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THE HYDRODYNAMIC FLOW OF NEMATIC LIQUID CRYSTALS IN R^3

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THE HYDRODYNAMIC FLOW OF NEMATIC LIQUID CRYSTALS IN $\mathbb{R}^3$

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Jay Lawrence Hineman
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

THE HYDRODYNAMIC FLOW OF NEMATIC LIQUID CRYSTALS IN $\mathbb{R}^3$

This manuscript demonstrates the well-posedness (existence, uniqueness, and regularity of solutions) of the Cauchy problem for simplified equations of nematic liquid crystal hydrodynamic flow in three dimensions for initial data that is uniformly locally $L^3(\mathbb{R}^3)$ integrable ($L^3_{\mathbb{U}}(\mathbb{R}^3)$). The equations examined are a simplified version of the equations derived by Ericksen and Leslie. Background on the continuum theory of nematic liquid crystals and their flow is provided as are explanations of the related mathematical literature for nematic liquid crystals and the Navier–Stokes equations.

KEYWORDS: nematic liquid crystals, Navier–Stokes equations, Ericksen–Leslie equations, harmonic maps

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This document and all the blood, sweat, and tears that came before it are dedicated to my wife, Sarah, and my parents. Without their love and support I would have not gotten this far!

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Chapter 1 Overview

1.1 Introduction

Liquid crystals and their mathematical description have a long and rich history dating back over 100 years (see [28], [5]). Liquid crystals can be thought of as materials that exhibit an intermediate phase between liquid and solid in the sense that while liquid crystals may flow like liquids they exhibit additional structural properties.

Many chemical compounds have liquid crystal phases. For example concentrated solutions of rigid polymers in suitable solvents, DNA, and certain viruses all exhibit liquid crystal phases. Since there are many possible microscopic structures there are accordingly many liquid crystal phases: nematic, smectic, cholesteric, for example.

The simplest example of a liquid crystal phase occurring in nature is the nematic phase for a single chemical species. Physically, in the single-species case, liquid crystals that exhibit a nematic phase, or nematic liquid crystals for short, are liquids that are uniformly composed of rod-like molecules whose structure induce a preferred average directional order. A historical example is the compound MBBA $N-(p-methoxybenzylidene)-p-butylaniline$ which is in the nematic phase for approximately the range temperatures $20°C$ to $47°C$ and whose length is on the scale of Angstroms.

For the modeling of a single species of nematic liquid crystal at a fixed temperature one can consider a continuum theory which disregards the individual molecular structure. Such a continuum assumption is valid since the distance over which directional order occurs is much larger than the molecular dimensions (that is proportional to μm versus proportional to Angstroms) see [5].

One of the major motivations for modeling nematic liquid crystals mathematically is to better understand their defect structure and its dynamics. Defects are discontinuities that appear in materials and break local symmetry. For example, in solids, such defects are dislocations that break translational symmetry and introduce plastic deformation. A continuum model for the hydrodynamic flow of nematic liquid crystals provides the first step in such an analysis of the dynamics of the defects for nematic liquid crystals.

This dissertation gives strong mathematical evidence for the validity (well-posedness) of commonly used continuum models for the three-dimensional hydrodynamic flow of nematic liquid crystals. By well-posedness, it is meant that the modeling equations have a unique solution that is sufficiently smooth. Since the full continuum equations for hydrodynamic flow of nematic liquid crystals proposed by Ericksen and Leslie are complicated this dissertation analyzes equations that retain the mathematical difficulties and physical properties of interest.

The system of equations analyzed can be intuitively thought of as a coupling between the three-dimensional Navier–Stokes equations and the equations for the transported heat flow of harmonic maps into spheres. Here, the Navier–Stokes equations give the velocity whereas the transported heat flow of harmonic maps into spheres give the preferred average directional order via a unit vector. Such systems have been widely analyzed throughout the literature, but the novel contributions of this work is that it includes three-dimensional flows and analyzes the transported heat flow of harmonic maps into spheres by directly enforcing the sphere constraint instead of analyzing an approximation of this system.

1.2 Past mathematical work

Since the model that is examined in manuscript in a coupling between the Navier–Stokes equations and the transported heat flow of harmonic maps into spheres the following section summarizes the relevant mathematical works on Navier–Stokes and nematic liquid crystals. The summary of past work relating to the Navier–Stokes equation is highly selective due to the sizable amount of work in the area. An attempt has been made to include the vital and motivating works for this dissertation.

Mathematical analysis of nematic liquid crystals

A rigorous mathematical analysis of the Oseen–Zocher–Frank continuum model for nematic liquid crystals described in Section 2.1 was completed by Hardt, Kinderlehrer and Lin in 1986 [13]. They established existence and partial regularity of minimizers of the Oseen–Zocher–Frank functional $\mathcal{F}$ (see (2.1)). To be precise, the major results of their work are summarized in the following two theorems.

Theorem 1.2.1 (Existence of minimizers)**.** *For $n_0 : \partial\Omega \to \mathbb{S}^2$ a Lipshitz map, the admissible class of minimizers*

$$\mathcal{A}(n_0) := \{u \in H^1(\Omega, \mathbb{S}^2) : n_0 = \text{ trace of } u \text{ on } \partial\Omega\}$$

is non-empty. Furthermore, for any $n_0 : \partial\Omega \to \mathbb{S}^2$ there exists $n \in \mathcal{A}(n_0)$ such that

$$\mathcal{F}[n] = \inf_{u \in \mathcal{A}(n_0)} \mathcal{F}[u].$$

Theorem 1.2.2 (Interior partial regularity)**.** *If $n \in H^1(\Omega, \mathbb{S}^2)$ is a minimizer of $\mathcal{F}$, then n is analytic on $\Omega \setminus Z$ for some relatively closed subset Z of Ω which has one-dimensional Hausdorff measure zero.*

Later, Lin and Liu [20] began mathematical study of the hydrodynamic flow of nematic liquid crystals. They examined a simplified version of the Ericksen–Leslie equations. They studied the equations

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nu\Delta\mathbf{u} + \nabla p &= -\lambda\nabla \cdot (\nabla\mathbf{d} \odot \mathbf{d})^{①} \\ \mathbf{d}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \gamma(\Delta\mathbf{d} - f(\mathbf{d})) \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{1.1}$$

It is noted that these equations can be derived in much the same way as the model equations studied in this dissertation (those being (1.7)-(1.9)).

① The product $\odot$ is given in components by $(\nabla\mathbf{d} \odot \nabla\mathbf{d})^{ij} = \nabla_i d^q \nabla_j d^q$.

The full Ericksen–Leslie equations describing the flow phenomena of nematic liquid crystals reduces to the Oseen–Zocher–Frank theory for nematics in the static case. Furthermore, minimizers of the Oseen–Zocher–Frank functional are simply harmonic maps into spheres provided the physical constants are properly chosen (see (2.31)). Hence, it is a helpful heuristic to consider the Ericksen–Leslie equations as a coupling between the Navier–Stokes equation and the transported heat flow of harmonic maps into spheres.

Lin and Liu motivated by the work on gradient flow of harmonic maps into spheres used the standard penalty approximation to relax the sphere constraint (see [24], [3], [4]). The difficulty with such approximations is that the convergence of solutions to penalized problem to original problem is only understood in a few cases (see again, [24], [3]).

These issues aside, Lin and Liu proved the following theorems for the system (1.1).

Theorem 1.2.3 (Existence global weak solutions). *Under the assumptions that* $\mathbf{u}_0(x) \in L^2(\Omega)$ *and* $\mathbf{d}_0(x) \in H^1(\Omega)$ *with* $\mathbf{d}_0|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$*, the system* (1.1) *has a global weak solution* $(\mathbf{u}, \mathbf{d})$ *such that*

$$\begin{aligned} \mathbf{u} &\in L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega)) \\ \mathbf{d} &\in L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \end{aligned}$$

for all $T \in (0, \infty)$

Theorem 1.2.4 (Wellposedness for large viscosity). *The problem* (1.1) *has a unique global classical solution* $(\mathbf{u}, \mathbf{d})$ *provided that* $\mathbf{u}_0(x) \in H^1(\Omega)$*,* $\mathbf{d}_0(x) \in H^2(\Omega)$*,* $\dim(\Omega) = 2, 3$*, and* $\nu \geq \nu_0(\lambda, \gamma, \mathbf{u}_0, \mathbf{d}_0)$.

Very recently, Lin, Lin, and Wang [19] revisited the nematic liquid crystal flow problem for two-dimensional flows. Namely, they examined the system:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\Delta\mathbf{u} + \nabla p &= -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\mathbf{d}) \\ \mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= |\nabla\mathbf{d}|^2\mathbf{d} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned} \tag{1.2}$$

for $\mathbf{u} : \Omega \times (0, +\infty) \to \mathbb{R}^2$ and $\mathbf{d} : \Omega \times (0, +\infty) \to \mathbb{S}^2$ and initial data

$$\mathbf{u}_0 \in \mathbf{H}, \mathbf{d}_0 \in H^1(\Omega, \mathbb{S}^2), \text{ and } d_0 \in C^{2,\beta}(\partial\Omega, \mathbb{S}^2)^{②} \text{ for some } \beta \in (0,1). \tag{1.3}$$

Where,

$$\begin{aligned} \mathbf{H} &= \text{closure of } C_0^\infty(\Omega,\mathbb{R}^2) \cap \{\mathbf{u} : \nabla\cdot\mathbf{u} = 0\} \text{ in } L^2(\Omega,\mathbb{R}^2), \\ \mathbf{J} &= \text{closure of } C_0^\infty(\Omega,\mathbb{R}^2) \cap \{\mathbf{u} : \nabla\cdot\mathbf{u} = 0\} \text{ in } H^1(\Omega,\mathbb{R}^2), \text{ and} \\ H^1(\Omega,\mathbb{S}^2) &= \{\mathbf{d} \in H^1(\Omega,\mathbb{R}^3) : \mathbf{d}(x) \in \mathbb{S}^2 \text{ a.e. } x \in \Omega\}. \end{aligned}$$

Lin, Lin and Wang succeeded in proving the following theorems:

② The Banach spaces of functions denoted with $C^{2,\beta}$ will be the set of functions which have second derivatives in the spatial variables that are Hölder β. See [10] or [29] for a more detailed explanation of such spaces.

Theorem 1.2.5 (Regularity)**.** *For* $0 < T < +\infty$ *assume* $\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{J})$ *and* $\mathbf{d} \in L^2([0,T], H^1(\Omega, \mathbb{S}^2))$ *is a suitable weak solution of (1.2) with (1.3). If in addition,* $\mathbf{d} \in L^2([0,T], H^2(\Omega))$*, then* $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0,T]) \cap C^{2,1}_\beta(\bar{\Omega}) \times (0,T])$[③]*.*

Theorem 1.2.6 (Global Weak Solutions of Partial Regularity)**.** *For data satisfying (1.3), there exist global suitably weak solutions* $\mathbf{u} \in L^\infty([0,\infty), \mathbf{H}) \cap L^2([0,\infty), \mathbf{J})$ *and* $\mathbf{d} \in L^\infty([0,\infty), H^1(\Omega, \mathbb{S}^2))$ *of the equations (1.2), which has the following properties:*

1. *There exists* $L \in \mathbb{N}$ *depending only on* $(\mathbf{u}_0, \mathbf{d}_0)$ *and* $0 < T_1 < \cdots < T_L$*,* $1 \le i \le L$*, such that*

$$(u, \mathbf{d}) \in C^\infty(\Omega \times ((0,\infty) \setminus \{T_i\}_{i=1}^L)) \cap C^{2,1}_\beta(\bar{\Omega} \times ((0,\infty) \setminus \{T_i\}_{i=1}^L)).$$

2. *Each singular time* T_i*,* $1 \le i \le L$*, can be characterized by*

$$\liminf_{t \uparrow T_i} \max_{x \in \bar{\Omega}} \int_{\Omega \cap B_r(x)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(y,t) dy \ge 8\pi, \ \forall r > 0.$$

   Moreover, there exist $x^i_m \to x^i_0 \in \Omega$*,* $t_i \uparrow T_i$*,* $r^i_m \downarrow 0$ *and a non-constant smooth harmonic map* $\omega_i : \mathbb{R}^2 \to \mathbb{S}^2$ *with finite energy such that as* $m \to \infty$*,*

$$(\mathbf{u}^i_m, \mathbf{d}^i_m) \to (0, \omega_i) \ \textit{in} \ C^2_{loc}(\mathbb{R}^2 \times [\infty, 0]),$$

   where

$$\mathbf{u}^i_m(x,t) = r^i_m \mathbf{u}(x^i_m + r^i_m x, t^i_m + (r^i_m)^2 t), \ \mathbf{d}^i_m(x,t) = \mathbf{d}(x^i_m + r^i_m x, t^i_m + (r^i_m)^2 t).$$

3. *Set* $T_0 = 0$*. Then, for* $0 \le i \le L-1$*,*

$$|\mathbf{d}_t| + |\nabla^2 \mathbf{d}| \in L^2(\Omega \times [T_i, T_{i+1} - \epsilon]), \ |\mathbf{u}_t| + |\nabla^2 \mathbf{u}| \in L^{4/3}(\Omega \times [T_i, T_{i+1} - \epsilon])$$

   for any $\epsilon > 0$*, and for any* $0 < T_L < T < \infty$*,*

$$|\mathbf{d}_t| + |\nabla^2 \mathbf{d}| \in L^2(\Omega \times [T_L, T]), \ |\mathbf{u}_t| + |\nabla^2 \mathbf{u}| \in L^{4/3}(\Omega \times [T_L, T]).$$

4. *There exist* $t_k \uparrow \infty$ *and a harmonic map* $\mathbf{d}_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\bar{\Omega}, \mathbb{S}^2)$ *with* $\mathbf{d}_\infty = \mathbf{d}_0$ *on* $\partial\Omega$ *such that* $\mathbf{u}(\cdot, t_k) \to 0$ *in* $H^1(\Omega)$*,* $\mathbf{d}(\cdot, t_k) \to \mathbf{d}_\infty$ *weakly in* $H^1(\Omega)$*, and there exist* $l \in \mathbb{N}$*, points* $\{x_i\}_{i=1}^l \in \Omega$*, and* $\{m_i\}_{i=1}^l \subset \mathbb{N}$ *such that*

$$|\nabla \mathbf{d}(\cdot, t_k)|^2 dx \to |\nabla \mathbf{d}_\infty|^2 dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i}.$$

[③]Banach spaces of functions denoted with $C^{2,1}_\beta$ are *anisotropic Hölder spaces* of functions that have second spatial derivatives and first time derivatives which are Hölder β. See, [29].

5. *If* $(\mathbf{u}_0, \mathbf{d}_0)$ *satisfies*

$$\int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \le 8\pi, \tag{1.4}$$

then $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0,\infty)) \cap C^{2,1}_\beta(\bar\Omega \times (0,\infty))$. *Moreover, there exist* $t_k \uparrow \infty$ *and* $\mathbf{d}_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\bar\Omega, \mathbb{S}^2)$ *with* $\mathbf{d}_\infty = d_0$ *on* $\partial\Omega$ *such that* $(\mathbf{u}(\cdot, t_k), \mathbf{d}(\cdot, t_k)) \to (0, \mathbf{d}_\infty)$ *in* $C^2(\Omega)$.

Remark 1.2.7. *In Theorem 1.2.6 item 4. illustrates the potential failure of strong convergence whereas item 5. gives sufficient conditions to guarantee strong convergence.*

Definition 1.2.8 (Suitable Weak Solutions)**.** *For* $0 < T \le \infty$, $\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{J})$ *and* $\mathbf{d} \in L^2([0,T], H^1(\Omega, \mathbb{S}^2))$ *is a suitable weak solution of (1.2) if the following local energy inequality holds*

$$\begin{aligned}
&-\int_{\Omega\times[0,T]} \langle \mathbf{u}, \psi'\phi\rangle + \int_{\Omega\times[0,T]} [\langle \mathbf{u}\cdot\nabla\mathbf{u}, \psi\phi\rangle + \nu\langle \nabla\mathbf{u}, \psi\nabla\phi\rangle] \\
&= -\psi(0)\int_\Omega \langle \mathbf{u}_0, \phi\rangle + \lambda \int_{\Omega\times[0,T]} \langle \nabla\mathbf{d}\odot\nabla\mathbf{d}, \psi\nabla\phi\rangle, \\
&-\int_{\Omega\times[0,T]} \langle \mathbf{d}, \psi'\phi\rangle + \int_{\Omega\times[0,T]} [\langle \mathbf{u}\cdot\nabla\mathbf{d}, \psi\phi\rangle + \nu\langle \nabla\mathbf{d}, \psi\nabla\phi\rangle] \\
&= -\psi(0)\int_\Omega \langle \mathbf{d}_0, \phi\rangle + \lambda \int_{\Omega\times[0,T]} |\nabla\mathbf{d}|^2 \langle \mathbf{d}, \psi\phi\rangle,
\end{aligned}$$

for any $\psi \in C^\infty([0,T])$ *with* $\psi(T) = 0$ *and* $\phi \in \mathbf{J} \cap H^1_0(\Omega, \mathbb{R}^3)$. *Moreover,* $(\mathbf{u}, \mathbf{d})$ *satisfies prescribed boundary data in the trace sense.*

Mathematical analysis of the Navier–Stokes equations

As mentioned earlier, due to the size and scope of the mathematical literature dedicated to the study of the Navier–Stokes equations it would be nearly impossible to give a complete overview of the literature. In what follows, the key literature pertaining to the new results presented subsequent chapters is summarized. One explicit goal of this subsection is to clarify the use and origin of the class $L^3_x L^\infty_t$[④]. Another goal is to examine some of the similarities between theoretical difficulties encountered in the analysis of the Navier–Stokes system and those encountered in the transported heat flow of harmonic maps into spheres.

The first major mathematical work on the well-posedness of the incompressible Navier–Stokes equations was by Jean Leray in his dissertation (published in [16]). Leray worked on the pure Cauchy problem for the incompressible Navier–Stokes equations in all of $\mathbb{R}^3$, namely,

$$\begin{aligned}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p = 0 \text{ in } \mathbb{R}^3 \times [0,\infty) \\
\nabla\cdot\mathbf{u} = 0 \text{ in } \mathbb{R}^3 \times [0,\infty) \\
\mathbf{u}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x}) \text{ in } \mathbb{R}^3.
\end{aligned} \tag{1.5}$$

[④] $L^p_x L^q_t(\Omega \times I) = L^p(I, L^q(\Omega))$ is a *mixed* or *anisotropic* L^p space.

for $\mathbf{a}$ a smooth divergence-free vector field in $\mathbb{R}^3$ which has sufficient decay at infinity. The results of Leray [16] are neatly summarized in [9]:

Theorem 1.2.9 (Leray 1934)**.** *1. There exists $T_* > 0$ such that the Cauchy problem (1.5) has a unique smooth solution with reasonable properties at infinity.*

2. *The problem (1.5) has at least one global weak solutions satisfying a local energy inequality. Moreover, the weak solutions correspond with the smooth (strong) solution in $\mathbb{R}^3 \times (0, T_*)$ (and hence are unique in that domain).*

3. *If $(0, T_*)$ is the maximal interval of existence of the smooth solution, then, for each $p > 3$, there exists $\epsilon_p > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u(x,t)| dx\right)^{1/p} \geq \frac{\epsilon_p}{(T_* - t)^{1/2(1-3/p)}}$$

as $t \uparrow T_*$.

4. *For a given weak solution, there exists a closed set* **Sing** $\subset (0, \infty)$ *of measure zero such that the solution is smooth in $\mathbb{R}^3 \times ((0,\infty) \setminus \mathbf{Sing})$. (Leray's proof actually gives* **Sing** *with $\mathcal{H}^{1/2}(\mathbf{Sing}) = 0$ [5]).*

An important generalization Theorem part 3. was provided collectively by Prodi [23], Ladyzhenskaya [15], and Serrin [25]. Namely, as summarized by [9], the following theorem is true.

Theorem 1.2.10 (Prodi-Ladyzhenskaya-Serrin)**.** *Suppose the initial data* $\mathbf{a}$ *for* (1.5) *is in the L^2 closure of the set of divergence free smooth vector fields. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two Leray-Hopf solutions of the Cauchy problem* (1.5)*. If for some $T > 0$ the solution* $\mathbf{u}$ *satisfies the so-called Ladyzhenskaya-Prodi-Serrin condition:*

$$\mathbf{u} \in L^s_x L^l_t(\mathbb{R}^3 \times (0,T)) \text{ with } \frac{3}{s} + \frac{2}{l} = 1, s \in (3, \infty), \tag{1.6}$$

then $\mathbf{u} = \mathbf{v}$ in $\mathbb{R}^3 \times (0,T)$ and moreover, $\mathbf{u}$ *is a smooth on $\mathbb{R}^3 \times (0,T)$.*

Later, Caffarelli, Kohn and Nirenberg expanded upon the idea of Theorem 1.2.9 part 4. by localizing in x. In [2], Caffarelli, Kohn and Nirenberg, building the work of Scheffer prove a local partial regularity result for a certain class of *suitable weak solutions*. Namely, they prove the following theorem for the parabolic Hausdorff measure $\mathcal{P}^s$ [6].

[5] For $A \in \mathbb{R}^n$, $0 \leq s < \infty$, $0 < \delta \leq \infty$ define the *s-dimensional Hausdorff measure of A* by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \inf \left\{ \sum |B_1(0)| \left(\frac{\operatorname{diam} C_j}{2}\right)^s : A \subset \bigcup C_j, \operatorname{diam} C_j \leq \delta \right\}.$$

[6] For $A \in \mathbb{R}^n$, $0 \leq s < \infty$, $0 < \delta \leq \infty$ define the *parabolic Hausdorff measure of A* by

$$\mathcal{P}^s(A) := \lim_{\delta \to 0} \inf \left\{ \sum r_j^s : A \subset \bigcup B_{r_j}(\mathbf{x}_0) \times [t_0 - r_j^2, t_0 + r_j^2], r_j \leq \delta \right\}.$$

Theorem 1.2.11 (Partial Regularity for Navier–Stokes)**.** *For any suitable weak solution of the Navier–Stokes system on an open set in space-time, the associated singular set* $\mathbf{S}$ *satisfies* $\mathcal{P}^1(\mathbf{S}) = 0$

Where, a suitable weak solution was defined by the author in the following way.

Definition 1.2.12. *The pair (u,p) is a suitable weak solution of the Navier–Stokes system*

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{aligned}$$

on an open set $D \subset \mathbb{R}^3 \times \mathbb{R}$ *with force* $\mathbf{f}$ *if the following conditions are satisfied*

1. Integrability hypotheses $\mathbf{u}$,p, *and* $\mathbf{f}$ *are measurable functions on* D *and*

   a) $\mathbf{f} \in L^q(D)$ *for some* $q > 5/2$ *and* $\nabla \cdot \mathbf{f} = 0$,

   b) $p \in L^{5/4}(D)$,

   c) *for some constants* E_0, $E_1 < \infty$,

   $$\int_{D_t} |\mathbf{u}|^2 dx \le E_0, \;\; D_t = D \cap (\mathbb{R}^3 \times \{t\}),$$

   for almost every t *such that* $D_t = \emptyset$, *and*

   $$\iint_D |\nabla \mathbf{u}|^2 dx dt \le E_1.$$

2. Equations $\mathbf{u}$, p, *and* $\mathbf{f}$ *satisfy the the above Naiver–Stokes systems in the sense of distributions on* D.

3. Generalized energy inequality *For each real-valued* $\phi \in C_0^\infty(D)$ *with* $\phi \ge 0$, *the following inequality is valid*

   $$2 \iint |\nabla \mathbf{u}|^2 \phi \le \iint [|u|(\phi_t + \Delta\phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2(u \cdot f)\phi].$$

Recently, Escauriaza, Seregin, and Sverak completed the end-point case for Serrin's uniqueness in their paper [9]. Namely, they proved:

Theorem 1.2.13 ($L_x^3 L_t^\infty$ Regularity)**.** *Consider two function* $\mathbf{u}$ *and* p *defined in the space-time cylinder* $Q = B_1(0) \times (0,1)$. *Assume that* $\mathbf{u}$ *and* p *satisfy the Navier–Stokes equation in* Q *in the sense of distributions and have the following differentiability properties*

$$\mathbf{u} \in L_x^2 L_t^\infty(Q) \cap L_t^2 H_x^1(B_1(0) \times (-1, 0)), \;\; p \in L^{3/2}(Q).$$

Let, in addition,

$$\|\mathbf{u}\|_{L_x^3 L_t^\infty(Q)} < \infty.$$

Then the function $\mathbf{u}$ *is Hölder continuous in the closure of the set*

$$B_{1/2}(0) \times (-(1/2)^2, 0).$$

Remark 1.2.14. *1. This result is the extension of Theorem 1.2.9 part 3. to* $p = 3$. *That is, of* $(0, T_*)$ *is the maximal interval of existence of the smooth solution to the problem (1.5) and* $T_* < \infty$, *then*

$$\limsup_{t \uparrow T_*} \int_{\mathbb{R}^3} |\mathbf{u}(x,t)|^3 = +\infty.$$

This means that if the solution $\mathbf{u}$ *develops a singularity (hence the existence of a* T_**), then the* L^3_x*-norm must blow-up.*

2. The class $L^3_x L^\infty_t$ *is special in the sense that finiteness in* $\|f\|_{L^3_x L^\infty_t(\mathbb{R}^3)}$ *does not imply* local smallness *of* f *despite the fact that* $\|f\|_{L^3_x L^\infty_t(\mathbb{R}^3)}$ *is still invariant under the natural scalings of the Navier–Stokes equations, namely, if* $(\mathbf{u}(x,t), p(x,t))$ *is a solution then so is*

$$(\lambda \mathbf{u}(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x \lambda^2 t)).$$

1.3 New contributions

The new results recorded in this dissertation are for solutions $\mathbf{u} : \mathbb{R}^3 \times [0,T) \to \mathbb{R}^3$ and $\mathbf{d} : \mathbb{R}^3 \times [0,T) \to S^2$ to

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \text{ in } \mathbb{R}^3 \times [0,T), \tag{1.7}$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \mathbb{R}^3 \times [0,T), \tag{1.8}$$
$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \Delta \mathbf{d} = |\nabla \mathbf{d}|^2 \mathbf{d} \text{ in } \mathbb{R}^3 \times [0,T), \tag{1.9}$$
$$(\mathbf{u}, \mathbf{d})|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0) \text{ in } \mathbb{R}^3 \times \{0\}. \tag{1.10}$$

This is the system examined by Lin, Lin and Wang in [19] for flow variable $\mathbf{u} : \Omega \times [0,T) \to \mathbb{R}^2$ for $\Omega \subset \mathbb{R}^2$. It bears repeating that this system should be thought of as a coupling being the Navier–Stokes equations and the equations for the transported heat flow of harmonic maps into spheres. It should also be noted that this system is the most basic form of the full continuum equations for the hydrodynamic flow of nematic liquid crystals given by Ericksen and Leslie. The relation between the system (1.7)-(1.9) and the Ericksen–Leslie equations will be examined in detail in the next chapter.

This work demonstrates the existence, uniqueness, and regularity of solutions to the Cauchy problem (1.7)-(1.10) for initial data $(\mathbf{u}_0, \mathbf{d}_0)$ satisfying $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3_{\mathbb{U}}(\mathbb{R}^3) \times L^3_{\mathbb{U}}(\mathbb{R}^3)$. The definition of $L^3_{\mathbb{U}}(\mathbb{R}^3)$ and the precise statement of the result are recorded now.

Definition 1.3.1. *Let* $\mathbf{f} \in L^3_{\mathbb{U}}(\mathbb{R}^3)$ *if*

$$\|\mathbf{f}\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} := \sup_{x \in \mathbb{R}^3} \|\mathbf{f}\|_{L^3(B_1(x))} < \infty.$$

The space is $L^3_{\mathbb{U}}(\mathbb{R}^3)$ *is called the space of* uniformly locally L^3 integrable functions *on* $\mathbb{R}^3$

Theorem 1.3.2 (Well-posedness)**.** *There exists* $\epsilon_0 > 0$ *and* $T_0 > 0$ *such that if* $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ *and* $\mathbf{d}_0 : \mathbb{R}^3 \to S^2$ *satisfy*

$$\|\mathbf{u}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \leq \epsilon_0, \tag{1.11}$$

then there exists a unique solution $(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times [0, T_0) \to \mathbb{R}^3 \times S^2$ *of* (1.7)-(1.10) *with the following properties:*

- $(\mathbf{u}, \nabla \mathbf{d}) \in C([0, T_0), L^3_{\mathbb{U}}(\mathbb{R}^3))$*;*
- $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, T_0))$*.*

Furthermore, if $T_0 < +\infty$ *is the maximum time interval, then*

$$\lim_{t \uparrow T_0} \|\mathbf{u}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} > \epsilon_0.$$

This theorem has many parts and so it is useful to make a rough outline of its proof and set down a plan for the later chapters of this manuscript. The proof of Theorem 1.3.2 has the following ingredients:

I. global and local energy inequalities for (1.7) -(1.9);

II. a priori gradient estimates for smooth solutions $(\mathbf{u}, p, \mathbf{d})$ that have small renormalized energies;

III. short-time existence of classical solutions proven in [19].

Through a weak-strong type argument analogous to Leray's proof of Theorem 1.2.9 part 2. these ingredients yield the desired well-posedness. Precise statements of the results of I.-III. are given in the following paragraphs.

I. Global and local energy inequalities

For the system (1.7)-(1.9) one has the following a priori estimates of smooth solutions.

Theorem 1.3.3 (Global L^3-energy inequality)**.** *If* $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3(\mathbb{R}^3)$*, then any smooth solution to* (1.7)-(1.9) *obeys:*

$$\begin{aligned} &\frac{d}{dt}\left\{\int_{\mathbb{R}^3} |\mathbf{u}|^3 + |\nabla \mathbf{d}|^3\right\} + \left[1 - C\|\mathbf{u}\|^2_{L^3(\mathbb{R}^3)}\right] \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 |\mathbf{u}| \\ &+ \left[1 - C(\|\mathbf{u}\|_{L^3(\mathbb{R}^3)} + \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)} + \|\nabla \mathbf{d}\|^2_{L^3(\mathbb{R}^3)})\right] \\ &\times \int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \leq 0. \end{aligned} \tag{1.12}$$

Theorem 1.3.4 (Local L^3 energy inequality)**.** *For* $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3_{\mathbb{U}}(\mathbb{R}^3)$, *let* $\phi \in C^1_0(\mathbb{R}^3)$, *then any smooth solution to* (1.7)-(1.9) *obeys:*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(|\mathbf{u}|^3+|\nabla \mathbf{d}|^3)\phi^2+\int_{\mathbb{R}^3}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2+|\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2]\\
&\leq C\int_{\mathbb{R}^3}(|\mathbf{u}|^3+|\nabla \mathbf{d}|^3)|\nabla\phi|^2+C\int_{\mathbb{R}^3}|\mathbf{u}||P-c|^2\phi^2\\
&+C\left[\left\{\int_{supp\ \phi}|\mathbf{u}|^3\right\}^{2/3}+\left\{\int_{supp\ \phi}|\nabla \mathbf{d}|^3\right\}^{2/3}\right]\\
&\times\int_{\mathbb{R}^3}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2+|\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2].
\end{aligned}\tag{1.13}$$

Lemma 1.3.5 (Local L^3-pressure estimate)**.** *There exist* $c_0 \in \mathbb{R}$ *and* $C_0 > 0$ *such that for any* $\phi \in C^1_0(\mathbb{R}^3)$,

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^3}|p-c_0|^3\phi^3\right\}^{1/3}\leq C_0\left\{\int_{\mathbb{R}^3}(|\mathbf{u}|^6+|\nabla \mathbf{d}|^6)\phi^3\right\}^{1/3}\\
&+\frac{C_0}{R}\left\{\int_{supp\ \phi}(|\mathbf{u}|^3+|\nabla \mathbf{d}|^3)\right\}^{2/3}+C_0R\sup_{z\in\mathbb{R}^3}\left\{\int_{B_R(z)}(|\mathbf{u}|^3+|\nabla \mathbf{d}|^3)\right\}^{2/3},
\end{aligned}$$

where $R > 0$ *is such that supp* $\phi \subset B_{2R}$.

II. A priori gradient estimates and regularity

The result in II. follows from building an appropriate framework for regularity. Following the work of Caffarelli, Kohn, and Nirenberg [2] one makes the following definition of suitable weak solutions (Definition 1.3.6) and can prove the corresponding generalized energy L^2 inequality (Theorem 1.3.7).

Definition 1.3.6. *A triple* $(\mathbf{u}, p, \mathbf{d}) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R} \times S^2$ *is called a* suitable weak solution *to* (1.7)-(1.9) *in* $\Omega \times (0, T)$ *if:*

1. $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega \times (0, T))$, $p \in L^{\frac{3}{2}}(\Omega \times (0, T))$ *and* $d \in L^2_t H^2_x(\Omega \times (0, T), S^2)$;
2. $(\mathbf{u}, p, \mathbf{d})$ *satisfies* (1.7)-(1.9) *in the weak sense;*
3. $(\mathbf{u}, p, \mathbf{d})$ *satisfies the local* L^2 *energy inequality* (1.3.7) *for* $\phi \in C^\infty_0(\Omega \times (0, T))$.

Theorem 1.3.7 (Generalized L^2-energy inequality)**.** *Suppose that* $(\mathbf{u}, p, \mathbf{d})$ *is a sufficiently regular solution to* (1.7)-(1.9) *on* $\Omega_T \equiv \Omega \times (0, T)$. *Then for any* $0 \leq \phi \in C^\infty_0(\Omega \times (0, T))$, *it holds*

$$\begin{aligned}
&2\int_{\Omega_T}\left(|\nabla \mathbf{u}|^2+|\Delta \mathbf{d}+|\nabla \mathbf{d}|^2\mathbf{d}|^2\right)\phi\\
&\leq\int_{\Omega_T}\left(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2\right)(\phi_t+\Delta\phi)+\int_{\Omega_T}(|\mathbf{u}|^2+|\nabla \mathbf{d}|^2+2p)\mathbf{u}\cdot\nabla\phi\\
&+2\int_{\Omega_T}\left(\nabla \mathbf{d}\odot\nabla \mathbf{d}-|\nabla \mathbf{d}|^2I_3\right):\nabla^2\phi+2\int_{\Omega_T}\nabla \mathbf{d}\odot\nabla \mathbf{d}:\mathbf{u}\otimes\nabla\phi.
\end{aligned}\tag{1.14}$$

With these results in hand, one can prove the following ϵ-regularity result (Theorem 4.2.6) for smooth solutions over the parabolic cylinder $P_r(x,t) = B_r(x) \times [t - r^2, t + r^2]$. The proof of this result involves a decay lemma, iteration, and Riesz potential estimates between Morrey spaces.

Theorem 1.3.8 (ϵ_0-regularity)**.** *Suppose that* $(\mathbf{u}, p, \mathbf{d})$ *is a suitable weak solution to* (1.7)-(1.9)*. There exists* $\epsilon > 0$ *such that if*

$$\left(r^{-2}\int_{P_r} |\mathbf{u}|^3\right)^{\frac{1}{3}} + \left(r^{-2}\int_{P_r} |p|^{\frac{3}{2}}\right)^{\frac{2}{3}} + \sup_{t-r^2\le t\le t+r^2}\left(\int_{B_r} |\nabla \mathbf{d}|^3(t)\right)^{1/3} \le \epsilon$$

where $P_r \equiv P_r(x,t)$*, then* $(\mathbf{u}, \mathbf{d}) \in C^\infty(P_{r/2}(x,t))$*. Moreover one has the estimate*

$$\|(\mathbf{u}, \mathbf{d})\|_{C^k(P_{r/2}(x,t))} \le C(k, \epsilon_0, r). \tag{1.15}$$

III. Short-time existence of classical solution and end-game

The end-game argument for Theorem 1.3.2 includes the following steps.

1. Approximate initial data $(\mathbf{u}_0, \mathbf{d}_0)$ by smooth initial data $(\mathbf{u}_0^k, \mathbf{d}_0^k)$ which also satisfy (1.11).

2. Short-time existence of smooth solutions $(\mathbf{u}^k, \mathbf{d}^k)$ is guaranteed by the contraction mapping principle (Theorem 3.1 of [19]) in $\mathbb{R}^3$ for initial data $(\mathbf{u}_0^k, \mathbf{d}_0^k)$.

3. Using the above local energy inequality and pressure estimate, one finds $T_0 > 0$, depending on $\epsilon_0 > 0$, such that

$$\sup_{0\le t\le T_0} \|\mathbf{u}^k\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}^k\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \le 2\epsilon_0.$$

4. Lastly, Theorem 4.2.6 implies $\|(\mathbf{u}^k, \mathbf{d}^k)\|_{C^2_{loc}(\mathbb{R}^3\times(\delta\times T_0))} \le C(\delta)$, $\delta > 0$. Hence, by passing to the limit via the theorem of Arzela-Ascoli, one proves the result.

1.4 Outline of the remainder of the manuscript

The plan for the rest of this manuscripts is as follows:

- Chapter 2, "Continuum models for nematic liquid crystals," gives a rapid introduction to the continuum theory for nematic liquid crystals.

- Chapter 3, "Energy Inequalities," is devoted to proving the energy inequalities (1.12) and (1.13).

- Chapter 4, "Regularity," contains the proof the generalized energy inequality (4.1) and Theorem 4.2.6.

- Chapter 5, "Well-posedness," brings together the facts from Chapters 2 and 3 and the short-time existence of [19] to complete the proof of Theorem 1.3.2.

The notation throughout the manuscript is standard. Since this manuscript will examine vector fields from $\mathbb{R}^3 \to \mathbb{R}^3$–vectors, tensors, and their products will appear. Loosely, bold symbols will be used represent vectors, capital bold symbols will represent tensors, components will be denoted with Latin superscripts, and partial derivatives will be denoted with Greek subscripts. There are cases where these convention must be violated, but care has been taken to leave few details out. As is often necessary, the *Einstein summation convention* is employed throughout–that is, repeated indices are summed. Reminders of other specific notations are made as footnotes as to not interrupt the flow of the manuscript.

Chapter 2 Continuum models for nematic liquid crystals

In this chapter the full continuum equations for the hydrodynamic flow of nematic liquid crystals are formally derived. To do so it is necessary to consider the static continuum theory for nematic liquid crystals based on work Oseen, Zocher, and Frank. It is then instructive to consider a short derivation of the Navier–Stokes equations before considering the derivation of the Ericksen–Leslie equations as both are derived from conservation laws. The critical difference between the two systems is that the Ericksen–Leslie equations require a third balance of angular momentum not present in the Navier–Stokes equations.

It is remarked during the derivation of the Ericksen–Leslie equations what simplifications must be made to arrive at the model equations analyzed in this manuscript, that is (1.7)-(1.9). The necessary simplifications are drastic and leave room for future investigation. Finally, it is noted that the derivations in this chapter do not include thermal effects.

2.1 Oseen–Zocher–Frank static theory

The simplest and earliest continuum theory for liquid crystals was put forth by Oseen [22], Zocher [30], and Frank [11] (1930-1960) (see also [28]). This model applies to both the nematic and cholesteric phase. Following Virga, to give a continuum description for liquid crystals (of any type) one must provide a free energy functional (hence making such a theory *variational*). Excluding dependence on temperature, such a free energy functional is the *Helmholtz* free energy. Let σ be the free energy density and so a free energy functional will be of the form

$$\mathcal{F}[\mathbf{n}] = \int_{\Omega} \sigma(\mathbf{n}, \nabla \mathbf{n}) dV. \tag{2.1}$$

Here $\mathcal{F}$ acts upon function $\mathbf{n} : \Omega \to \mathbb{S}^2$ where $\Omega \subset \mathbb{R}^3$ is body occupied by the liquid crystal and $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. Such functions $\mathbf{n}$ describe the average direction of molecules contained in Ω.

To build the theory one prescribes constitutive relations that describe the mechanical requirements of the material. For this manuscript it is prescribed that σ have the following properties.

1. The energy density σ must be *frame-indifferent.* $\sigma(\mathbf{Q}\mathbf{n}, \mathbf{Q}\mathbf{N}\mathbf{Q}^T) = \sigma(\mathbf{n}, \mathbf{N})$ for $\mathbf{n} \in \mathbb{S}^2$ and $\mathbf{N} \in L(\mathbf{n}, \mathcal{V})$ ① and $\mathbf{Q} \in \mathrm{SO}(\mathcal{V})$②. That is, the free energy per unit volume of the body must be the same in any two frames. It is noted that this property applies to both nematics and cholesterics–that is, it is not reliant on the material that occupies the body Ω. There are properties that do however rely upon material properties (material-symmetries).

①$L(\mathbf{n}, \mathcal{V}) = \{L : \mathcal{V} \to \mathcal{V} : L^T n = 0\}$, $\mathcal{V}$ a vector space.
②$\mathrm{SO}(\mathcal{V}) = \{\mathbf{Q} \in L(\mathbf{n}, \mathcal{V}) : \det(\mathbf{Q}) > 1, \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = I\}$.

2. The energy density σ must be *material-symmetric*. Nematic and cholesteric liquid crystals are differentiated at the microscopic level by which group transformations they obey. Nematics, which can be visualized as ellipsoids, are left unchanged after reflections. Cholesterics, having the shape of springs have a chirality or handedness that changes under reflection. It is expected, that such microscopic differences are also observed at the macroscopic scale. For nematics it is demanded that the energy remains the same under reflections whereas in cholesterics this is not required. Mathematically, the difference between nematics and cholestrics is that the condition that must hold for cholesterics is that $\sigma(\mathbf{Qn}, \mathbf{QNQ}^T) = \sigma(\mathbf{n}, \mathbf{N})$ for $\mathbf{n} \in \mathbb{S}^2$ and $\mathbf{N} \in L(\mathbf{n}, \mathcal{V})$ and $\mathbf{Q} \in \mathrm{O}(\mathcal{V})$ [3].

3. The energy density σ must be *even*. On the macroscopic level, the direction $\mathbf{n}$ at a material point p can be interpreted in a precise sense as the average orientation. So in this statistical sense, it is impossible to distinguish head from tail of a molecule–that is, one cannot distinguish $\mathbf{n}$ from $-\mathbf{n}$. So, it should be in either the nematic or cholesteric case that reversing the direction field leaves the energy density unchanged. That is, mathematically, $\sigma(-\mathbf{n}, -\nabla\mathbf{n}) = \sigma(\mathbf{n}, \nabla\mathbf{n})$.

4. The energy density σ must be *positive definite*. In the absence of external forces or boundary conditions (anchoring) that affect the orientation of the liquid crystal, the liquid crystal reaches and undistorted ground state. It is common assign this state zero, free energy and call such an orientation natural. This is manifested mathematically as, $\sigma(\mathbf{n}, \mathbf{N}) \geq 0$ for any $\mathbf{n} \in \mathbb{S}^2$ and $\mathbf{N} \in L(\mathbf{n}, \mathcal{V})$.

Frank's Formula

Under the constitutive relations discussed in the last paragraph one can determine which energy densities obey such relations. The final form of the energy density is known as Frank's formula–but, similar expressions can be found in the works of Zocher [30] and Oseen [22]. Frank gave a formal derivation of Theorem 2.1.3 using Taylor expansion. The rigorous proof of Theorem 2.1.3 recorded here is based upon the abstract representation Theorem 2.1.1 from the book of Virga [28].

The physical meaning and constraints on the constants appearing in Frank's formula, Theorem 2.1.3, follow the proof in a series of remarks.

Theorem 2.1.1 (Virga [28]). *For any* $\mathbf{n} \in \mathbb{S}^2$*, let* $\varphi(\mathbf{n}, \cdot)$ *be the scalar-valued function on* $L(\mathbf{n}, \mathcal{V})$ *defined by*

$$\varphi(\mathbf{n}, \mathbf{N}) := k(\mathbf{n}) + \mathbf{K}(\mathbf{n}) : \mathbf{N} + \mathbf{N} : \mathbb{K}(\mathbf{n})[\mathbf{N}],$$

where $k(\mathbf{n})$ *is a scalar,* $\mathbf{K}(\mathbf{n})$ *is a tensor in* $L(\mathbf{n}, \mathcal{V})$*, and* $\mathbb{K}(\mathbf{n})$ *is a symmetric tensor in* $L^2(\mathbf{n}, \mathcal{V})$ [4]*. The function* φ *satisfies*

$$\varphi(\mathbf{Qn}, \mathbf{QNQ}^T) = \varphi(\mathbf{n}, \mathbf{N}) \text{ for all } \mathbf{Q} \in SO(\mathcal{V})$$

[3] $\mathrm{O}(\mathcal{V}) = \{\mathbf{Q} \in L(\mathbf{n}, \mathcal{V}) : \mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = I\}$.
[4] $L^2(\mathbf{n}, \mathcal{V}) = \{\mathbb{L} : L(\mathbf{n}, \mathcal{V}) \to L(\mathbf{n}, \mathcal{V}) : \mathbb{L} \text{ Linear}\}$

if, and only if,

$$
\begin{aligned}
\varphi(\mathbf{n}, \mathbf{N}) = \alpha_0 &+ \alpha_2 P(\mathbf{n}) : \mathbf{N} + \alpha_3 \mathbf{W}(\mathbf{n}) : \mathbf{N} + \beta_1 (\mathbf{W}(\mathbf{n}) : \mathbf{N})^2 \\
&+ \beta_2 (P(\mathbf{n}) : \mathbf{N})^2 + \beta_3 (\mathbf{W}(\mathbf{n}) : \mathbf{N})(P(\mathbf{n}) : \mathbf{N}) \\
&+ \beta_4 \mathbf{N} : \mathbf{N} + \beta_5 P(\mathbf{n}) \mathbf{N} : \mathbf{N}(P(\mathbf{n})).
\end{aligned}
$$

where α_0*,* α_2*,* α_3 *and* $\beta_1, \ldots, \beta_5$*, are scalars, and* $P(\mathbf{n})$ *and* $\mathbf{W}(\mathbf{n})$ *are defined by*

$$
\begin{aligned}
P(\mathbf{n}) &:= I - \mathbf{n} \otimes \mathbf{n}^{⑤} \\
\mathbf{W}(\mathbf{n}) &:= \mathbf{n} \wedge \mathbf{v} \textit{ for all } \mathbf{v} \in \mathcal{V}.^{⑥}
\end{aligned}
$$

Lemma 2.1.2. *Let* $\mathbf{n} \in C^1(\Omega, \mathbb{S}^2)$*. The following equations hold identically in the domain of* $\mathbf{n}$*:*

$$(\nabla \mathbf{n})\mathbf{n} = -\mathbf{n} \wedge \operatorname{curl} \mathbf{n}; \tag{2.2}$$

$$\nabla \mathbf{n} : \nabla \mathbf{n} = \operatorname{tr}(\nabla \mathbf{n})^2 + (\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2 + |\mathbf{n} \wedge \operatorname{curl} \mathbf{n}|^2.^{⑦} \tag{2.3}$$

Proof. The action of every skew symmetric $\mathbf{W} \in L(\mathcal{V})$ [⑧] may be represented as

$$\mathbf{W} v = \mathbf{w} \wedge v$$

where $\mathbf{w}$ is the axial vector of $\mathbf{W}$. Additionally, one may prove by representing $\mathbf{w}\wedge$ as $\epsilon^{ikj} w^k$ that

$$\mathbf{W} \cdot \mathbf{W} = (\mathbf{w}\wedge) \cdot (\mathbf{w}\wedge) = \epsilon^{ikj} w^k \epsilon^{ilj} w^l = 2\mathbf{w} \cdot \mathbf{w}.$$

Then for the decomposition

$$\nabla \mathbf{n} = \mathbf{S} + \mathbf{W}, \mathbf{S} \text{ symmetric and } \mathbf{W} \text{ skew} \tag{2.4}$$

one has that the axial vector $\mathbf{w}$ of $\mathbf{W}$ is given as

$$w = \frac{1}{2} \operatorname{curl} \mathbf{n}. \tag{2.5}$$

Now, given that $(\nabla \mathbf{n})^T \mathbf{n} = 0$ one has that

$$0 = (\nabla \mathbf{n})^T \mathbf{n} = \mathbf{S}^T \mathbf{n} + \mathbf{W}^T \mathbf{n},$$

which is in turn,

$$\mathbf{W} \mathbf{n} = \mathbf{S} \mathbf{n}.$$

---

⑤The product $\otimes$ is the *tensor product* and is defined in components by $(\mathbf{a} \otimes \mathbf{b})^{ij} = a^i b^j$.

⑥The product $\wedge$ is the usual *cross product* which is defined in coordinates as $(\mathbf{a} \wedge \mathbf{b})^i = \epsilon^{ijk} a^j b^k$.

⑦The symbols $\cdot$ and $:$ denote the scalar (dot) products given by $\mathbf{a} \cdot \mathbf{b} = a^i b^i$ and $\mathbf{A} : \mathbf{B} = A^{ij} B^{ij}$.

⑧$L(\mathcal{V}) = \{L : \mathcal{V} \to \mathcal{V} : L \text{ linear }\}$ on $v \in \mathcal{V}$

Thus, collecting these facts, one has

$$(\nabla \mathbf{n})\mathbf{n} = \mathbf{S}\mathbf{n} + \mathbf{W}\mathbf{n} = 2\mathbf{W}\mathbf{n} = 2\left(\frac{1}{2}\operatorname{curl}\mathbf{n}\right) \wedge \mathbf{n} \tag{2.6}$$

which is precisely (2.2).

For (2.3), one may compute that

$$\nabla \mathbf{n} : \nabla \mathbf{n} = \mathbf{S} : \mathbf{S} + 2\mathbf{S} : \mathbf{W} + \mathbf{W} : \mathbf{W} = \mathbf{S} : \mathbf{S} + \mathbf{W} : \mathbf{W}$$

and

$$\begin{aligned} \operatorname{tr}(\nabla \mathbf{n} \nabla \mathbf{n}) &= \operatorname{tr}(\mathbf{S}^2 + \mathbf{S}\mathbf{W} + \mathbf{W}\mathbf{S} + \mathbf{W}^2) \\ &= \operatorname{tr}(\mathbf{S}^2) + \operatorname{tr}(\mathbf{S}\mathbf{W} + \mathbf{W}\mathbf{S}) + \operatorname{tr}(\mathbf{W}^2) \\ &= \operatorname{tr}(\mathbf{S}^2) + \operatorname{tr}(\mathbf{W}^2) \end{aligned}$$

from the decomposition $\nabla \mathbf{n} = \mathbf{S} + \mathbf{W}$ and the fact that $\mathbf{S}\mathbf{W} + \mathbf{W}\mathbf{S}$ is skew. Since, $\operatorname{tr}\mathbf{W}^2 = -\mathbf{W} : \mathbf{W}$ and $\operatorname{tr}\mathbf{S}^2 = \mathbf{S} : \mathbf{S}$, one has that

$$\nabla \mathbf{n} : \nabla \mathbf{n} = \operatorname{tr}(\nabla \mathbf{n})^2 + 2\mathbf{W} : \mathbf{W}$$

The proof of (2.3) is completed with the observation that

$$2\mathbf{W} : \mathbf{W} = 4\mathbf{w} \cdot \mathbf{w} = \operatorname{curl}\mathbf{n} \cdot \operatorname{curl}\mathbf{n} = |\operatorname{curl}\mathbf{n}|^2,$$

and the identity,

$$|\operatorname{curl}\mathbf{n}|^2 = (\mathbf{n} \cdot \operatorname{curl}\mathbf{n})^2 + |\mathbf{n} \wedge \operatorname{curl}\mathbf{n}|^2.$$

(The last equation may be verified quickly in component form). ☐

Theorem 2.1.3 (Frank [11],[28]). *Let* $\mathbf{n} \in C^1(\Omega, \mathbb{S}^2)$ *and* σ_F *be the scalar-valued function of the form*

$$\sigma_F(\mathbf{n}, \nabla \mathbf{n}) := k(\mathbf{n}) + \mathbf{K}(\mathbf{n}) : \nabla \mathbf{n} + \nabla \mathbf{n} : \mathbb{K}(\mathbf{n})[\nabla \mathbf{n}],$$

where $k(\mathbf{n})$, $\mathbf{K}(\mathbf{n})$, *and* $\mathbb{K}(\mathbf{n})$ *are as in Theorem 2.1.1. Then* σ_F *is frame indifferent if, and only if, there are five scalars* $k_1, \ldots, k_4$ *such that*

$$\begin{aligned} \sigma_F(\mathbf{n}, \nabla \mathbf{n}) = {} & k_1(\operatorname{div}\mathbf{n})^2 + k_2(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})^2 \\ & + k_3|\mathbf{n} \wedge \operatorname{curl}\mathbf{n}|^2 + (k_2 + k_4)(\operatorname{tr}(\nabla \mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2), \end{aligned}$$

Proof. It suffices to apply Theorem 2.1.1 to σ_F with $\mathbf{N} = \nabla \mathbf{n}$. For terms involving $P(\mathbf{n}) : \mathbf{N}$, observe, from the definition of $P(\mathbf{n})$, that

$$P(\mathbf{n}) : \nabla \mathbf{n} = I : \nabla \mathbf{n} + (\mathbf{n} \otimes \mathbf{n}) : \nabla \mathbf{n} = \operatorname{tr}(\nabla \mathbf{n}) = \operatorname{div}(\mathbf{n}). \tag{2.7}$$

Next, for terms involving $\mathbf{W}(\mathbf{n}) : \mathbf{N}$, recall for skew matrices $\mathbf{W}_1$ and $\mathbf{W}_2$, that

$$\mathbf{W}_1 : \mathbf{W}_2 = 2(\mathbf{w}_1 \wedge) : (\mathbf{w}_2 \wedge).$$

Then, making use of (2.4) and (2.5) one has that

$$\mathbf{W}(\mathbf{n}) : \nabla\mathbf{n} = \mathbf{W}(\mathbf{n}) : (\mathbf{S}+\mathbf{W}) = \mathbf{W}(\mathbf{n}) : \mathbf{W} = 2\mathbf{n}\cdot\left(\frac{1}{2}\operatorname{curl}\mathbf{n}\right) = \mathbf{n}\cdot\operatorname{curl}\mathbf{n}. \quad (2.8)$$

Lastly, since $(\nabla\mathbf{n})^T\mathbf{n} = 0$ and $(a\otimes b)A = (a\otimes A^T b)$, one has that

$$\begin{aligned} P(\mathbf{n})(\nabla\mathbf{n})\cdot(\nabla\mathbf{n})P(\mathbf{n}) &= (I-\mathbf{n}\otimes\mathbf{n})(\nabla\mathbf{n}) : (\nabla\mathbf{n})P(\mathbf{n}) \\ &= (\nabla\mathbf{n}) : (\nabla\mathbf{n})P(\mathbf{n}) \\ &= |\nabla\mathbf{n}|^2 - \nabla\mathbf{n}\cdot(\mathbf{n}\otimes\mathbf{n})\nabla\mathbf{n} \\ &= |\nabla\mathbf{n}|^2 - |(\nabla\mathbf{n})\mathbf{n}|^2. \end{aligned} \quad (2.9)$$

Where the last equality follows from the fact that $\operatorname{tr}(a\otimes b) = a\cdot b$ and the identity:

$$\begin{aligned} (\nabla\mathbf{n})\mathbf{n}\otimes(\nabla\mathbf{n})\mathbf{n} &= ((\nabla\mathbf{n})\mathbf{n}\otimes\mathbf{n})(\nabla\mathbf{n})^T \\ &= [\mathbf{n}\otimes(\nabla\mathbf{n})\mathbf{n}]^T(\nabla\mathbf{n})^T \\ &= \left[(\mathbf{n}\otimes\mathbf{n})(\nabla\mathbf{n})^T\right]^T(\nabla\mathbf{n})^T \\ &= \nabla\mathbf{n}(\mathbf{n}\otimes\mathbf{n})(\nabla\mathbf{n})^T. \end{aligned}$$

Applying Theorem 2.1.1 with (2.7) - (2.9) yields that

$$\begin{aligned} \sigma_F(\mathbf{n},\nabla\mathbf{n}) = \alpha_0 &+ (\alpha_2+\beta_3\mathbf{n}\cdot\operatorname{curl}\mathbf{n})\operatorname{div}\mathbf{n} + \alpha_3(\mathbf{n}\cdot\operatorname{curl}\mathbf{n}) \\ &+ k_1(\operatorname{div}\mathbf{n})^2 + k_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + k_3|\mathbf{n}\wedge\operatorname{curl}\mathbf{n}|^2 \\ &+ (k_2+k_4)\left[\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2\right] \end{aligned} \quad (2.10)$$

for

$$\beta_1 = -k_4,\ \ \beta_2 = k_1-k_2-k_4,\ \ \beta_4 = k_3,\ \ \beta_5 = k_2+k_4-k_3. \quad (2.11)$$

Now, for the desired evenness of the energy density σ_F one has in (2.10) that

$$\sigma_F(-\mathbf{n},-\nabla\mathbf{n}) - \sigma_F(\mathbf{n},\nabla\mathbf{n}) = -2(\alpha_2+\beta_4\mathbf{n}\cdot\operatorname{curl}\mathbf{n})\operatorname{div}\mathbf{n}$$

and so one must choose $\alpha_2 = \beta_4 = 0$. To obtain the energy density for a nematic one further chooses $\alpha_0 = \alpha_3 = 0$ and considers the admissible class

$$n \in \mathcal{N} = \text{ set of all constant fields } \Omega \to \mathbb{S}^2$$

so that σ_F vanishes on $\mathcal{N}$ whenever α_0 vanishes in (2.10). □

Remark 2.1.4.

- *The constants $k_1,\ldots,k_4$ in* (2.10) *are known as* Frank's Constants. *Some geometric interpretations of the constants will be given in Remark 2.1.5.*

- *The term* $[\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2]$ *in* (2.10) *is a null-Lagrangian, and thus, does not contribute to the free energy $\mathcal{F}$.*

- *Theorem 2.1.3 shows that an energy density that is frame-indifferent, appropriately materially symmetric. and even is of the form* (2.10). *For an appropriate choice of* $k_1, \ldots, k_4$ *it is also the case that* $\sigma_F(\mathbf{n}, \nabla\mathbf{n}) \geq 0$ *(postive definite). This is the conclusion of* Ericksen's inequalities *[28].*

Remark 2.1.5 (Special Orientations)**.** *For special choices of orientation* $\mathbf{n}$ *the energy density* σ_F *in* (2.10) *is proportional to each term of* σ_F *with* $k_1, \ldots, k_4$ *as a constant of proportionality.*

- Splay Field: *Using cylindrical coordinates* (r, θ, z) *with origin* o *and coordinate vectors* $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ [9] *consider the orientation field* $\mathbf{n}_s : \Omega \setminus e_z \to \mathbb{S}^2$

$$\mathbf{n}_s = \mathbf{e}_r$$

for $e_z = \{p \in \Omega : p - o = z\mathbf{e}_z, z \in \mathbb{R}\}$ *where* $p - o = r\mathbf{e}_r + z\mathbf{e}_z$ *for all* $p \in \Omega \setminus e_z$. *It is elementary that*

$$\begin{aligned} \nabla\mathbf{n}_s &= \frac{1}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta \\ \operatorname{div}\mathbf{n}_s &= \frac{1}{r} \\ \operatorname{curl}\mathbf{n}_s &= 0 \\ \operatorname{tr}(\nabla\mathbf{n}_s)^2 &= \frac{1}{r^2}. \end{aligned}$$

Thus, one concludes that

$$\sigma_F(\mathbf{n}_s, \nabla\mathbf{n}_s) = k_1\frac{1}{r^2}.$$

That is, for the orientation field $\mathbf{e}_r$, *the energy density* σ_F *is proportional to single term with proportionality constant* k_1. *One should call such an orientation field* $\mathbf{n}_s$ *the* splay field *since it is the field that is purely spreading or splaying in the plane* $(\mathbf{e}_r, 0, \mathbf{e}_z)$.

- Bend Field: *Using the same cylindrical coordinates, consider the orientation field* $\mathbf{n}_b : \Omega \setminus e_z \to \mathbb{S}^2$ *given by*

$$\mathbf{n}_b = \mathbf{e}_\theta.$$

Again, it is elementary that,

$$\begin{aligned} \nabla\mathbf{n}_b &= -\frac{1}{r}\mathbf{e}_r \otimes \mathbf{e}_\theta \\ \operatorname{div}\mathbf{n}_b &= 0 \\ \operatorname{curl}\mathbf{n}_b &= \frac{1}{r^2}\mathbf{e}_z \\ (\nabla\mathbf{n}_b)\mathbf{n}_b &= -\frac{1}{r}\mathbf{e}_r \\ (\nabla\mathbf{n}_b)^2 &= 0. \end{aligned}$$

[9] $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z) = (\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \mathbf{e}_z = \mathbf{e}_3)$

g Making use of Lemma 2.1.1 one has that

$$\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = k_3 \frac{1}{r^2}.$$

It is natural to call such an orientation field, the bend field *as it measures the bending out of the plane* $(\mathbf{e}_r, 0, \mathbf{e}_z)$.

- Twist Field: *Consider the orientation field*

$$\mathbf{n}_c(x_1, x_2, x_3) = \cos(\tau x_3 + \varphi_0)\mathbf{e}_1 + \sin(\tau x_3 + \phi_0)\mathbf{e}_2$$

for $(o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ *a coordinate frame, and* ϕ_0 *an arbitrary initial angle. By direct calculations, one has that*

$$\begin{aligned} \operatorname{div}\mathbf{n}_c &= 0 \\ \operatorname{curl}\mathbf{n}_c &= -\tau\mathbf{n}_c \\ (\nabla\mathbf{n}_C)^2 &= 0. \end{aligned}$$

Then,

$$\sigma_F(\mathbf{n}_c, \nabla\mathbf{n}_C) = k_2\tau^2$$

- Saddle-Splay Field: *Let each* $p \in \Omega$ *be represented in the coordinate frame*

$$p - o = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

for origin $o \in \mathcal{E}$. *Consider the level set* $\varphi(x_1, x_2, x_3) = 0$ *of saddle surface of* Ω *containing the origin given by*

$$\varphi(x_1, x_2, x_3) = x_3 - x_1x_2.$$

The field $\mathbf{n}_\varphi$ *given by*

$$\mathbf{n}_\varphi = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{1}{\sqrt{x_1^2 + x_2^2 + 1}}(-x_2\mathbf{e}_1 - x_1\mathbf{e}_2 + \mathbf{e}_3).$$

is the unit normal to the surface. In the cylinder $\mathcal{C}_\epsilon = \{p \in \mathcal{E} : x_1^2 + x_2^2 < \epsilon\}$ *one has*

$$\begin{aligned} \mathbf{n}_\varphi &= -x_2\mathbf{e}_1 - x_1\mathbf{e}_2 + \mathbf{e}_3 + o(\epsilon) \\ \nabla\mathbf{n}_\varphi &= -\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 - x_1\mathbf{e}_3 \otimes \mathbf{e}_1 - x_2\mathbf{e}_3 \otimes \mathbf{e}_2 + o(\epsilon) \\ \operatorname{div}\mathbf{n}_\varphi &= o(\epsilon) \\ \operatorname{curl}\mathbf{n}_\varphi \cdot \mathbf{n}_\varphi &= o(\epsilon) \\ (\nabla\mathbf{n}_\varphi)\mathbf{n}_\varphi &= -x_1\mathbf{e}_1 - x_2\mathbf{e}_2 + o(\epsilon) \\ \operatorname{tr}(\nabla\mathbf{n}_\varphi)^2 &= 2 + o(\epsilon^2). \end{aligned}$$

Thus,

$$\sigma_F(\mathbf{n}_\varphi, \nabla\mathbf{n}_\varphi) = 2(k_1 + k_4) + o(\epsilon).$$

2.2 The Navier–Stokes and Ericksen–Leslie equations

Navier–Stokes equations for isotropic fluids

Before examining the Ericksen–Leslie equations for the hydrodynamic flow of nematic liquid crystals it is instructive to give a brief derivation of the Navier–Stokes equations for isotropic fluids–that is fluids in which microstructure is homogeneous. The Navier–Stokes equations encapsulate the following physical laws:

1. mass is conserved,
2. the rate of change of momentum of a fluid parcel equals the force applied to it (Newton's second law),
3. energy is conserved.

Since the thermodynamic properties of liquid crystals and their flows will not be addressed in this manuscript, only the conservation of mass and balance of momentum are derived.

Conservation of mass is equivalent to the statement that

$$\frac{d}{dt}\int_{\Omega} \rho(x,t)dV = -\int_{\partial\Omega} \rho\mathbf{u}\cdot\nu dS$$

time change in mass in Ω = mass flowing across $\partial\Omega$.

By the divergence theorem, the former is equivalent to

$$\int_{\Omega} \left[\partial_t\rho + \nabla\cdot(\rho\mathbf{u})\right] dV = 0,$$

and since this holds for all Ω one has the corresponding differential or point-wise version of the conservation law,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0.$$

To derive the balance of momentum let $\mathbf{x}(t) = (x(t), y(t), z(t))$ be the trajectory of a fluid particle. Then, the velocity field is given by

$$\mathbf{u}(\mathbf{x}(t),t) = \mathbf{u}(x(t),y(t),z(t),t) = (\dot{x}(t),\dot{y}(t),\dot{z}(t)) = \dot{\mathbf{x}}(t)$$

and the acceleration of the fluid particle is given by

$$\ddot{\mathbf{x}}(t) = \dot{\mathbf{u}}(\mathbf{x}(t),t)$$

So, applying the chain rule one has

$$\begin{aligned}\ddot{\mathbf{x}}(t) &= \mathbf{u}_x\dot{x} + \mathbf{u}_y\dot{y} + \mathbf{u}_z\dot{z} + \mathbf{u}_t\dot{t}\\ &= \partial_t\mathbf{u} + (u\cdot\nabla)\mathbf{u}\\ &=: \frac{D\mathbf{u}}{Dt}\end{aligned}$$ [10]

the so called material derivative.

For any continuum, material forces are of two types: *stresses* where the material is acted upon by forces across its surface by the rest of the continuum and external or body forces which exert a force per unit volume on the entire continuum. For the derivation of the Navier–Stokes equation one may make the following physical assumption: for a surface of the continuum S with normal ν the forces on S per unit area are proportional to $\sigma = p(\mathbf{x}, t)\nu + \mathbf{f}(\mathbf{x}, t) \cdot \nu$ where $\mathbf{f}$ is the stress tensor.

An integral form of the balance of momentum is then

$$\frac{d}{dt}\int_{\Omega_t} \rho \mathbf{u} dV = -\int_{\partial\Omega_t} (p \cdot \nu - \mathbf{f} \cdot \nu) dA.$$

Furthermore, if one makes the standard assumptions on σ one arrives at the following form,

$$\begin{aligned}\mathbf{f} &= \lambda(\nabla \cdot \mathbf{u})I + 2\mu\frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2} \\ &= 2\mu\left[\frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2} - \frac{1}{3}(\nabla \cdot \mathbf{u})I\right] + \eta(\nabla \cdot \mathbf{u})I\end{aligned}$$

where μ and η are the first and second coefficients of viscosity. Applying Reynold's transport theorem and the divergence theorem to the integral form for the balance of momentum one has that

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u};$$

or if the fluid is incompressible one has that

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\Delta\mathbf{u}.$$

If $\rho(x, t) = \rho$ is constant, it is common to rewrite the last equation as

$$\frac{D\mathbf{u}}{Dt} = -\nabla\tilde{p} + \nu\Delta\mathbf{u},$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* and $\tilde{p} = p/\rho$. Any of the last three equations may be referred to at the Navier–Stokes equation.

It is common in applications to non-dimensionalize the Navier–Stokes equations. This is done by introducing a length scale L and a velocity scale U (and hence, by dimensional analysis a time scale $T = \frac{L}{U}$). Measuring $\mathbf{u}$, $\mathbf{x}$, and t as fractions of these scales yields the following dimensionless quantities:

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U},\ \tilde{p} = \frac{p}{U^2},\ \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L},\ \text{and } \tilde{t} = \frac{t}{T}.$$

[10] The *material* or *convective* derivative for a general vector field $\mathbf{v}$ is given by $\frac{D\mathbf{v}}{Dt} = \partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v}$ where $\mathbf{u}$ is a velocity gradient.

These, of course, induce variable transformations $\mathbf{u} = \tilde{\mathbf{u}}U$, $p = U^2\tilde{p}$, $\mathbf{x} = \tilde{\mathbf{x}}L$, and $t = \tilde{t}T$ and thus the Navier–Stokes equation in the new variables follows from the following elementary calculations:

$$\begin{aligned}
\partial_t U\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\tilde{t}) &= U\frac{\partial\tilde{\mathbf{u}}}{\partial(\tilde{\mathbf{x}},\tilde{t})}\frac{\partial(\tilde{\mathbf{x}},\tilde{t})}{\partial t} = \frac{U^2}{L}\frac{\partial\tilde{\mathbf{u}}}{\partial\tilde{t}};\\
(U\tilde{\mathbf{u}}\cdot\nabla)U\tilde{\mathbf{u}} &= U^2[(\nabla\tilde{\mathbf{u}})\tilde{\mathbf{u}}] = U^2\left[\left(\frac{\partial\tilde{\mathbf{u}}}{\partial(\tilde{\mathbf{x}},\tilde{t})}\frac{\partial(\tilde{\mathbf{x}},\tilde{t})}{\partial\tilde{\mathbf{x}}}\right)\tilde{\mathbf{u}}\right] = \frac{U^2}{L}(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}};\\
\Delta U\tilde{\mathbf{u}} &= U\frac{\partial}{\partial\mathbf{x}^j}\frac{\partial}{\partial\mathbf{x}^j}\tilde{\mathbf{u}} = \frac{U}{L}\frac{\partial}{\partial\mathbf{x}^j}\frac{\partial}{\partial\tilde{\mathbf{x}}^j}\tilde{\mathbf{u}} = \frac{U}{L^2}\frac{\partial}{\partial\tilde{\mathbf{x}}^j}\frac{\partial}{\partial\tilde{\mathbf{x}}^j}\tilde{\mathbf{u}} = \frac{U}{L^2}\Delta\tilde{\mathbf{u}};\\
\nabla U^2\tilde{p}(\tilde{\mathbf{x}},\tilde{t}) &= \frac{U^2}{L}\nabla\tilde{p}
\end{aligned}$$

where the chain rule is calculated in the sense of Jacobians. Thus, after division by $\frac{U^2}{L}$ the dimensionless quantities satisfy

$$\begin{aligned}
\tilde{\mathbf{u}}_t + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} &= -\nabla\tilde{p} + \frac{1}{Re}\Delta\tilde{\mathbf{u}}\\
\nabla\cdot\tilde{\mathbf{u}} &= 0
\end{aligned} \tag{2.12}$$

where $\frac{1}{Re} = \frac{\nu}{LU}$ defines the dimensionless constant, Re denoting the *Reynolds number*. For the mathematical analysis of the Navier–Stokes equation it is common to choose $U = \lambda$, $L = \frac{1}{\lambda}$. In this case, one has that whenever $(\mathbf{u}, p)$ is a solution to the incompressible Navier–Stokes equation, then so is

$$(\tilde{\mathbf{u}}, \tilde{p}) = (\lambda\mathbf{u}(\lambda\mathbf{x}, \lambda^2 t), \lambda^2 p(\lambda\mathbf{x}, \lambda^2 t)). \tag{2.13}$$

Ericksen–Leslie equations

One is now in the position to develop a continuum theory for the hydrodynamic flow of nematic liquid crystals. Though early dynamic theories exist for liquid crystals in the first widely accepted theory was given by Ericksen [6] based on the static theory (Oseen–Zocher–Frank) discussed in the earlier sections. Leslie [17] completed the dynamic theory by formulating appropriate constitutive relations. As discussed in previous sections, a constitutive relation describes the mechanical properties of the material in question (in this case different compounds exhibiting a nematic phase). The combined work on the hydrodynamic flow of nematic liquid crystals by Ericksen and Leslie is often referred to as the Ericksen–Leslie equations. Since only the isothermal case is considered in this manuscript the derivation is short (see [18], or [27]).

A nematic liquid crystal can be considered as a fluid with microstructure. It is the goal of this section to explain how the microstructure interacts with continuum. The Eulerian description of a fluid with microstructure employs two independent vector fields the usual velocity field $\mathbf{u}(x,t)$ and the axial vector $\hat{\mathbf{w}}(x,t)$. In the case of liquid crystals, $\hat{\mathbf{w}}$ is interpreted as the *local angular velocity*, that is, it represents the local angular velocity of the director $\mathbf{n}$. This differs from an ordinary continuum theory for a fluid in that only a velocity is required since the angular velocity is one-half the curl of the velocity. Denote, the usual angular velocity as $\bar{\mathbf{w}} = \frac{1}{2}\nabla \wedge \mathbf{u}$ and set the *relative angular velocity*

$$\mathbf{w} = \hat{\mathbf{w}} - \bar{\mathbf{w}} = \hat{\mathbf{w}} - \frac{1}{2}\nabla \wedge \mathbf{u}.$$

As in the static case, one considers orientation fields $\mathbf{n}$ with the property $\mathbf{n}\cdot\mathbf{n} = 1$. Consequently, the motion of $\mathbf{n}$ is rigid and one has that

$$\frac{D\mathbf{n}}{Dt} = \hat{\mathbf{w}} \wedge \mathbf{n}.$$

Define the following for the subsequent derivation:

$$\begin{aligned} S^{ij} &= \frac{1}{2}(\nabla_j u^i + \nabla_i u^j) = \text{ rate of strain tensor } = \text{ symmetric part of } \nabla\mathbf{u};^{[1]} \\ W^{ij} &= \frac{1}{2}(\nabla_j u^i - \nabla_i u^j) = \text{ skew part of } \nabla\mathbf{u}; \\ \mathbf{N} &= \mathbf{w} \wedge n = \text{ spin or angular velocity tensor.} \end{aligned}$$

A simple calculation in components shows that

$$\mathbf{N} = \frac{D\mathbf{n}}{Dt} - \mathbf{W}\mathbf{n}. \tag{2.14}$$

It can be shown, see [27] that

$$\mathbf{u}, \nabla\mathbf{u}, \mathbf{W}, \frac{D}{Dt}\mathbf{n} \text{ are not (material) frame indifferent}^{[2]}$$

and

$$\mathbf{n}, \mathbf{N}, \mathbf{S} \text{ are (material) frame indifferent.}$$

This indicates the variables that one should formulate constitutive relations for are $\mathbf{n}$, $\mathbf{N}$, $\mathbf{S}$.

As was done in the case of the Navier–Stokes equations earlier one seeks to find balance laws for the system. For a volume of nematic liquid crystal Ω one has the following conservation and balance laws:

1. *conservation of mass*

$$\frac{D}{Dt}\int_\Omega \rho dV = 0; \tag{2.15}$$

2. *balance of linear momentum*

$$\frac{D}{Dt}\int_\Omega \rho\mathbf{u} dV = \int_\Omega \rho\mathbf{f}_b dV + \int_{\partial\Omega} \mathbf{f}_s dS; \tag{2.16}$$

[1] Generally, the symbol ∇_α indicates the αth component of the gradient vector.

[2] In this context, *frame indifferent* means that under the motion $\mathbf{x}^*(t^*) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}(t)$, $t^* = t - a$ a vector $\mathbf{a}$ and second-order tensor $\mathbf{B}$ must obey $\mathbf{a}^* = \mathbf{Q}\mathbf{a}$ and $\mathbf{B}^* = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$ for $\mathbf{Q} \in \mathrm{SO}(\mathcal{V})$.

3. *balance of angular momentum*

$$\frac{D}{Dt}\int_{\Omega}\rho(\mathbf{x}\wedge\mathbf{u}) = \int_{\Omega}\rho(\mathbf{x}\wedge\mathbf{f}_b+\mathbf{k})dV + \int_{\partial\Omega}(\mathbf{x}\wedge\mathbf{f}_s+\mathbf{l})dS. \tag{2.17}$$

Where,

$$\begin{aligned}
\rho &= \text{ density,}\\
\mathbf{x} &= \text{ position,}\\
\mathbf{u} &= \text{ fluid velocity,}\\
\mathbf{f}_b &= \text{ external body force,}\\
\mathbf{f}_s &= \text{ surface force (stress),}\\
\mathbf{k} &= \text{ external body moment, and}\\
\mathbf{l} &= \text{ surface moment.}
\end{aligned}$$

From the conservation mass (2.15) one has using Reynolds transport theorem and assuming ρ is constant that

$$\nabla\cdot\mathbf{u} = 0,$$

and likewise,

$$S^{ii} = \mathrm{tr}(\nabla\mathbf{u}) = \nabla\cdot\mathbf{u} = 0.$$

For ν the normal vector to the surface S one has

$$f_s^i = F_s^{ij}\nu^j \text{ and } l^i = L^{ij}\nu^j$$

where $\mathbf{F}_s = F_s^{ij}$ is the *stress tensor* and $\mathbf{L} = L^{ij}$ is the *couple stress tensor.* A straightforward application of the divergence theorem and Reynolds transport theorem show that (2.16) may be written

$$\int_{\Omega}\left(\rho\frac{D\mathbf{u}}{Dt} - \rho\mathbf{f}_b - \nabla\cdot\mathbf{F}_s\right)dV = 0,$$

or in point-wise form,

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{f}_b + \nabla\cdot\mathbf{F}_s \text{ or } \rho\frac{Du^i}{Dt} = \rho f_b^i + \nabla_j F_s^{ij}. \tag{2.18}$$

On the other hand, applying Reynolds transport theorem to the left-hand side of (2.17) yields that

$$\begin{aligned}
\frac{D}{Dt}\int_{\Omega}\rho\mathbf{x}\wedge\mathbf{u}dV &= \int_{\Omega}\rho\epsilon^{ijk}\frac{Dx^j u^k}{Dt}dV\\
&= \int_{\Omega}\rho\epsilon^{ijk}x^j\frac{Du^k}{Dt}dV,
\end{aligned}$$

where the last equality follows from the identity,

$$\epsilon^{ijk}\frac{Dx^j}{Dt}u^k = \epsilon^{ijk}u^j u^k = 0,$$

since $u^j u^k = \mathbf{u} \otimes \mathbf{u}$ is symmetric. On the right-hand side of (2.17) one applies the divergence theorem to find

$$\int_{\partial\Omega} \epsilon^{ijk} x^j f_s^k dS = \int_\Omega \nabla_p(\epsilon^{ijk} x^j F_s^{kp}) = \int_\Omega (\epsilon^{ijk} x^j \nabla_p F_s^{kp} + \delta^{jp} F_s^{kp}) dV.$$

Using these two facts, the balance of angular momentum (2.17) may be written as

$$\int_\Omega \epsilon^{ijk} x^j \left(\rho \frac{Du^k}{Dt} - \rho f_b^k - \nabla_p F^{kp} \right) = \int_V (\rho K^i + \epsilon^{ijk} F_s^{kj} + \nabla_j L^{ij}) dV.$$

However, (2.18) implies that the left-hand side of the last equation is zero and so one has that

$$\int_\Omega (\rho K^i + \epsilon^{ijk} F_s^{kj} + \nabla_j L^{ij}) dV = 0, \tag{2.19}$$

or the point-wise equation,

$$\rho K^i + \epsilon^{ijk} F_s^{kj} + \nabla_j L^{ij} = 0. \tag{2.20}$$

The equations giving mass conservation (incompressiblity), balance of linear momentum, and balance of angular momentum can be thought of as kinematic equations. That is, they are equations that describe motion, but they do not explain the origin of the forces generating the motion. Dynamic equations would also include the constitutive relations for the material that describe the origins of forces. One constitutive hypothesis that can be made for the derivation of the Ericksen–Leslie equations is a *rate of work* assumption. Namely,

$$\begin{aligned} &\int_\Omega \rho(\mathbf{f}_b \cdot \mathbf{u} + \mathbf{k} \cdot \hat{\mathbf{w}}) dV + \int_{\partial\Omega} (\mathbf{f}_s \cdot \mathbf{u} + \mathbf{l} \cdot \hat{\mathbf{w}}) dS \\ &= \frac{D}{Dt} \int_\Omega \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \sigma \right) dV + \int_\Omega D dV \end{aligned} \tag{2.21}$$

where D is the rate of viscous dissipation per unit volume and σ_F is the Oseen–Zocher–Frank energy density.

In (2.21) one may apply the divergence theorem to find that

$$\begin{aligned} \int_{\partial\Omega} f_s^i u^i + l^i w^i dS &= \int_{\partial\Omega} F_s^{ij} u^i \nu^j + L^{ij} w^i \nu^j dS \\ &= \int_\Omega \nabla_j (F_s^{ij} u^i) + \nabla_j (L^{ij} w^i) dV \\ &= \int_\Omega F_s^{ij} \nabla_j u^i + L^{ij} \nabla_j w^i + u^i \nabla_j F_s^{ij} + w^i \nabla_j L^{ij} dV. \end{aligned}$$

Then, using the previously derived point-wise forms of the balances of linear and angular momentum, (2.18) and (2.20), one sees that

$$\begin{aligned} &\int_{\partial\Omega} f_s^i u^i + l^i w^i dS \\ &= \int_\Omega F_s^{ij} \nabla_j u^i + L^{ij} \nabla_j w^i + \rho u^i \left(\frac{Du^i}{Dt} - F_b^i \right) - \rho w^i (K^i + \epsilon^{ijk} F_s^{kj}) dV. \end{aligned} \tag{2.22}$$

Returning to (2.21), applying Reynolds transport theorem to the energy terms yields that

$$\frac{D}{Dt}\int_\Omega\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\sigma_F\right)dV=\int_\Omega \rho u^i\frac{Du^i}{Dt}+\frac{D\sigma_F}{Dt}dV.$$

Substituting the relations (2.22) and (2.2) into the rate of work balance (2.21) yields the point-wise relation

$$F_s^{ij}\nabla_j u^j + L^{ij}\nabla_j w^i - w^i\epsilon^{ijk}F_s^{kj} = \frac{D\sigma_F}{Dt}+D. \tag{2.23}$$

What remains is to find specific forms of the stress tensor F_s^{ij} and the couple stress tensor L^{ij}. This is done using Ericksen's identity [6]:

$$\epsilon^{ijk}\left(n^j\frac{\partial\sigma_F}{\partial n^k}+\nabla_p n^j\frac{\partial\sigma_F}{\partial\nabla_p n^k}+\nabla_p n^j\frac{\partial\sigma_F}{\partial\nabla_k n^p}\right)=0. \tag{2.24}$$

To apply Ericksen's identity one first need to calculate the material derivative of the Oseen–Zocher–Frank energy density σ_F. First, recall that

$$\begin{aligned}\frac{Dn^i}{Dt}&=\epsilon^{ijk}w^j n^k \text{ and}\\ \nabla_j\left[\frac{Dn^i}{Dt}\right]&=\partial_t\nabla_j n^i+\nabla_j u^k\nabla_k u^j+v^k\nabla_k\nabla_j n^i=\frac{D\nabla_j u^i}{Dt}+\nabla_j u^k\nabla_k n^i.\end{aligned}$$

Then for $\sigma_F=\sigma_F(\mathbf{n},\nabla\mathbf{n})$ one has by the chain rule, after relabeling, that

$$\begin{aligned}\frac{D\sigma_F}{Dt}&=\frac{\partial\sigma_F}{\partial n^p}\frac{Dn^p}{Dt}+\frac{\partial\sigma_F}{\partial\nabla_k n^p}\frac{D\nabla_k n^p}{Dt}\\ &=\epsilon^{ipq}\left[\left(n_q\frac{\partial\sigma_F}{\partial n^p}+\nabla_k n^q\frac{\partial\sigma_F}{\partial\nabla_k n^p}\right)\hat{w}^i+n^q\frac{\partial\sigma_F}{\partial\nabla_k n^p}\nabla_k\hat{w}^i\right]-\frac{\partial\sigma_F}{\partial\nabla_k n^p}\nabla_q n^p\nabla_k u^q.\end{aligned}$$

In the last expression, using (2.24), one replaces the terms in brackets and finds that

$$\frac{D\sigma_F}{Dt}=\epsilon^{ipq}\left[n^q\frac{\partial\sigma_F}{\partial\nabla_k n^p}\nabla_k\hat{w}^i-\nabla_q n^k\frac{\partial\sigma_F}{\partial\nabla_p n^k}\hat{w}^i\right]-\frac{\partial\sigma_F}{\partial\nabla_k n^p}\nabla_q n^p\nabla_k u^q. \tag{2.25}$$

With (2.25) and (2.23) one may express the dissipation as a function linear in $\hat{w}^i$, $\nabla_j\hat{w}^i$, and $\nabla_j u^i$, namely, one has that

$$\begin{aligned}D=&\left(F_s^{ij}+\frac{\partial\sigma_F}{\partial\nabla_j n^p}\nabla_i n^p\right)\nabla_j v^i\\ &+\left(L^{ij}-\epsilon^{iqp}n^q\frac{\partial\sigma_F}{\partial\nabla_j n^p}\right)\nabla_j\hat{w}^i+\hat{w}^i\epsilon^{ipq}\left(t^{pq}-\frac{\partial\sigma_F}{\partial\nabla_p n^k}\nabla_q n^k\right).\end{aligned} \tag{2.26}$$

Since the signs $\hat{w}^i$, $\nabla_j\hat{w}^i$, and $\nabla_j u^i$ are arbitrary it must be the case that the coefficients in (2.26) are identically zero. This in turn indicates that the stress tensor F_s^{ij} and the couple stress tensor L^{ij} may take the forms

$$\begin{aligned}F_s^{ij}&=-p\delta^{ij}-\frac{\partial\sigma_F}{\partial\nabla_j n^p}\nabla_i n^p+\tilde{F}_s{}^{ij}\\ L^{ij}&=\epsilon^{iqp}n^q\frac{\partial\sigma_F}{\partial\nabla_j n^p}+\tilde{L}^{ij}.\end{aligned} \tag{2.27}$$

Here, p is the pressure arising from incompressibility and $\tilde{F_s}^{ij}$ and $\tilde{L}^{ij}$ are the possible dynamic contributions. Inserting the expressions for F_s^{ij} and L^{ij} given in (2.27) into (2.26) yields that

$$\tilde{F_s}^{ij}\nabla_j u^i + \tilde{L}^{ij}\nabla_j \hat{w}^i + \hat{w}^i \epsilon^{ijk}\tilde{F_s}^{kj} = D. \tag{2.28}$$

To see this reduction, notice in the first term of (2.26) that $p\delta^{ij}\nabla_j v^i = p\nabla_i v^i = 0$ from incompressibility, and in the third term of (2.26) that $\epsilon^{ipq}\frac{\partial\sigma_F}{\partial\nabla_p n^k}\nabla_q n^k = 0$ since $\frac{\partial\sigma_F}{\partial\nabla_p n^k}\nabla_q n^k$ is symmetric in p, q.

To find constitutive relations for the dynamic terms $\tilde{F_s}^{ij}$, $\tilde{L}^{ij}$ it is assumed that the dissipation D is positive. In this case, one has from (2.28) that

$$\tilde{F_s}^{ij}\nabla_j u^i + \tilde{L}^{ij}\nabla_j \hat{w}^i + \hat{w}^i \epsilon^{ijk}\tilde{F_s}^{kj} = D \geq 0.$$

As much of this information will be discarded to get a workable model for analysis, namely (1.7)-(1.9), it is expedient to simply state one of the common expressions for the dynamic terms. The interested reader is invited consult [27] and the references therein for a derivation of the constitutive relations along these lines. From [27], the most widely-adopted and well-known form for the dynamic (or viscous) stress is

$$\begin{aligned}\tilde{F_s}^{ij} &= \alpha_1 n^k S^{kp} n^p n^i n^j + \alpha_2 N^i n^j + \alpha_3 n^i N^j + \alpha_4 S^{ij} \\ &\quad + \alpha_5 n^j S^{ik} n^k + \alpha_6 n^i S^{jk} n^k.\end{aligned} \tag{2.29}$$

Where, $\alpha_1, \ldots, \alpha_6$ are the Leslie viscosities. It is also easy to see that if $\tilde{L}^{ij}$ is assumed to not depend on $\nabla_j \hat{w}^i$, then it must be that

$$\tilde{L}^{ij} = 0 \tag{2.30}$$

since $\nabla_j \hat{w}^i$ does not necessarily have a single sign.

To arrive at model that retains the mathematically difficulties present in the Ericksen–Leslie equations, but is sufficiently concise for analysis, one may make the following simplifying assumptions. These assumptions are modeled after those made by Lin and Liu in [20], but there are critical differences. Namely, Lin and Liu use a Ginzburg–Landau type penalized energy for σ_F to enforce the sphere constraint $\mathbf{n}\cdot\mathbf{n} = 1$. For simplicity assume that the body force $\mathbf{f}_b$ and body moment $\mathbf{k}$ are zero, and that Frank elastic constants obey

$$k_1 = k_2 = k_3 = K \text{ and } k_4 = 0 \tag{2.31}$$

and the Leslie viscosities in (2.29) obey

$$\begin{aligned}&\alpha_1 = \alpha_5 = \alpha_6 = 0\\ &\alpha_3 - \alpha_2 \geq 0\\ &\alpha_3 + \alpha_2 = 0\end{aligned} \tag{2.32}$$

With the assumption (2.31) one may simplify the Oseen–Zocher–Frank energy density to

$$\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = \frac{K}{2}|\nabla\mathbf{n}|^2.$$

To derive balance of angular momentum that will be studied in the remainder of this manuscript one will need the following calculations for the energy density σ_F, the stress tensor F_s^{ij}, and the couple stress tensor L^{ij}:

$$\begin{aligned}
\frac{\partial \sigma_F}{\partial \nabla \mathbf{n}} &= \frac{\partial \sigma_F}{\partial \nabla_j n^q}\left(\frac{\nabla_j n^q \nabla_j n^q}{2}\right) = K\nabla_j u^q = K\nabla \mathbf{n};\\
\frac{\partial \sigma_F}{\partial \mathbf{n}} &= \frac{\partial \sigma_F}{\partial n^q}\left(\frac{\nabla_j n^q \nabla_j n^q}{2}\right) = \nabla_j \nabla_q n^q \nabla_j n^q = \mathbf{0};\\
\epsilon^{ijk} {F_s}^{kj} &= \epsilon^{ijk}\left[-p\delta^{ij} - \frac{\partial \sigma_F}{\partial \nabla_j n^p}\nabla_k n^p + \tilde{F_s}^{kj}\right]\\
&= \epsilon^{ijk}\left[-p\delta^{ij} - K\nabla_j n^p \nabla_k n^p + \tilde{F_s}^{kj}\right]\\
&= \epsilon^{ijk}\tilde{F_s}^{kj}\\
&= \epsilon^{ijk}(\alpha_2 N^k n^j + \alpha_3 n^k N^j + \alpha_4 S^{kj})\\
&= \epsilon^{ijk}(\alpha_2 - \alpha_3) n^k N^j;\\
\nabla_j L^{ij} &= \nabla_j \epsilon^{ipq}\left(n^q \frac{\partial \sigma_F}{\partial \nabla_j n^p}\right)\\
&= K\nabla_j \epsilon^{ipq}(n^q \nabla_j n^p)\\
&= K\epsilon^{ipq}(\nabla_j n^q \nabla_j n^p + n^q \nabla_j \nabla_j n^p)\\
&= \epsilon^{ipq}(n^q K\Delta n^p).
\end{aligned} \tag{2.33}$$

Using (2.33), the point-wise balance of angular momentum (2.20) with stress and couple stress given by (2.27) can be written

$$\begin{aligned}
0 &= -(\epsilon^{ijk}{F_s}^{kj} + \nabla_j L^{ij})\\
&= \epsilon^{ijk} n^k((\alpha_3 - \alpha_2)N^j - K\Delta u^j)\\
&= \epsilon^{ijk} n^k\left[(\alpha_3 - \alpha_2)\left(\frac{Dn^j}{Dt} - W^{jl}n^l\right) - K\Delta u^j\right].
\end{aligned}$$

After further simplification, one may write the last equation in the form that will be studied in the remainder of the manuscript (see Section 1.3), namely,

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} - \Delta\mathbf{d} = |\nabla\mathbf{d}|^2\mathbf{d}.$$

Similarly, to derive balance of linear momentum that results from the constitutive assumptions (2.31) and (2.32) one needs only calculate that

$$\begin{aligned}
\nabla\cdot\mathbf{F}_s &= \nabla_j {F_s}^{ij}\\
&= -\nabla_i p - \nabla_j(\nabla_i n^q \nabla_j n^q) + \nabla_j \tilde{F_s}^{ij};\\
&= -\nabla_i p - \nabla_j(\nabla_i n^q \nabla_j n^q) + \nabla_j\left[\alpha_2 N^i n^j + \alpha_3 n^i N^j + \alpha_4 S_{ij}\right];\\
&= -\nabla_i p - \nabla_j(\nabla_i n^q \nabla_j n^q) + \alpha_4\Delta u^j + \nabla_j\left[\alpha_2 N^i n^j + \alpha_3 n^i N^j\right];
\end{aligned}$$

and form,

$$\rho\frac{Du^i}{Dt} = -\nabla_i p - \nabla_j(\nabla_i n^q \nabla_j n^q) + \nabla_j\tilde{F_s}^{ij}.$$

Again, making further simplifying assumptions, one arrives at the form of the conservation of linear momentum that will be examined (see Section 1.3), namely,

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} = -\nabla p - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}).$$

Chapter 3 Energy inequalities

In this chapter inequalities that are similar to the standard energy inequalities common to the mathematical analysis of evolution problems are demonstrated for the system (1.7)-(1.9).

Since the following estimates involve gradients (tensors), component notations must be employed. The convention for this manuscript is to use superscripts with Latin characters for components. Similarly, partial derivatives are either denoted with subscripts or as components of the gradient via the notation ∇_α. Since constants are unimportant to the final result C represents and absolute constant, but it may vary from line-to-line of an estimate.

3.1 Global energy inequality

A global energy-type inequality for the system (1.7)-(1.9) is proven using standard PDE and harmonic analysis tools. Namely, the following proof uses the Sobolev embedding theorem and the Riesz transform. Note that to achieve estimates on on the L^3-norm of $\mathbf{u}$ and $\nabla \mathbf{d}$ one must choose appropriate test functions to multiply against the equations (1.7) and (1.9). Unlike the strategy usually employed to arrive at standard L^2 energy inequalities, it is not sufficient to just use a solution or gradient of a solution as a test function.

Theorem 3.1.1 (Global L^3-energy inequality). *If* $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3(\mathbb{R}^3)$*, then any smooth solution to* (1.7)-(1.9) *obeys:*

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_{\mathbb{R}^3} |\mathbf{u}|^3 + |\nabla \mathbf{d}|^3\right\} + \left[1 - C\|\mathbf{u}\|^2_{L^3(\mathbb{R}^3)}\right]\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2|\mathbf{u}| \\
&+ \left[1 - C(\|\mathbf{u}\|_{L^3(\mathbb{R}^3)} + \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)} + \|\nabla \mathbf{d}\|^2_{L^3(\mathbb{R}^3)})\right] \\
&\times \int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \leq 0.
\end{aligned} \tag{3.1}$$

Proof. Let d be a smooth solution of (1.9), then taking the gradient and integrating against $|\nabla \mathbf{d}|\nabla \mathbf{d}$ yields that

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla \mathbf{d}_t : |\nabla \mathbf{d}|\nabla \mathbf{d} = &\underbrace{\int_{\mathbb{R}^3} \nabla(\Delta \mathbf{d}) : |\nabla \mathbf{d}|\nabla \mathbf{d}}_{(3.2)_a} - \underbrace{\int_{\mathbb{R}^3} \nabla(\mathbf{u}\cdot\nabla \mathbf{d}) : |\nabla \mathbf{d}|\nabla \mathbf{d}}_{(3.2)_b} \\
&- \underbrace{\int_{\mathbb{R}^3} \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}) : |\nabla \mathbf{d}|\nabla \mathbf{d}}_{(3.2)_c}
\end{aligned} \tag{3.2}$$

On left-hand side of (3.2)

$$\int_{\mathbb{R}^3} \nabla \mathbf{d}_t : |\nabla \mathbf{d}| \nabla \mathbf{d} = \int_{\mathbb{R}^3} \partial_t \left\{ \frac{|\nabla \mathbf{d}|^2}{2} \right\} |\nabla \mathbf{d}| = \int_{\mathbb{R}^3} \frac{1}{3} \partial_t \left\{ |\nabla \mathbf{d}|^3 \right\}$$

since

$$\partial_t |\nabla \mathbf{d}|^3 = \partial_t (|\nabla \mathbf{d}|^2)^{3/2} = 3(|\nabla \mathbf{d}|^2)^{1/2} \partial_t \frac{|\nabla \mathbf{d}|^2}{2} = 3|\nabla \mathbf{d}| \partial_t \frac{|\nabla \mathbf{d}|^2}{2}. \tag{3.3}$$

Next, from $(3.2)_a$ one has that

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla(\Delta \mathbf{d}) : |\nabla \mathbf{d}| \nabla \mathbf{d} &= \int_{\mathbb{R}^3} \Delta d_\alpha \cdot |\nabla \mathbf{d}| \mathbf{d}_\alpha \\
&= -\int_{\mathbb{R}^3} \mathbf{d}_{\alpha\beta} \cdot (|\nabla \mathbf{d}| \mathbf{d}_\alpha)_\beta \\
&= -\int_{\mathbb{R}^3} \mathbf{d}_{\alpha\beta} \cdot (|\nabla \mathbf{d}| \mathbf{d}_{\alpha\beta} + |\nabla \mathbf{d}|_\beta \mathbf{d}_\alpha) \\
&= -\int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| + \int_{\mathbb{R}^3} \mathbf{d}_{\alpha\beta} \mathbf{d}_\alpha \cdot |\nabla \mathbf{d}|_\beta \\
&= -\int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| - \int_{\mathbb{R}^3} (\nabla^2 \mathbf{d} : \nabla \mathbf{d}) \cdot \left(\frac{\nabla^2 \mathbf{d} : \nabla \mathbf{d}}{|\nabla \mathbf{d}|} \right) \\
&= -\int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| - \int_{\mathbb{R}^3} \frac{|\nabla^2 \mathbf{d} : \nabla \mathbf{d}|^2}{|\nabla \mathbf{d}|}
\end{aligned} \tag{3.4}$$

where

$$\nabla |\nabla \mathbf{d}| = \nabla (|\nabla \mathbf{d}|^2)^{1/2} = \frac{1}{2(|\nabla \mathbf{d}|^2)^{1/2}} \nabla |\nabla \mathbf{d}|^2 = \frac{\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}}{|\nabla \mathbf{d}|}. \tag{3.5}$$

Integrating by parts in $(3.2)_b$ yields that

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla(\mathbf{u} \cdot \nabla \mathbf{d}) : |\nabla \mathbf{d}| \nabla \mathbf{d} &= -\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \nabla \cdot (|\nabla \mathbf{d}| \nabla \mathbf{d}) \\
&= -\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot ((\nabla |\nabla \mathbf{d}|) \nabla \mathbf{d} + \Delta \mathbf{d} |\nabla \mathbf{d}|)
\end{aligned} \tag{3.6}$$

Lastly, the term $(3.2)_c$ is

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}) : |\nabla \mathbf{d}| \nabla \mathbf{d} &= \int_{\mathbb{R}^3} (\nabla |\nabla \mathbf{d}|^2) \mathbf{d} : |\nabla \mathbf{d}| \nabla \mathbf{d} + \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^2 \nabla \mathbf{d} : |\nabla \mathbf{d}|^2 \nabla \mathbf{d} \\
&= \int_{\mathbb{R}^3} (\nabla_\alpha |\nabla \mathbf{d}|^2) |\nabla \mathbf{d}| \nabla_\alpha \mathbf{d} \cdot \mathbf{d} + \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^5 \\
&= \int_{\mathbb{R}^3} (\nabla_\alpha |\nabla \mathbf{d}|^2) |\nabla \mathbf{d}| \nabla_\alpha \left\{ \frac{|\mathbf{d}|^2}{2} \right\} + \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^5 \\
&= \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^5.
\end{aligned} \tag{3.7}$$

where the last holds since $|\mathbf{d}| = 1$. Collecting what has been shown in (3.4), (3.6) and (3.7) one has the following estimate from (3.2)

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 + \int_{\mathbb{R}^3} \frac{|\nabla^2 \mathbf{d}\cdot\nabla\mathbf{d}|}{|\nabla \mathbf{d}|} + |\nabla^2\mathbf{d}|^2|\nabla\mathbf{d}| \lesssim \int_{\mathbb{R}^3} |\nabla\mathbf{d}|^5 + |\mathbf{u}||\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|. \quad (3.8)$$

Next, observing that

$$\nabla|\nabla\mathbf{d}|^{3/2} = \frac{3}{2}|\nabla\mathbf{d}|^{1/2}\nabla|\nabla\mathbf{d}|$$

and observing from the Cauchy-Schwarz inequality that

$$|\nabla|\nabla\mathbf{d}|| = |\nabla\mathbf{d}|^{-1}|\nabla^2\mathbf{d} : \nabla\mathbf{d}| \leq |\nabla^2\mathbf{d}|$$

one has that

$$\int_{\mathbb{R}^3} |\nabla|\nabla\mathbf{d}|^{3/2}|^2 \lesssim^{[1]} \int_{\mathbb{R}^3} ||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}||^2 = \int_{\mathbb{R}^3} |\nabla\mathbf{d}||\nabla^2\mathbf{d}|^2. \quad (3.9)$$

The Sobolev embedding theorem and (3.9) imply

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^9 &= \int_{\mathbb{R}^3} (|\nabla\mathbf{d}|^{3/2})^6 \\ &\lesssim \left\{\int_{\mathbb{R}^3} |\nabla|\nabla\mathbf{d}|^{3/2}|^2\right\}^3 \\ &\lesssim \left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}||\nabla^2\mathbf{d}|^2\right\}^3.\end{aligned} \quad (3.10)$$

By Hölder's inequality and (3.10)

$$\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^5 \leq \left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^9\right\}^{1/3}\left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^3\right\}^{2/3} \lesssim \left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}||\nabla^2\mathbf{d}|^2\right\}\left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^3\right\}^{2/3}.$$

Substituting this in (3.8) with $\mathbf{u} = 0$ yields

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla\mathbf{d}|^3 + \left[1 - C\|\nabla\mathbf{d}\|^2_{L^3(\mathbb{R}^3)}\right]\int_{\mathbb{R}^3} |\nabla\mathbf{d}||\nabla^2\mathbf{d}|^2 \leq 0 \quad (3.11)$$

In the case that $\mathbf{u} \neq 0$ Hölder's inequality, the Sobolev embedding $W^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and (3.9) yield that

$$\begin{aligned}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}| &\leq \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{3/2}\|_{L^6(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla|\nabla\mathbf{d}|^{3/2}\|_{L^2(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\left\{\int_{\mathbb{R}^3} |\nabla\mathbf{d}||\nabla^2\mathbf{d}|^2\right\}^{1/2}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|_{L^2(\mathbb{R}^3)} \\ &= \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|_{L^2(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|_{L^2(\mathbb{R}^3)} \\ &= \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\||\nabla\mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

[1] The relation $A \lesssim B$ indicates that $A \leq CB$ for some constant C.

Inserting this estimate into (3.8) and using Hölder's inequality one has that

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla \mathbf{d}|^3+\left[1-C\left(\|\nabla \mathbf{d}\|^2_{L^3(\mathbb{R}^3)}+\|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\right)\right]\||\nabla \mathbf{d}|^{1/2}|\nabla^2\mathbf{d}|\|^2_{L^2(\mathbb{R}^3)}\leq 0. \quad (3.12)$$

Multiplying (1.7) by $|\mathbf{u}|\mathbf{u}$ and integrate over $\mathbb{R}^3$, one obtains

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbf{u}_t\cdot|\mathbf{u}|\mathbf{u}\\ &=\underbrace{\int_{\mathbb{R}^3}\Delta\mathbf{u}\cdot|\mathbf{u}|\mathbf{u}}_{(3.13)_a}-\underbrace{\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot|\mathbf{u}|\mathbf{u}}_{(3.13)_b}-\underbrace{\int_{\mathbb{R}^3}\nabla p\cdot|\mathbf{u}|\mathbf{u}}_{(3.13)_c}-\underbrace{\int_{\mathbb{R}^3}(\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}))\cdot|\mathbf{u}|\mathbf{u}}_{(3.13)_d}\end{aligned} \quad (3.13)$$

It is easy to see the left-hand side of (3.13) amounts to

$$\int_{\mathbb{R}^3}\mathbf{u}_t\cdot|\mathbf{u}|\mathbf{u}=\frac{1}{3}\int_{\mathbb{R}^3}\partial_t|\mathbf{u}|^3=\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{u}|^3.$$

For $(3.13)_a$ one has that

$$\begin{aligned}\int_{\mathbb{R}^3}(\Delta\mathbf{u})\cdot|\mathbf{u}|\mathbf{u}&=\int_{\mathbb{R}^3}\mathbf{u}_{\alpha\alpha}\cdot|\mathbf{u}|\mathbf{u}\\ &=-\int_{\mathbb{R}^3}\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha|\mathbf{u}|+\mathbf{u}_\alpha\cdot\mathbf{u}|\mathbf{u}|_\alpha\\ &=-\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2|\mathbf{u}|+\mathbf{u}_\alpha\cdot\mathbf{u}|\mathbf{u}|_\alpha\\ &=-\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2|\mathbf{u}|+\nabla_\alpha\left\{\frac{|\mathbf{u}|^2}{2}\right\}\nabla_\alpha|\mathbf{u}|\\ &=-\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2|\mathbf{u}|+|\mathbf{u}|\nabla|\mathbf{u}|\cdot\nabla|\mathbf{u}|\\ &=-\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2|\mathbf{u}|+|\mathbf{u}||\nabla|\mathbf{u}||^2.\end{aligned}$$

The term $(3.13)_b$ vanishes by (1.8), that is,

$$\begin{aligned}\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot|\mathbf{u}|\mathbf{u}&=\int_{\mathbb{R}^3}(u^\alpha u^i_\alpha)|\mathbf{u}|\mathbf{u}^i\\ &=\int_{\mathbb{R}^3}u^\alpha\nabla_\alpha\left\{\frac{|\mathbf{u}|^2}{2}\right\}|\mathbf{u}|\\ &=\int_{\mathbb{R}^3}u^\alpha\nabla_\alpha\left\{\frac{|\mathbf{u}|^3}{3}\right\}\\ &=0.\end{aligned}$$

For $(3.13)_c$ and $(3.13)_d$ one has that

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla p\cdot|\mathbf{u}|\mathbf{u}&=-\int_{\mathbb{R}^3}p(\nabla|\mathbf{u}|)\cdot\mathbf{u}+p|\mathbf{u}|\nabla\cdot\mathbf{u}\\ &=-\int_{\mathbb{R}^3}p(\nabla|\mathbf{u}|)\cdot\mathbf{u}\end{aligned}$$

and

$$
\begin{aligned}
-\int_{\mathbb{R}^3} (\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})) \cdot |\mathbf{u}|\mathbf{u} &= \int_{\mathbb{R}^3} (\nabla_j \cdot (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}))|\mathbf{u}|\mathbf{u}^i \\
&= \int_{\mathbb{R}^3} (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}) \nabla_j (|\mathbf{u}|u^i) \\
&= \int_{\mathbb{R}^3} (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}) \nabla_j |\mathbf{u}| u^i + (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d})|\mathbf{u}|\nabla_j u^i \\
&= \int_{\mathbb{R}^3} (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}) \frac{u^k \nabla_j u^k}{|\mathbf{u}|} u^i + (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d})|\mathbf{u}|\nabla_j u^i \\
&\leq \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^2 \frac{|\mathbf{u}||\nabla \mathbf{u}|}{|\mathbf{u}|}|\mathbf{u}| + |\nabla \mathbf{d}|^2 |\mathbf{u}||\nabla \mathbf{u}| \\
&= 2\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^2 |\mathbf{u}||\nabla \mathbf{u}|.
\end{aligned}
$$

Substituting the results for $(3.13)_a - (3.13)_d$ into (3.13) one arrives at the inequality

$$
\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^3 + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2|\mathbf{u}| + |\mathbf{u}||\nabla|\mathbf{u}||^2 \\
&\leq \int_{\mathbb{R}^3} |p||\nabla|\mathbf{u}|||\mathbf{u}| + 2\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^2|\mathbf{u}||\nabla \mathbf{u}|.
\end{aligned}
\tag{3.14}
$$

Next, using Kato's inequality $|\nabla|\mathbf{u}|| \leq |\nabla \mathbf{u}|$, Cauchy's inequality and Hölder's inequality in (3.14) may be written as

$$
\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^3 + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2|\mathbf{u}| \\
&\leq \int_{\mathbb{R}^3} (|p| + 2|\nabla \mathbf{d}|^2)|\mathbf{u}||\nabla \mathbf{u}| \\
&\leq \int_{\mathbb{R}^3} \frac{(|p| + 2|\nabla \mathbf{d}|^2)^2|\mathbf{u}|}{2} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2 \\
&\leq C(\|p\|^2_{L^3(\mathbb{R}^3)} + \||\nabla \mathbf{d}|^2\|^2_{L^3(\mathbb{R}^3)})\|\mathbf{u}\|_{L^3(\mathbb{R}^3)} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2.
\end{aligned}
$$

Therefore one obtains

$$
\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^3 + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2|\mathbf{u}| \lesssim (\|p\|^2_{L^3(\mathbb{R}^3)} + \||\nabla \mathbf{d}|^2\|^2_{L^3(\mathbb{R}^3)})\|\mathbf{u}\|_{L^3(\mathbb{R}^3)}. \tag{3.15}
$$

An estimate of the quantity $\|p\|_{L^3(\mathbb{R}^3)}$ is needed. Taking the divergence of (1.7) yields

$$
-\Delta p = \nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d})) \tag{3.16}
$$

Set

$$
g^{jk} := (\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d}).
$$

Then,

$$
p = -R_j R_k (g^{jk}) \tag{3.17}
$$

where R_j is the j^{th} Riesz transform. Thus, since $R_j : L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ is a bounded for $1 < q < \infty$ one has

$$\|p\|_{L^3(\mathbb{R}^3)} = \|R_j R_k(g^{jk})\|_{L^3(\mathbb{R}^3)} \lesssim \|g^{jk}\|_{L^3(\mathbb{R}^3)} \leq \||\mathbf{u}|^2\|_{L^3(\mathbb{R}^3)} + \||\nabla \mathbf{d}|^2\|_{L^3(\mathbb{R}^3)}. \quad (3.18)$$

Inserting (3.18) into (3.15) yields:

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^3 + \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2|\mathbf{u}| \lesssim (\||\mathbf{u}|^2\|^2_{L^3(\mathbb{R}^3)} + \||\nabla \mathbf{d}|^2\|^2_{L^3(\mathbb{R}^3)})\|\mathbf{u}\|_{L^3(\mathbb{R}^3)}. \quad (3.19)$$

Using Hölder's inequality, the Sobolev embedding $W^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and the point wise inequality $|\nabla|\mathbf{u}|^{3/2}| \lesssim |\nabla \mathbf{u}||\mathbf{u}|^{1/2}$ one has that

$$\begin{aligned}\||\mathbf{u}|^2\|^2_{L^3(\mathbb{R}^3)} &\leq \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\mathbf{u}\|^3_{L^9(\mathbb{R}^3)} \\ &\lesssim \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla|\mathbf{u}|^{3/2}\|^2_{L^2(\mathbb{R}^3)} \\ &= \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla|\mathbf{u}|^{3/2}|^2 \\ &\leq \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla|\mathbf{u}||^2|u|.\end{aligned} \quad (3.20)$$

Similarly one has that

$$\begin{aligned}\||\nabla \mathbf{d}|^2\|^2_{L^3(\mathbb{R}^3)} &\leq \|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)}\|\nabla \mathbf{d}\|^3_{L^9(\mathbb{R}^3)} \\ &\lesssim \|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)}\|\nabla|\nabla \mathbf{d}|^{3/2}\|^2_{L^2(\mathbb{R}^3)} \\ &\lesssim \|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^{1/2}.\end{aligned} \quad (3.21)$$

Substituting (3.20) and (3.21), into (3.19) one has that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{u}|^3 + \int_{\mathbb{R}^3}|\nabla \mathbf{u}|^2|\mathbf{u}| \\ &\lesssim (\||\mathbf{u}|^2\|^2_{L^3(\mathbb{R}^3)} + \||\nabla \mathbf{d}|^2\|^2_{L^3(\mathbb{R}^3)})\|\mathbf{u}\|_{L^3(\mathbb{R}^3)} \\ &\lesssim \|\mathbf{u}\|^2_{L^3(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla \mathbf{u}|^2|\mathbf{u}| + \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)}\int_{\mathbb{R}^3}|\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2.\end{aligned} \quad (3.22)$$

Combining (3.12) and (3.22) yields the general inequality:

$$\begin{aligned}&\frac{d}{dt}\left\{\int_{\mathbb{R}^3}|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3\right\} \\ &+ \left[1 - C\|\mathbf{u}\|^2_{L^3(\mathbb{R}^3)}\right]\int_{\mathbb{R}^3}|\nabla \mathbf{u}|^2|\mathbf{u}| \\ &+ \left[1 - C(\|\mathbf{u}\|_{L^3(\mathbb{R}^3)} + \|\mathbf{u}\|_{L^3(\mathbb{R}^3)}\|\nabla \mathbf{d}\|_{L^3(\mathbb{R}^3)} + \|\nabla \mathbf{d}\|^2_{L^3(\mathbb{R}^3)})\right]\int_{\mathbb{R}^3}|\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \\ &\leq 0.\end{aligned} \quad (3.23)$$

□

Lemma 3.1.2 (Case with $\mathbf{u} = 0$)**.** *There exists $\epsilon_0 > 0$ such that if*

$$\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3(0) \leq \epsilon_0^3, \tag{3.24}$$

then $t \mapsto \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3(t)$ is non-increasing. In particular, we have that,

$$\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3(t) \leq \epsilon_0^3 \text{ for all } t \in [0, T). \tag{3.25}$$

Proof. Set

$$\widetilde{E}(t) := \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3(t)$$

By continuity, there exists $t_0 \in (0, T)$ such that,

$$\widetilde{E}(t) \leq 2\widetilde{E}(0) = 2\epsilon_0^3 \tag{3.26}$$

for $t \in [0, t_0)$. Assume that $t_0 \in (0, T)$ is the maximal time such that (3.26) holds. Inserting (3.26) in (3.11) yields the inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 + \left[1 - C\epsilon_0^2\right] \int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \leq 0, \text{ for } t \in [0, t_0].$$

If $\epsilon_0 > 0$ is chosen such that

$$1 - C\epsilon_0^2 \geq 0,$$

then

$$\frac{d}{dt}\widetilde{E}(t) = \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 \leq 0 \text{ for } t \in [0, t_0].$$

Hence, $\widetilde{E}$ is non-increasing in $[0, t_0]$. This implies that

$$\widetilde{E}(t_0) \leq \widetilde{E}(0) \leq \epsilon_0^3 < 2\epsilon_0^3$$

and so t_0 is not the maximal time such that (3.26) holds. Thus, $t_0 = T$. □

Lemma 3.1.3 (Case with $\mathbf{u} \neq 0$)**.** *There exists $\epsilon_0 > 0$ such that if*

$$\int_{\mathbb{R}^3} |\mathbf{u}|^3(0) + |\nabla \mathbf{d}|^3(0) \leq \epsilon_0^3 \tag{3.27}$$

then $t \mapsto \int_{\mathbb{R}^3} |\mathbf{u}|^3(t) + |\nabla \mathbf{d}|^3(t)$ is non-increasing.

Proof. Set:

$$E(t) := \int_{\mathbb{R}^3} |\mathbf{u}|^3(t) + |\nabla \mathbf{d}|^3(t)$$

By continuity, there exists $t_0 \in (0, T)$ such that

$$E(t) \leq 2E(0) = 2\epsilon_0^3 \tag{3.28}$$

for $t \in [0, t_0)$. Then it is also true that

$$\|\nabla \mathbf{d}\|_{L^3}(t) \lesssim \epsilon_0 \text{ and } \|\mathbf{u}\|_{L^3}(t) \lesssim \epsilon_0. \tag{3.29}$$

for $t \in [0, t_0)$. Assume that $t_0 \in (0, T)$ is the maximal time such that (3.26) holds. Inserting (3.29) in (3.11) yields the inequality:

$$\frac{d}{dt}E(t) + \left[1 - C\epsilon_0^2\right] \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 |u| + \left[1 - C(\epsilon_0 + 2\epsilon_0^2)\right] \int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \leq 0.$$

If $\epsilon_0 > 0$ is chosen such that

$$1 - C\epsilon_0^2 \geq 0 \text{ and } 1 - C(\epsilon_0 + 2\epsilon_0^2) \geq 0,$$

then

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 + |\mathbf{u}|^2 \leq 0 \text{ for } t \in [0, t_0].$$

Hence, E is non-increasing in $[0, t_0]$. This implies that

$$E(t_0) \leq E(0) \leq \epsilon_0^3 < 2\epsilon_0^3$$

and so t_0 is not the maximal time such that (3.28) holds. Thus, $t_0 = T$. □

Using the monotonicity proven in Lemma 3.1.3 one has immediately

Theorem 3.1.4. *There exists an* $\epsilon_0 > 0$ *such that if* $(\mathbf{u}_0, \mathbf{d}_0) \in L^3(\mathbb{R}^3, \mathbb{R}^3) \times W^{1,3}(\mathbb{R}^3, S^2)$ *satisfies*

$$\int_{\mathbb{R}^3} |\mathbf{u}_0|^3 + |\nabla \mathbf{d}_0|^3 \leq \epsilon_0^3 \tag{3.30}$$

then there exists a unique global solution $(\mathbf{u}, \mathbf{d})$ *in* $C([0, \infty), L^3(\mathbb{R}^3) \times W^{1,3}(\mathbb{R}^3))$ *to the system of equations given by* (1.9) *and* (1.7)*. Moreover,* ϵ_3 *given by,*

$$\epsilon_3(t) := \int_{\mathbb{R}^3} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)(t) \tag{3.31}$$

is a non-increasing function of t*.*

3.2 Local L^3 energy inequality

In this section, a local L^3 energy inequality is proven. That is, a local L^3 norm inequality for smooth solutions of (1.9)-(1.7) with data in $L^3_{\mathbb{U}}(\mathbb{R}^3)$. One should note similarities between this inequality and the local L^2 inequality of Leray [16] or the local L^2 inequality of Lin, Lin, and Wang [19].

As with the global L^3 energy inequality, one must select test functions appropriately to gain local L^3 norm control.

Theorem 3.2.1 (Local L^3 energy inequality). *For* $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3_{\mathbb{U}}(\mathbb{R}^3)$*, let* $\phi \in C_0^\infty(\mathbb{R}^3)$*, then any smooth solution to* (1.7)-(1.9) *obeys*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(|\mathbf{u}|^3+|\nabla\mathbf{d}|^3)\phi^2+\int_{\mathbb{R}^3}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2+|\nabla(|\nabla\mathbf{d}|^{3/2}\phi)|^2]\\
&\le C\int_{\mathbb{R}^3}(|\mathbf{u}|^3+|\nabla\mathbf{d}|^3)|\nabla\phi|^2+C\int_{\mathbb{R}^3}|\mathbf{u}||P-c|^2\phi^2\\
&+C\left[\left\{\int_{supp\ \phi}|\mathbf{u}|^3\right\}^{2/3}+\left\{\int_{supp\ \phi}|\nabla\mathbf{d}|^3\right\}^{2/3}\right]\\
&\times\int_{\mathbb{R}^3}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2+|\nabla(|\nabla\mathbf{d}|^{3/2}\phi)|^2].
\end{aligned}\tag{3.32}$$

Here, supp ϕ *is support of the function* ϕ.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^3)$. To find norm estimates for $\nabla\mathbf{d}$ the equation (1.9) is differentiated and integrated against $|\nabla\mathbf{d}|(\nabla_\alpha\mathbf{d})\phi^2$ over $\mathbb{R}^3$ to obtain that

$$\begin{aligned}
\int_{\mathbb{R}^3}\nabla_\alpha\mathbf{d}_t\cdot|\nabla\mathbf{d}|(\nabla_\alpha\mathbf{d})\phi^2&=\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\mathbf{d}|^3\phi^2,\\
-\int_{\mathbb{R}^3}(\nabla_\alpha\Delta\mathbf{d})\cdot|\nabla\mathbf{d}|(\nabla_\alpha\mathbf{d})\phi^2&=-\int_{\mathbb{R}^3}[\nabla_\alpha(\nabla_\beta\nabla_\beta\mathbf{d})]\cdot|\nabla\mathbf{d}|(\nabla_\alpha\mathbf{d})\phi^2\\
&=\int_{\mathbb{R}^3}(\nabla_\alpha\nabla_\beta\mathbf{d})\nabla_\beta(|\nabla\mathbf{d}|\nabla_\alpha\mathbf{d}\phi^2).
\end{aligned}$$

Since $|\mathbf{d}|=1$ one has that

$$\begin{aligned}
&\int_{\mathbb{R}^3}[\nabla_\alpha(|\nabla\mathbf{d}|^2\mathbf{d})]\cdot|\nabla\mathbf{d}|(\nabla_\alpha\mathbf{d})\phi^2\\
&=\int_{\mathbb{R}^3}[\nabla_\alpha|\nabla\mathbf{d}|^2]|\nabla\mathbf{d}|\frac{\nabla_\alpha|\mathbf{d}|^2}{2}\phi^2+\nabla_\alpha\mathbf{d}|\nabla\mathbf{d}|^2\cdot|\nabla\mathbf{d}|\nabla_\alpha\mathbf{d}\phi^2\\
&=0+\int_{\mathbb{R}^3}|\nabla\mathbf{d}|^5\phi^2.
\end{aligned}$$

Combining the last two results yields the inequality:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\mathbf{d}|^3\phi^2+\underbrace{3\int_{\mathbb{R}^3}(\nabla_\alpha\nabla_\beta\mathbf{d})\cdot\nabla_\beta(|\nabla\mathbf{d}|\nabla_\alpha\mathbf{d}\phi^2)}_{(3.33)_a}\\
&\le 3\int_{\mathbb{R}^3}|\nabla\mathbf{d}|^5\phi^2+\underbrace{3\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla_\alpha(|\nabla\mathbf{d}|\nabla_\alpha\mathbf{d}\phi^2)}_{(3.33)_b}.
\end{aligned}\tag{3.33}$$

For $(3.33)_a$ one sees that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla_\alpha \nabla_\beta \mathbf{d}) \cdot \nabla_\beta (|\nabla \mathbf{d}| \nabla_\alpha \mathbf{d} \phi^2) \\
&= \underbrace{\int_{\mathbb{R}^3} (\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}) \cdot (\nabla |\nabla \mathbf{d}|) \phi^2}_{(3.34)_a} + \underbrace{\int_{\mathbb{R}^3} (\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}) \cdot (\nabla \phi^2) |\nabla \mathbf{d}|}_{(3.34)_b} + \underbrace{\int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| \phi^2}_{(3.34)_c}.
\end{aligned} \tag{3.34}$$

Since

$$\nabla |\nabla \mathbf{d}| = \frac{\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}}{|\nabla \mathbf{d}|}$$

one has that

$$\begin{aligned}
\int_{\mathbb{R}^3} (\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}) \cdot (\nabla |\nabla \mathbf{d}|) \phi^2 &= \int_{\mathbb{R}^3} \frac{|\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}|^2}{|\nabla \mathbf{d}|} \phi^2 \\
\int_{\mathbb{R}^3} (\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}) \cdot (\nabla \phi^2) |\nabla \mathbf{d}| &= \int_{\mathbb{R}^3} (\nabla |\nabla \mathbf{d}|) |\nabla \mathbf{d}|^2 \cdot (\nabla \phi^2).
\end{aligned}$$

By putting these two identities together with (3.34) one has that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla_\alpha \nabla_\beta \mathbf{d}) \cdot \nabla_\beta (|\nabla \mathbf{d}| \nabla_\alpha \mathbf{d} \phi^2) \\
&= \int_{\mathbb{R}^3} \frac{|\nabla^2 \mathbf{d} \cdot \nabla \mathbf{d}|^2}{|\nabla \mathbf{d}|} \phi^2 + \int_{\mathbb{R}^3} (\nabla |\nabla \mathbf{d}|) |\nabla \mathbf{d}|^2 \cdot (\nabla \phi^2) + \int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| \phi^2 \\
&\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| \phi^2 - C \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 |\nabla \phi|^2.
\end{aligned} \tag{3.35}$$

Estimating $(3.33)_b$ one has that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \nabla_\alpha (|\nabla \mathbf{d}| (\nabla_\alpha \mathbf{d}) \phi^2) \\
&\leq \int_{\mathbb{R}^3} |\mathbf{u}| |\nabla \mathbf{d}| \left(|\nabla |\nabla \mathbf{d}|| |\nabla \mathbf{d}| \phi^2 + |\nabla \mathbf{d}| |\Delta \mathbf{d}| \phi^2 + |\nabla \mathbf{d}|^2 |\nabla \phi^2| \right) \\
&\leq \int_{\mathbb{R}^3} |\mathbf{u}| \left(|\nabla^2 \mathbf{d}| |\nabla \mathbf{d}|^2 \phi^2 + |\nabla \mathbf{d}|^2 |\nabla^2 \mathbf{d}| \phi^2 + |\nabla \mathbf{d}|^3 |\nabla \phi^2| \right) \\
&\leq 2 \int_{\mathbb{R}^3} |\mathbf{u}| \left(|\nabla \mathbf{d}|^2 |\nabla^2 \mathbf{d}| \phi^2 + |\nabla \mathbf{d}|^3 \phi |\nabla \phi| \right) \\
&= \underbrace{2 \int_{\mathbb{R}^3} |\mathbf{u}| |\nabla \mathbf{d}|^2 |\nabla^2 \mathbf{d}| \phi^2}_{(3.36)_a} + \underbrace{2 \int_{\mathbb{R}^3} |\mathbf{u}| |\nabla \mathbf{d}|^3 \phi |\nabla \phi|}_{(3.36)_b}.
\end{aligned} \tag{3.36}$$

Applying Cauchy's inequality to $(3.36)_a$ one has that

$$2 \int_{\mathbb{R}^3} |\mathbf{u}| |\nabla \mathbf{d}|^2 |\nabla^2 \mathbf{d}| \phi^2 \leq \underbrace{\frac{1}{8} \int_{\mathbb{R}^3} |\nabla \mathbf{d}| |\nabla^2 \mathbf{d}|^2 \phi^2}_{(3.37)_a} + \underbrace{C \int_{\mathbb{R}^3} |\mathbf{u}|^2 |\nabla \mathbf{d}|^3 \phi^2}_{(3.37)_b}. \tag{3.37}$$

Applying Hölder's inequality to $(3.36)_b$ and $(3.37)_b$ yields that

$$\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{d}|^3 \phi |\nabla \phi| \le \left\{ \int_{\text{supp } \phi} |\mathbf{u}|^3 \right\}^{1/3} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{1/6} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 |\nabla \phi|^2 \right\}^{1/2},$$
$$\int_{\mathbb{R}^3} |\mathbf{u}|^2 |\nabla \mathbf{d}|^3 \phi^2 \le \left\{ \int_{\text{supp } \phi} |\mathbf{u}|^3 \right\}^{2/3} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{1/3} \tag{3.38}$$

After collecting the appropriate terms one obtains the following inequalities:

$$\begin{aligned} &\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \nabla_\alpha (|\nabla \mathbf{d}| (\nabla_\alpha \mathbf{d}) \phi^2) \\ &\le \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2 \phi^2 + C \left\{ \int_{\text{supp } \phi} |\mathbf{u}|^3 \right\}^{2/3} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{1/3} \\ &+ C \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 |\nabla \phi|^2, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 \phi^2 + \int_{\mathbb{R}^3} |\nabla^2 \mathbf{d}|^2 |\nabla \mathbf{d}| \phi^2 \\ &\le C \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^5 \phi^2 + |\nabla \mathbf{d}|^3 |\nabla \phi|^2 + C \left\{ \int_{\text{supp } \phi} |\mathbf{u}|^3 \right\}^{2/3} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{1/3} \\ &\le C \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^3 |\nabla \phi|^2 + C \left[\left\{ \int_{\text{supp } \phi} |\mathbf{u}|^3 \right\}^{\frac{2}{3}} + \left\{ \int_{\text{supp } \phi} |\nabla \mathbf{d}|^3 \right\}^{\frac{2}{3}} \right] \int_{\mathbb{R}^3} |\nabla (|\nabla \mathbf{d}|^{\frac{3}{2}} \phi)|^2. \end{aligned} \tag{3.40}$$

Where, in the last step the following two inequalities were used:

$$\left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{\frac{1}{3}} = \| |\nabla \mathbf{d}|^{\frac{3}{2}} \phi \|_{L^6(\mathbb{R}^3)}^2 \lesssim \| \nabla (|\nabla \mathbf{d}|^{\frac{3}{2}} \phi) \|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\nabla (|\nabla \mathbf{d}|^{\frac{3}{2}} \phi)|^2$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^5 \phi^2 &\le \left\{ \int_{\text{supp } \phi} |\nabla \mathbf{d}|^3 \right\}^{\frac{2}{3}} \left\{ \int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9 \phi^6 \right\}^{\frac{1}{3}} \\ &\lesssim \left\{ \int_{\text{supp } \phi} |\nabla \mathbf{d}|^3 \right\}^{\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla (|\nabla \mathbf{d}|^{\frac{3}{2}} \phi)|^2. \end{aligned}$$

To obtain norm estimates for $\mathbf{u}$ one multiplies (1.7) by $|\mathbf{u}|\mathbf{u}\phi^2$ and integrates over $\mathbb{R}^3$ to find that

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} |\mathbf{u}|^3 \phi^2 + \underbrace{3 \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla (|\mathbf{u}|\mathbf{u}\phi^2)}_{(3.41)_a} \\ &\lesssim \underbrace{\int_{\mathbb{R}^3} (\nabla \mathbf{d} \odot \nabla \mathbf{d}) : \nabla (|\mathbf{u}|\mathbf{u}\phi^2)}_{(3.41)_b} + \underbrace{\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\mathbf{u}|^3 \phi^2}_{(3.41)_c} + \underbrace{\int_{\mathbb{R}^3} |P - c||\nabla (|\mathbf{u}|\mathbf{u}\phi^2)|}_{(3.41)_d} \end{aligned} \tag{3.41}$$

where $c \in \mathbb{R}$ is an arbitrary constant. Estimating $(3.41)_a$ from below using Cauchy's inequality one finds that

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla(|\mathbf{u}|\mathbf{u}\phi^2) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 - 4\int_{\mathbb{R}^3} |\mathbf{u}|^3|\nabla\phi|^2. \end{aligned} \tag{3.42}$$

Whereas, estimating $(3.41)_c$ from above using Cauchy's, Hölder's and Sobolev's inequalities one finds that

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\mathbf{u}|^3\phi^2 \leq \frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 + C\left\{\int_{\text{supp }\phi} |\mathbf{u}|^3\right\}^{\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla(|\mathbf{u}|^{3/2}\phi)|^2. \tag{3.43}$$

Now, using integration by parts on $(3.41)_b$, one has that

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla(|\mathbf{u}|\mathbf{u}\phi^2) \\ &= \int_{\mathbb{R}^3} \nabla_i d^k \nabla_j d^k \nabla_i (|\mathbf{u}|\mathbf{u}^j\phi^2) \\ &= \int_{\mathbb{R}^3} \nabla_i d^k \nabla_j d^k \nabla_i u^j (|\mathbf{u}|\phi^2) + \int_{\mathbb{R}^3} \nabla_i d^k \nabla_j d^k \mathbf{u}^j \nabla_i (|\mathbf{u}|\phi^2) \\ &= \int_{\mathbb{R}^3} \nabla_i d^k \nabla_j d^k \nabla_i u^j (|\mathbf{u}|\phi^2) - \int_{\mathbb{R}^3} \nabla_i \nabla_i d^k \nabla_j d^k u^j |\mathbf{u}|\phi^2) \\ &- \int_{\mathbb{R}^3} \nabla_i d^k \nabla_i \nabla_j d^k u^j |\mathbf{u}|\phi^2) - \int_{\mathbb{R}^3} \nabla_i d^k \nabla_j d^k \nabla_i u^j |\mathbf{u}|\phi^2) \\ &\leq \underbrace{\int_{\mathbb{R}^3} 2|\nabla \mathbf{d}|^2|\nabla \mathbf{u}||\mathbf{u}|\phi^2}_{(3.44)_a} + \underbrace{\int_{\mathbb{R}^3} 2|\nabla \mathbf{d}||\nabla^2 \mathbf{d}||\mathbf{u}|^2\phi^2}_{(3.44)_b}. \end{aligned} \tag{3.44}$$

Now, estimate $(3.44)_a$ and $(3.44)_b$ as follows

$$\begin{aligned} \int_{\mathbb{R}^3} 2|\nabla \mathbf{d}|^2|\nabla \mathbf{u}||\mathbf{u}|\phi^2 &\leq \frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 \\ &+ C\left\{\int_{\text{supp }\phi} |\mathbf{u}|^3\right\}^{1/3} \left\{\int_{\text{supp }\phi} |\nabla \mathbf{d}|^3\right\}^{1/3} \left\{\int_{\mathbb{R}^3} |\nabla \mathbf{d}|^9\phi^6\right\}, \\ \int_{\mathbb{R}^3} 2|\nabla \mathbf{d}||\nabla^2 \mathbf{d}||\mathbf{u}|^2\phi^2 &\leq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2\phi^2 \\ &+ C\left\{\int_{\text{supp }\phi} |\nabla \mathbf{d}|^3\right\}^{1/3} \left\{\int_{\text{supp }\phi} |\mathbf{u}|^3\right\}^{1/3} \left\{\int_{\mathbb{R}^3} |\mathbf{u}|^9\phi^6\right\}^{1/3}. \end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45) and applying the Sobolev inequality yields that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla(|\mathbf{u}|\mathbf{u}\phi^2) \\
&\leq \frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2\phi^2 \\
&+ C\left\{\int_{\text{supp } \phi} |\nabla \mathbf{d}|^3\right\}^{1/3}\left\{\int_{\text{supp } \phi} |\mathbf{u}|^3\right\}^{1/3}\left[\int_{\mathbb{R}^3} |\nabla(|\mathbf{u}|^{\frac{3}{2}}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{\frac{3}{2}}\phi)|^2\right].
\end{aligned} \tag{3.46}$$

Finally estimating $(3.41)_d$ from above yields that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |P-c||\nabla \cdot (|\mathbf{u}|\mathbf{u}\phi^2)| \\
&\leq \frac{1}{8}\int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 + C\int_{\mathbb{R}^3} |P-c|^2|\mathbf{u}|\phi^2 + C\int_{\mathbb{R}^3} |\mathbf{u}|^3|\nabla\phi|^2.
\end{aligned} \tag{3.47}$$

Combining all these estimates yields that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^3\phi^2 + \int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 \\
&\leq C\int_{\mathbb{R}^3} |\mathbf{u}|^3|\nabla\phi|^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|\phi^2 + C\int_{\mathbb{R}^3} |P-c|^2|\mathbf{u}|\phi^2 \\
&+ C\left[\left\{\int_{\text{supp } \phi} |\mathbf{u}|^3\right\}^{2/3} + \left\{\int_{\text{supp } \phi} |\nabla \mathbf{d}|^3\right\}^{2/3}\right]\int_{\mathbb{R}^3} [|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2].
\end{aligned} \tag{3.48}$$

Noticing that

$$\begin{aligned}
|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 &\lesssim |\mathbf{u}||\nabla \mathbf{u}|^2\phi^2 + |\mathbf{u}|^3|\nabla\phi|^2 \\
|\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2 &\lesssim |\nabla \mathbf{d}||\nabla^2 \mathbf{d}|^2\phi^2 + |\nabla \mathbf{d}|^3|\nabla\phi|^2,
\end{aligned}$$

and, combining the estimates for $\mathbf{u}$ and ∇d in (3.40) and (3.48) yields the desired inequality:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)\phi^2 + \int_{\mathbb{R}^3} [|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2] \\
&\leq C\int_{\mathbb{R}^3} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)|\nabla\phi|^2 + C\int_{\mathbb{R}^3} |\mathbf{u}||P-c|^2\phi^2 \\
&+ C\left[\left\{\int_{\text{supp } \phi} |\mathbf{u}|^3\right\}^{2/3} + \left\{\int_{\text{supp } \phi} |\nabla \mathbf{d}|^3\right\}^{2/3}\right]\int_{\mathbb{R}^3} [|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2].
\end{aligned} \tag{3.49}$$

□

What remains is to estimate the term involving pressure–as it will be required later to have estimates involving only $\mathbf{u}$ and $\nabla \mathbf{d}$. It suffices to estimate $\|(P-c)\phi\|_{L^3(\mathbb{R}^3)}$ since from Hölder's inequality

$$\int_{\mathbb{R}^3} |\mathbf{u}||P-c|^2\phi^2 \leq \left\{\int_{\text{supp } \phi} |\mathbf{u}|^3\right\}^{1/3}\left\{\int_{\mathbb{R}^3} |P-c|^3\phi^3\right\}^{2/3}. \tag{3.50}$$

Estimating the commutator of the Riesz transforms R_iR_j one obtains the following lemma:

Lemma 3.2.2 (Local L^3-pressure estimate). *There exist $c_0 \in \mathbb{R}$ and $C_0 > 0$ such that for any $\phi \in C_0^1(\mathbb{R}^3)$*

$$\left\{\int_{\mathbb{R}^3} |p - c_0|^3 \phi^3\right\}^{1/3} \leq C_0 \left\{\int_{\mathbb{R}^3} (|\mathbf{u}|^6 + |\nabla \mathbf{d}|^6)\phi^3\right\}^{1/3}$$

$$+ \frac{C_0}{R}\left\{\int_{supp\ \phi} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)\right\}^{2/3} + C_0 R \sup_{z\in\mathbb{R}^3}\left\{\int_{B_R(z)} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)\right\}^{2/3}.$$

where $R > 0$ is such that supp $(\phi) \subset B_{2R}$.

Proof. Since

$$-\Delta p = \nabla^2_{jk} g^{jk} \text{ in } \mathbb{R}^3$$

where $g^{jk} := u^j u^k + \nabla_j d \cdot \nabla_k d$, one has that

$$p = -R_j R_k (g^{jk}).$$

for R_j is the Riesz transform. Hence, for $\phi \in C_0^\infty(\mathbb{R}^3)$, one has that

$$\begin{aligned}(p(\mathbf{x}) - c_0)\phi &= -R_j R_k(g^{jk})\phi - c_0\phi \\ &= -R_j R_k(g^{jk}\phi) - [\phi, R_j R_k](g^{jk}) - c_0\phi\end{aligned} \tag{3.51}$$

where $[\phi, R_j R_k]$ is the commutator, namely,

$$[\phi, R_j R_k](f) = \phi \cdot R_j R_k(f) - R_j R_k(f\phi) \ f \in C_0^\infty(\mathbb{R}^3).$$

Now $[\phi, R_j R_k](g^{jk})$ is estimated as follows

$$\begin{aligned}&[\phi, R_j R_k](g^{jk}) \\ &= \phi(\mathbf{x}) R_j R_k(g^{jk})(\mathbf{x}) - R_j R_k((g^{jk}\phi))(\mathbf{x}) \\ &= -\phi(\mathbf{x})\int_{\mathbb{R}^3} \frac{(x^j - y^j)(x^k - y^k)}{|\mathbf{x} - \mathbf{y}|^5} g^{jk}(\mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^3} \frac{(x^j - y^j)(x^k - y^k)}{|\mathbf{x} - \mathbf{y}|^5} \phi(\mathbf{y}) g^{jk}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \frac{(\phi(\mathbf{x}) - \phi(\mathbf{y}))(x^j - y^j)(x^k - y^k)}{|\mathbf{x} - \mathbf{y}|^5} g^{jk}(\mathbf{y}) d\mathbf{y}\end{aligned}$$

so that for any $\mathbf{x} \in$ supp ϕ,

$$\begin{aligned}&[\phi, R_j R_k](g^{jk})(\mathbf{x}) + c_0\phi(\mathbf{x}) \\ &= \int_{B_{2R}} \frac{(\phi(\mathbf{x}) - \phi(\mathbf{y}))(x^j - y^j)(x^k - y^k)}{|\mathbf{x} - \mathbf{y}|^5} g^{jk}(\mathbf{y}) d\mathbf{y} + c_0\phi(\mathbf{x}) \\ &+ \phi(\mathbf{x})\left[\int_{\mathbb{R}^3 \setminus B_{2R}} \frac{(x^j - y^j)(x^k - y^k)}{|\mathbf{x} - \mathbf{y}|^5} g^{jk}(\mathbf{y}) d\mathbf{y} + c_0\right] \\ &=: I + II.\end{aligned}$$

Since

$$|\phi(\mathbf{x}) - \phi(\mathbf{y})| \le \|\nabla\phi\|_{L^\infty}|\mathbf{x}-\mathbf{y}| \le CR^{-1}|\mathbf{x}-\mathbf{y}|$$

one has the estimate

$$\begin{aligned}
|I(\mathbf{x})| &\le \int_{\text{supp }\phi} \frac{|\phi(\mathbf{x})-\phi(\mathbf{y})||x^j-y^j||x^k-y^k|}{|\mathbf{x}-y|^5}|g^{jk}(\mathbf{y})|d\mathbf{y} \\
&\le CR^{-1}\int_{\text{supp }\phi}\frac{|\mathbf{x}-\mathbf{y}|^3}{|\mathbf{x}-\mathbf{y}|^5}(|\mathbf{u}|^2(\mathbf{y})+|\nabla\mathbf{d}|^2(\mathbf{y}))d\mathbf{y} \\
&= CR^{-1}I_1\left((|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\chi_{\text{supp }\phi}\right)(\mathbf{x}),
\end{aligned}$$

where I_1 is the Riesz potential of order 1 and $\chi_{\text{supp }\phi}$ is the characteristic function of the support of ϕ. Using the Hardy–Littlewood–Sobolev theorem we have that $I_1 : L^{3/2}(\mathbb{R}^3) \to L^3(\mathbb{R}^3)$ is bounded, that is,

$$\|I_1(f)\|_{L^3(\mathbb{R}^3)} \le C\|f\|_{L^{3/2}(\mathbb{R}^3)}. \tag{3.52}$$

Since $(|u|^2 + |\nabla\mathbf{d}|^2)\chi_{\text{supp }\phi} \in L^{3/2}(\mathbb{R}^3)$ we have

$$\begin{aligned}
\|I\|_{L^3(\mathbb{R}^3)} &\le CR^{-1}\|I_1\left((|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\chi_{\text{supp }\phi}\right)\|_{L^3(\mathbb{R}^3)} \\
&\le CR^{-1}\|(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\chi_{\text{supp }\phi}\|_{L^{3/2}(\mathbb{R}^3)} \\
&\le CR^{-1}\left\{\int_{\text{supp }\phi}|\mathbf{u}|^3+|\nabla\mathbf{d}|^3\right\}^{\frac{2}{3}}.
\end{aligned} \tag{3.53}$$

To estimate II, choose

$$c_0 = -\int_{\mathbb{R}^3\setminus B_{2R}}\frac{y^jy^k}{|\mathbf{y}|^5}g^{jk}(\mathbf{y})d\mathbf{y}$$

so that,

$$\begin{aligned}
|II(\mathbf{x})| &= \left|\phi(\mathbf{x})\int_{\mathbb{R}^3\setminus B_{2R}}\left(\frac{(x^j-y^j)(x^k-y^k)}{|\mathbf{x}-\mathbf{y}|^5}-\frac{y^jy^k}{|\mathbf{y}|^5}\right)g^{jk}(\mathbf{y})d\mathbf{y}\right| \\
&\le CR|\phi(\mathbf{x})|\int_{\mathbb{R}^3\setminus B_{2R}}\frac{1}{|\mathbf{x}-\mathbf{y}|^4}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(\mathbf{y})d\mathbf{y}.
\end{aligned}$$

In the last estimate, standard inequality of Stein (see [26])

$$\left|\frac{(x^j-y^j)(x^k-y^k)}{|\mathbf{x}-\mathbf{y}|^5}-\frac{y^jy^k}{|\mathbf{y}|^5}\right| \le \frac{C|\mathbf{x}|}{|\mathbf{x}-\mathbf{y}|^4} \text{ for } \mathbf{x}\in B_{3R/2} \text{ and } \mathbf{y}\in\mathbb{R}^3\setminus B_{2R}. \tag{3.54}$$

was employed. Thus, continuing the estimate,

$$\begin{aligned}
|II|(\mathbf{x}) &\leq CR \int_{\mathbb{R}^3 \setminus B_{2R}} \frac{1}{|\mathbf{x}-\mathbf{y}|^4} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\mathbf{y}) d\mathbf{y} \\
&\leq CR \int_{\mathbb{R}^3 \setminus B_{2R}} \frac{1}{|\mathbf{y}|^4} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\mathbf{y}) d\mathbf{y} \\
&\leq CR \sum_{k=2}^{\infty} \frac{1}{(kR)^4} \int_{B_{(k+1)R} \setminus B_{kR}} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\mathbf{y}) d\mathbf{y} \\
&=\leq \frac{C}{R} \left[\sum_{k=2}^{\infty} \frac{1}{k^2} \right] \sup_{z \in \mathbb{R}^3} \int_{B_R(z)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\mathbf{y}) d\mathbf{y} \\
&\leq C \sup_{z \in \mathbb{R}^3} \left(\int_{B_R(z)} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)(\mathbf{y}) d\mathbf{y} \right)^{2/3}.
\end{aligned}$$

Integrating II over B_{2R} yields

$$\|II\|_{L^3(\mathbb{R}^3)} \leq CR \sup_{z \in \mathbb{R}^3} \left(\int_{B_R(z)} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3)(\mathbf{y}) d\mathbf{y} \right)^{2/3}.$$

Finally,

$$\|R_j R_k (g^{jk} \phi)\|_{L^3(\mathbb{R}^3)} \leq C_0 \left\{ \int_{\mathbb{R}^3} (|\mathbf{u}|^6 + |\nabla \mathbf{d}|^6) \phi^3 \right\}^{1/3}$$

which implies the desired estimate. □

In this chapter it is demonstrated that suitable weak solutions exist exist for the system (1.7)-(1.9). Moreover, under the additional assumption that $(\mathbf{u}, \nabla \mathbf{d})$ is small in L^3, it is shown that such solutions are indeed smooth.

4.1 Suitable weak solutions

The notion of suitable weak solutions was introduced by Caffarelli, Kohn and Nirenberg [2]. In their study of the two dimension flow of nematic liquid crystals Lin, Lin, and Wang [19] also employed a similar notion of suitable weak solutions.

Definition 4.1.1. *A triple* $(\mathbf{u}, p, \mathbf{d}) : \Omega \times (0,T) \to \mathbb{R}^3 \times \mathbb{R} \times S^2$ *is called a* suitable weak solutions *to liquid crystal flow equations* (1.7)-(1.9) *in* $\Omega \times (0,T)$ *if:*

1. $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0,T))$, $p \in L^{\frac{3}{2}}(\Omega \times (0,T))$ *and* $\mathbf{d} \in L_t^2 H_x^2(\Omega \times (0,T), S^2)$;
2. $(\mathbf{u}, p, \mathbf{d})$ *satisfies the liquid crystal flow equations* (1.7)-(1.9) *in the weak sense;*
3. $(\mathbf{u}, p, \mathbf{d})$ *satisfies the generalized energy inequality* (4.1) *for* $\phi \in C_0^\infty(\Omega \times (0,T))$.

Lemma 4.1.2. *Suppose that* $(\mathbf{u}, p, \mathbf{d})$ *is a sufficiently regular solution to the equations* (1.7)-(1.9) *on* $\Omega \times (0,T)$. *Then for any* $0 \leq \phi \in C_0^\infty(\Omega \times (0,T))$, *it holds*

$$\begin{aligned} 2\int_{\Omega\times(0,T)} \left(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2\right) \leq & \int_{\Omega\times(0,T)} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\right)(\phi_t + \Delta\phi) \\ & + \int_{\Omega\times(0,T)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + 2p)\mathbf{u} \cdot \nabla\phi \\ & + 2\int_{\Omega\times(0,T)} \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3\right) : \nabla^2\phi \\ & + 2\int_{\Omega\times(0,T)} \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla\phi. \end{aligned} \tag{4.1}$$

Proof. To obtain the estimates involving $\mathbf{u}$ multiply (1.7) by $u\phi$ and integrate to find that

$$\begin{aligned} & \underbrace{\int_{\Omega\times(0,T)} \mathbf{u}_t \cdot \mathbf{u}\phi}_{(4.2)_a} + \underbrace{\int_{\Omega\times(0,T)} (\mathbf{u}\cdot\nabla\mathbf{u}) \cdot \mathbf{u}\phi}_{(4.2)_b} - \underbrace{\int_{\Omega\times(0,T)} \Delta\mathbf{u} \cdot \mathbf{u}\phi}_{(4.2)_c} + \underbrace{\int_{\Omega\times(0,T)} \nabla p \cdot \mathbf{u}\phi}_{(4.2)_d} \\ & = \underbrace{\int_{\Omega\times(0,T)} \nabla \cdot (\nabla\mathbf{d} \odot \nabla\mathbf{d})}_{(4.2)_e}. \end{aligned} \tag{4.2}$$

The terms $(4.2)_a - (4.2)_d$ simplify to

$$\begin{aligned}
\int_{\Omega\times(0,T)} \mathbf{u}_t \cdot \mathbf{u}\phi &= \int_{\Omega\times(0,T)} \frac{1}{2}\partial_t \left\{|\mathbf{u}|^2\phi\right\} - \frac{1}{2}|\mathbf{u}|^2\phi_t = -\int_{\Omega\times(0,T)} \frac{1}{2}|\mathbf{u}|^2\phi_t \\
\int_{\Omega\times(0,T)} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\phi &= \int_{\Omega\times(0,T)} \frac{\nabla|\mathbf{u}|^2}{2}\cdot\mathbf{u}\phi \\
&= -\int_{\Omega\times(0,T)} \frac{|\mathbf{u}|^2}{2}\left((\nabla\cdot\mathbf{u})\phi + \mathbf{u}\cdot\nabla\phi\right) \\
&= -\int_{\Omega\times(0,T)} \frac{|\mathbf{u}|^2}{2}\mathbf{u}\cdot\nabla\phi \\
\int_{\Omega\times(0,T)} \Delta\mathbf{u}\cdot\mathbf{u}\phi &= -\int_{\Omega\times(0,T)} \nabla\mathbf{u} : \nabla(\mathbf{u}\phi) \\
&= -\int_{\Omega\times(0,T)} (\nabla\mathbf{u}:\nabla\mathbf{u})\phi - \int_{\Omega\times(0,T)} \nabla\mathbf{u} : \mathbf{u}\otimes\nabla\phi \\
&= -\int_{\Omega\times(0,T)} |\nabla\mathbf{u}|^2\phi - \int_{\Omega\times(0,T)} \mathbf{u}\cdot\nabla_\alpha\mathbf{u}\nabla_\alpha\phi \\
&= -\int_{\Omega\times(0,T)} |\nabla\mathbf{u}|^2\phi + \int_{\Omega\times(0,T)} \frac{|\mathbf{u}|^2}{2}\Delta\phi \\
\int_{\Omega\times(0,T)} \nabla p\cdot\mathbf{u}\phi &= -\int_{\Omega\times(0,T)} p(\nabla\cdot\mathbf{u})\phi - \int_{\Omega\times(0,T)} p(\mathbf{u}\cdot\nabla\phi) \\
&= -\int_{\Omega\times(0,T)} p(\mathbf{u}\cdot\nabla\phi).
\end{aligned}$$

And for the term $(4.2)_e$ one has

$$\begin{aligned}
-\int_{\Omega\times(0,T)} \nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\mathbf{u}\phi &= \int_{\Omega\times(0,T)} (\nabla\mathbf{d}\odot\nabla\mathbf{d}) : \nabla\cdot(\mathbf{u}\phi) \\
&= \int_{\Omega\times(0,T)} (\nabla\mathbf{d}\odot\nabla\mathbf{d}) : \left((\nabla\mathbf{u})\phi + \mathbf{u}\otimes\nabla\phi\right).
\end{aligned}$$

Inserting the results for $(4.2)_a - (4.2)_e$ into (4.2) yields that

$$\begin{aligned}
&\int_{\Omega\times(0,T)} -\frac{1}{2}|\mathbf{u}|^2(\phi_t - \Delta\phi) - \left[\frac{|\mathbf{u}|^2}{2} + p\right](\mathbf{u}\cdot\nabla\phi) + \int_{\Omega\times(0,T)} |\nabla\mathbf{u}|^2\phi \\
&= \int_{\Omega\times(0,T)} (\nabla\mathbf{d}\odot\nabla\mathbf{d}) : \left((\nabla\mathbf{u})\phi + \mathbf{u}\otimes\nabla\phi\right).
\end{aligned} \tag{4.3}$$

Next, to obtain estimates involving $\nabla\mathbf{d}$ differentiate the equation (1.9) and integrate against $(\nabla \mathbf{d})\phi$, this yields that

$$\underbrace{\int_{\Omega\times(0,T)} (\nabla \mathbf{d})_t : (\nabla \mathbf{d})\phi}_{(4.4)_a} + \underbrace{\int_{\Omega\times(0,T)} \nabla(\mathbf{u}\cdot\nabla\mathbf{d}) : (\nabla \mathbf{d})\phi}_{(4.4)_b} = \underbrace{\int_{\Omega\times(0,T)} \nabla\left[\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d\right] : (\nabla\mathbf{d})\phi}_{(4.4)_c}. \tag{4.4}$$

In $(4.4)_a$ it is easy to see that

$$\int_{\Omega\times(0,T)} (\nabla \mathbf{d})_t : (\nabla \mathbf{d})\phi = -\int_{\Omega\times(0,T)} \frac{1}{2}|\nabla\mathbf{d}|^2\phi_t.$$

since ϕ has compact support. Using (1.8) the term $(4.4)_b$ is

$$\begin{aligned}
\int_{\Omega\times(0,T)} \nabla(\mathbf{u}\cdot\nabla\mathbf{d}) : (\nabla\mathbf{d})\phi &= \int_{\Omega\times(0,T)} \nabla_\alpha(u^j\mathbf{d}_j)\cdot\mathbf{d}_\alpha\phi \\
&= \int_{\Omega\times(0,T)} u^j_\alpha\mathbf{d}_j\cdot\mathbf{d}_\alpha\phi + \int_{\Omega\times(0,T)} u^j\mathbf{d}_{j\alpha}\cdot\mathbf{d}_\alpha\phi \\
&= \int_{\Omega\times(0,T)} \nabla\mathbf{u} : \nabla\mathbf{d}\otimes\nabla\mathbf{d}\phi + \int_{\Omega\times(0,T)} \mathbf{u}\cdot\nabla\left\{\frac{|\nabla\mathbf{d}|^2}{2}\right\}\phi \\
&= \int_{\Omega\times(0,T)} \nabla\mathbf{u} : \nabla\mathbf{d}\otimes\nabla\mathbf{d}\phi - \int_{\Omega\times(0,T)} (\mathbf{u}\cdot\nabla\phi)\frac{|\nabla\mathbf{d}|^2}{2}.
\end{aligned}$$

Before proceeding to the term $(4.4)_c$ recall the following point-wise identities for harmonic maps:

$$\begin{aligned}
0 &= \nabla\frac{|\mathbf{d}|^2}{2} = \nabla\mathbf{d}\cdot\mathbf{d} \\
0 &= \nabla\cdot(\nabla\mathbf{d}\cdot\mathbf{d}) = \nabla_\alpha\left\{\left(\nabla_\alpha d^i\right)d^i\right\} = (\Delta\mathbf{d})\cdot\mathbf{d} + |\nabla\mathbf{d}|^2.
\end{aligned} \tag{4.5}$$

Then, integrating by parts in $(4.4)_c$ yields that

$$\begin{aligned}
&\int_{\Omega\times(0,T)} \nabla\left[\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d\right]\cdot(\nabla\mathbf{d})\phi \\
&= -\int_{\Omega\times(0,T)} \left[\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d\right]\cdot\phi\left[(\Delta\mathbf{d})\phi + \nabla\mathbf{d}\cdot\nabla\phi\right] \\
&= -\int_{\Omega\times(0,T)} |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d|^2 \\
&- \int_{\Omega\times(0,T)} \left[\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d\right]\cdot\left[\nabla\mathbf{d}\cdot\nabla\phi\right] \\
&+ \int_{\Omega\times(0,T)} \left[\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d\right]\cdot\left[|\nabla\mathbf{d}|^2 d\right]\phi \\
&= -\int_{\Omega\times(0,T)} |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2 d|^2\phi - \int_{\Omega\times(0,T)} \Delta\mathbf{d}(\nabla\mathbf{d}\cdot\nabla\phi) + 0.
\end{aligned} \tag{4.6}$$

where (4.5) has been used in the last line to arrive at

$$\begin{aligned}\int_{\Omega\times(0,T)} \left[\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 d\right] \cdot \left[|\nabla \mathbf{d}|^2 d\right] \phi &= \int_{\Omega\times(0,T)} \left[-|\nabla \mathbf{d}|^2 (\Delta \mathbf{d}) \cdot \mathbf{d} + |\nabla \mathbf{d}|^4 |d|^2\right] \phi \\ &= \int_{\Omega\times(0,T)} \left[-|\nabla \mathbf{d}|^4 + |\nabla \mathbf{d}|^4\right] \phi = 0.\end{aligned}$$

Another integration by parts in (4.6), provides that

$$\begin{aligned}&-\int_{\Omega\times(0,T)} \Delta \mathbf{d}(\nabla \mathbf{d} \cdot \nabla \phi) \\ &= -\int_{\Omega\times(0,T)} \nabla_\alpha \left(\nabla_\alpha \mathbf{d} \cdot \nabla_\beta \mathbf{d}\right) \nabla_\beta \phi + \int_{\Omega\times(0,T)} \nabla_\alpha \mathbf{d} \cdot \nabla_{\alpha\beta} \mathbf{d} \nabla_\beta \phi \\ &= \int_{\Omega\times(0,T)} (\nabla \mathbf{d} \otimes \nabla \mathbf{d}) : \nabla^2 \phi - \int_{\Omega\times(0,T)} \frac{|\nabla \mathbf{d}|^2}{2} \Delta \phi \\ &= \int_{\Omega\times(0,T)} \left(\nabla \mathbf{d} \otimes \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3\right) : \nabla^2 \phi + \int_{\Omega\times(0,T)} \frac{|\nabla \mathbf{d}|^2}{2} \Delta \phi.\end{aligned}$$

Inserting these facts for $(4.4)_a - (4.4)_c$ into (4.4) yields that

$$\begin{aligned}&\int_{\Omega\times(0,T)} \left[-\frac{1}{2}|\nabla \mathbf{d}|^2 (\phi_t + \Delta \phi - \frac{1}{2}|\nabla \mathbf{d}|(\mathbf{u} \cdot \nabla \phi)\right] + \int_{\Omega\times(0,T)} \nabla \mathbf{u} : \nabla \mathbf{d} \otimes \nabla \mathbf{d} \phi \\ &= \int_{\Omega\times(0,T)} \left(\nabla \mathbf{d} \otimes \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3\right) : \nabla^2 \phi - \int_{\Omega\times(0,T)} |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 d|^2 \phi.\end{aligned} \tag{4.7}$$

Combining (4.3) and (4.7) one arrives at (4.1). □

Corollary 4.1.3. *Suppose that* $(\mathbf{u}, p, \mathbf{d})$ *is a suitable weak solution of the liquid crystal flow equations in* $\Omega \times (0, T)$*. Then, for each* $t \in (0, T)$ *and each smooth, compactly supported* $\phi \geq 0$ *on* $\Omega \times (0, T)$*, one has that*

$$\begin{aligned}&\int_{\Omega\times\{t\}} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)\phi + 2\int_{\Omega\times(0,t)} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 d|^2)\phi \\ &\leq \int_{\Omega\times(0,t)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2)(\phi_t + \Delta \phi) \\ &+ \int_{\Omega\times(0,t)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + 2p)\mathbf{u} \cdot \nabla \phi \\ &+ 2\int_{\Omega\times(0,t)} (\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 I_3) : \nabla^2 \phi \\ &+ 2\int_{\Omega\times(0,t)} \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla \phi.\end{aligned} \tag{4.8}$$

4.2 L^3 small solutions are smooth

In this section the Theorem 4.2.6 is proven. The technique that is employed has two basic parts. The first is proving a decay lemma for solutions of small renormalized energy L^3 energy (Lemma 4.2.1) using a *blow-up* argument. The second is to obtain estimates of Riesz potentials between Morrey spaces in the framework developed by Huang and Wang in [14]. These estimates lead smoothness of solutions to (1.7)-(1.9) with small renormalized L^3 energy.

Define *optimal Sobolev embedding constant* for the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ where $\Omega \subset \mathbb{R}^3$ by:

$$\mathcal{C}(3) = \inf \left\{ \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^6(\Omega)}} : f \neq 0, f \in C_0^\infty(\mathbb{R}^3) \right\}.$$

It is noted that given Ω one may explicitly calculate $\mathcal{C}(3)$.

Lemma 4.2.1 (decay)**.** *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $(\mathbf{u}, p, \mathbf{d})$ is a suitable weak solution on $P_r(\mathbf{x}_0, t_0) \equiv P_{r_0}$ with $\sup_{t_0 - r_0^2 \leq t \leq t_0} \|\nabla \mathbf{d}\|_{L^3(B_{r_0})}(t) < (\mathcal{C}(3))^{-2}$ and*

$$r_0^{-2/3}\|\mathbf{u}\|_{L^3(P_{r_0})} + r_0^{-4/3}\|p\|_{L^{3/2}(P_{r_0})} + r_0^{-2/3}\|\nabla \mathbf{d}\|_{L^3(P_{r_0})} \leq \epsilon_0 \tag{4.9}$$

then

$$\begin{aligned}(\theta_0 r_0)^{-2/3}\|\mathbf{u}\|_{L^3(P_{\theta_0 r_0})} + (\theta_0 r_0)^{-4/3}\|p\|_{L^{3/2}(P_{\theta_0 r_0})} + (\theta_0 r_0)^{-2}\|\nabla \mathbf{d}\|_{L^3(P_{\theta_0 r_0})} \\ \leq \frac{1}{2}\left[r_0^{-2/3}\|\mathbf{u}\|_{L^3(P_{r_0})} + r_0^{-4/3}\|p\|_{L^{3/2}(P_{r_0})} + r_0^{-2}\|\nabla \mathbf{d}\|_{L^3(P_{r_0})}\right].\end{aligned} \tag{4.10}$$

Proof. The equations (1.7)-(1.9) are invariant under translations and parabolic dilations. Recall that

$$P_r(\mathbf{x}, t) = (t - r^2, t + r^2) \times B_r(\mathbf{x}).$$

For $r_0 > 0$ and $z_0 = (\mathbf{x}_0, t_0)$ fixed, if $(\mathbf{u}, p, \mathbf{d})$ is a solutions of the liquid crystal flow equations on $P_{r_0}(z_0)$, then $(u_{z_0,r_0}, p_{z_0,r_0}, \mathbf{d}_{z_0,r_0})$ given by,

$$\begin{aligned}\mathbf{u}_{z_0,r_0}(x,t) &= r_0\mathbf{u}(x_0 + r_0 x, t_0 + r_0^2 t) \\ p_{z_0,r_0}(x,t) &= r_0^2 p(x_0 + r_0 x, t_0 + r_0^2 t) \\ \mathbf{d}_{z_0,r_0}(x,t) &= \mathbf{d}(x_0 + r_0 x, t_0 + r_0^2 t)\end{aligned}$$

is a solution on $P_1(0,0)$. Thus it suffices to consider $r_0 = 1$ and $z_0 = (x_0, t_0) = (0, 0)$.

Assume for contradiction that the conclusion were false. Then, there must exist a sequence of suitable weak solutions to the equations (1.7)=(1.9) on $P_1(\mathbf{0}, 0)$ such that

$$\left(\int_{P_1(0,0)} |\mathbf{u}_i|^3\right)^{\frac{1}{3}} + \left(\int_{P_1(0,0)} |p_i|^{\frac{3}{2}}\right)^{\frac{2}{3}} + \left(\int_{P_1(0,0)} |\nabla \mathbf{d}_i|^3\right)^{\frac{1}{3}} = \epsilon_i \to 0 \tag{4.11}$$

and where for any $\theta \in (0, \frac{1}{2})$ one has that

$$\begin{aligned} &\left[\left((\theta)^{-2} \int_{P_\theta(0,0)} |\mathbf{u}_i|^3 \right)^{\frac{1}{3}} + \left((\theta)^{-2} \int_{P_\theta(0,0)} |p_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left((\theta)^{-2} \int_{P_\theta(0,0)} |\nabla \mathbf{d}_i|^3 \right)^{\frac{1}{3}} \right] \\ &> \frac{1}{2} \left[\left(\int_{P_1(0,0)} |\mathbf{u}_i|^3 \right)^{\frac{1}{3}} + \left(\int_{P_1(0,0)} |p_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left(\int_{P_1(0,0)} |\nabla \mathbf{d}_i|^3 \right)^{\frac{1}{3}} \right]. \end{aligned} \tag{4.12}$$

Now define a new sequence $(\mathbf{v}_i, q_i, \mathbf{e}_i) : P_1(0,0) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$ by

$$\mathbf{v}_i(z) = \frac{\mathbf{u}_i(z)}{\epsilon_i}, \ q_i(z) = \frac{p_i(z)}{\epsilon_i}, \ \mathbf{e}_i(z) = \frac{\mathbf{d}_i(z) - (\mathbf{d}_i)_1}{\epsilon_i}$$

where $(\mathbf{d}_i)_1 = \frac{1}{|P_1(0,0)|} \int_{P_1(0,0)} \mathbf{d}_i$. Then $(\mathbf{v}_i, q_i, \mathbf{e}_i)$ satisfy

$$\begin{aligned} \partial_t \mathbf{v}_i - \Delta \mathbf{v}_i + \nabla q_i &= -\epsilon[\mathbf{v}_i \cdot \nabla \mathbf{v}_i + \nabla \cdot (\nabla \mathbf{e}_i \odot \nabla \mathbf{e}_i)] \\ \nabla \cdot \mathbf{v}_i &= 0 \\ \partial_t \mathbf{e}_i - \Delta \mathbf{e}_i &= \epsilon_i[|\nabla \mathbf{e}_i|^2 \mathbf{d}_i - \mathbf{v}_i \cdot \nabla \mathbf{e}_i]. \end{aligned} \tag{4.13}$$

It follows from (4.11) that

$$\left(\int_{P_1(0,0)} |\mathbf{v}_i|^3 \right)^{\frac{1}{3}} + \left(\int_{P_1(0,0)} |q_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left(\int_{P_1(0,0)} |\mathbf{e}_i|^3 \right)^{\frac{1}{3}} = 1. \tag{4.14}$$

Then for any $\theta \in (0, \frac{1}{2})$ it follows from (4.12) that

$$\left(\theta^{-2} \int_{P_\theta(0,0)} |\mathbf{v}_i|^3 \right)^{\frac{1}{3}} + \left(\theta^{-2} \int_{P_\theta(0,0)} |q_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left(\theta^{-2} \int_{P_\theta(0,0)} |\mathbf{e}_i|^3 \right)^{\frac{1}{3}} > \frac{1}{2}. \tag{4.15}$$

Since $(\mathbf{u}_i, p_i, \mathbf{d}_i)$ satisfy (4.8) one has that

$$\begin{aligned} &\sup_{\left(-\frac{3}{4}\right)^2 \le t \le 0} \int_{B_{3/4}(0)} (|\mathbf{u}_i|^2 + |\nabla \mathbf{d}_i|^2) + \int_{P_{3/4}(0,0)} (|\nabla \mathbf{u}_i|^2 + |\Delta \mathbf{d}_i + |\nabla \mathbf{d}_i|^2 \mathbf{d}_i|^2) \\ &\le C \int_{P_1(0,0)} (|\mathbf{u}_i|^2 + |\nabla \mathbf{d}_i|^2) + (|p_i| + |\mathbf{u}_i|^2 + |\nabla \mathbf{d}_i|^2)|\mathbf{u}_i|. \end{aligned} \tag{4.16}$$

Rescaling (4.16) and using Hölder's inequality with (4.14) and (4.11) one has that

$$\begin{aligned} &\sup_{\left(\frac{3}{4}\right)^2 \le t \le 0} \int_{B_{3/4}(0)} (|\mathbf{v}_i|^2 + |\nabla \mathbf{e}_i|^2) + \int_{P_{3/4}(0,0)} (|\nabla \mathbf{v}_i|^2 + \left|\Delta \mathbf{e}_i + \epsilon_i |\nabla \mathbf{d}_i|^2 d_i\right|^2) \\ &\le C \int_{P_1(0,0)} (|\mathbf{v}_i|^2 + |\nabla \mathbf{e}_i|^2) + (|q_i| + \epsilon_i |\mathbf{u}_i|^2 + \epsilon_i |\nabla \mathbf{e}_i|^2)|\mathbf{v}_i| \le C. \end{aligned} \tag{4.17}$$

On the other hand,

$$\int_{B_{1/2}(0)} |\nabla^2 \mathbf{e}_i|^2 \le C \int_{B_{3/4}(0)} |\nabla \mathbf{e}_i|^2 + C \int_{B_{3/4}(0)} |\Delta \mathbf{e}_i|^2.$$

Integrating the previous inequality in t yields that

$$\int_{P_{1/2}(0,0)} |\nabla^2 \mathbf{e}_i|^2 \lesssim \int_{P_{3/4}(0,0)} |\nabla \mathbf{e}_i|^2 + |\Delta \mathbf{e}_i|^2$$

Applying the point-wise inequality, $|\Delta \mathbf{e}_i|^2 \lesssim |\Delta \mathbf{e}_i + |\nabla \mathbf{e}_i|^2 d_i|^2 + |\nabla \mathbf{e}_i|^4$, one has that

$$\int_{P_{1/2}(0,0)} |\nabla^2 \mathbf{e}_i|^2 \lesssim \int_{P_{3/4}(0,0)} |\nabla \mathbf{e}_i|^2 + |\Delta \mathbf{e}_i + |\nabla \mathbf{e}_i|^2 d_i|^2 + \int_{P_{3/4}(0,0)} |\nabla \mathbf{e}_i|^4. \tag{4.18}$$

By Hölder's inequality and the Sobolev embedding $W^{1,2}(B_1) \hookrightarrow L^6(B_1)$ with optimal embedding constant $\mathcal{C}(3)$ one has that

$$\|\nabla \mathbf{e}_i\|_{L^4(B_1)} \leq \mathcal{C}(3) \|\nabla \mathbf{e}_i\|_{L^3(B_1)}^{1/2} \|\nabla^2 \mathbf{e}_i\|_{L^2(B_1)}^{1/2}.$$

Integrating this over t yields that

$$\int_{-1}^{0} \int_{B_1} |\nabla \mathbf{e}_i|^4 \leq \mathcal{C}(3)^4 \sup_{-1 \leq t \leq 0} \|\nabla \mathbf{e}_i\|_{L^3(B_1)}^2(t) \int_{-1}^{0} \int_{B_1} |\nabla^2 \mathbf{e}_i|^2. \tag{4.19}$$

Inserting the estimate (4.19) into (4.18) one has that

$$\begin{aligned} &\left[1 - \left(\mathcal{C}(3)^4 \sup_{-1 \leq t \leq 0} \|\nabla \mathbf{e}_i\|_{L^3(B_1)}^2(t)\right)\right] \int_{P_{1/2}(0,0)} |\nabla^2 \mathbf{e}_i|^2 \\ &\lesssim \int_{P_{3/4}(0,0)} |\nabla \mathbf{e}_i|^2 + |\Delta \mathbf{e}_i + |\nabla \mathbf{e}_i|^2 d_i|^2. \end{aligned} \tag{4.20}$$

So applying (4.17) to (4.20) one finds that

$$\int_{P_{1/2}(0,0)} |\nabla^2 \mathbf{e}_i|^2 \leq C. \tag{4.21}$$

Combining relevant estimates from (4.17) and (4.21) one has that

$$\int_{P_{1/2}(0,0)} |q_i|^{\frac{3}{2}} + \int_{P_{1/2}(0,0)} (|\mathbf{v}_i|^2 + |\nabla \mathbf{v}_i|^2) + \int_{P_{1/2}(0,0)} (|\nabla \mathbf{e}_i|^2 + |\nabla^2 \mathbf{e}_i|^2) \leq C. \tag{4.22}$$

From weak compactness and (4.22) one has that

$$\begin{aligned} &q_i \rightharpoonup q \text{ in } L^{\frac{3}{2}}(P_{\frac{1}{2}}(0,0)) \\ &\mathbf{v}_i \rightharpoonup \mathbf{v},\ \nabla \mathbf{v}_i \rightharpoonup \nabla \mathbf{v} \text{ in } L^2(P_{\frac{1}{2}}(0,0)) \\ &\mathbf{e}_i \rightharpoonup \mathbf{e},\ \nabla \mathbf{e}_i \rightharpoonup \nabla \mathbf{e},\ \nabla^2 \mathbf{e}_i \to \nabla^2 \mathbf{e} \text{ in } L^2(P_{\frac{1}{2}}(0,0)). \end{aligned}$$

where symbol $\rightharpoonup$ denotes weak convergence (likewise, the symbol $\to$ denotes strong convergence). Sending i to ∞ in (4.13) yields that

$$\begin{aligned} \partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla q &= 0 \\ \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{e} - \Delta \mathbf{e} &= 0. \end{aligned}$$

Using the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ and interpolations, (4.22) implies

$$\int_{P_{1/2}(0,0)} |\mathbf{v}|^3 + |q|^{\frac{3}{2}} + |\nabla e|^3 \leq C.$$

Furthermore, using a Cacciopoli-type estimate one has for $\theta \in (0, 1/2)$ that

$$\begin{aligned} &\theta^{-2} \int_{P_\theta(0,0)} (|\mathbf{v}|^3 + |q|^{3/2}) + \theta^{-2} \int_{P_\theta(0,0)} |\nabla e|^3 \\ &\leq C\theta^3 \left[\int_{P_1(0,0)} (|\mathbf{v}|^3 + |q|^{3/2}) + \int_{P_1(0,0)} |\nabla e|^3 \right] \\ &\leq C\theta^3 \end{aligned} \tag{4.23}$$

Recall the following lemma:

Lemma 4.2.2 (Aubin–Lions)**.** *Let $X_0 \subset X \subset X_1$ be Banach spaces such that X_0 is compactly embedded in X, X is continuously embedded in X_1, and X_0, X_1 are reflexive. Then for $1 < \alpha_0, \alpha_1 < \infty$*

$$\{\mathbf{u} \in L^{\alpha_0}(0,T;X_0) : \partial_t \mathbf{u} \in L^{\alpha_1}(0,T;X_1)\} \text{ is compactly embedded in } L^{\alpha_0}(0,T;X). \tag{4.24}$$

Claim: $\mathbf{v}_i \to \mathbf{v}$ in $L^2(P_{2/5}(0,0))$. From (4.22) one has that

$$\begin{aligned} &\|\mathbf{v}_i\|_{L^2_t H^1_x(P_{1/2})} \leq C \\ &\|\nabla \mathbf{e}_i\|_{L^2_t H^1_x(P_{1/2})} \leq C \\ &\|\mathbf{v}_i\|_{L^{10/3}_t L^{10/3}_x(P_{1/2})} \leq C \\ &\|\nabla \mathbf{e}_i\|_{L^{10/3}_t L^{10/3}_x(P_{1/2})} \leq C. \end{aligned} \tag{4.25}$$

So by Hölder's inequality

$$\int_{P_{1/2}} |\mathbf{v}_i \cdot \nabla \mathbf{v}_i|^{5/4} \leq \left\{ \int_{P_{1/2}} |\mathbf{v}_i|^{10/3} \right\}^{3/8} \left\{ \int_{P_{1/2}} |\nabla \mathbf{v}_i|^2 \right\}^{5/8} \leq C$$

and

$$\int_{P_{1/2}} |\nabla \cdot (\nabla \mathbf{e}_i \odot \nabla \mathbf{e}_i)|^{5/4} \leq \left\{ \int_{P_{1/2}} |\nabla^2 \mathbf{e}_i|^2 \right\}^{5/8} \left\{ \int_{P_{1/2}} |\nabla \mathbf{e}_i|^{10/3} \right\}^{3/8} \leq C.$$

The last inequalities imply that

$$\|\epsilon_i [\mathbf{v}_i \cdot \nabla \mathbf{v}_i + \nabla \cdot (\nabla \mathbf{e}_i \odot \nabla \mathbf{e}_i)]\|_{L^{5/4}_t L^{5/4}_x(P_{1/2})} \leq C. \tag{4.26}$$

From (4.26), the $W^{2,1}_\alpha$ estimate of the Stokes' equation implies that

$$\|\partial_t \mathbf{v}_i\|_{L^{5/4}(P_{2/5})} \leq C. \tag{4.27}$$

By (4.25) and (4.27) the sequence $\{\mathbf{v}_i\}$ is bounded in

$$\mathcal{Y}_1 = \left\{\mathbf{u} \in L_t^2 H_x^1(P_{2/5}) : \partial_t \mathbf{u} \in L_t^{5/4} L_x^{5/4}(P_{2/5})\right\}.$$

Then since $\mathcal{Y}_1$ is compactly embedded in $L_t^2 L_x^2(P_{2/5})$ by the Aubin–Lions Lemma, one has that

$$\mathbf{v}_i \to \mathbf{v} \text{ in } L_t^2 L_x^2(P_{2/5}). \tag{4.28}$$

Claim: $\nabla \mathbf{e}_i \to \nabla \mathbf{e}$ in $L^2(P_{2/5}(\mathbf{0}, 0))$. Using (4.25) and Hölder's inequality one has that

$$\int_{P_{1/2}} \left||\nabla \mathbf{e}_i|^2 \mathbf{d}_i\right|^2 = \int_{P_{1/2}} |\nabla \mathbf{e}_i|^4 \leq C$$

$$\|\mathbf{v}_i \cdot \nabla \mathbf{e}_i\|_{L^{20/11}(P_{1/2})} \leq \|\mathbf{v}_i\|_{L^{10/3}(P_{1/2})} \|\nabla \mathbf{e}_i\|_{L^4(P_{1/2})} \leq C.$$

Using interpolation the last inequalities imply that

$$\left\||\nabla \mathbf{e}_i|^2 d_i + \mathbf{v}_i \cdot \nabla \mathbf{e}_i\right\|_{L_t^{20/11} L_x^{20/11}(P_{1/2})} \leq C. \tag{4.29}$$

With (4.29) the $W_\alpha^{2,1}$ estimate for the heat equation implies that

$$\|\partial_t \mathbf{e}_i\|_{L_t^{20/11} L_x^{20/11}(P_{2/5})} \leq C.$$

Since $\nabla(L_x^{20/11}) = W_x^{-1,20/9}$ one has from the last inequality that

$$\|\partial_t \mathbf{e}_i\|_{L_t^{20/11} W_x^{-1,20/9}(P_{2/5})} \leq C. \tag{4.30}$$

By (4.25) and (4.30) the sequence $\{\nabla \mathbf{e}_i\}$ is bounded in

$$\mathcal{Y}_2 = \left\{\mathbf{u} \in L_t^2 H_x^1(P_{2/5}) : \partial_t \mathbf{u} \in L_t^{20/11} W_x^{-1,20/9}(P_{2/5})\right\},$$

and so by the Aubin–Lions lemma, one has that

$$\nabla \mathbf{e}_i \to \nabla \mathbf{e} \text{ in } L_t^2 L_x^2(P_{2/5}). \tag{4.31}$$

Additionally, by interpolation one can show that

$$\mathbf{v}_i \to \mathbf{v}, \nabla \mathbf{e}_i \to \nabla e \text{ in } L_t^3 L_x^3(P_{2/5}). \tag{4.32}$$

That is, for $\mathbf{v}_i$ one has by (4.28) that

$$\begin{aligned}\|\mathbf{v}_i - \mathbf{v}\|_{L_t^3 L_x^3(P_{2/5})} &\leq \|\mathbf{v}_i - \mathbf{v}\|_{L_t^2 L_x^2(P_{2/5})}^{1/6} \|\mathbf{v}_i - \mathbf{v}\|_{L_t^{10/3} L_x^{10/3}(P_{2/5})}^{5/6} \\ &\leq C \|\mathbf{v}_i - \mathbf{v}\|_{L_t^2 L_x^2(P_{2/5})}^{1/6} \to 0\end{aligned}$$

as $i \to \infty$. A similar result holds for $\nabla \mathbf{e}_i$ via (4.31).

From (4.32), for any $\theta \in (0, 1/4)$ and i sufficiently large, one has by (4.23) that

$$\begin{aligned} &\theta^{-2}\int_{P_\theta(0,0)}|\mathbf{v}_i|^3+\theta^{-2}\int_{P_\theta(0,0)}|\nabla\mathbf{e}_i|^3 \\ &\leq \theta^{-2}\int_{P_\theta(0,0)}|\mathbf{v}|^3+\theta^{-2}\theta^{-2}\int_{P_\theta(0,0)}|\nabla\mathbf{e}|^3+o(1)\leq C\theta^3. \end{aligned} \tag{4.33}$$

Finally using the pressure estimate (4.35) below with $\tau = \theta^2$, $r = \theta$ one has that

$$(\theta^2)^{-2}\int_{P_{\theta^2}(0,0)}|q_i|^{3/2}\leq C\left[\theta^{-4}\int_{P_\theta(0,0)}(|\mathbf{v}_i|^3+|\nabla\mathbf{e}_i|^3)+\theta\int_{P_\theta(0,0)}|q_i|^{3/2}\right]$$

for $\theta \in (0, 1/4)$. Using (4.32) and (4.22) the last inequality implies that

$$(\theta^2)^{-2}\int_{P_{\theta^2}(0,0)}|q_i|^{3/2}\leq C\left[\theta^{-2}\theta^3+\frac{\theta^3}{\theta^2}\right]\leq C\theta. \tag{4.34}$$

Combining (4.33) and (4.34) with sufficiently large $i = i(\theta)$ yields that

$$(\theta^{-2})^{-2}\int_{P_{\theta^2}(0,0)}|\mathbf{v}_i|^3+|\nabla\mathbf{e}_i|^3+|q_i|^{3/2}\leq C\theta.$$

This last inequality contradicts (4.15). □

Lemma 4.2.3. *Suppose that $(\mathbf{u}, p, \mathbf{d})$ is a suitable weak solution of the liquid crystal flow problem on $P_r(\mathbf{x}_0, t_0)$. Then for any $\tau \in (0, \frac{r}{2})$, it holds that*

$$\begin{aligned} &\frac{1}{\tau^2}\int_{P_\tau(x_0,t_0)}|p|^{3/2} \\ &\leq C\left[\left(\frac{r}{\tau}\right)^2\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)}(|\mathbf{u}-\mathbf{u}_{\mathbf{x}_0,r}(t)|^3+|\nabla\mathbf{d}|^3)+\left(\frac{\tau}{r}\right)^3\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)}|p|^{3/2}\right] \end{aligned} \tag{4.35}$$

where $\mathbf{u}_{\mathbf{x}_0,r}(t) = \frac{1}{|B_r(\mathbf{x}_0)|}\int_{B_r(\mathbf{x}_0)}\mathbf{u}(\mathbf{x}, t)d\mathbf{x}$ for $t_0 - r^2 \leq t \leq t_0$. In particular, it holds that

$$\begin{aligned} \frac{1}{\tau^2}\int_{P_\tau(\mathbf{x}_0,t_0)}|p|^{3/2}\leq{}& C\left(\frac{r}{\tau}\right)^2\left(\sup_{t_0-r^2\leq t\leq t_0}\frac{1}{r}\int_{B_r(\mathbf{x}_0)}|\mathbf{u}|^2\right)^{3/4}\left(\frac{1}{r}\int_{P_r(\mathbf{x}_0,t_0)}|\nabla\mathbf{u}|^2\right)^{3/4} \\ &+C\left[\left(\frac{r}{\tau}\right)^2\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)}|\nabla\mathbf{d}|^3+\left(\frac{\tau}{r}\right)^3\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)}|p|^{3/2}\right] \end{aligned} \tag{4.36}$$

Proof. By rescaling it is assumed without loss of generality that $r = 1$ and $(\mathbf{x}_0, t_0) = (0, 0)$. Using the divergence-free condition on $\mathbf{u}$

$$\begin{aligned}
&\operatorname{div}\operatorname{div}\{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t))\} \\
&= \nabla_j \nabla_i \left\{(\mathbf{u} - \mathbf{u}_{0,1}(t))^i (\mathbf{u} - \mathbf{u}_{0,1}(t))^j\right\} \\
&= \nabla_j \left\{(\mathbf{u} - \mathbf{u}_{0,1}(t))^i \nabla_i (\mathbf{u} - \mathbf{u}_{0,1}(t))^j\right\} \\
&= \nabla_j \left\{(\mathbf{u} - \mathbf{u}_{0,1}(t))^i (\nabla_i \mathbf{u}^j)\right\} \\
&= \left\{\nabla_j (\mathbf{u} - \mathbf{u}_{0,1}(t))^i (\nabla_i \mathbf{u}^j)\right\} + \left\{(\mathbf{u} - \mathbf{u}_{0,1}(t))^i (\nabla_j \nabla_i \mathbf{u}^j)\right\} \\
&= \left\{(\nabla_j \mathbf{u}^i)(\nabla_i \mathbf{u}^j)\right\} \\
&= \nabla_j \nabla_i (\mathbf{u}^i \mathbf{u}^j) \\
&= \operatorname{div}\operatorname{div}(\mathbf{u} \otimes \mathbf{u}).
\end{aligned}$$

Taking the divergence of (1.7) yields the pressure Poisson equation:

$$\Delta p = -\operatorname{div}\operatorname{div}\{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function of $B_{1/2}(0)$. That is, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{1/2}(0)$, $\eta \equiv 0$ outside $B_1(0)$ and $|\nabla \eta| \le C$. Define $\widetilde{p}$ by

$$\begin{aligned}
&\widetilde{p}(\mathbf{x}, t) = \\
&-\int_{\mathbb{R}^3} \nabla_{\mathbf{y}}^2 G(\mathbf{x} - \mathbf{y}) : \eta^2(\mathbf{y}) \{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\} (\mathbf{y}, t) d\mathbf{y}
\end{aligned}$$

where G is a fundamental solution of Laplace's equation on $\mathbb{R}^3$. Thus, integration by parts twice yields that

$$\begin{aligned}
&\Delta \widetilde{p} \\
&= -\int_{\mathbb{R}^3} \Delta G(\mathbf{x} - \mathbf{y}) \operatorname{div}\operatorname{div} \eta^2(\mathbf{y}) \{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\} (\mathbf{y}, t) d\mathbf{y} \\
&= -\int_{\mathbb{R}^3} \delta(\mathbf{x} - \mathbf{y}) \operatorname{div}\operatorname{div} \eta^2(\mathbf{y}) \{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\} (\mathbf{y}, t) d\mathbf{y} \\
&= \operatorname{div}\operatorname{div}\{(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\}.
\end{aligned}$$

Using the L^p Calderon-Zygmund inequality one has that

$$\begin{aligned}
\int_{B_\tau(0)} |\widetilde{p}(t)|^{3/2} &\le \int_{\mathbb{R}^3} |\widetilde{p}(t)|^{3/2} \\
&\le C \int_{\mathbb{R}^3} \eta^3 \left|(\mathbf{u} - \mathbf{u}_{0,1}(t)) \otimes (\mathbf{u} - \mathbf{u}_{0,1}(t)) + \nabla \mathbf{d} \odot \nabla \mathbf{d}\right|^{3/2} \\
&\le C \int_{B_1(0)} \left(|\mathbf{u} - \mathbf{u}_{0,1}(t)|^3 + |\nabla \mathbf{d}|^3\right).
\end{aligned}$$

By integrating the last inequality over $t \in (-\tau^2, 0)$ one has that

$$\frac{1}{\tau^2} \int_{P_\tau(0,0)} |\widetilde{p}|^{3/2} \le \frac{C}{\tau^2} \int_{P_1(0,0)} (|\mathbf{u} - \mathbf{u}_{0,1}(t)|^3 + |\nabla \mathbf{d}|^3). \tag{4.37}$$

The function $q := p - \widetilde{p} \in L^{3/2}(P_1(\mathbf{0}, 0))$ satisfies

$$\Delta q(t) = 0 \text{ in } B_{1/2}(0)$$

for $t \in [-1/4, 0]$. Now, by the Harnack inequality,

$$\begin{aligned}\frac{1}{\tau^2}\int_{B_\tau} |q|^{3/2} &\le C\tau^3 \int_{B_{1/2}} |q|^{3/2} \\ &\le C\tau^3 \left[\int_{B_1} |p|^{3/2} + \int_{B_1} |\widetilde{p}|^{3/2}\right] \\ &\le C\tau^3 \left[\int_{B_1} |p|^{3/2} + \int_{B_1} |\mathbf{u} - \mathbf{u}_{0,1}(t)|^3 + |\nabla \mathbf{d}|^3\right].\end{aligned}$$

Integrating this inequality over $t \in [-\tau^2, 0]$ implies that

$$\frac{1}{\tau^2}\int_{P_\tau(0,0)} |q|^{3/2} \le C\tau^3 \left[\int_{P_1(0,0)} |p|^{3/2} + \int_{P_1(0,0)} |\mathbf{u} - \mathbf{u}_{0,1}(t)|^3 + |\nabla \mathbf{d}|^3\right]. \tag{4.38}$$

The inequality (4.35) follows from adding the inequalities (4.37) and (4.38) and observing that $\tau \in (0, 1/2)$. That is, from (4.37) and (4.38) one has that

$$\begin{aligned}\frac{1}{\tau^2}\int_{P_\tau(0,0)} |p|^{3/2} &\lesssim \frac{1}{\tau^2}\int_{P_\tau(0,0)} |q|^{3/2} + |\widetilde{p}|^{3/2} \\ &\le C\tau^{-2}\int_{P_1(0,0)} |p|^{3/2} + C\tau^3 \int_{P_1(0,0)} |\mathbf{u} - \mathbf{u}_{0,1}(t)|^3 + |\nabla \mathbf{d}|^3.\end{aligned} \tag{4.39}$$

Using interpolation and the Sobolev inequality one has that

$$\begin{aligned}\int_{B_1} |\mathbf{u} - \mathbf{u}_{0,1}|^3 &\le \left(\int_{B_1} |\mathbf{u} - \mathbf{u}_{0,1}|^2\right)^{\frac{3}{4}} \left(\int_{B_1} |\mathbf{u} - \mathbf{u}_{0,1}|^6\right)^{\frac{3}{4}} \\ &\le C\left(\int_{B_1} |\mathbf{u}|^2\right)^{\frac{3}{4}} \left(\int_{B_1} |\nabla \mathbf{u}|^2\right)^{\frac{3}{4}}.\end{aligned} \tag{4.40}$$

Inserting, (4.40) into (4.39) yields the estimate (4.36). □

Corollary 4.2.4. *Under the same assumptions as Lemma 4.2.1, there exists* $\alpha \in (0, 1)$ *such that for any* $0 < \tau < r$*, it holds that*

$$\begin{aligned}&\left(\frac{1}{\tau^2}\int_{P_\tau(\mathbf{x}_0,t_0)} |\mathbf{u}|^3\right)^{1/3} + \left(\frac{1}{\tau^2}\int_{P_\tau(\mathbf{x}_0,t_0)} |p|^{3/2}\right)^{2/3} + \left(\frac{1}{\tau^2}\int_{P_\tau(\mathbf{x}_0,t_0)} |\nabla \mathbf{d}|^3\right)^{1/3} \\ &\le C\left(\frac{\tau}{r}\right)^\alpha \\ &\times \left[\left(\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)} |\mathbf{u}|^3\right)^{1/3} + \left(\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)} |p|^{3/2}\right)^{2/3} + \left(\frac{1}{r^2}\int_{P_r(\mathbf{x}_0,t_0)} |\nabla \mathbf{d}|^3\right)^{1/3}\right].\end{aligned} \tag{4.41}$$

Proof. Let $P_\rho = P_\rho(\mathbf{x}_0, t_0)$ and define Φ by

$$\Phi(\rho) := \left(\frac{1}{\rho^2}\int_{P_\rho} |\mathbf{u}|^3\right)^{1/3} + \left(\frac{1}{\rho^2}\int_{P_\rho} |p|^{3/2}\right)^{2/3} + \left(\frac{1}{\rho^2}\int_{P_\rho} |\nabla \mathbf{d}|^3\right)^{1/3}$$

Let ϵ_0 be given by Lemma 4.2.1. Then Lemma 4.2.1 implies that there exists $\theta_0 \in (0, 1/2)$ so that

$$\Phi(\theta_0 r) \leq \frac{1}{2}\Phi(r) \leq \frac{1}{2}\epsilon_0 \tag{4.42}$$

for $\theta \in (0, 1/2)$. Iterating (4.42) k-times yields:

$$\Phi(\theta_0^k r) \leq 2^{-k}\Phi(r). \tag{4.43}$$

Now setting $\tau = \theta_0^k r$ one has that

$$2^{-k} = \left(2^{\log_2(\tau/r)}\right)^{\frac{1}{\log_2(1/\theta)}} = \left(\frac{\tau}{r}\right)^{\frac{1}{\log_2(1/\theta)}}$$

where $\alpha = \frac{1}{\log_2(1/\theta)} \in (0, 1)$. Thus, from (4.43) one has the desired result. □

The last step in proving the smoothness of solutions to (1.7)-(1.9) will use estimates of Riesz potential between Morrey spaces. Here, the framework of Huang and Wang [14] is used. For other applications of Morrey spaces see the survey by Adams and Xiao [1], the book by Morrey [21], or the book of Giaquinta [12].

Definition 4.2.5 (Morrey Spaces)**.** *For $U \subset \mathbb{R}^{3+1}$, $1 \leq p < +\infty$, $0 \leq \lambda \leq 5$, define the* Morrey Space *$M^{p,\lambda}(U)$ by*

$$M^{p,\lambda}(U) := \left\{ f \in L^p_{loc}(U) : \|f\|^p_{M^{p,\lambda}(U)} \equiv \sup_{z \in U, r > 0} r^{\lambda - 5}\int_{P_r(z)\cap U} |f|^p < \infty \right\}. \tag{4.44}$$

Theorem 4.2.6. *Suppose that $(\mathbf{u}, p, d)$ is a suitable weak solution to* (1.7)-(1.9)*. There exists $\epsilon_0 > 0$ such that if*

$$\left(r_0^{-2}\int_{P_{r_0}(\mathbf{x}_0,t_0)} |\mathbf{u}|^3\right)^{\frac{1}{3}} + \left(r_0^{-2}\int_{P_{r_0}(\mathbf{x}_0,t_0)} |p|^{\frac{3}{2}}\right)^{\frac{2}{3}} + \left(r_0^{-2}\int_{P_{r_0}(\mathbf{x}_0,t_0)} |\nabla \mathbf{d}|^3\right)^{\frac{1}{3}} \leq \epsilon_0$$

and

$$\sup_{t_0 - r_0^2 \leq t \leq t_0} \left(\int_{B_{r_0}} |\nabla \mathbf{d}|^3(t)\right)^{1/3} < \mathcal{C}(3),$$

then $(\mathbf{u}, d) \in C^\infty(P_{r_0/2}(\mathbf{x}_0, t_0))$. Moreover one has the estimate:

$$\|(\mathbf{u}, d)\|_{C^k(P_{r_0/2}(\mathbf{x}_0,t_0))} \leq C(k, \epsilon_0, r_0). \tag{4.45}$$

Proof. By Corollary 4.2.4

$$\mathbf{u}, \nabla \mathbf{d} \in M^{3,3(1-\alpha)}\left(P_{r/2}(\mathbf{x}_0, t_0)\right), \text{ for some } 0 < \alpha < 1. \tag{4.46}$$

Writing the equation for the director $\mathbf{d}$ as

$$\partial_t \mathbf{d} - \Delta \mathbf{d} = \mathbf{f}, \ \mathbf{f} := (|\nabla \mathbf{d}|^2 d - \mathbf{u} \cdot \nabla \mathbf{d})$$

it is seen from (4.46) that

$$\mathbf{f} \in M^{3/2,3(1-\alpha)}\left(P_{r/2}(\mathbf{x}_0, t_0)\right), \text{ for some } 0 < \alpha < 1.$$

Now as in [14] let $\eta \in C_0^\infty(\mathbb{R}^{3+1})$ be a cutoff function of $P_{r/2}(\mathbf{x}_0, t_0)$ and set $\mathbf{w} = \eta^2 \mathbf{d}$. Then one has that

$$\partial_t \mathbf{w} - \Delta \mathbf{w} = \eta^2 \mathbf{f} + [\partial_t \eta^2 - \Delta \eta^2]\mathbf{d} - 2\nabla \eta^2 \cdot \nabla \mathbf{d} =: \mathbf{F} \tag{4.47}$$

One immediately has the that $\mathbf{F} \in M^{3/2,3(1-\alpha)}(\mathbb{R}^{3+1})$ with the estimate:

$$\|\mathbf{F}\|_{M^{3/2,3(1-\alpha)}(\mathbb{R}^{n+1})} \leq C\left[1 + \|\mathbf{f}\|_{M^{3/2,3(1-\alpha)}(P_{r/2}(\mathbf{x}_0,t_0))}\right].$$

Following [14] define the parabolic metric $\delta : \mathbb{R}^{3+1} \times \mathbb{R}^{3+1} \to \mathbb{R}$ by

$$\delta((\mathbf{x}, t), (\mathbf{y}, s)) = \max\left\{|\mathbf{x} - \mathbf{y}|, \sqrt{|t - s|}\right\}. \tag{4.48}$$

Immediately from Lemma 3.1 [14] one has that

$$|\nabla \Gamma(\mathbf{x}, t)| \leq \frac{C}{\delta((\mathbf{x}, t), (0, 0))^{3+1}} \text{ for all } (\mathbf{x}, t) \in \mathbb{R}^{3+1} \tag{4.49}$$

where Γ is the heat kernel in $\mathbb{R}^3$. By Duhamel's formula, and (4.49),

$$\begin{aligned} |\nabla \mathbf{w}(\mathbf{x}, t)| &\leq \int_0^t \int_{\mathbb{R}^3} |\nabla \Gamma(\mathbf{x} - \mathbf{y}, t - s)||\mathbf{F}(\mathbf{y}, s)| \\ &\leq C \int_0^t \int_{\mathbb{R}^4} \frac{|\mathbf{F}(\mathbf{y}, s)|}{\delta((\mathbf{x}, t), (\mathbf{y}, s))^{3+1}} \\ &= C\widetilde{I}_1(|\mathbf{F}|)(\mathbf{x}, t) \end{aligned} \tag{4.50}$$

where $\widetilde{I}_\beta$ is the Riesz potential of order $\beta \in [0, 5]$ on $\mathbb{R}^{3+1}$ defined by

$$\widetilde{I}_\beta(g) = \int_{\mathbb{R}^{3+1}} \frac{|g(\mathbf{y}, s)|}{\delta((\mathbf{x}, t), (\mathbf{y}, s))^{5-\beta}}, \ g \in L^p(R^{3+1}). \tag{4.51}$$

Now by Theorem 3.1 of [14] one concludes that $\nabla \mathbf{w} \in M^{\frac{3(1-\alpha)}{1-2\alpha}}(\mathbb{R}^{3+1})$ and that

$$\|\nabla \mathbf{w}\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}}(\mathbb{R}^{3+1})} \leq C\|\mathbf{F}\|_{M^{3/2,3(1-\alpha)}(\mathbb{R}^{n+1})}. \tag{4.52}$$

Observing that $\lim_{\alpha \uparrow \frac{1}{2}} \frac{3(1-\alpha)}{1-2\alpha} = +\infty$, one sees that

$$\nabla \mathbf{d} \in L^m \left(P_{r/2}(\mathbf{x}_0, t_0) \right) \text{ for all } 1 < m < \infty. \tag{4.53}$$

For equation (1.7), consider the auxiliary Stokes equation in $\mathbb{R}^{3+1}$, that is,

$$\begin{aligned} \partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla q = -\nabla \cdot [\eta^2 (\nabla \mathbf{d} \odot \nabla \mathbf{d} + \mathbf{u} \otimes \mathbf{u})] & \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\ \nabla \cdot v = 0 & \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\ \mathbf{v}(\cdot, t) = 0 & \quad \text{in } \mathbb{R}^3. \end{aligned} \tag{4.54}$$

A similar estimate to (4.50) is obtained for the Oseen kernel [16], namely,

$$|\mathbf{v}(\mathbf{x}, t)| \leq \int_0^t \int_{\mathbb{R}^{3+1}} \frac{|X(\mathbf{y}, s)|}{\delta((\mathbf{x}, t), (\mathbf{y}, s))^{3+1}} = \widetilde{I}_1(|X|)(\mathbf{x}, t), \tag{4.55}$$

where $\mathbf{X} = \eta^2 (\nabla \mathbf{d} \odot \nabla \mathbf{d} + \mathbf{u} \otimes \mathbf{u})$. As above, $X \in M^{3/2, 3(1-\alpha)}(\mathbb{R}^{3+1})$ and

$$\|\mathbf{X}\|_{M^{3/2,3(1-\alpha)}(\mathbb{R}^{3+1})} \leq C \left[\|\nabla \mathbf{d}\|_{M^{3,3(1-\alpha)}(P_{r/2}(\mathbf{x}_0, t_0))} + \|\mathbf{u}\|_{M^{3,3(1-\alpha)}(P_{r/2}(\mathbf{x}_0, t_0))} \right].$$

Again by Theorem 3.1 of [14], $\mathbf{v} \in M^{\frac{3(1-\alpha)}{1-2\alpha}}(\mathbb{R}^{3+1})$ and,

$$\|\mathbf{v}\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}}(\mathbb{R}^{3+1})} \leq C \|\mathbf{X}\|_{M^{3/2,3(1-\alpha)}(\mathbb{R}^{n+1})}. \tag{4.56}$$

In particular, $\mathbf{v} \in L^m(\mathbb{R}^{3+1})$ for $1 < m < \infty$. Note that $\mathbf{w} = \mathbf{u} - \mathbf{v}$ satisfies the homogeneous Stokes equation in $P_{r/2}(\mathbf{x}_0, t_0)$, that is,

$$\partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla(p - q) = 0, \ \nabla \cdot \mathbf{w} = 0 \text{ in } P_{r/2}(\mathbf{x}_0, t_0)$$

It is well-known that $\mathbf{w} \in L^\infty(P_{r/4}(\mathbf{x}_0, t_0))$. Therefore we conclude that

$$\mathbf{u} \in L^m \left(P_{r/4}(\mathbf{x}_0, t_0) \right) \text{ for } 1 < m < \infty. \tag{4.57}$$

The higher order regularity of $(\mathbf{u}, \mathbf{d})$ follows from linear theory, namely, the $W_p^{2,1}$-theory. □

In this final chapter, the proof of the main result of this manuscript is completed. The proof relies upon the short time existence and uniqueness theorem for smooth initial data of Lin, Lin, and Wang [19].

Theorem 5.0.7 (Lin–Lin–Wang, [19])**.** *For any $\alpha > 0$, if $\mathbf{u}_0 \in C_0^{2,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{d}_0 \in C^{2,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$, then there exists $T > 0$ depending on $\|\mathbf{u}_0\|_{C^{2,\alpha}(\mathbb{R}^3)}$, $\|\mathbf{d}_0\|_{C^{2,\alpha}(\mathbb{R}^3)}$ such that there is a unique smooth solution $(u, d) \in C_\alpha^{2,1}(\mathbb{R}^3 \times [0, T), \mathbb{R}^3 \times \mathbb{S}^2)$ to the initial value problem* (1.7)-(1.9)*.*

The existence and uniqueness claimed in Theorem 1.3.2 is proven by a weak-strong type argument. Such an argument exploits the relationship between weak and strong solutions. This line of reasoning was first employed by Leray in [16] for the Navier–Stokes equations and was also employed in the recent work of Lin, Lin, and Wang [19]. In the case at hand, Theorem 5.0.7 guarantees the existence of unique strong solutions to (1.7)-(1.9) for smooth data. One may then construct a sequence of smooth solutions arising from smooth data that converge to solutions of (1.7)-(1.10).

To start the weak-strong argument, let $(\mathbf{u}_0, \mathbf{d}_0)$ satisfy

$$\|\mathbf{u}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \leq \epsilon_0.$$

Let η be a standard mollifier and define the sequences

$$\mathbf{u}_0^k := \eta_{1/k} * \mathbf{u}_0 \text{ and } \mathbf{d}_0^k := \eta_{1/k} * \mathbf{d}_0. \tag{5.1}$$

It follows from Young's inequality that

$$(\mathbf{u}_0^k, \mathbf{d}_0^k) \to (\mathbf{u}_0, \mathbf{d}_0) \text{ in } L^3_{loc}(\mathbb{R}^3) \times W^{1,3}_{loc}(\mathbb{R}^3)$$

and

$$\|\mathbf{u}_0^k\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0^k\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \leq \|\mathbf{u}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \leq \epsilon_0.$$

For all $x \in \mathbb{R}^3$ one has that

$$\operatorname{dist}(\mathbf{d}_0^k(x), S^2) \leq |\eta_{1/k} * \mathbf{d}_0(x) - \mathbf{d}_0(y)| \text{ for all } y \in B_1(x).$$

Then by integrating over $B_1(x)$ and applying the modified Poincaré inequality one has that

$$\begin{aligned}
\operatorname{dist}(\mathbf{d}_0^k(x), S^2) &\leq \fint_{B_1(x)} |\eta_{1/k} * \mathbf{d}_0(x) - \mathbf{d}_0(y)| dy \\
&\lesssim \fint_{B_1(x)} |\nabla \mathbf{d}_0(y)| dy \\
&\lesssim \|\nabla \mathbf{d}_0\|_{L^3(B_1(x))} \\
&\leq \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \\
&\leq \epsilon_0.
\end{aligned}$$

Since $\mathbf{d}_0^k$ remains close to $\mathbb{S}^2$ it may be projected onto $\mathbb{S}^2$. Define $\widetilde{\mathbf{d}_0^k}$ by $\widetilde{\mathbf{d}_0^k} = \pi(\mathbf{d}_0^k)$ where $\pi : \mathbb{R}^3 \to \mathbb{S}^2$ by $\pi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$. One has that

$$|\nabla\widetilde{\mathbf{d}_0^k}| = |\nabla\pi(\mathbf{d}_0^k)\nabla\mathbf{d}_0^k| \le C|\nabla\mathbf{d}_0^k| \tag{5.2}$$

and so by the dominated convergence theorem

$$\nabla\widetilde{\mathbf{d}_0^k} \to \nabla\mathbf{d}_0 \text{ in } L^3_{loc}(\mathbb{R}^3).$$

Next, characterize the lifespan of smoothed initial data, that is:

Lemma 5.0.8 (Lifespan characterization)**.** *Let* $(\mathbf{u}_0, \mathbf{d}_0)$ *be smooth initial data of* (1.7)-(1.9) *satisfying*

$$\|\mathbf{u}_0\|^3_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla\mathbf{d}_0\|^3_{L^3_{\mathbb{U}}(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3}\int_{B_2(x)} |\mathbf{u}_0|^3 + |\nabla\mathbf{d}_0|^3 \le \epsilon_0^3.$$

There exists $T > 0$ *depending only on* ϵ_0 *and a unique solution* $(\mathbf{u}, \mathbf{d})$ *to* (1.7)-(1.9) *such that*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^3}\int_{B_1(x)} |\mathbf{u}|^3 + |\nabla\mathbf{d}|^3 \le 2\epsilon_0^3.$$

Proof. By Theorem 5.0.7 there is $T > 0$ and a smooth solution to (1.7)-(1.9) with initial data $(\mathbf{u}_0, \mathbf{d}_0)$. Let $0 < t^* \le T$ be the maximal time such that

$$\sup_{0\le t\le t^*}\sup_{\mathbf{x}\in\mathbb{R}^3}\int_{B_1(\mathbf{x})} (|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)(\cdot, t) \le 2\epsilon_0^3. \tag{5.3}$$

Since t^* is the maximal time

$$\sup_{\mathbf{x}\in\mathbb{R}^3}\int_{B_1(\mathbf{x})} (|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)(\cdot, t^*) = 2\epsilon_0^3. \tag{5.4}$$

For $\mathbf{x} \in \mathbb{R}^3$, let $\phi \in C_0^1(B_2(\mathbf{x}))$ be a cut-off function such that

$$0 \le \phi \le 1,\ \phi \equiv 1 \text{ on } B_1(\mathbf{x}),\ \phi \equiv 0 \text{ outside } B_2(\mathbf{x}), \text{ and } |\nabla\phi| \le 4 \tag{5.5}$$

Define

$$E_2^{\mathbf{x}}(t) := \int_{B_2(\mathbf{x})} \left[(|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)\phi\right](\cdot, t), \tag{5.6}$$

then the local energy inequality (3.32) implies that

$$\begin{aligned}
&\frac{d}{dt}E_2^{\mathbf{x}}(t) + (1 - C\epsilon_0^2)\int_{\mathbb{R}^3} [|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla\mathbf{d}|^{3/2}\phi)|^2] \\
&\le C\int_{\mathbb{R}^3} (|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)|\nabla\phi|^2 + C\int_{\mathbb{R}^3} |\mathbf{u}||P - c|^2\phi^2 \\
&\le C\epsilon_0^3 + C\|\mathbf{u}\|_{L^3(B_2(\mathbf{x}))}\|P - c\|^2_{L^3(B_2(\mathbf{x}))}
\end{aligned} \tag{5.7}$$

for $0 \le t \le t^*$. While Lemma 3.2.2 implies

$$\|P - c\|^2_{L^3(B_2(\mathbf{x}))} \le C\||\mathbf{u}| + |\nabla \mathbf{d}|\|^4_{L^6(B_4(\mathbf{x}))} + C\epsilon_0^4. \tag{5.8}$$

Using norm interpolation and the Sobolev inequality one has that

$$\begin{aligned}
&\||\mathbf{u}| + |\nabla \mathbf{d}|\|^4_{L^6(B_4(\mathbf{x}))} \\
&\le \||\mathbf{u}| + |\nabla \mathbf{d}|\|_{L^3(B_4(\mathbf{x}))}\||\mathbf{u}| + |\nabla \mathbf{d}|\|^3_{L^9(B_4(\mathbf{x}))} \\
&\lesssim \||\mathbf{u}| + |\nabla \mathbf{d}|\|_{L^3(B_4(\mathbf{x}))}\||\nabla(|\mathbf{u}|^{3/2})| + |\nabla(|\nabla \mathbf{d}|^{3/2})|\|^2_{L^2(B_4(\mathbf{x}))}.
\end{aligned} \tag{5.9}$$

From (5.8) and (5.9) one has in (5.7) that

$$\begin{aligned}
&\frac{d}{dt}E_2^{\mathbf{x}}(t) + (1 - C\epsilon_0^2)\int_{B_1(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2})|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2})|^2] \\
&\le C\epsilon_0^3 + C\||\mathbf{u}| + |\nabla \mathbf{d}|\|^2_{L^3_U(\mathbb{R}^3)}\||\nabla(|\mathbf{u}|^{3/2})| + |\nabla(|\nabla \mathbf{d}|^{3/2})|\|^2_{L^2(B_4(\mathbf{x}))}.
\end{aligned} \tag{5.10}$$

Integrating with respect to $t \in [0, t^*]$ yields that

$$\begin{aligned}
&E_2^{\mathbf{x}}(t^*) + (1 - C\epsilon_0^2)\int_0^{t^*}\int_{B_1(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2] \\
&\le C\epsilon_0^3 t^* + C\epsilon_0^2\int_0^{t^*}\int_{B_4(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2})|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2})|^2] + E_2^{\mathbf{x}}(0).
\end{aligned} \tag{5.11}$$

Taking the supremum over $\mathbf{x} \in \mathbb{R}^3$ in the last inequality,

$$\begin{aligned}
&\||\mathbf{u}(t^*)| + |\nabla \mathbf{d}(t^*)|\|^3_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \\
&+ (1 - C\epsilon_0^2)\sup_{\mathbf{x}\in\mathbb{R}^3}\int_0^{t^*}\int_{B_1(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2] \\
&\le C\epsilon_0^3 t^* + C\epsilon_0^2\sup_{\mathbf{x}\in\mathbb{R}^3}\int_0^{t^*}\int_{B_4(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2})|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2})|^2] + \epsilon_0^3.
\end{aligned} \tag{5.12}$$

Since $B_4(\mathbf{x})$ can be covered by finite number of balls of radius one,

$$\begin{aligned}
&\sup_{\mathbf{x}\in\mathbb{R}^3}\int_0^{t^*}\int_{B_4(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2] \\
&\le 4^4\sup_{\mathbf{x}\in\mathbb{R}^3}\int_0^{t^*}\int_{B_1(\mathbf{x})}[|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla \mathbf{d}|^{3/2}\phi)|^2].
\end{aligned} \tag{5.13}$$

With $\epsilon_0 > 0$ small enough, (5.12) yields that

$$\||\mathbf{u}(t^*)| + |\nabla \mathbf{d}(t^*)|\|^3_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \le C\epsilon_0^3 t^* + \epsilon_0^3. \tag{5.14}$$

By the choice of t^* the last inequality provides a lower bound for t^*, that is,

$$\epsilon_0^3 \le C\epsilon_0^3 t^* \;\Leftrightarrow\; \frac{1}{C} \le t^*. \tag{5.15}$$

□

Finally, all the pieces are in place to prove:

Theorem 5.0.9 (Well-posedness). *There exist $\epsilon_0 > 0$ and $T_0 > 0$ such that if $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{d}_0 : \mathbb{R}^3 \to S^2$ satisfy*

$$\|\mathbf{u}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \leq \epsilon_0,$$

then there exists a unique solution $(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times [0, T_0) \to \mathbb{R}^3 \times S^2$ of (1.7)-(1.9) *with the following properties:*

- $(\mathbf{u}, \nabla \mathbf{d}) \in C([0, T_0), L^3_{\mathbb{U}}(\mathbb{R}^3))$*;*
- $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, T_0))$*.*

Furthermore, if $T_0 < +\infty$ is the maximum time interval, then

$$\lim_{t \uparrow T_0} \|\mathbf{u}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} > \epsilon_0.$$

Proof. One may apply the Lemma 5.0.8 to mollified initial data described earlier in the section. The lemma implies that there exists a sequence of smooth solutions $(\mathbf{u}^k, \mathbf{d}^k)$ corresponding to the initial data $(\mathbf{u}^k_0, \mathbf{d}^k_0)$ existing on $[0, T_0]$ (T_0 independent of k) and such that

$$\sup_{0 \leq t \leq T_0} \sup_{\mathbf{x} \in \mathbb{R}^3} \int_{B_1(\mathbf{x})} |\mathbf{u}^k|^3 + |\nabla \mathbf{d}^k|^3 \leq 2\epsilon_0^3.$$

Theorem 4.2.6 implies that for all $R > 0$ and $0 < \delta < T_0$

$$\|(\mathbf{u}^k, \mathbf{d}^k)\|_{C^2(B_R \times (\delta, T_0))} \leq C(R, \delta).$$

The Arzela-Ascoli Theorem implies, after taking possible subsequences, that

$$(\mathbf{u}^k, \mathbf{d}^k) \to (\mathbf{u}, \mathbf{d}) \in C^2_{loc}(\mathbb{R}^3 \times (0, T_0)).$$

Since $(\mathbf{u}, \mathbf{d})$ must also satisfy the hypotheses of Theorem 4.2.6 one has that

$$(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, T_0)).$$

□

Chapter 6 Conclusions

It is worthwhile to reiterate the main result and discuss possible future directions related to systems discussed in the dissertation–namely the system (1.7)-(1.9) and the Ericksen–Leslie system. There are many other future directions in the mathematical analysis of liquid crystals that are not taken up here. For example, the mathematical analysis of model equations more tightly coupled with the molecular structure (microscopic structure) such as: Ericksen's variable degree of orientation theory [7], de Gennes' order-parameter (Q-tensor) theory [5], and the micropolar continuum theory of Eringen [8].

It has be proven in previous chapters that the system

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \text{ in } \mathbb{R}^3 \times [0, T), \tag{6.1}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \mathbb{R}^3 \times [0, T), \tag{6.2}$$

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \Delta \mathbf{d} = |\nabla \mathbf{d}|^2 \mathbf{d} \text{ in } \mathbb{R}^3 \times [0, T), \tag{6.3}$$

$$(\mathbf{u}, \mathbf{d})|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0) \text{ in } \mathbb{R}^3 \times \{0\}. \tag{6.4}$$

is well-posed for $\mathbf{u} : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ and $\mathbf{d} : \mathbb{R}^3 \times [0, T) \to S^2$ and moreover that one has the following theorem

Theorem 6.0.10 (Well-Posedness)**.** *There exists $\epsilon_0 > 0$ and $T_0 > 0$ such that if $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{d}_0 : \mathbb{R}^3 \to S^2$ satisfy*

$$\|\mathbf{u}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}_0\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} \le \epsilon_0, \tag{6.5}$$

then there exists a unique solution $(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times [0, T_0) \to \mathbb{R}^3 \times S^2$ of (6.1)-(6.4) with the following properties:

- $(\mathbf{u}, \nabla \mathbf{d}) \in C([0, T_0), L^3_{\mathbb{U}}(\mathbb{R}^3))$*;*
- $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, T_0))$*.*

Furthermore, if $T_0 < +\infty$ is the maximum time interval, then

$$\lim_{t \uparrow T_0} \|\mathbf{u}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} + \|\nabla \mathbf{d}(t)\|_{L^3_{\mathbb{U}}(\mathbb{R}^3)} > \epsilon_0.$$

The Cauchy problem (6.1)-(6.4) was chosen for a number of both physical and mathematical reasons. Furthermore, many compromises have been made between physical accuracy and mathematical tractability.

Physically, the most general fluid motions occur in three-dimension, hence the formulation in $\mathbb{R}^3$–however, for tractability, this work has been completed in all of $\mathbb{R}^3$ so to allow access to tools like the Riesz transform. Similarly, as has been already noted in many places throughout this manuscript, the system (6.1)-(6.3) is a drastic simplification of the Ericksen–Leslie equations. It does however, retain enough structure to suggest the plausibility of a similar treatment for Ericksen–Leslie equations.

Finally, this analysis was done with the equation (6.3) and not the penalized system of Lin and Liu [20], because though it is an extremely interesting problem to pursue the limits of their approximation, this problem seems much more difficult.

As for future directions the first is to tie up loose ends. Most glaringly is that the decay lemma (Lemma 4.2.1) required an assumption about the optimal Sobolev embedding constant. This was technical and for reason of controlling certain powers of the gradient of the of the rescaled director in the blow-up argument. This assumption should be able to be removed. After this, an obvious line of research would be to reintroduce enough terms to get the Ericksen–Leslie equations. This extension could be two-fold since strong simplifying assumptions have been made about both Frank's constants and the Leslie viscosities. In another direction, it would be very useful to revamp these solutions for bounded domains (here boundary conditions would have to be determined).

Bibliography

[1] D.R. Adams and J. Xiao. Morrey spaces in harmonic analysis. *Arkiv för Matematik*, pages 1–30, 2011.

[2] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the navier-stokes equations. *Communications on pure and applied mathematics*, 35(6):771–831, 1982.

[3] Y. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. *Mathematische Zeitschrift*, 201(1):83–103, 1989.

[4] YM Chen and FH Lin. Evolution of harmonic maps with dirichlet boundary conditions. *Comm. Anal. Geom*, 1(3-4):327–346, 1993.

[5] Pierre-Gilles de Gennes and J. Prost. *The Physics of Liquid Crystals*. International Series of Monographs on Physics. Clarendon Press, 1995.

[6] J. L. Ericksen. Conservation laws for liquid crystals. *Trans. Soc. Rheology*, 5:23–34, 1961.

[7] J.L. Ericksen. Liquid crystals with variable degree of orientation. *Archive for Rational Mechanics and Analysis*, 113(2):97–120, 1991.

[8] AC Eringen. A unified continuum theory of liquid crystals. *ARI-An International Journal for Physical and Engineering Sciences*, 50(2):73–84, 1997.

[9] L. Escauriaza, G.A. Seregin, and V. Sverak. $l^{3,\infty}$-solutions of the navier-stokes equations and backward uniqueness. *Russian Mathematical Surveys*, 58:211, 2003.

[10] L.C. Evans. *Partial differential equations*. Providence, Rhode Land: American Mathematical Society, 1998.

[11] F.C. Frank. I. liquid crystals. on the theory of liquid crystals. *Discuss. Faraday Soc.*, 25(0):19–28, 1958.

[12] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Princeton Univ Pr, 1983.

[13] R. Hardt, D. Kinderlehrer, and F.H. Lin. Existence and partial regularity of static liquid crystal configurations. *Communications in mathematical physics*, 105(4):547–570, 1986.

[14] Tao Huang and Changyou Wang. Notes on the regularity of harmonic map systems. *Proc. Amer. Math. Soc.*, 138(6):2015–2023, 2010.

[15] O. A. Ladyženskaja. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

[16] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta mathematica*, 63(1):193–248, 1934.

[17] F. M. Leslie. Some constitutive equations for liquid crystals. *Arch. Rational Mech. Anal.*, 28(4):265–283, 1968.

[18] F. M. Leslie. Continuum theory for nematic liquid crystals. *Contin. Mech. Thermodyn.*, 4(3):167–175, 1992.

[19] F. Lin, J. Lin, and C. Wang. Liquid crystal flows in two dimensions. *Archive for Rational Mechanics and Analysis*, 197(1):297–336, 2010.

[20] F.H. Lin and C. Liu. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Communications on Pure and Applied Mathematics*, 48(5):501–537, 1995.

[21] C.B. Morrey. *Multiple Integrals in the Calculus of Variations*. Classics in Mathematics. Springer, 2008.

[22] CW Oseen. The theory of liquid crystals. *Trans. Faraday Soc.*, 29(140):883–899, 1933.

[23] Giovanni Prodi. Un teorema di unicit per le equazioni di navier-stokes. *Annali di Matematica Pura ed Applicata*, 48:173–182, 1959. 10.1007/BF02410664.

[24] J. Rubinstein, P. Sternberg, and J.B. Keller. Fast reaction, slow diffusion, and curve shortening. *SIAM Journal on Applied Mathematics*, pages 116–133, 1989.

[25] J. Serrin. *Nonlinear problems (R. Langer ed.)*, chapter p. 69-98. University of Wisconsin Press, 1963.

[26] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series. Princeton University Press, 1970.

[27] I.W. Stewart. *The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction*. Liquid Crystals Book Series. Taylor & Francis, 2004.

[28] E.G. Virga. *Variational Theories for Liquid Crystals*. Applied Mathematics and Mathematical Computation. Chapman & Hall, 1994.

[29] Z. Wu, J. Yin, and C. Wang. *Elliptic & parabolic equations*. World Scientific Pub Co Inc, 2006.

[30] H. Zocher. The effect of a magnetic field on the nematic state. *Trans. Faraday Soc.*, 29(140):945–957, 1933.

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