

University of Kentucky UKnowledge

Theses and Dissertations--Mechanical Engineering

Mechanical Engineering

2012

### SPECIFIED MOTION AND FEEDBACK CONTROL OF ENGINEERING STRUCTURES WITH DISTRIBUTED SENSORS AND ACTUATORS

Amirhossien Ghasemi University of Kentucky, amir\_h\_ghasemi@yahoo.com

Right click to open a feedback form in a new tab to let us know how this document benefits you.

#### **Recommended Citation**

Ghasemi, Amirhossien, "SPECIFIED MOTION AND FEEDBACK CONTROL OF ENGINEERING STRUCTURES WITH DISTRIBUTED SENSORS AND ACTUATORS" (2012). *Theses and Dissertations--Mechanical Engineering*. 8. https://uknowledge.uky.edu/me\_etds/8

This Doctoral Dissertation is brought to you for free and open access by the Mechanical Engineering at UKnowledge. It has been accepted for inclusion in Theses and Dissertations--Mechanical Engineering by an authorized administrator of UKnowledge. For more information, please contact UKnowledge@lsv.uky.edu.

### STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained and attached hereto needed written permission statements(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine).

I hereby grant to The University of Kentucky and its agents the non-exclusive license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless a preapproved embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

### **REVIEW, APPROVAL AND ACCEPTANCE**

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's dissertation including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Amirhossien Ghasemi, Student Dr. T. Michael Seigler, Major Professor

Dr. James M. McDonough, Director of Graduate Studies

# SPECIFIED MOTION AND FEEDBACK CONTROL OF ENGINEERING STRUCTURES WITH DISTRIBUTED SENSORS AND ACTUATORS

Dissertation

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Collage of Engineering at the University of Kentucky

by

Amirhossien Ghasemi

Lexington, Kentucky

2012

Copyright  $^{\textcircled{o}}$  Amirhossein Ghasemi 2012

#### ABSTRACT OF DISSERTATION

#### SPECIFIED MOTION AND FEEDBACK CONTROL OF ENGINEERING STRUCTURES WITH DISTRIBUTED SENSORS AND ACTUATORS

This dissertation addresses the control of flexible structures using distributed sensors and actuators. The objective to determine the required distributed actuation inputs such that the desired output is obtained. Two interrelated facets of this problem are considered. First, we develop a dynamic-inversion solution method for determining the distributed actuation inputs, as a function of time, that yield a specified motion. The solution is shown to be useful for intelligent structure design, in particular, for sizing actuators and choosing their placement. Secondly, we develop a new feedback control method, which is based on dynamic inversion. In particular, filtered dynamic inversion combines dynamic inversion with a low-pass filter, resulting in a high-parameter-stabilizing controller, where the parameter gain is the filters cutoff frequency. For sufficiently large parameter gain, the controller stabilizes the closed-loop system and makes the  $\mathcal{L}_2$ -gain of the performance arbitrarily small, despite unknown-and-unmeasured disturbances. The controller is considered for both linear and nonlinear structural models.

KEYWORDS: Specified Motion, Intelligent Structures, Input-Output Linearizion, Dynamic Inversion, Piezoelectric

Amirhossein Ghasemi

30/06//2012

### Specified Motion and Feedback Control of Engineering Structures with Distributed Sensors and Actuators

 $\mathbf{b}\mathbf{y}$ 

Amirhossein Ghasemi

Dr. T. Michael Seigler

Director of Dissertation

Dr. James M. McDonough,

Director of Graduate Studies

30/06/2012,

### Acknowledgements

My deepest gratitude to Dr.T. Michael Seigler for his guidance and assistance throughout my entire PhD studies, and for continuously challenging my ideas to help expand my own knowledge.

I would like to thank my advisory committee Dr. Suzanne Weaver Smith, Dr. George Blandford, Dr. Louis M. Brock, and Dr. Tim Wu. I have great respect for each of them and always enjoy speaking with them. I owe a special gratitude to Dr. Babak Bazrgari for his friendship.

I would like to thank to all my friends in the Iranian Student Association at the University of Kentucky for their support, friendship and sense of community they have offered me.

Finally, I would also like to thank my family who have always offered their unconditional love and support to me which has been one of the greatest driving forces behind my work.

## Contents

Acknowledgements			iii
Li	List of Figures		
List of Tables			ix
1	Int	roduction	1
	1.1	Overview	1
	1.2	Distributed Sensing and Actuation	2
	1.3	The Servo-Constraint Problem	3
	1.4	Control of Flexible Structures	5
	1.5	Outline and Contribution	7
<b>2</b>	$\mathbf{Sp}$	ecified Motion of a Deformable Structure	9
	2.1	Problem description and general solution	11
	2.2	Linear Second Order System	16
	2.3	Applications to Flexible Structures	19
	2.4	Summary	25
3	$\mathbf{Dis}$	stributed actuation requirements of piezoelectric structures under servo-constrain	ts 29
	3.1	Piezoelectric shell model	30
	3.2	Actuation of servo-constraints	32
		3.2.1 Special case 1: non-zero motion constraint with zero disturbance	34
		3.2.2 Special case 2: zero motion constraint with non-zero disturbance	35
	3.3	Applications Examples	37
		3.3.1 Free-surface motion of a cantilever plate	37
		3.3.2 Localized vibration attenuation	41
	3.4	Summary	43

4	$\mathbf{Fil}$	tered Dynamic Inversion for Linear Systems	44
	4.1	Introduction	44
	4.2	Mathematical Background	48
	4.3	Filtered Dynamic Inversion	50
		4.3.1 Actuator and Sensor Collocation	54
		4.3.2 Modal Control	55
		4.3.3 Higher Order Filters	55
	4.4	Application Example: Control of a Cracked Beam	56
	4.5	Summary	61
5	Fil	tered Input-Output Linearization for Nonlinear Systems	62
	5.1	Introduction	62
	5.2	Filtered Input-Output Linearization	66
	5.3	Control of a Slewing Beam	69
		5.3.1 Vibration Control of Slewing Beam using Collocated Actuator/ Sensor $\ldots$ .	71
		5.3.2 Modal Control of a Slewing Flexible Beam	74
	5.4	Summary	77
6	Co	onclusions and Recommendations	78
	6.1	Conclusion	78
	6.2	Recommendations	79
$\mathbf{A}_{j}$	ppen	ndices	81
Bibliography			98
Vita			108

# List of Figures

1.1	Schematic of specific motion problem	1
1.2	Two mass spring system	4
1.3	Schematic of partly specified motion for an infinite dimensional servo-constraint problem	5
2.1	Schematic of partly specified motion problem	9
2.2	Schematic of partly specified motion problem	10
2.3	Inverted pendulum on a cart	15
2.4	(a) Pendulum angle, (b) Required actuation input force	16
2.5	Actuated Cantilever beam, case (1)	21
2.6	First required actuation input force for case 1 $(x_2 = L, x_{f1} = 0.1L, x_{f2} = 0.2L)$	23
2.7	Second required actuation input force for case 1 $(x_2 = L, x_{f1} = 0.1L, x_{f2} = 0.2L)$	23
2.8	First program actuation input force for case 1 $(x_2 = L, x_{f1} = 0.1L, x_{f2} = 0.6L)$	24
2.9	Second program actuation input force for case 1 $(x_2 = L, x_{f1} = 0.1L, x_{f_2} = 0.6L)$	24
2.10	Actuated Cantilever beam, case (2)	25
2.11	First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = L, x_{f1} = 0.1L, x_{f_2} = 0.2L$ )	26
2.12	Second program actuation input force for case 2 ( $x_2 = 0.8L, x_d = L, x_{f1} = 0.1L, x_{f2} =$	
	0.2L)	26
2.13	First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = 0.75L, x_{f1} = 0.1L, x_{f2} =$	
	0.2L)	27
2.14	First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = 0.75L, x_{f1} = 0.1L, x_{f_2} =$	
	0.2L)	27
3.1	Cantilever beam with three pairs of actuators	37
3.2	Components of transfer function $G_v$ for the cantilever plate	39
3.3	Components of inverse transfer function $G_v^{-1}$ for the cantilever plate	39
3.4	Piezoelectric actuation input requirements for Example 1, flapping $(c_1 = 2.5mm, c_2 =$	
	$2.5mm, c_3 = 2.5mm) \qquad \dots \qquad $	40

3.5	Piezoelectric actuation input requirements for Example 1, twisting $(c_1 = -2.5mm, c_2 =$		
	$0mm, c_3 = 2.5mm)$	40	
3.6	Piezoelectric actuation input requirements for Example 1, flapping and twisting ( $c_1 =$		
	$2.5mm, c_2 = 5mm, c_3 = 7.5mm$ )	40	
3.7	Simply supported damaged shell with three pairs of actuators.	41	
3.8	The unactuated response due to the disturbance input for Example 2		
3.9	Piezoelectric actuation input requirements for Example 2.	43	
3.10	3.10 Optimal piezoelectric actuation input requirements for Example 2		
4.1	Base excitation problem model	45	
4.2	Closed loop diagram of the base excitation model	48	
4.3	Feedback Connection	54	
4.4	A cantilever beam with a single-edge crack $\hfill \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	56	
4.5	A two-segment model with a massless rotational spring	56	
4.6	Tip-concentrated loading	56	
4.7	Cantilever beam with a piezoelectric actuator	57	
4.8	(a) Time response for the sensor output, (b) Time response for the deflection at the		
	crack, (c) Time response for the required actuation input $\hfill \ldots \ldots \ldots \ldots \ldots \ldots$	59	
4.9	Cantilever beam with two piezoelectric actuators	59	
4.10	Time response for th modal vibration	60	
4.11	Time response for the required actuation input $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	60	
4.12	Time response for th deflection at the crack	60	
5.1	A servo-actuated inverted pendulum	63	
5.2	A closed-loop system	64	
5.3	Feedback Connection	68	
5.4	Model of flexible spacecraft with a bonded piezoelectric actuator $\ldots \ldots \ldots \ldots$	69	
5.5	Model of flexible spacecraft with a collocated sensor and actuator	72	
5.6	Time response for attitude maneuver with Filtered Input-Output Linearization controller		
	for the collocated case	72	
5.7	Time response for sensor output with Filtered Input-Output Linearization controller for		
	the collocated case	73	
5.8	Time response for the required input with Filtered Input-Output Linearization controller		
	for the collocated case	73	

5.9	Time response for deflection of a tip of the beam with Filtered Input-Output Linearization	
	controller for the collocated case	73
5.10	The norm of the tip deflection for collocated case $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	74
5.11	Time response for attitude maneuver with Filtered Input-Output Linearization controller	
	for modal control	75
5.12	Time response for modal displacement with Filtered Input-Output Linearization con-	
	troller for modal control	75
5.13	Time response for the required input with Filtered Input-Output Linearization controller	
	for modal control	76
5.14	Time response for deflection of a tip of the beam with Filtered Input-Output Linearization	
	controller for modal control $\ldots \ldots \ldots$	76
5.15	The norm of the tip deflection for modal control	77

# List of Tables

2.1	Cantilever beam properties	21
3.1	Rectangular plate properties	38
3.2	Shell properties	42
4.1	Characteristic Solution of Intact and Damaged Cantilever Beam	58
5.1	Properties of the system	72

### Chapter 1

### Introduction

### 1.1 Overview

The focus of this dissertation is controlling specified motion of a flexible structure using distributed sensing and actuation. The general problem is depicted in Figure 1.1, where there is a region D of the structure for which the desired motion  $\xi(x, y, t)$  is specified. The objective is to determine the required distributed that produces the specified motion.



Figure 1.1: Schematic of specific motion problem

We consider two facet of this problem. First, we seek to determine actuation inputs, as a function of time, that produce a specified motion. Secondly, we seek to determine the actuation inputs, as a function of measured outputs that produce a specified motion. The first problem is a type of inverse problem, which is called the servo–constraint problem. The servo-constraint problem is useful in sizing actuators and choosing their placement. The second problem is a feedback control problem where the uncertainty is implied.

#### **1.2** Distributed Sensing and Actuation

The challenges of controlling flexible structures is greatly simplified by the use of distributed sensors and actuators. Distributed sensing provides more information about the structure than is obtained from sensing at a single point. For example, modal positions can be determined from a large number properly located of distributed sensors, thus making modal control possible. Distributed actuation is often the only feasible way to provide precise shape control. In vibration control, distributed actuators allow for greater control authority of modal positions.

Piezoelectric materials are often used as distributed actuators and sensors, because they are relatively small and lightweight, and can thus be attached to or embedded within flexible structures. As sensors, piezoelectric materials are highly sensitive to strain over a large frequency range. As actuators, piezoelectrics provide large blocking forces and high bandwidth control authority. Crawley and de Luis [1] are often cited as the first researchers to suggest the use of piezoelectric actuators as distributed sensors and actuators. Since then, piezoelectric materials have been used in numerous applications including [2–12].

In order to model a piezoelectric structure, two main techniques based on uniform strain assumption and the Bernoulli-Euler-Kirchhoff assumption are developed [13,14]. In the uniform strain assumption, it is assumed that a uniform strain exists in the mounted actuator while the strain is distributed linearly throughout the host structure. However, the Bernoulli-Euler-Kirchoff assumption considers a linear strain distribution throughout the actuator and host structure regardless of whether the actuator is embedded or laminated. The latter model has been widely applied on beams [15–18] and plates [19,20] and shells [21,22] to enhance the performance of these structures.

In this dissertation in order to model a flexible structure laminated or embedded with a piezoelectric actuator, a model based on Tzous form of Loves theory is used. Tzou introduced a generic shell model that showcased a double curvature, deep, flexible and elastic structure. He used four geometric equations (i.e., two Lame' parameter and two radii of curvature) to simplify the generic shell model to a variety of common shell structures such as paraboloidal shell [23], cylindrical shells [24, 25], conical shells [26], hemispherical shells [27] and toroidal shells [28]. Then based on the direct piezoelectricity, Gauss theorem, Maxwell principle, and the open circuit piezoelectricity assumptions, spatially distributed modal voltages and signals of piezoelectric patches were defined [29–31]. Piezoelectric forces and moments are derived, based on the converse piezoelectricy effect and Hook's law, as a function of modal voltages [32–34]. These controlling forces and moments are substituted into the generic shell equation to derive the equation of motion of piezoelectric shell.

Since the piezoelectric actuators and sensors are spatially distributed, the sensitivity of the these

types of sensors and actuators not only depends on the piezoelectric material properties but also on the location of the sensor or actuators. It is for this reason many focus on with the optimization of piezoelectric actuator location to determine the optimal location of sensors and actuators, in order to achieve the desired output with the minimum energy input [35–38].

Although the piezoelectric actuators have many advantages, one of the primary disadvantages of piezoelectric actuators are that the induced strains are severely limited (on the order of 1000  $\mu$ strain), consequentially limiting the amount of force that can be applied. Thus, an important consideration in the evaluation of piezoelectric actuators for a given application is whether they can supply sufficient input to achieve the desired objective. Seigler et al. [39, 40] suggested a general technique to evaluate piezoelectric voltages in order to achieve the desired motion, whether it be zero motion due to an external disturbance, such as vibration control or non-zero motion in regards to with active shape control, and to subsequently determine whether a distributed array of actuators is capable of supplying the required input .

### 1.3 The Servo-Constraint Problem

In the control of mechanical systems, it is often necessary to determine the forcing required to produce a desired motion. The designer of an intelligent structure should be confident that the distributed actuators are capable to accomplish the objectives, such as shape and vibration control. In addition to ensuring sufficient actuation authority, the relationship between force and motion helps the designer to place the actuators in the optimal locations so that the desired motion can be obtained with minimum actuation force.

Solving for the force from a specified motion is a type of inverse problem referred to in mechanics as the servo-constraint (or servo-control) problem. The simplest type of constraint is when the entire displacement field is specified; assuming that this motion is a solution to the equations of motion and the boundary conditions, the required forcing can be found by direct inverse solution. The more interesting circumstance is when the servo-constraint is such that only part of the displacement field is defined. Such a circumstance might arise for example when trying to control the trailing edge angle of a morphing wing [41], or localized vibration mitigation [42].

For finite-dimensional mechanical systems the servo-constraint problem is well illustrated by the two-mass system of Figure 1.2, taken from Ref. [43]. The positions of the two masses are labeled  $q_1$  and  $q_2$ , respectively; and a force, F, is applied to the first mass. A desired motion y(t), called the servo-constraint, is defined for the second mass. The program force, F(t), is sought such that  $q_2 = y(t)$ .



Figure 1.2: Two mass spring system

The servo-constraint problem is described by Chen [44, 45] as one of the few remaining frontiers of mechanics. It has been the subject of some recent interest, particularly in regards to finite dimensional systems [46–48]. For finite dimensional square systems (equal number of input and output), there are at least two known methods that are related to the servo-control problem. Perhaps the most well-known is a common technique of nonlinear control called *input-output linearization*.

However another method, perhaps less well-known, is the projection technique of Parciwizki and Blajer [43,49]. They discussed different types of servo -constraints where the constraint reaction due to the control input may be non-orthogonal or, in the extreme, tangent to the constraint manifold [50,51]. Blajer showed that the servo-constraint problem can be governed as a set of differential algebraic equations (DAE) [52–54]. The index of different DAE is a measure of singularity and can be defined as a number of times that an algebraic equation needs to be differentiated to obtain a standard form of an ordinary differential equation. The index of DAEs that describes the servo-constraint problem depends on the constraint that needs to be achieved. The role of an index in projection method closely related to the role of relative degree, which is key parameter in the input-output linearization. The controllability of partly specified motions problem, where the number of servo-constraints is less than the number of degree of freedom [55,56], is related to the concept of differential flatness [57,58]. Differential flatness denotes that the states and control input of a controllable system can be expressed as a desired out and the derivatives of the desired output. Blajer studied the required input actuation input when the servo-constraint problem is differentially flat [59–62]

In addition to the consideration of finite-dimensional systems, the servo-constraint problem is also applicable for infinite-dimensional mechanical systems. Some specific examples include adaptive structures applications such as morphing aircraft [41], the ultrasonic motor [63], and turbulent drag reduction via boundary wall vibration [64]. In each of these applications, it is necessary to produce a particular motion of a deformable structure. The corresponding servo-control problem is to determine the associated distributed actuation. The infinite-dimensional servo-constraint problem is usually more complex than the finite-dimensional case, especially since we must now deal with partial differential equations (see Ref. [65] for discussion). Of course, it is sometimes possible by various finite approximation methods to model a deformable structure by a finite set of ordinary differential equations; and then it becomes possible to apply finite-dimensional solution methods. This is the approach that is considered in this research.



Figure 1.3: Schematic of partly specified motion for an infinite dimensional servo-constraint problem

### 1.4 Control of Flexible Structures

Flexible structures have various applications in robotics [66], radar antenna [67] and flexible spacecraft [68,69]. Numerous research efforts have been conducted on the shape and vibration reduction of flexible systems [70–78]. The major challenges of controlling flexible structures are unknown and unmeasured external disturbances, high dimensionality, unknown system order, parametric uncertainty, and the nonlinearities.

Many flexible systems are usually lightweight which results in low frequency and low damping, if these structures are subjected to a disturbance, it might cause an unwanted vibration which can lead to the mechanical failure [79,80]. There are several classical control methods such as velocity and acceleration feedback to control the flexible structures [81,82]. Linear Quadratic Regulator (LQR) control [11,83] and Linear Quadratic Gaussian (LQG) control [84] are also proposed to control the vibrations of flexible structure. In the presence of known disturbance using internal model principle [85–87], the vibration of a flexible structure can be rejected; however for unknown disturbance some robust control methods such as adaptive control [88,89] must be applied for satisfactory performance.

In theory, flexible structures have an infinite dimensions; however in practice there are finite number of actuators and sensors. Consequently active flexible systems must be restricted to a few control modes. The effect of these residual (uncontrolled) modes brings uncertainty in the model, which might cause closed-loop instability in the closed loop system [90, 91]. Although methods such as LQR and velocity feedback control can attenuate the vibration of flexible structures, due to the high dimensionality of flexible structures, the application of these controllers have some limitations. In order to use LQR method, a full state feedback of a system is required. However, measurement of all states is not practically possible. To use the velocity feedback control an observer needs to be applied; thus this controller is not robust to the spillover phenomena.

Goh and Caughty [92] introduced positive position feedback (PFF) to attenuate the vibration of flexible structure. In this method, the second order nature of the filter causes the response to roll of quickly at high frequency, thus eliminating the usual spillover problem [93, 94]. Positive position feedback is a well-known method to control the vibration of structures. Fanson et al. [95] applied PFF strategy in space structures with piezoelectric materials. Baz and Poh [96] developed a modified independent modal space control (MIMSC) method. The MIMSC method used "time sharing" idea to control a large number of modes with a small numbers of actuators. Song and Agward [97] also used Pulse Width Pulse Frequency (PWPF) method to reduce the vibration of flexible systems. For the standard PFF controller, a second order filter in the feedback. Mahmoodi and Ahmadian [98,99] used two compensator in feedback to control the gain and damping simultaneously. One of the disadvantages of PPF is, in order to have a better performance and avoid instability, an approximate knowledge about the natural frequency of the system is required. Also, another disadvantageous of this method is that the closed loop of the system is more flexible than the open loop which may lead to a larger steady state error [100]. To solve this problem, a proportional term is added to the stiffness matrix of the system ensures that the steady state stiffness in the open and closed loop systems is the same. But adding this term will eliminate the advantages of robustness to spillover [101].

Modeling a flexible structure with a finite dimensional model, introduces uncertainty. This type of uncertainty plus other types of parametric uncertainties such as modal uncertainties and the uncertainty in the actuator dynamics can affect the performance of the structure significantly. To minimize effects of uncertainties in flexible structures different control methods such as neural network [102], genetic algorithm [103], adaptive control [104], fuzzy control [105], variable structure control [106, 107] and  $H_{\infty}$  [108] control have been proposed.

In practice, the flexible structure have nonlinearities. Sources of nonlinearity can involve the geometric nonlinearities and/or the material nonlinearities. Although in industrial applications, simple controller such as PD and PID, which are easy to implement are used, nonlinear control techniques in flexible systems have a significant importance. It is always desirable to reduce the control force without adding any actuators/sensors, while the same performance is the same. Inman [109] experimentally compare the PD control with a nonlinear feedback control. He showed that the system with a nonlinear feedback requires less input to achieve to desired motion. Kuo and Wang [110] also showed that the nonlinear controller can enhance the robustness of the two link manipulator. Popular powerful nonlinear control method include dynamic inversion [111–113], nonlinear adaptive control [114], Variable structure control [115, 116] and sliding mode control [117].

#### **1.5** Outline and Contribution

In chapter 2, The primary theoretical development of servo-constraint problem is addressed. The program constraint is defined by a general nonlinear relation of the motion of the structure. In this chapter, first a general solution that can be applied for the finite dimensional nonlinear model is introduced. Then the analysis is simplified by focusing on a linear vibrational model with the linear program constraints. To demonstrate the usefulness of this method, some examples are demonstrated.

In chapter 3, a piezoelectrically actuated shell structure with a servo-constraint is considered. Using the method explained in chapter 2, the required piezoelectric voltages that needs to be applied such that the servo-constraint is obtained will be determined. In the first step of this chapter, using the Tzou and Love's theory, the general piezoelectrically shell structure is modeled [118, 119]. Later, considering to have the same number of actuators as the number of servo-constraint, the required voltage is determined. In this chapter the problem is cast in the frequency domain. In order to show the effectiveness of the method, two examples of shape control and vibration mitigation for the shell structures is demonstrated in this chapter.

Chapter 4 presents a new feedback control strategy used to control the shape and vibration of a linear time-invariant structure. First, the basic idea behind this controller is shown through a simple example. First the open-loop control input to achieve the approximated desired motion is determined. The open loop control then will be augmented by a feedback loop that accounts for uncertainties due to modeling error and external vibration disturbances. The conditions on stability and performance of this controller will be discussed. Furthermore it is proven that if the ideal closed-loop system is asymptotically stable, and the filter cut-off frequency is sufficiently large, then the closed-loop system is stable. Also it is shown that if the ideal closed-loop system is asymptotically stable then the norm of the error can be made arbitrarily small by increasing the cut-off frequency. The numerical simulations demonstrate the effectiveness of the proposed controller in order to attenuate the vibration.

Chapter 5 focuses on the controller introduced in chapter 4. However in chapter 5, the controller is applied on the nonlinear system. The sufficient stability condition is discussed. It is proven that for the nonlinear if the ideal closed-loop system is  $\mathscr{L}_2$  stable, and the filter cut-off frequency is sufficiently large, then the closed-loop system is stable. Also it is shown that if the ideal closed-loop system is  $\mathscr{L}_2$  stable then the norm of the error can be made arbitrarily small by increasing the cut-off frequency. A slewing flexible is considered and it is shown that using the proposed controller the vibration of the beam can attenuated significantly while the the beam manurer a desired projection.

Finally, in chapter 6, the main results are discussed and suggestions for further research are provided.

### Chapter 2

# Specified Motion of a Deformable Structure

Solving for the force from a specified motion is a type of inverse problem referred to in mechanics as the *servo-constraint* (or *servo-control*) problem. To introduce the problem, consider a mass-spring system of Figure 2.1. The positions of the two masses are, respectively, labeled  $q_1$  and  $q_2$ . A force f, is applied to the first mass. The of motion given by

$$m_1 \ddot{q}_1 + k(2q_1 - q_2) = f, \tag{2.1}$$

$$m_2 \ddot{q}_2 + k(q_2 - q_1) = 0, (2.2)$$

where k is the stiffness and  $m_1$  and  $m_2$  are mass 1 and mass 2 respectively.



Figure 2.1: Schematic of partly specified motion problem

A desired motion y(t), which is called the program constraint, is defined for the second mass. That

is, the control objective is for all t > 0

$$q_2 = y(t). \tag{2.3}$$

We refer to (2.3), The objective is to determine the input force f that satisfies (2.3). The force f(t) that satisfies (2.3) is called the program force. To solve for the program force, we twice differentiate (2.3), and substituted  $\ddot{q}_2$  and  $q_2$  with $\ddot{y}$  and y, (2.2) takes the form of

$$\frac{m_1}{k}\ddot{y}(t) + y(t) = q_1.$$
(2.4)

(2.4) does not depend explicitly on f; however by differentiating (2.4) two more times respect to the time

$$\frac{m_1}{k}y^{(4)}(t) + \ddot{y}(t) = \ddot{q}_1.$$
(2.5)

Using (2.1-2.5), the required force f to satisfy the program constraint is

$$f = \frac{m_1 m_2}{k} y^{(4)}(t) + (m_1 + 2m_2) \ddot{y}(t) + k y(t).$$
(2.6)

We now consider a similar problem for flexible structures. The general problem addressed in this chapter is depicted in Figure 2.2. The objective is to determine the required input force F(x, z, t), such that the prescribed motion w(x, z, t) is obtained.



Figure 2.2: Schematic of partly specified motion problem

### 2.1 Problem description and general solution

Consider a general nonlinear model as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = \mathbf{f}(t) + \mathbf{B}\mathbf{u}(t), \qquad (2.7)$$

where  $\mathbf{M}$  is the  $n \times n$  generalized mass matrix,  $\mathbf{h}$  is a vector function quantifying the stiffness and damping properties of the structure and  $\mathbf{f}$  is an external disturbance input. The term  $\mathbf{Bu}$  is the applied control input. The term  $\mathbf{u}$  is the *m* vector control parameters and  $\mathbf{B}$  is the  $m \times n$  full rank matrix. In this research, we will limit ourselves to the linear dependence of  $\mathbf{u}$ ; however in general the dependence may be nonlinear. Sources of nonlinearity in  $\mathbf{h}$  can involve the geometric nonlinearities and/or the material nonlinearities.

Let P denote a material point of a deformable body, and let w(P, t) denote the location of the point P at time t. According to the series discretization method [120], it is assumed that w(P, t) can be approximated by the finite series

$$w(P,t) = \boldsymbol{\phi}^T(P)\mathbf{q}(t), \qquad (2.8)$$

where  $\boldsymbol{\phi} = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_n]^T$  is an *n*-vector of known functions, and  $\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$  is an *n*-vector of unknown generalized coordinates.

From the modal expansion of (2.8) it is often possible to construct a set of finite-dimensional ordinary differential equations to model the motion of the structure.

Let  $\{P_1, P_2, ..., P_s\}$  denote a set of points of the body for which motion is specified by the relation

$$\mathbf{g}(w(P_1, t), ..., w(P_s, t)) = \mathbf{y}(t), \tag{2.9}$$

where  $\mathbf{y} = [y_1 \quad \cdots \quad y_m]^T$  is an *m*-vector of continuous time-dependent functions. The relation of (2.9) is called the *program constraint* (or *servoconstraint*), and it indicates desired motion for the specified points of the body. Of course, these specified motions must also belong to the set of possible solutions of (2.7), and thus must not conflict with boundary conditions and initial conditions. It is also noted that the number of constraint equations in (2.9) and the number of control inputs are both equal to *m*. If the number of inputs were greater than *m*, some method of allocation should be defined.

While the program constraint of (2.9) expresses a general nonlinear relation between the motion of material points of the structure, it is worth mentioning the types of constraints that are commonly of interest. Applications involving structural control most typically fall into two categories: static/dynamic shape control [7], and vibration control. In the former application it is desired to actuate a particular shape,  $\eta(P,t)$ ; the corresponding constraint is  $w(P,t) = \eta(P,t)$ , which can be approximated by

$$w(P_1, t) = y_1(t)$$
  
$$\vdots$$
  
$$w(P_m, t) = y_m(t),$$

where  $y_i(t) = \eta(P_i, t)$ , i = 1, ..., m. Thus, *m* inputs are required to actuate *m* points of the structure. More generally, (2.9) can be used to construct any constraint that satisfies, at least approximately, the desired shape.

For vibration control, the objective is to suppress the motion of the structure; as such, the constraint might for example take the form

$$[w^{2}(P_{1},t) + \dots + w^{2}(P_{s},t)]^{1/2} = \epsilon,$$

where  $\epsilon$  is a constant that quantifies an acceptable level of vibration. The points  $P_1, ..., P_s$  could be the locations of maximum response for a given natural frequency, or simply "important" points of the structure where the vibration must be minimized (e.g., a crack in the structure).

Having defined a general form of the program constraint, we now seek a control input **u** that satisfies the program constraint. The solution method considered here, due to Parczewski and Blajer [49], is divided into two parts: analysis of the program motion, and synthesis of the control reaction. The general solution method is outlined as follows. With (2.8) the constraint of (2.9) can be written

$$\mathbf{g}(\mathbf{q}) = \mathbf{y}(t). \tag{2.10}$$

Differentiating (2.8) twice with respect to time results in

$$\Phi(\mathbf{q})\ddot{\mathbf{q}}(t) = \boldsymbol{\xi}(\mathbf{q}, \dot{\mathbf{q}}, t), \qquad (2.11)$$

where  $\mathbf{\Phi} = \partial \mathbf{g}/\partial \mathbf{q}$  and  $\boldsymbol{\xi}(t) = \ddot{\mathbf{y}} - (\partial \Phi/\partial \mathbf{q})\dot{\mathbf{q}}^2$ . Analysis of the program motion is conducted by defining a non-unique matrix  $\mathbf{B}^{\perp}$  such that  $\mathbf{B}^{\perp}\mathbf{B} = 0$ ; i.e.,  $\mathbf{B}^{\perp}$  is an orthogonal complement to  $\mathbf{B}$ . One method of constructing the orthogonal complement is to set  $\mathbf{B}^{\perp} = [\mathbf{v}_1 \cdots \mathbf{v}_{n-m}]^T$ , where the  $\mathbf{v}_i$  are the eigenvectors corresponding to the n - m zero eigenvalues  $\mathbf{BB}^T$ . Proving the aforementioned statement, the eigenvalue equation for a matrix  $\mathbf{B}^T \mathbf{B}$  can be expressed as

$$(\lambda \mathbf{I} - \mathbf{B}\mathbf{B}^T)\mathbf{v} = \mathbf{0}.$$
(2.12)

The eigenvector problem related t the zero eigen values is

$$\mathbf{B}\mathbf{B}^T\mathbf{v} = \mathbf{0}.\tag{2.13}$$

Pre-multiplying (2.13), with  $\mathbf{v}^T$ 

$$\mathbf{v}^T \mathbf{B} \mathbf{B}^T \mathbf{v} = 0. \tag{2.14}$$

Using matrix manipulation, (2.14) can be simplified as

$$(\mathbf{V}\mathbf{B}^T)^T(\mathbf{V}\mathbf{B}^T) = 0. \tag{2.15}$$

Considering  $(\mathbf{VB}^T)^T (\mathbf{VB}^T)$  is positive, it leads that eigenvectors  $\mathbf{v}$  corresponding to the zero eigenvalues  $\mathbf{B}^T \mathbf{B}$  is an orthogonal compliment of matrix  $\mathbf{B}$ . Pre-multiplying (2.7) by  $\mathbf{B}^{\perp}$  gives

$$\mathbf{B}^{\perp}\mathbf{M}\ddot{\mathbf{q}} = \mathbf{B}^{\perp}(\mathbf{f} - \mathbf{h}) \tag{2.16}$$

Hence, the constraint has been removed from the equations of motion by projection. The combined system of equations (2.11) and (2.16) is written

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}} = \tilde{\mathbf{h}},\tag{2.17}$$

where

$$ilde{\mathbf{M}} = \left[ egin{array}{c} \mathbf{B}^{\perp} \mathbf{M} \\ \mathbf{\Phi} \end{array} 
ight], \quad ilde{\mathbf{h}} = \left[ egin{array}{c} \mathbf{B}^{\perp} (\mathbf{f} - \mathbf{h}) \\ \mathbf{\xi} \end{array} 
ight].$$

The system of (2.17) consists of n equations with n unknown coordinates. In the case  $\tilde{\mathbf{M}}$  is invertible, (2.17) are ordinary differential equations (ODEs) that can be solved for  $\mathbf{q}$ . When  $\tilde{\mathbf{M}}$  is non-invertible, there are different methods that can be used to solve the problem [49, 121]. In this case where  $\tilde{\mathbf{M}}$  is singular, the program constraints needs to be differentiated until the control input shows up in the program constraint. Lets assume that  $k^{th}$  derivative of program constraint shown as

$$\mathbf{y}^{k} = \frac{d^{k}}{dt^{k}} \mathbf{\Phi} \mathbf{q}(t) = \mathbf{\Phi} \frac{d^{k}}{dt^{k}} \mathbf{M}^{-1} (\mathbf{f} - \mathbf{h} + \mathbf{B} \mathbf{u}).$$
(2.18)

Here k is the smallest positive number for which  $\frac{d^k}{dt^k} \Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{u} \neq \mathbf{0}$ . Defining  $\mathbf{D}^{\perp}$  such that  $\mathbf{D}^{\perp} \mathbf{M}^{-1} \mathbf{B} = \mathbf{0}$ , a new set of equations can be written as

$$\begin{bmatrix} \mathbf{D}^{\perp}\mathbf{M} \\ \mathbf{\Phi} \end{bmatrix} \mathbf{q}^{(k)}(t) = \begin{bmatrix} \mathbf{D}^{\perp}(\hat{\mathbf{f}} - \hat{\mathbf{h}}) \\ \mathbf{y}^{(k)}(t) \end{bmatrix}$$
(2.19)

where

$$\mathbf{y}^{k} = \mathbf{\Phi}\mathbf{M}^{-1}(\hat{\mathbf{f}} - \hat{\mathbf{h}}) + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{u}$$
(2.20)

where  $\hat{\mathbf{h}}$  is defined as

$$\hat{\mathbf{h}} = \frac{\partial^k}{\partial \dot{\mathbf{q}}^k} \mathbf{h}(\dot{\mathbf{q}}, \mathbf{q}, t) + \frac{\partial^k}{\partial \mathbf{q}^k} \mathbf{h}(\dot{\mathbf{q}}, \mathbf{q}, t) + \frac{\partial^k}{\partial t^k} \mathbf{h}(\dot{\mathbf{q}}, \mathbf{q}, t)$$

The set of (2.19) is an ordinary differential and the matrix  $\begin{bmatrix} \mathbf{D}^{\perp}\mathbf{M} & \mathbf{\Phi} \end{bmatrix}^T$  is an invertible matrix. For the control synthesis, upon solving for  $\mathbf{q}$  in (2.17) or in (2.19) and pre-multiplying  $\mathbf{B}^T$  to the 2.7, we have

$$\mathbf{B}^T \mathbf{M} \ddot{\mathbf{q}} + \mathbf{B}^T \mathbf{h} = \mathbf{B}^T \mathbf{f} + \mathbf{B}^T \mathbf{B} \mathbf{u}, \qquad (2.21)$$

and the input  ${\bf u}$  is found by

$$\mathbf{u} = \left(\mathbf{B}^T \mathbf{B}\right)^{-1} \mathbf{B}^T (\mathbf{M} \ddot{\mathbf{q}} + \mathbf{h} - \mathbf{f}).$$
(2.22)

The solution of (2.22) essentially proves the controllability of the specified motion.

#### Example 2.1

To demonstrate the control analysis discussed above, as shown in Figure 2.4 an inverted pendulum on the cart is considered.



Figure 2.3: Inverted pendulum on a cart

The of motion for this system is

$$\begin{bmatrix} M+m & ml\cos(\theta) \\ ml\cos(\theta) & I+ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -ml\dot{\theta}^2\sin(\theta) \\ -mgl\sin(\theta) \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$
(2.23)

where x is the cart position,  $\theta$  is the pendulum angle from vertical, M is the mass of the cart, l length to the pendulum center, f is the force applied to the cart and m, I are the mass and inertia of the pendulum; respectively. The control task is to keep the pendulum straight up. Hence the program constraint is

$$\theta = 0 \tag{2.24}$$

To obtain the program constraint in the form of (2.11), (2.24) needs to be differentiated twice. Here it needs to be noted that since the program constraint is differentiated, a number of integral constants needs to be determined. Thus the new program constraint will be

$$\ddot{\theta} + k_1 \dot{\theta} + k_2 \theta = 0, \tag{2.25}$$

where  $k_i$  are the positive integral constant. Here in this example, we assume  $k_i = 1$ . The orthogonal complement of matrix B would be matrix  $B^{\perp} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ . Pre-multiplying the of dynamics (2.23)

by  $B^{\perp}$  and considering the program constraints of (2.25), we will have

$$\begin{bmatrix} ml\cos(\theta) & I+ml^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} mgl\sin(\theta) \\ -k_1\dot{\theta} - k_2\theta \end{bmatrix}.$$
 (2.26)

, Solving (2.26), the states  $[x, \theta, \dot{x}, \dot{\theta}]$  will be determined. Then by implementing these states in to the (2.23), the required force can be found. Here it is assumed that M = 0.5kg, m = 0.5kg, l = 0.3m and  $I = 0.006kgm^2$ . In Figure 2.4, the required actuation force f and the pendulum angle  $\theta$  is shown.



Figure 2.4: (a) Pendulum angle, (b) Required actuation input force

### 2.2 Linear Second Order System

We now focus on an important subset vibration models described by (2.7), specifically linear systems of the form

$$M\ddot{q} + D\dot{q} + Kq = f + Bu, \qquad (2.27)$$

where  $\mathbf{M}$ ,  $\mathbf{D}$ ,  $\mathbf{K}$  are constant matrices; the latter two are the generalized damping and stiffness matrix, respectively. In addition, we consider a subset of program constraints of the form

$$\lim_{t \to \infty} \mathbf{E}[w(P_1, t) \cdots w(P_s, t)] = \mathbf{y}(t), \qquad (2.28)$$

where **E** is a constant  $m \times s$  matrix. This program constraint specifies a linear relation between the steady state motion of a finite number of material points of the structure. With (2.8), the the constraint of (2.28) becomes

$$\lim_{t \to \infty} \mathbf{Cq}(t) = \mathbf{y}(t), \tag{2.29}$$

where

$$\mathbf{C} = \mathbf{E} \begin{bmatrix} \phi(P_1) & \cdots & \phi(P_s) \end{bmatrix}^T,$$

is a constant  $m \times n$  matrix.

It is further assumed that  $\mathbf{y}(t)$  and  $\mathbf{f}(t)$  are periodic functions of the form

$$\mathbf{y}(t) = \sum_{i=1}^{p} (\mathbf{y}_{i}^{c} \cos \omega_{i} t + \mathbf{y}_{i}^{s} \sin \omega_{i} t), \qquad (2.30)$$

$$\mathbf{f}(t) = \sum_{i=1}^{p} (\mathbf{f}_{i}^{c} \cos \omega_{i} t + \mathbf{f}_{i}^{s} \sin \omega_{i} t), \qquad (2.31)$$

where the  $\mathbf{y}_i^c$ ,  $\mathbf{y}_i^s$ ,  $\mathbf{f}_i^c$ , and  $\mathbf{f}_i^s$  are specified constant *m*-vectors. The form of equations (2.30) and (2.31) are recognized as a sum of truncated Fourier series, and can thus be used to approximate any periodic function to a degree of precision corresponding to the choice of *p*. Note that  $\mathbf{y}(t)$  and  $\mathbf{f}(t)$  do not necessarily contain the same frequency content since any of the constant vectors of the summation can be set to zero for a given frequency.

We now seek the steady-state control input that satisfies equations (2.27) and (2.29). Together equations (2.29) and (2.30) imply that

$$\lim_{t \to \infty} \mathbf{q}(t) = \sum_{i=1}^{p} (\mathbf{q}_{i}^{c} \cos \omega_{i} t + \mathbf{q}_{i}^{s} \sin \omega_{i} t), \qquad (2.32)$$

where  $\mathbf{q}_i^c$  and  $\mathbf{q}_i^c$  are unknown constant *n*-vectors. Note that the deduction of (2.32) is independent of the vibration model in question, but results solely from the defined program constraint. However, conveniently (2.32) is a steady-state solution of (2.27), which is not necessarily the case for the nonlinear system of (2.7).

Substituting the steady-state solution of (2.32) along with forcing input of (2.31) into (2.27) results

$$\sum_{i=1}^{p} (\mathbf{\Lambda}_{M} \mathbf{q}_{i}^{c} - \omega_{i} \mathbf{D} \mathbf{q}_{i}^{s} - \mathbf{f}_{i}^{c}) \cos \omega_{i} t$$
$$+ \sum_{i=1}^{p} (\mathbf{\Lambda}_{M} \mathbf{q}_{i}^{s} + \omega_{i} \mathbf{D} \mathbf{q}_{i}^{c} - \mathbf{f}_{i}^{s}) \sin \omega_{i} t = \mathbf{B} \mathbf{u},$$

where

$$\mathbf{\Lambda}_M = \mathbf{K} - \omega_i^2 \mathbf{M}.$$

Thus, the steady state control input is apparently of the form

$$\lim_{t \to \infty} \mathbf{u} = \sum_{i=1}^{p} (\mathbf{u}_{i}^{c} \cos \omega_{i} t + \mathbf{u}_{i}^{s} \sin \omega_{i} t), \qquad (2.33)$$

where  $\mathbf{u}_{i}^{c}$  and  $\mathbf{u}_{i}^{s}$  are unknown *m*-vectors. It then follows that

$$\mathbf{\Lambda}_M \mathbf{q}_i^c - \omega_i \mathbf{D} \mathbf{q}_i^s = \mathbf{f}_i^c + \mathbf{B} \mathbf{u}_i^c, \qquad (2.34)$$

$$\mathbf{\Lambda}_M \mathbf{q}_i^s + \omega_i \mathbf{D} \mathbf{q}_i^c = \mathbf{f}_i^s + \mathbf{B} \mathbf{u}_i^s, \qquad (2.35)$$

for each i = 1, ..., p. Together equations (2.34) and (2.35) constitute 2n algebraic equations with 2nunknowns  $(n-m \text{ elements of both } \mathbf{q}_i^c$  and  $\mathbf{q}_i^s$ , and all m elements of both  $\mathbf{u}_i^c$  and  $\mathbf{q}_i^s$ ) for each i = 1, ..., p. This system of algebraic equations can be solved by a similar two-part process that was used for the system of (2.17) in the previous section. Define a matrix  $\mathbf{B}^{\perp}$  such that  $\mathbf{B}^{\perp}\mathbf{B} = 0$ . Pre-multiplying equations (2.34) and (2.35) by  $\mathbf{B}^{\perp}$  gives

$$\mathbf{B}^{\perp} \mathbf{\Lambda}_{M} \mathbf{q}_{i}^{c} - \omega_{i} \mathbf{B}^{\perp} \mathbf{D} \mathbf{q}_{i}^{s} = \mathbf{B}^{\perp} \mathbf{f}_{i}^{c}$$

$$(2.36)$$

$$\mathbf{B}^{\perp} \mathbf{\Lambda}_M \mathbf{q}_i^s + \omega_i \mathbf{B}^{\perp} \mathbf{D} \mathbf{q}_i^c = \mathbf{B}^{\perp} \mathbf{f}_i^s.$$
(2.37)

These constitute (for each i = 1, ..., p) a system of 2(n-m) equations with 2n unknowns. The remaining 2m relations come from the program constraint of (2.29). Combining equations (2.29) and (2.30) results in

$$\mathbf{C}\mathbf{q}_i^c = \mathbf{y}_i^c,\tag{2.38}$$

$$\mathbf{C}\mathbf{q}_i^s = \mathbf{y}_i^s,\tag{2.39}$$

We now have in equations (2.36-2.39) a system of n algebraic equations with n unknowns, which can be written

$$\begin{bmatrix} \mathbf{B}^{\perp} \mathbf{\Lambda}_{M} & -\omega_{i} \mathbf{B}^{\perp} \mathbf{D} \\ \omega_{i} \mathbf{B}^{\perp} \mathbf{D} & \mathbf{B}^{\perp} \mathbf{\Lambda}_{M} \\ \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{i}^{c} \\ \mathbf{q}_{i}^{s} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{\perp} \mathbf{f}_{i}^{c} \\ \mathbf{B}^{\perp} \mathbf{f}_{i}^{s} \\ \mathbf{y}_{i}^{c} \\ \mathbf{y}_{i}^{s} \end{bmatrix}.$$
(2.40)

The simplification of the (in general) DAEs of (2.17) to algebraic equations for the present case is a significant convenience since solving for the generalized coordinates is now trivial; (2.40) is of the convenient form  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , where the invertibility of  $\mathbf{A}$  indicates that the specified motion is possible. Solving (2.40) for the components of  $\mathbf{q}_i^c$  and  $\mathbf{q}_i^s$  gives the steady state response via (2.32). The components of  $\mathbf{u}_i^c$  and  $\mathbf{u}_i^s$  are then computed by the operation

$$\mathbf{u}_{i}^{c} = (\mathbf{B}^{T} \mathbf{\Lambda}_{M}^{-1} \mathbf{B})^{-1} \mathbf{B}^{T} \left[ \mathbf{q}_{i}^{c} - \omega_{i} \mathbf{\Lambda}_{M}^{-1} (\mathbf{D} \mathbf{q}_{i}^{s} - \mathbf{f}_{i}^{s}) \right]$$
(2.41)

$$\mathbf{u}_{i}^{s} = (\mathbf{B}^{T} \mathbf{\Lambda}_{M}^{-1} \mathbf{B})^{-1} \mathbf{B}^{T} \left[ \mathbf{q}_{i}^{s} + \omega_{i} \mathbf{\Lambda}_{M}^{-1} (\mathbf{D} \mathbf{q}_{i}^{c} - \mathbf{f}_{i}^{s}) \right].$$
(2.42)

The solutions of equations (2.41) and (2.42) provides the components of the control input per (2.33), thus solving the stated problem.

### 2.3 Applications to Flexible Structures

For the sake of clarity, we briefly review the construction of the finite dimensional model of (2.7). Since the sources of nonlinearity in (2.7) are too numerous to address in general, the present development is limited to linear vibration models. In particular, we consider infinite-dimensional vibration model of the form

$$\mathcal{L}w(x,t) + \mathcal{C}\dot{w}(x,t) + \mathcal{M}(x)\ddot{w}(x,t) = f(x,t) + f_c(x,t), \qquad (2.43)$$

where f(x,t) is the distributed forcing due to external disturbance,  $f_c(x,t)$  is the distributed control force,  $\mathcal{M}$  is the distributed mass density, and  $\mathcal{L}$  and  $\mathcal{C}$  are linear self-adjoint operators that identify the distributed stiffness and damping, respectively. The system of (2.43) is also subject to boundary conditions. It is noted that the model of (2.43) is itself a member of a larger set of distributed parameter vibration models that includes plates and shells. The generalized mass, stiffness, and damping matrices of (2.27) are found by [120]

$$\mathbf{M} = \int_{0}^{L} \mathcal{M} \boldsymbol{\phi} \boldsymbol{\phi}^{T} \, dx \tag{2.44}$$

$$\mathbf{D} = \int_0^L \phi \mathcal{C} \boldsymbol{\phi}^T \, dx \tag{2.45}$$

$$\mathbf{K} = \int_0^L \phi \mathcal{L} \phi^T \, dx, \qquad (2.46)$$

and the the external disturbance force and control force is

$$\mathbf{F} = \int_0^L \phi f(x,t) \, dx, \qquad (2.47)$$

$$\mathbf{B} = \int_0^L \phi f_c(x,t) \, dx. \tag{2.48}$$

Given the basic model structure, we now develop two examples that illustrate the application of the theory presented in the previous sections. The examples here are limited to the linear analysis of section 2.2 to clearly illustrate the main points without complexity introduced by nonlinearities. In particular, we consider an undamped Euler-Bernoulli beam model for which the components of (2.43) are

$$\mathcal{L} = EI(\partial^4 w / \partial x^4)$$
$$\mathcal{D} = 0$$
$$\mathcal{M} = \rho A,$$

where EI is the constant flexural rigidity,  $\rho$  is the constant density, A is the cross-sectional area, and L is the beam length. For cantilever boundary conditions the components of  $\phi$  are taken as the mode shapes [120]

$$\phi_i = \cosh \frac{\Omega_i x}{L} - \cos \frac{\Omega_i x}{L} - \frac{\sinh \Omega_i - \sin \Omega_i}{\cos \Omega_i + \cosh \Omega_i} \Big( \sinh \frac{\Omega_i x}{L} - \sin \frac{\Omega_i x}{L} \Big),$$

where the  $\Omega_i$  are solutions of the characteristic equation

$$\cos\Omega_i\cosh\Omega_i = -1.$$

It then follows that the components of (2.44-2.46) are

$$\mathbf{M} = (\rho A)\mathbf{I}$$
$$\mathbf{D} = 0$$
$$\mathbf{K} = (EI/L^4) \operatorname{diag}(\Omega_1^4, \cdots, \Omega_n^4),$$

where **I** is the identity matrix. For the following examples, we set n = 7. For the following examples the fist 7 modes of vibration of the beam is considered. For the following examples, the properties of the beam is expressed in the Table 2.1.

Table 2.1: Cantilever beam properties

Density $\rho$	$2700 \text{ kg/m}^3$
Young's Modulus of the plate $E$	70.0 Gpa
Beam dimensions $(L \times W \times h)$	$200\times10\times1~\mathrm{mm}$
Poisson's ratio $v$	0.3

Case 1: Dynamic shape control. As shown in Figure 2.5, two point force are located on the cantilever beam. There is no disturbance force, so that  $\mathbf{f} = 0$ . It is desired to actuate the motion

$$\lim_{t \to \infty} w(x, t) = \phi_1(x) \sin \omega t$$



Figure 2.5: Actuated Cantilever beam, case (1)

where  $\phi_1$  is the first mode shape. It is expected, since there is no damping, that actuation of the first mode shape requires no input at the first natural frequency since this is the natural response of the structure; at other frequencies, the magnitude of input is in question. As an approximation to this

desired motion, set

$$\lim_{t \to \infty} w(x_1, t) = \phi_1(x_1) \sin \omega t$$
$$\lim_{t \to \infty} w(x_2, t) = \phi_1(x_2) \sin \omega t,$$

where  $x_1$  and  $x_2$  are specified locations. Note that the desired motion could be approximated in this manner by more points if more actuators were used. Applying the expansion of (2.8), the program constraint is written

$$\lim_{t \to \infty} \left[ \begin{array}{ccc} \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \cdots & \phi_n(x_2) \end{array} \right] \left[ \begin{array}{c} q_1(t) \\ \vdots \\ q_n(t) \end{array} \right] = \left[ \begin{array}{c} \phi_1(x_1) \sin \omega t \\ \phi_1(x_2) \sin \omega t \end{array} \right],$$

which corresponds to . (2.29). Following the solution method of section (2.1), the steady state actuation inputs are computed for two locations of  $x = x_1$  and  $x_2$ , and for two actuator placements. The computed results are shown in figures (??) and (??) over a (non dimensional) frequency range of 20. As expected, at the first natural frequency the actuator input reduces to zero. At other frequencies the input magnitude shows peaks and valleys at frequency locations that depend on the actuator placement and the location of specified motion. The results suggests that the actuator locations and the specified motions can be optimally located to reduce the input requirements over a frequency range of interest.

Case 2: Localized vibration control. As with the previous example, two pair of actuators are attached to the surface of a cantilever beam. Additionally, as shown in Figure 2.10, a periodic disturbance force is located at  $x = x_d$ ; and two equal masses (m = 1Kg) are attached by massless links at locations  $x = x_1$  and  $x_2$ . The corresponding force is

$$f(x,t) = \delta(x - x_d) \sin \omega t - \delta(x - x_1) - \delta(x - x_2),$$

where  $\delta$  is the Dirac-delta function. Then, the force can be written as

$$\mathbf{f} = \begin{bmatrix} \phi_1(x_d) \\ \vdots \\ \phi_n(x_d) \end{bmatrix} \sin \omega t - \begin{bmatrix} \phi_1(x_1) + \phi_1(x_2) \\ \vdots \\ \phi_n(x_1) + \phi_n(x_2) \end{bmatrix}$$



Figure 2.6: First required actuation input force for case 1 ( $x_2 = L, x_{f1} = 0.1L, x_{f2} = 0.2L$ )



Figure 2.7: Second required actuation input force for case 1 ( $x_2 = L, x_{f1} = 0.1L, x_{f2} = 0.2L$ )



Figure 2.8: First program actuation input force for case 1 ( $x_2 = L, x_{f1} = 0.1L, x_{f_2} = 0.6L$ )



Figure 2.9: Second program actuation input force for case 1 ( $x_2 = L, x_{f1} = 0.1L, x_{f_2} = 0.6L$ )


Figure 2.10: Actuated Cantilever beam, case (2)

It is desired that the motion of the points of attachment is completely attenuated by the actuators. The program constraint is thus

$$\lim_{t \to \infty} w(x_1, t) = 0$$
$$\lim_{t \to \infty} w(x_2, t) = 0,$$

or with (2.8)

$$\lim_{t \to \infty} \left[ \begin{array}{ccc} \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \cdots & \phi_n(x_2) \end{array} \right] \left[ \begin{array}{c} q_1(t) \\ \vdots \\ q_n(t) \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right].$$

The actuation input required to satisfy this constrain is shown in figures (??) and (??) for two locations of  $x_1$  and  $x_2$ , and for two locations of the disturbance,  $x_d$ . As with the previous example, the peaks of the inputs depend on the location of the actuators and the material points to be controlled, and do not necessarily coincide with the natural frequencies of the structure.

#### 2.4 Summary

We have applied a projection method for determining the actuation input required to generate a specified relation between selected material points of a deformable structure. The method is dependent primarily on the assumption that the solution to the equations of motion can be sufficiently approximated by a modal expansion. Then under certain conditions on the actuation behavior, a specified number of independent actuator inputs is able to satisfy an equal number of program constraints. In the general



Figure 2.11: First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = L, x_{f1} = 0.1L, x_{f2} = 0.2L$ )



Figure 2.12: Second program actuation input force for case 2 ( $x_2 = 0.8L, x_d = L, x_{f1} = 0.1L, x_{f_2} = 0.2L$ )



Figure 2.13: First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = 0.75L, x_{f1} = 0.1L, x_{f_2} = 0.2L$ )



Figure 2.14: First program actuation input force for case 2 ( $x_2 = 0.8L, x_d = 0.75L, x_{f1} = 0.1L, x_{f_2} = 0.2L$ )

case where the model is nonlinear and the specified motion is of an arbitrary nonlinear form, the procedure produces a set of differential algebraic equations for which the solution not always trivial to compute. However, it was shown that for the conditions that the vibration model is linear, and for a class of linear periodic program constraints, the procedure reduces to a trivial set of algebraic equations. In the application examples, two potential applications for this analysis was demonstrated: static/dynamic shape control and localized vibration control.

# Chapter 3

# Distributed actuation requirements of piezoelectric structures under servo-constraints

In this chapter, we address the problem of specified motion for a flexible structure with distributed piezoelectric sensor and actuators. One of the primary advantages of piezoelectric actuators and sensors in the control of deformable structures is that they can be directly integrated into the structure and distributed throughout it. As discussed in the review of Crawley [14], this in situ distribution of sensing and control is extremely useful for the design and realization of "intelligent" structures. However, one of the primary disadvantages of piezoelectric is that the induced strains are severely limited (on the order of 1000  $\mu$ strain), consequentially limiting the amount of force that can be applied. Thus, an important consideration in the evaluation of piezoelectric actuators for a given application is whether they can supply sufficient input to achieve the desired objective. A general technique suggested here for evaluating piezoelectric actuators is to specify a desired motion that should be achieved, whether it be zero motion due to an external disturbance such as with vibration control or non-zero motion such as with active shape control, and subsequently determine whether a distributed array of actuators is capable of supplying the required input.

## 3.1 Piezoelectric shell model.

We consider the vibration of a elastic shell structure with a distributed array of m piezoelectric actuators. The undeflected mean plane of the shell is characterized by the curvilinear coordinates  $\alpha_1$  and  $\alpha_2$ . The position of a material point P of the mean plane relative to its initial location ( $\alpha_1, \alpha_2$ ) is quantified by the vector

$$\boldsymbol{w}(P,t) = \sum_{i=1}^{3} w_i(\alpha_1, \alpha_2, t) \boldsymbol{e}_i, \qquad (3.1)$$

where the  $e_i$  are orthogonal unit vectors,  $e_3$  being normal to the mean surface. It is assumed that the  $w_i$  can be written in the standard separated form

$$w_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} q_k(t) \Phi_{ik}(\alpha_1, \alpha_2),$$
(3.2)

where the  $\Phi_{ik}$  are known mode shape functions, and the  $q_k$  are unknown modal participation factors each governed by a second-order ordinary differential of the form

$$\ddot{q}_{k}(t) + 2\zeta_{k}\Omega_{k}\dot{q}_{k}(t) + \Omega_{k}^{2}q_{k}(t) = \hat{F}_{k}(t) + \hat{F}_{k}^{a}(t), \qquad (3.3)$$

where  $\Omega_k$  is the natural frequency,  $\zeta_k$  is the modal damping coefficient, and  $\hat{F}_k$  and  $\hat{F}_k^a$  are the modal forces due to the external mechanical excitations and distributed piezoelectric actuation, respectively. The modal forces due to external excitation is of the form [118]

$$\hat{F}_k(t) = \frac{1}{\rho h N_k} \int_{\alpha_1} \int_{\alpha_2} \left( \sum_{i=1}^3 F_i \Phi_{ik} \right) A_1 A_2 \ d\alpha_1 d\alpha_2, \tag{3.4}$$

where  $F_i(\alpha_1, \alpha_2, t)$  accounts for external distributed mechanical disturbance on the surface in the  $e_i$ direction,  $A_1$  and  $A_2$  are Lamé parameters,  $\rho$  is the mass density, h the thickness, and

$$N_k = \int_{\alpha_1} \int_{\alpha_2} \left( \sum_{i=1}^3 \Phi_{ik}^2 \right) A_1 A_2 \ d\alpha_1 d\alpha_2.$$

The disturbance  $F_i(\alpha_1, \alpha_2, t)$  is decomposed as the product of a spatial function that quantifies the location of the disturbance, and a time-dependent function. The modal force can then be written in

the general form

$$\hat{F}_k(t) = \sum_{i=1}^{\infty} D_{ki} d_i(t),$$
(3.5)

where  $D_{ki}$  is a constant that depends on the location of the disturbance, and  $d_i(t)$  quantifies the time-history of the disturbance.

Following the work of Tzou [5] the modal force due to piezoelectric actuation is

$$\hat{F}_{k}^{a}(t) = \frac{1}{\rho h N_{k}} \int_{\alpha_{1}} \int_{\alpha_{2}} \left( \sum_{i=1}^{3} L_{i}^{c}(\psi_{3}) \Phi_{ik} \right) A_{1} A_{2} \ d\alpha_{1} d\alpha_{2}, \tag{3.6}$$

where  $\psi_3(\alpha_1, \alpha_2, t)$  is the distributed transverse applied voltage, and  $L_i^c$  is Love's control operator given by

$$\begin{split} L_1^c\{\psi_3\} &= -\frac{1}{A_1A_2} \left\{ \frac{\partial}{\partial \alpha_1} (N_{11}^c A_2) - N_{22}^c \frac{\partial}{\partial \alpha_1} A_2 + \frac{1}{R_1} \left[ \frac{\partial}{\partial \alpha_1} (M_{11}^c A_2) - M_{22}^c \frac{\partial}{\partial \alpha_1} A_2 \right] \right\} \\ L_2^c\{\psi_3\} &= -\frac{1}{A_1A_2} \left\{ \frac{\partial}{\partial \alpha_2} (N_{22}^c A_1) - N_{11}^c \frac{\partial}{\partial \alpha_2} A_1 + \frac{1}{R_2} \left[ \frac{\partial}{\partial \alpha_2} (M_{22}^c A_1) - M_{11}^c \frac{\partial}{\partial \alpha_2} A_1 \right] \right\} \\ L_3^c\{\psi_3\} &= -\frac{1}{A_1A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left( \frac{1}{A_1} \frac{\partial (M_{11}^c A_2)}{\partial \alpha_1} - \frac{M_{22}^c}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right) \\ &+ \frac{\partial}{\partial \alpha_2} \left( \frac{1}{A_2} \frac{\partial (M_{22}^c A_1)}{\partial \alpha_2} - \frac{M_{11}^c}{A_2} \frac{\partial A_2}{\partial \alpha_2} \right) - A_1A_2 \left( \frac{N_{11}^c}{R_1} + \frac{N_{22}^c}{R_2} \right) \right\}. \end{split}$$

The actuator induced forces and moments, respectively  $N^c_{ii}$  and  $M^c_{ii},\,{\rm are}$ 

$$M_{ii}^c = r_{ii}d_{3i}E_p\psi_3(\alpha_1, \alpha_2, t)$$
$$N_{ii}^c = d_{3i}E_p\psi_3(\alpha_1, \alpha_2, t),$$

where  $E_p$  is the elastic modulus of the actuator,  $d_{3i}$  is the actuator constant, and  $r_{ii}$  is the moment arm measured from the plate neutral surface to the actuator mid-surface.

For a set of M actuators, the distributed control input  $\phi_3$  is decomposed as

$$\psi_3(\alpha_1, \alpha_2, t) = \sum_{i=1}^M v_i^a(t) b_i(\alpha_1, \alpha_2),$$
(3.7)

where  $v_i^a$  is the voltage input of the *i*-th actuator and the spatial function  $b_i(\alpha_1, \alpha_2)$  quantifies its size

and placement. The actuation input thus takes the general form

$$\hat{F}_{k}^{a}(t) = \sum_{i=1}^{M} B_{ki} v_{i}^{a}(t), \qquad (3.8)$$

where  $B_{ki}$  is a constant that is dependent on the placement, size, and electromechanical properties of the *i*-th actuator. With (3.6) and (3.8), the m + n differential relations of (3.3) can be written in the infinite-dimensional matrix form as

$$\ddot{\boldsymbol{q}}(t) + \boldsymbol{\Lambda}_D \dot{\boldsymbol{q}}(t) + \boldsymbol{\Lambda}_K \boldsymbol{q}(t) = \mathbf{B} \mathbf{v}(t) + \mathbf{F}, \mathbf{d}(t)$$
(3.9)

where  $\boldsymbol{q} = [q_1 \ q_2 \ \cdots]^T$ ,  $\boldsymbol{\Lambda}_D = \text{diag}\{2\zeta_1\omega_1, 2\zeta_2\omega_2, \cdots\}$ ,  $\boldsymbol{\Lambda}_K = \text{diag}\{\omega_1^2, \omega_2^2, \cdots\}$ ,  $\mathbf{d} = [d_1 \ d_2 \ \cdots]^T$ ,  $\mathbf{v} = [v_1^a \ \cdots \ v_m^a]^T$ , and  $\mathbf{F}$  and  $\mathbf{B}$  contain the  $D_{ki}$  and  $B_{ki}$  components of equations (3.5) and (3.8), respectively.

#### 3.2 Actuation of servo-constraints

Let **w** denote the mathematical vector function  $\mathbf{w} = [w_1 \ w_2 \ w_3]^T$ . Define a set of material points  $\{P_1, ..., P_s\}$  for which desired steady-state motion is specified by the relation

$$\lim_{t \to \infty} \mathbf{E} \begin{bmatrix} \mathbf{w}^T(P_1, t) & \cdots & \mathbf{w}^T(P_s, t) \end{bmatrix}^T = \mathbf{y}(t),$$
(3.10)

where **E** is a constant  $M \times 3s$  matrix, and **y** is a time-dependent *m*-vector. It is noted that the number of program constraints is equal to the number of actuators. The relation of (3.10) is called the *servoconstraint* (or *program constraint*). We seek the distributed piezoelectric actuation voltage **w** that enforces this constraint.

To further develop (3.10), the modal expansion of (3.2) is written in matrix form as

$$\mathbf{w}(P,t) = \begin{bmatrix} \mathbf{\Phi}_1(P) & \mathbf{\Phi}_2(P) & \mathbf{\Phi}_3(P) \end{bmatrix}^T \boldsymbol{q}(t)$$
(3.11)

where, and  $\mathbf{\Phi}_i = [\Phi_{i1} \ \Phi_{i2} \ \cdots ]^T$ , i = 1, 2, 3. Letting  $\mathbf{\Phi} = [\mathbf{\Phi}_1 \ \mathbf{\Phi}_2 \ \mathbf{\Phi}_3]$ , an  $\infty \times 3$  matrix function, (3.11) can be further simplified as

$$\mathbf{w}(P,t) = \mathbf{\Phi}^T(P)\mathbf{q}(t) \tag{3.12}$$

It then follows that (3.12) can be written

$$\lim_{t \to \infty} \mathbf{E} [ \mathbf{\Phi}(P_1) \cdots \mathbf{\Phi}(P_s) ]^T \mathbf{q}(t) = \mathbf{y}(t)$$
(3.13)

or concisely

$$\lim_{t \to \infty} \mathbf{C} \boldsymbol{q}(t) = \mathbf{y}(t), \tag{3.14}$$

where  $\mathbf{C} = \mathbf{E}[\mathbf{\Phi}(P_1) \cdots \mathbf{\Phi}(P_s)]^T$ .

Since we are interested in the steady-state response, it is convenient to work in the frequency domain. The equations of motion, (3.9), together with the servoconstraint, (3.14), are expressed in the frequency domain as

$$\boldsymbol{\alpha}(\omega)\hat{\boldsymbol{\eta}}(\omega) = \mathbf{B}\hat{\mathbf{v}}(\omega) + \mathbf{F}\hat{\mathbf{d}}(\omega)$$
(3.15)

$$\mathbf{C}\hat{\boldsymbol{\eta}}(\omega) = \hat{\mathbf{y}}(\omega) \tag{3.16}$$

where  $\hat{q}(\omega)$  denotes the frequency domain representation of q(t), etc., I is the identity matrix, and

$$\boldsymbol{\alpha}(\omega) = (\boldsymbol{\Lambda}_K - \omega^2 \mathbf{I} - i\omega \boldsymbol{\Lambda}_D)^{-1}$$

is called the *admittance*.

Given that equations (3.15) and (3.16) are in general complex equations, it is sometimes convenient to work with an equivalent real form of the equations of motion. By definition, the frequency domain transformation assumes that the inputs are of the form  $\mathbf{v}(t) = \mathbf{v}_c \cos \omega t + \mathbf{v}_s \sin \omega t$ , and  $\mathbf{d}(t) = \mathbf{d}_c \cos \omega t + \mathbf{d}_s \sin \omega t$ . Given that the system is linear it follows that in steady state, the response is of the form  $\mathbf{q}(t) = \mathbf{q}_c \cos \omega t + \mathbf{q}_s \sin \omega t$ . Substituting these into the equations of motion thus gives

$$\begin{bmatrix} \mathbf{\Lambda}_{K} - \omega^{2} \mathbf{I} & -\omega \mathbf{\Lambda}_{D} \\ \omega \mathbf{\Lambda}_{D} & \mathbf{\Lambda}_{K} - \omega^{2} \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{c}(\omega) \\ \mathbf{q}_{s}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{c}(\omega) \\ \mathbf{v}_{s}(\omega) \end{bmatrix} + \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{c}(\omega) \\ \mathbf{d}_{s}(\omega) \end{bmatrix}$$
(3.17)

Similarly, the output is

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{c}(\omega) \\ \mathbf{q}_{s}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{c}(\omega) \\ \mathbf{y}_{s}(\omega) \end{bmatrix}$$
(3.18)

The complex relations of equations. (3.15) and (3.16) will be used for subsequent development; however, since they contain the same structure as equations. (3.17) and (3.18), the two are considered interchangeable for all subsequent analysis.

We now seek the input  $\hat{v}$ , given  $\hat{r}$  and  $\mathbf{F}\hat{d}$ , such that (3.15) satisfies (3.16). Solving (3.18) for  $\hat{q}$  and substituting into (3.19) gives

$$(\mathbf{C}\boldsymbol{\alpha}\mathbf{B})\hat{\mathbf{v}} + (\mathbf{C}\boldsymbol{\alpha}\mathbf{D})\hat{\mathbf{d}} = \hat{\mathbf{y}}$$
(3.19)

Letting  $\mathbf{G}_v = \mathbf{C} \boldsymbol{\alpha} \mathbf{B}$  and  $\mathbf{G}_d = \mathbf{C} \boldsymbol{\alpha} \mathbf{F}$ , it follows that the control input that satisfies the servoconstraint is

$$\hat{\mathbf{v}} = \mathbf{G}_v^{-1} (\hat{\mathbf{y}} - \mathbf{G}_d \hat{\mathbf{d}}) \tag{3.20}$$

In steady state the actuation input of (3.20) is equivalent to any input provided by a measurement based feedback control algorithm satisfying the servo-constraint of (3.14). Hence, given known limits on the magnitude of  $\mathbf{v}(t)$ , (3.20) indicates whether a given motion is possible under the defined actuation scheme. Note that the static shape control solution is also contained in (3.20), corresponding to  $\omega = 0$ .

#### 3.2.1 Special case 1: non-zero motion constraint with zero disturbance.

Suppose that  $\hat{\mathbf{d}} = \mathbf{0}$ , and  $\hat{\mathbf{y}} \neq \mathbf{0}$ , in which case (3.20) reduces to

$$\hat{\mathbf{v}} = \mathbf{G}_v^{-1} \hat{\mathbf{y}}.\tag{3.21}$$

The natural frequencies of the system are contained in the admittance matrix,  $\alpha$ , which show up as poles of the component transfer functions of  $\mathbf{G}_{v}$ ; the kl component is

$$(\mathbf{G}_v)_{kl} = \sum_{j=1}^{\infty} \frac{C_{kj} B_{jl}}{\Omega_j^2 - \omega^2 - 2i\zeta_j \Omega_j \omega} = \frac{Z_{kl}(\omega)}{\prod_{j=1}^{\infty} (\Omega_j^2 - \omega^2 - 2i\zeta_j \Omega_j \omega)},$$

where the  $Z_{kl}$  are complex algebraic expressions in  $\omega$ ; assuming the series of (3.2) is truncated at i = n, the order of  $Z_{kl}$  in  $\omega$  is less than n, implying that the magnitude of the response "rolls off" for large  $\omega$ . The frequency  $\omega$  at which  $Z_{kl} = 0$  is a zero of  $Z_{kl}$ , meaning that the input  $\hat{v}_j$  does not affect the output  $\hat{y}_i$  at that frequency. The contribution of each pole to the actuation input is scaled by the components of **C** and **B**.

Now we have important point, although perhaps an obvious one, that the poles and zeros of  $\mathbf{G}_v^{-1}$  are not in general equal to those of  $\mathbf{G}_v$ ; and also, the poles of  $\mathbf{G}_v^{-1}$  are dependent on the matrices  $\mathbf{C}$  and  $\mathbf{B}$ . For example, the inverse for a system with two independent actuators is

$$\mathbf{G}_{v}^{-1} = \frac{\prod_{j=1}^{\infty} (\Omega_{j}^{2} - \omega^{2} - 2i\zeta_{j}\Omega_{j}\omega)}{Z_{11}(\omega)Z_{22}(\omega) - Z_{12}(\omega)Z_{21}(\omega)} \begin{bmatrix} Z_{22}(\omega) & -Z_{12}(\omega) \\ -Z_{21}(\omega) & Z_{11}(\omega) \end{bmatrix}.$$

Thus, the natural frequencies of the system show up as zeros of the inverse response, which can be canceled by a pole of  $\mathbf{G}_v^{-1}$ . The significance is that one can not expect that the actuation input is the smallest for servo-constraints near the natural frequencies of the system, nor necessarily that actuating a node of a natural frequency requires substantial input magnitudes. Generally speaking, the actuation input is the smallest when the motion to be controlled is close to the natural response of the structure, assuming the actuator does not lie on a modal line. These matters will be discussed further in the application example of section (3.3).

#### 3.2.2 Special case 2: zero motion constraint with non-zero disturbance.

Suppose now that  $\mathbf{y} = \mathbf{0}$  and  $\mathbf{d} \neq \mathbf{0}$ , in which case (3.20) reduces to

$$\hat{\mathbf{v}} = \mathbf{G}_{v}^{-1} \mathbf{G}_{d} \hat{\mathbf{d}}.$$
(3.22)

Similar to the previous special case, the poles of  $\mathbf{G}_v^{-1}\mathbf{G}_d$  are affected by the choice of  $\mathbf{C}$  and  $\mathbf{B}$ , and do not necessarily coincide with the natural frequencies of the system. Thus, it may not be that the actuation requirements are the greatest when the disturbance  $\mathbf{d}(t)$  is at one of the natural frequencies of the system. If, for example, a material point to be actuated is near a node of the natural frequency the input will decrease (assuming the actuator is placed at a node).

The most apparent purpose for this constraint is the desire to suppress motion of certain material elements in the presence of an external disturbance (i.e., localized vibration control). However, the complete suppression of motion is potentially a very restrictive constraint. A less restrictive constraint is  $\hat{\mathbf{y}}^T \hat{\mathbf{y}} \leq \epsilon^2$ , where  $\epsilon$  can be taken as a function of  $\omega$ . There are an infinite number of inputs  $\hat{\mathbf{v}}(\omega)$  that satisfy this constraint; we seek the one such that  $\hat{\mathbf{v}}(\omega)$  is a minimum. To solve the optimization problem, define the cost function

$$\mathcal{J} = \frac{1}{2} \hat{\mathbf{v}}^T \hat{\mathbf{v}},\tag{3.23}$$

and the equality constraint

$$\hat{\mathbf{y}}^T \hat{\mathbf{y}} - \epsilon^2 = 0. \tag{3.24}$$

Using the method of Lagrange multipliers, with (3.22) the function for which we seek a minimum is

$$\mathcal{L} = \frac{1}{2} (\mathbf{G}_v^{-1} \hat{\mathbf{y}} - \mathbf{G}_v^{-1} \mathbf{G}_d \hat{\mathbf{d}})^T (\mathbf{G}_v^{-1} \hat{\mathbf{y}} - \mathbf{G}_v^{-1} \mathbf{G}_d \hat{\mathbf{d}}) + \lambda (\hat{\mathbf{y}}^T \hat{\mathbf{y}} - \epsilon^2)$$
(3.25)

Taking the partial derivative of  $\mathcal{L}$  with respect to  $\hat{\mathbf{y}}$ , and setting it equal to zero results in

$$\hat{\mathbf{y}}(\epsilon) = (2\lambda \mathbf{I} + \mathbf{G}_v^{-T} \mathbf{G}_v^{-1})^{-1} \mathbf{G}_v^{-T} \mathbf{G}_v^{-1} \mathbf{G}_d \hat{\mathbf{d}}$$
(3.26)

where  $(\cdot)^{-T}$  indicates the transpose of the inverse (or vice versa). Substituting this into the equality constraint gives

$$[(2\lambda \mathbf{I} + \mathbf{G}_{v}^{-T}\mathbf{G}_{v}^{-1})^{-1}\mathbf{G}_{v}^{-T}\mathbf{G}_{v}^{-1}\mathbf{G}_{d}\hat{\mathbf{d}}]^{T}[(2\lambda \mathbf{I} + \mathbf{G}_{v}^{-T}\mathbf{G}_{v}^{-1})^{-1}\mathbf{G}_{v}^{-T}\mathbf{G}_{v}^{-1}\mathbf{G}_{d}\hat{\mathbf{d}}] - \epsilon^{2} = 0$$
(3.27)

This can be solved numerically for  $\lambda$ , which gives  $\hat{\mathbf{y}}$  per (??), and subsequently the minimum control input per (3.20).

Note that the smallest J independent of the constraint is J = 0, which corresponds to the unactuated response:  $\hat{\mathbf{y}} = \mathbf{G}_d \hat{\mathbf{d}}$ . The value of  $\epsilon$  corresponding to the unactuated response, denoted  $\epsilon^*$ , is thus

$$\epsilon^* = [\hat{\mathbf{d}}^T \mathbf{G}_d^T \mathbf{G}_d \hat{\mathbf{d}}]^{1/2}.$$

Since  $\mathcal{J}$  is a quadratic function in  $\hat{\mathbf{y}}$ , it follows that  $\mathcal{J}$  decreases as  $\epsilon$  decreases from 0 to  $\epsilon^*$ . Further increase of  $\epsilon$  above  $\epsilon^*$  results an increase  $\mathcal{J}$ . Hence, choosing  $0 < \epsilon < \epsilon^*$  ensures that the actuation input is less than that corresponding to  $\hat{\mathbf{y}} = 0$ . Setting  $\epsilon > \epsilon^*$  means that the desired motion is greater than the nominal response due to the disturbance input, and is thus not an appropriate constraint.

# 3.3 Applications Examples

To demonstrate the application of the results of the previous sections, here two application examples are presented corresponding to the two special cases of section (3.1). The first example is a thin cantilever plate for which it is desired to control material elements of the free edge. In the second example, we consider localized vibration attenuation of a half-cylindrical shell. The equations of motion for these examples are derived in the Appendix from the piezoelectric shell theory of section (3.1).

#### 3.3.1 Free-surface motion of a cantilever plate

We consider actuation of the thin cantilever plate shown in Figure 3.1. Three identical actuators are attached to the plate as shown. Actuators at the top and bottom of the plate are collocated (i.e., they are mirror imaged about the x - y plane), and an actuator at the bottom is given an identical actuation signals as its equivalent actuator at the top, only the polarity is reversed; hence, there are three actuator inputs for six actuators. The properties of both the plate and the piezoelectric actuators are specified in Table 3.1. The equations of motion for the structure corresponding to those given in general form in section (3.1) are developed in Appendix A.



Figure 3.1: Cantilever beam with three pairs of actuators.

The plate is assumed sufficiently thin so that only the transverse vibration,  $w_3(x, y, t)$ , is of conse-

$$w_3(L, W/2, t) = c_1 \sin \omega t m$$
$$w_3(L, 0, t) = c_2 \sin \omega t m$$
$$w_3(L, -W/2, t) = c_3 \sin \omega t m,$$

where  $c_1, c_2, c_3$  are positive constants, L is the length of the plate, and W is the width. The mode shapes given in the Appendix are labeled  $U_{mn}$ ; the indices are truncated as m = 1, 2, ..., M, n = 1, 2, ..., N so that (1) is of the form

$$\begin{bmatrix} \Phi_{11}(L, W/2) & \Phi_{12}(L, W/2) & \cdots & \Phi_{NM}(L, W/2) \\ \Phi_{11}(L, 0) & \Phi_{12}(L, 0) & \cdots & \Phi_{NM}(L, 0) \\ \Phi_{11}(L, -W/2) & \Phi_{12}(L, -W/2) & \cdots & \Phi_{NM}(L, -W/2) \end{bmatrix} \begin{bmatrix} q_{11}(t) \\ q_{12}(t) \\ \vdots \\ q_{NM}(t) \end{bmatrix} = \begin{bmatrix} c_1 \sin \omega t \\ c_2 \sin \omega t \\ c_3 \sin \omega t \end{bmatrix}$$

For these particular examples M = N = 9.

This constraint only specifies motion of three "characteristic" points, while disregarding the motion of all other material points of the structure. However, the basic motion can be deduced from the mode shapes for the cantilever plate. For example, for actuation frequencies within the first two natural frequencies (0.677 and 4.156 Hz for this example), only the first two bending modes will be excited, resulting in a combination of flapping and twisting (about the x-axis) of the plate. For larger natural frequencies, more complex motions will be produced.

Table 3.1: Rectangular plate properties

Density $\rho$	$2700 \text{ kg/m}^3$
Young's Modulus of the plate $E$	70.0 Gpa
Young's Modulus of the piezoelectric actuators $E_p$	63 Gpa
Plate dimensions $(L \times W \times h)$	$200\times50\times1~\mathrm{mm}$
Actuator dimensions $(L_p \times W_p \times t_p)$	$0.3L\times 0.2W\times 0.01h$
Poisson's ratio $v$	0.3
Piezoelectric constant $d_{31}, d_{32}$	$20 \times 10^{-11} \mathrm{m/V}$

In figures (3.2) and (3.3) are shown the components of  $\mathbf{G}_v$  and  $\mathbf{G}_v^{-1}$ , respectively, over a frequency range 5 Hz. Note that since there is no damping, the peaks and valleys are infinitely large and small, respectively, even though they are not shown as such. As previously discussed,  $\mathbf{G}_v$  contains the natural frequencies of the system. The first two natural frequencies are apparent except in the transfer functions involving second actuator, which is placed along a nodal line of the second mode (i.e., twisting about the x-axis). The zeros of  $\mathbf{G}_v^{-1}$  for this case are not the natural frequencies of the system. Hence, as previously discussed, actuation of specified motions at the natural frequencies does not necessarily require a "smaller" actuation input. This will depend on the shape of the actuation and location of the actuators.



Figure 3.2: Components of transfer function  $G_v$  for the cantilever plate.



Figure 3.3: Components of inverse transfer function  $G_v^{-1}$  for the cantilever plate

Based on the computations outlined in section (3.2), figures (3.4-3.6) show the steady-state actuator magnitude requirements for three different specifications of  $c_1$ ,  $c_2$ , and  $c_3$ , and for three different locations of  $x_a$ . The specifications of the constants respectively correspond to actuation of flapping (Fig. 3.4), twisting (Fig. 3.5), and flapping with twisting (Fig. 3.6). For the first two cases (Figs. 3.4, 3.5) it is not possible to produce pure flapping or pure twisting since only three points are being controlled; however, the motions can be described as "mostly flapping" and "mostly twisting", respectively. As shown in Figure 3.4, the least input is required to actuate flapping motion at around 0.7 Hz, slightly above the first natural frequency. It becomes more difficult to actuate the flapping motion as the frequency is increased above 0.7 Hz (the maximum is dependent on the placement  $x_a$ ), which is expected since the plate naturally wants to twist at these frequencies.



Figure 3.4: Piezoelectric actuation input requirements for Example 1, flapping  $(c_1 = 2.5mm, c_2 = 2.5mm, c_3 = 2.5mm)$ 



Figure 3.5: Piezoelectric actuation input requirements for Example 1, twisting  $(c_1 = -2.5mm, c_2 = 0mm, c_3 = 2.5mm)$ 



Figure 3.6: Piezoelectric actuation input requirements for Example 1, flapping and twisting  $(c_1 = 2.5mm, c_2 = 5mm, c_3 = 7.5mm)$ 

The twisting motion of Figure (3.5) requires the least input at the second natural frequency, which is also expected, since we are simply actuating a natural mode shape. Note at the first natural frequency and below the plate wants to solely flap, while we are commanding twisting. However, the actuation input does not have a peak at the first natural frequency. This is because the first mode is not excited due to the symmetry of both the material points and actuator placements.

As shown in Figure (3.6), the input requirements to actuate both flapping and twisting become substantially larger at higher frequencies. To limit the maximum voltage of each actuator to less than 100V, the maximum frequency that this motion can be actuated is around 3 Hz.

#### 3.3.2 Localized vibration attenuation

The second application example examines local vibration attenuation of the shell structure depicted in Figure 3.7, with coordinates  $(\alpha_1, \alpha_2) = (x, \beta)$  For the servo-constraint three material points  $(P_1, P_2, P_3)$  are chosen, and it is desired that these points have zero transverse motion in the presence of a point disturbance,  $d(t) = \sqrt{20 \times 10^3} \sin \omega t$  N, located at  $(x, \beta) = (0.75L, 0.75\pi)$ . The servo-constraint is thus

$$w_3(P_1,t) = w_3(P_2,t) = w_3(P_3,t) = 0.$$

where  $P_1 = (0.85L, 0.25\pi), P_2 = (0.80L, 0.3\pi), P_3 = (0.75L, 0.35\pi)$ . These points could represent for



Figure 3.7: Simply supported damaged shell with three pairs of actuators.

example a localized defect such as a crack, in which case it would be desirable to limit their motion to prevent propagation. The source of the disturbance that must be attenuated is located at a single point. Similar to the previous example, there are six actuators (three sets collocated at the top and bottom of the structure) with three independent voltage inputs. Properties of the shell and the piezoelectric actuators are listed in Table 3.2. The equations of motion for this example are derived in Appendix B.

Table 3.2: Shell properties

Young Modulus of the Shell, $E$	70 Gpa
Young Modulus of the piezoelectric patch, $E_p$	63 Gpa
Density, $\rho$	$2700 \text{ kg/m}^3$
Shell Length, $L$	0.1 m
Shell Thickness, $h$	1  mm
Modal damping, $\zeta_i$	0.01~%
Piezoelectric Thickness, $h^s$	$0.01 \ h$
Radius of Curvature, $R$	$0.5 \mathrm{m}$
Shell Curvature Angle	$\pi$
Poisson Ratio, $v$	0.3
Piezoelectric Constant, $d_{3i}$	$20 \times 10^{-11} \text{ m/V}$

The unactuated response due to the disturbance input is shown in Figure 3.8 for three different locations of  $x_a$ . The system has five natural frequencies between 240-270 Hz, and five more between 970-1000 Hz. In Figure 3.9 are shown the actuator inputs required to completely suppress the motion of the three points. There is peak in all of the actuation inputs at 246 Hz and 970 Hz, the second and sixth natural frequency, for all actuator placements. Note that there are several natural frequencies for which there is no substantial increase in the actuator input.



Figure 3.8: The unactuated response due to the disturbance input for Example 2.

As discussed in section (3.2), the constraint that the motion of all point be completely suppressed is a potentially restrictive constraint. For example, for the actuator placement of  $x_a = 0.6L$  there is a large peak on the order of 1000V at approximately 680 Hz. To relax this constraint we set  $\epsilon = 0.1 \ \mu m$ in the constraint of (24). The optimal results per the minimization of (25) are shown in Figure 3.10, here the actuator voltages for any placement is less than 263 V.



Figure 3.9: Piezoelectric actuation input requirements for Example 2.



Figure 3.10: Optimal piezoelectric actuation input requirements for Example 2.

### 3.4 Summary

It has been shown that a servo-constraint that takes the form a finite number of algebraic relations on the motion of a piezoelectric shell structure can be satisfied by an equivalent number of independent actuators. Such an analysis procedure is thought to be useful in evaluating the applicability of piezoelectric actuators for active structures. The application examples demonstrated the analysis for potential applications involving static and dynamic shape control, and localized vibration control. There are several interesting aspects of the problem that has been left unexplored, particularly with regards to optimization. It is clear that the actuator placement substantially effects the input requirements, and thus this placement can be optimized. Further, there is the matter of defining the servoconstraint. For the examples shown, the servoconstraint was based on actuating defined motion of a number of material points. However, the constraint can involve any number of material points; it is just that the number of constraints must be limited to the number of independent actuators. Thus, while this type of servoconstraint can not be made equivalent to every given continuous motion constraint, the two can be made "close".

# Chapter 4

# Filtered Dynamic Inversion for Linear Systems

The major challenges in the control of flexible structures are unknown and unmeasured disturbances, high dimensionality, unknown system order, parameter uncertainty and nonlinearities. The objective of this chapter is to design a robust feedback controller to address these challenges. A novel feedback controller called filtered dynamic inversion is introduced for multi-variable time-invariant minimum phase systems of unknown system order and with unknown and unmeasured disturbances.

# 4.1 Introduction

Consider the mass-damper-spring system mounted on a moving base which is shown in Figure 4.1. The control input u is the base motion. The equation of motion is given by

$$m\ddot{q} + c\dot{q} + kq = c\dot{u} + ku + c\dot{w} + kw, \tag{4.1}$$

where q is the position of the mass and m, c and k are the mass, damping and stiffness, respectively, and w is the external disturbance.

In order to write this problem in state space form, let  $x_1 = q$  and  $x_2 = \dot{q} - cu/m$ . It follows that



Figure 4.1: Base excitation problem model

(4.1) can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c/m \\ -c^2/m^2 + k/m \end{bmatrix} u + \begin{bmatrix} 0 \\ c\dot{w}/m + kw/m \end{bmatrix}.$$
 (4.2)

The output is defined as

$$y(t) = x_1. \tag{4.3}$$

It is desired that the system output y tracks a desired trajectory r. we first consider input-output linearization as a candidate controller. Following the usual procedure of input-output linearization [111], the output is differentiated until the input appears. Taking the first derivative of the output yields

$$\dot{y} = x_2 + \frac{c}{m}u. \tag{4.4}$$

Consider the imput-output linearization controller

$$u^* = \frac{m}{c} \Big[ -x_2 + v \Big],\tag{4.5}$$

where

$$v = \dot{r} + k_0(r - y). \tag{4.6}$$

Next, substituting  $u^*$  in to the (4.4) yields

$$\dot{e} + k_0 e = 0,$$
 (4.7)

where the error is defined

$$e \stackrel{\Delta}{=} r - y. \tag{4.8}$$

The error dynamics (4.7) are asymptotically stable if and only if  $k_0 > 0$ . Assuming  $k_0$  is positive, the error exponentially converges to zero, so that the output y tracks a desired trajectory r. The convergence rate is chosen by  $k_0$ .

Substituting  $u^*$  in (4.2), yields the closed loop system The closed loop system eigenvalues are  $\lambda_1 = -k_0$  and  $\lambda_2 = -k/c$ . Since the eigenvalues are in the open left half, the closed loop system is asymptotically stable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_0 & 0 \\ -k/m - kk_0/c + ck_0/m & -k/c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
(4.9)
$$+ \begin{bmatrix} \dot{r} + k_0r \\ +k\dot{r}/c - c\dot{r}/m + kk_0r/m - ck_0r/m + c/m\dot{w} + k/mw \end{bmatrix}.$$
(4.10)

implementation of  $u^*$  requires full state measurement and precise model information. However full state measurement and precise model information is not always possible, and model information is greatly uncertain. Next new controller is induced to address these challenges.

Consider passing  $u^*$  through a second order filter, that is, let u satisfy

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} + \omega_c^2 u = \omega_c^2 u^*, \tag{4.11}$$

where  $\zeta_c$  and  $\omega_c$  are the controller damping and filter cut-off frequency, respectively. Substituting the ideal control  $u^*$  from (4.5) into the (4.11) yields

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} + \omega_c^2 u = \frac{m\omega_c^2}{c} \Big[ -x_2 + v \Big].$$

$$(4.12)$$

Since  $x_2 = q - cu/m$ , (4.12) can be expressed as

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} + \omega_c^2 u = \frac{m\omega_c^2}{c} \Big[ -\dot{q} + \frac{c}{m}u + v \Big].$$
(4.13)

Canceling u from both sides of (4.13) yields the filtered dynamic inversion controller

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} = \frac{m\omega_c^2}{c} \Big[ \dot{e} + k_0 e \Big].$$
(4.14)

Taking the Laplace transform of the (4.14) the transfer function between the error e and the controller output u is

$$G_c(s) \stackrel{\Delta}{=} \frac{\hat{u}(s)}{\hat{e}(s)} = \frac{m\omega_c^2(s+k_0)}{c(s^2+2\zeta_c\omega_c s)}.$$
(4.15)

The controller (4.15), requires the knowledge of m/c, which can be obtained from the impulse response. Considering (??) and (4.3), the relation between the output y and the input u is

$$\hat{y}(s) = \frac{m(k+cs)}{c(ms^2 + cs + k)}\hat{u}(s)$$
(4.16)

The impulse response of (4.16), yields

,

$$\hat{y}(s) = \frac{m(k+cs)}{c(ms^2 + cs + k)}$$
(4.17)

Evaluating the inverse Laplace of (4.17) at t = 0 an output feedback yields m/c.

For large  $\omega_c$ , the controller (4.15) is lead controller with integral action. The control diagram is shown in figure 4.2, where it follows from (4.1) that

$$G_s(s) \stackrel{\Delta}{=} \frac{\hat{q}(s)}{\hat{u}(s)} = \frac{cs+k}{ms^2+cs+k}.$$
(4.18)

The closed loop transfer function between r and y is

$$G_{cl}(s) = \frac{y(s)}{r(s)} = \frac{G_s(s)G_c(s)}{1 + G_s(s)G_c(s)} = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{\alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$
(4.19)



Figure 4.2: Closed loop diagram of the base excitation model

where

$$\alpha_{0} = k_{0}km\omega_{c}^{2}, \qquad \beta_{0} = k_{0}km\omega_{c}^{2},$$

$$\alpha_{1} = 2\zeta_{c}\omega_{c}kc + m\omega_{c}^{2}(ck_{0} + k), \qquad \beta_{1} = (ck_{0} + k)m\omega_{c}^{2},$$

$$\alpha_{2} = kc + 2\zeta_{c}\omega_{c}c^{2} + m\omega_{c}^{2}c, \qquad \beta_{2} = mc\omega_{c}^{2},$$

$$\alpha_{3} = c^{2} + 2\zeta_{c}\omega_{c}cm,$$

$$\alpha_{4} = mc.$$

$$(4.20)$$

Using Routh-Hurwitz, it is shown in Appendix C that for sufficiently large cut-off frequency,  $G_{cl}$  is always asymptotically stable. Furthermore, regarding the performance, it can be shown that with increasing  $\omega_c$  the  $\mathscr{L}_2$  norm of the error can be artificially small. This will be proven chapter 4.3.

# 4.2 Mathematical Background

Before proceeding to the consideration of multi-variable systems, we briefly review important concepts related to the input-output stability. Consider a system with the input-output relation

$$y = Hu, \tag{4.21}$$

where u is a signal that maps the time interval  $t \in [0, \infty)$  to the space  $\mathbb{R}^m$ . Two major class of signals are

1. Piecewise continuous, bounded functions which satisfy

$$\sup_{t \ge 0} ||u(t)|| < \infty \tag{4.22}$$

2. Piecewise continuous, square integrable functions which satisfy

$$\int_0^\infty u^T(t)u(t) < \infty \tag{4.23}$$

To measure the size of signal, the following signal norm is introduced:

1. For a piecewise continuous, bounded function, the  $\mathscr{L}^m_\infty$  norm is defined as

$$||u||_{\mathscr{L}_{\infty}} = \sup_{t \ge 0} ||u(t)|| < \infty.$$

2. Let u be a piecewise contortions, square integral function. The  $\mathscr{L}_2^m$  norm is defined

$$||u||_{\mathscr{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)} < \infty.$$

**Definition 1:** Let  $f : \mathbb{R}^+ \to \mathbb{R}$ . Then for each  $T \in \mathbb{R}^+$ , the function  $f_T : \mathbb{R}^+ \to \mathbb{R}$  is defined by

$$f_T = \begin{cases} f(t) & 0 \le t < T \\ 0 & t \ge T \end{cases}$$

$$(4.24)$$

and is called the truncation of f to the interval [0, T].

Let H be a map from the extended space  $\mathscr{L}_{2e}^m$  to the extended space  $\mathscr{L}_{2e}^q,$  where

$$\mathscr{L}_{2e}^{m} = \Big\{ u | u_{\tau} \in \mathscr{L}_{2}^{m}, \forall \tau \in [0 \quad \infty) \Big\}.$$
(4.25)

**Definition 2:** An operator  $H : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^q$  is finite gain  $\mathscr{L}_2$  stable, if there exist a non negative constant  $\gamma$  and  $\beta$  such that

$$||(Hu)_{\tau}||_{\mathscr{L}_{2}^{q}} \leq \gamma ||u||_{\mathscr{L}_{2}^{m}} + \beta.$$

$$(4.26)$$

When the inequality of (4.26) is satisfied with some  $\gamma \ge 0$ , we say that the system has an  $L_p$  gain less than or equal to  $\gamma$ . **Theorem 1.** Consider the linear system

$$\dot{x} = Ax + Bu \tag{4.27}$$
$$y = Cx + Du$$

where A is Hurwitz. Let  $G = C(sI - A)^{-1}B + D$ . Then, the  $\mathscr{L}_2$  gain of the system is  $\sup_{\omega \in R} ||G(j\omega)||_2$ .

*Proof.* The proof is given in the theorem of 5.4 of Ref. [111]  $\Box$ 

## 4.3 Filtered Dynamic Inversion

Consider a structure that is modeled by the linear system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{u}(t) + \mathbf{F}\mathbf{d}(t)$$
(4.28)

$$\mathbf{y}(t) = \mathbf{\Phi}\mathbf{q}(t),\tag{4.29}$$

where **q** is the  $n \times 1$  generalized displacement, **M**, **C** and **K** are the constant  $n \times n$  mass, damping and stiffness matrices, **u** is the  $m \times 1$  input vector, **B** is the  $n \times m$  input matrix, **y** is the  $m \times 1$  output, **Φ** is  $n \times m$  output matrix, **F** is  $n \times s$  disturbance matrix and **d** is the  $s \times 1$  disturbance. Taking the second derivative of y yields

$$\ddot{\mathbf{y}}(t) = \mathbf{\Phi}\ddot{\mathbf{q}} = -\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} - \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{u} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{F}\mathbf{d}.$$
(4.30)

Assuming that for all  $\mathbf{q} \in \mathbb{R}^n$ ,  $\mathbf{H} = \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{B}$  is nonsingular, the dynamic inversion input is

$$\mathbf{u}^{*}(t) = \mathbf{H} - 1 \left( \ddot{\mathbf{r}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{C} \dot{\mathbf{q}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{K} \mathbf{q} - \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{F} \mathbf{d} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_{e} \mathbf{e} \right)$$
(4.31)

where the error is

$$\mathbf{e} \stackrel{\Delta}{=} \mathbf{r} - \mathbf{y}.\tag{4.32}$$

Substituting  $\mathbf{u}^*$  into (4.30)

$$\ddot{\mathbf{e}} + \mathbf{K}_{\dot{e}}\dot{\mathbf{e}} + \mathbf{K}_{e}\mathbf{e} = \mathbf{0} \tag{4.33}$$

The control objective is to make the error (4.32) small, so that the output **y** tracks the desired reference **r**. The feedback gain matrices  $\mathbf{K}_{\dot{e}}$  and  $\mathbf{K}_{e}$  are chosen such that  $\det(s^{2}I_{m} + \mathbf{K}_{\dot{e}}s + \mathbf{K}_{e})$  is Hurwitz, which implies that **e** converges exponentially to zero. Substituting  $u^{*}$  yields the closed loop system

$$\mathbf{M}\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \mathbf{B}\mathbf{H}^{-1}\ddot{\mathbf{r}} + \mathbf{K}_{\dot{e}}\dot{\mathbf{e}} + \mathbf{K}_{e}\mathbf{e} + \tilde{\mathbf{F}}\mathbf{d},$$
(4.34)

where

$$\tilde{\mathbf{C}} = \left[\mathbf{I} - \mathbf{B}\mathbf{H}^{-1}\mathbf{\Phi}\right]^{-1}\mathbf{C},$$

$$\tilde{\mathbf{K}} = \left[\mathbf{I} - \mathbf{B}\mathbf{H}^{-1}\mathbf{\Phi}\right]^{-1}\mathbf{K},$$

$$\tilde{\mathbf{F}} = \left[\mathbf{I} - \mathbf{B}\mathbf{H}^{-1}\mathbf{\Phi}\right]^{-1}\mathbf{F}.$$
(4.35)

The zero dynamics of (4.28) and (4.29) are given by

$$\mathbf{M}\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \mathbf{0}.$$
(4.36)

The system (4.36) has 2m zero eigenvalues. If the remaining 2n - 2m eigenvalues are contained in the open left half plane, then it follows that all of the eigenvalues of (4.34) are contained in the open left half plane and the system is minimum phase.

Even if the (4.28) is minimum phase and  $\mathbf{u}^*$  stabilize the system, still dynamic inversion method has some limitations. dynamic inversion requires full state feedback, and knowledge of model parameters  $\mathbf{C}, \mathbf{K}, \mathbf{B}$  and  $\mathbf{\Phi}$ , and knowledge or measurement of  $\mathbf{F}$ . However, in most of the mechanical structures, there is uncertainty in the system. The types of uncertainty include (but are not limited to) additive exogenous disturbances, structural uncertainties, and parametric uncertainty.

The objective is to design a controller that is robust to the uncertainties and disturbances. Consider  $\mathbf{u}^*$ , through a second order filter. Specifically, let  $\mathbf{u}$  satisfy

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{u}^*, \qquad (4.37)$$

where  $\zeta_c$  and  $\omega_c$  are the controller damping ratio and cut-off natural frequency, respectively. Substituting  $\mathbf{u}^*$  from (4.31) into (4.37) yields

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{r}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{C} \dot{\mathbf{q}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{K} \mathbf{q} - \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{F} \mathbf{d} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{e} \Big).$$
(4.38)

Substituting  $\mathbf{M}^{-1}\mathbf{C} + \mathbf{M}^{-1}\mathbf{K} = -\ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{B}\mathbf{u}$  from (4.28) into (4.38) yields

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{r}} b m \Phi \ddot{\mathbf{q}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{B} \mathbf{u} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{y} \Big).$$
(4.39)

Canceling the  $\mathbf{u}$  from both sides of equation, yields the filter dynamic inversion controller

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{e}} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{e} \Big).$$
(4.40)

The transfer function from  ${\bf e}$  to  ${\bf u}$  is

$$G_c(s) \stackrel{\Delta}{=} \frac{\hat{u}(s)}{\hat{y}(s)} = \mathbf{H}^{-1} \frac{s^2 \mathbf{I} + \mathbf{K}_1 s + \mathbf{K}_2}{s(s + 2\zeta_c \omega_c)}$$
(4.41)

Equations (4.40) and (4.41) show that filtered dynamic inversion does not require knowledge of disturbance, nor the model parameters  $\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{B}, \boldsymbol{\Phi}$ . The controller does however require the output feedback and the knowledge of the **H** which can be obtained from the impulse response. The state space form of (4.28) is given by

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \mathbf{u}.$$
(4.42)

Letting  $\mathbf{x} = [\mathbf{q}, \dot{\mathbf{q}}]^T$  the response of (4.42) is

$$\mathbf{x} = e^{(t-t_0)\mathbf{A}}\mathbf{x}(t_0) + \int_{t_0}^{\infty} e^{(t-T)\mathbf{A}}\mathbf{M}^{-1}\mathbf{B}\mathbf{u}(T)dT,$$
(4.43)

where  $\mathbf{x}_0$  is the initial condition, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}.$$
 (4.44)

Assume that  $\mathbf{x}(t_0) = 0$ . The response of (4.42) to an impulse input, at time t = 0 is

$$\mathbf{x}(0) = \mathbf{M}^{-1}\mathbf{B},\tag{4.45}$$

The output is given by

$$\mathbf{y} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{0}. \tag{4.46}$$

Thus we need to take a derivative of  $\mathbf{x}$ . Then the output would be

$$\dot{\mathbf{y}} = \begin{bmatrix} \Phi & \mathbf{0} \end{bmatrix} \dot{\mathbf{x}} = \Phi \mathbf{M}^{-1} \mathbf{B} = \mathbf{H}.$$
(4.47)

Combining (4.28) and (4.40), the closed loop system with filtered dynamic inversion controller is

$$\begin{bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \\ \dot{\mathbf{u}}(t) \\ \ddot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} & \mathbf{M}^{-1}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \omega_c^2\mathbf{H}^{-1}\mathbf{K}_e\mathbf{\Phi} & \omega_c^2\mathbf{H}^{-1}\mathbf{K}_{\dot{e}}\mathbf{\Phi} & \mathbf{0} & -2\zeta_c\omega_c \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F}\mathbf{d}(t) \\ \mathbf{0} \\ \omega_c^2\mathbf{H}^{-1}(\ddot{\mathbf{r}} + \mathbf{K}_{\dot{e}}\dot{\mathbf{r}} + \mathbf{K}_e\mathbf{r}) + \omega_c^2\mathbf{H}^{-1}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{F}\mathbf{d} \end{bmatrix}$$
(4.48)

Next, we investigate the stability and performance of (4.48). Define  $\mathbf{u} = \mathbf{u}^* + \Delta \mathbf{u}$ , so that (4.28) yields,

$$\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{f}) = \mathbf{B}\Delta \mathbf{u} + \mathbf{F}\mathbf{d}, \tag{4.49}$$

where  $\mathbf{N} = (\mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{B}\mathbf{u}^*). \text{Nest define the output}$ 

$$\mathbf{y}_1 = \dot{\mathbf{u}}^*(t). \tag{4.50}$$

Inserting  $\mathbf{u} = \mathbf{u}^* + \Delta \mathbf{u}$  into (4.37) yields

$$(\ddot{\mathbf{u}}^* + \Delta \ddot{\mathbf{u}}) + 2\zeta_c \omega_c (\dot{\mathbf{u}}^* + \Delta \dot{\mathbf{u}}) + \omega_c^2 (\mathbf{u}^* + \Delta \mathbf{u}) = \omega_c^2 \mathbf{u}^*, \qquad (4.51)$$

which becomes

$$\Delta \ddot{\mathbf{u}} + 2\zeta_c \omega_c \Delta \dot{\mathbf{u}} + \omega_c^2 \Delta \mathbf{u} = \ddot{\mathbf{u}}^* + 2\zeta_c \omega_c \dot{\mathbf{u}}^* \tag{4.52}$$

Define the output

$$\mathbf{y}_2 = \Delta \mathbf{u}.\tag{4.53}$$

Let  $S_1$  denote the system (4.49- 4.50) and let  $S_2$  denote the system (4.52-4.53). Therefore, (4.48) can be represented by the feedback connection shown in Figure 5.3. In order analyze the stability and the performance of the closed loop system the following theorem is established.



Figure 4.3: Feedback Connection

**Theorem 2.** Consider the system (4.49- 4.53), where the disturbances signal **f** and its derivative  $\dot{\mathbf{f}}$  belongs  $\mathscr{L}_{2e}^{n}$ , and suppose

i. **H** is invertible,

ii. The plant (4.28) and (4.29) is minimum phase.

For all  $\delta > 0$ , there exists a  $K_s$  such that for all  $\omega_c > K_s$ ,  $\gamma_1 < \delta$ , where  $\gamma_1$  is the finite  $\mathscr{L}_2$  gain of  $S_1$ .

*Proof.* The proof of Theorem 2 is given in Appendix D.

#### 4.3.1 Actuator and Sensor Collocation

In this section, it is assumed that the output matrix  $\Phi$  is related to **B** such that  $\mathbf{B} = \Gamma \Phi$  where  $\Gamma$  is the  $m \times r$  matrix. For the single input single output system, it is associated with the collocation of sensors

and actuators.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{\Phi}\mathbf{q}(t)$$
(4.54)

where  $\mathbf{M}, \mathbf{C}$  and  $\mathbf{K}$  are the constant  $n \times n$  mass, damping and stiffness matrices. Here  $\mathbf{q}$  is a  $n \times 1$ generalized vector,  $\mathbf{u}$  is a  $m \times 1$  input vector,  $\mathbf{B}$  is a  $n \times m$  input influence matrix,  $\mathbf{y}$  is  $m \times 1$  output and  $\boldsymbol{\Phi}$  is  $n \times m$  output influence matrix.

**Theorem 3.** Consider the system (4.28-4.29), where  $rank[\Phi] = m$  and  $rank[B] = r \leq m$ . If there exist an arbitrary  $m \times r$  matrix  $\Gamma$  such that  $\mathbf{B} = \Gamma \Phi$ , then (i) and (ii) in the theorem 2 is satisfied.

*Proof.* The proof of theorem 3 is shown in Ref. [122].

#### 4.3.2 Modal Control

Consider 4.28, where  $\mathbf{M}, \mathbf{C}, \mathbf{K}$  are diagonal matrices (the modal form) .In order to control the *m* modes of , the output matrix  $\mathbf{y}_m$  is defined as

$$\mathbf{y}_m = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{4.55}$$

**Proposition 1.** Consider the system (4.28-4.54), where **M**, **C**, and **K** are diagonal. Then the conditions in the Theorem 2 are satisfied.

*Proof.* The proof of proposition 1 is given in the Appendix E.

#### 4.3.3 Higher Order Filters

Previously, a second order filter is used to provide approximate of  $u^*$ . Since it was assumed the relative degree of the system is 2, the transfer function of (4.41) is exactly proper. In general, it is desirable to have a strictly proper transfer function, which rolls off at high frequencies. For the case that the system is high relative degree, the filter order should be greater than the relative degree to ensure a strictly proper controller.

A higher order low-pass filter can be be expressed as

$$u^{(\rho)} + \eta_{\rho-1,\omega_c} u^{(\rho-1)} + \dots + \eta_{2,\omega_c} \ddot{u} + \eta_{1,\omega_c} \dot{u} + \eta_{0,\omega_c} u = \eta_{0,\omega_c} u^*,$$
(4.56)

where  $\rho$  is the positive number larger than the relative degree (r) of the system  $(\rho \ge r)$ , and  $\eta_{\rho,\omega_c}$  is a positive constant which depends on  $\omega_c$ . We assume that  $\det(s^{\rho}I_m + ... + \eta_{1,\omega_c} + \eta_{0,\omega_c})$  is Hurwitz. Defining  $u = u^* + \Delta u$ , the transfer function between  $\dot{u}^*$  and  $\Delta u$  yields

$$G_{c}(s) \stackrel{\Delta}{=} \left\| \frac{s^{\rho-1}I_{m} + \dots + \eta_{1,\omega_{c}}}{s^{\rho}I_{m} + \dots + \eta_{1,\omega_{c}}s + \eta_{0,\omega_{c}}} \right\|$$
(4.57)

We assume that as cut-off frequency  $\omega_c$  increase the infinity norm of  $||G_c(s)||$  decreases. Specifically, we assume that for all  $\epsilon > 0$  there exists a  $k_f$  such that for  $\omega_c > k_f$ ,  $||G_c(s)||_{\infty} = \sup_{\omega \in \mathbb{R}} |G(j\omega)| < \epsilon$ 

## 4.4 Application Example: Control of a Cracked Beam

In this section we apply filtered dynamic inversion to mitigate the vibration of a crack beam shown in Fig 4.4. As shown in Fig 4.5 the crack is modeled by a massless rotational spring. The equivalent stiffness of the rotational spring is given by



Figure 4.4: A cantilever beam with a single-edge crack



Figure 4.5: A two-segment model with a massless rotational spring



Figure 4.6: Tip-concentrated loading

$$K_T = \frac{EIt_b}{6\pi \int_0^a af(a/t_b)da},\tag{4.58}$$

where  $t_b$  is the height of the beam, a is the depth of the crack, and

$$f\left(\frac{a}{t_b}\right) = \sqrt{\frac{2t_b}{\pi a} \tan\frac{\pi a}{2t_b}} \frac{0.923 + 0.199(1 - \sin\frac{\pi a}{2t_b})^4}{\cos\frac{\pi a}{2t_b}}.$$
(4.59)

The presence of the crack is represented by an additional rotation at the crack location  $(x = L_c)$ , resulting in a discontinuity in the slope of the beam. The transition of the slope is consequently expressed as

$$\frac{dY_I(L_c)}{dx} + \frac{EI}{K_T} \frac{d^2 Y_I(L_c)}{dx^2} = \frac{dY_{II}(L_c)}{dx}$$
(4.60)

where  $Y_I$  and  $Y_{II}$  are the amplitudes of the flexural deformation of the beam segments I and II. It should be noted that since the crack is assumed as massless rotational spring, the mass matrix  $\mathbf{M}$  of the system would remain the same but the damping  $\mathbf{C}$  and stiffness  $\mathbf{K}$  matrices will change.

**Example 4.1:** In this example a damaged beam is disturbed by a point force,  $10\sin(10t)$ , located at the tip of the beam. As shown in Fig 4.7, shows a pair of piezoelectric collocated sensor and actuator located at  $x_{a_1} = 0.1L_b$ .



Figure 4.7: Cantilever beam with a piezoelectric actuator

The piezoelectric objective is to suppress the vibration at the crack; however since in practice it may be difficult to determine the exact location of the crack, we attempt to control the vibration at the sensor. It is expected by controlling the sensor output, the amplitude of vibration at the damaged location will also be attenuated. Here it is assumed the ratio for crack depth and beam height is  $a/t_b = 0.5$ . The natural frequencies for the beam are

$$\omega_n = \lambda_n^2 \sqrt{EI/mL_b^4} \tag{4.61}$$

where  $\lambda_n$  is  $n^{th}$  the solution of characteristic for the beam, EI is the flexural rigidity, m is the mass per length and  $L_b$  is the beam length. The first four  $\lambda$ 's of the damaged and undamaged beam is shown in Table 4.1. The system parameters is shown in table We apply a second order filter where the cut-off frequency and the damping ratio for the compensator are  $\omega_c = 1600Hz$  and  $0\zeta_c = 0.05$ , respectively. The gain  $\mathbf{K}_e$  and  $\mathbf{K}_{\dot{e}}$  are also chosen as  $100\mathbf{I}_{2\times 2}$ , where  $\mathbf{I}_{2\times 2}$  is the identity matrix. Figure 4.8 shows the deflection at the sensor, and the crack, and the actuator input voltage. The filtered input-output controller significantly attenuated the vibration of both the output and the crack location.

Table 4.1: Characteristic Solution of Intact and Damaged Cantilever Beam

Mode	Intact	Damaged
1	1.89	1.71
2	4.61	4.50
3	7.89	7.88
4	10.14	10.14

**Example 4.2:** To control the vibration of a damaged beam, it is also possible to control the modes of vibration. Considering again the system of Example 4.1, the objective of this example is to control the first two modes of vibration. To make the system square (equal input and output), two piezoelectric patches, located at  $x_{a_1} = 0.1L_b$  and  $x_{a_2} = 0.3L_b$  are considered. In Figure 4.10, the first four modes of vibration are shown. It can be seen that the vibration of the first two modes significantly attenuated; however the control response for the  $3^{rd}$  and  $4^{th}$  modes are larger than the no control response. The required actuation force of the piezoelectric actuator is shown in the Figure 4.11. It should be noted that the amplitude of vibration for the third and fourth modes are negligible compared with the vibration of the first mode. In Figure 4.12, the vibration of the crack is shown. It is obvious modal vibration control can atunates the vibration at the crack.



Figure 4.8: (a) Time response for the sensor output, (b) Time response for the deflection at the crack, (c) Time response for the required actuation input



Figure 4.9: Cantilever beam with two piezoelectric actuators



Figure 4.10: Time response for th modal vibration



Figure 4.11: Time response for the required actuation input



Figure 4.12: Time response for th deflection at the crack
#### 4.5 Summary

This chapter presented filtered dynamic inversion control which was used to control a linear timeinvariant minimum phase structure. First, the basic of the controller was demonstrated through a simple example. the dynamic inversion control input to achieve the desired motion was determined. The open loop control was then augmented by a feedback loop that accounts for uncertainties due to modeling error and external vibration disturbances. The conditions on stability and performance of this controller were discussed. It was proven that if the ideal closed-loop system is asymptotically stable, and the filter cut-off frequency is sufficiently large, then the closed-loop system is stable. Also it was shown that if the system is minimum phase then the norm of the error can be made arbitrarily small by increasing the cut-off frequency. The numerical simulations demonstrated the effectiveness of the proposed controller in order to attenuate the vibration of a cracked beam.

# Chapter 5

# Filtered Input-Output Linearization for Nonlinear Systems

In this chapter, we extend the controller introduced in the previews section to address nonlinearities. Sources of nonlinearity in structures include the geometric nonlinearities and the material nonlinearities. The control objective is to design a controller to minimize the effects of uncertainty and disturbance in nonlinear structures. To demonstrate the effectiveness of the controller a slewing flexible beam example is considered.

### 5.1 Introduction

Consider the problem of stabilizing motor actuated inverted pendulum shown in Figure 5.1. The pendulum dynamics are given by

$$\ddot{\theta} + b\dot{\theta} - \frac{g}{l}\sin(\theta) = u + d, \tag{5.1}$$

where  $\theta$  is the angular displacement, u is the control input, b is the damping constant, l is the pendulum length and d is the disturbance. Let  $\mathbf{x} = [x_1, x_2]^T = [\theta, \dot{\theta}]^T$ , and it follows that (5.1) can be



Figure 5.1: A servo-actuated inverted pendulum

expressed as

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -bx_2 + \frac{g}{l}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u+d).$$
(5.2)

Next, consider the input-output linearization controller

$$u^* = bx_2 - \frac{g}{l}\sin(x_1) - d + k_1x_1 - k_2x_2.$$
(5.3)

Substituting (5.3) into (5.1), yields

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},\tag{5.4}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
(5.5)

This control  $u^*$  cancels the nonlinear dynamics of the system and replaces them with linear dynamics. However, implementation of  $u^*$ , requires full-state measurement and knowledge of the model.

In practice, there is always an uncertainty in the system. To address uncertainty, consider passing  $u^*$  through a second order filter. In particular, let u satisfy

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} + \omega_c^2 u = \omega_c^2 u^*, \tag{5.6}$$

where  $\zeta_c > 0$  and  $\omega_c > 0$ .

Substituting  $u^*$  from (5.1) into (5.6) yields

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} + \omega_c^2 u = \omega_c^2 (bx_2 - \frac{g}{l}\sin(x_1) - d + k_1 x_1 - k_2 x_2).$$
(5.7)

Substituting  $(bx_2 - \frac{g}{l}\sin(x_1) - d = u - \dot{x}_2$  from (5.1) into (5.7) yields the filtered input-output linearization controller

$$\ddot{u} + 2\zeta_c \omega_c \dot{u} = -\omega_c^2 (k_1 x_1 + k_2 x_2). \tag{5.8}$$

(5.8) shows that filtered input-output linearization controller does not need to have any knowledge about the model parameters nor disturbance. Next we consider the stability and performance of the closed loop system (5.1) and (5.8).Define  $u = u^* + \Delta u$ , (5.1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\Delta u \tag{5.9}$$

which we call the  $S_1$  system. Also (5.6) can be written as

$$\Delta \ddot{u} + 2\zeta_c \omega_c \Delta \dot{u} + \omega_c^2 \Delta u = \ddot{u}^* + 2\zeta_c \omega_c \dot{u}^* \tag{5.10}$$

which we call the  $S_2$  system. The closed loop system (5.1) and (5.8) is can be expressed by the feedback connection shown in Figure 5.2



Figure 5.2: A closed-loop system

The output of system  $S_1$  is  $\dot{u}^*$  and the output of system  $S_2$  is  $\Delta u$ . The  $\dot{u}^*$  is

$$\dot{u}^{*} = \frac{d}{dt}(bx_{2} - \frac{g}{l}\sin(x_{1}) - d + k_{1}x_{1} - k_{2}x_{2}) = \left(-\frac{g}{l}\cos(x_{1}) - k_{1}\right)\dot{x}_{1} + (b - k_{2})\dot{x}_{2}$$

$$\left(-\frac{g}{l}\cos(x_{1}) - k_{1}\right)\dot{x}_{1} + (b - k_{2})(-k_{1}x_{1} - k_{2}x_{2} + d + \Delta u)$$
(5.11)

It follows that

$$\begin{aligned} |\dot{u}^*| &= \left| \left( -\frac{g}{l} \cos(x_1) - k_1 \right) \dot{x}_1 + (b - k_2) (-k_1 x_1 - k_2 x_2 + d + \Delta u) \right| \\ &\leq \left| \left[ \begin{array}{c} -k_1 (b - k_2) \\ -\frac{g}{l} \cos(x_1) - k_1 - k_2 (b - k_2) \end{array} \right] \cdot \mathbf{x} \right| + |-k_2 + b| |d + \Delta u| \end{aligned}$$

$$\leq c_1 ||d|| + c_2 ||\Delta u||$$
(5.12)

where

$$c_{1} = \left\| \left[ \begin{array}{c} -k_{1}(b-k_{2}) \\ |-k_{1}-k_{2}(b-k_{2})| + \frac{g}{l} \end{array} \right] \right\|, \quad c_{2} = |-k_{2}+b|.$$
(5.13)

Thus from (5.12)

$$||\dot{u}^*|| \le c_1 \gamma ||\mathbf{x}||_{\mathscr{L}_2} + c_2 ||d||_{\mathscr{L}_2} + \beta$$

$$(5.14)$$

The finite  $\mathscr{L}_2$  gain for the system  $S_1$  is

$$\gamma_1 = c_1 \gamma + c_2 \tag{5.15}$$

Note that the open loop transfer function of (5.10) is

$$\hat{S}_2(s) = \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2}$$
(5.16)

which is finite gain  $\mathscr{L}_2$  stable [111], with

$$\gamma_2 = \sup_{\omega_R} \left\| \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2} \right\|.$$
(5.17)

Based on the small gain theorem [111], the closed loop system of Figure 5.2 is finite gain  $\mathcal{L}_2$  stable if

$$\gamma_1 \gamma_2 < 1 \tag{5.18}$$

Since  $\gamma_2$  is made arbitrary small by increasing  $\omega_c$ , it follows that for sufficiently large  $\omega_c$  the closed loop system (5.1) and (5.8) is finite gain  $\mathscr{L}_2$  stable.

#### 5.2 Filtered Input-Output Linearization

Consider a structure that is modeled by the system

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{u} + \mathbf{f}, \tag{5.19}$$

$$\mathbf{y} = \mathbf{\Phi} \mathbf{q},\tag{5.20}$$

where  $\mathbf{M}(\mathbf{q})$  is the  $n \times n$  mass matrix,  $\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$  is the  $n \times 1$  generalized forces which includes damping and stiffness forces,  $\mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})$  is the  $n \times m$  input matrix,  $\mathbf{u}$  is the  $m \times 1$  control forces,  $\mathbf{f}$  is the  $n \times 1$ disturbance signals,  $\mathbf{y}$  is the output signal and  $\boldsymbol{\Phi}$  is the output matrix. Assuming  $\mathbf{M}(\mathbf{q})$  is invertible, the second derivative of (5.20) is

$$\ddot{\mathbf{y}} = \mathbf{\Phi}\ddot{\mathbf{q}} = -\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{h} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{u} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{f}.$$
(5.21)

We assume that for all  $\mathbf{q} \in \mathbb{R}^n$ ,  $\mathbf{H} = \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{B}$  is nonsingular. Next, consider the input-output linearization controller

$$\mathbf{u}^{*}(t) = \mathbf{H}^{-1} \Big( \ddot{\mathbf{r}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{h} - \mathbf{\Phi} \tilde{\mathbf{M}}^{-1} \mathbf{f} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_{e} \mathbf{e} \Big),$$
(5.22)

where

$$\mathbf{e} \stackrel{\Delta}{=} \mathbf{r} - \mathbf{y} \tag{5.23}$$

is the error, and  $\mathbf{K}_{\dot{e}}$  and  $\mathbf{K}_{e}$  are  $m \times m$  gain matrices. We assume that the feedback gain matrices  $\mathbf{K}_{\dot{e}}$ and  $\mathbf{K}_{e}$  are chosen such that the det $(s^{2}\mathbf{I}_{m} + \mathbf{K}_{\dot{e}} + \mathbf{K}_{e})$  is Hurwitz (i.e. the ideal output dynamics are asymptotically stable). The control objective is to make the error (5.23) small, so that the output  $\mathbf{y}$ tracks the desired reference  $\mathbf{r}$ . Let (5.19)-(5.20) with  $\mathbf{u} = \mathbf{u}^{*}$  denote the ideal output dynamics, which are given by

$$\ddot{\mathbf{e}} + \mathbf{K}_{\dot{e}}\dot{\mathbf{e}} + \mathbf{K}_{e}\mathbf{e} = \mathbf{0}.\tag{5.24}$$

By choosing approximately, the desired response of the tracking error is obtained.

The objective is to design a controller that is robust to the uncertainties and disturbances. Consider passing the ideal control input in  $\mathbf{u}^*$  through a second order filter. Specifically, let  $\mathbf{u}$  satisfy

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{u}^*, \tag{5.25}$$

where  $\zeta_c$  and  $\omega_c$  are the controller damping ratio and cut-off natural frequency, respectively. We expect that as the cut-off frequency  $\omega_c$  is increased, **u** will approximate the ideal input **u**<sup>\*</sup>. Substituting **u**<sup>\*</sup> from (5.22) into the (5.25) yields

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{r}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{h} - \mathbf{\Phi} \tilde{\mathbf{M}}^{-1} \mathbf{f} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{e} \Big).$$
(5.26)

Substituting  $\mathbf{M}^{-1}\mathbf{h} - \mathbf{M}^{-1}\mathbf{B}\mathbf{u} = \mathbf{M}^{-1}\mathbf{f} - \ddot{\mathbf{q}}$  from (5.19) into (5.26) yields

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} + \omega_c^2 \mathbf{u} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{r}} - \mathbf{\Phi} \ddot{\mathbf{q}} + \mathbf{\Phi} \mathbf{M}^{-1} \mathbf{B} \mathbf{u} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{e} \Big).$$
(5.27)

Canceling  $\mathbf{u}$  from both sides of (5.27), yields the filter input-output linearization controller

$$\ddot{\mathbf{u}} + 2\zeta_c \omega_c \dot{\mathbf{u}} = \omega_c^2 \mathbf{H}^{-1} \Big( \ddot{\mathbf{e}} + \mathbf{K}_{\dot{e}} \dot{\mathbf{e}} + \mathbf{K}_e \mathbf{e} \Big)$$
(5.28)

Equations (5.28) show that filtered input-output linearization does not require knowledge of disturbance. The controller requires the knowledge of the  $\Phi M^{-1}B$ . Combining (5.19) and (5.28), the closed loop system with filtered input-output linearization controller is

$$\begin{bmatrix} \mathbf{M}\ddot{\mathbf{q}}(t) \\ \ddot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{B}\mathbf{u} + \mathbf{f}(t) \\ -2\zeta_c\omega_c\dot{\mathbf{u}} + \omega_c^2\mathbf{H}^{-1}\left(\ddot{\mathbf{e}} + \mathbf{K}_{\dot{e}}\dot{\mathbf{e}} + \mathbf{K}_e\mathbf{e}\right) \end{bmatrix}$$
(5.29)

Next, we investigate the stability and performance of (5.29). Define  $\mathbf{u} = \mathbf{u}^* + \Delta \mathbf{u}$ , then the closed loop

system (5.29) can be expressed as a feedback connection of the system  $S_1$ , defined by

$$\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{f}) = \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) \Delta \mathbf{u}$$

$$\mathbf{y}_1 = \dot{\mathbf{u}}^*(t)$$
(5.30)

and the system  $S_2$  defined by

$$\Delta \ddot{\mathbf{u}} + 2\zeta_c \omega_c \Delta \mathbf{u} + \omega_c^2 \Delta \mathbf{u} = \ddot{\mathbf{u}}^* + 2\zeta_c \omega_c \dot{\mathbf{u}}^*$$

$$\mathbf{y}_2 = \Delta \mathbf{u}$$
(5.31)

where  $\mathbf{N} = \mathbf{M}^{-1}(\mathbf{h} + \mathbf{B}\mathbf{u}^*)$  and  $\mathbf{g} = \mathbf{M}^{-1}\mathbf{B}$ . The feedback diagram is shown in Figure 5.3. The following theorem addresses the closed-loop stability and performance.



Figure 5.3: Feedback Connection

**Theorem 4.** Consider the system (5.31), where the disturbances signal  $\mathbf{f}$  and its derivative  $\dot{\mathbf{f}}$  belongs to the signal space L, where L could be any  $L_p$  space. Assume

- 1. **H** is invertible,
- 2. The system  $S_1$  is finite  $\mathscr{L}_2$  gain stable.

For all  $\delta > 0$ , there exists a  $K_s$  such that for all  $\omega_c > K_s$ ,  $\gamma_1 < \delta$ , where  $\gamma_1$  is the finite  $\mathscr{L}_2$  gain of  $S_1$ .

*Proof.* The proof of theorem 4 is given in the Appendix F.

### 5.3 Control of a Slewing Beam

A piezoelectric actuated flexible beam is shown in Figure (5.4), which consist of a rigid hub with radius r and a uniform cantilever flexible beam with an attached piezoelectric actuator. In Figure 5.4  $\theta$  is the attitude angle between the inertial frame XYZ and the body-fixed frame xyz, w(x,t) is the beam's deformation at point x with respect to the xyz frame and  $\tau$  is the applied control torque about the z axis. The kinetic energy and potential energy for a slewing beam can be expressed as [66, 123],



Figure 5.4: Model of flexible spacecraft with a bonded piezoelectric actuator

$$T = \frac{1}{2}J\dot{\theta}^{2} + \frac{1}{2}\rho \int_{0}^{L} \left\{ (w\dot{\theta})^{2} + [(x+b)\theta + \dot{w}]^{2} \right\} dx,$$
  

$$U = \frac{1}{2}EI \int_{0}^{L} (w^{''})^{2} dx,$$
(5.32)

where  $w'' = \frac{d^2\phi}{dx^2}$ ,  $\rho$  is the mass per unit length, J is the rotational inertia of the rigid body motion about the z axes, and EI is the uniform flexural rigidity of the beam. Using the modal expansion method the elastic deformation w(x,t) is assumed to be of the form

$$w(x,t) = \sum_{i=1}^{n} \phi_i q_i,$$
(5.33)

where  $\phi_i$  is the  $i^{th}$  mode shape function and  $q_i$  is the generalized coordinates corresponding to the  $i^{th}$  vibrational mode. The shape function for a cantilever beam is [124]

$$\phi_n = \sin\beta_n x - \sinh\beta_n x + \frac{\cos\beta_n L + \cosh\beta_n L}{\sin\beta_n L - \sinh\beta_n L} (\cos\beta_n x - \cosh\beta_n x), \tag{5.34}$$

where  $\beta_n > 0$  is the solution of

$$1 + \cos\beta_n \cosh\beta_n = 0. \tag{5.35}$$

The Lagrangian function L = T - U then can be expressed as

$$2L = J\dot{\theta}^2 + \sum_{i=1}^n \left( m_i \dot{q}_i^2 + I_i \dot{\theta} q_i + m_i \dot{\theta}^2 q_i^2 - k_i q_i^2 \right),$$
(5.36)

where

$$m_i = \rho S_b \int_0^L \phi_i^T \phi_i dx, \qquad (5.37)$$

$$I_{i} = \rho S_{b} \int_{0}^{L} (x+b)\phi_{i} dx, \qquad (5.38)$$

$$k_{i} = EI \int_{0}^{l} \phi_{i}^{''T} \phi_{i}^{''} dx, \qquad (5.39)$$

$$J = I_h + \rho S_b[(L+r)^3 - r^3]/3.$$
(5.40)

 $S_b$  is surface area of the beam, and  $I_h$  is the hub moment of inertia. The total work (W) done by the damping force, control torque, and piezoelectric patches can be expressed as

$$W = \sum_{i=1}^{n} \left( -\frac{1}{2} c_i \dot{q}_i^2 + q_i b_{a_i} v_{a_i} \right) + T_h \theta, \qquad (5.41)$$

where  $c_i$  is the damping constant of the beam,  $b_{a_i}$  is a components of the piezoelectric input matrix [40], and  $v_{a_i}$  is the piezoelectric voltage. Consequently, Lagrange's equations of motion can be written as

$$J\ddot{\theta} + \sum_{i=1}^{n} \left( m_i \ddot{\theta} q_i^2 + 2m_i \dot{\theta} q_i \dot{q}_i + m_i \ddot{\theta} q_i^2 \right) = \tau,$$

$$(5.42)$$

$$I_i \ddot{\theta} + m_i \ddot{q}_i - m_i \dot{\theta}^2 q_i + k_i q_i + c_i \dot{q}_i = b_{a_i} v_{a_i}.$$
(5.43)

The conditions in the Theorem 4 are the necessary condition. Although in the next example, these conditions of Theorem 4 can not be proved but the numerical results shows that the filtered input-output linearization controller can stabilized the closed loop. To demonstrate the application two application cases are presented. In the first case, collocated sensing and actuating is considered. In the second case we apply the modal control.

#### Control of a Slewing Flexible Beam with a Collocated Actuator/Sensor

Assume there is a pair of collocated piezoelectric sensors and actuator, and the goal of the controller is to achieve the desired angle  $(\theta_r)$  while suppressing the vibration at the sensor location. The output is defined as

$$\mathbf{y}_{c} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \alpha b_{1} & \dots & \alpha b_{n} \end{bmatrix} \begin{bmatrix} \theta \\ q_{1} \\ \vdots \\ q_{n} \end{bmatrix} = \begin{bmatrix} \theta_{r} \\ 0 \end{bmatrix}$$
(5.44)

where  $\alpha$  is a constant that depends on the piezoelectric sensor and actuators properties [33].

#### Modal Control of a Slewing Flexible Beam

In order to control the vibration of m modes of the beam while the desired angle is approaching  $\theta_r$ , the output of the system is defined as

$$\mathbf{y}_{m} = \begin{bmatrix} \theta \\ q_{1} \\ \vdots \\ q_{n} \end{bmatrix} = \begin{bmatrix} \theta_{r} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
(5.45)

## 5.3.1 Vibration Control of Slewing Beam using Collocated Actuator/ Sensor

In order to show the effectiveness of the proposed controller the system in Figure (5.5) is considered. It is assumed a collocated sensor and actuator pair is placed at the base of the beam,  $(x_s = 0.2L)$ , and the desired attitude angle is  $\theta_r = \pi/3$ . The parameters of the system are listed in Table 5.1. In this example the first two modes which have the natural frequencies of 26.3Hz and 164.9Hz are considered. The cut-off frequency and the damping ratio for the compensator are  $\omega_c = 1600$ Hz and  $\zeta_c = 0.05$ , respectively. The gain  $\mathbf{K}_e$  and  $\mathbf{K}_{\dot{e}}$  are assumed as  $100\mathbf{I}_{2\times 2}$ , where  $\mathbf{I}_{2\times 2}$  is the identity matrix.

Figure 5.6 shows that the desired rotation angle is achieved, while Figure 5.7 shows that the output of the sensor is suppressed. The required torque and piezoelectric voltages to achieve the desired motion are shown in Figure 5.8. Figure 5.9 shows the tip deflection of the beam. It can be seen that although the controller mitigates the sensor output, the piezoelectric actuator also attenuates the tip vibration.

A fundamental problem in control of flexible system is that in theory the flexible system is an infinitedimension system, but in practical applications there are a finite number of actuators and sensors. In order to show that  $\mathscr{L}_2$  gain of the system S<sub>1</sub> is bounded for different number of modes,  $||y_{tip}||_2$  of the



Figure 5.5: Model of flexible spacecraft with a collocated sensor and actuator

Table 5.1: Properties of the system

Density, $\rho$	$2700 kg/m^{3}$
Young Modulus of the Beam, $E$	70.0 Gpa
Young Modulus of the Piezoelectric Patch, $E_p$	63Gpa
Size of the Beam,	$250\times 20\times 2mm$
Size of the Actuato,r	$25\times20\times0.02mm$
Poisson Ratio, $v$	0.3
Piezoelectric Constant, $d_{3i}$	$20 \times 10^{-11} m/V$
Piezoelectric Constant , $h_{3i}$	$1 \times 10^{-5} V/m$
Mass of the hub, $m_h$	1Kg
Radius of the hub, $r_h$	6cm



Figure 5.6: Time response for attitude maneuver with Filtered Input-Output Linearization controller for the collocated case



Figure 5.7: Time response for sensor output with Filtered Input-Output Linearization controller for the collocated case



Figure 5.8: Time response for the required input with Filtered Input-Output Linearization controller for the collocated case



Figure 5.9: Time response for deflection of a tip of the beam with Filtered Input-Output Linearization controller for the collocated case

beam for different modes are shown in Figure 5.10. It can be seen that  $||y_{tip}||_2$  is bounded for increasing modes, which implies that  $\mathscr{L}_2$  gain of the system  $S_1$  is bounded. This implies that considering an increasing the dimension of the beam the cut-off frequency will not increasing proportionally and after a specific number of modes the required cut-off frequency is the same to stabilize the closed loop system



Figure 5.10: The norm of the tip deflection for collocated case

#### 5.3.2 Modal Control of a Slewing Flexible Beam

Next, we consider the modal control of the slewing beam. The system and control parameters are the same in Example (4.3) The objective is to control the vibration of the first mode in the presence of the point force disturbance  $F = 10 \sin(10t)$ , which is located located at  $x_f = 0.5L$ . The output matrix in this case can be written as

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_r \\ 0 \end{bmatrix}.$$
(5.46)

Figure 5.11 shows that the angular displacement goes to the desired value. The modal vibration of the generalized coordinates is shown in Figure 5.12. The controller attempts the vibration of the first mode significantly, however amplitude of the second generalized coordinates is increased. The required torque and piezoelectric voltages to obtain the desired motion are shown in Figure (5.13). As shown in Figure (5.14) the vibration at the tip mass is also significantly attached.

As discussed in previous Example, it can be shown that the residual modes will also not affect on the stability and performance of the augmented system in (5.31) for modal control case. Figure 5.15 shows the  $||y_{tip}||_2$  for the modal control. It is obvious that the  $||y_{tip}||_2$  is bounded for different number of modes, which implies that considering an increasing the dimension of the beam the cut-off frequency



Figure 5.11: Time response for attitude maneuver with Filtered Input-Output Linearization controller for modal control



Figure 5.12: Time response for modal displacement with Filtered Input-Output Linearization controller for modal control



Figure 5.13: Time response for the required input with Filtered Input-Output Linearization controller for modal control



Figure 5.14: Time response for deflection of a tip of the beam with Filtered Input-Output Linearization controller for modal control

will not increasing proportionally and after a specific number of modes the required cut-off frequency is the same to stabilize the closed loop system .



Figure 5.15: The norm of the tip deflection for modal control

### 5.4 Summary

This chapter focused on a nonlinear version of controller introduced in chapter 4. It was proven that if the system when controller by input-output linearization is finite gain  $\mathscr{L}_2$  stable, and the filter cut-off frequency is sufficiently large, then the closed-loop system is finite gain  $\mathscr{L}_2$  stable. Filtered input-output linearization was applied to a slewing beam example. It was demonstrated that, even though it could not be shown to satisfy the sufficient stability conditions, the controller was successful.

# Chapter 6

# **Conclusions and Recommendations**

#### 6.1 Conclusion

This dissertation was concerned with the distributed control of flexible engineering structures. In particular, the focus was on determining the distributed actuation required to produce specified motion of a material structure. In the first part of this work, we considered a type of inverse problem, called the servoconstraint problem, where the objective was to determine the actuation input as a function of time needed to produce a desired motion. The motivation for this solution was to have a tool for determining actuator sizing and placement.

We considered structures modeled by a finite number of ordinary second-order differential equations, having an equal number of control inputs and algebraic relations that defined the desired motion. In Chapter 2, a projection method was employed to solve this problem. For the case where the model was nonlinear and the desired motions were expressed as a general nonlinear relation between the system states, the method produces a set of differential algebraic equations for which the solution may not be trivial to compute. For the case that the model was linear and the desired motions were expressed as a linear relation between the system states, the method produces a set of trivial algebraic relations. To show the usefulness of solving the servoconstraint problem, we considered several examples in the actuation of beams.

Next, in Chapter 3 the servoconstraint problem was applied to the actuation of active (intelligent) structures with distributed actuators. In particular, we considered a general shell structure with a finite number of distributed piezoelectric actuators. Tzous form of Loves theory was used to model the piezoelectric structure, and the servoconstraint solution method was used to solve for the distributed

actuation voltage.

In the second part of this work, the objective was to determine the distributed actuation as a function of the measured output, that is, feedback control. In particular, the objective was to develop a method of feedback control that addressed common problems of structural control, including high-and-unknown dimensionality, spillover, parameter uncertainty, nonlinearities, and unknown-and-unmeasured disturbances.

In Chapter 4, a novel control method called Filtered Dynamic Inversion was introduced for linear structural models. Filtered Feedback Dynamic Inversion combines standard feedback dynamic inversion with a low-pass filter. The main features of the controller are that it provides an approximation of feedback dynamic inversion, however unlike standard feedback dynamic inversion it requires only output feedback (standard feedback dynamic inversion requires full-state measurement), it does not require knowledge of the model order (i.e., the number of elements or modes used to model the structure), it does not require detailed model knowledge (it requires only the high-frequency gain matrix), and it does not require knowledge or measurement of the disturbance. The key parameter of the controller is the filter cutoff frequency, which is increased to achieve better performance. In particular, we showed that for sufficiently large cutoff frequency the closed-loop system can be made finite-gain  $\mathcal{L}_2$  stable, and further increase of the cutoff frequency improves performance. Several examples were given to show how the controller can be used for structural vibration control.

In Chapter 5, Filtered Feedback Linearization was extended to address structural models with nonlinearities. It was shown that, for certain conditions, finite-gain  $\mathcal{L}_2$  stability and performance is achieved, as with the linear case, by selecting a sufficiently large filter cutoff frequency. However, for general nonlinear systems, the controller requires full-state feedback. To demonstrate the usefulness of the controller, we applied it to the slewing beam problem, where the objective is to control both the trajectory and the vibration of a rotating beam.

#### 6.2 Recommendations

The servo-constrain problem focuses on determining the required input force in order to obtain the voltage. In chapter 2 and 3, it was shown that based on the actuator location, the required voltage changes. One of the most interesting aspects of the servo-constraint problem is its relation to optimization. Since the actuator placement substantially effects the input requirements, this placement can be optimized such that with the minimum energy, the program constraints can be obtained. Furthermore, there is the matter of constraint optimization. Although the focus of the servo-constraint problem is

the motion of a single part of a structure, it is still important to be confident that the other parts of the flexible structure have an "acceptable motion". Thus, the motions of the other parts of the flexible structures also need to be considered in the program constraint.

In the chapter 4, it was shown that the filtered dynamic inversion controller does not require knowledge of the system or the external disturbance except the Markov parameter. We showed that the Markov parameter can be identified experimentally. However, uncertainties in measuring the Markov parameter may affect the performance of the controller and may cause instability. Even though for single input single output (SISO) systems, having the knowledge about the sign of the Markov parameter is sufficient, some additional research regarding the uncertainties of the Markov parameters for the cases with multi input multi output (MIMO) needs to be carried out.

Next, in order to use the filtered dynamic inversion controller, the system needs to be minimum phase. It was proven that for the cases of the collocated actuator and the sensor and modal control, the system is minimum phase. However, there are many applications in which the sensor and actuator are not collocated or modal control is not possible. To address limitations there are two general approaches that can be studied. In the first approach, the goal is to define a general solution between the actuator and the sensor location, so that by placing the actuator and sensor in the specific non-collocated locations the zero-dynamic will be stable. This approach may have its own advantages but there are still some cases in which the system is in the non-minimum phase. The other approach that could be the focus of future research is to extend the filtered dynamic inversion controller for non-minimum phase systems.

Finally, verifying the numerical results with experimental analysis would be the future goal of this research. There are some challenges regarding the experimental tests which need to be addressed. Sensor output produces noise. Therefore, because of the integration parts of the controller transfer function, and because of the bias in the output signal, the actuator input increases. To avoid actuator saturation, the system needs to be reset. Managing the resetting point is an issue that needs to be investigated more in the experimental study. The other expecting challenge is the matter of performance. In theory, by increasing the controller cut-off frequency, the norm of the error should decrease. However in real practice, due to the actuator saturation, there is a limit to the increase in the cut-off frequency. Thus, further investigation is needed in regards to the methods that can improve the performance of the controller.

#### A.Thin rectangular cantilevered plate

Let  $\alpha_1 = x$ ,  $\alpha_2 = y$  be the rectangular coordinates for a rectangular plate ( $0 \le x \le L$ ,  $0 \le y \le W$ ). The plate is considered to be sufficiently thin so that only the transverse vibration,  $w_3$ , is of consequence. The modal expansion is given by

$$w_3(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \eta_{mn}(t) \Phi_{3mn}(x, y)$$

For the boundary conditions, the plate is cantilevered along one edge while the remaining edges are free. The mode shapes are [124]

$$\Phi_{3} = \sum_{m=1}^{p} \left\{ C_{m1} + C_{m2} \left( 1 - 2\frac{y}{L} \right) + \sum_{n=3}^{q} C_{mn} \left[ \cosh \frac{\epsilon_{n}y}{W} + \cos \frac{\epsilon_{n}y}{W} - \frac{\cosh \epsilon_{n} - \cos \epsilon_{n}}{\sinh \epsilon_{n} - \sin \epsilon_{n}} \left( \sinh \frac{\epsilon_{n}y}{W} + \sin \frac{\epsilon_{n}y}{W} \right) \right] \right\}$$
$$\left[ \cosh \frac{\epsilon_{m}x}{L} - \cos \frac{\epsilon_{m}x}{L} - \frac{\sinh \epsilon_{m} - \sin \epsilon_{m}}{\cosh \epsilon_{m} + \cos \epsilon_{m}} \left( \sinh \frac{\epsilon_{m}x}{L} - \sin \frac{\epsilon_{m}x}{L} \right) \right]$$

where L is the length, W is the width, m is the longitudinal wave number, and n is the latitudinal wave number, and  $\epsilon_n$  and  $\epsilon_m$  are solutions of characteristic of equations [124]

$$\cos \epsilon_n \cosh \epsilon_n = 1$$
$$\cos \epsilon_m \cosh \epsilon_m = -1.$$

The first ten dimensionless natural frequencies for the structure are given in Table A.3 [125]. Considering the Lame parameters  $(A_1 = 1 \text{ and } A_2 = 1)$  and radii of curvature  $(R_1 = \infty \text{ and } R_2 = \infty)$  the Love's control operator would be

$$L_3^c\{\psi_3\} = -\left\{\frac{\partial^2 M_{11}^c}{\partial x^2} + \frac{\partial^2 M_{22}^c}{\partial y^2}\right\},\,$$

where  $M_{11}^c$  and  $M_{22}^c$  are piezoelectric induced moments are

$$M_{11}^c = r_{11}d_{31}E_p\psi^a(t)$$
$$M_{22}^c = r_{22}d_{32}E_p\psi^a(t),$$

where  $d_{31}$  and  $d_{32}$  are piezoelectric constants,  $r_{11} = r_{22}$  is the moment arm measured from the plate neutral surface to the actuator mid-surface, and  $E_p$  is the piezoelectric constant. The transverse actuating voltage  $\psi^a(x, y, t)$  applied to an actuator patch located from  $x = x_1$  to  $x = x_2$  in the longitudinal direction and  $y = y_1$  to  $y = y_2$  in the lateral direction is

$$\psi^a(x, y, t) = [H(x - x_1) - H(x - x_2)] [H(y - y_1) - H(y - y_2)] V(t)$$

where  $H(\cdot)$  is the unit step function. Thus the spatial derivatives of the transverse actuation signals are

$$\frac{\partial^2}{\partial x^2}\psi^a(x,y,t) = \left[\delta'(x-x_1) - \delta'(x-x_2)\right] \left[H(y-y_1) - H(y-y_2)\right] V(t)$$
$$\frac{\partial^2}{\partial y^2}\psi^a(x,y,t) = \left[H(x-x_1) - H(x-x_2)\right] \left[\delta'(y-y_1) - \delta'(y-y_2)\right] V(t),$$

where  $\delta'(\cdot)$  denotes the derivative of the Dirac function; note  $\int \delta'(x-x_0)f(x)dx = -f'(x_0)$ . Substituting the patch location and further calculations yields

$$\begin{split} F_{mn}^{c} &= \frac{-4r_{11}d_{31}E_{p}V(t)}{\rho hLW} \int_{y_{1}}^{y_{2}} \int_{x_{1}}^{x_{2}} \left\{ \left[ \delta^{'}(x-x_{1}) - \delta^{'}(x-x_{2}) \right] \left[ H(y-y_{1}) - H(y-y_{2}) \right] \right. \\ &+ \left[ H(x-x_{1}) - H(x-x_{2}) \right] \left[ \delta^{'}(y-y_{1}) - \delta^{'}(y-y_{2}) \right] \left\} U_{3mn} dx dy. \end{split}$$

where  $\rho$  is the density of the plate and  $D = Eh^3/[12(1-\nu^2)]$  is the bending stiffness.

Natural Frequency $(\omega_n L^2 \sqrt{\rho/D})$
3.4332
21.475
60.292
118.59
196.62
293.96
361.12
394.02
415.19
459.58

Table 1: Natural frequencies for cantilevered plate (L/W = 4)

#### B. Thin half-cylinder shell

Let  $\alpha_1 = x$ ,  $\alpha_2 = \theta$  be the rectangular coordinates for a half-cylindrical shell  $(0 \le x \le L, 0 \le \theta \le \pi)$ . The modal expansion is

$$w_3(x,\theta,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \eta_{mn}(t) \Phi_{3mn}(x,\theta),$$

where the mode shapes are

$$\Phi_{3mn} = \sin \frac{m\pi x}{L} \sin \frac{n\pi\theta}{\beta},$$

where  $\beta$  is the shell curvature angle, m is the longitudinal wave number and n is the latitudinal wave number. The natural frequencies for a thin shell are [118]

$$\omega_{mn}^2 = \frac{D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{\beta}\right)^2\right]^2 + \frac{K}{R^2}}{\rho h}$$

where  $K = Eh/(1 - v^2)$ , is the membrane stiffness,  $D = Eh^3/[12(1 - v^2)]$  is the bending stiffness, and R is the radius of curvature in the circumferential direction. The mode expansion used for the thin shell is

$$\ddot{\eta}_{mn} + C_{mn}\dot{\eta} + \Omega_{mn}^2\eta_{mn} = F_{mn}(t)$$

where  $C_{mn}$  is the constant damping matrix  $(C_{mn} = 0.001\Omega_{mn}^2)$ ,  $F_{mn}(t)$  is the modal force which consists two components of mechanical force  $(q_3(t))$  and control force  $(L_3^c\{\phi_3\})$ .

$$F_{mn}^{M} = \frac{1}{\rho h N_{mn}} \int_{x} \int_{\theta} q_{3} U_{3mn} A_{1} A_{2} d\alpha_{1} d\alpha_{2}$$

$$F_{mn}^c = \frac{1}{\rho h N_{mn}} \int_x \int_\theta L_3^c \{\psi_3\} U_{3mn} A_1 A_2 d\alpha_1 d\alpha_2$$

and

$$N_{mn} = \int_x \int_\theta U_{3mn}^2 d\alpha_1 d\alpha_2$$

where  $\rho$  is the mass density, h is the thickness of the shell,  $A_1$  and  $A_2$  are Lame parameters,  $\alpha_1$  and  $\alpha_2$ , respectively denote the directions of x and  $\theta$ . Substituting the Lame parameters (i.e.,  $A_1 = 1$  and  $A_2 = R$ ), radii of curvature (i.e.,  $R_1 = \infty$  and  $R_2 = R$ ) and the two principle directions  $\alpha_1 = x$  and  $\alpha_2 = \theta$  and also assuming the external force,  $q_3(t)$ , is a point force at  $(x^*, \theta^*)$  the mechanical force and the generic control force are

$$F_{mn}^{M} = \frac{4}{\rho h R L \beta} \int_{x} \int_{\theta} q_{3} \{\delta(x - x^{*})\delta(\theta - \theta^{*})\} \sin \frac{m\pi x}{L} \sin \frac{n\pi\theta}{\beta} dx R d\theta$$
$$= \frac{4q_{3}}{\rho h L \beta} \sin \frac{m\pi x^{*}}{L} \sin \frac{n\pi\theta^{*}}{\beta}$$

$$L_3^c\{\psi_3\} = -\left\{\frac{\partial^2 M_{xx}^c}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}^c}{\partial \theta^2} - \frac{1}{R}N_{\theta\theta}^c\right\}$$

where N and M, induced forces and moments by piezoelectric actuator with an applied voltage  $\psi^a$ , are

$$N_{ii} = d_{3i}E_p\psi^a(x,\theta,t)$$
$$M_{ii} = r_{ii}d_{3i}E_p\psi^a(x,\theta,t)$$

where  $d_{3i}$  is piezoelectric constant, (for a hexagonal structure it is assumed  $d_{3x} = d_{3\theta}$ ),  $r_{ii} = r_{xx} = r_{\theta\theta}$ is the moment arm measured from the shell neutral surface to the actuator mid-surface and  $E_p$  is the piezoelectric constant. The transverse actuating voltage  $\psi^a(x, \theta, t)$  applied to an actuator patch defined from  $x = x_1$  to  $x = x_2$  in longitudinal direction and  $\theta = \theta_1$  and  $\theta = \theta_2$  in the circumferential direction can be expressed by a spatial distribution part and time dependent part

$$\psi^{a}(x,\theta,t) = [H(x-x_{1}) - H(x-x_{2})] [H(\theta-\theta_{1}) - H(\theta-\theta_{2})] V(t)$$

where H(.) is the unit step function Thus the spatial derivatives of the transverse actuation signals is

$$\frac{\partial^2}{\partial x^2}\psi^a(x,\theta,t) = \left[\delta^{'}(x-x_1) - \delta^{'}(x-x_2)\right] \left[H(\theta-\theta_1) - H(\theta-\theta_2)\right]V(t)$$
$$\frac{\partial^2}{\partial \theta^2}\psi^a(x,\theta,t) = \left[H(x-x_1) - H(x-x_2)\right] \left[\delta^{'}(\theta-\theta_1) - \delta^{'}(\theta-\theta_2)\right]V(t)$$

where  $\delta'(.)$  is the derivative of a Dirac function:  $\int \delta'(x-x_0)f(x)dx = -f'(x_0)$ . Substituting the patch definition and further calculations yields

$$\begin{split} F_{mn}^{c} &= -\frac{4d_{31}E_{p}V(t)}{\rho hL\beta} \int_{x_{1}}^{x_{2}} \int_{\theta_{1}}^{\theta_{2}} \left\{ r_{xx} \left[ \delta^{'}(x-x_{1}) - \delta^{'}(x-x_{2}) \right] \left[ H(\theta-\theta_{1}) - H(\theta-\theta_{2}) \right] \right. \\ &+ \frac{r_{\theta\theta}}{R^{2}} \left[ H(x-x_{1}) - H(x-x_{2}) \right] \left[ \delta^{'}(\theta-\theta_{1}) - \delta^{'}(\theta-\theta_{2}) \right] \\ &+ \frac{1}{R} \left[ H(x-x_{1}) - H(x-x_{2}) \right] \left[ H(\theta-\theta_{1}) - H(\theta-\theta_{2}) \right] \right\} \\ &\left. \left( \sin \frac{m\pi x}{L} \sin \frac{n\pi \theta}{\beta} \right) dxd\theta \end{split}$$

$$F_{mn}^{c} = \frac{4d_{31}E_{p}V(t)}{\rho hL\beta} \left\{ \left[ r_{xx} \left(\frac{m\pi}{L}\right) \left(\frac{\beta}{n\pi}\right) \left(\cos\frac{m\pi x_{1}}{L} - \cos\frac{m\pi x_{2}}{L}\right) \left(\cos\frac{n\pi\theta_{2}}{\beta} - \cos\frac{n\pi\theta_{1}}{\beta}\right) \right] + \left[ \frac{r_{\theta\theta}}{R^{2}} \left(\frac{L}{m\pi}\right) \left(\frac{n\pi}{\beta}\right) \left(\cos\frac{m\pi x_{2}}{L} - \cos\frac{m\pi x_{1}}{L}\right) \left(\cos\frac{n\pi\theta_{1}}{\beta} - \cos\frac{n\pi\theta_{2}}{\beta}\right) \right] + \left[ \frac{1}{R} \left(\frac{L}{m\pi}\right) \left(\frac{\beta}{n\pi}\right) \left(\cos\frac{m\pi x_{2}}{L} - \cos\frac{m\pi x_{1}}{L}\right) \left(\cos\frac{n\pi\theta_{2}}{\beta} - \sin\frac{n\pi\theta_{1}}{\beta}\right) \right] \right\}$$

## C.Routh-Hurwitz Criterion

The characteristic polynomial of the closed loop transfer function of (4.19)

$$P(s) = \alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \tag{C.1}$$

where

$$\alpha_{0} = k_{0}km\omega_{c}^{2}$$

$$\alpha_{1} = 2\zeta_{c}\omega_{c}kc + m\omega_{c}^{2}(ck_{0} + k)$$

$$\alpha_{2} = kc + 2\zeta_{c}\omega_{c}c^{2} + m\omega_{c}^{2}c$$

$$\alpha_{3} = c^{2} + 2\zeta_{c}\omega_{c}cm$$

$$\alpha_{4} = mc$$
(C.2)

It has been mentioned in the (4.7), that in order to have a stable output dynamics,  $k_0$  needs to be positive. Considering this fact, it is obvious that all the coefficients of the polynomial have the same sign. Constructing the Routh-Hurwitz table as where

 Table 2: Completed Routh table

$s^4$	$lpha_4$	$\alpha_2$	$\alpha_0$
$s^3$	$lpha_3$	$\alpha_1$	
$s^2$ $s^1$ $s^0$	$\frac{-\begin{vmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{vmatrix}}{\begin{vmatrix} \alpha_3 & \alpha_1 \\ \psi_1 & 0 \end{vmatrix}} = \psi_1$ $\frac{-\begin{vmatrix} \alpha_3 & \alpha_1 \\ \psi_1 & 0 \end{vmatrix}}{\psi_1} = \gamma_1$	$lpha_0$	

$$\psi_1 = cm\omega_c^2 - \frac{-kc^2 + k_0cm^2\omega_c^2 + km^2\omega_c^2}{c + 2m\omega_c\zeta_c} + 2c^2\omega_c\zeta_c$$
$$\gamma_1 = \frac{\Lambda_5\omega_c^5 + \Lambda_4\omega_c^4 + \Lambda_3\omega_c^3 + \Lambda_2\omega_c^2 + \Lambda_1\omega_c}{\Lambda_3\omega_c^3 + \Lambda_2\omega_c^2 + \Lambda_1\omega_c + \Lambda_0}$$

where

$$\begin{split} \Lambda_{5} &= 2k_{0}\zeta_{c}c^{2}m^{3} + 2k\zeta_{c}cm^{3} \\ \Lambda_{4} &= 4c^{3}k_{0}m^{2}\zeta_{c}^{2} + c^{3}k_{0}m^{2} + 8c^{2}km^{2}\zeta_{c}^{2} + c^{2}km^{2} - c^{2}k_{0}^{2}m^{3} - 4ckk_{0}m^{3}\zeta_{c}^{2} - 2ckk_{0}m^{3} - k^{2}m^{3} \\ \Lambda_{3} &= 2k_{0}c^{4}m\zeta_{c} + 8c^{3}km\zeta_{c}^{3} + 4c^{3}km\zeta_{c} - 6k_{0}c^{2}km^{2}\zeta_{c} - 2ck^{2}m^{2}\zeta_{c} \\ \Lambda_{2} &= 4c^{4}k\zeta_{c}^{2} + mc^{2}k^{2} \\ \Lambda_{1} &= 2c^{3}k^{2}\zeta_{c} \\ \Lambda_{3} &= 2cm^{2}\zeta_{c} \\ \Lambda_{2} &= 4c^{2}m\zeta_{c}^{2} + c^{2}m - k_{0}cm^{2} - km^{2} \\ \Lambda_{1} &= 2c^{3}\zeta_{c} \\ \Lambda_{0} &= c^{2}k \end{split}$$

It is obvious that for large enough  $\omega_c,$  the limit of  $\psi_1$  and  $\gamma_1$  will be

$$\lim_{\omega_c \to \infty} \psi_1 = cm\omega_c^2$$
$$\lim_{\omega_c \to \infty} \gamma_1 = \frac{2k_0\zeta_c c^2 m^3 + 2k\zeta_c cm^3}{2cm^2\zeta_c}$$

Thus using Routh-Hurwitz criterion, it can be shown that for large enough  $\omega_c$  the closed loop systems of (4.7) is stable.

## D. Proof of the proposition 3

The linear vibration model can be expressed as

$$M\ddot{q} + C\dot{q} + Kq = Bu + Fd$$
 (G.1)

$$\mathbf{y} = \mathbf{\Phi}\mathbf{q} \tag{G.2}$$

and the ideal control input for this system is

$$\mathbf{u}^{*}(t) = \left(\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\right)^{-1} \left(\ddot{\mathbf{r}} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} - \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{F}\mathbf{d} - \mathbf{K}_{\dot{e}}\dot{\mathbf{e}} - \mathbf{K}_{e}\mathbf{e}\right)$$
(G.3)

and thus the derivatives of ideal control input is

$$\dot{\mathbf{u}}^{*}(t) = \left(\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\right)^{-1} \left(\mathbf{r}^{(3)} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\ddot{\mathbf{q}} + \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K}\dot{\mathbf{q}} - \mathbf{\Phi}\mathbf{M}^{-1}\mathbf{F}\dot{\mathbf{d}} + \mathbf{K}_{\dot{e}}\ddot{\mathbf{e}} + \mathbf{K}_{e}\dot{\mathbf{e}}\right)$$
(G.4)

Using (G.2) and considering  $\mathbf{u} = \mathbf{u}^* + \Delta \mathbf{u}$ , the system  $S_1$  can be written as

$$\begin{split} \dot{\mathbf{x}} &= \tilde{\mathbf{A}} \mathbf{x} + \tilde{\mathbf{B}} \boldsymbol{\lambda} \\ \mathbf{y} &= \tilde{\mathbf{C}} \mathbf{x} + \tilde{\mathbf{D}} \boldsymbol{\lambda} \end{split} \tag{G.5}$$

where  $\mathbf{x} = [\mathbf{q}, \dot{\mathbf{q}}], \, \mathbf{y} = \dot{\mathbf{u}}^*, \boldsymbol{\lambda} = [\Delta \mathbf{u}, \mathbf{d}, \dot{\mathbf{d}}, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \mathbf{r}^3]$ , and

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} + \mathbf{H}\Phi\mathbf{M}^{-1}\mathbf{K} - \mathbf{H}\mathbf{K}_{e}\Phi & -\mathbf{M}^{-1}\mathbf{C} + \mathbf{H}\Phi\mathbf{M}^{-1}\mathbf{C} - \mathbf{H}\mathbf{K}_{\dot{e}}\Phi \end{bmatrix}$$
(G.6)  
$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B} & \mathbf{M}^{-1}\mathbf{F} - \mathbf{M}^{-1}\mathbf{H}\Phi\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{M}^{-1}\mathbf{H}\mathbf{K}_{\dot{e}} & \mathbf{M}^{-1}\mathbf{H} & \mathbf{0} \end{bmatrix}$$

$$\begin{split} \tilde{\mathbf{C}} &= \left[ \begin{array}{c} \tilde{\mathbf{C}}_{\mathbf{q}} & \tilde{\mathbf{C}}_{\dot{\mathbf{q}}} \end{array} \right] \\ & \mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \mathbf{M}^{-1} \mathbf{B} - \mathbf{H} \mathbf{K}_{\dot{e}} \Phi \mathbf{M}^{-1} \mathbf{B} \\ & \mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \left[ -\mathbf{M}^{-1} \mathbf{B} \mathbf{H} \Phi \mathbf{F} + \mathbf{M}^{-1} \mathbf{F} \right] - \mathbf{H} \mathbf{K}_{\dot{e}} \left[ -\Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{F} + \mathbf{M}^{-1} \mathbf{F} \right] \\ & -\mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \left[ -\mathbf{M} - \mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\mathbf{e}} + \mathbf{H} \mathbf{K}_{\dot{e}} \Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\mathbf{e}} \\ & -\mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\mathbf{e}} + \mathbf{H} \mathbf{K}_{\dot{e}} \Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\mathbf{e}} \\ & -\mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\dot{e}} + \mathbf{H} \mathbf{K}_{\dot{e}} \Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{H} \mathbf{K}_{\mathbf{e}} \\ & -\mathbf{H} \Phi \mathbf{M}^{-1} \mathbf{C} \mathbf{M}^{-1} \mathbf{B} \mathbf{H} + \mathbf{H} \mathbf{K}_{\dot{e}} \Phi \mathbf{M}^{-1} \mathbf{B} \mathbf{H} + \mathbf{H} \mathbf{K}_{\mathbf{e}} \\ & \mathbf{H} \end{split} \end{split}$$

where

$$\begin{split} \mathbf{H} = & (\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B})^{-1} \\ \tilde{\mathbf{C}}_{\mathbf{q}} = & -\mathbf{H}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{K} + \mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{\Phi}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K} + \mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{K}e\mathbf{\Phi} \\ & +\mathbf{H}\mathbf{K}\dot{e}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K} - \mathbf{H}\mathbf{K}_{\dot{e}}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K} - \mathbf{H}\mathbf{K}_{\dot{e}}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{K}e\mathbf{\Phi} \\ \tilde{\mathbf{C}}_{\dot{\mathbf{q}}} = & -\mathbf{H}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{C} + \mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C} + \mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{K}\dot{e}\mathbf{\Phi} + \mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{K} \\ & +\mathbf{H}\mathbf{K}\dot{e}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C} - \mathbf{H}\mathbf{K}_{\dot{e}}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{C} - \mathbf{H}\mathbf{K}_{\dot{e}}\mathbf{\Phi}\mathbf{M}^{-1}\mathbf{B}\mathbf{H}\mathbf{K}\dot{e}\mathbf{\Phi} - \mathbf{H}\mathbf{K}_{e}\mathbf{\Phi} \end{split}$$

$$(\mathbf{G}.8)$$

Thus using theorem (4.5) of reference [111], it can be proved that  $S_1$  system in (G.5) is  $\mathscr{L}_2$  gain stable. Let  $\tilde{\mathbf{G}}(s) = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}$ . Then, the  $\gamma_1$  is  $\mathscr{L}_2$  gain of the system which  $\gamma_1 = \sup_{\omega \in \mathbb{R}} ||\tilde{\mathbf{G}}(j\omega)||_2$ .

Also since  $\mathrm{S}_2$  is Hurwitz, theorem 4.5 from reference [111] implies that

$$||\Delta \mathbf{u}||^{2} \le (\sup ||G(j\omega)||_{2})^{2} \, ||\dot{\mathbf{u}}^{*}||_{2}^{2} \tag{G.9}$$

where

$$G(s) = \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2} \tag{G.10}$$

Thus form equations (D.1-D.3), it can be shown that

$$||\Delta \mathbf{u}|| \le \gamma^* ||\Delta \mathbf{u} + \mathbf{d} + \dot{\mathbf{d}}|| + \beta^*$$
(G.11)

where

$$\gamma^* = \sup ||(G(j\omega))||_2 \gamma$$

$$\beta^* = \sup ||(G(j\omega))||_2 \beta$$
(G.12)

(D.4) can be written as

$$||\Delta \mathbf{u}|| \le \frac{1}{1 - \gamma^*} [\gamma^* ||\mathbf{d} + \dot{\mathbf{d}}|| + \beta^*]$$
(G.13)

From (D.6) it follows that for a large enough cut-off frequency  $(\omega_c \to \infty)$ , then the norm of a transform function G would be as small as possible  $(sup||G(j\omega)||_2 \to 0)$ , Hence it is obvious that  $\gamma^* \leq 1$  and thus  $\Delta \mathbf{u}$  is bounded and it is going to zero [111].

$$G_k(s) = \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2}.$$
(G.14)

Let  $\gamma_k = ||G_{\omega_c}(s)||_{\infty}$ . It follows that

$$\gamma_{\omega_{c}} \leq \left\| \frac{s}{s^{2} + 2\zeta_{c}\omega_{c}s + \omega_{c}^{2}} \right\|_{\infty} + \left\| \frac{2\zeta_{c}k}{s^{2} + 2\zeta_{c}\omega_{c}s + \omega_{c}^{2}} \right\|_{\infty},$$
  
$$= \sup_{\omega \in R} \left[ \frac{\omega^{2}}{(\omega_{c}^{2} - \omega^{2})^{2} + (2\zeta_{c}\omega_{c}\omega)^{2}} \right]^{1/2} + \sup_{\omega \in R} \left[ \frac{(2\zeta_{c}\omega_{c})^{2}}{(\omega_{c}^{2} - \omega^{2})^{2} + (2\zeta_{c}\omega_{c}\omega)^{2}} \right]^{1/2}, \quad (G.15)$$

where

$$\sup_{\omega \in R} \left[ \frac{\omega^2}{(\omega_c^2 - \omega^2)^2 + (2\zeta_c \omega_c \omega)^2} \right]^{1/2} = \frac{1}{2\zeta_c \omega_c},\tag{G.16}$$

$$\sup_{\omega \in R} \left[ \frac{(2\zeta_c \omega_c)^2}{(\omega_c^2 - \omega^2)^2 + (2\zeta_c k \omega)^2} \right]^{1/2} = \begin{cases} \frac{2\zeta_c}{\omega_c}, & \zeta_c > \frac{1}{\sqrt{2}}, \\ \frac{1}{\omega_c} \sqrt{\frac{1}{1 - \zeta_c^2}}, & 0 < \zeta_c \le \frac{1}{\sqrt{2}} \end{cases}$$
(G.17)

It follows from (D.8)-(D.10) that

$$\gamma_k \le \frac{1}{\omega_c} (0.5\zeta_c + \max(2\zeta_c, \sqrt{2})). \tag{G.18}$$

Since the zero dynamics for the system is stable, it follows that

$$\ddot{\mathbf{X}} + \mathbf{N}(\mathbf{X}, \dot{\mathbf{X}}) = 0 \tag{G.19}$$

is stable. Hence the system  $S_1$ , is input to state stable [111] which means the bound on the states  $\mathbf{X} = [\theta, \mathbf{q}]$  is proportional to the bound on the input  $\Delta \mathbf{u}$ . It implies

$$||\mathbf{X}_{i}|| \leq \frac{\hat{\gamma}}{1 - \gamma^{*}} [\gamma^{*} ||\mathbf{d} + \dot{\mathbf{d}}|| + \beta^{*}] + \hat{\beta}$$
(G.20)

It is interesting to note that the right hand side of the (G.20) approaches  $\hat{\beta}$ , which shows that for a enough large cut-off frequency the disturbance attenuation can be achieved.

#### E. Proof of the proposition 1

The modal of a linear systems can be expressed as

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}_D \dot{\mathbf{q}} + \mathbf{\Lambda}_K \mathbf{q} = \mathbf{B} \mathbf{u}(t) + \mathbf{F} \mathbf{d}(t)$$
(F.1)

where  $q \in \mathbb{R}^n$  and  $\mathbf{M}, \mathbf{C}$  and  $\mathbf{K}$  are  $n \times n$  matrices. Lets consider the output matrix as

$$\mathbf{y}(t) = \mathbf{\Phi}\mathbf{q}(t) \tag{F.2}$$

where  $y \in \mathbb{R}^m$  and  $\mathbf{\Phi}$  is the  $m \times n$  matrix in the form of

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \phi_{mn} \end{bmatrix}$$
(F.3)

Hence the zeros dynamics of the system are defined as

$$\hat{\mathbf{M}}\ddot{\mathbf{q}} + \hat{\mathbf{C}}\dot{\mathbf{q}} + \hat{\mathbf{K}}\mathbf{q} = \mathbf{0}$$
(F.4)

where

$$\mathbf{M} = \mathbf{I}$$
$$\hat{\mathbf{C}} = (\mathbf{I} - \mathbf{B}(\mathbf{\Phi}\mathbf{B})^{-1}\mathbf{\Phi})\mathbf{\Lambda}_D$$
(F.5)
$$\hat{\mathbf{K}} = (\mathbf{I} - \mathbf{B}(\mathbf{\Phi}\mathbf{B})^{-1}\mathbf{\Phi})\mathbf{\Lambda}_K$$

In order to find the eigenvalues of the system, the traditional approach is to assume a solution of the form

$$\mathbf{q} = \mathbf{\Psi} e^{\lambda t} \tag{F.6}$$

where **q** is an N- dimensional vector, and  $\Psi$  is an N-dimensional vector of constants. Substitution of (F.6) and it derivatives in to (F.4) yields

$$(\hat{\mathbf{M}}\lambda^2 + \hat{\mathbf{C}}\lambda + \hat{\mathbf{K}})\Psi e^{\lambda t} = 0$$
(F.7)

which is satisfied for all t when

$$(\hat{\mathbf{M}}\lambda^2 + \hat{\mathbf{C}}\lambda + \hat{\mathbf{K}})\Psi = 0 \tag{F.8}$$

Nontrivial solutions to require that the following determinant is equal to zero

$$|\hat{\mathbf{M}}\lambda^2 + \hat{\mathbf{C}}\lambda + \hat{\mathbf{K}}| = 0 \tag{F.9}$$

which yields the polynomial

$$d_{2N}\lambda^{2N} + d_{2(N-1)}\lambda^{2(N-1)} + \dots + d_2\lambda^2 + d_0 = 0$$
 (F.10)

which  $d_{2r}, r = 1, \dots, N$  as constants coefficient. Solving (F.10) for  $\lambda$  results in N complex conjugate pairs, each pair in the form

$$\lambda_r = i\omega_r, r = 1, \cdots, N \tag{F.11}$$

and

$$\lambda_r^* = i\omega_r^*, r = 1, \cdots, N \tag{F.12}$$

Each complex conjugate pair of eigenvalues constitute two out of 2N possible eigenvalues, all of which satisfy (F.7). For each eigenvalue  $\lambda_r$  or  $\lambda_r^*$ , there is eigenvector  $\Psi_r$  which is obtained from (F.8). The eigenvalues are typically presented in the form of

$$[\boldsymbol{\Psi}] = [\{\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \cdots \boldsymbol{\Psi}_N\}] \tag{F.13}$$

and corresponding eigenvalues can also be represented in vector form as

$$\{\mathbf{\Lambda}\} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix}$$
(F.14)

(F.8) can be written in the form of

$$(\hat{\mathbf{M}}\lambda_r^2 + \hat{\mathbf{C}}\lambda_r + \hat{\mathbf{K}})\Psi_r = 0 \tag{F.15}$$

Pre-multiplying both sides of (F.15) by the transpose of the eigenvector  $\Psi_r^{T}$  gives

$$\boldsymbol{\Psi}_{\boldsymbol{r}}^{\boldsymbol{T}}(\hat{\mathbf{M}}\lambda_{\boldsymbol{r}}^2 + \hat{\mathbf{C}}\lambda_{\boldsymbol{r}} + \hat{\mathbf{K}})\boldsymbol{\Psi}_{\boldsymbol{r}} = \boldsymbol{0}$$
(F.16)

Although  $\lambda_r$  is complex valued. it is still a scalar quantity. Therefore (F.8) can also be written as

$$\Psi_{\boldsymbol{r}}^{\boldsymbol{T}}\hat{\mathbf{M}}\Psi_{\boldsymbol{r}}\lambda_{\boldsymbol{r}}^{2} + \Psi_{\boldsymbol{r}}^{\boldsymbol{T}}\hat{\mathbf{D}}\Psi_{\boldsymbol{r}}\lambda_{\boldsymbol{r}} + \Psi_{\boldsymbol{r}}^{\boldsymbol{T}}\hat{\mathbf{K}}\Psi_{\boldsymbol{r}} = 0$$
(F.17)

A vector-matrix-vector multiplication , such as  $\Psi_r^T \hat{\mathbf{M}} \Psi_r$ , always results in a scalar quantity. Therefore, assign

$$\hat{M}_{r} = \Psi_{r}^{T} \hat{\mathbf{M}} \Psi_{r}$$

$$\hat{C}_{r} = \Psi_{r}^{T} \hat{\mathbf{C}} \Psi_{r}$$

$$\hat{K}_{r} = \Psi_{r}^{T} \hat{\mathbf{K}} \Psi_{r}$$
(F.18)

The scalar constants  $M_r, C_r$  and  $K_r$  are associated with the  $r^{th}$  mode of multi-degree of freedom system. Therefore,  $M_r$  is referred to as the modal mass constant,  $C_r$  is referred to as the modal damping constant and  $K_r$  is referred to as the modal stiffness constant for the  $r^{th}$  mode. The modal mass, damping and stiffness constants can also be represented in matrix form, by assigning

$$\hat{\mathbf{M}} = \boldsymbol{\Psi}^{T} \hat{\mathbf{M}} \boldsymbol{\Psi}$$
$$\hat{\mathbf{C}} = \boldsymbol{\Psi}^{T} \hat{\mathbf{C}} \boldsymbol{\Psi}$$
$$\hat{\mathbf{K}} = \boldsymbol{\Psi}^{T} \hat{\mathbf{K}} \boldsymbol{\Psi}$$
(F.19)

where  $[\mathbf{M}], [\mathbf{C}]$  and  $[\mathbf{K}]$  are diagonal matrices consisting of values of  $M_r, C_r$  and  $K_r$  respectively. Substitution of the  $r^{th}$  modal mass, damping and stiffness constants in to (F.17) gives

$$\hat{M}_r \lambda_r^2 + \hat{C}_r \lambda + K_r = 0 \tag{F.20}$$

Solving for  $\lambda_r$  gives

$$\lambda_r = \frac{-\hat{C}_r \pm \sqrt{(\hat{C}_r)^2 - 4\hat{K}_r}}{2M_r}$$
(F.21)

Now we need to show  $\hat{C}_r$  is positive, hence the eigenvalues of the system is always negative. From (F.5)  $\hat{C}$ 

$$\hat{\mathbf{C}} = (\mathbf{I} - \underbrace{\mathbf{B}(\mathbf{\Phi}\mathbf{B})^{-1}\mathbf{\Phi}}_{\mathbf{C}^{\star}})\Lambda_D$$
(F.22)

where  $\mathbf{C} = 2\zeta_i \omega_i$ . Here we are trying to show  $\mathbf{C}^*$  has the form of

$$\mathbf{D}^{\star} = \begin{bmatrix} \mathbf{I}_{m \times m} & \mathbf{0}_{(n-m) \times m} \\ \mathbf{x}_{(n-m) \times m} & \mathbf{0}_{(n-m) \times (n-m)} \end{bmatrix}$$
(F.23)

Hence matrix  $(\mathbf{I}-\mathbf{C}^{\star})$  has the form of

$$\mathbf{I} - \mathbf{C}^{\star} = \begin{bmatrix} \mathbf{0}_{m \times m} & \mathbf{0}_{(n-m) \times m} \\ -\mathbf{x}_{(n-m) \times m} & \mathbf{I}_{(n-m) \times (n-m)} \end{bmatrix}$$
(F.24)

and thus  $\hat{\mathbf{C}}$  is in the form of

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{0}_{m \times m} & \mathbf{0}_{(n-m) \times m} \\ \mathbf{0}_{(n-m) \times m} & \mathbf{2} \zeta_{(n-m)(n-m)} \omega_{(n-m)(n-m)} \end{bmatrix}$$
(F.25)

and from (F.25), it is obvious that the eigenvalues of zero-dynamics are either zeros or negative and the system is minimum phase. To show that (F.23) is right and it has the same form, we use Einsteins notations. in this notation the matrix inverse of a general  $N \times N$  matrix A can be written in the form of

$$\left(A^{-1}\right)_{ji} = \frac{\epsilon_{ii_2\cdots i_N}\epsilon_{jj_2\cdots j_N}A_{i_2j_2\cdots i_Nj_N}}{(N-1)!detA}$$
(F.26)

where det(A) is

$$\det(A) = \epsilon_{k1k2\cdots k_N} A_{1i_1} A_{2i_2} \cdots A_{Ni_N} \tag{F.27}$$

or another involving the determinant

$$\epsilon_{q_1q_2\cdots q_N} = \epsilon_{k1k2\cdots k_N} A_{q_1i_1} A_{q_2i_2} \cdots A_{q_Ni_N} \tag{F.28}$$

To show that  $C^{\star}$  has the form in (F.23), we write

$$C_{kl}^{\star} = B_{kj} \left( \Phi \mathbf{B} \right)_{ji}^{-1} \phi_{il} \quad i, j = 1, 2, \cdots, m$$
 (F.29)

but from (F.2), we know

$$\left( \mathbf{\Phi} \mathbf{B} \right)_{ji}^{-1} = \frac{\epsilon_{ii_2 \cdots i_m} \epsilon_{jj_2 \cdots j_m} \phi_{i2i2} B_{i_2j_2} \cdots \phi_{i_m i_m} B_{i_m j_m}}{(m-1)! det(\mathbf{\Phi} \mathbf{H})}$$

$$= \frac{\epsilon_{ii_2 \cdots i_m} \epsilon_{jj_2 \cdots j_m} \phi_{i2i2} B_{i_2j_2} \cdots \phi_{i_m i_m} B_{i_m j_m}}{(m-1)! \epsilon_{q_1q_2 \cdots q_m} \phi_{11} B_{1q1} \cdots \phi_{mm} B_{mq_m}}$$
(F.30)

Hence  $C_{kl}^{\star}$  can be written in the form of

$$C_{kl}^{\star} = \frac{B_{kj}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{il}}{(m-1)!\epsilon_{q_1q_2\cdots q_m}\phi_{11}B_{1q1}\cdots\phi_{mm}B_{mq_m}}$$
(F.31)

for the fist m diagonal terms in matrix shown in (F.23), we have k = l, k, l < m, and also the only  $\phi_{ii}$ are non-zeros (Eq.(F.2)), thus (F.31) can be written in the form of

$$C_{ii}^{\star} = \frac{B_{ij}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{ii}}{(m-1)!\epsilon_{q_1q_2\cdots q_m}\phi_{11}B_{1q_1}\cdots\phi_{mm}B_{mq_m}}$$
(F.32)

from (F.28)

$$C_{ii}^{\star} = \frac{\epsilon_{s_1 s_2 \cdots s_m} B_{ij} \epsilon_{ii_2 \cdots i_m} \epsilon_{jj_2 \cdots j_m} \phi_{i_2 i_2} B_{i_2 j_2} \cdots \phi_{i_m i_m} B_{i_m j_m} \phi_{ii}}{(m-1)! \epsilon_{q_1 q_2 \cdots q_m} \phi_{s_1 s_1} B_{s_1 q_1} \cdots \phi_{s_m s_m} B_{s_m q_m}}$$
(F.33)

now if we replace  $s_2 \cdots s_m$  with  $i_2 \cdots i_m$  and also change  $s_1$  with i, (F.33) yields to

$$C_{ii}^{\star} = \frac{\epsilon_{ii_2\cdots i_m} B_{ij}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2} B_{i_2j_2}\cdots\phi_{i_mi_m} B_{i_mj_m}\phi_{ii}}{(m-1)!\epsilon_{q_1q_2\cdots q_m}\phi_{ii} B_{iq_1}\cdots\phi_{i_mi_m} B_{i_mq_m}}$$
(F.34)

and now replace  $q_2 \cdots q_m$  with  $j_2 \cdots j_m$  and also change  $q_1$  with j, (F.33) yields to

$$C_{ii}^{\star} = \frac{\epsilon_{ii_2\cdots i_m} B_{ij} \epsilon_{ii_2\cdots i_m} \epsilon_{jj_2\cdots j_m} \phi_{i_2i_2} B_{i_2j_2} \cdots \phi_{i_m i_m} B_{i_m j_m} \phi_{ii}}{(m-1)! \epsilon_{jj_2\cdots j_m} \phi_{ii} B_{ij} \cdots \phi_{i_m i_m} B_{i_m j_m}} = 1$$
(F.35)

for the off-diagonal terms, where k,l < m, we can write  $C_{kl}$  as

$$C_{kl}^{\star} = \frac{B_{kj}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{il}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})}$$
(F.36)

or since l < m we can write it as

$$C_{ki}^{\star} = \frac{B_{kj}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{ii}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})}$$
(F.37)

but since  $k \neq i$ , it is either of  $i_2, \dots, i_m$ . Without losing any generality, lets assume we are interested in  $C_{i_2i}^{\star}$ , hence

$$C_{i_2i}^{\star} = \frac{B_{i_2j}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{ii}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})}$$
(F.38)

but if  $\epsilon_{jj_2...j_m}$ , is replace by  $\epsilon_{j_2j...j_m}$ , (F.38) would be

$$C_{i_{2}i}^{\star} = \frac{B_{i_{2}j_{2}}\epsilon_{ii_{2}\cdots i_{m}}\epsilon_{j_{2}j\cdots j_{m}}\phi_{i_{2}i_{2}}B_{i_{2}j}\cdots\phi_{i_{m}i_{m}}B_{i_{m}j_{m}}\phi_{ii}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})} = -C_{i_{2},i}^{\star} = 0$$
(F.39)

for the cases when l > m, it is obvious from (F.2), that  $\phi_{il} = 0$ , thus

$$C_{kl}^{\star} = 0 \quad l > m \tag{F.40}$$

and finally for the case, where k > m but l < m,  $C_{kl}$  is

$$C_{kl}^{\star} = \frac{B_{kj}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}B_{i_2j_2}\cdots\phi_{i_mi_m}B_{i_mj_m}\phi_{il}}{(m-1)!\epsilon_{q_1q_2\cdots q_m}\phi_{11}B_{1q_1}\cdots\phi_{mm}B_{mq_m}}$$
(F.41)

Hence it proved that the matrix  $\mathbf{C}^{\star}$  is in the form

$$\mathbf{C}^{\star} = \begin{bmatrix} \mathbf{I}_{m \times m} & \mathbf{0}_{(n-m) \times m} \\ \mathbf{x}_{(n-m) \times m} & \mathbf{0}_{(n-m) \times (n-m)} \end{bmatrix}$$
(F.42)

but all above is when the matrix  $\Phi B$  is invertible. if it is not then we need to differentiate (F.2) as many times as input **u** appears. The zero-dynamics for that case would

$$\hat{\mathbf{M}} = \mathbf{I}$$

$$\hat{\mathbf{C}} = (\mathbf{I} - \mathbf{B}(\mathbf{\Phi}\mathbf{\Lambda}_D^s \mathbf{B})^{-1}\mathbf{\Phi}\mathbf{\Lambda}_D^s)\mathbf{\Lambda}_D$$
(F.43)
$$\hat{\mathbf{K}} = (\mathbf{I} - \mathbf{B}(\mathbf{\Phi}\mathbf{\Lambda}_K^s \mathbf{B})^{-1}\mathbf{\Phi}\mathbf{\Lambda}_K^s)\mathbf{\Lambda}_K$$

where s is the number of derivative we need to take from (F.2) until **u** appears. In the same manner we used before we can show this new zero-dynamics is stable. Here we know that matrices  $\Lambda_D$  and  $\Lambda_K$  are diagonal matrices and hence when they power s times,  $\Lambda_D^s$ ,  $\Lambda_K^s$ , they still would be diagonal. Again for the first m diagonal terms,

$$C_{ii}^{\star} = \frac{\epsilon_{ii_2\cdots i_m} B_{ij} \epsilon_{ii_2\cdots i_m} \epsilon_{jj_2\cdots j_m} \phi_{i_2i_2} \Lambda_{D_{i_2i_2} B_{i_2j_2}}^s \cdots \phi_{i_m i_m} B_{i_m j_m} \phi_{ii} \Lambda_{D_{ii}}^s}{(m-1)! \epsilon_{jj_2\cdots j_m} \phi_{ii} \Lambda_{D_{ii} B_{ij}}^s \cdots \phi_{i_m i_m} \Lambda_{D_{i_m i_m}}^s B_{i_m j_m}} = 1$$
(F.44)

and for off-diagonal case where k,l < m

$$C_{i_{2}j}^{\star} = \frac{B_{i_{2}j}\epsilon_{ii_{2}\cdots i_{m}}\epsilon_{jj_{2}\cdots j_{m}}\phi_{i_{2}i_{2}}\Lambda_{D_{i_{2}i_{2}}B_{i_{2}j_{2}}}^{s}\cdots\phi_{i_{m}i_{m}}\Lambda_{D_{i_{m}i_{m}}}^{s}B_{i_{m}j_{m}}\phi_{ii}\Lambda_{D_{i_{i}}}^{s}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})}$$

$$= \frac{B_{i_{2}j_{2}}\epsilon_{ii_{2}\cdots i_{m}}\epsilon_{j_{2}j\cdots j_{m}}\phi_{i_{2}i_{2}}\Lambda_{D_{i_{2}i_{2}}}^{s}B_{i_{2}j}\cdots\phi_{i_{m}i_{m}}}\Lambda_{D_{i_{m}i_{m}}}^{s}B_{i_{m}j_{m}}\phi_{ii}}\Lambda_{D_{i_{i}}}^{s}}{(m-1)!\det(\mathbf{\Phi}\mathbf{B})}$$
(F.45)

for the cases when l > m, it is obvious from (F.2), that  $\phi_{il} = 0$ , thus

$$C_{kl}^{\star} = 0 \quad l > m \tag{F.46}$$
and finally for the case, where k > m but l < m,  $D_{kl}$  is

$$C_{kl}^{\star} = \frac{B_{kj}\epsilon_{ii_2\cdots i_m}\epsilon_{jj_2\cdots j_m}\phi_{i_2i_2}\Lambda_{D_{i_2i_2}B_{i_2j_2}}^s \cdots \phi_{i_mi_m}\Lambda_{D_{i_mi_m}B_{i_mj_m}}^s\phi_{il}\Lambda_{l,l}^s}{(m-1)!\epsilon_{q_1q_2\cdots q_m}\phi_{11}B_{1q_1}\cdots \phi_{mm}B_{mq_m}}$$
(F.47)

Thus it is proved that the zero dynamics for a modal control of a linear system is stable. Following the same procedure, shown in the appendix E, it can be that the  $S_1$  is  $\mathscr{L}_2$  gain stable for this case.

## F. Proof of the Theorem 5.

Since the system  $S_1$ , is finite  $\mathscr{L}_2$  gain stable, it follows that

$$||\dot{\mathbf{u}}^*|| \le \gamma ||\Delta \mathbf{u} + \mathbf{d} + \dot{\mathbf{d}}|| + \beta \tag{D.1}$$

where  $\gamma$  and  $\beta$  are nonnegative constants. Also since S<sub>2</sub> is Hurwitz, theorem 4.5 from reference [111] implies that

$$||\Delta \mathbf{u}||^2 \le (\sup ||G(j\omega)||_2)^2 ||\dot{\mathbf{u}}^*||_2^2$$
(D.2)

where

$$G(s) = \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2}$$
(D.3)

Thus form equations (D.1-D.3), it can be shown that

$$||\Delta \mathbf{u}|| \le \gamma^* ||\Delta \mathbf{u} + \mathbf{d} + \dot{\mathbf{d}}|| + \beta^* \tag{D.4}$$

where

$$\gamma^* = \sup ||(G(j\omega))||_2 \gamma$$
  

$$\beta^* = \sup ||(G(j\omega))||_2 \beta$$
(D.5)

(D.4) can be written as

$$||\Delta \mathbf{u}|| \le \frac{1}{1 - \gamma^*} [\gamma^* ||\mathbf{d} + \dot{\mathbf{d}}|| + \beta^*]$$
(D.6)

From (D.6) it follows that for a large enough cut-off frequency  $(\omega_c \to \infty)$ , then the norm of a transform function G would be as small as possible  $(sup||G(j\omega)||_2 \to 0)$ , Hence it is obvious that  $\gamma^* \leq 1$  and thus  $\Delta \mathbf{u}$  is bounded and it is going to zero [111].

$$G_k(s) = \frac{s + 2\zeta_c \omega_c}{s^2 + 2\zeta_c \omega_c s + \omega_c^2}.$$
 (D.7)

Let  $\gamma_k = ||G_{\omega_c}(s)||_{\infty}$ . It follows that

$$\gamma_{\omega_{c}} \leq \left\| \frac{s}{s^{2} + 2\zeta_{c}\omega_{c}s + \omega_{c}^{2}} \right\|_{\infty} + \left\| \frac{2\zeta_{c}k}{s^{2} + 2\zeta_{c}\omega_{c}s + \omega_{c}^{2}} \right\|_{\infty},$$
  
$$= \sup_{\omega \in R} \left[ \frac{\omega^{2}}{(\omega_{c}^{2} - \omega^{2})^{2} + (2\zeta_{c}\omega_{c}\omega)^{2}} \right]^{1/2} + \sup_{\omega \in R} \left[ \frac{(2\zeta_{c}\omega_{c})^{2}}{(\omega_{c}^{2} - \omega^{2})^{2} + (2\zeta_{c}\omega_{c}\omega)^{2}} \right]^{1/2}, \quad (D.8)$$

where

$$\sup_{\omega \in R} \left[ \frac{\omega^2}{(\omega_c^2 - \omega^2)^2 + (2\zeta_c \omega_c \omega)^2} \right]^{1/2} = \frac{1}{2\zeta_c \omega_c},\tag{D.9}$$

$$\sup_{\omega \in R} \left[ \frac{(2\zeta_c \omega_c)^2}{(\omega_c^2 - \omega^2)^2 + (2\zeta_c k\omega)^2} \right]^{1/2} = \begin{cases} \frac{2\zeta_c}{\omega_c}, & \zeta_c > \frac{1}{\sqrt{2}}, \\ \frac{1}{\omega_c} \sqrt{\frac{1}{1 - \zeta_c^2}}, & 0 < \zeta_c \le \frac{1}{\sqrt{2}} \end{cases}$$
(D.10)

It follows from (D.8)-(D.10) that

$$\gamma_k \le \frac{1}{\omega_c} (0.5\zeta_c + \max(2\zeta_c, \sqrt{2})). \tag{D.11}$$

## Bibliography

- E. F. Crawley and J. de Luis. Use of piezoelectric actuators as elements of intelligent structures. AIAA Journal, 25(10):1373–1385, 1987.
- [2] E. H. Anderson and N. W. Hagood. Simultaneous piezoelectric sensing-actuation: analysis and application to controlled structures. *Journal of Sound and Vibration*, 174:617–639, 1994.
- [3] R. Ganguli and K. Umesh. Shape and vibration control of a smart composite plate with matrix cracks. *Journal of Smart Materials and Structures*, 18, 2009.
- [4] I. Chopra. Review of state of art of smart structures and integrated system. Journal of AIAA, 40:2145–2187, 2002.
- [5] H. S. Tzou. Piezoelectric Shells: Distributed Sensing and Control of Continua. Springer, 1993.
- [6] H. S. Tzou. Distributed modal identification and vibration control of continua: theory and applications. Journal of Dynamic Systems, Measurement, and Control, 113:494–499, 1991.
- [7] H. Irschik. A review on static and dynamic shape control of structures by peizoelectric actuation. Journal of Engineeing Structure, 24:5–11, 2002.
- [8] Jalili N. Liu Z. Dadfarnia, M. and D.M. Dawson. An observer-based piezoelectric control of flexible cartesian robot arms: theory and experiment. *Journal of Control Engineering Practice*, 12:1041–1053, 2004.
- [9] X. Wang and J. Tang. Damage detection using piezoelectric admittance approach with inductive circuitry. Journal of Intelligent Material Systems and Structures, 21:667–676, 2010.
- [10] H.S. Tzou and W.K. Chai. Design and testing of a hybrid polymeric electrostrictive/ piezoelectric beam with bangbang control. *Journal of Mechanical Systems and Signal Processing*, 21:417–429, 207.

- [11] Sulla. J.L. Sparks Jr. D.W. Won, C.C. and W.K. Belvin. Application of piezoelectric devices to vibration suppression. *Journal of Guidance, Control, and Dynamics*, 17("6"), 1994.
- [12] A. Kawamura and N. takeda. Linear ultrasonic piezoelectric actuator. *IEEE*, 1:203–206, 1999.
- [13] E. F. Crawley. Detailed models of piezoceramic actuation of beams. Journal of Intellignet Materials Systems and Strucutres, 1(1):1689–1699, 1990.
- [14] E. F. Crawley. Intelligent structures for aerospace: a technology overview and assessment. AIAA Journal, 32(8):1689–1699, 1994.
- [15] T. Bailey and Hubbard. J.E. Distributed piezoelectric-polymer active vibration control of a cantilever beam. *Journal of Guidance Control*, 8(5):605–611, 1985.
- [16] R. L. Clark, C. R. Fuller, and A. Wicks. Characterization of multiple piezoelectric actuators for structural excitation. *Journal of the Acoustic Society of America*, 90:346–357, 1991.
- [17] W.H. Hogood, N.W. Chung and A. Von Flotow. Modeling of piezoelectric actuator dynamics for active structural control. *Journal of Intelligent Materials Systems and Structures*, 1(3):237–354, 1990.
- [18] C.K. Lee and F.C. Moon. Modal sensors/ actuators. Journal of Applied Mechanics, 57:434–441, 1990.
- [19] E. K. Dimitriadis, C. R. Fuller, and C. A. Rogers. Piezoelectric actuators for distributed vibration excitation of thin plates. *Journal of Vibration and Acoustics*, 113:100–107, 1991.
- [20] Lazarus K.B. Crawley, E. F. Induced strain actuation of isotropic and antisotropic plates. AIAA Journal, 29(6):558–5571, 1991.
- [21] H. S. Tzou and M. Gadre. Theoretical analysis of multi-layered thin shell actuation for distributed vibration controls. *Journal of Sound and Vibration*, 132(3):433–450, 1989.
- [22] H.S. Tzou and J. Ding. Smart materials, precision sensors/ actuators, smart structures, and structronic structures. Journal of Advanced Materials and Structures, 11:367–393, 2004.
- [23] H.S. Tzou and J.H. Ding. Distributed modal signals of nonlinear paraboloidal shells with distributed neurons. *Journal of Vibration and Acoustics*, ""(JVA 00-161), 2002.
- [24] H.S. Tzou and J.P. Zhong. Spatially filtered vibration control of cylinderical shelss. Journal of Shock and Vibration, 3(4), 1995.

- [25] H.S. Tzou and Bao. Y. Parametric study of segmented transducers laminated on cylindrical shells part 1: sensor patches. *Journal of Sound and Vibration*, 197(2), 1996.
- [26] Chai W.K. Tzou, H.S. and D.W. Wang. Modal voltages and distributed signal analysis of conical shells of revolution. *Journal of Sound and Vibration*, 260(4), 2003.
- [27] Smithmaitrie P. Tzou, H.S. and J.H. Ding. Sensor electromechanics and distributed signal analysis of piezo (electric)- elastic spherical shells. *Journal of Mechanical Systems and Signal*, 16(2/3), 2002.
- [28] H.S. Tzou and J.H. Ding. Distributed dynamic signal analysis of piezoelectric laminated linear and nonlinear toroidal shells. *Journal of Sound and Vibration*, 254(2), 2002.
- [29]
- [30] Chai W.K. Howard, R.V. and H.S Tzou. Modal voltages of linear and nonlinear structures using distributed artificial neurons. *Journal of Mechanical Systems and Signal Processing*, 15(3), 2001.
- [31] Bao Y. Tzou, H.S. and C.S. Chou. Precision shape control of nonlinear shells and plates. Journal of Wave-Material Interaction, 14, 1999.
- [32] Chai W.K. Tzou, H.S. and D.W. Wang. Micro control actions and location sensitivity of actuator patches laminated on toroidal shell. *Journal of Sound and Vibration*, 126, 2004.
- [33] Han Y. Higuchi K. Chai, W.K and H.S. Tzou. Micro-actuation characteristic of rocket conical shell section actions of segmented actuator patches laminated on deep parabolodial shells. *Journal* of Sound and Vibration, 293:286–298, 2005.
- [34] Chai W.K. Tzou, H.S. and M. Hanson. Dynamic actuation and quadratic magnetoelastic coupling of thin magnetostrictive shells. *Journal of vibration and Acoustic*, 128:385–391, 2006.
- [35] S.O.R. Halim, D. Moheimani. An optimization approach to optimal placement of collocated piezoelectric actuators and sensors on a thin plate. *Journal of Mechatronic*, 13:27–47, 2003.
- [36] O. Ibidapo-Obe. Optimal actuator placement for the active control flexible structure. Journal of Mathematical Analysis and Applications, 105:12–25, 1985.
- [37] Abou-Kandil H. Leleu, S., and Y. Bonnassieux. Piezoelectric actuators and sensors location for active control of flexible structures. *IEEE Transaction*, 50(6), 2001.

- [38] Galiimard-L. Bruant, I. and S. Nikoukar. Optimal piezoelectric actuator and sensor location for active vibration control, using genetic algorithm. *Journal of Sound and Vibration*, 329, 2010.
- [39] Ghasemi-A.H. Seigler, T.M and A. Salehian. Distributed actuation requirements of piezoelectric structures under servoconstraints. *Journal of Intelligent Materials Systems and Structures*, 22(11), 2011.
- [40] T.M Seigler and A.H. Ghasemi. Specified motion of piezoelectrically actuated structures. Journal of Vibration and Acoustic, 134:1–7, 2012.
- [41] B. Sanders, F. E. Eastep, and E. Foster. Aerodynamic and aeroelastic considerations of wings with conformal control surfaces for morphing aircraft. *Journal of Aircraft*, 40:94–99, 2007.
- [42] J. H. Su and C. E. Ruckman. Aerodynamic and aeroelastic considerations of wings with conformal control surfaces for morphing aircraft. *Journal of Intelligent Material Systems and Structures*, 6:801–808, 1995.
- [43] W. Blajer. Dynamics and control of mechanical systems in partly specified motion. Journal of the Franklin Institute, 334B:407–426, 1997.
- [44] Y-H. Chen. Mechanical systems under servo constraints: the lagrange's approach. *Mechatronics*, 15:317–337, 2004.
- [45] Y-H. Chen. Equations of motion of mechanical systems under servo constraints: The maggi approach. *Mechatronics*, 18:208–217, 2007.
- [46] T. R. Kane and D. A. Levinson. Dynamics: Theory and Applications. McGraw-Hill, 1985.
- [47] R. L. Huston. Multibody Dynamics. Butteworth-Heinemann, 1990.
- [48] A. Rosen. Applying the lagrange method to solve problems of control constraints. Journal of Applied Mechanics, 66:1013–1015, 1999.
- [49] J. Parczewski and W. Blajer. On realization of program constraints: Part I theory, part II practical implications. *Journal of Applied Mechanics*, 56:676–684, 1989.
- [50] W. Blajer and A. Czaplicki. Contact modeling and identification of planar somersaults on the trampoline. Journal of Multibody System Dynamics, 10:289–312, 2003.
- [51] W. Blajer. On the determination of joint reactions in multibody mechanisms. Journal of Mechanical Design, 126:341–350, 2004.

- [52] W. Blajer and k. Kolodziejczyk. A projection method approach to constrained dynamic analysis. Journal of Applied Mechanics, 59:643–649, 1992.
- [53] W. Blajer. Structure of differential-algebraic equations for control problems in mechanics. International Journal of Systems Science, 24:2367–2377, 1993.
- [54] Bestle-D. Blajer, W. and W. Schiehlen. An orthogonal complement matrix formulation for constrained multibody systems. *Journal of Mechanical Design*, 16:423–428, 1994.
- [55] W. Blajer and K. Kolodziejczyk. A geometric approach to solving problems of control constraints: theory and dae framework. *Multibody System Dynamics*, 11:343–364, 2004.
- [56] W. Blajer. Modeling of under-actuated mechanical system in partly specified motion. Journal of Theoretical and Applied Mechanics, 126:341–350, 2004.
- [57] Martin P. Fliess M., Levine J. and International Journal of Control 61 1327-1361 Rouchon P., 1995. Flatness and defect of nonlinear systems: introductory theory and examples. *International Journal of Control*, 61:1327–1361, 1995.
- [58] P. Rouchen. Flatness based control of oscillators. Journal of Applied Mathematics and Mechanics, 85:411–421, 2005.
- [59] Dziewiecki K Blajer, W and Z. Mazur. Multibody modeling of human body for the inverse dynamics analysis of sagittal plane movements. *Journal of Multibody System Dynamics*, 18:217– 232, 2007.
- [60] Kolodziejczyk k. Blajer, W. Control of under-actuated mechanical systems with servo-constraints. Journal of Nonlinear Dynamics, 50:781–791, 2007.
- [61] W. Blajer and K. Kolodziejczyk. Index of differential algebraic equations governing the dynamics of constrained mechanical systems. *Application of Math and Modeling*, 46(2):383–394, 2008.
- [62] Graffstein J. Blajer, W. and M. Krawczyk. Modeling of aircraft prescribed trajectory flight as an inverse simulation problem. Journal of Modeling Simulation and Control of Nonlinear Engineering Dynamical Systems, pages 153–162, 2009.
- [63] T. Sashida and T. Kenjo. An introduction to ultrasonic motors. Oxford University Press Inc., Oxford, NY, 1993.
- [64] H. Zhao, J. Wu, and J. S. Luo. Turbulent drag reduction by traveling wave of flexible wall. *Fluid Dynamics Research*, 34:175–198, 2004.

- [65] A. A. Shabana. Flexible multibody dynamics: review of past and recent developments. Multibody System Dynamics, 1:189–222, 1997.
- [66] Q. Hu and G. Ma. Maneuvering and vibration reduction of a flexible spacecraft integrating optimal sliding mode controller and distributed piezoelectric sensors actuators. *Journal of Adaptive Control and Signal Processing*, 21:452–476, 2007.
- [67] Seigler T. M. Salehian, A. and D.J. Inman. Dynamic effects of a radar panel mounted on a truss satellite. AIAA Journal, 45(7):1642–1654, 2007.
- [68] Neil D. A. Bae J. S. Seigler, T. M. and D.J. Inman. Modeling and control of large-scale morphing aircraft. AIAA Journal, 44(4):1077–1087, 2007.
- [69] Q. Hu and Ga Ma. Maneuver and vibration suppression of flexible spacecrafts with input nonlinearities. Journal of Aircraft Engineering and Aerospace Technology, 77(5):388–400, 2005.
- [70] M.H Kim. Simultaneous structural health monitoring and vibration control of adaptive structures using smart materials. *Journal of Shock and Vibration*, 9:329–339, 2002.
- [71] Buckham B.J. Gilardi, G. and E.J Park. Finite element of a slewing non-linear flexible beam for active vibration control with arrays of sensors and actuators. *Journal of Intelligent Material Systems and Structures*, 20:1941–1958, 2009.
- [72] H.H. Lee. A new trajectory control of a flexible link robot based on a distributed paramter dynamic model. *International Journal of Control*, 77(15):546–553, 2004.
- [73] Pergher R. Bottega, V. and J.S.O. Fonseca. Simultaneous control and piezoelectric insert optimization for manipulators with flexible link. *Journal of the Brazilian Society of Mechanical Sciences*, 31(2):105–116, 2009.
- [74] Q. Hu. Robust adaptive sliding mode attitude maneuvering and vibration damping of three-axisstabilized flexible spacecraft with actuator saturation limits. *Nonlinear Dynamics*, 55(4):301–322, 2009.
- [75] L. R. Ray and L. Tian. Damage detection in smart structures through sensitivity enhancing feedback control. Journal of Sound and Vibration, 227:987–1002, 1999.
- [76] Koh B. H. Ray, L. R. and L. Tian. Damage detection and vibration control in smart plates: Towards multifunctional smart structures. *Journal of Intelligent Materials Systems and Structures*, 11:725–739, 2000.

- [77] F.E. Udwadia and P. Phohomsiri. Active control of structures using time delayed positive feedback proportional control design. Journal of Structural Control and Health Monitoring, 13:536–552, 2006.
- [78] Lew J.S. Ahmad, S.S. and L.H. Kell. Robust control of flexible structures against structural damage. *IEEE Transaction*, 8(1):170–182, 2000.
- [79] Qui J. Zhang, W. and J. Tani. Robust vibration control of plate using self sensing actuators of piezoelectric patches. *Journal of Intelligent Material Systems and Structures*, 15(12):923–931, 2004.
- [80] Han J.d. Zhang X.M. Wang Y.C. Qiu, Z.C. and Z.W. Wu. Active vibration control of a flexible beam using a non-collocated acceleration sensor and piezoelectric patch actuator. *Journal of Sound and Vibration*, 326:438–455, 2009.
- [81] Pergher R. Bottega, V. and J.S.O. Fonseca. Simultaneous control and piezoelectric insert optimization for manipulators with flexible link. *Journal of the Brazilian Society of Mechanical Sciences*, 31(2):105–116, 2009.
- [82] A. Preumont. Vibration Control of Active Structures. Kluwar Academics, 1997.
- [83] E. Garcia and D.J. Inman. Advantages of slewing an active structure. Journal of Intelligent Materials Systems and Structures, 1:261–272, 1990.
- [84] A.E. Bryson Jr. Control of Spacecraft and Aircraft. Prentice University Press, New Jersey, 1994.
- [85] B.A. Francis and W.M. Wonham. The internal model princeiple of control theory. Journal of Automatica, 12(5):457–456, 1976.
- [86] J. Huang and C.F. Lin. Internal model principle and robust control of nonlinear systems. *IEEE Transaction*, 2:1501–1506, 1993.
- [87] Constantinescu A. Landau, A.D. and A. TITLE = Rey.
- [88] Santosuosso G.L. Marino, R. and P. Tomei. robust adaptive compensation of biased sinusoidal disturbances with unknown frequency. *Journal of Automatica*, 39:1755–1761, 2003.
- [89] N. Forov. adaptive non-linear tracking with complete compensation of unknown disturbances. Journal of Control, 4:132–139, 1998.

- [90] M.J. Balas. Active control of flexible systems. Journal of Optimization Theory and Applications, 25(3):415–436, 1978.
- [91] L. Meirovitch. Dynamics and Control of Structures. John Wiley, 1990.
- [92] C.J. Goh and T.K. Caughey. On the stability problem caused by finite actuator dynamics in the collocated control of large space structures. *International Journal of Control*, 41(3):787–802, 1985.
- [93] H.C. Shin and S.B. Choi. Position control of a two link flexible manipulator featuring piezoelectric actuators and sensors. *Journal of Mechatronics*, 11:707–729, 2001.
- [94] Liu H.T. Shan, J. and D. Sun. Slewing and vibration cotrol of a single flexible manipulator by positive position feedback (ppf). *Journal of Mechatronics*, 15:487–503, 2005.
- [95] Caughey T.K. Fanson, J.L. Positive position feedback control for large space structures. AIAA Journal, 28(7):717–724, 1987.
- [96] S. Poh and A. Baz. Active control of flexible structures using a modal positive position feedback controller. Journal of Intelligent Material Systems and Structures, 1:273–288, 1990.
- [97] S.N. Mahmoodi and M. Ahmadian. Vibration suppression of flexible spacecraft during attitude control. Journal of Acta Astronautica, 49(2):73–83, 2001.
- [98] S.N. Mahmoodi and M. Ahmadian. Active vibration control with modified positive position feedback. Journal of Dynamic Systems, Measurement, and Control, 131, 2009.
- [99] S.N. Mahmoodi and M. Ahmadian. Modified acceleration feedback for active vibration control of aerospace structures. *Journal of Smart Materials and Structures*, 19, 2010.
- [100] Inman D. J. Friswell, M. I. and R.W. Reitz. Active damping of thermally induced vibrations. Journal of Intelligent Materials and Structures, 8:678–685, 1997.
- [101] M. I. Friswell and D. J. Inman. The relation between positive position feedback and output feedback and output feedback controllers. *Journal of Smart Materials and Structures*, 8:285–291, 1999.
- [102] R. Jha and J. Rower. Experimental investigation of active vibration control using neural networks and piezoelectric actuators. *Journal of Smart Materials and Structures*, 11(1):115–121, 2002.

- [103] R. Rodrigues J. Vaz M. Silva, S. Ribeiro and J.T. Monteiro. The application of genetic algorithms for shape control with piezoelectric patches.
- [104] K.G. Ma. Vibration control of smart structures with bonded pzt patches: novel adaptive ltering algorithm and hybrid control scheme. Journal of Smart Materials and Structures, 12(6):473–482, 2003.
- [105] P. Mayhan and G. Washington. Fuzzy model reference learning control: a new control paradigm for smart structures. *Journal of Smart Materials and Structures*, 19:874–884, 1998.
- [106] Q. Hu and Ga Ma. Variable structures control and active vibration suppression of flexible spacecraft during attitude maneuver. Journal of Aerospace Science and Technology, 9:307–317, 2005.
- [107] Eghtesad M Mirzaee, E. and S.A. Fazelzadeh. Maneuver control and active vibration suppression of a two-link flexible arm using a hybrid variable structures/ lyapunov control design. *Journal of Acta Astroautica*, 67:1218–1232, 2010.
- [108] Richer E. Hurmuzlu Y. Shimon, P. Theoretical and experimental study of efficient control of vibration in clamped square plate. *Journal of Sound and Vibrationl*, 282(1-2):453-473, 2005.
- [109] R.W. Reitz and D.J Inman. Comparison of linear and nonlinear of a slewing beam. Journal of Vibration and Control, 6:309–322, 2000.
- [110] C.Y. Kuo and S.P.T. Wang. Nonlinear robust hybrid control of robotic manipulators. Journal of Dynamics Systems, Measurments, and Controll, 112:49–54, 1990.
- [111] H. Khalil. Nonlinear Systems. Prentice Hall, Upper Saddle River, New Jersey, 2002.
- [112] S. Sastry. Nonlinear Systems; Analysis, Stability and Control. Springer, 1999.
- [113] V.J. Nagata, T. Modi and H. Matsuo. Dynamics and control of flexible multibody systems: part ii : simulation code and parametric studies with nonlinear control. *Journal of Acta Astronautica*, 49(11):595–610, 2001.
- [114] Lian F.L. Yang, J.H. and L.C. Fu. Nonlinear adaptive control for flexible-link manipulators. *IEEE Transaction*, 13(1):140–148, 1997.
- [115] L. Slotine. Applied Nonlinear Systems. Prentice Hall, Upper Saddle River, New Jersey,, 1991.
- [116] W.B. Hung, J.Y. Cao and J.C. Hung. Varibale strunture control. *IEEE Transaction*, 40(1):2–22, 1993.

- [117] S.B. Choi and J.W. Sohm. Chattering alleviation in vibration control of a smart beam structures using peizofim actuators : experimental verification. *Journal of Sound and Vibration*, 294(3):640– 649, 2006.
- [118] W. Sodel. Vibrations of shells and plates. Marcel Dekker, 2004.
- [119] H.S. Tzou and J. Ding. Micro-control actions of segmented actuator patches laminated on deep paraboloidal shells. *Journal of JSME*, 45:8–15, 2002.
- [120] L. Meirovitch. Principles and Techniques of Vibrations. Prentice Hall, 1996.
- [121] S.H. Lam. On lagrangian dynamics and its control formulation. Applied Mathematics and Computations, 91:259–284, 1998.
- [122] J.L. Lin and J.N. Juang. Sufficient conditions for minimum phase second order linear systems. Journal of Vibration and Control, 1:183–199, 1995.
- [123] A. Zuyev. Partial asymptotic stabilization of nonlinear distributed parameter systems. Journal of automatica, 41:1–10, 2005.
- [124] L. Meirovitch. Principles and Techniques of Vibrations. Prentice-Hall Inc. Uppers., 1997.
- [125] A.W. Leissa. Vibration of plates. U.S. Government Printing Office, 1969.

## Vita

Amirhossein Ghasemi was born in Mashhad, Iran on January 8,1983. He received the Bachelor of Science degree in Mechanical Engineering from Ferdowsi University of Mashhad in 2005. He then attended Amirkabir Institute of Technology where he received the Master of Science in Mechanical Engineering in 2008. He graduated from University of Kentucky with PhD in Mechanical Engineering in 2012.