2008

MODELING AND ANALYSIS OF SPLIT AND MERGE PRODUCTION SYSTEMS

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ABSTRACT OF THESIS

MODELING AND ANALYSIS OF SPLIT AND MERGE PRODUCTION SYSTEMS

Many production systems have split and merge operations to increase production capacity and variety, improve product quality, and implement product control and scheduling policies. This thesis presents analytical methods to model and analyze split and merge production systems with Bernoulli and exponential reliability machines under circulate, priority and percentage policies. The recursive procedures for performance analysis are derived, and the convergence of the procedures and uniqueness of the solutions, along with the structural properties, are proved analytically, and the accuracy of the estimation is justified numerically with high precision. In addition, comparisons among the effects of different policies in system performance are carried out.

KEYWORDS: Split, Merge, Production Rate, Bernoulli Reliability, Exponential Reliability

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PRODUCTION SYSTEMS

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RULES FOR THE USE OF THESIS

THESIS

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MODELING AND ANALYSIS OF SPLIT AND MERGE PRODUCTION SYSTEMS

THESIS

A thesis submitted in partial fulfillment of the requirements
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Chapter 1

INTRODUCTION

Substantial amount of research effort has been devoted to performance analysis of production systems for decades. Most of the studies emphasize on serial production lines or assembly systems (see reviews by Dallery and Gershwin 1992, Papadopoulos and Heavey 1996, Li *et al.* 2006, 2008 and monographs by Viswanadham and Narahari 1992, Buzacott and Shanthikumar 1993, Papadopoulos *et al.* 1993, Gershwin 1994, Altiok 1997 and Li and Meerkov 2007). In modern manufacturing systems, split and merge operations are typically used to increase production capacity and variety, improve product quality, and implement product control and scheduling policies. For example, parallel operations/lines are used to increase production volumes, defective parts are separated from main line to be either repaired or scraped, dedicated operations may be carried out for specific products, etc. To implement such operations, different split and merge policies have been adopted to pursue desired system performance. In recent years, a few performance analysis methods have been developed to model such systems (see reviews by Li *et al.* 2006, 2008 and representative papers by Helber 2000, Tan 2001, Helber and Jusic 2004, Diamantidis *et al.* 2004, Li and Huang 2005, Colledani *et al.* 2005, Li 2004a, b, c, 2005 and Diamantidis and Papadopoulos 2006). Despite of these efforts, the split and merge systems with different policies have not been studied thoroughly. Comparisons among different split and merge policies and system-theoretic properties have not been discussed.

In this thesis, Bernoulli and exponential reliability production systems with split and merge operations are considered. Three most widely used split and merge policies are considered: circulate, priority and percentage. In circulate policy, the split machine sends a part to downstream branches in circulation when it is not blocked by any of these branches. A branch will be ignored if its buffer is full, thus blocks the split machine. Similar scenario occurs in merge operations, where the merge station takes parts from all upstream branches circularly while ignoring an empty buffer branch. In priority policy, one branch has higher priority so that the split machine always dispatches parts to this branch first. Parts are sent to lower priority branches only when the split machine is blocked by the one with higher priority. Similarly, the merge station takes parts from higher priority upstream branch first.

In percentage policy, parts are dispatched to downstream branches or loaded from upstream branches based on given percentages. This thesis presents recursive analytical algorithms to analyze the performance of split and merge systems with Bernoulli and exponential reliability machines under different scheduling policies, investigates the structural properties, and compares the effectiveness of different policies.

The remaining of the thesis is structured as follows: Chapter 2 reviews the available literature. Chapter 3 gives the details about split/merge system with Bernoulli reliability machines, and Chapter 4 gives the details about split/merge system with exponential reliability machines, where the problems are formulated in Sections 3.1 and 4.1 for Bernoulli and exponential models, respectively. Detailed analysis for split and merge policies are introduced in Sections 3.2 and 3.3 for Bernoulli models respectively, and 4.2, 4.3 for exponential models respectively. Structural properties are discussed in Sections 3.4, 4.4 for Bernoulli and exponential models, respectively. Chapter 5 extends the methods introduced to split and merge systems with longer lines and multiple branches. Finally, Chapter 6 formulates the conclusions and presents some possible future research directions. All proofs are provided in the Appendix.

Chapter 2

LITERATURE REVIEW

2.1 Split and Merge

Most of the studies on modeling and analysis of production systems emphasize on serial lines and assembly systems. A few relatively recent papers address more complex structures, such as split and merge. Specifically, a transfer line with split operations is introduced in Helber 2000, and three station merge system with a shared buffer are discussed in Tan 2001, Helber and Jusic 2004, Diamantidis *et al.* 2004 and Diamantidis and Papadopoulos 2006. Papers by Li and Huang 2005 and Colledani *et al.* 2005 study multiple product systems where different products are processed at the dedicated machines or buffers. Rework split and merge are discussed in Li 2004a, b, for single and multiple loops, respectively. Li 2004c analyzes parallel systems where parallel lanes are split from a common buffer. After additional processing, the parallel lanes merge into another shared buffer. A general approach to model complex production systems, referred to as overlapping decomposition, is introduced in Li 2005. Although significant advancement has been achieved in such studies, the split and merge systems with different policies have not been studied thoroughly. In particular, more complete study on split and merge system under different policies is needed. To our best knowledge, no study has compared the impact of different split or merge policies. The structural properties of the split and merge system have not been fully investigated. Therefore, this thesis is intended to contribute in this direction.

2.2 Re-entrant Line

Re-entrant lines are widely encountered in semiconductor and electronics manufacturing industries. Most of the studies in re-entrant lines concentrates on scheduling and control policies, using queueing models, Petri net and simulation approaches.

In Kumar 1993, the author first describes a re-entrant line with multiple buffers before each service center. Under the assumption that the machines do not fail, the arrivals are described by a Possion process, the service times at service center are all exponentially distributed and the scheduling policy at each service center is First Come First Serve (FCFS).

In order to reduce the mean delay (cycle-time), the Last Buffer First Serve (LBFS) policy is introduced. However, some buffer priority policies can be unstable. Then the stability of the Last Buffer First Serve (LBFS) policy is given. It's a non-trivial fact that the LBFS policy is stable. It also shows that the FBFS policy is stable. Since reduction of the variance of the cycle-time is also often described as an important goal in semiconductor manufacturing, a Least Slack (LS) policy is introduced.

Queueing theory has been extensively used to model the manufacturing systems with re-entrant lines (Shanthikumar *et al.* 2007, Bramson and Dai 1999). In re-entrant lines, the parts which are going to be processed by different machines at different stages are similar to the customers in a multi-class queueing model. Bramson and Dai 1999 studies the queue limit at high traffic load, and proves the heavy traffic limit theorem for re-entrant lines with FBFS and LBFS policies.

Fluid model is also used to study multi-class queueing network in re-entrant lines. The fluid approximation and the stability for a multi-class queueing network are given in Chen 1995 and Dai 1995. It has been proved that a scheduling policy is stable if the corresponding fluid model is stable. The stability of First Buffer First Serve (FBFS) and Last Buffer First Served (LBFS) policies for re-entrant lines are addressed in Dai 1995.

Due to the complexity of re-entrant lines, queueing theory faces changelings. According to the literature, most of the analysis is cumbersome and is limited to the study of different scheduling policies. However, typically, only a performance bound can be obtained using queueing and fluid models, the production rate of the whole system has not been analyzed accurately.

Petri net approach provides another way of modeling re-entrant lines. Choi and Reveliotis 2003 presents an analytical framework for modeling flexibly automated re-entrant lines. However, a limitation of this approach is that it requires the enumeration of the state space, which explodes fast when the production system becomes more complex.

Since the queueing model and Petri net approach are limited to provide accurate analysis, simulations are widely applied in cycle time estimation and performance analysis of re-entrant lines (Shanthikumar *et al.* 2007). Running the computer model after many iterations could give relatively accurate result to help design and schedule the production lines, and validate any analytical model. However, intensive computation is required for simulation. Furthermore, simulation is a case study method. Therefore, it could not give insight for general scenarios.

The future research work intends to develop novel methods to evaluate re-entrant lines analytically using the results obtained in the Bernoulli and exponential split/merge system analysis.

Chapter 3

BERNOULLI PRODUCTION SYSTEMS

3.1 Problem Formulation

The typical structures of Bernoulli split and merge systems are shown in Figures 3.1 and 3.2 respectively, where the circles represent the machines and the rectangles represent the buffers. The following assumptions address the machines, the buffers, and their interactions.

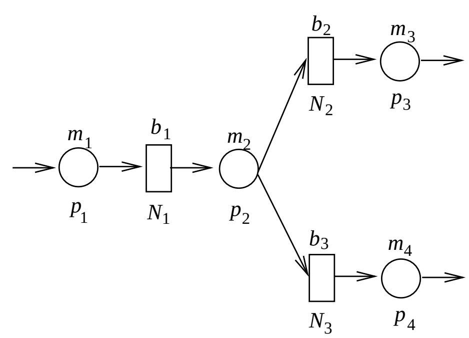

Figure 3.1: Bernoulli production system with split

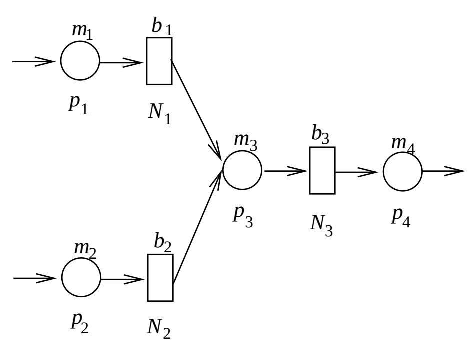

Figure 3.2: Bernoulli production system with merge

i) All machines have identical processing times. The time is slotted as cycle time.

ii) Each machine m_i, $i = 1, \dots, 4$, is characterized by its reliability p_i, i.e., at each cycle, m_i has probability p_i to be up and $1 - p_i$ to be down. When it is up, it is capable of processing a part. When the machine is down, no production takes place.

Remark 1 Assumptions *i*) and *ii*) formulate the Bernoulli reliability model of the machines. Many production systems can be characterized by this reliability model, where the machine downtime is comparable to machine cycle time. For example, in automotive assembly systems, the majority of the machine breakdowns is due to pallet jam, push button stop, etc., and only a short period of time is needed to correct these problems. In Li and Meerkov 2007, an exp-B transformation is introduced to transform exponential machine reliability models, where machines may have different speeds, up- and downtimes, into Bernoulli models with acceptable accuracy. For instance, the slower machines in the split branches would be transformed into Bernoulli machine with a smaller p_i. Li *et al.* (2006a) show that the differences in throughput using Bernoulli and other reliability models are typically small.

iii) Each buffer b_k, $k = 1, 2, 3$, has capacity N_k, $0 < N_k < \infty$.

iv) A machine is blocked if it is up, downstream buffer is full and downstream machine does not take a part from the buffer at the beginning of the time slot. In split system, machines m_3 and m_4 are never blocked. In merge system, m_4 is never blocked.

v) A machine is starved if it is up, and upstream buffer is empty. Machine m_1 in split system is never starved, and m_1 and m_2 in merge system are never starved.

vi) Machine m_2 in split system (correspondingly, m_3 in merge system) will send a part to downstream buffers b_2 and b_3 (respectively, take a part from upstream buffers b_1 and b_2) based on the following policies:

- *Circulate policy.* m_2 will send a part to buffers b_2 and b_3 circularly if it is not blocked (respectively, m_3 takes part from b_1 and b_2 circularly when it is not starved). If it is blocked by one buffer, m_2 will send the part to another buffer (respectively, m_3 will take part from another buffer if it is starved by one).
- *Priority policy.* m_2 will keep sending parts to buffer b_2 whenever it has space, i.e., b_2 has higher priority (respectively, m_3 takes part from b_1 if it has available parts). m_2 sends parts to b_3 only when it is blocked by b_2 (respectively, m_3 takes parts from b_2 only when it is starved by b_1).
- *Percentage policy (split only).* m_2 will send a part to buffers b_1 and b_2 based on pre-designed percentage, $\alpha \cdot 100\%$, i.e., $\alpha \cdot 100\%$ to b_1 and $(1 - \alpha) \cdot 100\%$ to b_2.

Remark 2 In practice, circulate and priority policies are used more often in production than other policies due to relatively easy implementation. For example, circulate policy is often used in parallel operations, and priority policy are typical in rework and re-entrant lines. Percentage policy has also been studied in the literature, however, it is less popular due to implementation difficulty. In addition, the application of percentage merge is not widely encountered, therefore, only percentage split policy is included in discussion.

The system under consideration is defined by assumptions $i) - vi)$, which define a stationary, ergodic Markov chain in the time scale of the time slot. We consider the steady state of the chain in this thesis and refer to this steady state as the normal system operations.

Let PR be the production rate of the system, i.e., the average number of parts produced by the last machines (m_3 and m_4 in split system and m_4 in merge case) per time slot. The problem addressed is formulated as follows: *Given production system $i) - vi)$, develop a method for evaluating the production rate as a function of the system parameters.*

3.2 Modeling and Analysis of Bernoulli Split Systems

3.2.1 Idea of the approach

The main difficulty of analyzing split system is that the split machine has to allocate capacity to different downstream branches and all machines and buffers are interfering with each other and impact such allocation. This makes the exact analysis all but impossible. Therefore, approximation is pursued. The idea of the approximation is based on overlapping decomposition (Li 2005), and is illustrated as follows (Figure 3.3):

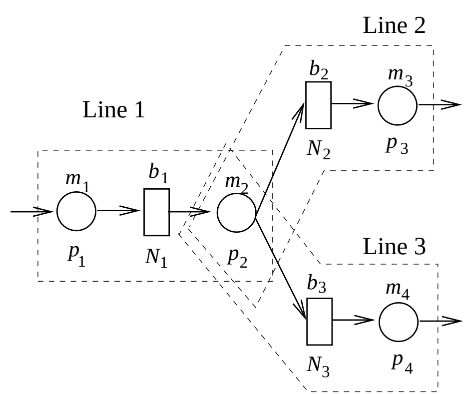

Figure 3.3: Overlapping decomposition of Bernoulli split system

Consider the split system depicted in Figure 3.3.

Assume the probabilities that m_2 is blocked by b_2 and b_3 are known, machine m_2 can be modified as m_2' to take into account these effects. Denote this line as Line 1 (m_1, b_1 and m_2'). Then the probability that m_2 is starved by b_1 can be calculated. Now consider machine m_2 with capacity allocated only to buffer b_2 and m_3, modify m_2 into m_2'' to include only such capacity and its starvation probability by b_1, we obtain Line 2 (m_2'', b_2 and m_3). Again, the probability that m_2 is blocked by b_2 can be calculated. Analogously, m_2 again can be modified into m_2''' to take into account the starvation probability and the only capacity allocated to b_3 and m_4, Line 3 (m_2''', b_3 and m_4) is composed and the probability that m_2 is blocked by b_3 can be obtained. Using these probabilities, we carry out the analysis for Line 1 again, and the procedure is repeated anew. When the procedure is convergent, we obtain the production rates of Lines 1-3. The specific split policies (priority or circulate) will be taken into account when modifications of m_2 are carried out.

3.2.2 Recursive procedures for Bernoulli split system

Introduce operator $PR(p_1, p_2, N_1)$ to denote the production rate calculation of a two-machine serial line, where p_i and N_1 represent the machine reliability and buffer capacity, respectively (see Li and Meerkov 2007 for details). Then the formula for $PR(p_1, p_2, N_1)$ is:

$$\begin{aligned}
\alpha(p_1, p_2) &=: \frac{p_1(1-p_2)}{p_2(1-p_1)}, \\
P_0 &= \begin{cases} \frac{(1-p_1)(1-\alpha(p_1,p_2))}{1-\frac{p_1}{p_2}\alpha^{N_1}(p_1,p_2)}, & \text{if } p_1 \neq p_2, \\ \frac{1-p}{N_1+1-p}, & \text{if } p_1 = p_2 = p, \end{cases} \\
PR(p_1, p_2, N_1) &= p_2(1-P_0),
\end{aligned} \tag{1}$$

where P_0 represents the probability of that the buffer occupancy is 0.

Using this operator, the recursive procedures to analyze split systems with different policies are developed.

Circulate policy Consider the split system in Figure 3.1. The rationale behind the modification of m_2 is that, in Line 1, m_2 is available to b_1 if it is blocked by neither b_2 nor b_3. In Line 2, when m_2 is not starved, it is available to b_2 50% of time if b_3 is not full, and 100% of time otherwise. Similar argument applies to Line 3. Thus, the recursive procedure is introduced as follows:

Procedure 1

$$\begin{aligned}
&\textit{Line 1} \\
p_2'(s+1) &= p_2(1-\widehat{X}_{2N_2}(s)\widehat{X}_{3N_3}(s)), \\
\widehat{pr}_1(s+1) &= PR(p_1, p_2'(s+1), N_1), \\
\widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_1(s+1)}{p_2'(s+1)},
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\textit{Line 2} \\
p_2''(s+1) &= p_2(0.5(1-\widehat{X}_{3N_3}(s))+\widehat{X}_{3N_3}(s))(1-\widehat{X}_{10}(s+1)), \\
\widehat{pr}_2(s+1) &= PR(p_2''(s+1), p_3, N_2), \\
\widehat{X}_{2N_2}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{p_2''(s+1)},
\end{aligned} \tag{3}$$

Line 3

$$
\begin{aligned}
p_2'''(s+1) &= p_2(0.5(1-\widehat{X}_{2N_2}(s+1))+\widehat{X}_{2N_2}(s+1))(1-\widehat{X}_{10}(s+1)),\\
\widehat{pr}_3(s+1) &= PR(p_2'''(s+1),p_4,N_3),\\
\widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{p_2'''(s+1)},\\
s &= 0,1,2,\dots,\\
\widehat{X}_{2N_2}(0) &= \widehat{X}_{3N_3}(0)=0,
\end{aligned}
\tag{4}
$$

where $\widehat{X}_{10}$, $\widehat{X}_{2N_2}$, $\widehat{X}_{3N_3}$ *denote the estimates of the probabilities that* b_1 *is empty,* b_2 *and* b_3 *are full, respectively.*

Priority policy Assuming buffer b_2 has higher priority than b_3. Then, in Line 2, m_2 is always available to b_2 when b_1 is not empty. m_2 is available to b_3 only when it is not starved by b_1, but blocked by b_2. The recursive procedure is modified as follows:

Procedure 2

Line 1

$$
\begin{aligned}
p_2'(s+1) &= p_2(1-\widehat{X}_{2N_2}(s)\widehat{X}_{3N_3}(s)),\\
\widehat{pr}_1(s+1) &= PR(p_1,p_2'(s+1),N_1),\\
\widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_1(s+1)}{p_2'(s+1)},
\end{aligned}
\tag{5}
$$

Line 2

$$
\begin{aligned}
p_2''(s+1) &= p_2(1-\widehat{X}_{10}(s+1)),\\
\widehat{pr}_2(s+1) &= PR(p_2''(s+1),p_3,N_2),\\
\widehat{X}_{2N_2}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{p_2''(s+1)},
\end{aligned}
\tag{6}
$$

Line 3

$$
\begin{aligned}
p_2'''(s+1) &= p_2(1-\widehat{X}_{10}(s+1))\widehat{X}_{2N_2}(s+1),\\
\widehat{pr}_3(s+1) &= PR(p_2'''(s+1),p_4,N_3),\\
\widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{p_2'''(s+1)},\\
s &= 0,1,2,\dots,\\
\widehat{X}_{2N_2}(0) &= \widehat{X}_{3N_3}(0)=0.
\end{aligned}
\tag{7}
$$

Percentage policy To ensure that the final products consisting of parts $100\cdot\alpha\%$ produced by m_3 and $100\cdot(1-\alpha)\%$ by m_4, which agrees with the expectation of percentage policy, assume machine m_2 has probability β to be available on parts to buffer b_2, and probability $1-\beta$ on parts to b_3. This implies that during the uptime of machine m_2, it has probability

β intended to send parts to buffer b_2 and $1-\beta$ to b_3. However, it may be blocked by b_2 or b_3. When it is blocked, it has to wait until the route is clear. Thus, the actual probability sending parts to b_2 and b_3 will be α and $1-\alpha$, respectively. Therefore, we need

$$p_2(1-\widehat{X}_{10})\beta(1-\widehat{X}_{2N_2}) = p_2(1-\beta\widehat{X}_{2N_2}-(1-\beta)\widehat{X}_{3N_3})(1-\widehat{X}_{10})\cdot\alpha,$$

i.e., products to branch b_2 are $\alpha \cdot 100\%$ of products going through b_1, which leads to

$$\beta(1-\widehat{X}_{2N_2}) = (1-\beta\widehat{X}_{2N_2}-(1-\beta)\widehat{X}_{3N_3})\cdot\alpha.$$

It follows that

$$\beta(1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha) = (1-\widehat{X}_{3N_3})\alpha.$$

Therefore, we obtain

$$\beta = \frac{(1-\widehat{X}_{3N_3})\alpha}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}. \tag{8}$$

Finally, the recursive procedure is introduced as follows:

Procedure 3

$$\beta(s+1) = \frac{(1-\widehat{X}_{3N_3}(s))\alpha}{1-\widehat{X}_{2N_2}(s)+\widehat{X}_{2N_2}(s)\alpha-\widehat{X}_{3N_3}(s)\alpha},$$

Line 1

$$\begin{aligned} p_2'(s+1) &= p_2(1-\beta(s+1)\widehat{X}_{2N_2}(s)-(1-\beta(s+1))\widehat{X}_{3N_3}(s)), \\ \widehat{pr}_1(s+1) &= PR(p_1, p_2'(s+1), N_1), \\ \widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_1(s+1)}{p_2'(s+1)}, \end{aligned} \tag{9}$$

Line 2

$$\begin{aligned} p_2''(s+1) &= \beta(s+1)p_2(1-\widehat{X}_{10}(s+1)), \\ \widehat{pr}_2(s+1) &= PR(p_2''(s+1), p_3, N_2), \\ \widehat{X}_{2N_2}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{p_2''(s+1)}, \end{aligned} \tag{10}$$

Line 3

$$\begin{aligned} p_2'''(s+1) &= (1-\beta(s+1))p_2(1-\widehat{X}_{10}(s+1)), \\ \widehat{pr}_3(s+1) &= PR(p_2'''(s+1), p_4, N_3), \\ \widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{p_2'''(s+1)}, \\ s &= 0,1,2,\ldots, \\ \widehat{X}_{2N_2}(0) &= \widehat{X}_{3N_3}(0) = 0. \end{aligned} \tag{11}$$

3.2.3 Convergence

Let $\widehat{pr}_i$, $i = 1,2,3$, denote the production rates obtained for Line i if Procedures 1-3 are convergent. It is shown below that these procedures lead to convergent results.

Theorem 1 *Under assumptions i)-vi), Procedures 1 and 2 are convergent, therefore, the following limits exist:*

$$\lim_{s\to\infty} \widehat{pr}_i(s) := \widehat{pr}_i, \qquad i = 1, 2, 3. \tag{12}$$

Proof: See Appendix. ■

The analytical proof of the convergence of Procedure 3 is not available. However, the procedure converges in all the examples we tested. Therefore, we formulate it as a numerical fact below.

Numerical Fact 1 *Under assumptions i)-vi), Procedure 3 is convergent and the following limits exist:*

$$\lim_{s\to\infty} \widehat{pr}_i(s) := \widehat{pr}_i, \qquad i = 1, 2, 3. \tag{13}$$

Corollary 1 *Under assumptions i)-vi) and Numerical Fact 1, the steady state equations of Procedures 1-3 have unique solutions.*

Proof: See Appendix. ■

Therefore, we obtain an estimate of the production rates, $\widehat{PR}_{s,c}$ for circulate policy, $\widehat{PR}_{s,p}$ for priority policy, and $\widehat{PR}_{s,\%}$ for percentage policy, of the split systems in steady state. Such estimates equal to $\widehat{pr}_2 + \widehat{pr}_3$ in their corresponding procedures.

3.2.4 Accuracy

The accuracy of the estimation is investigated numerically. Specifically, we randomly and equiprobably select machine and buffer parameters from the following sets, and construct 50 split systems for circulate and priority policies, and 20 split systems for percentage policy with three different percentages for each line.

$$\begin{aligned} p_1, p_2 &\in [0.75, 0.95], \\ p_3, p_4 &\in [0.4, 0.6], \\ N_i &\in \{1, 2, 3\}, \\ \alpha &\in \{10\%, 30\%, 50\%\}. \end{aligned} \tag{14}$$

For each of these lines, both analytical method using Procedures 1-3 and simulation approach using *Simul8* (Haige and Paige 2001) are pursued to evaluate system production rates. In each simulation, 10,000 cycles of warmup time are assumed, and the next 100,000 cycles are used for collecting steady state statistics. 20 replications are carried out to obtain the average production rate, with 95% confidence intervals consistently ranging within ± 0.0006. Typically, the computation time for Procedures 1-3 is within a fraction of second, and is around 5 minutes for simulation on a PC with 3.4GHz processor and 2GB RAM. The differences between analytical and simulation results are evaluated as

$$\begin{aligned} \epsilon_{s,c} &= \frac{\widehat{PR}_{s,c} - PR_{s,c}}{PR_{s,c}} \cdot 100\%, \\ \epsilon_{s,p} &= \frac{\widehat{PR}_{s,p} - PR_{s,p}}{PR_{s,p}} \cdot 100\%, \\ \epsilon_{s,\%} &= \frac{\widehat{PR}_{s,\%} - PR_{s,\%}}{PR_{s,\%}} \cdot 100\%, \end{aligned} \tag{15}$$

where $PR_{s,c}$, $PR_{s,p}$ and $PR_{s,\%}$ are the production rates obtained by simulation for circulate, priority and percentage split policies, respectively.

Remark 3 Assumption iv) defines a block before service (BBS) convention, i.e., a machine will not load a part if it is blocked. In Simul8, a block after service (BAS) is typically used, where a part is still loaded and processed even if no downstream buffer is available. The capacity of buffers under BBS and BAS schemes are related as

$$N_i^{BBS} = N_i^{BAS} + 1, \quad i = 1, 2, 3.$$

The results of this investigation are illustrated in Figures 3.4-3.6 for Procedures 1-3, respectively. It is shown that in all cases we studied, the error is less than 2%. Therefore, Procedures 1-3 provide an accurate approximation for system production rates.

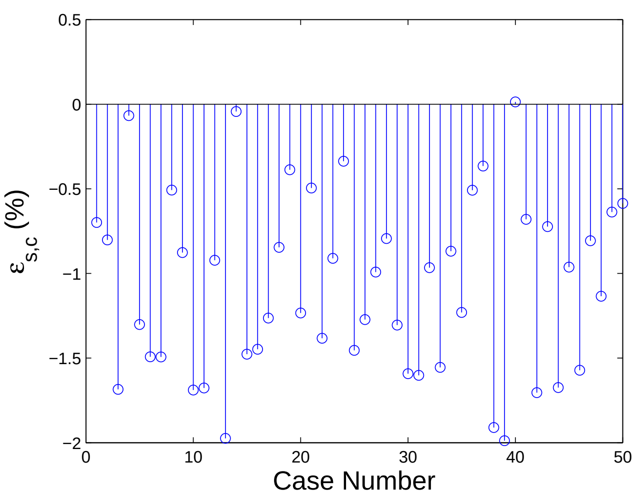

Figure 3.4: Accuracy of Procedure 1

3.3 Modeling and Analysis of Bernoulli Merge Systems

3.3.1 Idea of the approach

The idea similar to that of the split system can be applied to the merge system (Figure 3.7) as well. Machine m_1, buffer b_1 and a pseudo machine m'_3, which takes into the account of

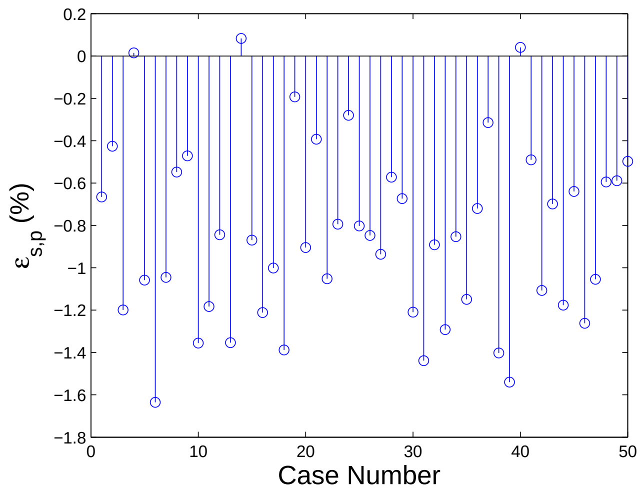

Figure 3.5: Accuracy of Procedure 2

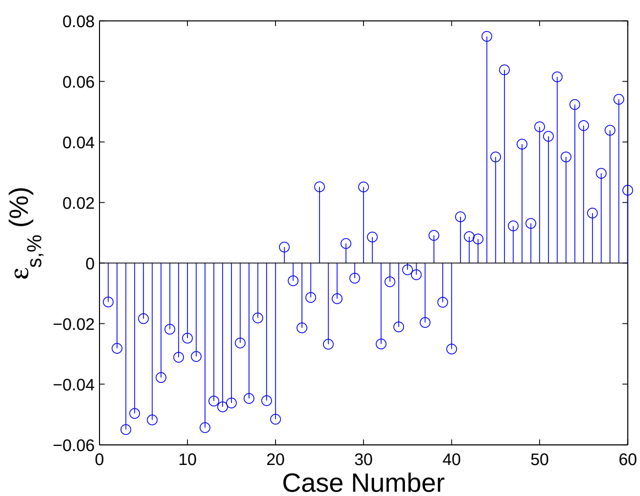

Figure 3.6: Accuracy of Procedure 3

blockage of buffer b_3 and capacity allocation to buffer b_1, consists of Line 1. Analogously, Line 2 is composed of m_2, b_2 and pseudo machine m_3'', which considers the blockage of b_3 and capacity allocated to b_2. Finally, Line 3 has pseudo machine m_3''', which includes starvation probabilities from b_1 and b_2, buffer b_3, and last machine m_4. The recursive procedures are again introduced to update the blockage and starvation probabilities of machine m_3 until they are convergent.

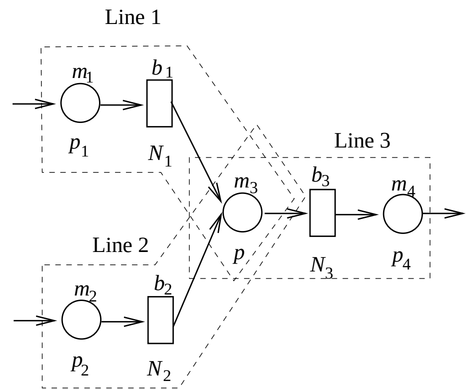

Figure 3.7: Overlapping decomposition of Bernoulli merge system

3.3.2 Recursive procedures for Bernoulli merge system

Circulate policy Consider the merge system in Figure 3.2. Similar to the rationale in circulate split policy, by replacing the blockage with starvation, and vice versa, we obtain

Procedure 4

Line 3

$$\begin{aligned} p_3'''(s+1) &= p_3(1-\widehat{X}_{10}(s)\widehat{X}_{20}(s)), \\ \widehat{pr}_3(s+1) &= PR(p_3'''(s+1), p_4, N_3), \\ \widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{p_3'''(s+1)}, \end{aligned} \tag{16}$$

Line 1

$$\begin{aligned} p_3'(s+1) &= p_3(0.5(1-\widehat{X}_{20}(s))+\widehat{X}_{20}(s))(1-\widehat{X}_{3N_3}(s+1)), \\ \widehat{pr}_1(s+1) &= PR(p_1, p_3'(s+1), N_1), \\ \widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_1(s+1)}{p_3'(s+1)}, \end{aligned} \tag{17}$$

Line 2

$$\begin{aligned} p_3''(s+1) &= p_3(0.5(1-\widehat{X}_{10}(s+1))+\widehat{X}_{10}(s+1))(1-\widehat{X}_{3N_3}(s+1)), \\ \widehat{pr}_2(s+1) &= PR(p_2, p_3''(s+1), N_2), \\ \widehat{X}_{20}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{p_3''(s+1)}, \\ s &= 0,1,2,\dots, \end{aligned} \tag{18}$$

$$\widehat{X}_{10}(0) = \widehat{X}_{20}(0) = 0,$$

where $\widehat{X}_{10}$, $\widehat{X}_{20}$, $\widehat{X}_{3N_3}$ denote the estimates of the probabilities that b_1 and b_2 are empty, and b_3 is full, respectively.

Priority policy Assuming buffer b_1 has higher priority than b_2. Analogously to Procedure 2, we have

Procedure 5

Line 3

$$\begin{aligned} p_3'''(s+1) &= p_3(1-\widehat{X}_{10}(s))\widehat{X}_{20}(s)), \\ \widehat{pr}_3(s+1) &= PR(p_3'''(s+1), p_4, N_3), \\ \widehat{X}_{3N_3}(s+1) &= 1-\frac{pr_3(s+1)}{p_3'''(s+1)}, \end{aligned} \tag{19}$$

Line 1

$$\begin{aligned} p_3'(s+1) &= p_3(1-\widehat{X}_{3N_3}(s+1)), \\ \widehat{pr}_1(s+1) &= PR(p_1, p_3'(s+1), N_1), \\ \widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_1(s+1)}{p_3'(s+1)}, \end{aligned} \tag{20}$$

Line 2

$$\begin{aligned} p_3''(s+1) &= p_2\widehat{X}_{10}(s+1)(1-\widehat{X}_{3N_3}(s+1)), \\ \widehat{pr}_2(s+1) &= PR(p_2, p_3''(s+1), N_2), \\ \widehat{X}_{20}(s+1) &= 1-\frac{pr_2(s+1)}{p_3''(s+1)}, \\ s &= 0,1,2,\ldots, \\ \widehat{X}_{10}(0) &= \widehat{X}_{20}(0) = 0. \end{aligned} \tag{21}$$

3.3.3 Convergence

Again let $\widehat{pr}_i$, $i=1,2,3$, denote the production rates obtained for Line i when Procedures 4 and 5 are convergent. We show below that both procedures are convergent.

Theorem 2 *Under assumptions i)-vi), Procedures 4 and 5 are convergent, therefore, the following limits exist:*

$$\lim_{s\to\infty} pr_i(s) := \widehat{pr}_i, \quad i=1,2,3. \tag{22}$$

Proof: See Appendix. ∎

Corollary 2 *Under assumptions i)-vi), the steady state equations of Procedures 4 and 5 have unique solutions.*

Proof: See Appendix. ∎

Therefore, we obtain the estimates of the production rates, $\widehat{PR}_{m,c}$ for circulate policy, $\widehat{PR}_{m,p}$ for priority policy, of the merge systems in steady state, which are equal to $\widehat{pr}_3$ in their corresponding procedures.

3.3.4 Accuracy

The accuracy of the estimation is again investigated numerically. By reversing the split systems, and applying the corresponding parameters, we obtain 50 merge lines. We apply both circulate and priority merge policies to these lines and same simulation setups are carried out. The differences between analytical and simulation results are evaluated as

$$\begin{aligned} \epsilon_{m,c} &= \frac{\widehat{PR}_{m,c} - PR_{m,c}}{PR_{m,c}} \cdot 100\%, \\ \epsilon_{m,p} &= \frac{\widehat{PR}_{m,p} - PR_{m,p}}{PR_{m,p}} \cdot 100\%, \end{aligned} \tag{23}$$

where $PR_{m,c}$, $PR_{m,p}$ are the production rates obtained by simulation for circulate and priority merge policies, respectively.

The results of this investigation are illustrated in Figures 3.8 and 3.9 for circulate and priority merge policies, respectively. Again it is shown that both procedures provide accurate approximation for production rates, with errors less than 2% (except two cases where errors go up to 2.5%).

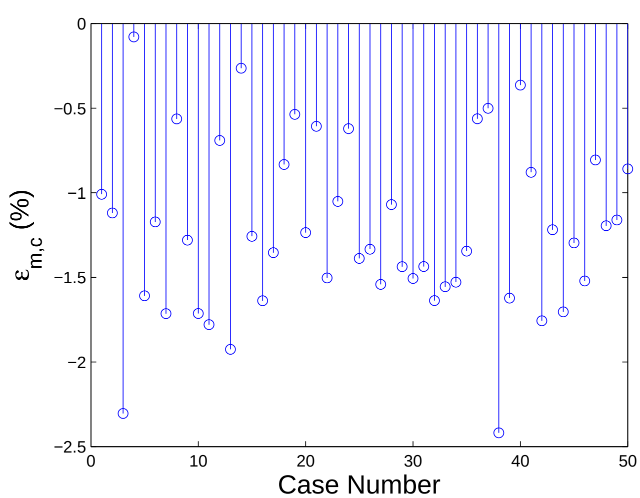

Figure 3.8: Accuracy of Procedure 4

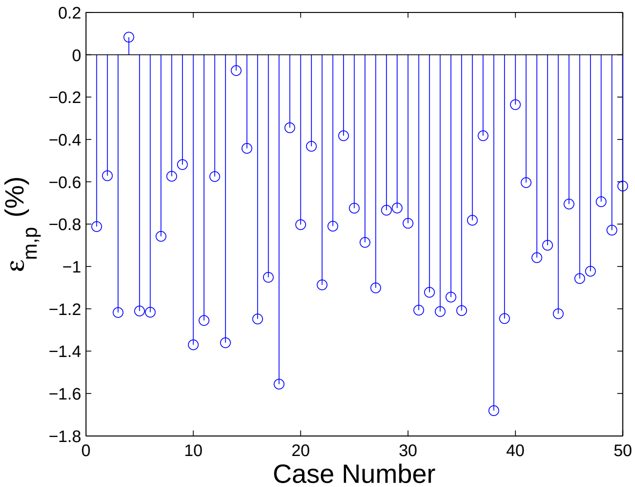

Figure 3.9: Accuracy of Procedure 5

3.4 Structural Properties

3.4.1 Conservation of flow

The conservation of flow holds for both split and merge systems.

Corollary 3 *Under assumptions i)-vi), the production rates of Lines 1-3 in split system satisfy the following property:*

$$\widehat{pr}_1 = \widehat{pr}_2 + \widehat{pr}_3. \tag{24}$$

In particular, for percentage split policy,

$$\widehat{pr}_2 = \alpha\widehat{pr}_1, \quad \widehat{pr}_3 = (1-\alpha)\widehat{pr}_1. \tag{25}$$

Similarly, for Lines 1-3 in merge system,

$$\widehat{pr}_3 = \widehat{pr}_1 + \widehat{pr}_2.$$

Proof: See Appendix. ■

3.4.2 Monotonicity

It has been shown in Li and Meerkov 2007 that monotonicity holds in serial lines and assembly systems, i.e., improving machine reliability and/or increasing buffer capacity lead to improvement of system production rate. Similar properties are observed in split and merge systems for all policies as well.

Corollary 4 *Under assumptions i)-vi), the system production rates in split and merge systems are monotonically increasing with respect to p_i, $i = 1, \ldots, 4$, and N_i, $i = 1, 2, 3$.*

Proof: See Appendix. ■

3.4.3 Reversibility

It has been shown that the reversibility exists in Bernoulli serial production lines (Li and Meerkov 2007). For the split and merge systems with circulate and priority policies considered in this thesis, such property still holds. To illustrate this, denote the machine and buffer parameters in Figure 3.1 as p_i^s, $i = 1, \ldots, 4$, N_i^s, $i = 1, 2, 3$, and in Figure 3.2 as p_i^m, and N_i^m. In addition,

$$\begin{aligned} p_1^s &= p_4^m, \qquad p_2^s = p_3^m, \\ p_3^s &= p_1^m, \qquad p_4^s = p_2^m, \\ N_1^s &= N_3^m, \qquad N_2^s = N_1^m, \\ N_3^s &= N_2^m. \end{aligned} \tag{26}$$

Corollary 5 *Under assumptions i)-vi) and condition (26), the system production rates in split and merge systems with circulate and priority policies have identical production rates. In other words*

$$\widehat{PR}_{m,c} = \widehat{PR}_{s,c}, \qquad \widehat{PR}_{m,p} = \widehat{PR}_{s,p}. \tag{27}$$

Proof: See Appendix. ■

3.4.4 Comparisons

A comparison between the circulate and priority policies has been carried out. The results show that the difference in system production rates between systems with circulate and priority policies is typically small. In other words,

$$|\widehat{PR}_{s,c} - \widehat{PR}_{s,p}| \ll 1, \qquad |\widehat{PR}_{m,c} - \widehat{PR}_{m,p}| \ll 1. \tag{28}$$

Comparing with percentage policy with 50% split, we obtain

$$\widehat{PR}_{s,c} > \widehat{PR}_{s,50\%}, \qquad \widehat{PR}_{s,p} > \widehat{PR}_{s,50\%}. \tag{29}$$

The reason for smaller production rate in split systems under percentage policy is due to the fact that parts that are intended to be sent to buffer b_i have to wait for the availability of b_i if they are blocked, in order to ensure a 50% split.

In addition, numerical results suggest that it is always beneficial to assign more reliable machine with higher priority. In other words, if $p_1 > p_2$, then a merge system with machine m_1 having higher priority will achieve better production rate than a system where m_2 has higher priority. It is formulated as the following numerical fact.

Numerical Fact 2 *Under assumptions i)-vi), for Bernoulli merge system with priority policy where the first branch has higher priority, if* $p_1 > p_2$*, then*

$$\widehat{PR}(p_1, p_2, p_3, p_4, N_1, N_2, N_3) > \widehat{PR}(p_2, p_1, p_3, p_4, N_1, N_2, N_3). \tag{30}$$

Based on reversibility, similar argument applies to split system as well.

Chapter 4

EXPONENTIAL PRODUCTION SYSTEMS

4.1 Problem Formulation

The typical structures of exponential split and merge systems are shown in Figures 4.1 and 4.2, respectively.

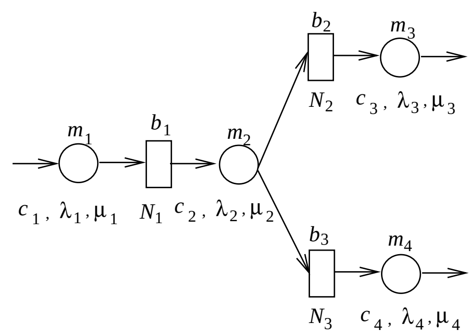

Figure 4.1: Exponential production system with split

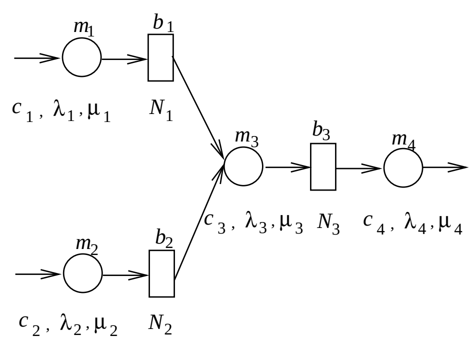

Figure 4.2: Exponential production system with merge

The following assumptions address the machines, the buffers, and their interactions.

i') Each machine's processing speed is c_k parts/unit of time, $k = 1, \ldots, 4$, which implies the production system is asynchronous.

ii′) Each machine m_i, $i = 1, \ldots, 4$, is characterized by its breakdown rate λ_i and repair rate μ_i. When it is up, it is capable of processing parts. When the machine is down, no production takes place.

iii′) Each buffer b_k, $k = 1, 2, 3$, has capacity N_k, $0 < N_k < \infty$.

iv′) A machine is blocked if it is up, downstream buffer is full and downstream machine does not take a part from the buffer. In split system, machines m_3 and m_4 are never blocked. In merge system, m_4 is never blocked.

v′) A machine is starved if it is up, and upstream buffer is empty. Machine m_1 in split system is never starved, and m_1 and m_2 in merge system are never starved.

vi′) Machine m_2 in split system (correspondingly, m_3 in merge system) will send a part to downstream buffers b_2 and b_3 (respectively, take material from upstream buffers b_1 and b_2) based on the following policies:

- *Circulate policy.* m_2 will send a part to buffers b_2 and b_3 circularly if it is not blocked (respectively, m_3 takes part from b_1 and b_2 circularly when it is not starved). If it is blocked by one buffer, m_2 will send the part to another buffer (respectively, m_3 will take part from another buffer if it is starved by one).
- *Priority policy.* m_2 will keep sending parts to buffer b_2 whenever it has space, i.e., b_2 has higher priority (respectively, m_3 takes part from b_1 if it has available parts). m_2 sends parts to b_3 only when it is blocked by b_2 (respectively, m_3 takes parts from b_2 only when it is starved by b_1).

Remark 4 *Percentage policy in exponential lines is still under investigation and will be part of the future work.*

The system under consideration is defined by assumptions $i') - vi')$, which define a stationary, ergodic Markov chain. We still consider the steady state of the chain and refer to this steady state as the normal system operations.

The problem is similar to that in Section 3.1: *Given production system $i') - vi')$, develop a method for evaluating the production rate as a function of the system parameters.*

Solutions to the problem are presented in Sections 4.2 and 4.3 for split and merge systems, respectively.

4.2 Modeling and Analysis of Exponential Split Systems

4.2.1 Idea of the approach

Again, similar to the Bernoulli case, due to the complexity in the split and merge systems, exact analysis is all but impossible. Therefore, approximation is pursued. The idea of the approximation is again based on overlapping decomposition, and is illustrated as follows

(Figure 4.3):

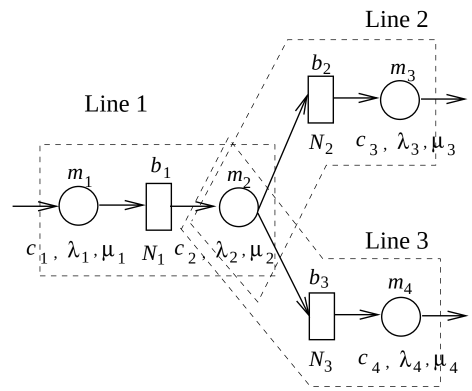

Figure 4.3: Overlapping decomposition of exponential split system

The split system is decomposed into 3 overlapped two-machine lines, and m_2 is the overlapping machine. Line 1 has machines m_1 and m_2', where m_2' includes blockage effects from buffers b_2 and b_3; line 2 consists of machines m_2'' and m_3, and the starvation of b_1 and the blockage of b_3 are embedded in m_2''; finally, m_2''' and m_4 compose line 4, and m_2''' has starvation impact from b_1 and blockage impact from b_2. Then the recursive procedures are introduced to estimate the blockage and starvation probabilities. When the procedure is convergent, we obtain the production rates of Lines 1-3. The specific split policies (priority or circulate) will be taken into account when modifications of m_2 are carried out.

4.2.2 Recursive procedures for exponential split system

Introduce operator $PR([\lambda_1, \lambda_2], [\mu_1, \mu_2], [c_1, c_2], N_1)$ to denote the production rate calculation of a two-machine serial line. The time is continuous and the machines obey the exponential reliability model, i.e., having processing speed c_i, break down rate and repair rate λ_i and μ_i, $i = 1, 2$, and N_1 represent the buffer capacity (see Li and Meerkov 2007 for details). The formula for $PR([\lambda_1, \lambda_2], [\mu_1, \mu_2], [c_1, c_2], N_1)$ is as follows:

- If $c_1 < c_2$,

$$PR = \frac{c_2 e_2 A e^{k_1 N_1} + c_1 e_1 B e^{k_2 N_1} + c_1 e_1 C e^{-k_2 N_1}}{A e^{k_1 N_1 + B e^{k_2 N_1} + C_1 e^{-k_2 N_1}}}, \tag{1}$$

where

$$\begin{aligned}
e_i &= \frac{\mu_i}{\lambda_i + \mu_i}, i = 1, 2 \\
R &= \sqrt{[c_1(\mu_1 + \mu_2 + \lambda_2) - c_2(\mu_1 + \mu_2 + \lambda_1)]^2 + 4c_1c_2\lambda_1\lambda_2}, \\
k_1 &= \frac{1}{2c_1c_2(\mu_1 + \mu_2)(c_1 - c_2)}[\mu_1 c_1^2(\mu_1 + \mu_2 + \lambda_2) - c_1c_2[(\mu_1 + \mu_2)^2 \\
&\quad +((\mu_1 + \mu_2)(\lambda_1 + \lambda_2) + (\mu_1\lambda_2 + \mu_2\lambda_1)] + \mu_2 c_2^2(\mu_1 + \mu_2 + \lambda_1)], \\
k_2 &= \frac{(c_1\mu_1 + c_2\mu_2)R}{2c_1c_2(\mu_1 + \mu_2)(c_2 - c_1)},
\end{aligned}$$

$$
\begin{aligned}
A &= \mu_1 R^2 + \mu_1 R[c_1(\mu_1+\mu_2+\lambda_2) - c_2(\mu_1+\mu_2+\lambda_1)], \\
B &= \mu_2\lambda_1 c_2[(c_1-c_2)(\mu_1-\mu_2) - (c_2\lambda_1 + c_1\lambda_2) - R], \\
C_1 &= \frac{e_2(c_2-c_1e_1)A + c_1e_1(1-e_2)B}{c_1e_1(e_2-1)}.
\end{aligned}
$$

- If $c_1 > c_2$,

$$
PR = \frac{c_1e_1Ae^{k_1N_1} + c_2e_2Be^{k_2N_1} + c_2e_2Ce^{-k_2N_1}}{Ae^{k_1N_1+Be^{k_2N_1}+C_2e^{-k_2N_1}}}, \tag{2}
$$

where

$$
\begin{aligned}
e_i &= \frac{\mu_i}{\lambda_i+\mu_i}, i = 1, 2 \\
R &= \sqrt{[c_1(\mu_1+\mu_2+\lambda_2) - c_2(\mu_1+\mu_2+\lambda_1)]^2 + 4c_1c_2\lambda_1\lambda_2}, \\
k_1 &= \frac{1}{2c_1c_2(\mu_1+\mu_2)(c_2-c_1)}[\mu_1c_1^2(\mu_1+\mu_2+\lambda_2) - c_1c_2[(\mu_1+\mu_2)^2 \\
&\quad +((\mu_1+\mu_2)(\lambda_1+\lambda_2) + (\mu_1\lambda_2+\mu_2\lambda_1)] + \mu_2c_2^2(\mu_1+\mu_2+\lambda_1)], \\
k_2 &= \frac{(c_1\mu_1+c_2\mu_2)R}{2c_1c_2(\mu_1+\mu_2)(c_2-c_1)}, \\
A &= \mu_1R^2 + \mu_1R[c_1(\mu_1+\mu_2+\lambda_2) - c_2(\mu_1+\mu_2+\lambda_1)], \\
B &= \mu_1\lambda_2c_1[(c_1-c_2)(\mu_1-\mu_2) - (c_2\lambda_1+c_1\lambda_2) + R], \\
C_2 &= \frac{e_1(c_1-c_2e_2)A + c_2e_2(1-e_1)B}{c_2e_2(e_1-1)}.
\end{aligned}
$$

Using this operator, the recursive procedures to analyze split systems with different policies are developed.

Circulate policy Consider the split system in Figure 4.1. The rationale behind the modification of m_2 is that, in Line 1, m_2 is available to b_1 if it is not blocked by both b_2 and b_3. In Line 2, when m_2 is not starved, it is available to b_2 50% of time if b_3 is not full, and 100% of time otherwise. Similar argument applies to Line 3. Thus, the recursive procedure is introduced as follows:

Procedure 6

Line 1

$$
\begin{aligned}
\mu_2'(s+1) &= \mu_2(1-\widehat{X}_{2N_2}(s)\widehat{X}_{3N_3}(s)), \\
\lambda_2'(s+1) &= \lambda_2 + \mu_2 - \mu_2'(s+1), \\
\widehat{pr}_1(s+1) &= PR([\lambda_1, \lambda_2'(s+1)], [\mu_1, \mu_2'(s+1)], [c_1, c_2], N_1), \\
e_2'(s+1) &= \mu_2'(s+1)/(\lambda_2'(s+1) + \mu_2'(s+1)), \\
\widehat{X}_{10}(s+1) &= 1 - \frac{\widehat{pr}_1(s+1)}{c_2e_2'(s+1)},
\end{aligned} \tag{3}
$$

Line 2

$$
\mu_2''(s+1) = \mu_2(0.5(1-\widehat{X}_{3N_3}(s)) + \widehat{X}_{3N_3}(s))(1-\widehat{X}_{10}(s+1)),
$$

$$
\begin{aligned}
\lambda_2''(s+1) &= \lambda_2 + \mu_2 - \mu_2''(s+1), \\
\widehat{pr}_2(s+1) &= PR([\lambda_2''(s+1), \lambda_3], [\mu_2''(s+1), \mu_3], [c_2, c_3], N_2), \\
e_2''(s+1) &= \mu_2''(s+1)/(\lambda_2''(s+1) + \mu_2''(s+1)), \\
\widehat{X}_{2N_2}(s+1) &= 1 - \frac{\widehat{pr}_2(s+1)}{c_2 e_2''(s+1)},
\end{aligned} \tag{4}
$$

Line 3

$$
\begin{aligned}
\mu_2'''(s+1) &= \mu_2(0.5(1 - \widehat{X}_{2N_2}(s+1)) + \widehat{X}_{2N_2}(s+1))(1 - \widehat{X}_{10}(s+1)), \\
\lambda_2'''(s+1) &= \lambda_2 + \mu_2 - \mu_2'''(s+1), \\
\widehat{pr}_3(s+1) &= PR([\lambda_2'''(s+1), \lambda_4], [\mu_2'''(s+1), \mu_4], [c_2, c_4], N_3), \\
e_2'''(s+1) &= \mu_2'''(s+1)/(\lambda_2'''(s+1) + \mu_2'''(s+1)), \\
\widehat{X}_{3N_3}(s+1) &= 1 - \frac{\widehat{pr}_3(s+1)}{c_2 e_2'''(s+1)}, \\
s &= 0, 1, 2, \dots, \\
\widehat{X}_{2N_2}(0) &= 0, \\
\widehat{X}_{3N_3}(0) &= 0,
\end{aligned} \tag{5}
$$

where $\widehat{X}_{10}$, $\widehat{X}_{2N_2}$, $\widehat{X}_{3N_3}$ denote the estimates of the probabilities that b_1 is empty, b_2 and b_3 are full, respectively.

Priority policy Assuming buffer b_2 has higher priority than b_3. Then, in Line 2, m_2 is always available to b_2 when b_1 is not empty. m_2 is available to b_3 only when it is not starved by b_1, but blocked by b_2. The recursive procedure is modified as follows:

Procedure 7

Line 1

$$
\begin{aligned}
\mu_2'(s+1) &= \mu_2(1 - \widehat{X}_{2N_2}(s)\widehat{X}_{3N_3}(s)), \\
\lambda_2'(s+1) &= \lambda_2 + \mu_2 - \mu_2'(s+1), \\
\widehat{pr}_1(s+1) &= PR([\lambda_1, \lambda_2'(s+1)], [\mu_1, \mu_2'(s+1)], [c_1, c_2], N_1), \\
e_2'(s+1) &= \mu_2'(s+1)/(\lambda_2'(s+1) + \mu_2'(s+1)), \\
\widehat{X}_{10}(s+1) &= 1 - \frac{\widehat{pr}_1(s+1)}{c_2 e_2'(s+1)},
\end{aligned} \tag{6}
$$

Line 2

$$
\begin{aligned}
\mu_2''(s+1) &= \mu_2(1 - \widehat{X}_{10}(s+1)), \\
\lambda_2''(s+1) &= \lambda_2 + \mu_2 - \mu_2''(s+1), \\
\widehat{pr}_2(s+1) &= PR([\lambda_2''(s+1), \lambda_3], [\mu_2''(s+1), \mu_3], [c_2, c_3], N_2), \\
e_2''(s+1) &= \mu_2''(s+1)/(\lambda_2''(s+1) + \mu_2''(s+1)),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\widehat{X}_{2N_2}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{c_2 e_2''(s+1)}, \\
\textit{Line 3} & \\
\mu_2'''(s+1) &= \mu_2 \widehat{X}_{2N_2}(s+1)(1-\widehat{X}_{10}(s+1)), \\
\lambda_2'''(s+1) &= \lambda_2+\mu_2-\mu_2'''(s+1), \\
\widehat{pr}_3(s+1) &= PR([\lambda_2'''(s+1),\lambda_4],[\mu_2'''(s+1),\mu_4],[c_2,c_4],N_3), \\
e_2'''(s+1) &= \mu_2'''(s+1)/(\lambda_2'''(s+1)+\mu_2'''(s+1)), \\
\widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{c_2 e_2'''(s+1)}, \\
s &= 0,1,2,\ldots, \\
\widehat{X}_{2N_2}(0) &= 0, \\
\widehat{X}_{3N_3}(0) &= 0.
\end{aligned} \tag{8}
$$

The definitions of $\widehat{X}_{10}$, $\widehat{X}_{2N_2}$, $\widehat{X}_{3N_3}$ are the same as in Procedure 6.

4.2.3 Convergence

Let $\widehat{pr}_i$, $i = 1, 2, 3$, denote the production rates obtained for Line i if Procedures 6 and 7 are convergent. It is shown below that these procedures lead to convergent results.

Theorem 3 *Under assumptions i')-vi'), Procedures 6 and 7 are convergent, therefore, the following limits exist:*

$$\lim_{s\to\infty} \widehat{pr}_i(s) := \widehat{pr}_i, \qquad i = 1, 2, 3. \tag{9}$$

Proof: See Appendix. ■

Corollary 6 *Under assumptions i')-vi'), the steady state equations of Procedures 6 and 7 have unique solutions.*

Proof: See Appendix. ■

Therefore, we obtain an estimate of the production rates, $\widehat{PR}_{s,c}$ for circulate policy, $\widehat{PR}_{s,p}$ for priority policy of the split systems in steady state. Such estimates equal to $\widehat{pr}_2 + \widehat{pr}_3$ in their corresponding procedures.

4.2.4 Accuracy

The accuracy of the estimation is investigated numerically. Specifically, we randomly and equiprobably select machine and buffer parameters from the following sets, and construct 30 split systems for circulate and priority policies for each line.

$$
\begin{aligned}
e_i &\in [0.75, 0.95], i = 1, \ldots, 4, \\
T_{down,i} &\in [2, 10], i = 1, \ldots, 4, \\
c_i &\in [1, 1.2], i = 1, 2, \\
c_i &\in [0.6, 0.8], i = 3, 4, \\
N_i &\in [1, 3] \cdot T_{down,i}, i = 1, \ldots, 4,
\end{aligned} \tag{10}
$$

where e_i and $T_{down,i}$ represent machine m_i's efficiency and average downtime respectively. The following equations show how λ_i and μ_i are obtained through e_i and $T_{down,i}$.

$$
\begin{aligned}
\lambda_i &= \frac{1}{T_{up,i}}, \\
\mu_i &= \frac{1}{T_{down,i}}, \\
e_i &= \frac{T_{up,i}}{T_{up,i} + T_{down,i}} = \frac{\mu_i}{\lambda_i + \mu_i}.
\end{aligned} \tag{11}
$$

For each of these lines, both analytical method using Procedures 6 and 7 and simulation approach using *Simul8* are pursued to evaluate system production rates. The simulation settings are the same as that in the Bernoulli case. Typically, the computation time for Procedures 6 and 7 is within a fraction of second, and is around 7 minutes for simulation on a PC with 3.4GHz processor and 2GB RAM. The differences between analytical and simulation results are evaluated as

$$
\begin{aligned}
\epsilon_{s,c} &= \frac{\widehat{PR}_{s,c} - PR_{s,c}}{PR_{s,c}} \cdot 100\%, \\
\epsilon_{s,p} &= \frac{\widehat{PR}_{s,p} - PR_{s,p}}{PR_{s,p}} \cdot 100\%,
\end{aligned} \tag{12}
$$

where $PR_{s,c}$ and $PR_{s,p}$ are the production rates obtained by simulation for circulate and priority split policies, respectively. Note that again the block before service (BBS) convention is used in the exponential models.

The results of this investigation are illustrated in Figures 4.4 and 4.5 for circulate and priority split policies, respectively. Again it is shown that both procedures provide accurate approximation for production rates, with errors less than 10% (except two cases where errors go up to 12.5%).

4.3 Modeling and Analysis of Exponential Merge Systems

4.3.1 Idea of the approach

An idea similar to that of the split exponential system is applied to the merge exponential system (Figure 4.6).

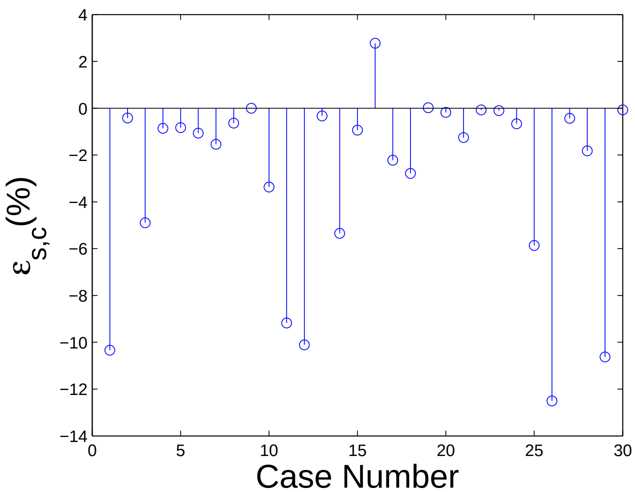

Figure 4.4: Accuracy of Procedure 6

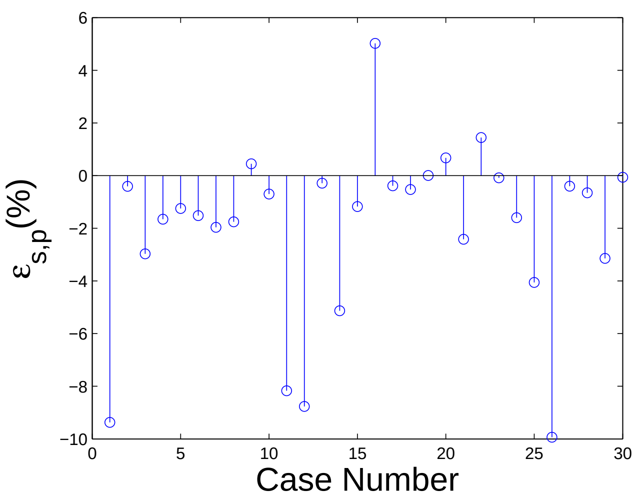

Figure 4.5: Accuracy of Procedure 7

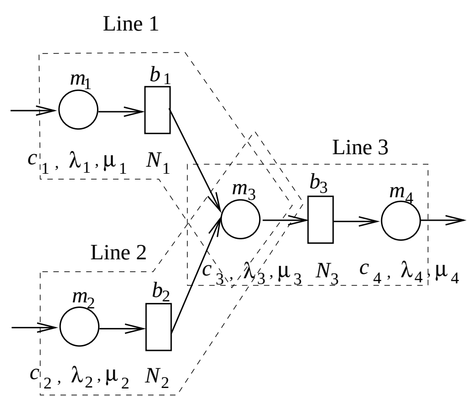

Figure 4.6: Overlapping decomposition of exponential merge system

The system is again decomposed into three overlapped serial lines and pseudo machines m_2', m_2'' and m_2''' are introduced to represent the effects of blockages and starvations due to other lines. The recursive procedures are again introduced to update the blockage and starvation probabilities of machine m_3 until they are convergent.

4.3.2 Recursive procedures for exponential merge system

Circulate policy Consider the merge system in Figure 4.2. Similar to the rationale in circulate split policy, by replacing the blockage with starvation, and vice versa, we obtain

Procedure 8

$$\begin{aligned}
&\textit{Line 1} && \\
\mu_3'(s+1) &= \mu_3(0.5(1-\widehat{X}_{20}(s)) + \widehat{X}_{20}(s))(1-\widehat{X}_{3N_3}(s)), \\
\lambda_3'(s+1) &= \lambda_3 + \mu_3 - \mu_3'(s+1), \\
\widehat{pr}_1(s+1) &= PR([\lambda_1, \lambda_3'(s+1)], [\mu_1, \mu_3'(s+1)], [c_1, c_3], N_1), \qquad (13) \\
e_3'(s+1) &= \mu_3'(s+1)/(\lambda_3'(s+1) + \mu_3'(s+1)), \\
\widehat{X}_{10}(s+1) &= 1 - \frac{\widehat{pr}_1(s+1)}{c_3 e_3'(s+1)}, \\
&\textit{Line 2} && \\
\mu_3''(s+1) &= \mu_3(0.5(1-\widehat{X}_{10}(s+1)) + \widehat{X}_{10}(s+1))(1-\widehat{X}_{3N_3}(s)), \\
\lambda_3''(s+1) &= \lambda_3 + \mu_3 - \mu_3''(s+1), \\
\widehat{pr}_2(s+1) &= PR([\lambda_2, \lambda_3''(s+1)], [\mu_2, \mu_3''(s+1)], [c_2, c_3], N_2), \qquad (14) \\
e_3''(s+1) &= \mu_3''(s+1)/(\lambda_3''(s+1) + \mu_3''(s+1)), \\
\widehat{X}_{20}(s+1) &= 1 - \frac{\widehat{pr}_2(s+1)}{c_3 e_3''(s+1)}, \\
&\textit{Line 3} && \\
\mu_3'''(s+1) &= \mu_3(1-\widehat{X}_{10}(s+1)\widehat{X}_{20}(s+1)), \\
\lambda_3'''(s+1) &= \lambda_3 + \mu_3 - \mu_3'''(s+1), \\
\widehat{pr}_3(s+1) &= PR([\lambda_3'''(s+1), \lambda_4], [\mu_3'''(s+1), \mu_4], [c_3, c_4], N_2), \qquad (15) \\
e_3'''(s+1) &= \mu_3'''(s+1)/(\lambda_3'''(s+1) + \mu_3'''(s+1)), \\
\widehat{X}_{3N_3}(s+1) &= 1 - \frac{\widehat{pr}_3(s+1)}{c_3 e_3'''(s+1)}, \\
\widehat{X}_{20}(0) &= 0, \\
\widehat{X}_{3N_3}(0) &= 0,
\end{aligned}$$

where $\widehat{X}_{10}$, $\widehat{X}_{20}$, $\widehat{X}_{3N_3}$ denote the estimates of the probabilities that b_1 and b_2 are empty, and b_3 is full, respectively.

Priority policy Assuming buffer b_1 has higher priority than b_2. Analogously to Procedure 7, we have

Procedure 9

$$\begin{aligned}
& \textit{Line 1} \\
\mu_3'(s+1) &= \mu_3(1-\widehat{X}_{3N_3}(s)), \\
\lambda_3'(s+1) &= \lambda_3+\mu_3-\mu_3'(s+1), \\
\widehat{pr}_1(s+1) &= PR([\lambda_1,\lambda_3'(s+1)],[\mu_1,\mu_3'(s+1)],[c_1,c_3],N_1), \qquad (16)\\
e_3'(s+1) &= \mu_3'(s+1)/(\lambda_3'(s+1)+\mu_3'(s+1)), \\
\widehat{X}_{10}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{c_3e_3'(s+1)}, \\
& \textit{Line 2} \\
\mu_3''(s+1) &= \mu_3\widehat{X}_{10}(s+1)(1-\widehat{X}_{3N_3}(s)), \\
\lambda_3''(s+1) &= \lambda_3+\mu_3-\mu_3''(s+1), \\
\widehat{pr}_2(s+1) &= PR([\lambda_2,\lambda_3''(s+1)],[\mu_2,\mu_3''(s+1)],[c_2,c_3],N_2), \qquad (17)\\
e_3''(s+1) &= \mu_3''(s+1)/(\lambda_3''(s+1)+\mu_3''(s+1)), \\
\widehat{X}_{20}(s+1) &= 1-\frac{\widehat{pr}_2(s+1)}{c_3e_3''(s+1)}, \\
& \textit{Line 3} \\
\mu_3'''(s+1) &= \mu_3(1-\widehat{X}_{10}(s+1)\widehat{X}_{20}(s+1)), \\
\lambda_3'''(s+1) &= \lambda_3+\mu_3-\mu_3'''(s+1), \\
\widehat{pr}_3(s+1) &= PR([\lambda_3'''(s+1),\lambda_4],[\mu_3'''(s+1),\mu_4],[c_3,c_4],N_3), \qquad (18)\\
e_3'''(s+1) &= \mu_3'''(s+1)/(\lambda_3'''(s+1)+\mu_3''(s+1)), \\
\widehat{X}_{3N_3}(s+1) &= 1-\frac{\widehat{pr}_3(s+1)}{c_3e_3'''(s+1)}, \\
\widehat{X}_{10}(0) &= 0, \\
\widehat{X}_{3N_3}(0) &= 0.
\end{aligned}$$

The definitions of $\widehat{X}_{10}, \widehat{X}_{20}, \widehat{X}_{3N_3}$ are the same as in Procedure 8.

4.3.3 Convergence

Again let $\widehat{pr}_i$, $i=1,2,3$, denote the production rates obtained for Line i when Procedures 8 and 9 are convergent. We show below that both procedures are convergent.

Theorem 4 *Under assumptions i')-vi'), Procedures 8 and 9 are convergent, therefore, the following limits exist:*

$$\lim_{s\to\infty} pr_i(s) := \widehat{pr}_i, \qquad i=1,2,3. \qquad (19)$$

Proof: See Appendix. ■

Corollary 7 *Under assumptions i')-vi'), the steady state equations of Procedures 8 and 9 have unique solutions.*

Proof: See Appendix. ■

Therefore, we obtain the estimates of the production rates, $\widehat{PR}_{m,c}$ for circulate policy, $\widehat{PR}_{m,p}$ for priority policy, of the merge systems in steady state, which are equal to $\widehat{pr}_3$ in their corresponding procedures.

4.3.4 Accuracy

The accuracy of the estimation is again investigated numerically. By reversing the split systems, and applying the corresponding parameters, we obtain 30 merge lines. We apply both circulate and priority merge policies to these lines and same simulation setups are carried out. The differences between analytical and simulation results are evaluated as

$$\begin{aligned} \epsilon_{m,c} &= \frac{\widehat{PR}_{m,c} - PR_{m,c}}{PR_{m,c}} \cdot 100\%, \\ \epsilon_{m,p} &= \frac{\widehat{PR}_{m,p} - PR_{m,p}}{PR_{m,p}} \cdot 100\%, \end{aligned} \tag{20}$$

where $PR_{m,c}$, $PR_{m,p}$ are the production rates obtained by simulation for circulate and priority merge policies, respectively.

The results of this investigation are illustrated in Figures 4.7 and 4.8 for circulate and priority merge policies, respectively. Again it is shown that both procedures provide accurate approximation for production rates, with errors less than 10% (except two cases where errors go up to 12.5%).

4.4 Structural Properties

4.4.1 Conservation of flow

The conservation of flow holds again for both split and merge systems in exponential models.

Corollary 8 *Under assumptions i')-vi'), the production rates of Lines 1-3 in split system satisfy the following property:*

$$\widehat{pr}_1 = \widehat{pr}_2 + \widehat{pr}_3. \tag{21}$$

Similarly, for Lines 1-3 in merge system,

$$\widehat{pr}_3 = \widehat{pr}_1 + \widehat{pr}_2.$$

Proof: See Appendix. ■

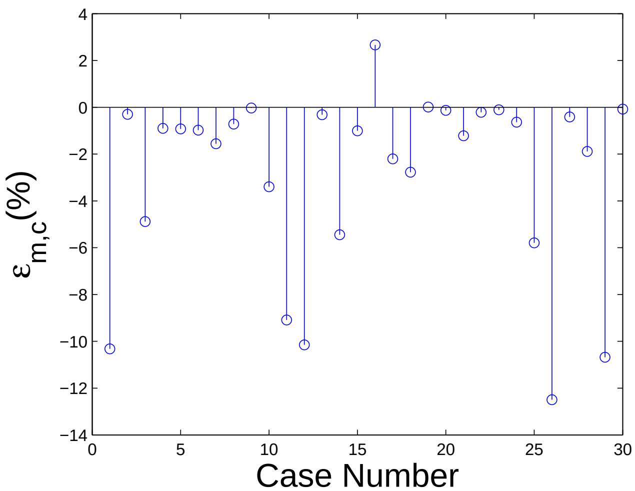

Figure 4.7: Accuracy of Procedure 8

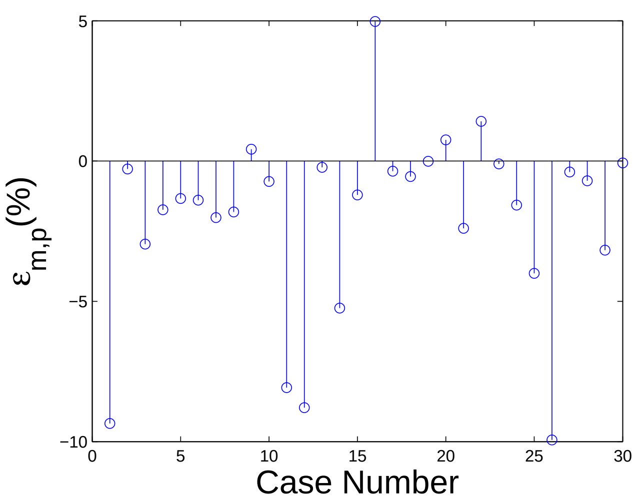

Figure 4.8: Accuracy of Procedure 9

4.4.2 Monotonicity

It has been shown in Li and Meerkov 2007 that monotonicity holds in serial lines and assembly systems, i.e., improving machine reliability and/or increasing buffer capacity lead to improvement of system production rate. Similar properties are observed in split and merge systems for all policies as well.

Corollary 9 *Under assumptions i')-vi'), the system production rates in split and merge systems are monotonically increasing with respect to μ_i and c_i, $i = 1, \dots, 4$, N_i, $i = 1, 2, 3$; and are monotonically decreasing with respect to λ_i.*

Proof: See Appendix. ■

4.4.3 Reversibility

For the split and merge exponential systems with circulate and priority policies considered in this thesis, reversibility property still holds. To illustrate this, denote the machine and buffer parameters in Figure 4.1 as λ_i^s, μ_i^s, c_i^s, $i = 1, \dots, 4$, N_i^s, $i = 1, 2, 3$, and in Figure 4.2 as λ_i^m, μ_i^m, c_i^m, and N_i^m. In addition,

$$\begin{aligned} \lambda_1^s &= \lambda_4^m, & \lambda_2^s &= \lambda_3^m, \\ \lambda_3^s &= \lambda_1^m, & \lambda_4^s &= \lambda_2^m, \\ \mu_1^s &= \mu_4^m, & \mu_2^s &= \mu_3^m, \\ \mu_3^s &= \mu_1^m, & \mu_4^s &= \mu_2^m, \\ c_1^s &= c_4^m, & c_2^s &= c_3^m, \\ c_3^s &= c_1^m, & c_4^s &= c_2^m, \\ N_1^s &= N_3^m, & N_2^s &= N_1^m, \\ N_3^s &= N_2^m. \end{aligned} \tag{22}$$

Corollary 10 *Under assumptions i')-vi') and condition (22), the system production rates in split and merge systems with circulate and priority policies have identical production rates. In other words*

$$\widehat{PR}_{m,c} = \widehat{PR}_{s,c}, \qquad \widehat{PR}_{m,p} = \widehat{PR}_{s,p}. \tag{23}$$

Proof: See Appendix. ■

4.4.4 Comparisons

Again, a comparison between the circulate and priority policies has been carried out. Similar to Bernoulli case, the results show that the difference in system production rates between systems with circulate and priority policies is typically small. In other words,

$$|\widehat{PR}_{s,c} - \widehat{PR}_{s,p}| \ll 1, \quad |\widehat{PR}_{m,c} - \widehat{PR}_{m,p}| \ll 1. \tag{24}$$

In addition, numerical results suggest that it is always beneficial to assign more reliable machine with higher priority.

Based on reversibility, similar argument applies to split system as well.

Chapter 5

EXTENSIONS

The methods introduced here can be easily extended to split and merge systems with longer lines and multiple branches. Please see Figures (see Figures 5.1-5.4 for illustrations).

The preliminary studies have been carried out to analyze such systems. Overlapping decomposition procedures have been implemented to evaluate the system performance. For example, for long split lines, overlapped Lines 1-3 become $(m_{11}, \ldots, m'_{1M_1}, b_{11}, \ldots, b_{1M_1-1})$, $(m''_{1M_1}, m_{21}, \ldots, m_{2M_2}, b_{21}, \ldots, b_{2M_2})$ and $(m'''_{1M_1}, m_{31}, \ldots, m_{3M_3}, b_{31}, \ldots, b_{3M_3})$. Long serial line analysis procedure (Li and Meerkov 2007) will be applied here. For multiple split lines, M lines are introduced, (m_1, m'_2, b_1), (m''_2, m_3, b_2), $\ldots$, $(m_2^{\prime\cdots\prime}, m_{M+1}, b_M)$.

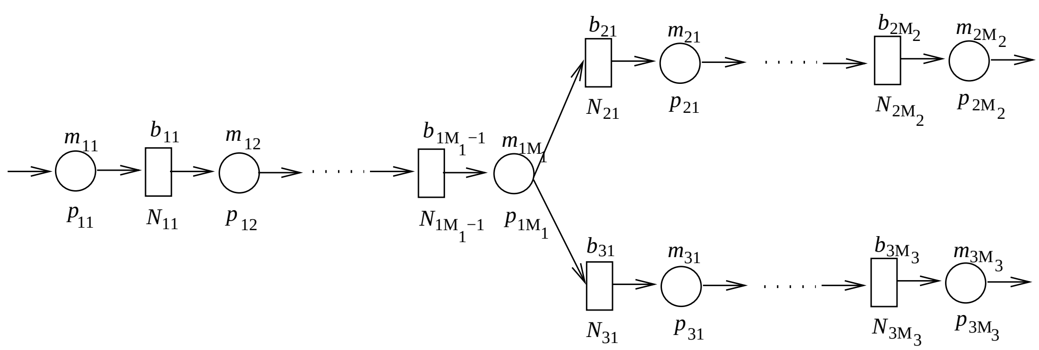

Figure 5.1: Long split lines

The preliminary results show that the proposed methods still achieve acceptable accuracy in production rate estimation. All structural properties hold for such systems as well.

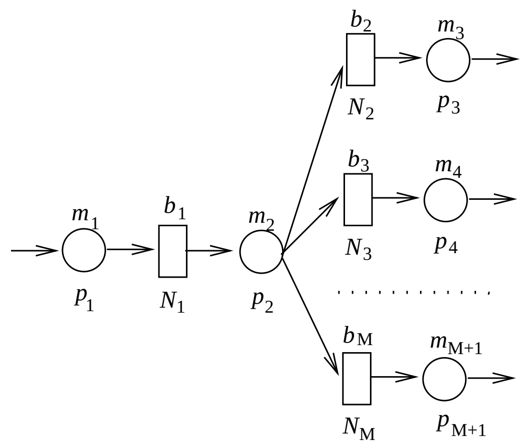

Figure 5.2: Multiple split lines

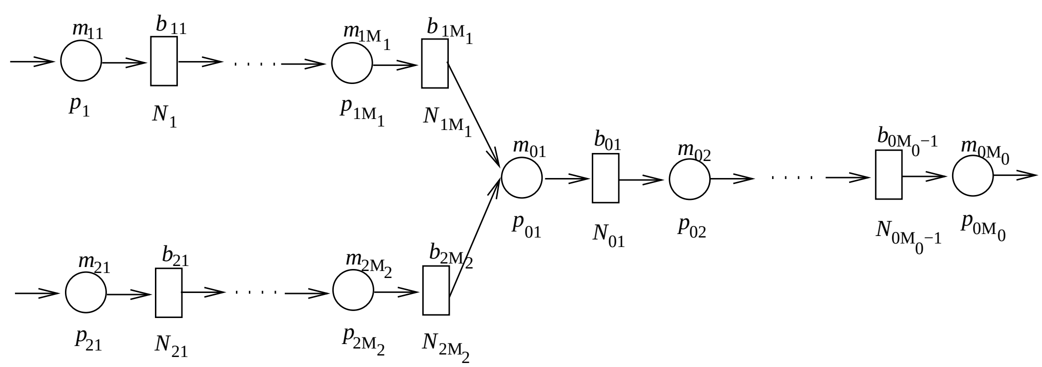

Figure 5.3: Long merge lines

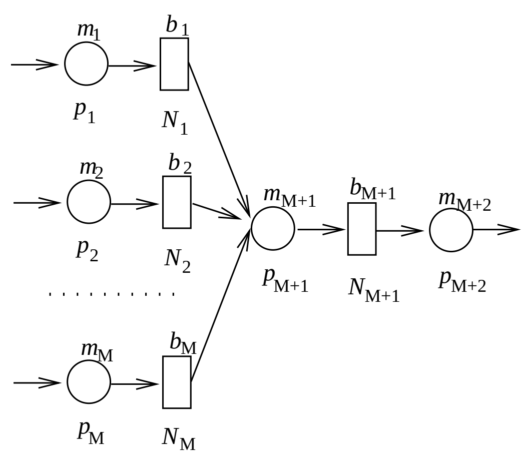

Figure 5.4: Multiple merge lines

Chapter 6

CONCLUSIONS AND FUTURE WORK

Split and merge are widely used in many manufacturing systems. This thesis presents analytical methods to approximate the system production rates of split and merge systems with Bernoulli and exponential reliability machines. Three split and merge policies are addressed: circulate, priority and percentage. It is shown that these methods can provide an accurate precision for system production rate estimation. The successful development of such methods will provide production engineers a quantitative tool for design and continuous improvement of such complex production systems.

In future work, we plan to extend the work to performance analysis of re-entrant lines. Figure 6.1 provides an example of a re-entrant line with Bernoulli reliability machines. The system consists of M machines and $2M-1$ buffers and jobs need to be processed twice before releasing. The first time jobs are processed at machines m_i, $i = 1, \ldots, M$, and buffers b_{1i}, $i = 1, \ldots, M-1$, between two consecutive machines. After first time processing at machine m_M, all jobs are sent to buffer b_0, waiting for second time processing. Then the jobs are reprocessed at machines m_i, $i = 1, \ldots, M$, but through buffers b_{2i}, $i = 1, \ldots, M-1$. Jobs leave the system after being processed at m_M for the second time.

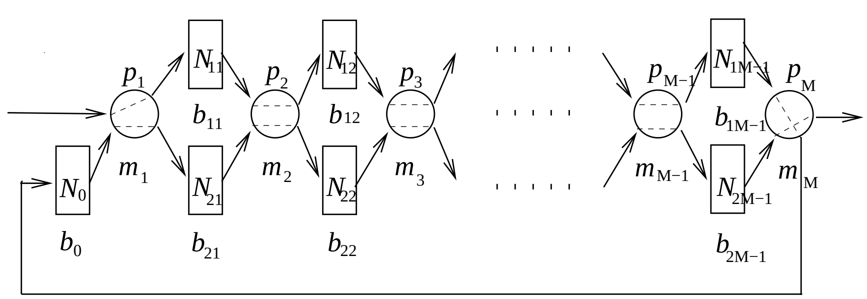

Figure 6.1: Re-entrant lines

As pointed out in Chapter 2, no accurate analysis of re-entrant lines is available. Developing a method to analyze the performance of re-entrant lines is of importance. Using the method on split/merge systems developed here, we can consider re-entrant lines as a group of multiple split/merge lines. Therefore, a recursive procedure could be developed to analyze them iteratively.

The successful development of this work could provide production engineers and managers a quantitative tool for design, analysis and continuous improvement of complex production systems.

APPENDIX: PROOFS

To prove Theorem 1, we need the following facts:

Lemma A.1 *Consider the sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$, defined in procedures 1 and 2. If $\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1)$ and $\widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1)$, then $\widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s)$.*

Proof: First we consider circulate policy. If

$$\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1), \quad \widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1),$$

from (2) and monotonicity in serial line (Jacobs and Meerkov 1995) we obtain

$$p_2'(s+1) < p_2'(s), \quad \widehat{pr}_1(s+1) < \widehat{pr}_1(s), \quad \widehat{X}_{10}(s+1) < \widehat{X}_{10}(s).$$

It follows from (3) and monotonicity in serial line that

$$p_2''(s+1) > p_2''(s), \quad \widehat{pr}_2(s+1) > \widehat{pr}_2(s), \quad \widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s).$$

Analogously, due to (4), we obtain

$$p_2'''(s+1) > p_2'''(s), \quad \widehat{pr}_3(s+1) > \widehat{pr}_3(s), \quad \widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s).$$

Using similar logic, the statement is true for priority policy. ■

Lemma A.2 *Sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$ defined in Procedures 1 and 2 are monotonically increasing.*

Proof: By induction. First we consider circulate policy. For $s = 0$, from (2)

$$p_2'(1) = p_2, \quad \widehat{X}_{10}(1) < 1.$$

It follows from (3) and (4) that

$$p_2''(1) = 0.5p_2(1 - \widehat{X}_{10}(1)), \quad \widehat{X}_{2N_2}(1) > 0,$$
$$p_2'''(1) = 0.5p_2(1 + \widehat{X}_{2N_2}(1))(1 - \widehat{X}_{10}(1)), \quad \widehat{X}_{3N_3}(1) > 0.$$

This leads to

$$p'_2(2) < p_2 = p'_2(1), \quad \widehat{X}_{10}(2) < \widehat{X}_{10}(1).$$

Thus, we obtain

$$p''_2(2) > p''_2(1), \quad p'''_2(2) > p'''_2(1),$$

which implies that

$$\widehat{X}_{2N_2}(2) > \widehat{X}_{2N_2}(1), \quad \widehat{X}_{3N_3}(2) > \widehat{X}_{3N_3}(1).$$

Now assume for $s > 0$, $\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1)$ and $\widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1)$. Then from Lemma A.1, we obtain

$$\widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s) \quad \widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s).$$

Similar arguments apply to priority policy. ■

Proof of Theorem 1: Since the sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$ are monotonic (Lemma A.2) and are bounded from above and below (Jacobs and Meerkov 1995), they are convergent. Therefore, the limits of $\widehat{pr}_i(s)$, $i = 1, 2, 3$, exist. ■

It turns out it is easy to prove Corollary 3 first then to prove Corollaries 1 and 2.

Proof of Corollary 3: First we prove the property for split system shown in Figure 3.3.

- **Circulate policy.** The steady state equations of Procedure 1 are as follows:

$$\begin{aligned}
p'_2 &= p_2(1 - \widehat{X}_{2N_2}\widehat{X}_{3N_3}), \\
\widehat{pr}_1 &= PR(p_1, p'_2, N_1), \qquad (A.1) \\
\widehat{X}_{10} &= 1 - \frac{\widehat{pr}_1}{p'_2}, \\
p''_2 &= 0.5p_2(1 + \widehat{X}_{3N_3})(1 - \widehat{X}_{10}), \\
\widehat{pr}_2 &= PR(p''_2, p_3, N_2), \qquad (A.2) \\
\widehat{X}_{2N_2} &= 1 - \frac{\widehat{pr}_2}{p''_2}, \\
p'''_2 &= 0.5p_2(1 + \widehat{X}_{2N_2})(1 - \widehat{X}_{10}), \\
\widehat{pr}_3 &= PR(p'''_2, p_4, N_3), \qquad (A.3) \\
\widehat{X}_{3N_3} &= 1 - \frac{\widehat{pr}_3}{p'''_2}.
\end{aligned}$$

Then the production rates of Lines 1-3 can be written as

$$\begin{aligned}
\widehat{pr}_1 &= p'_2(1 - \widehat{X}_{10}) = p_2(1 - \widehat{X}_{2N_2}\widehat{X}_{3N_3})(1 - \widehat{X}_{10}), \\
\widehat{pr}_2 &= p''_2(1 - \widehat{X}_{2N_2}) = 0.5p_2(1 + \widehat{X}_{3N_3})(1 - \widehat{X}_{10})(1 - \widehat{X}_{2N_2}), \\
\widehat{pr}_3 &= p'''_2(1 - \widehat{X}_{3N_3}) = 0.5p_2(1 + \widehat{X}_{2N_2})(1 - \widehat{X}_{10})(1 - \widehat{X}_{3N_3}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{pr}_2 + \widehat{pr}_3 &= 0.5p_2(1+\widehat{X}_{3N_3})(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}) + 0.5p_2(1+\widehat{X}_{2N_2})(1-\widehat{X}_{10})(1-\widehat{X}_{3N_3}) \\
&= 0.5p_2(1-\widehat{X}_{10})(1+\widehat{X}_{3N_3}-\widehat{X}_{2N_2}-\widehat{X}_{2N_2}\widehat{X}_{3N_3}+1-\widehat{X}_{3N_3}+\widehat{X}_{2N_2}-\widehat{X}_{2N_2}\widehat{X}_{3N_3}) \\
&= p_2(1-X_{10})(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}) \\
&= \widehat{pr}_1.
\end{aligned}$$

- **Priority policy.** Similarly, the steady state equations of Procedure 2 are

$$\begin{aligned}
p_2' &= p_2(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}), \\
\widehat{pr}_1 &= PR(p_1, p_2', N_1), && \text{(A.4)} \\
\widehat{X}_{10} &= 1-\frac{\widehat{pr}_1}{p_2'}, \\
p_2'' &= p_2(1-\widehat{X}_{10}), \\
\widehat{pr}_2 &= PR(p_2'', p_3, N_2), && \text{(A.5)} \\
\widehat{X}_{2N_2} &= 1-\frac{\widehat{pr}_2}{p_2''}, \\
p_2''' &= p_2(1-\widehat{X}_{10})\widehat{X}_{2N_2}, \\
\widehat{pr}_3 &= PR(p_2''', p_4, N_3), && \text{(A.6)} \\
\widehat{X}_{3N_3} &= 1-\frac{\widehat{pr}_3}{p_2'''}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\widehat{pr}_1 &= p_2(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3})(1-\widehat{X}_{10}), \\
\widehat{pr}_2 &= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}), \\
\widehat{pr}_3 &= p_2(1-\widehat{X}_{10})\widehat{X}_{2N_2}(1-\widehat{X}_{3N_3}).
\end{aligned}$$

Again it follows that

$$\begin{aligned}
\widehat{pr}_2 + \widehat{pr}_3 &= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}) + p_2\widehat{X}_{2N_2}(1-\widehat{X}_{10})(1-\widehat{X}_{3N_3}) \\
&= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}-\widehat{X}_{2N_2}\widehat{X}_{3N_3}) \\
&= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}) \\
&= \widehat{pr}_1.
\end{aligned}$$

- **Percentage policy.** The steady state equations of Procedure 3 will be

$$\begin{aligned}
\beta &= \frac{(1-\widehat{X}_{3N_3})\alpha}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}, \\
p_2' &= p_2(1-\beta\widehat{X}_{2N_2}-(1-\beta)\widehat{X}_{3N_3}), \\
\widehat{pr}_1 &= PR(p_1, p_2', N_1), && \text{(A.7)}
\end{aligned}$$

$$
\begin{aligned}
\widehat{X}_{10} &= 1 - \frac{\widehat{pr}_1}{p_2'}, \\
p_2'' &= \beta p_2(1-\widehat{X}_{10}), \\
\widehat{pr}_2 &= PR(p_2'', p_3, N_2), && \text{(A.8)} \\
\widehat{X}_{2N_2} &= 1 - \frac{\widehat{pr}_2}{p_2''}, \\
p_2''' &= (1-\beta)p_2(1-\widehat{X}_{10}), \\
\widehat{pr}_3 &= PR(p_2''', p_4, N_3), && \text{(A.9)} \\
\widehat{X}_{3N_3} &= 1 - \frac{\widehat{pr}_3}{p_2'''}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\widehat{pr}_1 &= p_2(1-\beta\widehat{X}_{2N_2} - (1-\beta)\widehat{X}_{3N_3})(1-\widehat{X}_{10}), \\
\widehat{pr}_2 &= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2})\beta, \\
\widehat{pr}_3 &= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{3N_3})(1-\beta).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\widehat{pr}_2 + \widehat{pr}_3 &= p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2})\beta + p_2(1-\widehat{X}_{10})(1-\widehat{X}_{3N_3})(1-\beta) \\
&= p_2(1-\widehat{X}_{10})(\beta - \widehat{X}_{2N_2}\beta + 1 - \widehat{X}_{3N_3} - \beta + \widehat{X}_{3N_3}\beta) \\
&= \widehat{pr}_1.
\end{aligned}
$$

In addition, it can be shown that

$$
\begin{aligned}
\frac{\widehat{pr}_2}{\widehat{pr}_1} &= \frac{p_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2})\beta}{p_2(1-\beta\widehat{X}_{2N_2} - (1-\beta)\widehat{X}_{3N_3})(1-\widehat{X}_{10})} \\
&= \frac{\beta(1-\widehat{X}_{2N_2})}{1-\beta\widehat{X}_{2N_2} - (1-\beta)\widehat{X}_{3N_3}} \\
&= \frac{(1-\widehat{X}_{2N_2})\frac{\alpha(1-\widehat{X}_{3N_3})}{1-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}-\alpha\widehat{X}_{3N_3}}}{1-\frac{\alpha(1-\widehat{X}_{3N_3})}{1-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}-\alpha\widehat{X}_{3N_3}}\widehat{X}_{2N_2} - (1-\frac{\alpha(1-\widehat{X}_{3N_3})}{1-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}-\alpha\widehat{X}_{3N_3}})\widehat{X}_{3N_3}} \\
&= \frac{\alpha(1-\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})}{(1-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}-\alpha\widehat{X}_{3N_3})(1-\widehat{X}_{3N_3}) - \alpha(1-\widehat{X}_{3N_3})(\widehat{X}_{2N_2}-\widehat{X}_{3N_3})} \\
&= \alpha.
\end{aligned}
$$

Similarly

$$
\frac{\widehat{pr}_3}{\widehat{pr}_1} = 1-\alpha.
$$

Analogously, the same logic can be applied to the merge system. The corollary is readily obtained. ■

To prove Corollary 1, the following lemma is needed.

Lemma A.3 *Introduce quantity γ,*

$$\gamma = 1 - \beta \widehat{X}_{2N_2} - (1-\beta)\widehat{X}_{3N_3}. \tag{A.10}$$

Quantity γ is monotonically decreasing with respect to $\widehat{X}_{2N_2}$ and $\widehat{X}_{3N_3}$.

Proof:

$$\begin{aligned}
\gamma &= 1 - \beta \widehat{X}_{2N_2} - (1-\beta)\widehat{X}_{3N_3} \\
&= 1 - \frac{(1-\widehat{X}_{3N_3})\alpha}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}\widehat{X}_{2N_2} - \left(1 - \frac{(1-\widehat{X}_{3N_3})\alpha}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}\right)\widehat{X}_{3N_3} \\
&= \frac{1}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}\Big(1-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}-\alpha\widehat{X}_{3N_3}-\alpha\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2}\widehat{X}_{3N_3}-\widehat{x}_{3N_3} \\
&\quad +\widehat{X}_{2N_2}\widehat{X}_{3N_3}-\alpha\widehat{X}_{2N_2}\widehat{3N_3}+\alpha\widehat{X}^2_{3N_3}+\alpha\widehat{X}_{3N_3}-\alpha\widehat{X}^2_{3N_3}\Big) \\
&= \frac{(1-\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial\gamma}{\partial\widehat{X}_{2N_2}} &= \frac{-(1-\widehat{X}_{3N_3})}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha} - \frac{(1-\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2}(-1+\alpha) \\
&= \frac{(1-\widehat{X}_{3N_3})(-1+\widehat{X}_{2N_2}-\widehat{X}_{2N_2}\alpha+\widehat{X}_{3N_3}\alpha+1-\alpha-\widehat{X}_{2N_2}+\alpha\widehat{X}_{2N_2})}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2} \\
&= \frac{-\alpha(1-\widehat{X}_{3N_3})^2}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2} \\
&< 0, \\
\frac{\partial\gamma}{\partial\widehat{X}_{3N_3}} &= \frac{-(1-\widehat{X}_{2N_2})}{1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha} - \frac{(1-\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2}(-\alpha) \\
&= \frac{(1-\widehat{X}_{2N_2})(-1+\widehat{X}_{2N_2}-\alpha\widehat{X}_{2N_2}+\alpha\widehat{X}_{3N_3}+\alpha-\alpha\widehat{X}_{3N_3})}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2} \\
&= \frac{-\alpha(1-\widehat{X}_{2N_2})^2}{[1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}\alpha-\widehat{X}_{3N_3}\alpha]^2} \\
&< 0.
\end{aligned}$$

Therefore, γ is monotonically decreasing with respect to $\widehat{X}_{2N_2}$ and $\widehat{X}_{3N_3}$. ■

Proof of Corollary 1: This corollary is proved by contradiction.

- **Circulate policy**. Assume there exists another solution $\widetilde{\widehat{pr}}_1 \neq \widehat{pr}_1$.
  - **Case 1.** If $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, then from (A.1), due to monotonicity of serial lines (Jacobs and Meerkov 1995) and by monotonicity of $Q(\cdot)$ function (Jacobs and Meerkov 1995), it follows that

$$\widetilde{p}'_2 > p'_2, \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{X}}_{10} > \widehat{X}_{10}. \tag{A.11}$$

Now consider the following cases:

* **Case 1.1.** If $\widetilde{\widehat{pr}}_2 \leq \widehat{pr}_2$, from (A.2) and monotonicity of $Q(\cdot)$ function, we must have
$$\widetilde{p}_2'' \leq p_2'', \qquad \widetilde{\widehat{X}}_{2N_2} \leq \widehat{X}_{2N_2}.$$
Due to (A.3) and (A.11), it implies that
$$\widetilde{p}_2''' < p_2''', \qquad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3.$$
It follows that
$$\widetilde{\widehat{pr}}_2 + \widetilde{\widehat{pr}}_3 < \widehat{pr}_2 + \widehat{pr}_3.$$
From Corollary 3, we have $\widetilde{\widehat{pr}}_1 \leq \widehat{pr}_1$, which is a contradiction.
* **Case 1.2.** If $\widetilde{\widehat{pr}}_2 > \widehat{pr}_2$, again using (A.2) and monotonicity of $Q(\cdot)$ function, we obtain
$$\widetilde{p}_2'' > p_2'', \qquad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$
Due to (A.11), we must have
$$\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}.$$
From (A.2) it implies that
$$\widetilde{p}_2'' < p_2'', \qquad \widetilde{\widehat{pr}}_2 < \widehat{pr}_2.$$
Again it is a contradiction to the assumption. Therefore, $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$ is not possible.

- **Case 2.** If $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, analogously we can show that it also leads to a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, i.e., there is a unique solution.

- **Priority policy.** Again assume there exists another solution $\widetilde{\widehat{pr}}_1 \neq \widehat{pr}_1$.

  - **Case 1.** If $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, then equation (A.11) is obtained. From (A.5), we obtain
$$\widetilde{p}_2'' < p_2'', \qquad \widetilde{\widehat{pr}}_2 < \widehat{pr}_2, \qquad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}. \tag{A.12}$$
By (A.6), it implies that
$$\widetilde{p}_2''' < p_2''', \qquad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3.$$
Thus, we obtain $\widetilde{\widehat{pr}}_2 + \widetilde{\widehat{pr}}_3 < \widehat{pr}_2 + \widehat{pr}_3$, which leads to a contradiction. Hence $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$ is not possible.
  - **Case 2.** If $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, analogously we can show that it also leads to contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, i.e., there is a unique solution.

- **Percentage policy.** Assume again there exists another solution $\widetilde{\widehat{pr}}_1 \neq \widehat{pr}_1$.

  – **Case 1.** If $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, then we obtain equation (A.11). From Corollary 3 we have

  $$\widetilde{\widehat{pr}}_2 = \alpha\widetilde{\widehat{pr}}_1 > \alpha\widehat{pr}_1 = \widehat{pr}_2, \quad \widetilde{\widehat{pr}}_3 = (1-\alpha)\widetilde{\widehat{pr}}_1 > (1-\alpha)\widehat{pr}_1 = \widehat{pr}_3.$$

  It follows that

  $$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \quad \widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}.$$

  From Lemma A.3 it leads to

  $$\widetilde{\gamma} < \gamma.$$

  It follows that

  $$\widetilde{p}_2' < p_2',$$

  which is contradictory to (A.11). Hence $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$ is not possible.

  – **Case 2.** If $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, analogously we can show that it also leads to contradiction.

  Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, i.e., there is a unique solution.

■

Proof of Corollary 2: By applying the similar logic in the proof of Corollary 1, the corollary can be approved. ■

Proof of Corollary 4: This corollary is also proved by contradiction. First, we consider the split system.

- **Circulate policy.** We begin with monotonicity with respect to p_i, $i = 1, \ldots, 4$, then extent to the cases of N_i, $i = 1, 2, 3$.

  Monotonicity with respect to p_1. Assume that if $\widetilde{p}_1 > p_1$, it leads to $\widetilde{\widehat{pr}}_1 \leq \widehat{pr}_1$. Then we must have

  $$\widetilde{p}_2' < p_2', \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{X}}_{10} < \widehat{X}_{10}. \tag{A.13}$$

  * **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.2) we obtain

    $$\widetilde{p}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}. \tag{A.14}$$

    Using (A.3) this implies that

    $$\widetilde{p}_2''' > p_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3, \quad \widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}. \tag{A.15}$$

    It follows that $\widetilde{\widehat{pr}}_2 + \widetilde{\widehat{pr}}_3 > \widehat{pr}_2 + \widehat{pr}_3$, which leads to a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, from (A.3) we must have

  $$\widetilde{\widehat{pr}}_3 < \widehat{pr}_3, \quad \widetilde{p}_2''' < p_2''', \quad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2},$$

  which implies that

  $$\widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{2N_2}\widehat{X}_{3N_3}.$$

  Again it is a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, i.e., there is a unique solution.

- **Monotonicity with respect to p_2.** Again assume that $\widetilde{p}_2 > p_2$ leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain equation (A.13).

  * **Case 1.** If $\widetilde{p}_2'' > p_2''$, we obtain

    $$\widetilde{\widehat{pr}}_2 > \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

    This leads to (A.29). In other words, we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction.

  * **Case 2.** If $\widetilde{p}_2'' < p_2''$, following the similar logic we can obtain $\widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}$. Then due to (A.13) we must have $\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}$, which implies that

    $$\widetilde{\widehat{pr}}_3 > \widehat{pr}_3, \quad \widetilde{p}_2''' > p_2'''.$$

    However, this results in $\widetilde{p}_2'' > p_2''$, which is a contradiction.

  * **Case 3.** If $\widetilde{p}_2'' = p_2''$, this implies from (A.2) and (A.3) that

    $$\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}, \quad \widetilde{p}_2''' < p_2''',$$

    and

    $$\widetilde{\widehat{pr}}_3 < \widehat{pr}_3, \quad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}.$$

    This will be contradict to (A.13). Therefore, $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$ is impossible.

  Now we consider the possibility of $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. Similarly, we can show that this implies

  $$\widetilde{\widehat{X}}_{10} = \widehat{X}_{10}, \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{pr}}_2 = \widehat{pr}_2.$$

  * **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, we have

    $$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \quad \widetilde{p}_2''' > p_2''',$$

    this leads to $\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}$, which is a contradiction.

  * **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, it implies (A.29) holds and it follows that

    $$\widetilde{p}_2''' > p_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

    In other words, we obtain $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which contradicts the assumption.

Therefore, the only possibility is $\widehat{\widetilde{pr}}_1 > \widehat{pr}_1$, which implies the monotonicity.

– **Monotonicity with respect to** p_3**.** Again Assume that $\widetilde{p}_3 > p_3$ results in $\widehat{\widetilde{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13).

  * **Case 1.** If $\widehat{\widetilde{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.2) we have

$$\widetilde{p}_2'' \geq p_2'', \quad \widehat{\widetilde{pr}}_2 > \widehat{pr}_2, \quad \widehat{\widetilde{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

    Using (A.3) this leads to

$$\widetilde{p}_2''' > p_2''', \quad \widehat{\widetilde{pr}}_3 > \widehat{pr}_3.$$

    In other words, we have $\widehat{\widetilde{pr}}_1 > \widehat{pr}_1$, which is a contradiction.

  * **Case 2.** If $\widehat{\widetilde{X}}_{3N_3} < \widehat{X}_{3N_3}$, then from (A.13) we must have

$$\widehat{\widetilde{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

    This results in

$$\widetilde{p}_2''' > p_2''', \quad \widehat{\widetilde{X}}_{3N_3} > \widehat{X}_{3N_3},$$

    which is a contradiction.

  Now if $\widehat{\widetilde{pr}}_1 = \widehat{pr}_1$. Similarly, we can show that this implies

$$\widehat{\widetilde{X}}_{10} = \widehat{X}_{10}, \quad \widehat{\widetilde{X}}_{2N_2}\widehat{\widetilde{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widehat{\widetilde{pr}}_2 = \widehat{pr}_2.$$

  * **Case 1.** If $\widehat{\widetilde{X}}_{3N_3} > \widehat{X}_{3N_3}$, again it implies (A.29) and it follows that

$$\widetilde{p}_2''' > p_2''', \quad \widehat{\widetilde{pr}}_3 > \widehat{pr}_3.$$

    Again this implies $\widehat{\widetilde{pr}}_1 > \widehat{pr}_1$, and is a contradiction.

  * **Case 2.** If $\widehat{\widetilde{X}}_{3N_3} < \widehat{X}_{3N_3}$, due to (A.13) we must have

$$\widehat{\widetilde{X}}_{2N_2} > \widehat{X}_{2N_2},$$

    and it leads to

$$\widetilde{p}_2''' < p_2''', \quad \widehat{\widetilde{pr}}_3 < \widehat{pr}_3, \quad \widehat{\widetilde{X}}_{3N_3} < \widehat{X}_{3N_3}. \tag{A.16}$$

    A contradiction is also arrived.

  Therefore, the only possibility is $\widehat{\widetilde{pr}}_1 > \widehat{pr}_1$.

– **Monotonicity with respect to** p_4**.** Using the same logic to prove the monotonicity with respect to p_3, we can obtain the conclusion analogously.

– **Monotonicity with respect to** N_1**.** If $\widetilde{N}_1 > N_1$ results in $\widehat{\widetilde{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13).

* **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.2) we obtain (A.29). Using (A.3) this leads to

$$\widetilde{p}_2''' > p_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

In other words, we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, then due to (A.13) we must have

$$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

From (A.3) this results in

$$\widetilde{p}_2''' > p_2''', \quad \widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3},$$

which is again a contradiction.

Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. Following the similar logic, we can show that this also leads to a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

– **Monotonicity with respect** N_2**.** Still we assume $\widetilde{N}_2 > N_2$ leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then (A.13) is obtained.

* **Case 1.** If $\widetilde{\widehat{X}}_{2N_2} \geq \widehat{X}_{2N_2}$, then (A.15) holds and this leads to

$$\widetilde{p}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2.$$

Thus we have a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}$, then from (A.13) we must have

$$\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}.$$

This results in

$$\widetilde{p}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2,$$

which implies that we must have $\widetilde{\widehat{pr}}_3 < \widehat{pr}_3$. In other words,

$$\widetilde{p}_2''' < p_2''', \quad \widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3},$$

which is also a contraction.

Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, it implies

$$\widetilde{p}_2' = p_2', \quad \widetilde{\widehat{X}}_{10} = \widehat{X}_{10}, \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} = \widehat{X}_{2N_2}\widehat{X}_{3N_3}. \tag{A.17}$$

* **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.2) we have

$$\widetilde{p}_2''' \geq p_2''', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2.$$

It follows that

$$\widetilde{\widehat{pr}}_3 < \widehat{pr}_3, \quad \widetilde{p}_2''' < p_2'''.$$

Thus, a contradiction is arrived.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, then

$$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \quad \widetilde{p}_2''' \geq p_2'''.$$

It implies that $\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}$, which is again a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

– **Monotonicity with respect to N_3.** This property can be proved analogously to the above proof.

• **Priority policy.**

– **Monotonicity with respect to p_1.** Assume that if $\widetilde{p}_1 > p_1$, it leads to $\widetilde{\widehat{pr}}_1 \leq \widehat{pr}_1$. Then from (A.4) we must have (A.13). By (A.5) it results in (A.29). From (A.6) this implies that (A.15) holds. It follows that

$$\widetilde{\widehat{pr}}_2 + \widetilde{\widehat{pr}}_3 > \widehat{pr}_2 + \widehat{pr}_3,$$

which leads to a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

– **Monotonicity with respect to p_2.** Again assume that if $\widetilde{p}_2 > p_2$, it leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13) again. Similarly, (A.29) and (A.15) hold. In other words, we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction. If $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. we can show that this implies

$$\widetilde{\widehat{X}}_{10} = \widehat{X}_{10}, \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{p}_2' = p_2'.$$

Then from (A.5) we obtain (A.29), and this leads to (A.15), which will result in $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, i.e., a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

– **Monotonicity with respect to p_3.** Assume that if $\widetilde{p}_3 > p_3$, it leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we again obtain (A.13). From (A.5) it leads to

$$\widetilde{p}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2,$$

Then we must have $\widetilde{\widehat{pr}}_3 < \widehat{pr}_3$. From (A.6) this implies that

$$\widetilde{p}_2''' < p_2''', \quad \widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}.$$

It follows that we need to have

$$\widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}.$$

This leads to $\widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}$, which is a contradiction. Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. We can show that this implies (A.17) holds. It follows from (A.5) that

$$\widetilde{p}_2'' = p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

Thus (A.15) holds. Again this implies $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, and is a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect to** p_4**.** Assume again that if $\widetilde{p}_4 > p_4$, it leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13). It follows that (A.29) and (A.15) hold and this leads to $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction. Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. We can show that this implies (A.17) holds. Due to (A.5) it implies

$$\widetilde{p}_2'' = p_2'', \quad \widetilde{\widehat{pr}}_2 = \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} = \widehat{X}_{2N_2}.$$

  Thus from (A.6) we have

$$\widetilde{p}_2''' = p_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

  Again this implies $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, and is a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect to** N_1**.** If $\widetilde{N}_1 > N_1$ results in $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13). Similarly it follows that (A.29) and (A.15) hold and we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction. If $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, following the similar logic, we can show that this also leads to a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect** N_2**.** Still we assume $\widetilde{N}_2 > N_2$ results in $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then (A.13) is obtained. Thus from (A.5) we have

$$\widetilde{p}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2.$$

  Then we must have (A.16) and it leads to $\widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}$. Thus we have a contradiction. Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, it implies (A.17) holds. Then it follows from (A.5) that

$$\widetilde{p}_2'' = p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}.$$

  Due to (A.6) we obtain (A.16), which implies that

$$\widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{2N_2}\widehat{X}_{3N_3}.$$

  Thus, a contradiction is arrived. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect to** N_3**.** Assume $\widetilde{N}_3 > N_3$ results in $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.13) and (A.29) again. Then it follows that

$$\widetilde{p}_2''' > p_2''', \quad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3,$$

  which leads to a contradiction. Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, it implies (A.17) holds and

$$\widetilde{p}_2'' = p_2'', \quad \widetilde{\widehat{pr}}_2 = \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} = \widehat{X}_{2N_2}.$$

  It follows from (A.6) that

$$\widetilde{p}_2''' = p_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

  It results in contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Percentage policy.**
  - **Monotonicity with respect to p_1.** Assume that $\widetilde{p}_{>}p_2$ leads to $\widetilde{\widehat{pr}}_1 \leq \widehat{pr}_1$. From (A.7) it follows that $\widetilde{p}_2' < p_2'$. In addition, from Corollary 3 we obtain

    $$\widetilde{\widehat{pr}}_2 \leq \widehat{pr}_2, \quad \widetilde{\widehat{pr}}_3 \leq \widehat{pr}_3.$$

    Using (A.8) and (A.9) we have

    $$\widetilde{p}_2'' \leq p_2'', \quad \widetilde{p}_2''' \leq p_2''',$$

    and it follows that

    $$\widetilde{\widehat{X}}_{2N_2} \leq \widehat{X}_{2N_2}, \quad \widetilde{\widehat{X}}_{3N_3} \leq \widehat{X}_{3N_3}.$$

    From Lemma A.3, we obtain $\widetilde{p}_2' \geq p_2'$, which is a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$
  - **Monotonicity with respect to p_i, $i = 2, 3, 4$ and to N_i, $i = 1, 2, 3$.** Following the similar logic, these monotonic properties can be proved.

The monotonic properties in merge system can be proved analogously. ■

Proof of Corollary 5: This corollary is also proved by contradiction.

- **Circulate policy.** If $\widehat{pr}_1^s > \widehat{pr}_3^m$, from (A.1) we obtain

  $$p_2^{s\prime} > p_3^{m\prime}, \quad \widehat{X}_{10}^s > \widehat{X}_{3N}^m.$$

  - **Case 1.** If $\widehat{pr}_2^s < \widehat{pr}_1^m$, we will have

    $$p_2^{s\prime\prime} < p_3^{m\prime\prime}, \quad \widehat{X}_{2N}^s < \widehat{X}_{10}^m.$$

    It follows from (A.3) that

    $$p_2^{s\prime\prime\prime} < p_3^{m\prime\prime\prime}, \quad \widehat{pr}_3^s < \widehat{pr}_2^m.$$

    It implies that $\widehat{pr}_1^s < \widehat{pr}_3^m$, which is a contradiction.
  - **Case 2.** If $\widehat{pr}_3^s < \widehat{pr}_2^m$, by following the similar logic we can show that this also leads to a contradiction.
  - **Case 3.** If $\widehat{pr}_2^s \geq \widehat{pr}_1^m$, $\widehat{pr}_3^s \geq \widehat{pr}_2^m$, we obtain that

    $$p_2^{s\prime\prime} \geq p_3^{m\prime\prime}, \quad \widehat{X}_{2N}^s \geq \widehat{X}_{10}^m,$$

    and

    $$p_2^{s\prime\prime\prime} < p_3^{m\prime\prime\prime}, \quad \widehat{X}_{3N}^s < \widehat{X}_{20}^m.$$

    It follows from (A.1) that

    $$p_2^{s\prime} < p_3^{m\prime}, \quad \widehat{pr}_1^s < \widehat{pr}_3^m.$$

    Again a contradiction is arrived.

Analogously we can prove that $\widehat{pr}_1^s < \widehat{pr}_3^m$ also implies contradictory results. Therefore, the only possibility is $\widehat{pr}_1^s = \widehat{pr}_3^m$, i.e, $\widehat{PR}_{s,c} = \widehat{PR}_{m,c}$.

- **Priority policy.** If $\widehat{pr}_1^s > \widehat{pr}_3^m$, from (A.4) we obtain

$$p_2^{s\prime} > p_3^{m\prime}, \quad \widehat{X}_{10}^s > \widehat{X}_{3N}^m.$$

Using (A.5) we have

$$p_2^{s\prime\prime} < p_3^{m\prime\prime}, \quad \widehat{pr}_2^s < \widehat{pr}_1^m, \quad \widehat{X}_{2N}^s < \widehat{X}_{10}^m.$$

From (A.6) it follows that

$$p_2^{s\prime\prime\prime} < p_3^{m\prime\prime\prime}, \quad \widehat{pr}_3^s < \widehat{pr}_2^m, \quad \widehat{X}_{3N}^s < \widehat{X}_{20}^m.$$

It implies that $\widehat{pr}_1^s < \widehat{pr}_3^m$, which is a contradiction. Analogously, $\widehat{pr}_1^s > \widehat{pr}_3^m$ also leads to a contradiction. Therefore, the only choice is $\widehat{pr}_1^s = \widehat{pr}_3^m$, i.e, $\widehat{PR}_{s,p} = \widehat{PR}_{m,p}$.

■

To prove Theorem 3, we need the following lemmas:

Lemma A.4 *Consider the sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$, defined in procedures 6 and 7. If $\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1)$ and $\widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1)$, then $\widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s)$.*

Proof: First we consider circulate policy. If

$$\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1), \quad \widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1),$$

from (3) and monotonicity in serial line (Chiang 1999) we obtain

$$\mu_2'(s+1) < \mu_2'(s), \quad \lambda_2'(s+1) > \lambda_2'(s), \quad \widehat{pr}_1(s+1) < \widehat{pr}_1(s), \quad \widehat{X}_{10}(s+1) < \widehat{X}_{10}(s).$$

It follows from (4) and monotonicity in serial line that

$$\mu_2''(s+1) > \mu_2''(s), \quad \lambda_2''(s+1) < \lambda_2''(s), \quad \widehat{pr}_2(s+1) > \widehat{pr}_2(s), \quad \widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s).$$

Analogously, due to (5), we obtain

$$\mu_2'''(s+1) > \mu_2'''(s), \quad \lambda_2'''(s+1) < \lambda_2'''(s), \quad \widehat{pr}_3(s+1) > \widehat{pr}_3(s), \quad \widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s).$$

Using similar logic, the statement is true for priority policy. ■

Lemma A.5 *Sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$ defined in Procedures 6 and 7 are monotonically increasing.*

Proof: By induction. First we consider circulate policy. For $s = 0$, from (3)

$$\mu_2'(1) = \mu_2, \quad \lambda_2'(1) = \lambda_2, \quad \widehat{X}_{10}(1) < 1.$$

It follows from (4) and (5) that

$$\mu_2''(1) = 0.5\mu_2(1 - \widehat{X}_{10}(1)), \quad \lambda_2''(1) = \lambda_2 + \mu_2 - \mu_2''(1), \quad \widehat{X}_{2N_2}(1) > 0,$$
$$\mu_2'''(1) = 0.5\mu_2(1 + \widehat{X}_{2N_2}(1))(1 - \widehat{X}_{10}(1)), \quad \lambda_2'''(1) = \lambda_2 + \mu_2 - \mu_2'''(1), \quad \widehat{X}_{3N_3}(1) > 0.$$

Thus, we obtain

$$\widehat{X}_{2N_2}(1) > \widehat{X}_{2N_2}(0), \quad \widehat{X}_{3N_3}(1) > \widehat{X}_{3N_3}(0).$$

Now assume for $s > 0$, $\widehat{X}_{2N_2}(s) > \widehat{X}_{2N_2}(s-1)$ and $\widehat{X}_{3N_3}(s) > \widehat{X}_{3N_3}(s-1)$. Then from Lemma A.4, we obtain

$$\widehat{X}_{2N_2}(s+1) > \widehat{X}_{2N_2}(s) \quad \widehat{X}_{3N_3}(s+1) > \widehat{X}_{3N_3}(s).$$

Similar arguments apply to priority policy. ■

Proof of Theorem 3: Since the sequences $\widehat{X}_{2N_2}(s)$ and $\widehat{X}_{3N_3}(s)$ are monotonic (Lemma A.5) and are bounded from above and below (Chiang 1999), they are convergent. Therefore, the limits of $\widehat{pr}_i(s)$, $i = 1, 2, 3$, exist. ■

Proof of Corollary 8: First we prove the property for split system shown in Figure 4.3.

- **Circulate policy.** The steady state equations of Procedure 6 are as follows:

$$\begin{aligned}
\mu_2' &= \mu_2(1 - \widehat{X}_{2N_2}\widehat{X}_{3N_3}), \\
\lambda_2' &= \lambda_2 + \mu_2 - \mu_2', \\
\widehat{pr}_1 &= PR([\lambda_1, \lambda_2'], [\mu_1, \mu_2'], [c_1, c_2], N_1), \qquad \text{(A.18)} \\
e_2' &= \mu_2'/(\lambda_2 + \mu_2), \\
\widehat{X}_{10} &= 1 - \frac{\widehat{pr}_1}{c_2 e_2'}, \\
\mu_2'' &= 0.5\mu_2(1 + \widehat{X}_{3N_3})(1 - \widehat{X}_{10}), \\
\lambda_2'' &= \lambda_2 + \mu_2 - \mu_2'', \\
\widehat{pr}_2 &= PR([\lambda_2'', \lambda_3], [\mu_2'', \mu_3], [c_2, c_3], N_2), \qquad \text{(A.19)} \\
e_2'' &= \mu_2''/(\lambda_2 + \mu_2), \\
\widehat{X}_{2N_2} &= 1 - \frac{\widehat{pr}_2}{c_2 e_2''}, \\
\mu_2''' &= 0.5\mu_2(1 + \widehat{X}_{2N_2})(1 - \widehat{X}_{10}), \\
\lambda_2''' &= \lambda_2 + \mu_2 - \mu_2'',
\end{aligned}$$

$$\begin{aligned}
\widehat{pr}_3 &= PR([\lambda_2''', \lambda_4], [\mu_2''', \mu_4], [c_2, c_4], N_3), \quad (A.20)\\
e_2''' &= \mu_2'''/(\lambda_2+\mu_2),\\
\widehat{X}_{3N_3} &= 1 - \frac{\widehat{pr}_3}{c_2 e_2'''}.
\end{aligned}$$

Then the production rates of Lines 1-3 can be written as

$$\begin{aligned}
\widehat{pr}_1 &= c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3})/(\lambda_2+\mu_2),\\
\widehat{pr}_2 &= 0.5c_2\mu_2(1-\widehat{X}_{2N_2})(1+\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2),\\
\widehat{pr}_3 &= 0.5c_2\mu_2(1+\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{pr}_2 + \widehat{pr}_3 &= 0.5c_2\mu_2(1-\widehat{X}_{2N_2})(1+\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2)\\
&\quad +0.5c_2\mu_2(1+\widehat{X}_{2N_2})(1-\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2)\\
&= 0.5c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}+\widehat{X}_{3N_3}-\widehat{X}_{2N_2}\\
&\quad +1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}-\widehat{X}_{3N_3}+\widehat{X}_{2N_2})/(\lambda_2+\mu_2)\\
&= c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3})/(\lambda_2+\mu_2)\\
&= \widehat{pr}_1.
\end{aligned}$$

- **Priority policy.** Similarly, the steady state equations of Procedure 7 are

$$\begin{aligned}
\mu_2' &= \mu_2(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3}),\\
\lambda_2' &= \lambda_2+\mu_2-\mu_2',\\
\widehat{pr}_1 &= PR([\lambda_1, \lambda_2'], [\mu_1, \mu_2'], [c_1, c_2], N_1), \quad (A.21)\\
e_2' &= \mu_2'/(\lambda_2+\mu_2),\\
\widehat{X}_{10} &= 1 - \frac{\widehat{pr}_1}{c_2 e_2'},\\
\mu_2'' &= \mu_2(1-\widehat{X}_{10}),\\
\lambda_2'' &= \lambda_2+\mu_2-\mu_2'',\\
\widehat{pr}_2 &= PR([\lambda_2'', \lambda_3], [\mu_2'', \mu_3], [c_2, c_3], N_2), \quad (A.22)\\
e_2'' &= \mu_2''/(\lambda_2+\mu_2),\\
\widehat{X}_{2N_2} &= 1 - \frac{\widehat{pr}_2}{c_2 e_2''},\\
\mu_2''' &= \mu_2\widehat{X}_{2N_2}(1-\widehat{X}_{10}),\\
\lambda_2''' &= \lambda_2+\mu_2-\mu_2'',\\
\widehat{pr}_3 &= PR([\lambda_2''', \lambda_4], [\mu_2''', \mu_4], [c_2, c_4], N_3), \quad (A.23)\\
e_2''' &= \mu_2'''/(\lambda_2+\mu_2),\\
\widehat{X}_{3N_3} &= 1 - \frac{\widehat{pr}_3}{c_2 e_2'''}.
\end{aligned}$$

Then we obtain

$$\begin{aligned} \widehat{pr}_1 &= c_2\mu_2(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2), \\ \widehat{pr}_2 &= c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2})/(\lambda_2+\mu_2), \\ \widehat{pr}_3 &= c_2\mu_2(1-\widehat{X}_{10})\widehat{X}_{2N_2}(1-\widehat{X}_{3N_3})/(\lambda_2+\mu_2)). \end{aligned}$$

Again it follows that

$$\begin{aligned} \widehat{pr}_2+\widehat{pr}_3 &= c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2})/(\lambda_2+\mu_2)+c_2\mu_2(1-\widehat{X}_{10})\widehat{X}_{2N_2}(1-\widehat{X}_{3N_3})/(\lambda_2+\mu_2)) \\ &= c_2\mu_2(1-\widehat{X}_{10})(1-\widehat{X}_{2N_2}+\widehat{X}_{2N_2}-\widehat{X}_{2N_2}\widehat{X}_{3N_3})/(\lambda_2+\mu_2) \\ &= c_2\mu_2(1-\widehat{X}_{2N_2}\widehat{X}_{3N_3})(1-\widehat{X}_{10})/(\lambda_2+\mu_2) \\ &= \widehat{pr}_1. \end{aligned}$$

Analogously, the same logic can be applied to the merge system. The corollary is readily obtained.

■

Proof of Corollary 6: This corollary is proved by contradiction.

- **Circulate policy**. Assume there exists another solution $\widehat{\widetilde{pr}}_1 \neq \widehat{pr}_1$.
  - **Case 1.** If $\widehat{\widetilde{pr}}_1 > \widehat{pr}_1$, then from (A.18), due to monotonicity of serial lines (Chiang 1999), it follows that

    $$\widetilde{\lambda}_2' > \lambda_2', \quad \widetilde{\mu}_2' < \mu_2', \quad \widehat{\widetilde{X}}_{2N_2}\widehat{\widetilde{X}}_{3N_3} < \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widehat{\widetilde{X}}_{10} > \widehat{X}_{10}. \tag{A.24}$$

    Now consider the following cases:
    * **Case 1.1.** If $\widehat{\widetilde{pr}}_2 \leq \widehat{pr}_2$, from (A.19), we must have

      $$\widetilde{\lambda}_2'' \geq \lambda_2'', \quad \widetilde{\mu}_2'' \leq \mu_2'', \quad \widehat{\widetilde{X}}_{2N_2} \leq \widehat{X}_{2N_2}.$$

      Due to (A.20) and (A.24), it implies that

      $$\widetilde{\mu}_2''' \leq \mu_2''', \quad \widetilde{\lambda}_2''' \geq \lambda_2''', \quad \widehat{\widetilde{pr}}_3 \leq \widehat{pr}_3.$$

      It follows that

      $$\widehat{\widetilde{pr}}_2+\widehat{\widetilde{pr}}_3 \leq \widehat{pr}_2+\widehat{pr}_3.$$

      From Corollary 8, we have $\widehat{\widetilde{pr}}_1 \leq \widehat{pr}_1$, which is a contradiction.
    * **Case 1.2.** If $\widehat{\widetilde{pr}}_2 > \widehat{pr}_2$, again using (A.19), we obtain

      $$\widetilde{\lambda}_2'' < \lambda_2'', \quad \widetilde{\mu}_2'' > \mu_2'', \quad \widehat{\widetilde{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

      Due to (A.24), we must have

      $$\widehat{\widetilde{X}}_{3N_3} < \widehat{X}_{3N_3}.$$

From (A.19) it implies that

$$\widetilde{\mu}_2'' < \mu_2'', \quad \widetilde{\widehat{pr}}_2 < \widehat{pr}_2.$$

Again it is a contradiction to the assumption. Therefore, $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$ is not possible.

- **Case 2.** If $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, analogously we can show that it also leads to a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, i.e., there is a unique solution.

- **Priority policy.** Again assume there exists another solution $\widetilde{\widehat{pr}}_1 \neq \widehat{pr}_1$.

  - **Case 1.** If $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, then equation (A.24) is obtained. From (A.22), we obtain

  $$\widetilde{\mu}_2'' < \mu_2'', \quad \widetilde{\widehat{pr}}_2 < \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}. \tag{A.25}$$

  By (A.23), it implies that

  $$\widetilde{\mu}_2''' < \mu_2''', \quad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3.$$

  Thus, we obtain $\widetilde{\widehat{pr}}_2 + \widetilde{\widehat{pr}}_3 < \widehat{pr}_2 + \widehat{pr}_3$, which leads to a contradiction. Hence $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$ is not possible.

  - **Case 2.** If $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, analogously we can show that it also leads to contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$, i.e., there is a unique solution.

■

Proof of Corollary 7: By applying the similar logic in the proof of Corollary 6, the corollary can be approved. ■

Proof of Corollary 9: This corollary is also proved by contradiction. First, we consider the split system.

- **Circulate policy.** We begin with monotonicity with respect to λ_i, $i = 1, \ldots, 4$, then extend to the cases of μ_i, c_i, $i = 1, \ldots, 4$ and N_i, $i = 1, 2, 3$.

  Monotonicity with respect to λ_1. Assume that if $\widetilde{\lambda}_1 > \lambda_1$, it leads to $\widetilde{\widehat{pr}}_1 \geq \widehat{pr}_1$. Then we must have

  $$\widetilde{\lambda}_2' < \lambda_2', \quad \widetilde{\mu}_2' > \mu_2', \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{X}}_{10} > \widehat{X}_{10}. \tag{A.26}$$

  * **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.20) we obtain

  $$\widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2}, \quad \widetilde{\mu}_2''' < \mu_2''', \quad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3, \quad \widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}. \tag{A.27}$$

  This leads to a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, from (A.19) we must have

$$\widetilde{\mu}_2'' < \mu_2'', \quad \widetilde{\widehat{pr}}_2 < \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} < \widehat{X}_{2N_2},$$

which implies that

$$\widetilde{\mu}_2''' < \mu_2''', \quad \widetilde{\widehat{pr}}_3 < \widehat{pr}_3, \quad \widetilde{\widehat{pr}}_1 < \widehat{pr}_1.$$

Again it is a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, i.e., there is a unique solution.

– **Monotonicity with respect to λ_i, $i = 2, \ldots, 4$.** Using similar logic, we can obtain the conclusion analogously.

– **Monotonicity with respect to μ_1.** Assume that if $\widetilde{\mu}_1 > \mu_1$, it leads to $\widetilde{\widehat{pr}}_1 \leq \widehat{pr}_1$. Then we must have

$$\widetilde{\lambda}_2' > \lambda_2', \quad \widetilde{\mu}_2' < \mu_2', \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{X}}_{10} < \widehat{X}_{10}. \tag{A.28}$$

* **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \leq \widehat{X}_{3N_3}$, then from (A.20) we obtain

$$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3, \quad \widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}. \tag{A.29}$$

This leads to a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3}$, from (A.19) we must have

$$\widetilde{\mu}_2'' > p_2'', \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2, \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2},$$

which implies that

$$\widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3, \quad \widetilde{\widehat{pr}}_1 > \widehat{pr}_1.$$

Again it is a contradiction.

Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, i.e., there is a unique solution.

– **Monotonicity with respect to μ_i, $i = 2, \ldots, 4$.** Using similar logic, we can obtain the conclusion analogously.

– **Monotonicity with respect to c_1.** We assume that $\widetilde{c_1} > c_1$ leads to $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$, then we obtain

$$\widetilde{\lambda}_2' > \lambda_2', \quad \widetilde{\mu}_2' < \mu_2', \quad \widetilde{\widehat{X}}_{2N_2}\widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{2N_2}\widehat{X}_{3N_3}, \quad \widetilde{\widehat{X}}_{10} < \widehat{X}_{10}. \tag{A.30}$$

* **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.19), we obtain

$$\widetilde{\mu}_2'' > \mu_2'', \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2.$$

Using (A.20) this leads to

$$\widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

In other words, we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction.

* **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, then due to (A.30), we must have

$$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

From (A.20) this results in

$$\widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3},$$

which is again a contradiction.

Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. Following the similar logic, we can show that this also leads to a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect** c_i**,** $i = 2, \ldots, 4$**.** This property can be proved analogously to the above proof.
- **Monotonicity with respect to** N_1**.** If $\widetilde{N}_1 > N_1$ results in $\widetilde{\widehat{pr}}_1 < \widehat{pr}_1$. Then we obtain (A.30).

  * **Case 1.** If $\widetilde{\widehat{X}}_{3N_3} \geq \widehat{X}_{3N_3}$, then from (A.19), we obtain

  $$\widetilde{\mu}_2'' > \mu_2'', \quad \widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}, \quad \widetilde{\widehat{pr}}_2 > \widehat{pr}_2.$$

  Using (A.20) this leads to

  $$\widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{pr}}_3 > \widehat{pr}_3.$$

  In other words, we have $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$, which is a contradiction.

  * **Case 2.** If $\widetilde{\widehat{X}}_{3N_3} < \widehat{X}_{3N_3}$, then due to (A.30), we must have

  $$\widetilde{\widehat{X}}_{2N_2} > \widehat{X}_{2N_2}.$$

  From (A.20) this results in

  $$\widetilde{\mu}_2''' > \mu_2''', \quad \widetilde{\widehat{X}}_{3N_3} > \widehat{X}_{3N_3},$$

  which is again a contradiction.

  Now if $\widetilde{\widehat{pr}}_1 = \widehat{pr}_1$. Following the similar logic, we can show that this also leads to a contradiction. Therefore, the only possibility is $\widetilde{\widehat{pr}}_1 > \widehat{pr}_1$.

- **Monotonicity with respect** N_i**,** $i = 2, 3$**.** This property can be proved analogously to the above proof.

• **Priority policy.** The proof for priority policy is similar to circulate policy.

Proof of Corollary 10:

- **Circulate policy.** In Procedure 6, change the name of the parameters as follows:

$$\begin{aligned} \lambda_1^s &\Rightarrow \lambda_4^m, & \lambda_2^s &\Rightarrow \lambda_3^m, \\ \lambda_3^s &\Rightarrow \lambda_1^m, & \lambda_4^s &\Rightarrow \lambda_2^m, \\ \mu_1^s &\Rightarrow \mu_4^m, & \mu_2^s &\Rightarrow \mu_3^m, \\ \mu_3^s &\Rightarrow \mu_1^m, & \mu_4^s &\Rightarrow \mu_2^m, \\ c_1^s &\Rightarrow c_4^m, & c_2^s &\Rightarrow c_3^m, \\ c_3^s &\Rightarrow c_1^m, & c_4^s &\Rightarrow c_2^m, \\ N_1^s &\Rightarrow N_3^m, & N_2^s &\Rightarrow N_1^m, \\ N_3^s &\Rightarrow N_2^m. \end{aligned}$$

  Also change the serial number of the three lines as follows:

$$\begin{aligned} Line_1^s &\Rightarrow Line_3^m, \\ Line_2^s &\Rightarrow Line_1^m, \\ Line_3^s &\Rightarrow Line_2^m. \end{aligned}$$

  Finally, change the estimate of the probability that a buffer is empty to the estimate of the probability that a buffer is full, and vice versa.

  After these three name changes, Procedure 6 becomes Procedure 8. Therefore, we get $\widehat{pr}_1^s = \widehat{pr}_3^m$, i.e, $\widehat{PR}_{s,c} = \widehat{PR}_{m,c}$.

- **Priority policy.** After applying the same method, we get the conclusion that $\widehat{pr}_1^s = \widehat{pr}_3^m$, i.e, $\widehat{PR}_{s,p} = \widehat{PR}_{m,p}$.

■

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