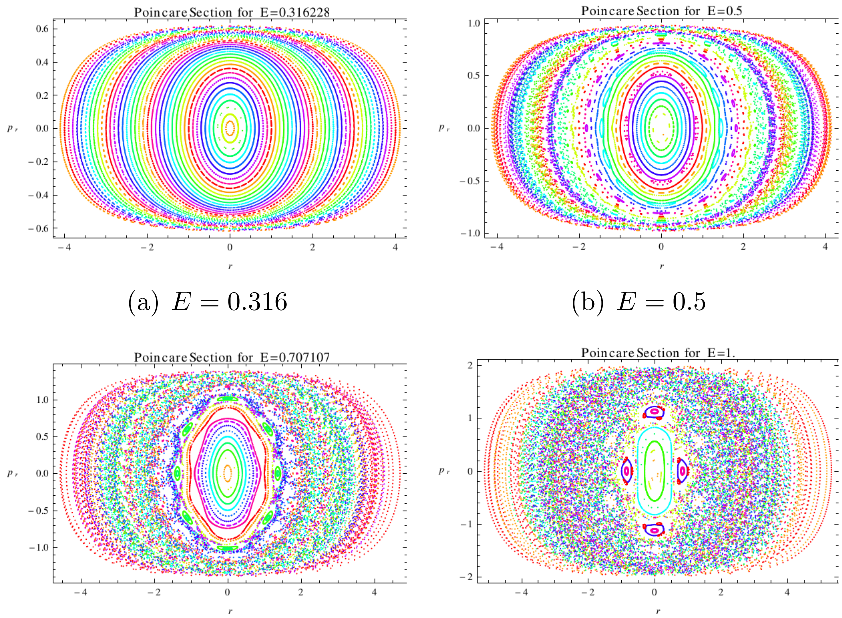
The only parameter in the dynamical system is thus E which we might refer as the energy (being related to the conserved quantity (8.2.25)). Note that this was precisely the case in the analysis of spinning strings in the AdS soliton [194]. Indeed we see that for smaller value of energies (E) which is playing the role of the strength of the non-integrable perturbation in the language of KAM theory, a distinct foliation structure exists in the phase space [Fig.8.1(a)], as at smaller energy the system may be thought as two decoupled oscillators in r and R. Recall that the oscillator with $r_0 = 0$ corresponds to the Regge trajectory as discussed previously. However as we increase the energy some tori get dissolved [Figs.8.1(b),8.1(c).8.1(d)]. Although there is no water tight definition of chaos, this destruction of the the KAM tori is one of the strongest indicators of chaotic behavior. The tori which are destroyed sometimes get broken down into smaller tori [Figs.8.1(b),8.1(c).8.1(d)]. Eventually the tori disappear and become a collection of scattered points known as cantori. However the breadths of these cantori are restricted by the undissolved tori and other dynamical elements. Usually they do not span the whole phase space [Figs.8.1(d)]. The mechanism is very similar to what happens in well known chaotic systems like Hénon-Heiles models; our figures are very typical and we refer the reader to the standard text books in this field, for example, [178, 179] for qualitative comparison.

8.4.2 Lyapunov exponent

Let us discuss another important indicator of chaos – the largest Lyapunov exponent. Sensitivity to the initial conditions is one of the most intuitive characteristics of chaotic systems. More precisely, sensitive dependence on initial conditions means that for some points X in phase space, there is (at least) one point arbitrarily close to X that diverges from X. The separation between the two is also a function of the initial location and has the form $\Delta X(X_0, \tau)$. The largest Lyapunov exponent is a quantity that characterizes the rate of separation of such infinitesimally close trajectories. Formally it is defined as,

$$\lambda = \lim_{\tau\to\infty} \lim_{\Delta X_0 \to 0} \frac{1}{\tau} \ln \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)} \tag{8.4.3}$$

In practice we use an algorithm by Sprott [183], which calculates λ over short intervals and then takes a time average. We should expect to observe that, as time τ is increased, λ settles down to oscillate around a given value. For trajectories belonging to the KAM tori, λ is zero, whereas it is expected to be non-zero for a chaotic orbit. We verify such expectations for our case. We calculate λ with various initial conditions and parameters. For apparently chaotic orbits we observe a nicely convergent positive λ [Fig.8.4.2]. Our emphasis in not so much in the precise value which might require extensive use of numerical techniques as done in [193], rather, we are content with showing that the largest Lyapunov exponent is positive. In figure (8.4.2) we present a calculation following (8.4.3) of the largest Lyapunov exponent. We consider a



(a) $E = 0.316$ (b) $E = 0.5$

(c) $E = 0.71$ (d) $E = 1.0$

Figure 8.1: Poincaré sections demonstrate breaking of the KAM tori en route to chaos. Each color represents a different initial condition. For smaller values of E the sections of the KAM tori are intact curves, except for the resonant ones. The tori near the resonant ones start breaking as E is increased. For very large values of E all the colors get mixed – this indicates that all the tori get broken and they fill the entire phase space.

trajectory with $r(0) = 2, R(0) = 1.0, p_r(0) = 0, p_R(0) = 0$ and its neighbor which differs in phase space by $r(0) = r(0) + \epsilon$ with $\epsilon = 10^{-3}$.

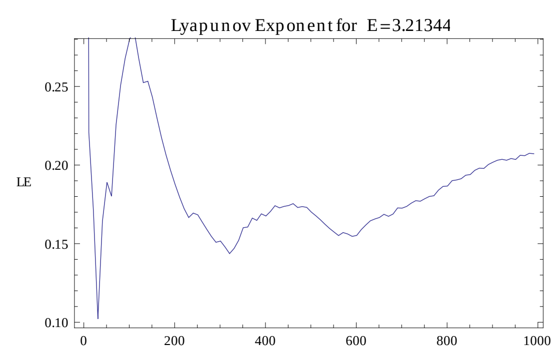


Figure 8.2: The Lyapunov Exponent converges to a positive value of about 0.2.

8.5 Conclusions

We have established that the motion of certain classical strings in the general class of backgrounds dual to confining theory is chaotic. We have shown analytically, by means of Hamiltonian techniques, that such systems are non-integrable. One important result of our paper is that non-integrability in confining backgrounds is a direct consequence of the Wilson loop area law. The conditions (8.2.17) on the metric, that lead to an area-law behavior of the dual gauge theory thereby implying confinement [189, 190], are precisely the conditions required to prove that the string moving in such backgrounds is non-integrable. Non-integrability is thus central to the approach of AdS/CFT to realistic theories.

Furthermore, we have also shown numerically that in the case of the MN background the Poincaré sections and the largest Lyapunov exponent return positive tests for chaotic behavior. Identical results for the AdS soliton background have already been obtained in [194]. Although, these are the simplest examples in this class of backgrounds that we can explicitly demonstrate to be chaotic, the same result should apply to all theories in the class.

There are various topics that we find particularly deserving of further attention. We have established that the classical string trajectory corresponding to the Regge trajectory in field theory is an attractor point in the dynamical system. This same system contains the GKP string which is dual to twist-two operators. It would be interesting to explore in full detail the connection between these two trajectories.

Along similar lines we established in an appendix that Ansatz I can not be chaotic as the effective dynamical "time" is periodic. There is *a priori* nothing surprising except from the fact that the difference between Ansatzë I and II is largely due to $r(\sigma) \to r(\tau)$ which conspicuously looks like a T-duality. This topic is certainly worth exploring.

Lastly, it would be interesting to understand the implication of this classical chaos on the Regge trajectories themselves. Recall that the spectrum of quantum systems obtained as the quantization of systems that in their classical limit are chaotic is quite different from the spectrum of quantum systems obtained from the quantization of integrable classical systems. This is particularly interesting due to the potential implications for the spectrum of hadronic matter.

Appendix A

Adiabatic and scaling analysis of a toy model

In this appendix, we consider a (0 + 1)-dimensional toy model which follows the equation

$$2i\mu\dot{\phi} + (m^2 - \mu^2)\phi + \phi|\phi|^2 = J(t) \ . \tag{A.0.1}$$

The function $J(t)$ asymptotes to constants at early and late times and passes through zero in a linear fashion at some intermediate time, e.g.

$$J(t) = J_0 \tanh(vt) \ . \tag{A.0.2}$$

A.1 Adiabaticity

We first derive conditions for breakdown of adiabaticity near the critical point $m^2 = \mu^2$ and $J(t) = 0$. We carry out adiabatic expansion as following:

$$\partial_t \to \epsilon\partial_t \ , \quad \phi \to \phi_0(J(t)) + \epsilon\phi_1(t) + \cdots \ , \tag{A.1.1}$$

where $\phi_0(J(t))$ is the (real) adiabatic solution given by

$$\phi_0(J(t)) = [J(t)]^{1/3} \ . \tag{A.1.2}$$

The solution to $O(\epsilon^2)$ is ,

$$\phi = \phi_0[J(t)] + \epsilon\, i\,\frac{2\mu}{\phi_0^2}\dot{\phi}_0 + \epsilon^2\frac{1}{3\phi_0^2}\left(8\mu^2\Big(\frac{\dot{\phi}_0^{\,2}}{\phi_0^3} - \frac{\ddot{\phi}_0}{2\phi_0^2}\Big) - \frac{4\mu^2}{\phi_0^3}\dot{\phi}_0^{\,2}\right) + O(\epsilon^3) \ . \tag{A.1.3}$$

where The breakdown of adiabaticity happens when,

$$\frac{2\mu}{\phi_0^2}\dot{\phi}_0 \ \sim \ \phi_0 \ , \tag{A.1.4}$$

$$\frac{1}{3\phi_0^2}\left(8\mu^2\Big(\frac{\dot{\phi}_0^{\,2}}{\phi_0^3} - \frac{\ddot{\phi}_0}{2\phi_0^2}\Big) - \frac{4\mu^2}{\phi_0^3}\dot{\phi}_0^{\,2}\right) \ \sim \ \phi_0 \ , \tag{A.1.5}$$

For $J(t) = \tanh(vt) \sim vt$ the above two equations translate into,

$$\mu \ \sim \ t^{5/3}v^{2/3} \ . \tag{A.1.6}$$

Thus if μ is of $O(1)$ then the above equations give us,

$$t \ \sim \ v^{2/5} \ . \tag{A.1.7}$$

A.2 Scaling behavior

Now sitting at the critical point we study the behavior of the scaling solution with $\mu = O(1)$. From the adiabatic analysis we expect scaled time, $\bar{t} = v^{2/5}t$.
We write the field ϕ as $\chi + i\xi$ and the source as $J_{\rm R} + iJ_{\rm Im}$, where both $J_{\rm R}$ and $J_{\rm Im}$ go as vt. To find the scaling exponents we extract the v dependencies as,

$$t = v^{\alpha}\bar{t}\ , \qquad \chi = v^{\beta}\bar{\chi}\ , \qquad \xi = v^{\gamma}\bar{\xi}\ . \tag{A.2.1}$$

Consistency of the equations demand,

$$\alpha = -\frac{2}{5}\ , \quad \beta = \frac{1}{5}\ , \quad \gamma = \frac{1}{5}\ . \tag{A.2.2}$$

This determines the scaling behavior of the field ϕ at $m^2 = \mu^2$ and with μ of $O(1)$ as,

$$\phi(t, v) = v^{1/5}\phi(v^{2/5}t, 1)\ . \tag{A.2.3}$$

This agrees with our expectation from adiabatic analysis, and also has been confirmed numerically in Section 4.

A.3 Late time behavior

At late times the source $J(t)$ can be treated to be a constant and the solution can be obtained by perturbing around the static solution, $\phi_{static} = J^{1/3}$. It is then straightforward to see that the solution $\phi(t)$ is oscillatory with a frequency $\omega = \frac{J^{2/3}}{\sqrt{2}\mu}$.

Appendix B

Validity of the small v expansion

To argue for the small v expansion of $\bar{\xi}^k(\eta)$ (3.4.16)) we need to consider the eigenvalue problem

$$[-\partial_{\bar{\rho}}^2 + V_0(\bar{\rho})]\chi_k = k^2\chi_k \tag{B.0.1}$$

The above potential $V_0(\bar{\rho}) \to -e^{-\bar{\rho}}$ as $\bar{\rho} \to \infty$.

The basic features of the eigenfunctions can be understood from a simpler problem in which we replace the potential by the following potential which has the same qualitative features.

$$U(\bar{\rho}) = \begin{cases} V_0 & \text{for } \bar{\rho} = 0 \\ -U_0 & \text{for } 0 \leq \bar{\rho} \leq 1 \\ 0 & \text{for } 1 \leq \bar{\rho} \leq \infty \end{cases} \tag{B.0.2}$$

with $U_0, V_0 > 0$. The eigenfunctions of the Schrodinger operator with eigenvalue $k^2 > 0$ are

$$\begin{aligned} \psi_k(\bar{\rho}) &= \frac{A(k)}{\sqrt{\pi}}\left(\sin(\sqrt{k^2+U_0}\;\bar{\rho}) + \kappa\cos(\sqrt{k^2+U_0}\;\bar{\rho})\right) \qquad 0 \leq \bar{\rho} \leq 1 \\ \psi_k(\bar{\rho}) &= \frac{1}{\sqrt{\pi}}\sin(k\bar{\rho} + \theta(\bar{\rho})) \qquad 1 \leq \bar{\rho} \leq \infty \end{aligned} \tag{B.0.3}$$

Here κ plays the role of the double trace deformation of our original problem, in the spirit that here too it dictates the modified boundary condition at $\bar{\rho} = 0$. The constants $A(k)$ and $\theta(k)$ are determined by matching at $\bar{\rho} = 1$,

$$\begin{aligned} A(k) &= \frac{k}{\sqrt{k^2(1+\kappa^2) + \left(\cos(\sqrt{k^2+U_0}) - \kappa\sin(\sqrt{k^2+U_0})\right)^2 U_0}} \\ \theta(k) &= \cos^{-1}\left(\frac{\left(\cos(\sqrt{k^2+U_0}) - \kappa\sin(\sqrt{k^2+U_0})\right)\sqrt{k^2+U_0}}{\sqrt{k^2(1+\kappa^2) + \left(\cos(\sqrt{k^2+U_0}) - \kappa\sin(\sqrt{k^2+U_0})\right)^2 U_0}}\right) \end{aligned} \tag{B.0.4}$$

The solution for $k = 0$ is

$$\begin{aligned} \psi_0(\bar{\rho}) &= \frac{B}{\sqrt{\pi}}\left(\sin(\sqrt{U_0}\;\bar{\rho}) + \kappa\cos(\sqrt{U_0}\;\bar{\rho})\right) \qquad 0 \leq \bar{\rho} \leq 1 \\ \psi_0(\bar{\rho}) &= \frac{1}{\sqrt{\pi}}(a\bar{\rho} + b) \qquad 1 \leq \bar{\rho} \leq \infty \end{aligned} \tag{B.0.5}$$

The matching conditions at $\bar{\rho} = 1$ now yield

$$
\begin{aligned}
B\left(\sin(\sqrt{U_0}) + \kappa\cos(\sqrt{U_0})\right) &= a + b \\
B\sqrt{U_0}\left(\cos(\sqrt{U_0}) - \kappa\sin(\sqrt{U_0}\right) &= a
\end{aligned} \tag{B.0.6}
$$

For any $a \neq 0$ the solution blows up at $\bar{\rho} = \infty$. Thus regular solutions require $a = 0$. However the second equation in (B.0.6) then imply that

$$
\sqrt{U_0} = \cot^{-1}\kappa \tag{B.0.7}
$$

These are the zero modes. In the context of our model this is the potential where we have a critical point.

The small k behavior of $A(k)$ and $\theta(k)$ can be read off from the expressions (B.0.4). For a generic U_0 these are

$$
\begin{aligned}
A(k) &\sim \frac{k}{\left(\cos(\sqrt{U_0}) - \kappa\sin(\sqrt{U_0}\right)\sqrt{U_0}} + O(k^3) \\
\theta(k) &\sim k[\frac{\kappa + \tan\sqrt{U_0}}{\sqrt{U_0}\left(1 - \kappa\tan\sqrt{U_0}\right)} - 1] + O(k^3)
\end{aligned} \tag{B.0.8}
$$

whereas for critical potentials we have

$$
\begin{aligned}
A(k) &\sim \frac{1}{\sqrt{1+\kappa^2}}\left(1 - \frac{k^2}{8} + O(k^4)\right) \\
\theta(k) &\sim \frac{\pi}{2} - \frac{k}{2}
\end{aligned} \tag{B.0.9}
$$

This has implications for the coefficients like $(\mathcal{B}_{k0} - \mathcal{B}_{00})$ and $(\bar{C}_{k000} - \bar{C}_{0000})$ of equation(3.4.18). Consider the quantity, $\mathcal{B}_{k0}$. We have,

$$
\mathcal{B}_{k0} = \int d\bar{\rho}\ \bar{\chi}_k(\bar{\rho})\partial_{\bar{\rho}}\bar{\chi}_0(\bar{\rho})
$$

If we replace the true eigenfunctions by those of our simplified problem, we get

$$
\begin{aligned}
\mathcal{B}_{k0} &= A(k)\int_0^1 d\bar{\rho}\ \left(\sin(\sqrt{k^2 + U_0}\ \bar{\rho}) + \kappa\cos(\sqrt{k^2 + U_0}\ \bar{\rho})\right)B\sqrt{U_0} \\
&\times \left(\cos(\sqrt{U_0}\bar{\rho}) - \kappa\sin(\sqrt{U_0}\bar{\rho})\right)
\end{aligned} \tag{B.0.10}
$$

Using (B.0.8) and (B.0.9) we therefore see that

$$
\mathcal{B}_{k0} - \mathcal{B}_{00} \sim k \qquad k \to 0 \tag{B.0.11}
$$

for generic potentials, whereas

$$
\mathcal{B}_{k0} - \mathcal{B}_{00} \sim k^2 \qquad k \to 0 \tag{B.0.12}
$$

for critical potentials. The behavior for $(\bar{C}_{k000} - \bar{C}_{0000})$ is similar.

Going back to (3.4.18) we therefore see that the small v expansion is generically *not valid* since the corrections diverge at small k. However for the critical potential, $\tilde{\xi}_k$ remain finite as $k \to 0$ and the expansion in powers of $v^{1/2}$ makes sense.

Appendix C

de Sitter "bluewall" details

We give some details on the dS_4-"bluewall" $\frac{ds^2}{R^2} = -\frac{d\tau^2}{\tau^2(1-\tau_0^3/\tau^3)} + \tau^2(1-\tau_0^3/\tau^3)dw^2 + \tau^2 dx_i^2$ here. We can analyse the vicinity of $\tau = \tau_0$ as for the Schwarzschild black hole, defining a "tortoise" τ-coordinate: for the dS_4-solution, this is

$$\tau_* = \int \frac{d\tau}{\tau^2(1-\frac{\tau_0^3}{\tau^3})} = \frac{1}{3\tau_0}\Big(\log \frac{\tau-\tau_0}{\sqrt{\tau^2+\tau\tau_0+\tau_0^2}} \ + \ \sqrt{3}\tan^{-1}\frac{2\frac{\tau}{\tau_0}+1}{\sqrt{3}}\Big). \qquad \text{(C.0.1)}$$

Analogs of Kruskal-Szekeres coordinates can then be defined as

$$\tilde{u} = e^{3(\tau_*-w)\tau_0/2}\,, \quad \tilde{v} = e^{3(\tau_*+w)\tau_0/2}\,, \qquad \tilde{u}\tilde{v} = e^{3\tau_*\tau_0} = \frac{\tau-\tau_0}{\sqrt{\tau^2+\tau\tau_0+\tau_0^2}}\, e^{\sqrt{3}\tan^{-1}\frac{2\frac{\tau}{\tau_0}+1}{\sqrt{3}}}\,, \qquad \text{(C.0.2)}$$

and $u = \tilde{u} - \tilde{v} = -2e^{3\tau_*\tau_0/2}\sinh\frac{3w\tau_0}{2}$, $v = \tilde{u}+\tilde{v} = 2e^{3\tau_*\tau_0/2}\cosh\frac{3w\tau_0}{2}$, giving (6.3.2) and the Penrose diagram Figure 6.1. With $T = \int d\tau/(\tau\sqrt{1-\tau_0^3/\tau^3})$, this is recast in FRW-form as an accelerating cosmology $ds^2 = -dT^2 + e^{-2T}(e^{3T}+\tau_0^3)^{4/3}\frac{(e^{3T}-\tau_0^3)^2}{(e^{3T}+\tau_0^3)^2}dw^2 + e^{-2T}(e^{3T}+\tau_0^3)^{4/3}dx_i^2$ with w-anisotropy. Further redefining $T = \log\eta$, we can obtain a Fefferman-Graham expansion for this asymptotically-dS_4 spacetime near the boundary $\tau \to \infty$. Following Einstein-Rosen's description [213] of the "bridge" in the Schwarzschild black hole (using $\rho^2 = r - 2m$), define $t^2 = \tau - \tau_0$. This coordinate has the range $t : -\infty \to \infty$ as $\tau : \infty \to \tau_0$ and then $\tau : \tau_0 \to \infty$, giving two t-sheets of the asymptotic deSitter region,

$$ds^2 = \frac{-4(t^2+\tau_0)dt^2}{(t^2+\tau_0)^2+\tau_0(t^2+\tau_0)+\tau_0^2} + \Big(\frac{(t^2+\tau_0)^2+\tau_0(t^2+\tau_0)+\tau_0^2}{t^2+\tau_0}\Big)t^2dw^2 + (t^2+\tau_0)^2dx_i^2.$$

Thus the two asymptotic universes are connected by a timelike Einstein-Rosen bridge. At $\tau = \tau_0$, we have $g_{ww} = 0$ so the w-direction shrinks to vanishing size. Near $\tau = \tau_0$, the metric is approximated as $ds^2 \sim -\frac{d\tau^2}{k(\tau-\tau_0)} + (\tau-\tau_0)\tau_0^2dw^2 + \tau_0^2dx_i^2 \sim -dt^2 + t^2d\tilde{w}^2 + d\tilde{x}_i^2$, which is flat space with the (t,w)-plane in Milne coordinates.

Null geodesics $ds^2 = 0$, in the (τ, w)-plane defining lightcones and causal structure, are $dw = \pm d\tau_*$, $\frac{dw}{d\tau} = \pm\frac{1}{\tau^2(1-\tau_0^3/\tau^3)}$, with τ_* given in (C.0.1). Near the horizon, the trajectories approach $w = \pm\tau_* + const \sim \pm\frac{1}{3\tau_0}\log\frac{|\tau-\tau_0|}{3\tau_0}$, *i.e.* $w \to \pm\infty$. These null rays intersect the horizon and hit the singularity in the interior at $w_0 \pm \frac{1}{3\tau_0}\log 3$.

Note that $\tau = const$ surfaces are spacelike hyperbolic hypersurfaces with $v^2 - u^2 = const$ in the region outside the horizons, using (C.0.2). In these exterior regions,

$$w = const \ \text{ path} \quad \Rightarrow \quad \frac{u}{v} = -\tanh\frac{3w\tau_0}{2} = const\ , \quad i.e.\ \ \frac{\tilde{u}}{\tilde{v}} = \frac{1-\tanh\frac{3w\tau_0}{2}}{1+\tanh\frac{3w\tau_0}{2}} \equiv k\ , \qquad \text{(C.0.3)}$$

i.e. straight lines passing through the bifurcation region, crossing over from the past asymptotic region II to the future one I. The induced worldline metric on such a $w, x_i = const$ trajectory and associated proper time are $dl^2 = \frac{d\tau^2}{\tau^2 - \tau_0^3/\tau} \equiv dT^2$, $T = \frac{2}{3}\log(\tau^{3/2} + \sqrt{\tau^3 - \tau_0^3})$. The spatial metric on a $\tau = const$ hypersurface orthogonal to these constant-w, x_i trajectories is $\frac{d\sigma^2}{R^2} = \tau^2\left(1 - \frac{\tau_0^3}{\tau^3}\right)dw^2 + \tau^2 dx_i^2$. We see that at the bridge $\tau \to \tau_0$, the spatial metric degenerates and the cross-sectional 3-area $V_{w,x_1,x_2} = \Delta w \Delta^2 x_i \tau^3 \sqrt{1 - \tau_0^3/\tau^3}$ vanishes. The proper time T is consistent with the equations for timelike geodesics at constant x_i in the (τ, w)-plane, and is finite along such geodesic paths between the horizon $\tau = \tau_0$ and any point $\tau < \infty$. It can be seen by studying geodesic deviation for a congruence of such timelike geodesic static observers with const-w, x_i that there are no diverging tidal forces as one crosses the bifurcation region from the past universe to the future one.

Consider now the spacetime in Kruskal form (6.3.2) written as $ds^2 = -2f(\tilde{u}, \tilde{v})d\tilde{u}d\tilde{v} + g(\tilde{u}, \tilde{v})dx_i^2$. A family of generic timelike paths in the Penrose diagram is $\tilde{u} = k\tilde{v} + c$, which are obtained by translating sideways the $w = const$ paths (C.0.3). For $c = 0$, these are geodesics passing through the bifurcation region (without intersecting the horizon). Parametrizing these timelike paths as $x^\alpha(\lambda)$ in the $\tilde{u}, \tilde{v}$-plane ($x_i = const$), it can be shown that the acceleration components $a^{\tilde{u}} = \ddot{\tilde{u}} + \Gamma^{\tilde{u}}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ and similarly $a^{\tilde{v}}$ are finite as $\tau \to \tau_0$, as is the covariant acceleration norm $g_{\mu\nu}a^\mu a^\nu$. Any arbitrary smooth timelike trajectory can be approximated as a straight line in the neighbourhood of any point, in particular near the horizon. Thus the acceleration vanishes for any timelike path crossing the horizons. This is perhaps not surprising since the near horizon geometry is essentially Milne.

Appendix D

Straight line solution and NVE in Confining Backgrounds

In this appendix we show explicitly that the prototypical supergravity backgrounds in the gauge/gravity correspondence conform to the analysis presented in the main text. We consider the KS and MN backgrounds explicitly.

D.0.1 The Klebanov-Strassler background

We begin by reviewing the KS background, which is obtained by considering a collection of N regular and M fractional D3-branes in the geometry of the deformed conifold [214]. The 10-d metric is of the form:

$$ds_{10}^2 = h^{-1/2}(\tau) dX_\mu dX^\mu + h^{1/2}(\tau) ds_6^2 \ , \tag{D.0.1}$$

where ds_6^2 is the metric of the deformed conifold:

$$ds_6^2 = \frac{1}{2}\varepsilon^{4/3}K(\tau)\left[\frac{1}{3K^3(\tau)}(d\tau^2+(g^5)^2)+\cosh^2\left(\frac{\tau}{2}\right)[(g^3)^2+(g^4)^2]+\sinh^2\left(\frac{\tau}{2}\right)[(g^1)^2+(g^2)^2]\right]. \tag{D.0.2}$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}, \tag{D.0.3}$$

and

$$\begin{aligned}
g^1 &= \frac{1}{\sqrt{2}}\big[-\sin\theta_1 d\phi_1 - \cos\psi\sin\theta_2 d\phi_2 + \sin\psi d\theta_2\big],\\
g^2 &= \frac{1}{\sqrt{2}}\big[d\theta_1 - \sin\psi\sin\theta_2 d\phi_2 - \cos\psi d\theta_2\big],\\
g^3 &= \frac{1}{\sqrt{2}}\big[-\sin\theta_1 d\phi_1 + \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\big],\\
g^4 &= \frac{1}{\sqrt{2}}\big[d\theta_1 + \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\big],\\
g^5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2.
\end{aligned} \tag{D.0.4}$$

The warp factor is given by an integral expression for h is

$$h(\tau) = \alpha\frac{2^{2/3}}{4}I(\tau) = (g_s M\alpha')^2 2^{2/3}\varepsilon^{-8/3}I(\tau) \ , \tag{D.0.5}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3} \ . \tag{D.0.6}$$

The above integral has the following expansion in the IR:

$$I(\tau \to 0) \to a_0 - a_2\tau^2 + \mathcal{O}(\tau^4) \ , \tag{D.0.7}$$

where $a_0 \approx 0.71805$ and $a_2 = 2^{2/3}\,3^{2/3}/18$. The absence of a linear term in τ reassures us that we are really expanding around the end of space, where the Wilson loop will find it more favorable to arrange itself.

D.0.1.1 The straight line solution in KS

We consider the quadratic fluctuations and their influence on the Regge trajectories (8.2.20). In the notation used in the bulk of the paper we have:

$$\begin{aligned} a^2(r) &= h^{-1/2}(r), \\ b^2(r) &= \frac{\varepsilon^{4/3}}{6K^2(r)} h^{1/2}(r). \end{aligned} \tag{D.0.8}$$

Let us first consider the metric. The part of the metric perpendicular to the world volume, which is the deformed conifold metric, does not enter in the classical solution which involves only world volume fields. Noting that the value r_0 of section 8.2.1 is $\tau = 0$, we expand the deformed conifold up to quadratic terms in the coordinates:

$$ds_6^2 = \frac{\varepsilon^{4/3}}{2^{2/3}3^{1/3}}\left[\frac{1}{2}g_5^2 + g_3^2 + g_4^2 + \frac{1}{2}d\tau^2 + \frac{\tau^2}{4}(g_1^2 + g_2^2)\right]. \tag{D.0.9}$$

Let us further discuss the structure of this metric. It is known on very general grounds that the deformed conifold is a cone over a space that is topologically $S^3 \times S^2$ [215]. We can see that the S^3 roughly spanned by (g_3, g_4, g_5) has finite size, while the S^2 spanned by (g_1, g_2) shrinks to zero size at the apex of the deformed conifold. More importantly for us is the fact that, if we do not allow non-trivial behavior in the directions (g_1, g_2) they cannot contribute to the NVE around the straight line solution characterized by $\tau = 0$. Therefore, we have that the NVE equation for the spinning string in the KS background is precisely of the form (8.3.4).

D.0.2 The Maldacena-Nùñez background

The MN background [216] whose IR regime is associated with $\mathcal{N} = 1$ SYM theory is that of a large number of D5 branes wrapping an S^2. To be more precise: (i) the dual field theory to this SUGRA background is the $\mathcal{N} = 1$ SYM contaminated with KK modes which cannot be de–coupled from the IR dynamics, (ii) the IR regime is described by the SUGRA in the vicinity of the origin where the S^2 shrinks to zero size. The full MN SUGRA background includes the metric, the dilaton and the RR three-form. It can also be interpreted as uplifting to ten dimensions a solution of seven dimensional gauged supergravity [217]. The metric and dilaton of the background are

$$ds^2 = e^{\phi}\left[dX^a dX_a + \alpha' g_s N(d\tau^2 + e^{2g(\tau)}(e_1^2 + e_2^2) + \frac{1}{4}(e_3^2 + e_4^2 + e_5^2))\right],$$

$$
\begin{aligned}
e^{2\phi} &= e^{-2\phi_0}\frac{\sinh 2\tau}{2e^{g(\tau)}},\\
e^{2\,g(\tau)} &= \tau\coth 2\tau - \frac{\tau^2}{\sinh^2\, 2\tau} - \frac{1}{4},
\end{aligned}
\tag{D.0.10}
$$

where,

$$
\begin{aligned}
e_1 &= d\theta_1, \qquad e_2 = \sin\theta_1 d\phi_1,\\
e_3 &= \cos\psi\, d\theta_2 + \sin\psi\sin\theta_2\, d\phi_2 - a(\tau)d\theta_1,\\
e_4 &= -\sin\psi\, d\theta_2 + \cos\psi\sin\theta_2\, d\phi_2 - a(\tau)\sin\theta_1 d\phi_1,\\
e_5 &= d\psi + \cos\theta_2\, d\phi_2 - \cos\theta_1 d\phi_1, \quad a(\tau) = \frac{\tau^2}{\sinh^2\tau}.
\end{aligned}
\tag{D.0.11}
$$

where $\mu = 0, 1, 2, 3$, we set the integration constant $e^{\phi_{D_0}} = \sqrt{g_s N}$.

Note that we use notation where x^0, x^i have dimension of length whereas ρ and the angles $\theta_1, \phi_1, \theta_2, \phi_2, \psi$ are dimensionless and hence the appearance of the α' in front of the transverse part of the metric.

D.0.2.1 The straight line solution in MN

The position referred to as r_0 in section (8.2) is $\tau = 0$. Therefore, we will expand the metric around that value. Let us first identify some structures in the metric that are similar to the deformed conifold considered in the previous subsection. Notice that $e_1^2 + e_2^2$ is precisely an S^2. Moreover, near $\tau = 0$ we have that $e^{2g} \approx \tau^2 + \mathcal{O}(\tau^4)$. Thus (τ, e_1, e_2) span an $\mathbb{R}^3$ in the limit

$$
d\tau^2 + e^{2g(\tau)}(e_1^2 + e_2^2).
\tag{D.0.12}
$$

This means that without exciting the KK modes corresponding to (e_1, e_2) in our Ansatz II, the NVE equation is precisely of the form (8.3.4). Certainly $e_3^2 + e_4^2 + e_5^2$ parametrizes a space that is topologically a three sphere fibered over the S^2 spanned by (e_1, e_2). However, near $\tau = 0$ we have a situation very similar to the structure of the metric in the deformed conifold. Namely, at $\tau = 0$ there we have that: $e_5 \to g_5$, $e_3 \to \sqrt{2}g_4$, $e_4 \to \sqrt{2}g_3$ (up to a trivial identification $\theta_1 \to -\theta_1$, $\phi_1 \to -\phi_1$). This allows us to identify this combination as a round S^3 of radius 2 and therefore can not alter the form of the NVE (8.3.4).

D.0.3 The Witten QCD background

The ten-dimensional string frame metric and dilaton of the Witten QCD model are given by

$$
\begin{aligned}
ds^2 &= (\frac{u}{R})^{3/2}(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{4R^3}{9u_0}f(u)d\theta^2) + (\frac{R}{u})^{3/2}\frac{du^2}{f(u)} + R^{3/2}u^{1/2}d\Omega_4^2\,,\\
f(u) &= 1 - \frac{u_0^3}{u^3}\,, \qquad\qquad R = (\pi N g_s)^{\frac{1}{3}}\alpha'^{\frac{1}{2}}\,,
\end{aligned}
$$

$$e^{\Phi} = g_s \frac{u^{3/4}}{R^{3/4}} . \quad \text{(D.0.13)}$$

The geometry consists of a warped, flat 4-d part, a radial direction u, a circle parameterized by θ with radius vanishing at the horizon $u = u_0$, and a four-sphere whose volume is instead everywhere non-zero. It is non-singular at $u = u_0$. Notice that in the $u \to \infty$ limit the dilaton diverges: this implies that in this limit the completion of the present IIA model has to be found in M-theory. The background is completed by a constant four-form field strength

$$F_4 = 3R^3 \omega_4 , \quad \text{(D.0.14)}$$

where ω_4 is the volume form of the transverse S^4.

We will be mainly interested in classical string configurations localized at the horizon $u = u_0$, since this region is dual to the IR regime of the dual field theory. In this case the coordinate u is not suitable because the metric written in this coordinate looks singular at $u = u_0$. Then, as a first step, let us introduce the radial coordinate

$$r^2 = \frac{u - u_0}{u_0} , \quad \text{(D.0.15)}$$

so that the metric expanded to quadratic order around $r = 0$ becomes

$$ds^2 \approx (\frac{u_0}{R})^{3/2}[1 + \frac{3r^2}{2}](\eta_{\mu\nu}dx^{\mu}dx^{\nu}) + \frac{4}{3}R^{3/2}\sqrt{u_0}(dr^2 + r^2 d\theta^2) + R^{3/2}u_0^{1/2}[1 + \frac{r^2}{2}]d\Omega_4^2 . \quad \text{(D.0.16)}$$

D.0.3.1 The straight line solution in WQCD

In this section we consider the closed string configuration corresponding to the glueball Regge trajectories. The relevant closed folded spinning string configuration dual to the Regge trajectories and constituting the straight line solution in our analysis is

$$X^0 = k\tau , \qquad X^1 = k \cos\tau \sin\sigma , \qquad X^2 = k \sin\tau \sin\sigma , \quad \text{(D.0.17)}$$

and all the other coordinates fixed.

To understand the NVE around the straight line solution given above, we need only look at (D.0.16) and realize that the only possible contribution to the NVE given in (8.3.4) can come only from KK modes in the S^4 of equation (D.0.16). We conclude that, in this case, as well the NVE is precisely of the form given in (8.3.4).

Appendix E

Comments on Ansatz I

For confining backgrounds we have that the conditions on g_{00} described in (8.2.17) imply that:

$$a(r) \approx a_0 - a_2(r - r_0)^2, \tag{E.0.1}$$

where a_0 is the nonzero minimal value of $g_{00}(r_0)$ and the absence of a linear terms indicates that the first derivative at r_0 vanishes.

In this region is easy to show that both equations above can be satisfied. The equation for $r(\sigma)$ is satisfied by $r = r_0$ and $dr/d\sigma = 0$. The equation for $R(\sigma)$ is simply

$$\frac{d^2}{d\sigma^2}R(\sigma) + \omega^2 R(\sigma) = 0, \longrightarrow R(\sigma) = A\sin(\omega\sigma + \phi_0). \tag{E.0.2}$$

We can now write down the NVE equation by considering an expansion around the *straight line* solution, that is,

$$r = r_0 + \eta(\sigma). \tag{E.0.3}$$

We obtain

$$\eta'' - \frac{a_2 E^2}{2b_0^2 a_0^3}\left[1 - \frac{2\omega^2 A^2 a_0^4}{E^2}\cos 2\omega\sigma\right]\eta = 0. \tag{E.0.4}$$

The question of integrability of the system (8.2.27) has now turned into whether or not the NVE above can be solved in quadratures. The above equation can be easily recognized as the Mathieu equation. The analysis above has naturally appeared in the context of quantization of Regge trajectories and other classical string configurations. For example, [191, 192] derived precisely such equation in the study of quantum corrections to the Regge trajectories, those work went on to compute one-loop corrections in both, fermionic and bosonic sectors. Our goal here is different, for us the significance of (E.0.4) is as the Normal Variational Equation around the dynamical system (8.2.27) whose study will inform us about the integrability of the system. The general solution to the above equation is

$$\eta(\sigma) = c_1\ C(-\frac{\alpha}{\omega^2}, -\frac{\alpha\beta}{2\omega^2}, \omega\,\sigma) + c_2\ S(-\frac{\alpha}{\omega^2}, -\frac{\alpha\beta}{2\omega^2}, \omega\,\sigma), \tag{E.0.5}$$

where c_1 and c_2 are constants and

$$\alpha = \frac{a_2 E^2}{2b_0^2 a_0^3}, \qquad \beta = \frac{2\omega^2 A^2 a_0^4}{E^2}. \tag{E.0.6}$$

Notice, crucially, that although the system obtain here is similar to the one discussed in the main text there is a key difference. Namely, that the effective "time" variable σ is now periodic.

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   arXiv:1404.2633 [hep-th].
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2. **"dS/CFT at uniform energy density and a de Sitter 'bluewall' "**
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3. **"Quantum Quench and Double Trace Couplings"**
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4. **"Double Trace Flows and Holographic RG in dS/CFT correspondence"**
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