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GAUGE-GRAVITY DUALITY AND ITS APPLICATIONS TO COSMOLOGY AND FLUID DYNAMICS

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial
fulfillment of the requirements for
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This thesis is devoted to the study of two important applications of gauge-gravity duality: the cosmological singularity problem and conformal fluid dynamics. Gauge-gravity duality is a concrete dual relationship between a gauge theory (such as electromagnetism, the theories of weak and strong interactions), and a theory of strings which contains gravity. The most concrete application of this duality is the AdS/CFT correspondence, where the theory containing gravity lives in the bulk of an asymptotically anti-de-Sitter space-time, while the dual gauge theory is a deformation of a conformal field theory which lives on the boundary of anti-de-Sitter space-time(AdS).

Our first application of gauge-gravity duality is to the cosmological singularity problem in string gravity. A cosmological singularity is defined as a spacelike region of space-time which is highly curved so that Einstein's gravity theory can be no longer applied. In our setup the bulk space-time has low curvature in the far past and the physics is well described by supergravity (which is an extension of standard Einstein gravity). The cosmological singularity is driven by a time dependent string coupling in the bulk theory. The rate of change of the coupling is slow, but the net change of the coupling can be large. The dual description of this is a time dependent coupling of the boundary gauge theory. The coupling has a profile which is a constant in the far past and future and attains a small but finite value at intermediate times. We construct the supergravity solution, with the initial condition that the bulk space-time is pure AdS in the far past and show that the solution remains smooth in a derivative expansion without formation of black holes. However when the intermediate value of the string coupling becomes weak enough, space-time becomes highly curved and the supergravity approximation breaks down, mimicking a spacelike singularity. The resulting dynamics is analyzed in the dual gauge theory with a time dependent coupling constant which varies slowly. We develop an appropriate adiabatic expansion in the gauge theory in terms of coherent states and show that the time evolution continues to be smooth. We cannot, however, arrive at a definitive conclusion about the fate of the system at very late times when the coupling has again risen and supergravity again applies. One possibility is that the energy which has been supplied to the universe is simply extracted out and the space-time goes back to its initial state. This could provide a model for a bouncing cosmology. A second possibility is that dissipation leads to a thermal state at late time. If this possibility holds, we show that such a thermal state will be described either by a gas of strings or by a small black hole, but not by a big black hole. This means that in either case, the future space-time is close to AdS.

We then apply gauge-gravity duality to conformal fluid dynamics. The long wavelength behavior of any strongly coupled system with a finite mean free path is described by an appropriate fluid dynamics. The bulk dual of a fluid flow in the boundary theory is a black hole with a slowly varying horizon. In this work we consider certain fluid flows which become supersonic in some regions. It is well known that such flows present acoustic analogs of ergoregions and horizons, where acoustic waves cannot propagate in certain directions. Such acoustic horizons are expected to exhibit thermal radiation of acoustic waves with temperature essentially given by the gradient of the velocity at the acoustic horizon. We find acoustic analogs of black holes in charged conformal fluids and use gauge-gravity duality to construct dual gravity solutions. A certain class of gravitational quasinormal wave modes around these gravitational backgrounds perceives a horizon. Upon quantization, this implies that these gravitational modes should have a thermal spectrum.

The final issue that we study is fluid-gravity duality at zero temperature. The usual way of constructing gravity duals of fluid flows is by means of a small derivative expansion, in which the derivatives are much smaller than the temperature of the background black hole. Recently, it has been reported that for charged fluids, this procedure breaks down in the zero temperature limit. More precisely, corrections to the small derivative expansion in the dual gravity of charged fluid at zero temperature have singularities at the black hole horizon. In this case, fluid-gravity duality is not understood precisely. We explore this problem for a zero temperature charged fluid driven by a low frequency, small amplitude and spatially homogeneous external force. In the gravity dual, this force corresponds to a time dependent boundary value of the dilaton field. We calculate the bulk solution for the dilaton and the leading backreaction using a modified low frequency expansion. The resulting solutions are regular everywhere, establishing fluid-gravity duality to this order.

KEYWORDS: String theory, Gauge-gravity duality, Matrix big-bang, Conformal fluid dynamics

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By
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RULES FOR THE USE OF DISSERTATIONS

DISSERTATION

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The most of the parts of Chapter3,4, and 5 are appeared in the following papers respectively.

- Adel Awad, Sumit Das, Archisman Ghosh, K. Narayan, Jae-Hyuk Oh, Sandip Trivedi, Slowly Varying Dilaton Cosmologies and their Field Theory Duals, Phys.Rev.D 80:126011,2009 [arXiv:hep-th/0906.3275][72]. In this paper, I have contributed to the calculation of the normalization factor $F(2n)$ in Appendix A.3.
- Sumit R. Das, Archisman Ghosh, Jae-Hyuk Oh, Alfred D. Shapere, On Dumb Holes and their Gravity Duals, [arXiv:hep-th/1011.3822][73]. In [73], I have contributed to the set up of the whole frame works of the supersonic fluid which contain constructing fluid flows in the dual fluid on the particular curved geometry and the sound waves on that geometry which show the properties of the acoustic black hole. I also have contributed to development of its gravity duals perceiving some of the features of the acoustic black holes.
- Jae-Huyk Oh, Small Amplitude Forced Fluid Dynamics from Gravity at T=0, [arXiv:hep-th/1012.1040][74]. I have worked on this paper for myself. I stress that in this paper, I firstly calculated the gravitational back reaction and its resolution of singularities appearing in the dual gravity system.

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Chapter 1 Introduction

In nature there are four different fundamental interactions: the electromagnetic, strong, weak, and gravitational interactions. It is well known that the first three forces have been successfully described by quantum field theory. For most practical purposes it is enough to describe gravity classically by Einstein's General Theory of Relativity. However, at a very high energy scale known as the *Planck* scale, it is necessary to treat the gravitational interaction quantum mechanically. Up to now attempts to find a field theoretic formulation of quantum gravity have been unsuccessful.

String theory has been widely regarded as a potentially complete theory of quantum gravity[1]. In string theory, the fundamental constituents are one-dimensional strings rather than zero-dimensional point particles. The infinitely many vibrational modes of the string correspond to various kinds of particles, which include ordinary matter particles as well as force-carrying particles like photons and gravitons. String theories may include both open and closed strings. The massless spectrum of open strings describes a gauge theory (which could include electromagnetism and the strong and weak forces), whereas the closed string massless spectrum includes gravity.

One of the most important developments in string theory is "gauge-gravity duality" [2, 3, 4, 5]. This is a concrete relationship between a gauge theory and a theory of strings containing gravity. When the interactions in the gauge theory are strong enough(which means that we cannot deal with this theory by usual quantum field theory methods), the dual string theory can be approximated by Einstein gravity. On the other hand, when the string theory containing gravity is in a regime in which the space-time is highly curved, the dual gauge theory may be in a regime in which we can apply perturbative quantum field theory techniques. The chief merit of this relationship is that one can analyze one theory in the regime where the dual theory is intractable.

There are huge classes of applications of gauge-gravity duality. In this thesis, we will restrict ourselves to two types of applications: to the cosmological singularity problem and to fluid dynamics. The former attempts to understand cosmological singularities by mapping them to a dual gauge theory. If we extrapolate the standard cosmological picture of the expanding universe backward in time, we are led to the conclusion that at some initial time, the entire universe was concentrated at a point in space, an infinite-curvature singularity of the space-time metric known as the Big Bang. In Einstein's gravity theory, the Big Bang is a genuine singularity, a time when gravitational forces become infinitely large. Within this classical framework there is no sense in which we can talk of a time prior to the big bang: it represents a "beginning of time". A similar situation is a"big crunch" where the universe can collapse into a small region of infinite curvature. While our universe most likely will not have a big crunch, this is a theoretical possibility. Big bangs and big crunches are examples of cosmological singularities.

Cosmological singularities troublesome because Einstein's theory of gravity breaks down close to them and quantum gravity effects must be taken into account. Before the advent of String Theory there was no consistent quantum theory of gravity, which is why the nature of cosmological singularities has been a major puzzle for many decades. Understanding cosmological singularities is not only a theoretical problem, but could very well be necessary to interpret observational results in cosmology in the coming decades.

Recently, there has been some progress in understanding the nature of cosmological singularities using gauge-gravity duality[16, 17, 18, 20, 21]. When the gauge coupling is large, the dual string theory can be approximated by classical gravity. On the other hand, in the case that the gauge coupling is small enough, this dual string theory is strongly coupled and quantum effects dominate. Regions of space-time close to singularities could thus be dual to weakly coupled gauge theories, and analyzing these dual gauge theories could provide useful information about cosmological singularities. Several research groups have used this approach to provide toy models of cosmological singularities [16, 17, 18, 20, 21]. In these models, the dual gauge theory is driven by time dependent parameter(s), for example, the string coupling. The idea is to see if the gauge coupling can be used to compute the time evolution of the system in a regime where Einstein gravity fails.

In our research[72], we have studied one class of toy models for cosmological singularities in which the gauge coupling is time-dependent. At early times and late times, we consider the gauge coupling to be strong but at intermediate times we suppose that it becomes small. We suppose that we start in a ground state at early times, to which one can apply usual space-time notions. We can analyze and evolve these grounds states from the point of view of the dual gravitational theory, which incorporates the time-dependent coupling as a dilaton background. However, at the intermediate times, when the space-time is highly curved – which means the energy density is huge – Einstein gravity cannot be applied any longer. We regard this regime as a model of a cosmological singularity. The cosmological singularity in our model is rather special in the sense that the space-time is spoiled not by quantum mechanical effects but by string effects. In this sense, our model is not a realistic model, but it retains the essential feature of the problem in that the notion of space-time breaks down near the singularities.

To deal with this, we have developed a new framework which allows us to study the time dependence of this system. In the dual description of the string theory by a gauge theory, time evolution is governed by a time-dependent quantum Hamiltonian operator. We assume that the time-dependence of the parameters in the Hamiltonian is slow, and that its parameters return at late times to their values at early times. The assumption of slow time-dependence plays a crucial technical role by allowing us to apply an "adiabatic approximation". Then, if the Hamiltonian operator is slowly varying with time, the expectation values of operators at late times will differ from their initial values by exponentially small corrections.

In our work, we try to determine the late time behavior of the string theory system after passing the cosmological singularity, by studying the hamiltonian evolution in the dual gauge theory. Here we do not have a definitive conclusion, but we are able to narrow down the outcome to a few possibilities. One such possibility is that the energy which has been supplied to the universe (by the time dependent coupling) during the contracting phase is simply extracted out during a future expanding phase. The second possibility is that the energy does not return, but gets dissipated, leading to a thermal state at late times. Typically such a thermal state in the gauge theory is described in the gravitational dual as a black hole. Our calculations show that if such a thermal state indeed occurs, the corresponding black hole would be very small compared to the overall size of the universe: so that most of the universe would resemble a normal expanding universe. This is distinct from several other scenarios that we examine, involving rapidly varying couplings, where almost the entire universe is engulfed in a big black hole.

To determine which of the above possibilities actually occurs we need to understand the process by which some amount of energy gets thermalized. While thermalization is a common phenomenon, its theoretical understanding is rather primitive. In the future, I plan to attack this problem by applying some models of thermalization borrowed from other areas of physics.

In the second part of this thesis, we apply gauge-gravity duality to obtain a better understanding of fluid dynamics. A famous example of a physical system described by fluid dynamics is a strongly coupled gauge field theory of quarks and gluons. In heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC), the quarks and gluons produced in collisions come rapidly into a state of local thermal equilibrium, known as the quark-gluon plasma (QGP). In this regime, until the quarks and gluons cool and combine to produce heavier particles, hadrons, one can apply hydrodynamics to describe the QGP.

Fluid dynamics has been widely studied using gauge-gravity duality[25, 42, 43, 44, 45, 60, 61, 68, 69], which has been used to obtain unexpected relationships between thermodynamic quantities describing the fluid. The most celebrated example from dual gravity calculations is that the ratio of shear viscosity to the entropy density of a conformal fluid has a lower bound [42, 43].

Most research on fluids has been done in the local rest frame, in which all thermodynamic quantities describing the fluid are locally constant. However, it is interesting to study the global structure of the an inhomogeneous fluid (i.e., a fluid in which the thermodynamics quantities are varying in space and time). An interesting example of an inhomogeneous fluid flows is the so-called acoustic black hole (or dumb hole), which was firstly found by Unruh[47] in 1995. To illustrate this example, let us suppose that there is an inhomogeneous and steady (time-independent) fluid flow in which the fluid velocity is varying, so that in certain “supersonic” regions the fluid velocity becomes faster than the speed of sound waves, and in other regions the speed is subsonic. In this case, any acoustic waves created in those supersonic regions cannot propagate back in the direction from which the fluid is flowing. Such supersonic regions are called acoustic ergoregions. In certain cases, an acoustic horizon forms, bounding a region out of which the sound waves cannot escape. This is very similar to the physics of ergoregions and horizons in general relativity. Ergoregions and the horizons are generic features of black holes, which prevent light, rather than sound, from traveling in certain directions.

Black holes also exhibit thermal radiation, known as Hawking radiation, and behave like thermal objects with a certain temperature. The same physics that leads to Hawking radiation from black holes also leads to a Hawking-like radiation of sound waves (or phonons) with thermal spectrum from acoustic horizons. The temperature of this acoustic Hawking radiation is given by the same mathematical expression from black hole physics, in which the black hole temperature is proportional to the gravitational acceleration at the black hole horizon. Such fluid configurations have been proposed as possible experimentally realizable systems for testing the physics of Hawking radiations in the laboratory[53].

In our current research [73], we construct dual gravitational descriptions of such flows as non-static black holes. The duals of sound waves around those inhomogeneous fluid flows are then gravitational wave modes which satisfy certain boundary conditions — namely, that the gravitational waves become purely incoming waves at the black hole horizon and decay sufficiently rapidly as the gravitational wave modes approach the boundary of the space-time of the gravity system(Such wave modes are called "Quasi-normal Modes"). It follows from fluid-gravity duality that upon quantization one should find Hawking-like radiation into these quasi-normal modes with an approximately thermal spectrum. We will find that the bulk gravity theory contains a "quasi-normal mode horizon" which is dual to the acoustic horizon in the fluid.

The precise set up is that we have constructed fluid flows in 4-dimensional space time (three spatial dimensions with coordinates x,y and z and one time dimension with t) in which the fluid velocity vanishes at $z = -\infty$ and $z = \infty$. The fluid velocity is entirely in the z-direction and is a function of z only. At intermediate values of z the fluid speed passes the speed of sound at several different values of z, producing a planar acoustic horizon at each such z. We expect Hawking-like thermal radiation for each acoustic horizon, and corresponding Hawking-like radiation from each corresponding "quasi-normal mode horizon" in the dual gravity theory.

It is important to note that there are two different temperatures in this system. On one hand, the fluid itself has a temperature, and the quantum modes of the fluid are populated thermally. On the other hand, the approximate thermal radiation from the dumb hole horizon is at a temperature distinct from the temperature of the fluid itself. In the dual gravity system, the former corresponds to Hawking radiation from the black hole horizon whereas the latter corresponds to the Hawking-like radiation of quasi-normal modes.

To observe thermal radiation from these quasi-normal mode horizons, it is desirable for the temperature of the black hole not to be comparable with that of quasi-normal horizon. If this condition is not satisfied, it may be difficult to distinguish the quasi-normal mode radiation from the thermal radiation of the black hole. To avoid such a situation, one can adjust the black hole temperature (which is the same as the temperature of the dual fluid) to be nearly zero. For example, one may consider a Reissner-Nordstrom black hole (an electromagnetically charged black hole) and tune its charge towards near extremality.

The last subject of this thesis is the resolution of a problem of fluid-gravity duality in the zero-temperature limit. The usual way of obtaining gravitational solutions dual to inhomogeneous fluids is to solve Einstein's equations perturbatively under the condition that each thermodynamic quantity in the inhomogeneous fluid is slowly varying. This perturbative expansion in Einstein's equations is effectively the same as the small frequency (and small momentum) expansion. The method has been nicely applied to construct gravity duals of inhomogeneous uncharged fluids[25, 44, 45, 60, 61, 68, 69]. For charged fluids however, the gravity dual involves a Reissner-Nordstrom black brane, and in regions where the local temperature approaches zero, the solutions become singular at the horizon at leading order(and possibly up to certain subleading orders) in perturbation theory. This singularity spoils the smoothness of physical quantities like the electric field and gravitational force in the gravitational solution.

We point out that the failure of this method of obtaining the gravity dual solutions does not agree with expectations from the dual fluid dynamics. In the dual fluid dynamics, there is an effective length scale called mean free path. It is well understood that in fluid dynamics as long as the mean free path is finite, the fluid dynamics is well-defined. As was shown recently[25, 42, 43, 44, 45, 57, 60, 61, 68, 69, 70], it is certain that the mean free path in the charged fluid is finite even in the zero temperature limit, and so physical quantities should be well-behaved in both the fluid and its gravity dual.

Similar types of divergences in the derivative expansion appear in the linearized limit (which assumes that both the amplitudes and frequencies of the fields are small). Recently, this problem has been considered for a linearized scalar field in a zero-temperature black hole background[64]. The usual sort of small frequency expansion of the scalar field is not straightforward at zero temperature, and leads to nonanalytic solutions. However, this problem is resolved by the so-called modified low frequency expansion[64], for which we introduce new coordinate variables near the horizon. The new coordinate variables are obtained by a nonlocal (frequency-dependent) transformation from the original coordinates. With these new variables, we can reorganize the small frequency expansion near the horizon. It turns out that the solution in this modified expansion is completely regular.

In my recent research[74], I have solved the corresponding problem for fluid-gravity duality using similar techniques. Specifically, I explore a charged fluid in the limit of zero temperature, driven by an external force which is spatially homogeneous and has small amplitude and low frequency. The gravity dual of this fluid is given by a gravity system containing an electric field and a massless, slowly-varying scalar field with negative cosmological constant. Some recent papers have considered a similar setup[64, 65, 66]. However, we go beyond these works by calculating the bulk solution for the dilaton and the leading back-reactions of the background space-time and electric field, using the modified low frequency expansion. We find that the back-reacted gravity solution is regular everywhere (up to certain order in the modified small frequency expansion), as expected from fluid-gravity duality. This is the first paper which shows explicitly that a modified expansion of this type can lead to regular solutions once back-reactions are taken into account.

Chapter 2 Strings and Gauge-Gravity Duality

2.1 Strings, Supergravity and Black p-Branes

Quantum field theory describes nature successfully up through experimentally accessible energies. However at the order of the *Planck* energy scale, we need to consider quantum gravity effects which no one could have dealt with successfully up to nowdays. It has not been possible to quantize gravity theories by usual path integral methods (canonical quantization). One way to deal with quantum gravity is by introducing strings[1]. Strings are one-dimensional objects which have no substructures. What they carry are string tensions like rubber bands and the scale of the string tensions is denoted by $\frac{1}{\alpha'} \equiv m_s^2 \equiv \frac{1}{l_s^2}$. m_s is string energy scale and l_s is called string length scale.(Tension has a dimension of energy per unit length, then the string tension has a dimension of $[E/L]$, where E is unit of energy and L is unit of length. When we introduce natural units in which we set speed of light and *Planck* constant to be 1, the energy dimension is inversely proportional to the length dimension: $[E] \sim \frac{1}{[L]}$. Therefore, the dimension of tension becomes $[E^2]$ or $\frac{1}{[L^2]}$, which is the dimension of $\frac{1}{\alpha'}$). These strings can oscillate and with these oscillations, strings can get mass and spin spectrum. There is an infinite number of the oscillation modes in this string spectrum, by which strings can become various types of particles with different masses and spins. There are two types of strings: open string and closed string. The oscillation modes of the both types of strings can have massless spectrums. The massless spectrum of the closed string has spin 2, technically of which modes provide fields with two space-time indices. The particles of such kinds in nature are gravitons. Therefore, the closed string theory contains gravity. Massless modes of the open string oscillations have spin 1, which are the fields having one space-time index. These can be identified to photons or gauge bosons in nature because they are spin 1 particles.

For closed string, the oscillation modes become standing waves on them whose boundary conditions are either periodic or anti-periodic ones. For open string(it has two end points), there can be two different boundary conditions: Neumann boundary condition and Dirichlet boundary condition. The Neumann boundary condition allows that the two open string end points are freely moving with speed of light whereas the Dirichlet one fixes its two end points at some spatial points. The Dirichlet boundary condition naturally gives a concept of D-branes on which the open string end points are attached for them not to move to the perpendicular directions of the branes. The massless spectrum of open strings on the D-branes provides Gauge Field Theory like theory of photons and gauge bosons. The most simple case is that we have single D-brane. In this case, we have Abelian Gauge Field Theory. One can generalize the situation to that open strings are attached to N-stacks of D-branes. In this case, we have Non-Abelian Gauge Field Theory.

In low energy, the closed string theory provides Supergravity Theory. In the string spectrum, there are finite number of massless excitations and infinite number of massive ones. In the low energy limit, one can obtain the low energy effective action of massless fields by integrating out of the massive modes (Wilsonian effective action). It is well known that the effective action can be represented by systematic expansion in the number of derivatives(It is effectively a low frequency expansion). The small expansion parameter becomes $\frac{E}{m_s}$, where E is energy(frequency) of the massless fields. The effects from massive fields are contained in higher order terms in this low frequency expansion. Then, the precise meaning of low energy limit is that the energy of the string fields is much less than string energy scale m_s. m_s can be treated as a bookkeeping parameter in the low frequency expansion. To get low energy theory without higher order terms, one can take a limit as $m_s \to \infty$.

Now, let us discuss supergravity solutions. Consider type IIB supergravity theory in 10-dimension and get a black hole solution which is called black p-brane solution[5, 6, 7]. Type IIB supergravity theory is the low energy effective theory of type IIB string theory, where D-branes carry Ramond-Ramond charges(R-R charge). These R-R charges become electric charges in the IIB supergravity theory. Then, D-branes act as electrons or other charged particles in nature. The only difference is that electrons are zero-dimensional point particles, but D-branes are spatially extended objects. If these D-branes are extended to p-spatial directions, they are called Dp-branes. Dp-branes in the string theory couple to $p+2$-form field strength in the supergravity as electrons do to electromagnetic field(which is $0+2$-form field strength). For each Dp-brane, there is its magnetically charged dual brane which interacts with dual $(8-p)$-form field strength. For example, D3-brane couples to 5-form field strength, F_5 and its magnetic dual is also 5-form field strength. Black p-brane solution is obtained from the type IIB supergravity action as

$$S_{IIB} = \frac{m_s^8}{(2\pi)^7} \int d^{10}x\sqrt{-g} \left(e^{-2\phi}(R + 4(\nabla\phi)^2) - \frac{2}{(8-p)!} F_{p+2}^2 \right), \tag{2.1}$$

where p is odd number for this to become IIB supergravity action, R is curvature scalar, ϕ is a massless scalar field which is called dilaton and F_{p+2} is $p+2$-form field strength from $p+1$ form gauge field, A_{p+1}. The other supergravity fields are simply turned off.

The resulting solutions of metric, $p+2$-form field strength and dilaton fields are given by

$$\begin{aligned} ds^2 &= -\frac{f_+(r)}{\sqrt{f_-(r)}}dt^2 + \sqrt{f_-(r)}dx_p^2 + \frac{f_-(r)^{-\frac{1}{2}-\frac{5-p}{7-p}}}{f_+(r)}dr^2 \\ &+ r^2\left(f_-(r)\right)^{\frac{1}{2}-\frac{5-p}{7-p}} d\Omega_{8-p}^2, \\ \phi(r) &= \frac{p-3}{4} ln(f_-(r)) \text{ and } *F_{p+2} = Q\epsilon_{8-p}. \end{aligned} \tag{2.2}$$

where

$$f_\pm(r) = 1 - (\frac{r_\pm}{r})^{7-p}. \tag{2.3}$$

$dx_p^2 \equiv \sum_{i=1}^p dx^i dx^i$, p-dimensional Euclidean space and $d\Omega_{8-p}^2$ is $8-p$-dimensional sphere. Q is an electric charge which couples to $p+2$ form field strength. $Q = \frac{N}{V_{8-p}}$, where N is number of Dp-branes and $V_{8-p} = (2\pi)^{\frac{p+3}{2}}/\frac{p+1}{2}!$, the volume of $8-p$-sphere. ϵ_{8-p} is $8-p$-dimensional anti-symmetric tensor and $*F_{p+2}$ denotes magnetic dual of F_{p+2}. The solution is asymptotically flat and has inner and outer horizons, $r_\pm$ which are determined by mass of the black p-brane and N. r_+ should not be less than r_-, otherwise the black brane has naked singularity. When $r_+ > r_-$, the black brane present finite Hawking temperature which is given by $T_H = \frac{7-p}{4\pi r_+}\left(1-\left(\frac{r_-}{r_+}\right)^{7-p}\right)^{\frac{5-p}{2(7-p)}}$. The extremal limit(zero temperature limit) of the black brane can be achieved by taking $r_+ = r_-$. The dialton solution has non-trivial r-dependence but when $p = 3$, the dilaton becomes arbitrary constant. In the following section, we use black p-brane solution to discuss conceptual derivation of gauge-gravity duality, in particular extremal limit of the solution in $p = 3$ case.

The last subject to discuss is grey-body factors of black p-branes, which is important to discuss the decoupling limits in the follwing section. The classical calculation of black hole radiation by Hawking's semi-classical approximation[8] is

$$d\Gamma_{black\ hole} = \frac{v\sigma_{absorption}}{e^{\frac{\omega}{T_H}} \pm 1}\frac{d^d k}{(2\pi)^d}, \tag{2.4}$$

where v, $\vec{k}$ and ω are the transverse directional velocity, d-dimensional momentum and frequency of the emitted particle from the black hole, and T_H is Hawking temperature of the black hole. $+$ sign at the denominator is for fermionic particle and $-$ sign is for bosonic one. $\sigma_{absorption}$ is the absorption cross section of the black hole for the particle coming from infinity. For the ideal black-body, the absoption cross section is a constant. However black hole is not an ideal black-body. Then $\sigma_{absorption}$ has non-trivial frequency dependence, so it is called grey-body. This feature is qualitatively the same with that of the black p-brane solution. For simple example, let us suppose a massless scalar field coming in from infinity to the black 3-brane. To simplify the calculation, we assume that the the scalar field is s-wave(spherically symmetric wave) and its frequency is low. Then, the absorption cross section is given by

$$\sigma_{absorption} \sim \omega^3. \tag{2.5}$$

In the extremely low frequency, $\sigma_{absorption}$ vanishes and the low energy fields from infinity decouple from the black 3-brane. This result will be used to discuss decoupling limit in the following section.

2.2 AdS/CFT Correspondence

In this section we briefly review a conceptual derivation of AdS/CFT correspondence, the most concrete example of gauge-gravity duality[2, 5]. Consider parallel $D3$-brane stacks in 10-dimensional space-time. There are fundamental open and closed strings in this background too. The closed string massless spectrum provides type IIB-Super string theory whereas the open string massless spectrum on $D3$-brane does $\mathcal{N}$=4 $U(N)$ Super-Yang Mills Theory(SYM).

In the low energy, the effective action of this system can be witten as

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{2.6}$$

where S_{bulk}, S_{brane} and S_{int} are 10-dimensional supergravity action, Yang-Mills action on $D3$-branes and interaction between the fields in 10-dimensional bulk and on the branes respectively. This effective action is constructed by integrating out of all the massive fields keeping massless ones as dynamical fields. In this action, we still keep higher order terms of the low frequency expansion(See Sec.2.1), contributions from massive string spectrum.

In the limit of $m_s \to \infty (l_s \to 0)$, S_{int} disappears because it does not have any leading order terms in the low frequency expansion. The higher derivative terms in S_{bulk} and S_{brane} also disappear, then they become supergravity theory in 10-dimensional flat space-time and pure Super Yang-Mills theory on the branes respectively. These two theories are decoupled because the interaction vanishes. This regime is called decoupling limit(*decoupling limit* 1).

There are another decoupling limit. As mentioned previously, stacks of N $D3$-branes become a source of black hole to produce following geometry:

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{f}}(-dt^2 + \sum_{i=1}^{3} dx_i^2) + \sqrt{f}(dr^2 + r^2 d\Omega_5^2), \\ F_5 &= (1+*)dtdx_1dx_2dx_3dx_4df^{-1}, \\ f &= 1 + \frac{R_{AdS}^4}{r^4}, \quad \text{and} \quad R_{AdS}^4 = g_s l_s^4 N, \end{aligned} \tag{2.7}$$

where R_{AdS} is the radius of AdS space. In this background, there is infinite red shift for the particles to climb up from the horizon of the geometry($r = 0$) to the AdS boundary($r = \infty$). The energy measured on the AdS boundary $E_B(= \frac{\partial}{\partial t})$ is infinitely suppressed by red shift factor, $\sqrt{g_{tt}}$, for any particles with arbitrary high energy, $E_H(= \frac{\partial}{\partial \tau} = f^{\frac{1}{4}} \frac{\partial}{\partial t}$ at $r \sim 0$) near horizon. Therefore, any particles near horizon can be observed as low energy excitations to the boundary observers. These near horizon fields decouple from the low energy modes around far region(the far region is asymptotically flat) in the bulk, because the absorption cross section of black hole vanishes in the low frequency limit(See Eq(2.5)). Again, we have another decoupling limit between any excitations near horizon and the low energy excitations around the far region in the bulk. In this low energy decoupling limit, the former becomes string theory on the near horizon geometry,

$$ds^2 = \frac{u^2}{R_{AdS}^2}(-dt^2 + \sum_{i=1}^{3} dx_i^2) + \frac{R_{AdS}^2 du^2}{u^2} + R_{AdS}^2 d\Omega_5^2, \tag{2.8}$$

whereas the later does supergravity theory on the flat background(*decoupling limit* 2). The near horizon limit of the geometry is obtained by taking a double scaling limit as

$$l_s \to 0, \quad r \to 0, \quad \text{and} \quad u \equiv \frac{r}{l_s^2} = \text{ finite.} \tag{2.9}$$

The near horizon geometry is called $AdS_5 \times S^5$. In Eq(2.8), $d\Omega_5^2$ denotes 5-dimensional sphere and the other factors together constitute 5-dimensional negatively curved space: AdS space-time.

In each decoupling limit, one of the decoupling systems is supergravity in the flat background. Thus, it is natural to argue that the other systems are the same. These two other systems are $\mathcal{N} = 4$ $U(N)$ Super Yang-Mills theory in 4-dimensional space-time and type IIB Super String theory on $AdS_5 \times S^5$ geometry: AdS/CFT correspondence[5].

There are several quantities in both theories. In $\mathcal{N} = 4$ SYM theory, there are gauge coupling, g_{YM} and rank of $U(N)$ gauge group, N. On the other hand, string theory on AdS space contains string coupling g_s, string length scale l_s(or string energy scale m_s), AdS radius R_{AdS} and flux of $D3$-branes N. These quantities have relationships between one and another by the correspondence. The string coupling and SYM's coupling have a relationship as $g_s = g_{YM}^2$. The flux N in the string theory can be identified to the rank N in the SYM's theory. Finally, we have relation as $R_{AdS}^4 = g_s l_s^4 N$ by the supergravity solution(2.7).

Now, we take a limit as $N \to \infty$, $g_s \to 0$ but $\lambda \equiv g_s N$ is kept to be fixed, where λ is called 't Hooft coupling. When 't Hooft coupling becomes much smaller than 1, $\lambda \ll 1$, large N expansion in $N = 4$ SYM is valid and we trust the perturbative analysis in the gauge theory. However, in this case, the curvature radius of AdS space becomes so much smaller than string length scale l_s as

$$\frac{R_{AdS}^4}{l_s^4} = g_s N = \lambda \ll 1. \tag{2.10}$$

that we cannot trust supergravity in the bulk any longer due to huge amount of string excitations. This is because when $\lambda \ll 1$, supergravity modes have enough energy to create any massive string excitations. On the other hand, if $\lambda \gg 1$, in the bulk supergravity description can stay in the valid regime, but large N expansion in the dual gauge theory cannot be applied. As we mentioned in introduction, the chief merit from gauge-gravity duality is that one can analyze one theory in the regime that cannot be studied by usual techniques that we know by its dual theory. The regime of difficulty or easiness can be determined by how large the 't Hooft coupling becomes.

2.3 Bulk Fields-CFT Operators Correspondence

$\mathcal{N} = 4$ $U(N)$ SYM's theory is a conformal field theory, in which the strengths of coupling constants do not change as the energy scale changes. The conformal field theory does not have asymptotic freedom, so the most natural objects to deal with are operators and their correlation functions. According to AdS/CFT correspondence, to each local operator O^i in SYM's theory defined on the AdS boundary, there is a corresponding field ϕ_i in its dual string theory in the bulk[5]. This field ϕ_i is one of the string excitations. There are roughly $O(N^2)$ numbers of such fields in the bulk. Boundary values of these bulk fields at $u = \infty$ can be treated as sources to be turned on which couples to the corresponding operators in the boundary conformal field theory. Therefore, turning on a bulk field ϕ_i can add terms as $\int d^4x\phi_{0i}(x^\mu)O^i(x^\mu)$ to the conformal field theory action, where $\phi_{0i}(x^\mu)$ is boundary value of $\phi_i(u,x^\mu)$. More precisely, it is naturally proposed that

$$< e^{\int d^4x\phi_{0i}(x^\mu)O^i(x^\mu)} >_{CFT} = Z_{Bulkfields}\left|_{\phi_i(r,x^\mu)|_{r=\infty}=\phi_{0i}(x^\mu)}\right. , \tag{2.11}$$

where $Z_{Bulkfields}$ is the partition function of bulk fields from string excitations as

$$Z_{Bulkfields} = \int \Pi_i(D\phi_i)e^{iS_{IIB\ string}(\phi_i)} \tag{2.12}$$

which satisfies the boundary condition, $\phi_i(u,x^\mu)|_{u=\infty} = \phi_{0i}(x^\mu)$ on the AdS boundary, $u=\infty$. The left hand side of Eq(2.11) is expectation value of operator $e^{\int d^4x\phi_{0i}(x^\mu)O^i(x^\mu)}$ in the dual conformal field theory. In particular, in the supergravity regime, $\lambda \gg 1$, we have roughly $O(1)$ number of massless supergravity fields only. In this case, the string partition function $Z_{Bulkfields}$ is approximated to $e^{iS_{sugra}}$, where S_{sugra} is the supergravity action. In this approximation, we ignore all the string massive modes($l_s \to 0$) and loop contributions($N \to \infty$).

The correlation functions of the operator O^i can be obtained by applying usual field theory techniques. To get n-point correlation functions, we act functional derivatives on the string theory action as

$$\begin{aligned} < O^{i_1}(x^\mu)...O^{i_n}(x^\mu) > &= \frac{1}{n!}\frac{\delta}{\delta\phi_{0i_1}(x^\mu)}...\frac{\delta}{\delta\phi_{0i_n}(x^\mu)} < e^{\int d^4x\phi_{0i}(x^\mu)O^i(x^\mu)} >_{CFT} \\ &= \frac{1}{n!}\frac{\delta}{\delta\phi_{0i_1}(x^\mu)}...\frac{\delta}{\delta\phi_{0i_n}(x^\mu)} Z_{Bulkfields}\left|_{\phi_i(u,x^\mu)|_{u=\infty}=\phi_{0i}(x^\mu)}\right. . \end{aligned} \tag{2.13}$$

The second equality comes from the duality(2.11). In the supergravity limit, Eq(2.13) becomes

$$< O^1(x^{\mu_1})O^2(x^{\mu_2})...O^n(x^{\mu_3}) >= \frac{1}{n!}\frac{\delta}{\delta\phi_{0i_1}(x^{\mu_1})}\frac{\delta}{\delta\phi_{0i_2}(x^{\mu_2})}...\frac{\delta}{\delta\phi_{0i_3}(x^{\mu_n})}S_{sugra}(\phi_{0i}). \tag{2.14}$$

In the following, we would discuss a simple example for computing the correlation functions of massive scalar field.

Massive Scalar field in AdS Space

We start with massive scalar field action in AdS_{d+1} space-time[9], which has a form of

$$S_\phi = \int d^{d+1}x\sqrt{-g}\left(g^{MN}\partial_M\phi(x^P)\partial_N\phi(x^P) + m^2\phi^2(x^P)\right), \tag{2.15}$$

where M,N and P are d+1 dimensional bulk space-time indices. The equation of motion is

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\phi(x^P)\right) - m^2\phi(x^P) = 0 \tag{2.16}$$

In the following calculations, we use Poincare coordinate for the background metric g_{MN}, of which form is

$$ds^2 = \frac{1}{z^2}(-dt^2 + dx_{d-1}^2 + dz^2), \tag{2.17}$$

where $dx_{d-1}^2 \equiv \sum_{i=1}^{d-1} dx_i^2$ and the AdS radial coordinate $z \sim \frac{1}{u}$(See AdS metric(2.8) and the scaling limit(2.9)). Then, the AdS boundary is at $z = 0$ and Poincare horizon is at $z = \infty$. To solve the scalar field equation, it is convenient to separate the AdS radial variable z from the boundary coordinate $x^\mu \equiv (t, \vec{x})$ in which we set the solution to the boundary direction to be plane waves as

$$\phi(r, x^\mu) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}} \psi(z) \tag{2.18}$$

Substituting this into Eq(2.16), we get

$$0 = z^2 \partial_z^2 \psi(z) + z\partial_z \psi(z) - \left(m^2 + \frac{d^2}{4} + q^2\right) \psi(z), \tag{2.19}$$

where $q^\mu = (\omega, \vec{k})$ and $q^2 = q_\mu q_\nu \eta^{\mu\nu} = -\omega^2 + \vec{k}^2$. The solutions of Eq(2.19) can be sorted by the signs of q^2. When $q^2 > 0$, there are two independent solutions as

$$\phi_1(x^M) = e^{-i\omega t + i\vec{k}\vec{x}} z^{\frac{d}{2}} K_\nu(qz), \text{ and } \phi_2(x^M) = e^{-i\omega t + i\vec{k}\vec{x}} z^{\frac{d}{2}} I_\nu(qz), \tag{2.20}$$

where $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2}$. The solution, ϕ_2, is ill-behaved in the interior, which blows up exponentially as $z \to \infty$ whereas ϕ_1 is well-behaved. Thus, ϕ_1 is the only acceptable solution. As ϕ_1 approach the AdS boundary, it behaves as $\phi_1 \sim z^{2h_-}$, where

$$h_\pm = \frac{d}{4} \pm \frac{\nu}{2}. \tag{2.21}$$

ϕ_1 is divergent as $z \to 0$ unless $m = 0$. When $m = 0$, $\phi_1 = \text{const}$ asymptotically. These kinds of asymptotically non-vanishing solutions are called $non-normalizable$ solutions. When $q^2 < 0$, there are also two linearly independent solutions as

$$\phi^\pm(x^M) = e^{-i\omega t + i\vec{k}\vec{x}} z^{\frac{d}{2}} J_{\pm\nu}(|q|z), \tag{2.22}$$

if ν is non-integral or

$$\phi^+(x^M) = e^{-i\omega t + i\vec{k}\vec{x}} z^{\frac{d}{2}} J_\nu(|q|z) \text{ and } \phi^-(x^M) = e^{-i\omega t + i\vec{k}\vec{x}} z^{\frac{d}{2}} Y_\nu(|q|z), \tag{2.23}$$

if ν is an integral number. These solutions are well-behaved in the interior. The solution, $\phi^+ \sim z^{2h_+}$ which vanishes as $z \to 0$. This kind of solution is called *normalizable* solution whereas ϕ^- is non-normalizable solution.

Operator Expectation Values

In this section, we precisely compute correlation functions of CFT operator O. In particular, it would be pointed out that the normalizable and non-normalizable solutions in the bulk supergravity are duals to states and sources in the dual conformal field theory respectively by analyzing the massive scalar field solutions obtained in the previous section. It is convenient to deal with the scalar field solutions in momentum space on the boundary because the boundary derivatives ∂_μ acting on the scalar field can be replaced by boundary momentum q_μ multiplying that. Therefore, we perform Fourier transform as $\phi(z,q_\alpha)=\int_{-\infty}^{\infty}d^4xe^{-iq_\nu x^\nu}\phi(z,x^\alpha)$. We set boundary of AdS space to be at $z=\epsilon$ and take a limit $z\rightarrow 0$ in the last step not to lose any information. With this, the Dirichlet boundary condition on the AdS boundary becomes $\phi(q,z)|_{z=\epsilon}=\epsilon^{2h_-}e^{iq_\alpha x^\alpha}\phi_0(q_\alpha)$. The extra factor of ϵ^{2h_-} comes from the radial coordinate dependence of the scalar field solution. We choose solution with $q^2<0$ case because for $q^2>0$ there is no normalizable solution. For $q^2<0$, the most general solution with Dirichlet boundary condition is given by

$$\phi(q_\alpha,z)=\Phi(q_\alpha)e^{iq_\alpha x^\alpha}\phi^+(q_\alpha,z)+K(q_\alpha,\epsilon,z)\epsilon^{2h_-}e^{iq_\alpha x^\alpha}\phi_0(q_\alpha), \tag{2.24}$$

where

$$K(q_\alpha,\epsilon,z)=\frac{\phi^-(q_\alpha,z)+A(q_\alpha)\phi^+(q_\alpha,z)}{\phi^-(q_\alpha,\epsilon)+A(q_\alpha)\phi^+(q_\alpha,\epsilon)}. \tag{2.25}$$

$K(q_\alpha,\epsilon,z)$ is boundary-bulk propagator, which becomes 1 at $z=\epsilon$. The first term in Eq(2.24) is normalizable solution of which Fourier coefficient is $\Phi(q_\alpha)$. The second term is non-normalizable solution in which there is an ambiguity from normalizable solutions with arbitrary coefficient $A(q_\alpha)$. This is because one can freely add such a term to non-normalizable solution without changing the Dirichlet boundary condition. To compute correlation functions, we substitute Eq(2.24) into the supergravity action(2.15). Then, the action becomes

$$\begin{aligned} S_\phi(\Phi(q_\alpha),\phi_0(q_\alpha)) &= \frac{1}{2}\int_{z=\epsilon}d^dqd^dq'\delta^d(q_\alpha+q'_\alpha)z^{1-d} \\ &\times\ \partial_z\left(\Phi(q_\alpha)\phi^+(q_\alpha,z)K(q'_\alpha,\epsilon,z)\phi_0(q'_\alpha)+\phi_0(q_\alpha)K(q'_\alpha,\epsilon,z)\phi_0(q'_\alpha)\right). \end{aligned} \tag{2.26}$$

1-point function can be read off from the action by taking functional derivative respect to ϕ_0 once(See Eq(2.13)), which is given by

$$< O(q_\alpha) >_{\phi_0}=\partial_z\left(\Phi(-q_\alpha)\phi^+(-q_\alpha,z)K(q_\alpha,\epsilon,z)+\phi(-q_\alpha)K_A(q_\alpha,\epsilon,z)\right)|_{z\rightarrow\epsilon}, \tag{2.27}$$

where the subscript on the left hand side denotes that we do not turn off the source terms. There are two different contributions to this 1-point function. The first term is proportional to the Fourier coefficient of normalizable solution whereas the second term does to that of non-normalizable one which is source term of the dual operator O in the boundary conformal field theory. If we turn off the non-normalizable solution, there become no sources at all. Without any sources, vacuum expectation values of 1-point functions should become zero, but the first term in Eq(2.27) still survives. Non-zero expectation value of 1-point function can be obtained only when the states that sandwich on the corresponding operator become excited states. This means that turning on the normalizable solutions in the bulk correspondes to excited states in the dual conformal field theory. Let us summarize this as

- Non-normalizable solutions correspond to turning on sources,

- Normalizable solutions correspond to turning on excited states.

By the same method, 2-point correlation function can be computed, which has a form of

$$< O(q_\alpha)O(k_\alpha) >_{\phi_0}=< O(q_\alpha) >_{\phi_0}< O(k_\alpha) >_{\phi_0} +\delta^d(q_\alpha + k_\alpha)\partial_z K(q_\alpha, \epsilon, z)|_{z\to\epsilon}. \quad (2.28)$$

2.4 Retarded Green's function and Finite Temperature Field Theory

In this section, we extend our discussion of the correlation functions to more specific correlators. The most natural extension of n-point correlators of operator O is computing those correlation functions in the thermal background. Then, we will discuss how to compute one of the correlators in the conformal field theory, retarded green's function from its dual gravity theory, in particular in the thermal background[10].

Retarded Correlators

In the linear response theory, if there is an external source S which couples to an operator O, then the response function of this source, $< \delta O >$ is given by

$$< \delta O >=< O >_S - < O >_0= \int_{-\infty}^{\infty} dt' \int d^3x' G_R(x, t; x', t')S(x', t') \quad (2.29)$$

where

$$G_R(x, t; x', t') = \Theta(t - t') < [O(x, t), O(x', t')] >, \quad (2.30)$$

where G_R is called Retarded Green's Function. $< O >_S$ indicates expectation value of the operator O with the source term S whereas $< O >_0$ does that without turning on S. $\Theta(t - t')$ is step function, which is 1 when $t > t'$ and becomes zero otherwise, and the square bracket denotes commutator. It is firstly proposed by Dam T. Son and Andrei O. Starinets that the retarded correlator in conformal field theory can be calculated from its dual gravity theory[10]. In the previous section, we have computed a massive scalar field solution in AdS background. In this case, two different boundary conditions are imposed: one is that solutions are regular in the interior, another is Dirichlet boundary condition of the solutions as the solutions approach AdS boundary. However, the regularity condition cannot completely fix the solutions. As seen in Eq(2.24) and Eq(2.25), the solutions still have ambiguity with arbitrary coefficients as $\Phi(q_\alpha)$ and $A(q_\alpha)$ even after requesting the boundary conditions. Therefore, the correlation functions that we get previously are the most general type of those. What Son and Starinets have shown is that types of the correlators depend on specific boundary conditions in the interior. In this case, the most natural boundary condition is incoming boundary condition at the horizon, $z = \infty$. This incoming boundary condition is that the waves can only travel into the black hole but cannot come out from there. It is shown that a correlation function obtained by the method in Sec.2.3 obeying incoming boundary condition at the horizon of black hole correspondes to the retarded green's function in the dual conformal field theory. One can also get advanced green's function by imposing the outgoing wave boundary condition at the horizon.

The precise way of computing retarded green's function is following. To simplify the situation, let us deal with massive scalar field. The general calculations obtaining correlators are already demonstrated in Sec.2.3. Therefore, we just provide crucial intermediate steps to obtain retarded green's function. Firstly, solve the massive scalar field equation and request Dirichlet boundary condition as before. For the zero temperature case, we just use the scalar field solutions in Sec.2.3. Secondly, we request incoming boundary condition of the solutions at the horizon. The precise boundary condition at the horizon is

- For $q^2 > 0$, there is no normalizable solution. Therefore, we just request regularity at the black hole horizon.

- For $q^2 < 0$, there are both normalizable and non-normalizable solutions, $\phi^{\pm}$. In this case, we request incoming boundary condition at the black hole horizon. The solutions, $\phi^{\pm}$, do not individually satisfy incoming boundary condition at the horizon, but linear combination of those can do. ξ_1 and ξ_2, linear combinations of $\phi^{\pm}$ as

$$\xi_1(x^M) = \phi^+(x^M) + i\phi^-(x^M) \text{ for } \omega > 0, \tag{2.31}$$
$$\xi_2(x^M) = \phi^+(x^M) - i\phi^-(x^M) \text{ for } \omega < 0. \tag{2.32}$$

  can satisfy the incoming boundary condition. As we approach black hole horizon, $z \to \infty$, $\xi_1 \sim z^{\frac{d-1}{2}} e^{-i|\omega|t+i|q|z}$ and $z \to \infty$, $\xi_2 \sim z^{\frac{d-1}{2}} e^{i|\omega|t-i|q|z}$. They are both ingoing waves at the horizon.

Substitute ξ_1 and ξ_2 into the scalar field action(2.15) and take functional derivatives. Then 2-point correlation functions are obtained as

$$\begin{aligned} G_R &\sim q^4 ln(q^2) \text{ for } q^2 > 0, \\ G_R &\sim q^4(ln(q^2) - i\pi \text{ sign}(\omega)) \text{ for } q^2 < 0, \end{aligned} \tag{2.33}$$

in the momentum space on the boundary.

Quasi-normal Modes of Solutions

Quasi-normal modes in the AdS black hole background are defined as fields satisfying equations of motion to obey following two boundary conditions: those become purely incoming waves at the black hole horizon and normalizable solutions(as $z \to 0$, the solutions vanish). To be more precise, let us consider previous calculations of the scalar field. Suppose that a scalar field solution satisfying incoming wave boundary condition at the black hole horizon has a form of

$$\xi(q_\alpha, z) = \Theta(q_\alpha)\phi^+(z) + \Xi(q_\alpha)\phi^-(z). \tag{2.34}$$

Near the AdS boundary, the scalar field solution behaves as

$$\xi(q_\alpha, z \to 0) = \Theta(q_\alpha)z^{h_+} + ... + \Xi(q_\alpha)z^{h_-} + ... \tag{2.35}$$

Plug the asymptotic form of ξ into the scalar field action(2.15), and compute retarded Green's function. Then, one can obtain

$$G_R(q_\alpha) \sim (h_+ - h_-)\frac{\Theta(q_\alpha)}{\Xi(q_\alpha)}\epsilon^{h_+-h_--1} + \text{ some contact terms } + O(\epsilon). \tag{2.36}$$

Ignoring the contact terms, the leading term of G_R is proportional to the ratio of the coefficients between the normalizable solution and the non-normalizable one. In the usual field theory, poles of the correlator provide on-shell dispersion relation of certain propagating waves. For example, if we have acoustic waves as a low frequency modes in the conformal fluid dynamics on the boundary, then the retarded correlator provides response functions of these acoustic waves. The retarded correlator has poles when the phase velocity of this acoustic waves satisfy $\frac{\omega}{k} = v_s$, where v_s is the velocity of the sound waves[44]. The poles of the retarded green's function correspond to zeros of the non-normalizable coefficient, $\Xi(q_\alpha)$. $\Xi(q_\alpha) = 0$ is precisely vanishing Dirichlet boundary condition on the AdS boundary, which defines quasi normal modes of solutions. Therefore, we conclude that turning on quasi-normal modes correspond to turning on certain classical modes(which satisfy certain equation of motion on the boundary, similarly a certain dispersion relation) in the boundary conformal field theory.

Retarded Green's Function in the Thermal Background

In this section, we discuss calculating retarded green's function in the thermal background. The basic idea is that introducing temperature in boundary conformal field theory is the dual of non-extremal AdS black hole geometry in the bulk gravity theory, which is given by

$$ds^2 = \frac{R_{AdS}^2}{z^2}\left(-f(z)dt^2 + \sum_{i=1}^{3} dx_i^2 + \frac{dz^2}{f(z)}\right) + R_{AdS}^2 d\Omega_5^2, \tag{2.37}$$

where

$$f(z) = 1 - \frac{z_h^4}{z^4}. \tag{2.38}$$

The horizon of this black hole is located at $z = z_h$. One can also calculate Hawking temperature of this black hole geometry, which is given by $T_H = \frac{1}{\pi z_h}$. The black hole temperature is identified to that in the conformal field theory, then we have thermal field theory as a dual of this non-extremal black hole. The way of calculations of retarded green's function is the same with the steps shown in the previous sections, except switching pure AdS geometry to AdS black holes.

Chapter 3 Slowly Varying Dilaton Cosmologies and their Field Theory Duals

The AdS/CFT correspondence [2, 3, 4] provides us with a non-perturbative formulation of quantum gravity. One hopes that it will shed some light on the deep mysteries of quantum gravity, in particular on the question of singularity resolution.

Motivated by this hope we consider a class of time dependent solutions in this paper which can be viewed as deformations of the $AdS_5 \times S^5$ background in IIB string theory. These solutions are obtained by taking the boundary value of the dilaton in AdS space to become time dependent [1] . We are free to take the boundary value of the dilaton to be any time dependent function. To keep the solutions under analytical control though we take the rate of time variation of the dilaton to be small compared to the radius of AdS space, R_{AdS}. This introduces a small parameter ϵ and we construct the bulk solution in perturbation theory in ϵ. The resulting solutions are found to be well behaved. In particular one finds that no black hole horizon forms in the course of time evolution. The metric and dilaton respond on a time scale of order R_{AdS} which is nearly instantaneous compared to the much slower time scale at which the boundary value of the dilaton varies. For dilaton profiles which asymptote to a constant in the far future one finds that all the energy that is sent in comes back out and the geometry settles down eventually to that of AdS space. What makes these solutions non-trivial is that by waiting for a long enough time, of order $\frac{R_{AdS}}{\epsilon}$, a big change in the boundary dilaton can occur. The solutions probe the response of the bulk to such big changes.

Consider an example of this type where the boundary dilaton undergoes a big change making the 'tHooft coupling[2] of order unity or smaller at intermediate times,

$$\lambda \equiv g_s N \leq O(1), \tag{3.1}$$

when [3] $t \simeq 0$, before becoming large again in the far future. As was mentioned above, the bulk responds rapidly to the changing boundary conditions and within a time of order R_{AdS} the dilaton everywhere in the bulk then becomes small and meets the condition, Eq.(3.1). Now the supergravity solution receives α' corrections in string theory, these are important when R_{AdS} becomes of order the string scale. Using the well known relation,

$$R_{AdS}/l_s \sim (g_s N)^{\frac{1}{4}} \tag{3.2}$$

we then find that once Eq.(3.1) is met the curvature becomes of order the string scale everywhere along a space-like slice which intersects the boundary. As a result the

[1]It is important in the subsequent discussion that we work in global AdS_5 with the boundary $S^3 \times R$.

[2]When we refer to the 'tHooft coupling we have the gauge theory in mind and accordingly by the dilaton in this context we will always mean its boundary value.

[3]Here N is the number of units of flux in the bulk and the rank of the gauge group in the boundary theory.

supergravity approximation breaks down along this slice and the higher derivative corrections becomes important for the subsequent time development. This break down of the supergravity approximation is the sense in which a singularity arises in these solutions.

In contrast the curvature in units of the 10-dim. Planck scale l_{Pl} (or the 5-dim Planck scale) remains small for all time. The radius R_{AdS} in l_{Pl} units is given by,

$$R_{AdS}/l_{Pl} \sim N^{\frac{1}{4}} \tag{3.3}$$

We keep N to be fixed and large throughout the evolution, this then keeps the curvature small in Planck units [4]. The solutions we consider can therefore be viewed in the following manner: the curvature in Planck units in these solutions stays small for all time, but for a dilaton profile which meets the condition Eq.(3.1) the string scale in length grows and becomes of order the curvature scale at intermediate times. At this stage the geometry gets highly curved on the string scale. We are interested in whether a smooth spacetime geometry can emerge again in the future in such situations.

It is worth relating this difference in the behavior of the curvature as measured in string and Planck scales to another fact. We saw that when the curvature becomes of order the string scale α' corrections become important. The second source of corrections to the supergravity approximation are quantum loop corrections. Their importance is determined by the parameter $1/N$. Since N is kept fixed and large these corrections are always small. From Eq.(3.3) we see that this ties into the fact that the AdS radius stays large in Planck units.

To understand the evolution of the system once the curvature gets to be of order the string scale we turn to the dual gauge theory. The gauge theory lives on an S^3 of radius R and the slowly varying dilaton maps to a Yang-Mills coupling which varies slowly compared to R. Since these are the only two length scales in the system the slow time variation suggests that one can understand the resulting dynamics in terms of an adiabatic approximation.

In fact we find it useful to consider two different adiabatic perturbation theories. The first, which we call quantum adiabatic perturbation theory is a good approximation when the parameter ϵ satisfies the condition,

$$N\epsilon \ll 1. \tag{3.4}$$

Once this condition is met the rate of change of the Hamiltonian is much smaller than the energy gap between the ground state and the first excited state in the gauge theory. As a result the standard text book adiabatic approximation in quantum mechanics applies and the system at any time is, to good approximation, in the ground state of the *i*nstantaneous Hamiltonian. In the far future, when the time dependence turns off, the state settles into the ground state of the resulting $\mathcal{N} = 4$ SYM theory, and admits a dual description as a smooth AdS space.

[4] The backreaction corrects the curvature but these corrections are suppressed in ϵ.

Note that this argument holds even when the 'tHooft coupling at intermediate times becomes of order unity or smaller. The fact that the states of the time independent $\mathcal{N} = 4$ SYM theory furnish a unitary representation of the conformal group guarantees that the spectrum has a gap of order $1/R$ for all values of the Yang Mills coupling, [11], see also, [12], [5]. Thus as long as Eq.(3.4) is met the conditions for this perturbation theory apply. As a result, we learn that for very slowly varying dilaton profiles which meet the condition, Eq.(3.4), the geometry after becoming of order the string scale at intermediate times, again opens out into a smooth AdS space in the far future.

The supergravity solutions we construct are controlled in the approximation,

$$\epsilon \ll 1. \tag{3.5}$$

This is different, and much less restrictive, than the condition stated above in Eq.(3.4) for the validity of the quantum adiabatic perturbation theory. In fact one finds that a different perturbation theory can also be formulated in the gauge theory. This applies when the conditions,

$$N\epsilon \gg 1, \quad \epsilon \ll 1 \tag{3.6}$$

are met. This approximation is classical in nature and arises because the system is in the large N-limit (otherwise Eq.(3.6) cannot be met). We will call this approximation the "Large N Classical Adiabatic Perturbation Theory" (LNCAPT) below. The behavior of the system in this approximation reproduces the behavior of the supergravity solutions for cases where the 'tHooft coupling is large for all times.

Let us now discuss this approximation in more detail. Each gauge invariant operator in the boundary theory gives rise to an infinite tower of coupled oscillators whose frequency grows with growing mode number. The gauge invariant operators are dual to bulk modes. The infinite tower of oscillators which arises for each operator is dual to the infinite number of modes, with different radial wave functions and different frequency, which arise for each bulk field. Of particular importance is the operator dual to the dilaton $\hat{\mathcal{O}}$ and the modes which arise from it. The time varying boundary dilaton results in a driving force for these oscillators. When $N\epsilon \gg 1$, these oscillators are excited by the driving force into a coherent state with a large mean occupation number of quanta, of order $N\epsilon$, and therefore behave classically. This is a reflection of the fact that at large N, the system behaves classically : coherent states of these oscillators correspond to classical configurations (see e.g. Ref [13]).

Usually a reformulation of the boundary theory in terms of such oscillators is not very useful, since these oscillators would have a nontrivial operator algebra which would signify that the bulk modes are interacting. Simplifications happen in low dimensional situations like Matrix Quantum Mechanics [14] where one is led to a collective field theory in 1+1 dimensions as an explicit construction of the holographic map [15]. Even in this situation, the collective field theory is a nontrivial interacting theory, i.e. the oscillators are coupled. In our case there are an infinite number of collective fields which would seem to make the situation hopeless.

In our setup, however, the slowness of the driving force simplifies the situation drastically. The source couples directly to the dilaton in the bulk, and when $\epsilon \ll 1$, to lowest order the response of the dilaton as well as the other fields is *linear* and independent of each other. This will be clear in the supergravity solutions we present below. This implies that to lowest order in ϵ, the oscillators which are dual to these modes are really *harmonic oscillators* which are decoupled from each other.

The resulting dynamics is then well approximated by the classical adiabatic perturbation theory, which we refer to as the LNCAPT as mentioned above. The criterion for its applicability is that the driving force varies on a time scale much slower than the frequency of each oscillator. In particular if the frequency of the driving force is of order that of the oscillators one would be close to resonance and the perturbation theory would break down. In our case this condition for the driving force to vary slowly compared to the frequency of the oscillators, becomes Eq.(3.5). When this condition is met, the adiabatic approximation is valid for all modes - even those with the lowest frequency. The expectation value of the energy and the operator dual to the dilaton, $\hat{\mathcal{O}}$, can then be calculated in the resulting perturbation theory and we find that the leading order answers in ϵ agree with the supergravity calculations [5].

Having understood the supergravity solutions in the gauge theory language we turn to asking what happens if the 'tHooft coupling becomes of order unity or smaller at intermediate times (while still staying in the parametric regime Eq.(3.6)). The new complication is that additional oscillators now enter the analysis. These oscillators correspond to string modes in the bulk. When the 'tHooft coupling becomes of order unity their frequencies can become small and comparable to the oscillators which are dual to supergravity modes.

At first sight one is tempted to conclude that these additional oscillators do not change the dynamics in any significant manner and the system continues to be well approximated by the large N classical adiabatic approximation. The following arguments support this conclusion. First, the anharmonic terms continue to be of order ϵ and thus are small, so that the oscillators are approximately decoupled. Second, the existence of a gap of order $1/R$ for all values of the 'tHooft coupling, which we referred to above, ensures that the driving force varies much more slowly than the frequency of the additional oscillators, thus keeping the system far from resonance. Finally, one still expects that in the parametric regime, Eq.(3.6), an $O(N\epsilon)$ number of quanta are produced keeping the system classical. These arguments suggest that the system should continue to be well approximated by the LNCAPT. In fact, since the additional oscillators do not directly couple to the driving force produced by the time dependent dilaton, but rather couple to it only through anharmonic terms which are subdominant in ϵ, their effects should be well controlled in an ϵ expansion. If these arguments are correct the energy which is pumped into the system initially should then get completely pumped back out and the system should settle into the ground state of the final $\mathcal{N} = 4$ theory in the far future. The dual description in the far future would then be a smooth AdS_5 space-time.

However, further thought suggests another possibility for the resulting dynamics

[5] More precisely, both the supergravity and the forced oscillator calculations need to be renormalized to get finite answer. One finds that after the counter terms are chosen to get agreement for the standard two point function (which measures the response for a small amplitude dilaton perturbation) the expectation value of the energy and $\hat{O}$, agree.

which is of a qualitatively different kind. This possibility arises because, as was mentioned above, when the 'tHooft coupling becomes of order unity string modes can get as light as supergravity modes. This means that the frequency of some of the oscillators dual to string modes can become comparable to oscillators dual to supergravity modes, and thus the string mode oscillators can get activated. Now there are many more string mode oscillators than there are supergravity mode oscillators, since the supergravity modes correspond to chiral operators in the gauge theory which are only $O(1)$ in number, while the string modes correspond to non-chiral operators which are $O(N^2)$ in number. Thus once string mode oscillators can get activated there is the possibility that many new degrees of freedom enter the dynamics.

With so many degrees of freedom available the system could thermalized at least in the large N limit. In this case the energy which is initially present in the oscillators that directly couple to the dilaton would get equi-partitioned among all the degrees of freedom. The subsequent evolution would be dissipative and this energy would not be recovered in the far future. At late times, when the 'tHooft coupling becomes big again, the gravity description of the dissipative behavior depends on how small is ϵ. From the calculations done in the supergravity regime one knows that the total energy that is produced is of order $N^2\epsilon^2$. When $N\epsilon \gg 1$, but $\epsilon \ll (g_{YM}^2 N)^{-7/8}$ the result is likely to be a gas of string modes. However if $\epsilon > (g_{YM}^2 N)^{-7/8}$, the energy is sufficient to form a small black hole (with horizon radius smaller than R_{AdS}). A big black hole cannot form since this would require an energy of the order of N^2, and $\epsilon \ll 1$ always. Thus, in the far future, once the 'tHooft coupling becomes large again, the strongest departure from normal space-time would be the presence of a small black hole in AdS space. The small black hole would eventually disappear by emitting Hawking radiation but that would happen on a much longer time scale of order $N^2 R_{AdS}$.

It is difficult for us to settle here which of the two possibilities discussed above, either adiabatic non-dissipative behavior well described by the LNCAPT, or dissipative behavior with organized energy being lost in heat, is the correct one. One complication is that the rate of time variation which is set by ϵ is also the strength of the anharmonic couplings between the oscillators. In thermodynamics, working in the microcanonical ensemble, it is well known that with energy of order $N^2\epsilon^2$ the configuration which entropically dominates is a small black hole [6]. This suggests that if the time variation in the problem were much smaller than the anharmonic terms a small black hole would form. However, in our case their being comparable makes it a more difficult question to decide. One should emphasize that regardless of which possibility is borne out our conclusion is that most of the space time in the far future is smooth AdS, with the possible presence of a small black hole.

Let us end with some comments on related work. The spirit of our investigation is close to the work on AdS cosmologies in [16] and related work in [17] - [20]. See also [21], [22], [23] for additional work. Discussion of cosmological singularities in the context of Matrix Theory appears in [24].

The supergravity analysis we describe is closely related to the strategy which was

[6] At least when the 'tHooft coupling is big enough so that supergravity can be trusted.

used in the paper [25], for finding forced fluid dynamics solutions; in that case one worked with an infinite brane at temperature T and the small parameter was the rate of variation of the dilaton (or metric) compared to T. Our regime of interest is complementary to that in [26] where the dilaton was chosen to be small in amplitude, but with arbitrary time dependence and which leads to formation of black holes in supergravity for a suitable regime of parameters.

This paper is organized as follows. In section §2 we find the supergravity solutions and use them to find the expectation value of operators in the boundary theory like the stress energy and $\hat{\mathcal{O}}$ in §3. The quantum adiabatic perturbation theory is discussed in §4. A forced harmonic oscillator is discussed in §5. This simple system helps illustrate the difference between the two kinds of perturbation theory and sets the stage for the discussion of the The Large N classical adiabatic approximation in §6. Conclusions and future directions are discussed in §7. There are three appendices which contains details of derivation of some of the formula in the main text.

3.1 The Bulk Response

In this section we will calculate the deformation of the supergravity solution in the presence of a slowly varying time dependent but spatially homogeneous dilaton specified on the boundary. This will be a reliable description of the time evolution of the system so long as $e^{\Phi(t)}$ never becomes small.

Some General Considerations

IIB supergravity in the presence of the RR five form flux is well known to have an $AdS_5 \times S^5$ solution. In global coordinates this takes the form,

$$ds^2 = -(1 + \frac{r^2}{R_{AdS}^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{AdS}^2}} + r^2 d\Omega_3^2 + R_{AdS}^2 d\Omega_5^2. \tag{3.7}$$

Here R_{AdS} is given by,

$$R_{AdS} = (4\pi g_s N)^{1/4} l_s \sim N^{1/4} l_{pl} \tag{3.8}$$

where l_s is the string scale and $l_{pl} \sim g_s^{1/4} l_s$ is the ten dimensional Planck scale. g_s is the value of the dilaton, which is constant and does not vary with time or spatial position,

$$e^{\Phi} = g_s. \tag{3.9}$$

In the time dependent situations we consider below N will be held fixed. Let us discuss some of our conventions before proceeding. We will find it convenient to work in the 10-dim. Einstein frame. Usually one fixes l_{Pl} to be of order unity in this frame. Instead for our purposes it will be convenient to set

$$R_{AdS} = 1. \tag{3.10}$$

From Eq(3.8) this means setting $l_{Pl} \sim 1/N^{1/4}$. The $AdS_5 \times S^5$ solution then becomes,

$$ds^2 = -(1 + r^2)dt^2 + \frac{1}{(1 + r^2)} dr^2 + r^2 d\Omega_3^2 + d\Omega_5^2, \tag{3.11}$$

for any constant value of the dilaton, Eq.(3.9). Let us also mention that when we turn to the boundary gauge theory we will set the radius R of the S^3 on which it lives to also be unity.

The essential idea in finding the solutions we describe is the following. Consider a situation where Φ varies with time slowly compared to R_{AdS}. Since the solution above exists for any value of g_s and the dilaton varies slowly one expects that the resulting metric at any time t is well approximated by the $AdS_5 \times S^5$ metric given in Eq.(3.11). This zeroth order metric will be corrected due to the varying dilaton which provides an additional source of stress energy in the Einstein equations. However these changes should be small for a slowly varying dilaton and should therefore be calculable order by order in perturbation theory.

Let us make this more precise. Consider as the starting point of this perturbation theory the AdS_5 metric given in Eq.(3.11) and a dilaton profile,

$$\Phi = \Phi_0(t) \tag{3.12}$$

which is a function of time alone. We take $\Phi_0(t)$ to be of the form,

$$\Phi_0 = f(\frac{\epsilon t}{R_{AdS}}) \tag{3.13}$$

where $f(\frac{\epsilon t}{R_{AdS}})$ is dimensionless function of time and ϵ is a small parameter,

$$\epsilon \ll 1. \tag{3.14}$$

The function f satisfies the property that

$$f'(\frac{\epsilon t}{R_{AdS}}) \sim O(1) \tag{3.15}$$

where prime indicates derivative with respect to the argument of f.

When $\epsilon = 0$, the dilaton is a constant and the solution reduces to $AdS_5 \times S^5$. When ϵ is small,

$$\frac{d\Phi_0}{dt} = \frac{\epsilon}{R_{AdS}} f'(\frac{\epsilon t}{R_{AdS}}) \sim \frac{\epsilon}{R_{AdS}} \tag{3.16}$$

so that the dilaton is varying slowly on the scale R_{AdS}, and the contribution that the dilaton makes to the stress tensor is parametrically suppressed [7]. In such a situation the back reaction can be calculated order by order in ϵ. The time dependent solutions we consider will be of this type and ϵ will play the role of the small parameter in which we carry out the perturbation theory. A simple rule to count powers of ϵ is that every time derivative of Φ_0 comes with a factor of ϵ.

The profile for the dilaton we have considered in Eq.(3.12) is S^5 symmetric. It is consistent to assume that the back reacted metric will also be S^5 symmetric with the radius of the S^5 being equal to R_{AdS}. The interesting time dependence will then

[7]The more precise statement for the slowly varying nature of the dilaton, as will be discussed in a footnote before Eq.(3.84), is that its Fourier transform has support at frequencies much smaller than $1/R_{AdS}$.

unfold in the remaining five directions of AdS space and we will focus on them in the following analysis.

The zeroth order metric in these directions is given by,

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_3^3. \tag{3.17}$$

And the zeroth order dilaton is given by Eq.(3.12),

$$\Phi_0 = f(\epsilon t). \tag{3.18}$$

We can now calculate the corrections to this solution order by order in ϵ.

Let us make two more points at this stage. First, we will consider a dilaton profile Φ_0 which approaches a constant as $t \to -\infty$. This means that in the far past the corrections to the metric and the dilaton which arise as a response to the time variation of the dilaton must also vanish. Second, the perturbation theory we have described above is a derivative expansion. The solutions we find can only describe slowly varying situations. This stills allows for a big change in the amplitude of the dilaton and the metric though, as long as such changes accrue gradually. It is this fact that makes the solutions non-trivial.

Corrections to the Dilaton

Let us first calculate the corrections to the dilaton. We can expand the dilaton as,

$$\Phi(t) = \Phi_0(t) + \Phi_1(r,t) + \Phi_2(r,t)\cdots, \tag{3.19}$$

where Φ_0 is the zeroth order profile we start with, given in Eq.(3.13). Φ_1 is of order ϵ, Φ_2 is of order ϵ^2 and so on. The metric can be expanded as,

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)} + \cdots \tag{3.20}$$

where $g_{ab}^{(0)}$ is the zeroth order metric given in Eq.(3.17) and $g_{ab}^{(1)}, g_{ab}^{(2)}$... are the first order, second order etc corrections.

The dilaton satisfies the equation,

$$\nabla^2\Phi = 0. \tag{3.21}$$

Expanding this we find that to order ϵ^2,

$$\nabla_0^2\Phi_0 + \nabla_0^2\Phi_1 + \nabla_1^2\Phi_0 + \nabla_1^2\Phi_1 + \nabla_0^2\Phi_2 = 0. \tag{3.22}$$

Here ∇_0^2 is the Laplacian which arises from the zeroth order metric, and ∇_1^2, ∇_2^2 are the corrections to the Laplacian to order ϵ, ϵ^2 respectively, which arise due to the corrections in the metric. The first term on the left hand side is of order ϵ^2, since it involves two time derivatives acting on Φ_0. The second term is of order [8] ϵ, and so is

[8] It is easy to see that Φ_1, if non-vanishing, must depend on the radial coordinate, this makes $\nabla_0^2\Phi_1$ of order ϵ. Φ_1 would be r dependent for the same reason that Φ_2 in Eq(3.25) is.

the third term. However, we see in §2.3 that the $O(\epsilon)$ correction to the metric and thus ∇_1^2 vanishes. So the second term is the only one of $O(\epsilon)$ and we learn that

$$\Phi_1 = 0. \tag{3.23}$$

The first correction to the dilaton therefore arises at $O(\epsilon^2)$. Eq.(3.22) now becomes,

$$\nabla_0^2 \Phi_0 + \nabla_0^2 \Phi_2 = 0. \tag{3.24}$$

Since Φ_0 preserves the S^3 symmetry of AdS_5, Φ_2 will also be S^3 symmetric and must therefore only be a function of t, r. Further since Φ_2 is $O(\epsilon^2)$ any time derivative on it would be of higher order and can be dropped. Solving Eq.(3.24) then gives,

$$\Phi_2(r,t) = \int^r \frac{dr'}{(r')^3(1+(r')^2)} \left[\int^{r'} \frac{y^3}{1+y^2} dy \;\; \ddot{\Phi}_0(t) + a_1(t) \right] + a_2(t). \tag{3.25}$$

Here $a_1(t), a_2(t)$ are two functions of time which arise as integration "constants".

The integrations in (3.25) can be performed, leading to

$$\begin{aligned} \Phi_2(r,t) &= \frac{1}{4}\ddot{\Phi}_0(t) \left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \ \mathrm{dilog}(1+r^2) \right] \\ &\quad + a_1(t)\frac{1}{2}\left[\log(1+r^2) - \frac{1}{r^2} - 2\log\ r \right] \ + a_2(t). \end{aligned} \tag{3.26}$$

The first term in Φ_2 is regular at $r = 0$, while the term multiplying $a_1(t)$ diverges here. To find a self-consistent solution in perturbation theory Φ_2 must be small compared to Φ_0 for all values of r, we therefore set $a_1 = 0$. The first term in $\Phi_2(r,t)$ has the following expansion for large values of r,

$$\ddot{\Phi}_0(t) \left[\frac{\pi^2}{24} - \frac{1}{4r^2} + \left(\frac{3}{16} + \frac{1}{4}\log\ r \right) \frac{1}{r^4} + \cdots \right]. \tag{3.27}$$

Since we are solving for the dilaton with a specified boundary value $\Phi_0(t)$, $\Phi_2(r,t)$ should vanish at the boundary. This determines $a_2(t)$ to be,

$$a_2(t) = -\frac{\pi^2}{24}\ddot{\Phi}_0(t), \tag{3.28}$$

leading to the final solution

$$\Phi_2(r,t) = \frac{1}{4}\ddot{\Phi}_0(t) \left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \ \mathrm{dilog}(1+r^2) - \frac{\pi^2}{6} \right]. \tag{3.29}$$

The solution is regular everywhere. Since $\mathrm{Lim}_{t\to-\infty}\dot{\Phi}_0(t), \ddot{\Phi}_0(t) = 0$, the correction vanishes in the far past, as required.

Corrections to the Metric

The time varying dilaton provides an additional source of stress energy. The lowest order contribution due to this stress energy is $O(\epsilon)^2$ as we will see below. It then follows, after a suitable coordinate transformation if necessary, that the $O(\epsilon)$ corrections to the metric vanish and the first non-vanishing corrections to it arise at order ϵ^2. The essential point here is that any $O(\epsilon)$ correction to the metric must be r dependent and thus would lead to a contribution to the Einstein tensor of order ϵ, which is not allowed. This is illustrated by the dilaton calculation above, where a similar argument lead to the $O(\epsilon)$ contribution, Φ_1, vanishing. In this subsection we calculate the leading $O(\epsilon^2)$ corrections to the metric.

Before we proceed it is worth discussing the boundary conditions which must be imposed on the metric. As was discussed in the previous subsection we consider a dilaton source, Φ_0, which approaches a constant value in the far past, $t \to -\infty$. The corrections to the metric that arise from such a source should also vanish in the far past. Thus we see that as $t \to -\infty$ the metric should approach that of AdS_5 space-time. Also the solutions we are interested in correspond to the gauge theory living on a time independent $S^3 \times R$ space-time in the presence of a time dependent Yang Mills coupling (dilaton). This means the leading behavior of the metric for large r should be that of AdS_5 space. Changing this behavior corresponds to turning on a non-normalizable component of the metric and is dual to changing the metric of the space-time on which the gauge theory lives.

We expect that these boundary conditions, which specify both the behavior as $t \to -\infty$ and as $r \to \infty$ should lead to a unique solution to the super gravity equations. The former determine the normalizable modes and the latter the non-normalizable modes. This is dual to the fact that in the gauge theory the response should be uniquely determined once the time dependent Lagrangian is known (this corresponds to the fixing the non-normalizable modes) and the state of the system is known in the far past(this corresponds to fixing the normalizable modes).

Since Φ_0 is S^3 symmetric, we can consistently assume that the corrections to the metric will also preserve the S^3 symmetry. The resulting metric can then be written as,

$$ds^2 = -g_{tt}(t,r)dt^2 + g_{rr}(t,r)dr^2 + 2g_{tr}(t,r)dtdr + R^2 d\Omega^2. \tag{3.30}$$

Now as is discussed in Appendix A up to $O(\epsilon^2)$ we can consistently set $g_{tr} = 0$. In addition we can to this order set $R^2 = r^2$. Below we also use the notation,

$$g_{tt} \equiv e^{2A(t,r),} \tag{3.31}$$

$$g_{rr} \equiv e^{2B(t,r)}. \tag{3.32}$$

The metric then takes the form,

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + r^2 d\Omega^2. \tag{3.33}$$

The trace reversed Einstein equation are:

$$R_{AB} = \Lambda g_{AB} + \frac{1}{2}\partial_A\Phi\partial_B\Phi. \tag{3.34}$$

In our conventions,

$$\Lambda = -4. \tag{3.35}$$

To order ϵ^2 we can set $\Phi = \Phi_0$ in the second term on the RHS.

A few simple observations make the task of computing the curvature components to $O(\epsilon^2)$ much simpler. As we mentioned above the first corrections to the metric should arise at $O(\epsilon^2)$. To order ϵ^2 the metric is then

$$g_{ab}(t,r) = g^{(0)}_{ab}(r) + g^{(2)}_{ab}(t,r). \tag{3.36}$$

Now the zeroth order metric, $g^{(0)}_{ab}$, is time independent. The time derivatives of $g^{(2)}_{ab}$ are non-vanishing but of order ϵ^3 and thus can be neglected for calculating the curvature tensor to this order. As a result for calculating the curvature components to order ϵ^2 we can neglect all time derivatives of the metric, Eq.(3.36).

Before proceeding we note that the comments above imply that the equations determining the second order metric components schematically take the form,

$$\hat{O}(r) g^{(2)}_{ab} = f_{ab}(r)\dot{\Phi}_0^2 \tag{3.37}$$

where $\hat{O}(r)$ is a second order differential operator in the radial variable, r. As a result the solution will be of the form,

$$g^{(2)}_{ab} = \mathcal{F}(r)_{ab}\dot{\Phi}_0^2, \tag{3.38}$$

where $\mathcal{F}(r)$ are functions of r which arise by inverting $\hat{O}(r)$. We see that the corrections to the metric at time t are determined by the dilaton source Φ_0 at the same instant of time time t. Note also that since we are only considering a dilaton source Φ_0 which vanishes in the far past, the solution Eq.(3.38) correctly imposes the boundary condition that $g^{(2)}_{ab}$ vanishes in far past and the metric becomes that of AdS_5.

Bearing in mind the discussion above, the curvature components are now easy to calculate. The $t-t$ component of Eq.(3.34) gives,

$$\frac{(A'e^{(A-B)})'}{e^{(A+B)}} + 3\frac{A'e^{-2B}}{r} = \frac{\dot{\Phi}_0^2}{2}e^{-2A} + 4. \tag{3.39}$$

The $r-r$ component gives,

$$-\frac{(A'e^{(A-B)})'}{e^{(A+B)}} + 3\frac{B'e^{-2B}}{r} = -4. \tag{3.40}$$

The component with legs along the S^3 gives,

$$\frac{B'-A'}{e^{2B}r} + \frac{2}{r^2}(1-e^{-2B}) = -4. \tag{3.41}$$

In these equations primes indicates derivative with respect to r and dot indicates derivative with respect to time.

Adding the $t-t$ and $r-r$ equations gives,

$$3(A'+B')\frac{e^{-2B}}{r}=\frac{\dot{\Phi}_0^2}{2}e^{-2A}. \tag{3.42}$$

Eq.(3.41) and Eq.(3.42) then lead to

$$\frac{2B'e^{-2B}}{r}-\frac{1}{6}\dot{\Phi}_0^2e^{-2A}+\frac{2}{r^2}(1-e^{-2B})=-4. \tag{3.43}$$

This is a first order equation in B. Integrating we get to order ϵ^2,

$$e^{-2B}=1+r^2+\frac{c_1}{r^2}-\frac{1}{6}\frac{\dot{\Phi}_0^2}{r^2}[\int_0^r e^{-2A_0}r^3dr]. \tag{3.44}$$

Here c_1 is an integration constant and $e^{2A_0}=1+r^2$ is the zeroth order value of e^{2A}. We require that the metric become that of AdS_5 space as $t\to-\infty$ this sets $c_1=0$ [9]. A negative value of c_1 would mean starting with a black hole in AdS_5 in the far past.

The integral within the square brackets on the RHS in Eq.(3.44) is given by,

$$\int_0^r e^{-2A_0}r^3dr=\frac{1}{2}[r^2-\ln(1+r^2)+d_1]. \tag{3.45}$$

This gives,

$$e^{-2B}=1+r^2-\frac{1}{12}\frac{\dot{\Phi}_0^2}{r^2}[r^2-\ln(1+r^2)+d_1]. \tag{3.46}$$

A solution which is regular for all values of r, is obtained by setting d_1 to vanish. This gives,

$$e^{-2B}=1+r^2-\frac{1}{12}\dot{\Phi}_0^2[1-\frac{1}{r^2}\ln(1+r^2)]. \tag{3.47}$$

We can obtain e^{2A} from Eq.(3.42). To second order in ϵ^2 this equation becomes,

$$A'=\frac{1}{6}r\dot{\Phi}_0^2e^{-2(A_0-B_0)}-B', \tag{3.48}$$

which gives,

$$A=-B+\frac{1}{12}\dot{\Phi}_0^2[-\frac{1}{1+r^2}+d_3], \tag{3.49}$$

with d_3 being a general function of time. Eq.(3.49) and Eq.(3.47) leads to

$$e^{2A}=1+r^2+\dot{\Phi}_0^2[-\frac{1}{4}+\frac{1}{12}\frac{\ln(1+r^2)}{r^2}+\frac{d_3}{6}(1+r^2)]. \tag{3.50}$$

The last term on the right hand side changes the leading behavior of e^{2A} as $r\to\infty$, if d_3 does not vanish, and therefore corresponds to turning on a non-normalizable

[9] Note that c_1 could be a function of time and still solve Eq.(3.43), recall though that the equations above were derived by neglecting all time derivatives of the metric, Eq.(3.36). Only a time independent constant c_1 is consistent with this assumption. A similar argument will also apply to the other integration constants we obtain as we proceed.

mode of the metric. As was discussed above we want solutions where this mode is not turned on, and we therefore set d_3 to vanish.

This gives finally,

$$e^{2A} = 1 + r^2 - \frac{1}{4}\dot{\Phi}_0^2 + \frac{1}{12}\dot{\Phi}_0^2 \frac{\ln(1+r^2)}{r^2}. \tag{3.51}$$

Eq.(3.47), (3.51) are the solutions to the metric, Eq.(3.33), to second order. Note that the Einstein equations gives rise to three equations, Eq.(3.39), Eq.(3.40), Eq.(3.41). We have used only two linear combinations out of of these to find A, B. One can show that the remaining equation is also solved by the solution given above.

In summary we note that the Einstein equations can be solved consistently to second order in ϵ^2. The resulting solution is horizon-free and regular for all values of the radial coordinate and satisfies the required boundary conditions discussed above. The second order correction to the metric is parametrically suppressed by ϵ^2 compared to the leading term for all values of r, thereby making the perturbation theory self consistent.

Let us end by commenting on the choice of integration constants made in obtaining the solution above. The boundary conditions, as $t \to -\infty$ and $r \to \infty$, determine most of the integration constants. One integration constant d_1 which appears in the solution for e^{2B}, Eq.(3.46) is fixed by regularity at $r \to 0$ [10]. For $d_1 = 0$ the second order correction is small compared to the leading term, and the use of perturbation theory is self-consistent. Moreover we expect that the boundary conditions imposed here lead to a unique solution to the supergravity equations, as was discussed at the beginning of this subsection. Thus the solution obtained by setting $d_1 = 0$ should be the correct one.

The solution above is regular and has no horizon. It has these properties due to the slowly varying nature of the boundary dilaton. The dual field theory in this case is in a non-dissipative phase. Once the dilaton begins to change sufficiently rapidly with time we expect that a black hole is formed, corresponding to the formation of a strongly dissipative phase in the dual field theory. In [26] the effect of a *small amplitude* time dependent dilaton with arbitrary time dependence was studied. Indeed it was found that when the time variation is fast enough there are no regular horizon-free solutions and a black hole is formed.

Finally, the analysis of this section holds when e^{Φ} is large enough to ensure applicability of supergravity. The fact that a black hole is not formed in this regime does not preclude formation of black holes from stringy effects when e^{Φ} becomes small enough. In fact we will argue in later sections that the latter is a distinct possibility.

Effective decoupling of modes

An important feature of the lowest order calculation of this section is that the perturbations of the dilaton and the metric are essentially linear and do not couple to each other. To this order, the dilaton perturbation is simply a solution of the linear

[10] Similarly in solving for the dilaton perturbation the integration constant a_1 is fixed by requiring regularity at $r = 0$, Eq.(3.25).

d'Alembertian equation in AdS_5. Similarly the metric perturbations also satisfy the linearized equations of motion in AdS, albeit in the presence of a source provided by the energy momentum tensor of the dilaton. This is a feature present only in the leading order calculation. As explained above, this arises because of the smallness of the parameter ϵ. We will use this feature to compare leading order supergravity results with gauge theory calculations in a later section.

3.2 Calculation of Stress Tensor and Other Operators

In this section we calculate the boundary stress tensor and the expectation value of the operator dual to the dilaton, staying in the supergravity approximation. This will be done using standard techniques of holographic renormalization group [27, 28, 29, 30, 56, 32, 33, 34].

The Energy-Momentum Tensor

The metric is of the form, Eq.(3.33), Eq.(3.47), Eq.(3.51). For calculating the stress tensor a boundary is introduced at large and finite radial location, $r = r_0$. The induced metric on the boundary is,

$$ds_B^2 \equiv h_{\mu\nu}dx^\mu dx^\nu = -e^{2A}dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2). \tag{3.52}$$

The 5 dim. action is given by

$$S_5 = \frac{1}{16\pi G_5}\int_M d^5x\sqrt{-g}\big(R + 12 - \frac{1}{2}(\nabla\Phi)^2\big) - \frac{1}{8\pi G_5}\int_{r=r_0} d^4x\sqrt{-h}\Theta. \tag{3.53}$$

Here $h_{\mu\nu}$ is the induced metric on the boundary , and Θ is the trace of the extrinsic curvature of the boundary. In our conventions, with $R_{AdS} = 1$,

$$G_5 = \frac{\pi}{2N^2}. \tag{3.54}$$

A counter term needs to be added, it is,

$$S_{ct} = -\frac{1}{8\pi G_5}\int_{\partial M} d^4x\sqrt{-h}\big[3 + \frac{\mathcal{R}}{4} - \frac{1}{8}(\nabla\Phi)^2 - log(r_0)a_{(4)}\big]. \tag{3.55}$$

The last term is needed to cancel logarithmic divergences which arise in the action, it is well known and is discussed in e.g. [27, 33]. From Eq(24) of [33] we have [11] that

$$a_{(4)} = \frac{1}{8}R_{\mu\nu}R^{\mu\nu} - \frac{1}{24}R^2 - \frac{1}{8}R^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{24}Rh^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{16}(\nabla^2\Phi)^2 + \frac{1}{48}\{(\nabla\Phi)^2\}^2 \ . \tag{3.56}$$

Here ∇ is a covariant derivative with respect to the metric $h_{\mu\nu}$.

[11] Note that our definition of the dilaton Φ is related to $\phi_{(0)}$ in [33] by $\phi_{(0)} = \Phi/2$.

Varying the total action $S_T = S_5 + S_{ct}$ gives the stress energy,

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-h}}\frac{\delta S_T}{\delta h_{\mu\nu}} \\ &= \frac{1}{8\pi G_5}\big[\Theta^{\mu\nu} - \Theta h^{\mu\nu} - 3h^{\mu\nu} + \frac{1}{2}G^{\mu\nu} - \frac{1}{4}\nabla^\mu\Phi\nabla^\nu\Phi + \frac{1}{8}h^{\mu\nu}(\nabla\Phi)^2 + \cdots\big]. \end{aligned} \tag{3.57}$$

Here $G^{\mu\nu}$ is the Einstein tensor with respect to the metric $h_{\mu\nu}$. The ellipses stand for extra terms obtained by varying the last term in Eq.(3.55) proportional to $a_{(4)}$. While these terms are not explicitly written down in Eq.(3.57), we do include them in the calculations below.

The expectation value of the stress tensor in the boundary theory is then given by,

$$< T^\mu_\nu >= r_0^4 T^\mu_\nu \tag{3.58}$$

Carrying out the calculation gives a finite answer,

$$\begin{aligned} < T^t_t > &= \frac{N^2}{4\pi^2}[-\frac{3}{8} - \frac{\dot{\Phi}_0^2}{16}] \\ < T^\theta_\theta >=< T^\psi_\psi >=< T^\phi_\phi > &= \frac{N^2}{4\pi^2}[\frac{1}{8} - \frac{\dot{\Phi}_0^2}{16}] \end{aligned} \tag{3.59}$$

where we have used Eq.(3.54). We remind the reader that in our conventions the radius of the S^3 on which the boundary gauge theory lives has been set equal to unity. The first term on the right hand side of (3.59) arises due to the Casimir effect. The second term is the additional contribution due to the varying Yang Mills coupling.

From Eq.(3.59) the total energy in the boundary theory can be calculated. We get,

$$E = - < T^t_t > V_{S^3} = \frac{3N^2}{16} + \frac{N^2\dot{\Phi}_0^2}{32}. \tag{3.60}$$

where $V_{S^3} = 2\pi^2$ is the volume of a unit three-sphere. Note that the varying dilaton gives rise to a positive contribution to the mass, as one would expect. Moreover this additional contribution vanishes when the $\dot{\Phi}$ vanishes. In particular for a dilaton profile which in the far future, as $t \to \infty$, again approaches a constant value (which could be different from the starting value it had at $t \to -\infty$) the net energy produced due to the varying dilaton vanishes.

Expectation value of the Operator Dual to the Dilaton

The operator dual to the dilaton has been discussed explicitly in [4], [35], [16].

It's expectation value is given by,

$$< \hat{O}_{l=0} >= \frac{\delta S_T}{\delta \Phi_B}|_{\Phi_B \to 0} \tag{3.61}$$

Here S_T is the total action including the boundary terms, Eq. (3.55). Since Φ_B is a function of t alone the LHS is the $l = 0$ component of the operator dual to the dilaton which we denote by, $\hat{O}_{l=0}$.

The steps involved are analogous to those above for the stress tensor and yield,

$$< \hat{\mathcal{O}}_{l=0} >= -\frac{N^2}{16}\ddot{\Phi}_0 \tag{3.62}$$

Note that the LHS refers to the expectation value for the dual operator integrated over the boundary S^3. In obtaining Eq.(3.62) we have removed all the divergent terms and only kept the finite piece. A quadratically divergent piece is removed by the third term in Eq.(3.55) proportional to $(\nabla\Phi)^2$, and a log divergence is removed by a contribution from the last term in Eq.(3.55) proportional to $a_{(4)}$.

Additional Comments

Let us end this section with a few comments.

The only source for time dependence in the boundary theory is the varying Yang Mills coupling. A simple extension of the usual Noether procedure for the energy, now in the presence of this time dependence, tells us that

$$\frac{dE}{dt} = -\dot{\Phi}_0 < \hat{\mathcal{O}}_{l=0} > . \tag{3.63}$$

It is easy to see that the answers obtained above in Eq.(3.60), Eq.(3.62) satisfy this relation. The relation Eq.(3.63) is a special case of a more general relation which applies for a dilaton varying both in space and time, this was discussed in Appendix A of [25].

In general, for a slowly varying dilaton one can expand $< \hat{\mathcal{O}}_{l=0} >$ in a power series in $\dot{\Phi}_0$. For constant dilaton, the solution is AdS_5 where one knows that the $< \hat{\mathcal{O}}_{l=0} >$ vanishes. Thus one can write,

$$< \hat{\mathcal{O}}_{l=0} >= c_1\dot{\Phi}_0 + c_2\ddot{\Phi}_0 + c_3(\dot{\Phi}_0)^2 \cdots \tag{3.64}$$

where the ellipses stand for higher powers of derivatives of the dilaton. Comparing with the answer in Eq.(3.62) one sees that in the supergravity limit c_1 and c_3 vanish. As a result $\frac{dE}{dt}$ is a total derivative, and as was discussed above if the dilaton asymptotes to a constant in the far future there is no net gain in energy.

It is useful to contrast this with what happens in the case of an infinite black brane at temperature T subjected to a time dependent dilaton which is slowly varying compared to the temperature T. This situation was analyzed extensively in [25]. In that case (see Eq(2.13), Eq(3.20) and section 7.2 of the paper) the leading term in Eq.(3.64) proportional to $\dot{\Phi}_0$ does not vanish. The temperature then satisfies an equation,

$$\frac{dT}{dt} = \frac{1}{12\pi}\dot{\Phi}_0^2 \tag{3.65}$$

As a result any variation in the dilaton leads to a net increase in the temperature, and the energy density. Note the first term in Eq.(3.64) contains only one derivative with respect to time and breaks time reversal invariance. It can only arise in a dissipative system. In the case of a black hole the formation of a horizon breaks time reversal invariance and turns the system dissipative allowing this term to arise. In the solution we construct no horizon forms and consistent with that the first term is absent.

We see in the solution discussed above that the second order corrections to the dilaton and metric arise in an instantaneous manner - at some time t, and for all values of r, they are determined by the boundary value of the dilaton at the same instant of time t. This might seem a little puzzling at first since one would have expected the effects of the changing boundary conditions to be felt in a retarded manner. Note though that in AdS space a light ray can reach any point in the bulk from the boundary within a time of order R_{AdS}. When $\epsilon \ll 1$ this is much smaller than the time taken for the boundary conditions to change appreciably. This explains why the leading corrections arise in an instantaneous manner. Some of the corrections which arise at higher order would turn this instantaneous response into a retarded one.

From the solution and the expectation values of the energy and $\hat{\mathcal{O}}_{l=0}$ it follows that in the far future the system settles down into an AdS_5 solution again. The near instantaneous nature of the solution means that this happens quickly on the times scale of order R_{AdS}. This agrees with general expectations. The supergravity modes carry an energy of order $1/R_{AdS}$ and should give rise to a response time of order R_{AdS}.

Also note that in our units, where $R_{AdS} = 1$, each supergravity mode carries an energy of order unity. The total energy at intermediate times is of order $N^2\epsilon^2$, so we see that an $O(N^2\epsilon^2)$ number of quanta are excited by the time varying boundary dilaton. This can be a big number when $N\epsilon \gg 1$. In fact the energy is really carried by the various dilaton modes. The metric perturbations are S^3 symmetric and thus contain no gravitons (in the sense of genuine propagating modes). One can think of this energy as being stored in a spatial region of order R_{AdS} in size located at the center of AdS space. This is what one would expect, since the supergravity modes which are produced by the time varying boundary dilaton have a size of order R_{AdS} and their gravitational redshift is biggest at the center of AdS space [12].

In summary, the response in the bulk to the time varying boundary dilaton is characteristic of a non-dissipative adiabatic system which is being driven much more slowly than its own fast internal time scale of response.

3.3 Gauge Theory : Quantum Adiabatic Approximation

We now turn to analyzing the behavior of the system in the dual field theory. The motivation behind this is to be able to extend our understanding to situations in which the 'tHooft coupling at intermediate time becomes of order one or smaller, so that the geometry in the bulk becomes of order the string scale. In such situations the supergravity calculation presented in the previous section breaks down and higher derivative corrections become important. The gauge theory description continues to

[12] AdS is of course a homogeneous space-time, but our boundary conditions pick out a particular notion of time. The center of AdS, where the energy is concentrated, is the region as mentioned above where the redshift in the corresponding energy is the biggest.

be valid, however. Using this description one can then hope to answer how the system evolves in the region of string scale curvature, and in particular whether by waiting for enough time a smooth geometry with small curvature emerges again on the gravity side.

We saw in the previous subsection that the bulk response was characteristic of an adiabatic system which was being driven slowly compared to the time scale of its own internal response. This suggests that in the gauge theory also an adiabatic perturbation theory should be valid and should prove useful in understanding the response. A related observation is the following. The bulk solutions we have considered correspond to keeping the radius R of the S^3 on which the gauge theory lives to be constant and independent of time. We will choose conventions in which $R = R_{AdS} = 1$. The Yang Mills coupling is related to the boundary dilaton by,

$$g_{YM}^2 = e^{\Phi_0(t)}. \tag{3.66}$$

The dilaton profile Eq.(3.18) also means that Yang Mills coupling in the gauge theory varies slowly compared to the radius R. Since this is the only other scale in the system, this also suggests that an adiabatic approximation should be valid in the boundary theory.

We will discuss two different kinds of adiabatic perturbation theory below. The first, which we call Quantum adiabatic perturbation theory, is studied in this section. This is the adiabatic perturbation theory one finds discussed in a standard text book of quantum mechanics, see [36],[37]. Its validity, we will see below requires the condition, $N\epsilon \ll 1$, to be met. We will argue that once this condition is met the gauge theory analysis allows us to conclude that, even in situations where the curvature becomes of order the string scale at intermediate times, a dual smooth AdS_5 geometry emerges as a good approximation in the far future.

The supergravity calculations, however, required only the condition $\epsilon \ll 1$, which is much less restrictive than the condition $N\epsilon \ll 1$. Understanding the supergravity regime on the gauge theory side leads us to formulate another perturbation theory, which we call "Large N Classical Adiabatic Perturbation Theory" (LNCAPT). To explain this we find it useful to first discuss the example of a driven harmonic oscillator, as considered in §5. Following this, we discuss LNCAPT in the gauge theory in §6. We find that its validity requires that the conditions Eq.(3.6) are met. Using it we will get agreement with the supergravity calculations of sections §2, §3, when the 'tHooft coupling remains large for all times.

Towards the end of §6, we discuss what happens in the gauge theory when conditions Eq.(3.6) are met but with the 'tHooft coupling becoming small at intermediate times. Two qualitatively different behaviors are possible, and we will not be able to decide between them here. Either way, at late times a mostly smooth AdS description becomes good on the gravity side, with the possible presence of a small black hole.

In the discussion below we will consider the following type of profile for the boundary dilaton: it asymptotes in the far past and future to constant values such that the initial and final values of the 'tHooft coupling, λ, are big, and attains its minimum value near $t = 0$. If this minimum value of $\lambda \leq 1$ the supergravity approximation will break down. We will also take the initial state of the system to be the ground state of the $\mathcal{N} = 4$ theory, on S^3 the spectrum of the gauge theory is gapped and this state is well defined.

The Quantum Adiabatic Approximation

- General Features

It is well known that the spectrum of the $\mathcal{N} = 4$ theory on S^3 has a gap between the energy of the lowest state and the first excited state. This gap is of order $1/R$ and thus is of order unity in our conventions. The existence of this gap follows very generally just from the fact that the spectrum must provide a unitary representation of the conformal group, [11], and the gap is therefore present for *all* values of the Yang Mills coupling constant. In the supergravity approximation the spectrum can be calculated using the gravity description and is consistent with the gap, the lowest lying states have an energy $E = 2$. This is also true at very weak 'tHooft coupling.

Now for a slowly varying dilaton Eq.(3.18) we see that the Yang Mills coupling and therefore the externally imposed time dependence varies slowly compared to this gap. There is a well known adiabatic approximation which is known to work in such situations, see e.g. [36],[37] whose treatment we closely follow. We will refer to this as the quantum adiabatic approximation below and study the Yang Mills theory in this approximation.

The essential idea behind this approximation is that when a system is subjected to a time dependence which is slow compared to its internal response time, the system can adjust itself very quickly and as a result to good approximation stays in the ground state of the instantaneous Hamiltonian.

More precisely, consider a time dependent Hamiltonian $H(\zeta(t))$, where $\zeta(t)$ is the time varying parameter. Now consider the one parameter family of time $i$ndependent Hamiltonians given by $H(\zeta)$. To make our notation clear, a different value of ζ corresponds to a different Hamiltonian in this family, but each Hamiltonian is time independent. Let $|\phi_m(\zeta)>$ be a complete set of eigenstates of the Hamiltonian $H(\zeta)$ satisfying,

$$H(\zeta)|\phi_m(\zeta)>= E_m(\zeta)|\phi_m(\zeta)>, \tag{3.67}$$

in particular let the ground state of $H(\zeta)$ be given by $|\phi_0(\zeta)>$. We take $|\phi_m(\zeta)>$ to have unit norm. Then the adiabatic theorem states that if $\zeta \to \zeta_0$ in the far past, and we start with the state $|\phi_0>$ which is the ground state of $H(\zeta_0)$ in the far past, the state at any time t is well approximated by,

$$|\psi^0(t)>\simeq |\phi_0(\zeta)> e^{-i\int_{-\infty}^{t} E_0(\zeta)dt}. \tag{3.68}$$

Here $|\phi_0(\zeta)>$ is the ground state of the time independent Hamiltonian corresponding to the value $\zeta = \zeta(t)$. Similarly in the phase factor $E_0(\zeta)$ is the value of the ground state energy for $\zeta = \zeta(t)$.

Corrections can be calculated by expanding the state at time t in a basis of energy eigenstates at the instantaneous value of the parameter ζ. The first corrections take the form,

$$|\psi^1(t)>=\sum_{n\neq 0} a_n(t)|\phi_n(\zeta)> e^{-i\int_{-\infty}^{t} E_n dt} \tag{3.69}$$

where the coefficient $a_n(t)$ is,

$$a_n(t) = -\int_{-\infty}^{t} dt' \frac{<\phi_n(\zeta)|\frac{\partial H}{\partial \zeta}|\phi_0(\zeta)>}{E_0 - E_n} \; \dot{\zeta} \; e^{-i\int_{-\infty}^{t'}(E_0-E_n)dt'} \tag{3.70}$$

In the formula above on the RHS $|\phi_n(\zeta)>, \frac{\partial H}{\partial \zeta}, E_n(\zeta)$, are all functions of time, through the time dependence of ζ.

- Conditions For Validity

For the adiabatic approximation to be good the first corrections must be small. To ensure this we impose the condition,

$$|<\phi_n|\frac{\partial H}{\partial \zeta}|\phi_0>\dot{\zeta}| \ll (E_1 - E_0)^2 \tag{3.71}$$

where $(E_1 - E_0)$ is the energy gap between the ground state and the first excited state and $|\phi_n>$ is any excited state. (This would then imply that the LHS in Eq.(3.71) is smaller than $(E_n - E_0)^2$ for all n.) This condition is imposed for all time for the adiabatic approximation to be valid [13] .

In our case the role of the parameter ζ is played by the dilaton Φ_0(with the gauge coupling $g_{YM}^2 = e^{\Phi_0}$). Thus Eq.(3.71) takes the form,

$$|<\phi_n|\frac{\partial H}{\partial \Phi_0}|\phi_0>\dot{\Phi}_0| \ll (E_1 - E_0)^2. \tag{3.72}$$

Now, as we will see below in subsection §4.3, $\frac{\partial H}{\partial \Phi_0}$ is, up to a sign, exactly the operator $\hat{\mathcal{O}}_{l=0}$ which is dual to the modes of the dilaton which are spherically symmetric on the S^3. Therefore Eq.(3.72) becomes

$$|<\phi_n|\hat{\mathcal{O}}_{l=0}|\phi_0>\dot{\Phi}_0| \ll (E_1 - E_0)^2. \tag{3.73}$$

We have argued above that the RHS is of order unity in our conventions due to the existence of a robust gap. On the LHS, $\dot{\Phi}_0 \sim O(\epsilon)$, and as we will argue below the matrix element, $|<\phi_n|\hat{\mathcal{O}}_{l=0}|\phi_0>\sim O(N)$. Thus Eq.(3.73) becomes,

$$N\epsilon \ll 1. \tag{3.74}$$

[13] The actual condition is that the corrections to $|\psi^0>$ must be small. This means that at first order $<\psi^1|\psi^1>$ should be small. When Eq.(3.71) is met $|a_n|$ is small, but in some cases that might not be enough and the requirement that the sum $\sum |a_n|^2$ is small imposes extra restrictions. There could also be additional conditions which arise at second order etc.

Highly Curved Geometries

Eq.(3.74) is the required condition then for the applicability of quantum perturbation theory. When this condition is met, we can continue to trust the quatum adiabatic approximation in the gauge theory even when the 'tHooft coupling becomes of order unity or smaller at intermediate times. All the conditions which are required for the validity of this approximation continue to be hold in this case. First, as was discussed above the gap of order unity continues to exist. Second, the matrix elements which enter are in fact independent of λ since they correspond to the two-point function of dilaton which is a chiral operator. Thus the system continues to be well described in the quantum adiabatic approximation so long as Eq.(3.74) is met. It follows then that in the far future the state of the system to good approximation is the ground state of the $\mathcal{N} = 4$ theory. This implies that the dual description in the far future is a smooth AdS_5 geometry.

There is one important caveat to the above conclusion. It is possible that at $\lambda \sim O(1)$ there are several states in the spectrum, scaling as a positive power of N, which accumulate near the first excited state. This does not happen for $\lambda \gg 1$ and for $\lambda \ll 1$ (where the spectrum of the free theory is of course known) but it remains a logical possibility. If this is true the conditions for the adiabatic approximation will have to be revised so that the dilaton varies even more slowly as a power of N. This is a question which can be settled in principle once the spectrum of the $\mathcal{N} = 4$ theory is known for all λ. Similarly, the possibility for unexpected surprises at higher orders can also be examined once enough is known about the $\mathcal{N} = 4$ theory. The point is simply that in this approximation all matrix elements and conditions can be phrased as statements in the time independent $\mathcal{N} = 4$ theory. As our knowledge of the $\mathcal{N} = 4$ theory grows we will be able to check for any such unexpected surprises.

Let us also mention before proceeding that when the condition Eq.(3.74) is met and for a dilaton profile where the 'tHooft coupling stays large for all time, the metric is to good approximation smooth AdS_5 for all time. However the small corrections to this metric and dilaton cannot be calculated reliably in the classical approximation used in section 2. This is because in this regime it is very difficult to even produce one supergravity quantum as an excitation above the adiabatic vaccum. Therefore quantum effects are important in calculating these corrections.

More Comments

We close this section by discussion two points relevant to the analysis leading up to condition, Eq.(3.74).

First, let us argue why $\frac{\partial H}{\partial \Phi_0} = -\hat{\mathcal{O}}_{l=0}$. The argument is sketched out below, more details can be found in [16]. The action of the $\mathcal{N} = 4$ theory is given by,

$$S = \int dt\ d\Omega_3\ \sqrt{-g}(-\frac{1}{4e^{\Phi_0}})TrF_{\mu\nu}F^{\mu\nu} + \cdots \tag{3.75}$$

where the ellipses indicate extra terms coming from scalars and fermions. Varying with respect to Φ_0 gives us the operator dual to the dilaton,

$$\hat{\mathcal{O}} = \sqrt{-g}(\frac{1}{4e^{\Phi_0}})TrF_{\mu\nu}F^{\mu\nu} + \cdots \tag{3.76}$$

where the ellipses denote extra terms which arise from the terms left out in Eq.(3.75). Henceforth, to emphasize the key argument we neglect the additional terms coming from the ellipses.

Working in $A_0 = 0$ gauge, the Hamiltonian density $\mathcal{H}$ is given by,

$$\mathcal{H} = e^{\Phi_0}\frac{\pi_i\pi^i}{2} + \frac{e^{-\Phi_0}}{4}F_{ij}F^{ij} \tag{3.77}$$

where

$$\pi_i = e^{-\Phi_0}\partial_0 A_i \tag{3.78}$$

is the momentum conjugate to A_i. Varying with respect to Φ_0 gives,

$$\frac{\partial\mathcal{H}}{\partial\Phi_0} = \frac{\pi_i\pi^i}{2}e^{\Phi_0} - \frac{e^{-\Phi_0}}{4}F_{ij}F^{ij}. \tag{3.79}$$

Substituting from Eq.(3.78) one sees that this agrees (up to a sign) with the operator $\hat{\mathcal{O}}$ given in Eq.(3.76). When the dilaton depends on time alone we can integrate the above equations over S^3, which leads to the relation $\frac{\partial H}{\partial\Phi_0} = -\hat{\mathcal{O}}_{l=0}$, where H now stands for the hamiltonian (rather than the hamiltonian density).

Second, we estimate how the matrix element, $< \phi_n|\hat{\mathcal{O}}_{l=0}|\phi_0 >$, which appears in Eq.(3.73), scales with N. It is useful to first recall that the $\mathcal{N} = 4$ theory, which is conformally invariant, has an operator state correspondence. The states $|\phi_n >$ can be thought of as being created from the vacuum by the insertion of a local operator. This makes it clear that the only states having a non-zero matrix element, $< \phi_n|\hat{\mathcal{O}}_{l=0}|\phi_0 >$, are those which can be created from the vacuum by inserting $\hat{\mathcal{O}}_{l=0}$, since the only operator with which $\hat{\mathcal{O}}_{l=0}$ has a non-zero two point function is $\hat{\mathcal{O}}_{l=0}$ itself.

Now in terms of powers of N the two-point function scales like,

$$< \hat{\mathcal{O}}_{l=0} \;\; \hat{\mathcal{O}}_{l=0} >\sim N^2. \tag{3.80}$$

The state $|\phi_n >$ which appears in the matrix element in Eq.(3.73) has unit norm and is therefore created from the vacuum by the operator,

$$|\phi_n >\sim \frac{1}{N}\hat{\mathcal{O}}_{l=0}|0 > \tag{3.81}$$

From Eq.(3.80), Eq.(3.81), we then see that the matrix element scales like,

$$< \phi_n|\hat{\mathcal{O}}_{l=0}|\phi_0 >\sim N \tag{3.82}$$

as was mentioned above.

Our discussion leading up to the estimate of the matrix element has been imprecise in some respects. First, strictly speaking the operator state correspondence we used is a property of the Euclidean theory on R^4, where as we are interested in the Minkowski theory on $S^3 \times R$. However, this is a technicality which can be taken care of by first relating the matrix element in the Minkowski theory to that in Euclidean $S^3 \times R$ space and then relating the latter to that on R^4 by a conformal transformation.

More importantly, the state created by $\hat{\mathcal{O}}_{l=0}$ is not an eigenstate of energy, but is in fact a sum over an infinite number of states labelled by an integer n with energies $\omega_n = 4 + 2n$. This can be understood as follows. The operator $\hat{\mathcal{O}}$ can be expanded into positive and negative frequency modes, $A_n, A_n^\dagger$ respectively, for an infinite set n, and acting with any of the $A_n^\dagger$'s gives a state,

$$|\varphi_n> \simeq A_n^\dagger|0> . \tag{3.83}$$

One must therefore worry about the dependence on the mode number n in the matrix element and the effects of summing up the contributions for all these modes. We will return to address this issue in more detail in subsections 6.2 and 6.3, when we describe the operators $A_n, A_n^\dagger$ more explicitly and discuss renormalization. For now, let us state that after the more careful treatment we will find that the condition for the quantum adiabatic approximation Eq.(3.74) goes through unchanged. The physical reason is simply this: we are interested here in the very low-frequency response of the system and its very high frequency modes are not relevant for this.

3.4 The Slowly Driven Harmonic Oscillator

The supergravity calculations required the condition $\epsilon \ll 1$. To understand this regime in the dual gauge theory it is first useful to consider a quantum mechanical Harmonic oscillator with frequency ω_0 driven by a time dependent source $J(t)$. We will see that in this case a classical adiabatic perturbation theory becomes valid when[14]

$$\frac{\ddot{J}}{\dot{J}\omega_0} \ll 1, \tag{3.84}$$

$$\dot{J} \gg \omega_0^{5/2}. \tag{3.85}$$

Having understood this system we then return to the gauge theory in the following subsection.

The Hamiltonian is given by

$$H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega_0^2(X + \frac{J(t)}{\omega_0^2})^2. \tag{3.86}$$

In the quantum adiabatic approximation one considers the instantaneous Hamiltonian. At time t_0 this is given by,

$$H_0 = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega_0^2(X + \frac{J(t_0)}{\omega_0^2})^2 \tag{3.87}$$

[14]Eq.(3.84), (3.85), clearly cannot hold when $\dot{J}$ vanishes. The more precise versions of these conditions are as follows. Eq.(3.84) is really the requirement that J is slowly varying. By this one means that the Fourier transform of J has support, up to say exponentially small corrections, only for small frequencies compared to ω_0. Eq.(3.85) is the requirement that the coherent state parameter, $\lambda(t)$ given in Eq.(3.99), is large.

where $J(t_0)$ is to regarded as a time independent constant in H_0.

The ground state of H_0 is a coherent state. Define,

$$X = \frac{a + a^\dagger}{\sqrt{2\omega_0}}, \quad P = -i\sqrt{\omega_0}(\frac{a - a^\dagger}{\sqrt{2}}) \tag{3.88}$$

to be the conventional creation and destruction operators. Here,

$$P = \dot{X} \tag{3.89}$$

is the conjugate momentum. The ground state is

$$|\phi_0 >= N_\alpha e^{\alpha a^\dagger}|0 > . \tag{3.90}$$

Here N_α is a normalization constant, determined by requiring that $< \phi_0|\phi_0 >= 1$. The state $|0>$ is the vacuum annihilated by a, i.e.,

$$a|0 >= 0, \tag{3.91}$$

and

$$\alpha = -\frac{J}{\sqrt{2\omega_0^3}}. \tag{3.92}$$

The ground state energy is

$$E_0 = \frac{1}{2}\omega_0, \tag{3.93}$$

it is independent of time.

A quick way to derive these results is to work with the shifted creation and destruction operators,

$$\tilde{a} = a - \alpha, \tilde{a}^\dagger = a^\dagger - \alpha \tag{3.94}$$

where α is given in Eq.(3.92). The Hamiltonian takes the form,

$$H = \omega_0(\tilde{a}^\dagger \tilde{a}) + \frac{1}{2}\omega_0 \tag{3.95}$$

It is clear then that the ground state is annihilated by $\tilde{a}$, leading to Eq.(3.90) and the ground state energy is Eq.(3.93).

For the quantum adiabatic theorem to be valid, the condition in Eq.(3.71) must hold. For the harmonic oscillator it is easy to see that this gives,

$$\dot{J} \ll \omega_0^{5/2}. \tag{3.96}$$

In fact the time evolution in this case can be exactly solved. We consider the case where $J(t) \to 0, t \to -\infty$. Starting with the state $|0>$ in the far past, which is the vacuum of the Hamiltonian in the far past, we then find that the state at any time t is given by,

$$|\psi(t) >= N(t)e^{\lambda(t)a^\dagger}|\phi_0 > \tag{3.97}$$

where $|\phi_0>$ is the adiabatic vaccum given in Eq.(3.90), $N(t)$ is a normalization constant and the coherent state parameter is $\lambda(t)$. Imposing Schrodinger equation one gets

$$i\dot{\lambda} = i\frac{\dot{J}}{\sqrt{2\omega_0^3}} + \omega_0\lambda. \tag{3.98}$$

The solution for $\lambda(t)$ with initial condition $\lambda(-\infty) = 0$ is given by,

$$\lambda(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0^3}} \int_{-\infty}^{t} \dot{J}(t')e^{i\omega_0 t'}dt'. \tag{3.99}$$

Some details leading to Eq. (3.98) are given in Appendix A.2. This state will behave like a classical state when the coherent state parameter is big in magnitude, i.e., when

$$|\lambda| \gg 1. \tag{3.100}$$

The integral on the RHS of Eq.(3.99) can be done by parts (we set $J(-\infty) = 0$),

$$\int_{-\infty}^{t} dt' \ \dot{J}e^{i\omega_0 t'} = \dot{J}(t)\frac{e^{i\omega_0 t}}{i\omega_0} - \int_{-\infty}^{t} dt' \ddot{J}\frac{e^{i\omega_0 t'}}{i\omega_0}. \tag{3.101}$$

Subsequent iterations obtained by further integrations by parts gives rise to a series expansion[15] for λ in terms of higher derivatives of J. The higher order terms are small if J is slowly varying compared to the frequency of the oscillator ω_0. Evaluating the second term which arises in his expansion for example and requiring it to be smaller than the first term in Eq.(3.101) gives,

$$\frac{\ddot{J}}{\dot{J}\omega_0} \ll 1 \tag{3.102}$$

We assume now that J is slowly varying and the first term on the RHS of Eq.(3.101) is a good approximation to the integral. This tells us that for Eq.(3.100) to be true the condition which must be met is,

$$\dot{J} \gg \omega_0^{5/2}. \tag{3.103}$$

Note that this condition is opposite to the one needed for the quantum adiabatic theorem to apply Eq.(3.96).

The answer for the $< X >$ can be easily obtained by inserting the expression for λ obtained in Eq.(3.99) in the wave function, Eq.(3.97). Let us obtain it here in a slightly different manner. When Eq.(3.100) is true the system behaves classically and its response to the driving force can be obtained by solving the classical equation of motion for the forced oscillator. In terms of the Fourier transform of J this gives,

$$X(t) = \int \frac{J(\omega)}{\omega^2 - \omega_0^2}e^{-i\omega t}d\omega \tag{3.104}$$

[15] In general one expects this to be an asymptotic rather than convergent series.

The correct pole prescription on the RHS is that for a retarded propagator.

When the source is slowly varying compared to ω_0, the denominator $\omega^2 - \omega_0^2$ in Eq.(3.104) can be expanded in a power series in $\frac{\omega^2}{\omega_0^2}$ and the resulting Fourier transforms can be expressed as time derivatives of J. The first two terms give,

$$X = -\frac{J(t)}{\omega_0^2} + \frac{\ddot{J}}{\omega_0^4} + \cdots \tag{3.105}$$

The first term on the RHS is the location of the instantaneous minimum. The second term is the first correction due to the time dependent source. Subsequent corrections are small if the source is slowly varying and condition Eq.(3.102) is met. It is useful to express this result as,

$$X + \frac{J(t)}{\omega_0^2} = \frac{\ddot{J}}{\omega_0^4} + \cdots . \tag{3.106}$$

The left hand side is the expectation value of X after adding a shift to account for the instantaneous minimum of the potential. The right hand side we see now only contains time derivatives of J. Before proceeding let us note that the expanding the denominator in Eq.(3.104) in a power series in $\frac{\omega^2}{\omega_0^2}$ gives a good approximation only if $J(\omega)$ has most of its support for $\omega \ll \omega_0$. This is how the more precise condition mentioned in the footnote before Eq.(3.84) arises.

It is also useful to discuss the energy. From Eq.(3.105) and the Hamiltonian we see that the leading contribution comes from the Kinetic energy term and is given to leading order by,

$$E = \frac{1}{2}\frac{\dot{J}^2}{\omega_0^4} \tag{3.107}$$

(strictly speaking this is the energy above the ground state energy).

The external source driving the oscillator changes its energy. Noether's argument in the presence of the time dependent source leads to the conclusion that

$$\frac{\partial H}{\partial t} = \dot{J}(X + \frac{J}{\omega_0^2}) \tag{3.108}$$

(this also directly follow from the Hamiltonian, Eq.(3.86)). From Eq.(3.106) and Eq.(3.107) we see that this condition is indeed true. Let us also note that the rate of change in energy can be expressed in terms of the shifted operators, Eq.(3.94), as,

$$\frac{\partial H}{\partial t} = \dot{J}(\frac{\tilde{a} + \tilde{a}^\dagger}{\sqrt{2\omega_0}}), \tag{3.109}$$

this form will be useful in our discussion below.

To summarize, we find that when the conditions Eq.(3.103), Eq.(3.102), are met the driven harmonic oscillator behaves like a classical system. Its response, for example, $< X >$, and the energy, E, can be calculated in an expansion in time derivatives of J, which is controlled when Eq.(3.102) is valid and the source is slowly varying. We will refer to this perturbation expansion as the classical adiabatic perturbation approximation below. Note that the condition, Eq.(3.103) is opposite to the one required for the quantum adiabatic perturbation theory to hold. In the next subsection we will discuss how a similar classical adiabatic approximation arises in the gauge theory.

3.5 Gauge Theory: Large N Classical Adiabatic Perturbation Theory

We now return to the gauge theory and formulate a large N classical adiabatic approximation based on coherent states in this theory. This will allow us to obtain results in the gauge theory which agree with those obtained using supergravity in §2, §3.

Adiabatic Approximation in terms of Coherent States

The supergravity solution in §2 describes *classical solutions* rather than states which contain a small number of bulk particles. The AdS/CFT correspondence implies that bulk classical solutions corresponds to coherent states in the boundary gauge theory with a large number of particles in which operators like $\hat{\mathcal{O}}$ have nontrivial expectation values. On the other hand, states obtained by the action of a few factors of $\hat{\mathcal{O}}$ on the vaccum are few-particle states in the bulk. The quantum adiabatic approximation described in §4 attempts to determine the wave function in a basis formed out of such single particle states and does not apply to the supergravity solution in §2.

We, therefore, need to formulate an adiabatic approximation in terms of coherent states of gauge invariant operators in the boundary theory to try and understand the supergravity solutions of §2 in a dual description. As is well known, these coherent states become classical in a smooth fashion in the $N \to \infty$ limit. (See e.g. [13]). Consider a complete (usually overcomplete) set of gauge invariant operators in the Schrodinger picture, $\hat{\mathcal{O}}^I$. A general coherent state is of the form

$$|\Psi(t) >= \exp \left[i\chi(t) + \sum_I \lambda^I(t) \hat{\mathcal{O}}^I_{(+)} \right] |0>_A . \qquad (3.110)$$

Here $\hat{\mathcal{O}}^I_{(+)}$ denotes the creation part of the operator and $|0>_A$ denotes the adiabatic vacuum corresponding to some instantaneous value of the dilation Φ_0,

$$H[\Phi_0]|0>_A = E_{\Phi_0}|0>_A \qquad (3.111)$$

with the ground state energy E_{Φ_0}.

The algebra of operators $\hat{\mathcal{O}}^I$, together with the Schrodinger equation then leads to a differential equation which determines the time evolution of the coherent state parameters $\lambda^I(t)$ in terms of the time dependent source $\Phi_0(t)$. The idea is then to solve this equation in an expansion in time derivatives of $\Phi_0(t)$. This is the coherent state adiabatic approximation we are seeking.

In general it is almost impossible to implement this program practically, since the operators $\hat{\mathcal{O}}^I$ have a non-trivial operator algebra which mixes all of them. The coherent state (3.110) is in the co-adjoint orbit of this algebra [13]. The resulting theory of fields conjugate to these operators would be in fact the full interacting string field theory in the bulk. In our case, however, the situation drastically simplifies for large 't Hooft coupling at the lowest order of an expansion in $\dot{\Phi}_0$. This is because these various operators decouple and their algebra essentially reduces to free oscillator algebras.

We have already found this decoupling in our supergravity calculation. The departure of the solution from $AdS_5 \times S^5$ is due to the time-dependence of the boundary value of the dilaton, and are small when the time variations are small, controlled by the parameter ϵ. To lowest order in ϵ (which is $O(\epsilon^2)$) the deformation of the bulk dilaton in fact satisfied a linear equation in the AdS_5 background in the presence of a source provided by the boundary value $\Phi_0(t)$. This equation does not involve the deformation of the metric. Similarly, the equation for metric deformation does not involve the dilaton deformation to lowest order.

This allows us to treat each supergravity field and its dual operator separately. With this understanding we will now consider the coherent state (3.110) with only the operator dual to the dilaton, $\hat{\mathcal{O}}$. Since our source is spherically symmetric and higher point functions of the operators are not important in this lowest order calculation, we can restrict this operator to its spherically symmetric part.

Large N Classical Adiabatic Perturbation Theory (LNCAPT)

Let us now elaborate in more detail on the LNCAPT.

The linearized approximation in the gravity theory means that only the two point function is non-trivial and all connected higher point functions vanish. The non-linear terms correspond to nontrivial higher order correlations. In this approximation the gauge theory simplifies a great deal. Each gauge invariant operator- which is dual to a bulk mode- gives rise to a tower of harmonic oscillators. The response of the gauge theory can be understood from the response of these oscillators.

In fact in the quadratic approximation the only oscillators which are excited are those which couple directly to the dilaton and so we only have to discuss their dynamics. We have already discussed the operator dual to the dilaton in section §4.3. The dilaton excitations we consider are S^3 symmetric and correspondingly the only modes of $\hat{\mathcal{O}}$ which are excited are S^3 symmetric. Here we denote these by $\hat{\mathcal{O}}_{l=0}$.

In the Heisenberg picture $\hat{\mathcal{O}}_{l=0}$ can be expanded in terms of time dependent modes, this is dual to the fact that the S^3 symmetric dilaton can be expanded in terms of modes with different radial and related time dependence in the bulk. One finds, as is discussed in Appendix A.3, that only even integer frequencies appear in the time dependence giving,

$$\hat{\mathcal{O}}_{l=0} = N \sum_{n=1}^{\infty} F(2n)[A_{2n} e^{-i2nt} + A_{2n}^{\dagger} e^{i2nt}]. \tag{3.112}$$

Here $A_{2n}, A_{2n}^{\dagger}$ are canonically normalized creation and destruction operators satisfying the relations,

$$[A_m, A_n] = [A_m^{\dagger}, A_n^{\dagger}] = 0 \qquad [A_m, A_n^{\dagger}] = \delta_{m,n}. \tag{3.113}$$

Their commutators with the gauge theory hamiltonian are

$$[H, A_{2n}^\dagger] = (2n)A_{2n}^\dagger \qquad\qquad [H, A_{2n}] = -(2n)A_{2n} \tag{3.114}$$

The normalization factor $F(2n)$ may be computed by comparing with the standard the 2-point function as is detailed in Appendix A.3. The result is

$$|F(2n)|^2 = \frac{A\pi^4}{3}\, n^2(n^2-1) \tag{3.115}$$

for $n \geq 2$. $F(0)$ and $F(2)$ vanish, so this means that the sum in Eq.(3.112) receives its first contribution at $n = 2$. It also means that the lowest energy state which can be created by acting with $\hat{\mathcal{O}}_{l=0}$ on the vacuum has energy equal to 4. This is what we expect on general grounds, since the energies of states created by an operator with conformal dimension Δ are given by

$$\omega(n,l) = \Delta + 2n + l(l+2) \qquad n = 0, 1, 2\cdots \tag{3.116}$$

The constant A in Eq.(3.115) is the normalization of the 2-point function which may be determined e.g. from a bulk calculation. Before proceeding let us also note that $F(2n)$ grows like $F(2n) \sim n^2$, Eq.(3.115), for large mode number n. This enhances the coupling of the higher frequency modes to the dilaton and will be important in our discussion of renormalization below.

From now onwards we will find it convenient to work in the Schrodinger representation, in which operators are time independent. The operator $\hat{\mathcal{O}}_{l=0}$ in this representation is given by,

$$\hat{\mathcal{O}}_{l=0} = N \sum_n F(2n)[A_{2n} + A_{2n}^\dagger]. \tag{3.117}$$

From Eq.(3.114) it follows that the Hamiltonian for $A_{2n}, A_{2n}^\dagger$ modes can be written as,

$$H = \sum_n 2n A_{2n}^\dagger A_{2n}. \tag{3.118}$$

Note this Hamiltonian measures the energy above that of the ground state.

The operators, $A_{2n}^\dagger, A_{2n}$ create and destroy a single quantum of excitation when acting on the vaccum of the $\mathcal{N} = 4$ theory with the instantaneous value of $g_{YM}^2 = e^{\Phi_0}$. Thus they are the analogue of the shifted creation and destruction operators we had defined in the harmonic oscillator case, $\tilde{a}, \tilde{a}^\dagger$. The Hamiltonian, Eq.(3.118), is the analogue of the Hamiltonian, Eq.(3.95) in the harmonic oscillator case.

The time dependence of the Hamiltonian due to the varying dilaton can be expressed as follows,

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial \Phi}\dot{\Phi}_0 = -\hat{\mathcal{O}}_{l=0}\dot{\Phi}_0 \tag{3.119}$$

leading to,

$$\frac{\partial H}{\partial t} = -\hat{\mathcal{O}}_{l=0}\dot{\Phi}_0 = -N\sum_n F(2n)[A_{2n} + A_{2n}^\dagger]\dot{\Phi}_0, \tag{3.120}$$

where we have used Eq.(3.117). It is useful to write this as

$$\frac{\partial H}{\partial t} = -N \sum F(2n)\sqrt{4n}\dot{\Phi}_0 [\frac{A_{2n} + A^\dagger_{2n}}{\sqrt{4n}}], \tag{3.121}$$

which is analogous to the time dependence in the forced oscillator system, Eq.(3.109).

So we see that the gauge theory, in the quadratic approximation maps to a tower of oscillators, with frequencies, $\omega_n = 2n$. Comparing with Eq.(3.109) we see that the oscillator with energy $2n$ couples to a source,

$$\dot{J}_n = -NF(2n)\sqrt{4n}\dot{\Phi}_0. \tag{3.122}$$

The analysis of the harmonic oscillator now directly applies. The resulting state is a coherent state,

$$|\psi >= \hat{N}(t) e^{(\sum_n \lambda_n A^\dagger_{2n})} |\phi_0 > . \tag{3.123}$$

Here $|\phi_0>$ is the adiabatic vacuum, which in is the ground state of the $\mathcal{N} = 4$ theory with coupling $g^2_{YM} = e^{\Phi_0}$. $\hat{N}(t)$ is a normalization constant and the coherent state parameter λ_n is given from Eq.(3.99) by,

$$\lambda_n = \frac{e^{-i\omega_n t}}{\sqrt{2\omega_n^3}} \int_{-\infty}^{t} \dot{J}_n(t') e^{i\omega_n t'} dt'. \tag{3.124}$$

The condition that the source is varying slowly, Eq.(3.102), becomes,

$$|\frac{\ddot{\Phi}_0}{n\dot{\Phi}_0}| \ll 1 \qquad \forall n. \tag{3.125}$$

It is clearly sufficient to satisfy this condition for $n = 1$,

$$|\frac{\ddot{\Phi}_0}{\dot{\Phi}_0}| \sim \epsilon \ll 1. \tag{3.126}$$

This condition is met for the dilaton profile we have under consideration [16]. When this condition is true λ_n can be evaluated by keeping the first term in Eq.(3.101). The condition that the state is classical, is that $\lambda_n \gg 1$, this gives[17],

$$|NF(2n)\sqrt{4n}\dot{\Phi}_0| \gg (2n)^{5/2}. \tag{3.127}$$

Noting from Eq.(3.115) that $F(2n) \sim n^2$ for large n we see that the factors of n cancell out on both sides, leading to the conclusion that when,

$$|N\dot{\Phi}_0| \sim N\epsilon \gg 1 \tag{3.128}$$

[16] This condition is analogous to Eq.(3.84) for the driven harmonic oscillator. As discussed in that context in the footnote before Eq.(3.84) there is a more precise version of this condition. It is the statement that for all modes, n, the Fourier transform of J_n must have essentially all its support at frequencies much smaller than the oscillator frequency, $2n$.

[17] The more precise condition is simply that $\lambda_n \gg 1, \ \ \forall n$. This gives, Eq.(3.127) provided that the integral in Eq.(3.124) can be approximated by the first term of the derivative approximation.

all the oscillators are in a classical state. In this way we recover the first condition discussed in Eq.(3.6).

The summary is that when the two conditions,

$$\epsilon \ll 1, N\epsilon \gg 1 \tag{3.129}$$

are both valid, the gauge theory is described to leading order in ϵ as a system of harmonic oscillators. The oscillators which couple to the dilaton are excited by it and are in a classical state.

This description can be used to calculate the resulting expectation value of operators. The calculation for $< \frac{A_{2n}+A^{\dagger}_{2n}}{\sqrt{4n}} >$ is analogous to that for $< X + \frac{J}{\omega^2} >$ in the harmonic oscillator case (since the $A_{2n}, A^{\dagger}_{2n}$ are analogous to the shifted operators, $\tilde{a}, \tilde{a}^{\dagger}$ Eq.(3.94)). From Eq.(3.106) and Eq.(3.122) we get that to leading order in ϵ,

$$< \frac{A_{2n}+A^{\dagger}_{2n}}{\sqrt{4n}} >= -N\frac{F(2n)\sqrt{4n}}{(2n)^4}\ddot{\Phi}_0. \tag{3.130}$$

Substituting in Eq.(3.117) next gives,

$$< \hat{\mathcal{O}}_{l=0} >= -CN^2\ddot{\Phi}_0 \tag{3.131}$$

where C is

$$C = \sum \frac{F(2n)^2}{4n^3}. \tag{3.132}$$

The functional dependence on Φ_0 and N in Eq.(3.130) agrees with what we found in the supergravity calculation, Eq.(3.62). The constant of proportionality C is in fact quadratically divergent. This follows from noting that for large n, $F(2n) \sim n^2$.

A little thought tells us that the divergence should in fact have been expected. The supergravity calculation also had a divergence and the finite answer in Eq.(3.62) was obtained only after regulating this divergence and renormalizing. Therefore it is only to be expected that a similar divergence will also appear in the description in terms of the oscillators. In the subsection which follows we will discuss the issue of renormalization in more detail. The bottom line is that counter terms can be chosen so that the coefficient in Eq.(3.62) agrees with that in the supergravity calculation.

It is also important to discuss how the energy behaves. From Eq.(3.107) and Eq.(3.122) we see that the energy above the ground state is

$$< E > -E_{gnd} = \frac{1}{2}CN^2\dot{\Phi}_0^2 \tag{3.133}$$

We note that the functional dependence on $\dot{\Phi}_0, N$ match with those obtained in the supergravity calculations, Eq.(3.60). The constant of proportionality which is obtained by summing over the oscillator modes in the case of the energy is the same as C defined above, Eq.(3.132). It is also therefore quadratically divergent.

The fact that the two constants of proportionality in Eq.(3.133) and Eq.(3.131) are the same follows on general grounds. Noether's argument in the presence of the time dependence means that each oscillator satisfies the relation, Eq.(3.108). On summing over all of them we then get the relation

$$< \frac{dE}{dt} >= -\dot{\Phi}_0 < \hat{\mathcal{O}}_{l=0} > \tag{3.134}$$

leading to the equality of the two constants. Earlier we had also seen that the supergravity calculation satisfies this relation, Eq.(3.63). It follows from these observations that if after renormalization the answer for $< \hat{\mathcal{O}}_{l=0} >$ agrees between the supergravity theory and the oscillator description developed here, then the expectation value for E will also agree in the two cases.

Here we have analyzed the gauge theory to leading order in ϵ. Going to higher orders introduces anharmonic couplings between the different oscillators. These couplings arise because of connected three-point and higher point correlations in the gauge theory. The three point function for example is suppressed by $1/N$, the four point function by $1/N^2$ and so on. For computations in the ground state these would therefore be suppressed in the large N limit. However as we have seen here the time dependence results in a coherent state which contains $O(N\epsilon)^2$ quanta being produced. The 3- pt function in such a state is suppressed by $O(\epsilon)$ and not by $O(1/N)$. Since $\epsilon \ll 1$, this is still enough though to justify our neglect of the cubic terms to leading order in ϵ. Similarly the effect of 4-pt correlators in the coherent state are suppressed by $O(\epsilon)^2$ etc. This is in agreement with the supergravity calculation, where the cubic terms in the equations of motion are suppressed by $O(\epsilon)$ etc.

To go to higher orders in ϵ using the oscillator description the effect of the anharmonic couplings induced by the higher order correlations would have to be introduced. In addition one would have to keep the contributions from the quadratic approximation to the required order in ϵ. As long as the 'tHooft coupling stays big for all times and the supergravity approximation is valid, there is no reason to believe that these effects will be significant and the behavior of the system should be well described by the leading harmonic oscillator description, in agreement with what we saw in supergravity. When the 'tHooft coupling begins to get small though the anharmonic couplings could potentially significantly change the behavior of the system, as we will discuss in section 6.4.

Renormalization

Let us now return to the constant C Eq.(3.132). One would like to know if it can be made to agree with the supergravity answer Eq.(3.62). Since the mode sum in C diverges, at first sight it would seem that by suitably removing the infinities this can always be done. To be explicit, imposing a cutoff on the mode sum in C one gets from Eq.(3.132),

$$C = \sum \frac{F(2n)^2}{4n^3} = c_1 n_{max}^2 + c_2 \ln(n_{max}) + \text{finite term} \tag{3.135}$$

(A term linear in n_{max} can always be removed by shifting n_{max}). Removing the infinities would mean removing the first two terms, but by changing n_{max} by a finite amount the finite term left over will clearly change and can be made equal to any answer we want.

However this seems too superficial an answer. One would like to ensure that the freedom to adjust C corresponds to the freedom to add local counterterms in the theory, and also that once the counter terms are chosen so that C agrees no other discrepancy appears with supergravity.

This is in fact true and can be easily seen by relating the calculation for $< \hat{\mathcal{O}} >$ in Eq.(3.131) to the two-point function for the dilaton. In fact we will only need the two point function of the S-wave dilaton which is equal to the two-point function of $< \hat{\mathcal{O}}_{l=0}\hat{\mathcal{O}}_{l=0} >$ in the gauge theory. Since the S-wave dilaton couples directly to $\hat{\mathcal{O}}_{l=0}$, we have

$$< \hat{\mathcal{O}}_{l=0}(t) >= \int dt' \ < \hat{\mathcal{O}}_{l=0}(t)\hat{\mathcal{O}}_{l=0}(t') > \Phi(t') \tag{3.136}$$

Using Eq.(3.112) we find that

$$< \hat{\mathcal{O}}_{l=0}(t)\hat{\mathcal{O}}_{l=0}(t') >= N^2 \sum_n F(2n)^2(4n) \int \frac{d\omega}{2\pi i} \frac{e^{-i(t-t')\omega}}{(\omega^2 - (2n)^2)} \tag{3.137}$$

where we have expressed the answer in terms of a Fourier transform in frequency space. We are not being explicit about the pole prescription here, this will determine which propagator (Feynman, Retarded etc) one requires. From Eq.(3.137) the propagator in frequency space can be read off to be,

$$G(\omega) = N^2 \sum_n \frac{F(2n)^2(4n)}{(\omega^2 - (2n)^2)} \tag{3.138}$$

Since $F(2n) \sim n^2$ the sum over modes on the RHS is *q*uartically divergent.

For purposes of comparing with the adiabatic approximation we expand this propagator in powers in ω^2. This gives,

$$\frac{G(\omega)}{N^2} = -\sum \frac{F(2n)^2(4n)}{(2n)^2} - \omega^2 \sum \frac{F(2n)^2(4n)}{((2n)^2)^2} - \omega^4 \sum \frac{F(2n)^2(4n)}{((2n)^2)^3} + \cdots \tag{3.139}$$

The terms within the ellipses contain powers higher than ω^4 and are not divergent. The first term on the RHS must be set to zero after renormalization to preserve conformal invariance, otherwise the vacuum expectation value for $< \hat{\mathcal{O}} >$ in the $\mathcal{N} = 4$ theory with constant coupling would not vanish. The leading contribution to $< \hat{\mathcal{O}} >$ in the adiabatic approximation then arises from the second term which is quadratically divergent. After Fourier transforming the ω^2 dependence of this term gives rise to the second derivative with respect to the time of the dilaton. And the sum over modes is the same as that in C, Eq.(3.132).

Now the point is that all divergences in the two-point function can be removed by local counterterms since they correspond to contact terms. In fact the gravity calculation also needed counterterms and from our discussion in §3.1 we know that these counterterms are of the form given in Eq.(3.55). In particular the third term in Eq.(3.55) proportional to $(\nabla\Phi)^2$ cancels the quadratic divergence while the last term in Eq.(3.55), $a_{(4)}$, contains terms which cancel the subleading logarithmic divergence. Also once the counter terms are chosen so that C agrees no other discrepancy can appear. The point here is that the leading order in ϵ calculations are only sensitive to the two-point function. And the finite terms in the two-point function are well known to agree between the gravity and gauge theory sides. In fact the finite two point function is just determined by conformal invariance and since the anomalous dimension of $\hat{O}$ does not get renormalized, it can be calculated in the free field limit itself.

The bottom line then is that using the freedom to adjust the counter terms, C can be made to agree with the supergravity calculations in §3.

Let us end by pointing out that the supergravity value for C, Eq.(3.62) is,

$$C_{sugra} = \frac{1}{16} \tag{3.140}$$

which means that the effect of renormalization is to only include the contributions of modes with mode number $n \sim O(1)$. This makes good physical sense, we are dealing with the low frequency response of the system here, and the high frequency modes should not be relevant for this purpose.

This last comment also has a bearing on our discussion in §4 of the quantum adiabatic perturbation theory. The criterion for the validity of this approximation was stated in Eq.(3.74). Now what this condition really ensures is that the amplitude to excite the system to a state $|\phi_n>= A_n^\dagger|0>$ containing any one single oscillator excitation is small. However there are an infinite number of such single excitation states, corresponding to the infinite number of values that n takes, and one might be worried that this condition is not sufficient. Even though the amplitude to excite the system into any given state $|\phi_n>$ is small the sum of these amplitudes, more correctly the norm of the first order correction of the wave function $<\psi^1|\psi^1>$, Eq.(3.69), is still be large and in fact would diverge when summed over all the modes. This would invalidate the approximation. The reason this concern does not arise is tied to our discussion above. After renormalization only a few low frequency modes contribute to the response of the system and one is only interested in how the wave function changes for these modes. For this purpose the condition in Eq.(3.74) is enough and we see that when it is met the quantum adiabatic approximation is indeed valid.

Highly Curved Geometry

So far we have considered what happens in the parametric regime, Eq.(3.129), when the 'tHooft coupling stays big all times. In this case the supergravity description is always valid. We saw above that the gauge theory can be described in this regime in terms of approximately decoupled classical harmonic oscillators and this reproduces the supergravity results.

Now let us consider what happens when the dilaton takes a larger excursion so that the 'tHooft coupling at intermediate times becomes of order unity or even smaller. Some of the resulting discussion is already contained in the introduction above.

A natural expectation is that description in terms of classical adiabatic system of weakly coupled oscillators should continue to apply even when the 'tHooft coupling becomes small. There are several reasons to believe this. First, anharmonic terms continue to be of order ϵ and thus are small. The leading anharmonic terms arise from three -point correlations, $< \hat{O}_1\hat{O}_2\hat{O}_3 >$. In the vaccum these go like $1/N$. In the coherent state produced by the time dependence these go like ϵ. The enhancement by $N\epsilon$ arises because the coherent state contains $O((N\epsilon)^2)$ quanta, so that the probability goes as $(N\epsilon)^2/N^2 \sim \epsilon^2$. Four-point functions give rise to terms going like $O(\epsilon^2)$ and so on, these are even smaller. In the absence of anharmonic terms the theory should reduce to a system of oscillators. Second, the existence of a gap of order $1/R$ means that for each oscillator the time dependence is slow compared to its frequency. Therefore the system continues to be very far from resonance and should evolve adiabatically. Finally, in the parametric regime, Eq.(3.129) the analysis of the previous subsections should then apply leading to the conclusion that an $O(N\epsilon) \gg 1$ quanta are produced making the coherent state a good classical state.

If this expectation is borne out the system should settle back into the ground state of the final $\mathcal{N} = 4$ theory in the far future and should have a good description in terms of smooth AdS space then.

However, as discussed in the introduction, there are reasons to worry that this expectation is not borne out. New features could enter the dynamics when the 'tHooft coupling becomes small at intermediate times, and these could change the qualitative behavior of the system. These new features have to do with the fact that string modes can start getting excited in the bulk when the curvature becomes of order the string scale. These modes correspond to non-chiral operators in the gauge theory and the corresponding oscillators have a time dependent frequency. When the 'tHooft coupling is big these frequencies are much bigger than those of the supergravity modes and as a result the string mode oscillators are not excited. But when the 'tHooft coupling becomes of order unity some of the frequencies of these string modes become of order the supergravity modes and hence these oscillators can begin to get excited [18]. In fact the string modes are many more in number than the supergravity modes, since there are an order unity worth of chiral operators in the gauge theory and an $O(N^2)$ worth of non-chiral ones.

The worry then is that if a significant fraction of these string oscillators get excited the correct picture which could describe the ensuing dynamics is one of thermalization rather than classical adiabatic evolution. In this case the energy pumped into the system initially would get equipartitioned among all the different degrees of freedom. Subsequent evolution would then be dissipative, and the energy would increases in a monotonic manner, as it does for a large black hole, Eq.(3.65).

Due to the dissipative behavior the energy which is initially pumped in would not be recovered in the future. Rather one would expect that when the 'thooft coupling becomes large again, the energy, which is of order $N^2\epsilon^2$ remains in the system. The gravity description of the resulting thermalized state depends on the value of ϵ relative to $\lambda \equiv g_{YM}^2 N$ and N. In this late time regime of large 't Hooft coupling, the various possibilities can be figured out from entropic considerations in supergravity (see e.g.

[18]The primary reason for them getting excited are the anharmonic terms which couple them to the modes dual to the dilaton.

section 3.4 of [5]). The result in our case is the following. For $\epsilon \ll (g_{YM}^2 N)^{5/4}/N$ a gas of supergravity modes is favored. For $(g_{YM}^2 N)^{5/4}/N < \epsilon \ll (g_{YM}^2 N)^{-7/8}$ one would have a gas of massive string modes. For $(g_{YM}^2 N)^{-7/8} < \epsilon \ll 1$ one gets a small black hole, i.e. a black hole whose size is much smaller than R_{AdS}. A big black hole requires $O(N^2)$ energy which is parametrically much larger. Thus, the strongest departure from AdS space-time in the far future would be presence of small black holes. Such black holes would eventually evaporate by emitting Hawking radiation. However this takes an $O(N^2 R_{AdS})$ amount of time which is much longer than the time scale $O(R_{AdS}/\epsilon)$ on which the 'tHooft coupling evolves. As a result for a long time after the 'tHooft coupling has become big again the gravity description would be that of a small black hole in AdS space.

An important complication in deciding between these two possibilities is that the rate of time variation is ϵ which is also the strength of the anharmonic couplings between the supergravity oscillators and string oscillators. If the rate of time variation could have been made much smaller, thermodynamics would become a good guide for how the system evolves. In the microcanonical ensemble, which is the correct one to use for our purpose, with energy $N^2\epsilon^2$ the entropically dominant configurations are as discussed in the previous paragraph, and this would suggest that dissipation would indeed set in. However, as emphasized above this conclusion is far from obvious here since the time variation is parametrically identical to the strength of the anharmonic couplings.

In fact we know that the guidance from thermodynamics is misleading in the supergravity regime, where the 'tHooft coupling stays large for all times. In this case we have explicitly found the solution in §2. It does not contain a black hole. Moreover, it does not suffer from any tachyonic instability - since it is a small correction from AdS space which does not have any tachyonic instability [19]. The only way a black hole could form is due to a tunneling process but this would be highly suppressed in the supergravity regime.

One reason for this suppression is that the energy in the supergravity solution discussed in §2 is carried by supergravity quanta which have a size of order R_{AdS}. This energy would have to be concentrated in much smaller region of order the small black hole's horizon to form the black hole and this is difficult to do. In contrast, away from the supergravity regime this could happen more easily. When the 'tHooft coupling becomes small at intermediate times, strings become large and floppy, of order R_{AdS}, at intermediate times. If a significant fraction of the energy gets transferred to these strings at intermediate times it could find itself concentrated within a small black hole horizon once the 'tHooft coupling becomes large again.

In summary we do not have a clean conclusion for the future fate of the system in the parametric regime, Eq.(3.129). Note however that in both possibilities discussed above most of space-time in the far future is smooth AdS space, with the possible presence of a small black hole. Hopefully, the framework developed here will be useful to think about this issue further.

[19]Note that we are working on S^3 here.

3.6 Conclusions

In this paper we examined the behavior of the $AdS_5 \times S^5$ solution of IIB supergravity when it is subjected to a time dependent boundary dilaton. This is dual to the behavior of the $\mathcal{N} = 4$ Super Yang -Mills theory subjected to a time dependent gauge coupling. The AdS_5 solution was studied in global coordinates and the dual field theory lives on an S^3 of fixed radius R. We worked in units where $R_{AdS} = R = 1$. Three parameters are relevant for describing the resulting dynamics:

1. N - which is the number of units of flux and is dual to the rank of the gauge group. This was held fixed during the evolution.
2. $\lambda = e^{\Phi(t)}N$ - which determines the value of R_{AdS} in string units is the 'tHooft coupling in the gauge theory. Especially relevant is its minimum value λ_{min} during the time evolution. When $\lambda_{min} \gg 1$ supergravity is a good approximation for all times. When $\lambda_{min} \leq O(1)$ supergravity breaks down at intermediate times.
3. $\epsilon \sim \dot{\Phi}$ - which determines the rate of change of the boundary dilaton in units of R_{AdS}. Throughout the analysis we worked in the slowly varying regime where $\epsilon \ll 1$.

Our results are as follows:

- When $N\epsilon \ll 1$ the dynamics can be described by a quantum adiabatic approximation. The gauge theory stays in the ground state of the instantaneous Hamiltonian to good approximation. At late times the system is well described by smooth AdS_5 spacetime. This is true even when $\lambda_{min} \leq 1$ as discussed in §4.
- When $N\epsilon \gg 1$ and $\lambda_{min} \gg 1$, the system is well described by a supergravity solution, which consists of AdS_5 spacetime with corrections which are suppressed in ϵ. The gauge theory provides an alternate description in terms of weakly coupled harmonic oscillators which are modes of gauge invariant operators dual to supergravity modes. These oscillators are subjected to a driving force that is slowly varying compared to their frequency. A classical adiabatic perturbation theory, the LNCAPT, describes the dynamics of the system. This dual description reproduces the supergravity answers for the energy and $< \hat{\mathcal{O}} >$, as discussed in §6.1, §6.2.
- When $N\epsilon \gg 1$, and $\lambda_{min} \leq O(1)$, supergravity breaks down. In this case we do not have a clean conclusion for the final state of the system. Additional oscillators which correspond to string modes can now get activated. There are two possibilities : either the description in terms of classical adiabatic dynamics for the oscillators continues to apply, or a qualitative new feature of thermalization sets in. In the former case spacetime in the far future is well approximated by smooth AdS space. In the latter case the gravity description depends on the

value of ϵ and may consist of a string gas or small black holes. This is discussed in §6.3.

- We have not addressed here what happens when the dilaton begins to vary more rapidly and ϵ becomes $\sim O(1)$. It is natural to speculate that a black hole forms eventually in this case. The oscillators in the gauge theory now become strongly coupled with $O(1)$ anharmonic couplings.

  If $\lambda_{min} \gg 1$ this parametric regime can be studied in supergravity itself. When $\epsilon \ll 1$ the calculations in §2 showed that no black hole forms. As ϵ increases the natural expectation is that eventually a black hole should begin to form at some critical value. The size of this black hole should then grow with ϵ, leading to a big black hole with radius bigger than AdS scale. Very preliminary indications for this come from the calculations in §2 where we see that as ϵ increases the value of $|g_{tt}|$ becomes smaller at the center of AdS Eq.(3.51), suggesting that a horizon would eventually form at $\epsilon \sim O(1)$. Better evidence comes from studying a region of parameter space where $\epsilon \gg 1$ but where the total amplitude of the dilaton variation is small. In this case [20] one finds that a boundary variation of the dilaton, which is sufficiently fast compared to its amplitude, always produces a black hole.

  When $\lambda_{min} \leq O(1)$, and ϵ becomes $\sim O(1)$, supergravity breaks down at intermediate times. If thermalization has already set in in the parametric regime, $N\epsilon \gg 1, \epsilon \ll 1$, as discussed above, then one expects that the small black hole which has formed for $\epsilon \ll 1$ would grow and become of order the AdS scale or bigger when $\epsilon \geq O(1)$. If thermalization does not set in when $\epsilon \ll 1$, then at some critical value $\epsilon \sim O(1)$ one would expect that this does happen leading to the formation of a black hole whose mass then grows as ϵ further increases.

  It will be interesting to try and analyze this regime further in subsequent work.

- Finally one can consider a regime where $\epsilon \to \infty$ at time $t \to 0$. This regime was considered in [16] where the dilaton was taken to vanishes like $e^{\Phi} \sim (t)^p$ as $t \to 0$, leading to a diverging value for $\dot{\Phi}$. In a toy quantum mechanics model it was argued that the response of the system in this case is singular, suggesting that this singularity is a genuine pathology which is not smoothened out. However the conclusions for the toy model do not directly apply to the field theory. Important questions regarding the renormalization of this time dependent field theory remain and could invalidate this conclusion.

One is hesitant to try and draw general conclusions about the possibility of emergence of a smooth spacetime from string scale curved regions on the basis of the very limited analysis presented here. One lesson which has emerged is that, at least for the kind of time dependence studied in this paper, AdS space has a tendency to form a black hole [21]. This fate can be avoided (as in the case when $N\epsilon \ll 1$) but it requires

[20] The results reported in [26] are for the case of AdS_{d+1} spacetimes with d odd.

[21] AdS space is of course homogeneous so the reader might be puzzled about where the black hole forms. The point is that the time dependence imposed on the boundary picks out a particular

slow time variation or perhaps more generally rather finally tuned conditions. To understand in greater detail when this fate of black hole formation can be avoided requires a deeper understanding of the process of thermalization in the dual field theory.

In this paper we analyzed the effects of a time dependent dilaton. It will be interesting to extend this to other supergravity modes as well by making their boundary values time dependent - e.g, making the radius of the S^3 on which the gauge theory lives time dependent or introducing time dependence along the other exactly flat directions in the $\mathcal{N}=4$ theory besides the dilaton. Also, we have kept the parameter N fixed in this work. As was discussed in the introduction N measures the strength of quantum corrections and is also the value of R_{AdS} in Planck units Eq.(3.3). It would be interesting to consider cases where N changes and become smaller thereby increasing the strength of quantum effects and making the curvature of order l_{Pl}. One way to do this might be by introducing time dependence that moves the system onto the Coulomb branch. This could reduce the effective value of N in the interior. For recent interesting work see, [38], also the related earlier work, [39], [40]. Finally, a length scale was introduced in the gauge theory by working on S^3 here. Instead one could consider a confining gauge theory like the Klebanov-Strassler kind [22], [41], which has a mass gap on R^3. In this case one could consider the response of the system to time dependence slow compared to the confining scale and hope to use an adiabatic approximation to understand this response.

notion of time and the black hole forms where the redshift factor for this time is smallest, this is the "center of AdS space" in global coordinates.

[22] We thank M. Mulligan for a related discussion.

Chapter 4 On Dumb Holes and their Gravity Duals

In recent years, gauge-string duality [2, 5] has been useful in exploring properties of strongly coupled field theories in regimes where their duals may be truncated to classical gravity. In particular, application of gauge-string duality to the hydrodynamic regime of these field theories has led to a "fluid-gravity correspondence" [25, 42, 43, 44, 45, 46]. Properties of solutions of classical gravity then lead to predictions for interesting properties of the dual fluid, the most celebrated example being the ratio of shear viscosity to the entropy density of a conformal fluid [42, 43].

In this note we use the fluid–gravity correspondence in the opposite direction. We use properties of supersonic fluid flows to predict interesting properties of fluctuations around a class of deformed black brane spacetimes in asymptotically AdS spacetimes. These spacetimes are duals of inhomogeneous flows of conformal fluids where the fluid velocity exceeds the speed of sound in some region. Unruh [47] showed that such flows lead to the formation of an "acoustic ergoregion" and, under suitable conditions, to an "acoustic horizon". The same physics which leads to Hawking radiation from black holes in General Relativity now leads to a Hawking-like radiation of quantized sound waves (or phonons) with a thermal spectrum, the temperature being proportional to the gradient of the velocity field at the acoustic horizon [49, 50]. Even when an acoustic horizon is not present, the presence of an ergoregion leads to characteristic properties like superradiance [52] . Fluid configurations with such acoustic horizons have been termed "dumb holes", and have been proposed as possible experimentally realizable systems for testing the physics of Hawking radiation in the laboratory [53] .

We will show that the gravity duals of such supersonic flows are non-static black holes. The duals of sound waves are then certain quasinormal modes around such black holes, and it follows from the fluid-gravity correspondence that at the quantum level one should find a Hawking-like radiation of these modes with an approximately thermal spectrum [44]. This Hawking-like radiation is distinct from the usual Hawking radiation associated with the black hole horizon, and would be present even when the background black hole is extremal and hence at zero temperature. The temperature of this quasinormal mode radiation depends on the properties of a "quasinormal mode horizon", which is an extension into the bulk of the acoustic horizon of the boundary fluid.

It should be emphasized that this phenomenon could have been found purely in General Relativity (or its supergravity extensions relevant to our considerations) by studying the fluctuation problem around these non-static black holes. However, without the fluid gravity correspondence and knowledge of acoustic Hawking radiation, there would not have been an obvious motivation to look for quasinormal mode horizons in non-static black brane backgrounds.

While we believe that the phenomenon of Hawking radiation or super-radiance of quasinormal modes is quite general, it turns out to be rather difficult to come up with examples within a controlled approximation scheme. The simplest and perhaps most interesting background where such a phenomenon could be present is a Kerr black hole in asymptotically AdS_5 spacetime. The dual of such a background is a rotating conformal fluid on S^3 [56, 57]. There is a regime of parameters of the black hole geometry for which the dual rotating fluid has supersonic velocities in a band around the equator of the S^3, thus producing an ergoregion for sound modes. The physics of sound waves around such a rotating fluid background would have a dual description in terms of quasinormal modes of gravitational perturbations around the Kerr black hole in AdS_5. However, this flow has vorticity, and the existence of acoustic Hawking radiation has been demonstrated mostly for irrotational flow. In the presence of nonzero vorticity, the sound modes get mixed up badly with other modes and analysis becomes difficult [58].

The situation simplifies, however, if the flow is irrotational. As shown in [47] for non-relativistic perfect fluids, and extended to relativistic perfect fluids in [59], the velocity potential then obeys the wave equation for a minimally coupled massless scalar field propagating on a curved background, the metric of which is determined by the underlying flow. The mathematical problem of quantizing sound waves or phonons around such a flow is then quite similar to that of quantizing a massless scalar field in an ordinary black hole background. This implies the existence of an acoustic analog of Hawking radiation.

Known examples of supersonic flows of perfect fluids often lead to infinite "acoustic surface gravity" (which is proportional to the gradient of the velocity at the acoustic horizon). The presence of viscosity usually regulates this divergence and renders it finite [55]. The incorporation of viscosity, however, makes the analysis complicated.

In this paper we find simple examples of acoustic horizons in *ideal* relativistic conformal fluids with finite acoustic Hawking temperature. The simplest example involves a fluid moving in a background spacetime of the form

$$ds_B^2 = -dt^2 + dz^2 + R(z)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

where $R(z)$ is a slowly varying function which has the behavior $R(z) \to |z|$ as $|z| \to \infty$. The fluid flow is steady and the only nonzero component of the fluid velocity is $v_z(z)$, with all derivatives bounded. Starting with $v_z = 0$ at $z = -\infty$ we will show that v_z reaches the speed of sound – producing an acoustic horizon – at minima of $R(z)$. If the function $R(z)$ has only one minimum, *e*.g. $R(z) = \sqrt{(z^2 + z_0^2)}$, the assumption of a smooth solution of the fluid equations of motion implies that the fluid velocity continues to increase beyond the acoustic horizon until it reaches the speed of light at $z = \infty$. However, if $R(z)$ has multiple extrema, *e*.g. two minima separated by a maximum, we will find smooth flows where the fluid velocity reaches a maximum supersonic value at the maximum of $R(z)$ and then decreases, then turns subsonic at the second minimum, and finally reaches zero at $z = \infty$. Sound waves cannot escape to the asymptotic region $z = -\infty$ from beyond the first acoustic horizon (which is therefore like a black hole horizon), and cannot cross the second acoustic horizon (which behaves more like a white hole horizon) from the $z = \infty$ asymptotic region .

We also study flows in a warped $R^{1,1} \times T^2$ geometry

$$ds^2 = -dt^2 + R(z)^2(d\theta_1^2 + d\theta_2^2) + dz^2 \tag{4.2}$$

and find very similar phenomena.

The acoustic Hawking radiation that arises when the sound modes of these flows are quantized will be detectable only if the acoustic Hawking temperature T_H is larger than the ambient temperature of the fluid, $T_H > T$. For an uncharged conformal fluid, the only scale is the temperature, so that hydrodynamics is valid only when all derivatives are small compared to the temperature. However T_H itself is proportional to the gradient of the velocity field, $T_H \sim \frac{dv_z}{dz}|_{z=\bar{z}}$. Thus it is not possible to discuss this phenomenon consistently in an uncharged conformal fluid.

This leads us to consider conformal fluids which carry a global charge density q with a conjugate chemical potential μ. For such a fluid in equilibrium, the ambient temperature can be made to vanish provided the charge density is chosen properly. In analogy with the gravitational duals of such fluids considered below, we will call such a fluid "extremal". We will consider *isentropic* flows of such a fluid where the charge density q, energy density ϵ and the velocity vary slowly (in a sense defined precisely below) but whose variations can themselves be $O(1)$. In an isentropic flow, the entropy density per unit charge density is, however, a constant. For such flows $q \propto \epsilon^{3/4}$ where ϵ is the energy density. Then $\mathcal{T} \propto \epsilon^{\frac{1}{4}}$ is the only independent energy scale in the theory. $\mathcal{T}$ is in general a function of the chemical potential μ and the temperature T. As shown below, the isentropic condition allows us to keep the local temperature T to be always much smaller than $\mathcal{T}$, even though the other hydrodynamic quantities can change by $O(1)$. In the limit of very small temperature $\mathcal{T} \propto \mu$. One would expect that the hydrodynamic approximation is valid so long as all gradients are small compared to $\mathcal{T}$. We will show that it is consistently possible to construct fluid flows described above with all gradients $\frac{dv_z}{dz}$ and $\frac{1}{R(z)}\frac{dR}{dz}$ much smaller than $\mathcal{T}$, thus ensuring $\mathcal{T} \gg T_H \gg T$.

The spacetimes (4.1) and (4.2) may be regarded as the boundary of an asymptotically AdS_5 near-extremal charged black brane geometry which is deformed due to a nonzero boundary curvature. This deformation can be large though slowly varying, *i*.e. $\frac{1}{R(z)}\frac{dR}{dz}$ is much smaller than the radius of the outer horizon R_+ in AdS units. Very close to extremality, $R_+ \sim \mathcal{T} \sim \mu$, so that this condition is in fact the condition for validity of hydrodynamics in the boundary theory. In the second example, the boundary metric has two compact directions. This means that the nature of the dual geometry depends on the size of the compact directions compared to R_+ [62]. We will choose $R(z) \gg R_+$ so that the dual is a near-extremal black brane rather than an AdS soliton with a small temperature.

A fluid flow profile in the boundary theory is then described by a normalizable deformation of this bulk metric. We construct the deformed bulk metric using a derivative expansion, following [45], [60] and [61]. The straightforward derivative expansion breaks down in "tubes" of constant retarded time where the geometry becomes exactly extremal; therefore, we consider fluid flows where the local temperature is small but nonzero. We then consider the class of linearized fluctuations around this background geometry - quasinormal modes - which are dual to sound waves in the presence of the corresponding fluid flow. Note that while the deformations of the bulk metric due to a nontrivial $R(z)$ and a nontrivial velocity profile $v_z(z)$ are typically large, the quasinormal mode amplitudes are small.

The behavior of the quasinormal modes clearly shows that at leading order in the derivative expansion, the acoustic horizon of the fluid extends into the bulk in the following sense. Let r denote the radial coordinate in the AdS space and let $z = \bar{z}$ be the location of the acoustic horizon in the boundary flow. We find that for any value of r, these quasinormal modes suffer an infinite blue-shift as we approach $z = \bar{z}$: modes which travel along the direction of the fluid flow are smooth at $z = \bar{z}$, while the modes which travel in the direction opposite to the flow have rapid oscillations. Thus, in an eikonal approximation these quasinormal modes cannot cross the quasinormal mode horizon at $z = \bar{z}$, which extends radially from the acoustic horizon into the bulk.

Standard arguments imply that upon quantization[1], one would find a thermal distribution of these quasinormal modes with a temperature T_H, which is the gravity dual of the acoustic Hawking radiation in the fluid. Only these specific quasinormal modes perceive the quasinormal mode horizon; other modes can cross it with ease. By the same token, the thermal distribution will be made up only of these quasinormal modes; it exists independently of (and at a different temperature from) the usual Hawking radiation associated with the event horizon of the background black brane.

Our discussion is restricted to the lowest non-trivial order in the derivative expansion, which is consistent with the perfect fluid approximation. However we expect that the physical consequences should survive higher derivative corrections. Furthermore our discussion of *fluctuations*, both in the boundary fluid and in bulk gravity, is restricted to the linearized limit. We do not address the effect of nonlinear interactions of the sound waves and other modes.

Admittedly, our setup is a bit contrived and is meant to provide a simple toy model in which this novel gravitational phenomenon can be studied in a controlled fashion. We expect, however, that the phenomenon is quite general and would be present in more interesting situations (*e*.g. the Kerr black hole mentioned above).

The paper is organized as follows. In Section 4.1, we give a self-contained discussion of acoustic metrics and dumb holes for conformal relativistic fluids. In Section 4.2, we describe the bulk dual. In Section 4.3 we discuss the regime of validity of our solutions.

4.1 Acoustic metric for relativistic conformal fluid

In this section we derive the equation governing the propagation of sound around gradient flows of a perfect relativistic conformal fluid. For such fluids, the pressure p and the energy density ϵ are related by

$$p = \frac{\epsilon}{3}. \tag{4.3}$$

For a charged conformal fluid with charge density q, there is an additional equation of state $\epsilon = \epsilon(s, q)$, or equivalently $s = s(\epsilon, q)$, that relates the energy, entropy and

[1]Quantization of bulk modes corresponds to $1/N$ corrections in the $SU(N)$ gauge theory on the boundary.

charge densities. [2] We will eventually be considering charged fluids with gravity duals, and will write down an explicit equation of state for such fluids in Section (4.2).

The first law of thermodynamics reads

$$d\epsilon = T\,ds + \mu\,dq\,, \tag{4.4}$$

where T and μ are the intensive quantities temperature and chemical potential respectively and can be obtained from the equation of state by taking derivatives:

$$T = \left.\frac{\partial\epsilon(s,q)}{\partial s}\right|_q, \qquad \mu = \left.\frac{\partial\epsilon(s,q)}{\partial q}\right|_s. \tag{4.5}$$

For a homogeneous system, it follows from extensivity that all thermodynamic variables are related by a Gibbs-Duhem relationship, which using (4.3) may be written as

$$\tfrac{4}{3}\epsilon = Ts + \mu q\,. \tag{4.6}$$

In the following, we will define a quantity $\mathcal{T}$ with dimensions of energy by

$$p = \frac{\epsilon}{3} = c\mathcal{T}^4\,, \tag{4.7}$$

where c is a dimensionless constants depending on the underlying system. $\mathcal{T}$ will be a function of T and μ (or q), which reduces to T in the uncharged limit. Our fluids will also admit a zero-temperature, finite-μ limit, close to which $\mathcal{T}$ is proportional to μ. $\mathcal{T}$ sets the energy scale of our conformal fluid, and will play an important role in defining limits in which our approximations are valid.

The equations of motion of fluid dynamics are conservation of the energy momentum tensor and conservation of the currents associated with any conserved charges, including the conserved particle number:

$$\begin{aligned} &\nabla_\mu T^{\mu\nu} = 0 \\ \text{and} \qquad &\nabla_\mu j_i^\mu = 0\,. \end{aligned} \tag{4.8}$$

The stress tensor $T^{\mu\nu}$ depends on the 3 independent components of velocity $v(x^\mu)$, the energy density, the pressure and their derivatives. The currents j_i^μ additionally depend on the densities q_i of the conserved charges. To leading order in the derivative expansion,

$$T^{\mu\nu} = p\,g^{\mu\nu} + (\epsilon + p)u^\mu u^\nu = c\mathcal{T}^4\left(g^{\mu\nu} + 4u^\mu u^\nu\right) \tag{4.9}$$

$$j_i^\mu = q_i u^\mu\,. \tag{4.10}$$

Here $u^\mu \equiv (\gamma, \gamma v)$ and $\gamma = \frac{1}{\sqrt{1-v^2}}$. Conformal invariance implies that the stress tensor is traceless. In a general curved background there is a trace anomaly; however this is a higher order effect in the derivative expansion and we have ignored it. We have also ignored the viscous and diffusive terms – which are again higher order in the

[2] For an uncharged conformal fluid, such a relation is trivial and $s \sim \epsilon^{\frac{4}{3}}$.

derivative expansion – and we therefore work in the perfect fluid limit. In addition we will also restrict attention to the case of a single charge of density $q(x^\mu)$.

The parallel component of the equations of motion (4.8), $u_\nu \nabla_\mu T^{\mu\nu} = 0$, leads to the conservation law

$$\nabla_\mu \left(\mathcal{T}^3 u^\mu\right) = 0\,. \tag{4.11}$$

In the uncharged case $\mathcal{T}^3$ is proportional to the entropy density of the fluid and the above equation is the conservation of the entropy current. The perpendicular component $P^\lambda_\nu \nabla_\mu T^{\mu\nu} = 0$ (where the projector $P^\lambda_\nu \equiv \delta^\lambda_\nu + u^\lambda u_\nu$) gives

$$u^\mu \nabla_\mu (\mathcal{T} u^\nu) = -\nabla^\nu \mathcal{T} \tag{4.12}$$

which can be manipulated to yield

$$\nabla_\mu(\mathcal{T} u_\nu) - \nabla_\nu(\mathcal{T} u_\mu) = -\mathcal{T}\omega_{\mu\nu}\,, \tag{4.13}$$

where $\omega_{\mu\nu} \equiv P^\lambda_\mu P^\kappa_\nu \left(\partial_\lambda u_\kappa - \partial_\kappa u_\lambda\right) = 0$ is the vorticity of the fluid.[3] Therefore, for an irrotational flow, we can define a potential ϕ such that

$$\mathcal{T} u_\mu = \partial_\mu \phi\,. \tag{4.14}$$

Thus to solve for irrotational flows of an uncharged fluid, it is sufficient to solve (4.14) and (4.11), along with an additional equation like (4.10) for every conserved charge. Note that since $u^\mu u_\mu = -1$, the equation (4.14) may be used to express $\mathcal{T}$ in terms of the potential ϕ

$$\mathcal{T}^2 = -(\partial_\mu \phi)(\partial^\mu \phi) \tag{4.15}$$

so that ϕ determines both u^μ and $\mathcal{T}$.

In general, the charge density q is not related to $\mathcal{T}$. We will, however, restrict ourselves to solutions where $q/\mathcal{T}^3$ is a constant. The current conservation equations (4.8) are then automatically solved once the equation (4.11) is solved. For such flows $\mathcal{T}(x)$ is the only independent dimensionful quantity that governs the flow. $\phi(x)$ determines all the hydrodynamic quantities once the ratio $q/\mathcal{T}^3$ is specified. In fact, substituting (4.15) in (4.14) and finally in (4.11) one gets a single complicated nonlinear differential equation for $\phi(x^\mu)$.

Although the restriction that $q \sim \mathcal{T}^3$ might seem quite ad hoc at this stage, for fluids with gravity duals that we will be considering in Section (4.2), this will turn out to imply that the flow is *isentropic*. Isentropic flows also allow us to parametrically control the temperature of the fluid. As discussed above, we need to consider fluids at low temperatures. In the flows we consider, derivatives of the velocity, entropy etc. are small, even though their values can and should change by $O(1)$. The equation (4.65) shows that once we fix the ratio q/s so that T is small at some time, it remains parametrically small at all times, since the change of s is of order 1.

Isentropic sound waves in such a gradient flow are described by small amplitude fluctuations of the velocity potential $\phi \to \phi + \delta\phi$. This induces variations of $u^\mu, \mathcal{T}$

[3] It can be shown that the condition of vanishing vorticity is identical to the condition $\partial_\mu(f u_\nu) = \partial_\nu(f u_\mu)$ for any scalar function f. In (4.14) f is simply the temperature T.

and q^i, $\mathcal{T} \to \mathcal{T} + \delta\mathcal{T},\ u^\mu \to u^\mu + \delta u^\mu, q_i \to q_i + \delta q_i$. Plugging these into the equations of motion (4.14), we get

$$\begin{aligned}(\mathcal{T}+\delta\mathcal{T})(u^\mu+\delta u^\mu) &= \partial^\mu\phi + \partial^\mu\delta\phi \\ \text{or} \qquad u^\mu\delta\mathcal{T} + \mathcal{T}\delta u^\mu &= \partial^\mu\delta\phi\,. \end{aligned} \tag{4.16}$$

Using $u_\mu\delta u^\mu = 0$ we get

$$\begin{aligned} \delta\mathcal{T} &= -u^\mu\partial_\mu\delta\phi \\ \text{and} \qquad \mathcal{T}\delta u^\mu &= P^{\mu\nu}\partial_\nu(\delta\phi)\,. \end{aligned} \tag{4.17}$$

Plugging (4.17) in (4.11)

$$\begin{aligned} &\nabla_\mu\left[\left(3\mathcal{T}^2\delta\mathcal{T}u^\mu + \mathcal{T}^3\delta u^\mu\right)\right] = 0 \\ \Longrightarrow \qquad &\partial_\mu\left[\sqrt{-g}\mathcal{T}^2\left(g^{\mu\nu} - 2u^\mu u^\nu\right)\partial_\nu\right](\delta\phi) = 0 \end{aligned} \tag{4.18}$$

For sound waves in a static equilibrium fluid in flat spacetime, this gives $(-3\partial_t^2 + \partial_i^2)(\delta\phi) = 0$, from which we can read off the speed of sound $c_s = \frac{1}{\sqrt{3}}$.

More generally (4.18) is the Klein-Gordon equation of motion of a *massless* scalar field in a non-trivial background metric,

$$\partial_\mu\left[\sqrt{-G}G^{\mu\nu}\partial_\nu\right](\delta\phi) = 0 \tag{4.19}$$

where

$$\begin{aligned} \sqrt{-G}G^{\mu\nu} &= \sqrt{-g}\mathcal{T}^2\left(g^{\mu\nu} - 2u^\mu u^\nu\right) \\ G_{\mu\nu} &= \sqrt{3}\mathcal{T}^2\left(g_{\mu\nu} + \frac{2}{3}u_\mu u_\nu\right)\,. \end{aligned} \tag{4.20}$$

The metric $G_{\mu\nu}$ above, termed the "acoustic metric", is described by the line element [4]

$$d\tilde{s}^2 = \sqrt{3}\mathcal{T}^2\left\{-(1-\frac{2}{3}\gamma^2)dt^2 - \frac{4}{3}\gamma^2 v_i dx^i dt + \left(g_{ij} + \frac{2}{3}\gamma^2 v_i v_j\right)dx^i dx^j\right\}\,. \tag{4.21}$$

If we make the transformation $d\tau = dt + \frac{\frac{2}{3}\gamma^2 v_i}{1-\frac{2}{3}\gamma^2}dx^i$, the metric becomes

$$d\tilde{s}^2 = \sqrt{3}\mathcal{T}^2\left\{-(1-\frac{2}{3}\gamma^2)d\tau^2 + \left(g_{ij} + \frac{\frac{2}{3}\gamma^2}{1-\frac{2}{3}\gamma^2}v_i v_j\right)dx^i dx^j\right\}\,. \tag{4.22}$$

This is the most general form of the acoustic metric for a relativistic conformal fluid. Note that the metric factors vanish or become singular at $\gamma = \sqrt{\frac{3}{2}}$ which precisely corresponds to the speed of sound $v = c_s = \frac{1}{\sqrt{3}}$. This indicates that an "acoustic horizon" is formed where the flow becomes supersonic and sound waves do not emerge from out of that horizon.

[4] We will use $G_{\mu\nu}$ and $d\tilde{s}^2$ for the acoustic metric to distinguish it from the spacetime metric. Here the spacetime metric has the form $ds^2 = -dt^2 + g_{ij}dx^i dx^j$

Steady flows leading to acoustic horizons

In this section we will find steady fluid flows with acoustic horizons when the background spacetime has a metric of the form

$$ds^2 = -dt^2 + dz^2 + R(z)^2 d\Omega_2^2 \,. \tag{4.23}$$

$R(z) = z$ corresponds to flat spacetime; we will later consider more general functions $R(z)$ and work on spacetimes that are asymptotically flat. We also assume that the thermodynamic quantities depend only on z and that $v_z(z)$ is the only nonzero component of the velocity. The form of the acoustic metric is then

$$\begin{aligned} d\tilde{s}^2 &= \sqrt{3}\mathcal{T}^2 \left\{ -(1 - \frac{2}{3}\gamma(z)^2) d\tau^2 + \frac{dz^2}{3(1-\frac{2}{3}\gamma(z)^2)} + R(z)^2 d\Omega_2^2 \right\} \qquad (4.24) \\ &= \sqrt{3}\mathcal{T}^2 \left\{ -c_s^2\,\gamma(z)^2 (1 - \frac{v_z(z)^2}{c_s^2}) d\tau^2 + \frac{dz^2}{\gamma(z)^2(1-\frac{v_z(z)^2}{c_s^2})} + R(z)^2 d\Omega_2^2 \right\} . \end{aligned}$$

Up to an overall conformal factor the metric is remarkably similar to that of a Schwarzschild black hole and a horizon is present at the radius where the flow becomes supersonic. An acoustic Hawking temperature T_H and a surface gravity κ can be defined by the standard process of Euclidean continuation of the acoustic metric near the horizon; then

$$T_H = \frac{\kappa}{2\pi} = \frac{3}{4\pi} \left| \frac{dv_z}{dz} \right|_{z_h} . \tag{4.25}$$

Thermal radiation of quantized phonons is expected from the horizon since Hawking radiation is a purely kinematic effect independent of the underlying dynamics [54].

In order to get an explicit solution, we need to solve the equations of motion (4.13) and (4.11)

$$\partial_z(\mathcal{T}\gamma) = 0 \quad \Longrightarrow \quad \mathcal{T}\gamma = \mathcal{T}_\infty \tag{4.26}$$

$$\partial_z\left(R(z)^2\mathcal{T}^3\gamma v_z\right) = 0 \quad \Longrightarrow \quad R(z)^2\mathcal{T}^3\gamma v_z = \Phi_S \tag{4.27}$$

where we have identified the integration constants as the "asymptotic temperature" $\mathcal{T}_\infty$ and the "entropy flux" Φ_S. From (4.26) and (4.27)

$$v_z(1 - v_z^2) = \frac{\Phi_S}{\mathcal{T}_\infty^3}\frac{1}{R(z)^2} \,. \tag{4.28}$$

From the isentropic condition $q \sim \mathcal{T}^3$ it follows that

$$q(z) = \frac{q_\infty}{\gamma^3(z)} \,. \tag{4.29}$$

Singular radially symmetric solution If we take $R(z) = z$ with $z \in (0, \infty)$, the metric (4.23) describes flat spacetime. The LHS of Eq(4.28) has a maximum value

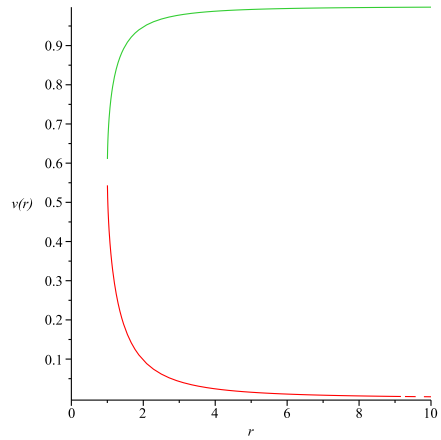

Figure 4.1: Plot of $v_z(z)$ for the spherically symmetric case given by Eq.(4.31) with $z_h = 1$. There are two physical branches; neither is valid for $z < z_h$. The third branch is superluminal.

of $\frac{2}{3\sqrt{3}}$ for $v = c_s = \frac{1}{\sqrt{3}}$, and therefore there is no solution for z below a minimum value of

$$z_h = \left(\frac{\Phi_S}{\mathcal{T}_\infty^3}\frac{3\sqrt{3}}{2}\right)^{\frac{1}{2}} . \tag{4.30}$$

$$\Rightarrow v_z(1 - v_z^2) = \frac{\Phi_S}{\mathcal{T}_\infty^3}\frac{1}{z^2} = \frac{2}{3\sqrt{3}}\frac{z_h^2}{z^2} . \tag{4.31}$$

This cubic equation has two physical branches, one subsonic and the other supersonic; the third branch is superluminal. The physical branches are plotted in Fig. 1. In the subsonic branch, as $z \to \infty$, $v \to 0$ which is consistent with the identification (4.26). At the horizon the derivatives blow up and the hydrodynamic description breaks down:

$$-\frac{dv_z}{dz} = \frac{\Phi_S}{\mathcal{T}_\infty^3}\frac{2}{3z^3(1 - 3v_z^2)} \to \infty \text{ at } \ z = z_h. \tag{4.32}$$

Solution in more general geometries To obtain solutions that are valid globally, we can choose a more general $R(z)$ such that

- the spacetime is asymptotically flat, $R(z) \sim |z|$ for $z \to \pm\infty$
- the maximum value of the RHS of (4.28) is $\frac{2}{3\sqrt{3}}$, the same as the maximum possible value of the LHS. This condition implies that the minimum value attained by $R(z)$ is

$$R_{\text{min}} = \left(\frac{3\sqrt{3}}{2}\frac{\Phi_S}{\mathcal{T}_\infty^3}\right)^{1/2} . \tag{4.33}$$

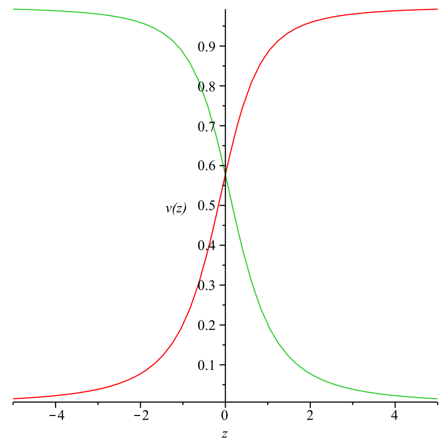

Figure 4.2: Plot of $v_z(z)$ for the wormhole geometry Eq(4.35) with $z_0 = 1$. There are two physical branches, making a subsonic to supersonic (supersonic to subsonic) transition at the acoustic horizon at $z = 0$.

Then we can construct a smooth solution that changes over from subsonic to supersonic or vice-versa every time $R(z)$ attains the above minimum value. For a generic velocity profile, the derivatives at the horizon do not blow up because the divergence of $\frac{dv_z}{dR(z)}$ at the horizon is canceled by the fact that

$$\frac{dR(z)}{dz} = -R(z)\frac{1-3v_z^2}{2v_z(1-v_z^2)}\frac{dv_z}{dz} = 0 \text{ at } z = z_h. \tag{4.34}$$

The simplest example is the wormhole geometry given by

$$R(z) = R_{\min}\sqrt{1+\frac{z^2}{z_0^2}}\,. \tag{4.35}$$

There are two asymptotically flat sheets as $z \to \pm\infty$ which are connected by a throat at $z = 0$, where $R(z) = R_{\min}$. Using (4.33), we get for (4.28)

$$v_z(1-v_z^2) = \frac{\Phi_S}{\mathcal{T}_\infty^3}\frac{1}{R(z)^2} = \frac{2}{3\sqrt{3}}\frac{z_0^2}{z^2+z_0^2}\,. \tag{4.36}$$

Again there are two physical branches of the cubic equation. One of them smoothly increases from $v = 0$ to $v = 1$ for $z \in (-\infty, \infty)$ while the other one smoothly decreases. Both solutions have acoustic horizons at $z = 0$. A plot of the velocity is given in Fig. 2. The velocity and its derivatives remain finite near the horizon, as seen from the near-horizon expansion of the increasing solution:

$$v_z = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{3}\frac{z}{z_0} \tag{4.37}$$

and the acoustic Hawking temperature is

$$T_H = \frac{3}{4\pi}\left|\frac{dv_z}{dz}\right|_{z=0} = \frac{1}{2\sqrt{2}\pi z_0}\,. \tag{4.38}$$

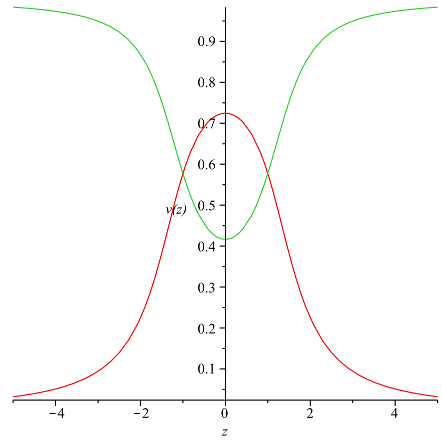

Figure 4.3: Plot of $v_z(z)$ for the geometry with two throats given by Eq.(4.39) with $z_0 = 1$. There are two horizons located at $z_h = \pm z_0$. The red branch remains subluminal for all z.

An obvious problem with this solution is that it reaches the speed of light asymptotically on one of the sheets.

To fix this problem, we can choose, for example

$$R(z) = R_{\min} \frac{(z^4 - 2z^2 z_0^2 + 2z_0^4)^{\frac{1}{4}}}{z_0} \tag{4.39}$$

which has minima at $z = \pm z_0$ where $R(z) = R_{\min}$. This $R(z)$ again corresponds to a wormhole with two asymptotically flat regions. Again using (4.33) we have

$$v_z(1 - v_z^2) = \frac{\Phi_S}{\mathcal{T}_\infty^3} \frac{1}{R(z)^2} = \frac{2}{3\sqrt{3}} \frac{z_0^2}{\sqrt{z^4 - z^2 z_0^2 + 2z_0^4}} \,. \tag{4.40}$$

Now we can find a solution that crosses over from subsonic to supersonic and back to subsonic at the horizons and remains subluminal for all z, as shown in Fig. 3.

Nozzle geometry We can also choose the spatial section of the metric to be asymptotically cylindrical with $R(z) \to$ constant at $z \to \pm\infty$. For variety we consider geometries with a toroidal cross-section

$$ds^2 = -dt^2 + dz^2 + R(z)^2(d\theta_1^2 + d\theta_2^2). \tag{4.41}$$

The form of the acoustic metric (4.23) remains very similar with the replacement of $d\Omega_2^2$ by $d\theta_1^2 + d\theta_2^2$. The acoustic Hawking temperature is still given by (4.25). As an example, we can take the following profile for $v(z)$ and then solve (4.28) for $R(z)$:

$$v_z \;\; = \;\; v_{\min} + (v_{\max} - v_{\min}) \,\text{sech}\left(\frac{z}{z_0}\right) \,. \tag{4.42}$$

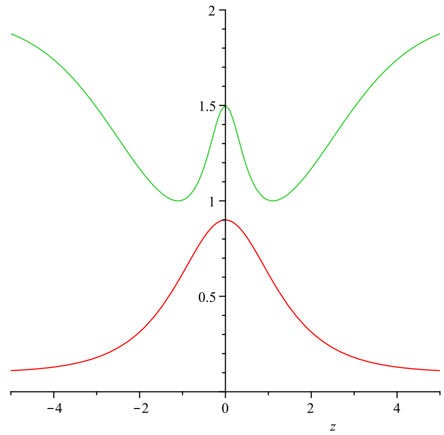

Figure 4.4: Velocity profile (in red) for the nozzle geometry given by Eq.(4.42) with $v_{\,\rm min} = 0.1$, $v_{\,\rm max} = 0.9$, $z_0 = 1$ and the corresponding $R(z)/R_{\,\rm min}$ (in green). The minima in $R(z)$ at $z = \pm z_0$ correspond to $v = c_s$.

The $v_{\,\rm min}$ term ensures that $R(z) \to$ constant for large $|z|$. The profiles of $v(z)$ and $R(z)$ are shown in Fig. 4. There are two horizons located at

$$z_\pm = \pm z_0 \cosh^{-1} \frac{v_{\,\rm max} - v_{\,\rm min}}{c_s - v_{\,\rm min}} \,. \tag{4.43}$$

Assuming $v_z > 0$, the fluid passes the speed of sound at the left horizon z_- and again returns to subsonic speeds as it crosses the right horizon z_+.

Sound waves

In this subsection we examine the behavior of sound waves around the background flows described in the previous subsections. Sound waves are fluctuations of the velocity potential ϕ which satisfy a massless Klein-Gordon equation in the background acoustic metric (4.24). We will consider background flows on a simple wormhole geometry - more complicated wormholes or nozzle geometries can be treated along similar lines.

Near the acoustic horizon (chosen at $z = 0$), the metric is similar to the usual Schwarzschild metric, so we expect a large blueshift effect for outgoing modes. To display this, it is sufficient to consider modes in the s-wave. Let us first rewrite the metric in terms of null coordinates u, v as follows

$$d\tilde{s}^2 = \sqrt{3}\mathcal{T}^2(z) \left\{ -(1 - \frac{2}{3}\gamma^2) du dv + R^2(z)(d\theta^2 + \sin^2\theta d\phi^2) \right\} , \tag{4.44}$$

where

$$\begin{aligned} du &= d\tau - \frac{dz}{\sqrt{3}(1-\frac{2}{3}\gamma^2)} = dt + \frac{\frac{2}{3}\gamma^2 v\, dz}{1-\frac{2}{3}\gamma^2} - \frac{dz}{\sqrt{3}(1-\frac{2}{3}\gamma^2)} = dt - \frac{dz}{v_+(z)} \\ dv &= d\tau + \frac{dz}{\sqrt{3}(1-\frac{2}{3}\gamma^2)} = dt + \frac{\frac{2}{3}\gamma^2 v\, dz}{1-\frac{2}{3}\gamma^2} + \frac{dz}{\sqrt{3}(1-\frac{2}{3}\gamma^2)} = dt - \frac{dz}{v_-(z)} \end{aligned} \tag{4.45}$$

Here $v_\pm(z) = \frac{v(z)\pm c_s}{1\pm v(z)c_s}$ is the relativistic sum of the local fluid velocity and the velocity of sound $c_s = 1/\sqrt{3}$. Then close to the acoustic horizon $R(z) \to R(0)$ is finite, and the s-wave solutions of the Klein-Gordon equation are approximately

$$\psi_+ \sim e^{-i\omega u} \qquad \psi_- \sim e^{-i\omega v} . \tag{4.46}$$

In the asymptotic region, where the velocity becomes constant, we obtain the usual spherical Bessel functions. To analyze the near-horizon behavior, we can expand the velocity field near the horizon as $v(z) \approx c_s + \frac{2}{3}\kappa z + \ldots$, where $\kappa = \frac{3}{2}\left|\frac{dv}{dz}\right|_{z_h}$ is the surface gravity at the horizon. Using this in (4.46), we find

$$\psi_+ \sim e^{-i\omega\left[t - \frac{2z}{\sqrt{3}} + O(z^2)\right]} \tag{4.47}$$

$$\psi_- \sim e^{-i\omega\left[t - \frac{1}{\kappa}\ln|z| + \frac{z}{\sqrt{3}} + O(z^2)\right]} \tag{4.48}$$

The ψ_+ mode is continuous at the horizon and is right moving with a velocity $\frac{\sqrt{3}}{2}$, which is the relativistic sum of $\frac{1}{\sqrt{3}}$ with itself. The ψ_- mode has rapid oscillations near the horizon:

$$z \lesssim 0 \; : \quad \psi_- \sim e^{-i\omega\left[t - \frac{1}{\kappa}\ln(-z)\right]} \quad \text{(left-moving)}$$

$$z \gtrsim 0 \; : \quad \psi_- \sim e^{-i\omega\left[t - \frac{1}{\kappa}\ln(z)\right]} \quad \text{(right-moving)}$$

indicating that inside the horizon, both modes are right-moving.

To extend these modes away from the horizon, we employ an eikonal approximation. We decompose the sonic fluctuation into a rapidly varying phase or "eikonal" (here $\lambda \gg 1$) times a slowly varying envelope

$$\psi(x^\mu) = A(x^\mu) e^{-i\lambda S(x^\mu)} \tag{4.49}$$

and plug it into the wave equation $\partial_\mu\left[\sqrt{-G}G^{\mu\nu}\partial_\nu\right]\psi = 0$ with the acoustic metric (4.21) written out in the original t-z coordinates

$$d\tilde{s}^2 = \mathcal{T}^2\left\{-(1-\frac{2}{3}\gamma^2)dt^2 - \frac{4}{3}\gamma^2 v_z dt dz + (1+\frac{2}{3}\gamma^2 v_z^2)dz^2 + R(z)^2(d\theta^2 + \sin^2\theta d\phi^2)\right\} \tag{4.50}$$

We then get a sequence of differential equations for $S(x^\mu)$ and $A(x^\mu)$ by expanding the wave equation order by order in λ:

$$O(\lambda^2) \; : \; \partial_\mu S(x^\alpha)\partial^\mu S(x^\alpha) = 0 \tag{4.51}$$

$$O(\lambda) \; : \; 2\partial_\mu S(x^\alpha)\partial^\mu A(x^\alpha) + A(x^\alpha)\nabla^2 S(x^\alpha) = 0 \tag{4.52}$$

$$\vdots$$

The leading equation (4.51) can be used to solve for $S(x^\alpha)$. Let us consider s-wave solutions independent of θ, ϕ. With the ansatz

$$\lambda S(x^\alpha) = \omega t - f(z) \tag{4.53}$$

we can solve for the phase,

$$\lambda S_{\pm}(x^\alpha) = \omega t + \omega \int dz\, \frac{\frac{2}{3}\gamma^2 v \mp \frac{1}{\sqrt{3}}}{1-\frac{2}{3}\gamma^2} = \omega t - \omega \int dz\, \frac{1 \pm \frac{v}{\sqrt{3}}}{v \pm \frac{1}{\sqrt{3}}} \tag{4.54}$$

The momenta of the wavepackets are given by derivatives of the eikonal[5] $p_\mu \equiv -\partial_\mu[\lambda S(x^\alpha)]$ giving

$$\begin{aligned} p_t &= \omega \\ p_z &= \omega\, \frac{1 \pm v c_s}{v \pm c_s} = \frac{\omega}{v_\pm}\,, \end{aligned} \tag{4.55}$$

where $v_\pm$ has been defined above. The upper and the lower signs would normally correspond to right- and left-moving sound modes. However, if the fluid (assumed to be right moving) has a velocity greater than the speed of sound, then both the modes become right-moving. Using the ansatz $A(x^\alpha) = A(z)$ in the subleading equation (4.52), a full solution is obtained:

$$\psi_\pm = \frac{A_0}{\mathcal{T} R(z)} e^{-i\omega\left[t - \int \frac{dz}{v \pm (z)}\right]} \tag{4.56}$$

4.2 Gravity dual of acoustic solution

The fluid-gravity correspondence [42, 43, 44, 45, 46, 25, 60, 61] provides a correspondence between solutions of certain fluids with classical solutions of suitable Einstein-Maxwell equations in a $4+1$ dimensional spacetime with a negative cosmological constant. In our case the fluid is a conformal fluid with a global U(1) charge and the higher dimensional bulk theory is given by the five-dimensional action

$$S = \frac{1}{16\pi G} \int d^5x\, \sqrt{-g} \left[R + 12 - F_{AB}F^{AB} - \frac{4\kappa}{3} \epsilon^{EABCD} A_E F_{AB} F_{CD} \right] \tag{4.57}$$

where G is the five dimensional Newton constant, the indices A, B run from 0 to 4, A_B is a U(1) gauge field, and we have chosen units in which the cosmological constant is $\Lambda = -6$. The above action is a consistent truncation of IIB supergravity for $\kappa = 1/(2\sqrt{3})$. We will, however, allow arbitrary values of κ.

A uniform charged black brane solution of this action is given in a boosted reference frame by

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 V(r,m,\tilde{q}) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ A &= \frac{\sqrt{3}\tilde{q}}{2r^2} u_\mu dx^\mu \end{aligned} \tag{4.58}$$

where u_μ are constant 4-velocities (the indices $\mu, \nu = 0 \cdots 3$) and $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu n_\nu$ is the spatial projection operator. The function $V(r, m, \tilde{q})$ is given by

$$V(r,m,\tilde{q}) = 1 - \frac{m}{r^4} + \frac{\tilde{q}^2}{r^6} \tag{4.59}$$

[5] The leading order equation (4.51) is thus a null geodesic equation, $p_\mu p^\mu = 0$ for the phonons.

where m and $\tilde{q}$ are parameters of the solution and we are using the notation of [61].

This solution is dual to a charged fluid in equilibrium living on the flat boundary of the five-dimensional spacetime. The fluid is strongly-coupled $\mathcal{N} = 4$ SU(N) Yang-Mills theory, viewed in a boosted frame with coordinates x^μ. The temperature T, charge density q, energy density ϵ and entropy density s of the fluid are given by [61]

$$T = \frac{R_+}{2\pi}\left(2 - \frac{\tilde{q}^2}{R_+^6}\right), \quad q = \sqrt{3}\alpha\tilde{q}, \quad \epsilon = 3\alpha m, \quad s = 4\pi\alpha R_+^3, \quad \alpha \equiv \frac{1}{16\pi G} \tag{4.60}$$

where R_+ denotes the radius of the outer horizon, i.e. the largest root of the equation $V(r, m, \tilde{q}) = 0$. The energy momentum tensor $T_{\mu\nu}$ and the charge current J_μ of the fluid are given by

$$T_{\mu\nu} = \frac{\epsilon}{3}(\eta_{\mu\nu} + 4u_\mu u_\nu) \qquad\qquad J_\mu = q u_\mu \tag{4.61}$$

The expressions (4.60) and (4.61) involve the bulk parameter G. In our units, G is related to the rank of the gauge group of the boundary theory by

$$G = \frac{\pi}{2N^2}, \qquad \alpha = \frac{1}{16\pi G} = \frac{N^2}{8\pi^2}. \tag{4.62}$$

With the substitutions in (4.60), the equation of state $\epsilon(s, q)$ becomes identical to the condition $V(R_+(s), m(\epsilon), \tilde{q}(q)) = 0$. The equation of state for a charged conformal fluid with a gravity dual is thus

$$1 - \frac{\epsilon}{3\alpha R_+^4} + \frac{q^2}{3\alpha^2 R_+^6} = 0, \text{ with } R_+ = \left(\frac{s}{4\pi\alpha}\right)^{\frac{1}{3}} \tag{4.63}$$

$$\Rightarrow \qquad \epsilon(s, q) = 3\alpha\left(\frac{s}{4\pi\alpha}\right)^{\frac{4}{3}} + \frac{q^2}{\alpha}\left(\frac{4\pi\alpha}{s}\right)^{\frac{2}{3}} \tag{4.64}$$

The temperature and chemical potential can be obtained by taking derivatives of $\epsilon(s, q)$ using (4.5):

$$\begin{aligned} T &= \frac{1}{\pi}\left(\frac{s}{4\pi\alpha}\right)^{1/3}\left(1 - \frac{8\pi^2 q^2}{3s^2}\right) \\ \mu &= 2\left(\frac{q}{\alpha}\right)\left(\frac{4\pi\alpha}{s}\right)^{2/3} \end{aligned} \tag{4.65}$$

This temperature reproduces the value quoted in (4.60). In the uncharged limit $R_+ = \pi T$, and one can fix the value of c defined in (4.7) by comparing it with (4.63) and requiring that $\mathcal{T} = T$ for uncharged fluids; this gives $c = \alpha\pi^4$ and hence

$$\epsilon_{\mu=0} = 3\alpha(\pi T)^4\,, \qquad s_{\mu=0} = 4\pi\alpha(\pi T)^3\,. \tag{4.66}$$

For charged fluids at finite μ there is a zero-temperature limit, reached when $R_+ = \mu/(2\sqrt{6})$ and $\mathcal{T} = \mu/(192^{\frac{1}{4}}\pi)$. In this limit,

$$q_{T=0} = \frac{\alpha}{48}\mu^3\,, \qquad \epsilon_{T=0} = \frac{\alpha}{64}\mu^4\,, \qquad s_{T=0} = \frac{\pi\alpha}{12\sqrt{6}}\mu^3\,. \tag{4.67}$$

As discussed in Section 2 we restrict our attention to *isentropic* flows. For such flows $q/\mathcal{T}^3$ is constant and since $\epsilon = 3\alpha\pi^4\mathcal{T}^4$, it follows from the equation of state (4.63) that $\mathcal{T}/R_+$ is a constant, and therefore that for such flows the entropy per unit charge s/q is constant. The first equation in (4.65) shows that by choosing

$$1 - \frac{8\pi^2}{3}\left(\frac{s}{q}\right)^2 \ll 1 \tag{4.68}$$

we can keep the temperature $T \ll R_+$ everywhere and at all times.

The gravity dual of a general fluid motion is then constructed in a derivative expansion as follows. First, we replace the parameters of the solution by functions of the boundary coordinates x^μ, $u^\mu \to u^\mu(x), m \to m(x), \tilde{q} \to \tilde{q}(x)$ which respectively represent the velocity field, energy density field and the charge density field of the fluid. We also replace the flat boundary metric $\eta_{\mu\nu}$ with a curved metric $g_{\mu\nu}(x)$. With these replacements, (4.58) is no longer a solution of the bulk equations of motion. Second we need to add correction terms to the metric and the gauge field so that the full metric and the gauge field now solve the equations of motion. This second step is of course impossible to perform in an exact fashion. However, these corrections can be calculated systematically in a *derivative expansion*, provided that the derivatives of $u^\mu(x),\ m(x),\ \tilde{q}(x)$ with respect to x^ν are small compared to the outer horizon radius R_+. To lowest nontrivial order in the derivative expansion, the modified metric and gauge fields are

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 V(r,m,\tilde{q}) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ &+ \frac{2}{3} r(\nabla_\alpha u^\alpha) u_\mu u_\nu dx^\mu dx^\nu + \frac{2r^2}{R_+}\sigma_{\mu\nu} F_2(\rho,M) dx^\mu dx^\nu \\ &- 2r u_\mu u^\alpha (\nabla_\alpha u_\nu) dx^\mu dx^\nu \\ &- 2u_\mu \left(\frac{\sqrt{3}\kappa\tilde{q}^3}{mr^4} l_\nu + \frac{6r^2}{R_+^7}(P_\nu^\lambda \partial_\lambda \tilde{q} + 3(u^\lambda\nabla_\lambda u_\nu)\tilde{q}) F_1(\rho,M) \right) dx^\mu dx^\nu, \end{aligned} \tag{4.69}$$

$$A = \left[\frac{\sqrt{3}\tilde{q}}{2r^2} u_\mu + \frac{3\kappa\tilde{q}^2}{2mr^2} l_\mu - \frac{\sqrt{3}r^5}{2R_+^8}(P_\mu^\lambda \partial_\lambda \tilde{q} + 3(u^\lambda \nabla_\lambda u_\mu)\tilde{q}) \right] dx^\mu \tag{4.70}$$

where we have defined the quantities

$$M = \frac{m}{R_+^4} \qquad Q = \frac{\tilde{q}}{R_+^3} \qquad \rho = \frac{r}{R_+}\,. \tag{4.71}$$

∇_μ is a covariant derivative with boundary metric $g_{\mu\nu}$, and

$$l_\mu = g_{\mu\gamma}\epsilon^{\nu\alpha\beta\gamma} u_\nu \nabla_\alpha u_\beta\,, \qquad \sigma_{\mu\nu} = \frac{1}{2}P^{\mu\alpha}P^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{3}P^{\mu\nu}(\nabla_\alpha u^\alpha)\,. \tag{4.72}$$

The functions $F_1(\rho,M,Q)$ and $F_2(\rho,M)$ are defined as

$$F_1(\rho,M,Q) = \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right) \int_\rho^\infty dp \frac{1}{(1-\frac{M}{\rho^4}+\frac{Q^2}{\rho^6})^2} \left(\frac{1}{p^8} - \frac{3}{4p^7}(1+\frac{1}{M})\right), \tag{4.73}$$

$$F_2(\rho,M) = \int_\rho^\infty dp \frac{p(p^2+p+1)}{(p+1)(p^4+p^2-M+1)} \tag{4.74}$$

Note that in the above expressions $m, \tilde{q}, R_+, M, Q, \rho$ are also functions of the boundary coordinates x^μ since m and $\tilde{q}$ are functions of x^μ.

This is a solution of the bulk equations of motion, provided that $m(x)$, $\tilde{q}(x)$ and $u_\mu(x)$ are such that the energy momentum tensor and current

$$T_{\mu\nu} = \frac{m(x)}{16\pi G}(g_{\mu\nu}(x) + 4u_\mu(x)u_\nu(x)) \qquad J_\mu = \frac{\sqrt{3}\tilde{q}(x)}{16\pi G}u_\mu(x) \tag{4.75}$$

are covariantly conserved,

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu J^\mu = 0 \tag{4.76}$$

Thus every solution of fluid dynamics leads to a bulk solution.

Gravity duals of dumb holes

We now apply the results of the preceding subsection to construct gravitational duals of the fluid flows with acoustic horizons that were studied in Section 2. These flows are special in several ways: first, the background spacetime metric of the fluid is of the form (4.23) or (4.41) where the only inhomogeneity is in the z direction. Second, both the background flow and the sound wave fluctuations have vanishing vorticity. Third, the background flows as well perturbations around them are isentropic.

It follows from the isentropic condition that the quantities M and Q are constants. As argued above (see discussion following equation (4.15)), for isentropic flows there is just one length scale, and all quantities are related to this length scale by dimensional analysis. In particular, the dimensionless quantities M and Q must be constant. In addition, the inhomogeneous parts of all quantities which appear in the bulk metric and the gauge field are determined in terms of a single scalar field $\phi(x)$.

As discussed in the introduction, in order for the acoustic Hawking radiation to be detectable we need to consider fluids which have a very small ambient temperature. This means that the constant quantities Q and M need to be close to their extremal values, $Q \approx \sqrt{2}$ and therefore $M \approx 3$. While the various quantities like $\epsilon(x), q(x)$ can change by $\mathcal{O}(1)$ amounts (only their derivatives are small), the isentropic condition ensures that if the fluid temperature is initially small it will remain small (see the discussion in Section 2 above).

To construct the background fluid flow, we simply need to insert the velocity potential ϕ_0 for the solutions of Section 2 into the general metric and gauge field in (4.69) and (4.70). The conditions of vanishing vorticity and isentropic flow simplify these general expressions somewhat. The most drastic simplification appears in the expression for the gauge field, equation (4.70). In fact the first order corrections (in the derivative expansion) to the gauge field vanish for isentropic gradient flows. To see this we note first that $\nabla_\alpha u_\beta$ can be replaced by $\partial_\alpha u_\beta$ in the expression l_μ of (4.72). Then using $\mathcal{T} u_\mu = \partial_\mu \phi$ we get

$$l_\mu = g_{\mu\gamma}\epsilon^{\nu\alpha\beta\gamma}\frac{\partial_\nu\phi}{\mathcal{T}}[-\frac{1}{\mathcal{T}^2}\partial_\alpha\mathcal{T}\partial_\beta\phi + \frac{1}{\mathcal{T}}\partial_\alpha\partial_\beta\phi] = 0 \tag{4.77}$$

due to antisymmetry of the epsilon symbol. The third term on the RHS of (4.70) also vanishes, as can be seen by applying the isentropic condition $q/\mathcal{T}^3$ =constant and the relations (4.14) and (4.15) to the expression

$$\begin{aligned} P^\lambda_\mu \nabla_\lambda \tilde{q} + 3(u^\lambda \nabla_\lambda u_\mu)\tilde{q} &= 3a\mathcal{T}\left[\mathcal{T}\partial_\mu \mathcal{T} + \mathcal{T}u^\lambda \nabla_\lambda(\mathcal{T}u_\mu)\right] \\ &= 3a\mathcal{T}\left[-\frac{1}{2}\partial_\mu(\partial_\alpha\phi\partial^\alpha\phi) + \partial^\lambda\phi\nabla_\lambda\partial_\mu\phi\right] = 0 \end{aligned} \quad (4.78)$$

Thus, to first order in the derivative expansion, the bulk gauge field is given by the first term of the right hand side of (4.70), which is just the term which would have resulted from a simple boost of the original black brane solution. In our case the charge density and the 4-velocity appearing in (4.70) are functions of z, as determined by the fluid flow on the boundary. So there is a nonzero electric field component along the z direction, given by

$$F_{0z} = -\frac{\sqrt{3}\tilde{q}_\infty}{r^2} v_z \partial_z v_z \quad (4.79)$$

where we have used (4.29) to express $q(z)$ in terms of the velocity v_z.

The expression for the bulk metric simplifies as well. In the fourth line of (4.69), the first term is proportional to l_μ which vanishes for our flows. The second term is proportional to H_μ defined in (4.78) and vanishes as shown above.

In the derivative expansion, the relationship between the boundary and the bulk becomes essentially local. The bulk solution can in fact be constructed approximately by patching together tube geometries obtained by extending the boundary data in a given region of the boundary to the bulk using the radial equations of motion. Consequently we expect that the acoustic horizon of the fluid flow on the boundary extends trivially into the bulk. We will explicitly verify this in the next subsection.

However, this tubular approximation breaks down in regions where the local geometry is exactly extremal. This is apparent in the results of [60] and [61]. Furthermore, recent work on perturbations around extremal black holes shows that the relevant low energy expansion is different from the naive derivative expansion [63, 64, 65, 66, 74]. Our results hold close to extremality, but not exactly at extremality.

Gravity duals of phonons

The gravity duals of phonons in the fluid are quasinormal modes of metric and gauge field perturbations. Once again, construction of these modes is trivial. We need to write

$$\phi(x^\mu) = \phi_0(x^\mu) + \beta\delta\phi(x^\mu), \quad (4.80)$$

compute $\mathcal{T}(x)$ and hence $m(x), q(x), u_\mu(x)$ in terms of $\delta\phi$, substitute into (4.69) and (4.70), and consider the terms which are linear in β. By construction, these modes satisfy ingoing boundary conditions at the bulk horizon.

The fluctuations of the gauge field A_μ obtained by this procedure have a particularly simple form

$$\delta A_\mu = -\beta\frac{\sqrt{3}a}{2r^2}[(\partial_\beta\phi_0)(\partial^\beta\phi_0)\delta^\alpha_\mu + (\partial_\mu\phi_0)(\partial^\alpha\phi_0)]\partial_\alpha(\delta\phi) \quad (4.81)$$

For the background flows considered in Section 2, we have found solutions to the wave equation (4.18) for $\delta\phi$ in the region close to the acoustic horizon. Upon inserting these solutions into (4.81), we see that the fluctuations δA_μ have a characteristic behavior near the acoustic horizon, viz. ingoing waves are smooth while outgoing waves have rapid fluctuations. This is the precise sense in which the fluctuations perceive the acoustic horizon, which has now extended into the bulk. From the nature of the solution that the extension of the acoustic horizon into the bulk is rather trivial - i.e. the horizon perceived by these modes is at the same value of z as the acoustic horizon on the boundary, and for all values of r.

The fluctuations for the components of the metric can be similarly worked out and also see a horizon structure at the same value of z. We therefore conclude that there are certain quasinormal modes of the bulk metric and the gauge field which perceive a horizon. If these bulk modes are quantized, one should find a thermal bath of such modes characterized by the temperature of the acoustic horizon on the boundary.

4.3 Regime of validity

It is important to check that the fluid flow described above is consistent with the standard conditions for validity of hydrodynamics. Roughly speaking, hydrodynamics is valid when the gradients of velocities, temperature and charge densities are small compared to the inverse mean free path l_m. For charged conformal fluids considered above, there are two scales - the temperature T and the chemical potential $\mu \equiv \nu T$, so that $l_m \sim f(\nu)/T$. The function $f(\nu)$ is of order one for generic values of ν, but there is an upper bound on ν, ν_c where $f(\nu)$ has a simple zero. It is possible to take the limit of $\nu \to \nu_c$ simultaneously with $T \to 0$ such that l_m is finite - the dual of this is in fact the extremal black hole. For the flow described in the previous section, in this limit we have

$$\begin{aligned} \left|\frac{dv_z(z)}{dz}\right| &\ll \frac{1}{l_m} \\ \left|\frac{1}{\mathcal{T}}\frac{d\mathcal{T}(z)}{dz}\right| &\ll \frac{1}{l_m} \end{aligned} \tag{4.82}$$

In particular, since the acoustic Hawking temperature T_H is $\frac{3}{4\pi}\left|\frac{dv_z}{dz}\right|_{z_\mp}$, this implies that

$$T_H \ll \frac{1}{l_m} \tag{4.83}$$

For observable acoustic Hawking radiation, the Hawking temperature should be higher than the fluid temperature. So we require

$$T \lesssim T_H \tag{4.84}$$

Furthermore, the frequency of the sound waves should also be small compared to the basic scale, $\omega \ll 1/l_m$. However, the finite Hawking temperature is going to introduce an upper bound on the allowed wavelengths, due to periodicity in Euclidean time;

thus $\omega > T_H$. Thus we need

$$T \lesssim T_H < \omega \ll \frac{1}{l_m} \tag{4.85}$$

For fluids with no conserved charge, there is only one energy scale, namely, the temperature T; thus $1/l_m \sim T$ and (4.84) cannot be satisfied. Although the solution is otherwise valid, the Hawking radiation, at a temperature much lower than the ambient temperature, is not going to be observable. For fluids with a conserved charge the condition (4.84) does not pose a problem because now we have two length scales, the temperature T and the chemical potential μ. For fluids very close to zero temperature, the mean free path will be governed only by μ. We can thus have

$$0 \approx T \lesssim T_H < \omega \ll \frac{1}{l_m} \approx \mathcal{T} \approx \mu \tag{4.86}$$

The ability to construct a gravity dual using a derivative expansion imposes further conditions. In the presence of a nonconstant $R(z)$, the validity of the derivative expansion of the solutions of the bulk equations of motion requires

$$\left| \frac{1}{R(z)} \frac{dR(z)}{dz} \right| \ll \frac{1}{l_m} \tag{4.87}$$

We get an additional condition if some of the boundary directions are compactified as in the nozzle geometry of Section (4.1). If one boundary direction of a $AdS \times S$ geometry is made compact with a radius R, the dual is an AdS soliton [67] which caps off the geometry at a value of the radial coordinate $r = 1/(2R)$. For a black brane geometry, compactification of a boundary direction would lead to a similar modification of the usual black brane geometry. However if $R \gg 1/(2R_+)$, where R_+ is the location of the black brane horizon, the place where the bulk geometry would cap off is far inside the black brane horizon. In this situation we can continue to use the standard black brane geometry with a compact longitudinal direction. We will therefore require that for all z,

$$R(z) \gg 1/R_+ \tag{4.88}$$

for the solution of Section (4.1). Finally we require for the nozzle solution that $R(z)$ be finite for large z. The geometry is then asymptotically $\mathbb{R} \times T^2$ and has an AdS dual.

Validity of our solutions

Finally, we determine the range of parameters for which our approximations are valid, for the specific flows studied in Section 2. Let us first discuss the wormhole solution of equations (4.39) and (4.40). The solution has four parameters, $\mathcal{T}_\infty$, q_∞, z_0 and $R_{\min}$. Φ_S is fixed by (4.33) once $R_{\min}$ is chosen. Since $v < 1$, γ remains finite for all z. $\mathcal{T} = \frac{\mathcal{T}_\infty}{\gamma} > 0$ and we can have a valid derivative expansion w.r.t. $\mathcal{T}_\infty$. All the derivatives, $\frac{dv_z}{dz}$, $\frac{1}{\mathcal{T}}\frac{d\mathcal{T}}{dz}$, $\frac{1}{R(z)}\frac{dR(z)}{dz}$ are proportional to $\frac{1}{z_0}$. q_∞ can be chosen such that we are always at very low temperatures, following the discussion around equation (4.68). Thus we require

$$T \to 0, \qquad \frac{1}{z_0} < \omega \ll \mathcal{T}_\infty \,. \tag{4.89}$$

For the nozzle solution described by (4.42), the parameters are z_0, $v_{\,\mathrm{max}}$, $v_{\,\mathrm{min}}$, $\mathcal{T}_\infty$, q_∞ and Φ_S. As in the previous case, all derivatives in the solution are proportional to $\frac{1}{z_0}$. We need $v_{\,\mathrm{max}} < 1$ for derivative expansion and we obtain the same conditions as (4.89). In addition, we require from (4.88) that $R_{\,\mathrm{min}} \gg 1/\mathcal{T}_\infty$. Since $R_{\,\mathrm{min}}$ is given by (4.33), this implies $\Phi_S \gg \mathcal{T}_\infty$. Moreover, we need $v_{\,\mathrm{min}} > 0$ in order that $R(z)$ be finite at large z – the asymptotic geometry remains $\mathbb{R} \times T^2$, and we have an asymptotically AdS gravity dual. Summarizing, for the nozzle solution, we require

$$T \to 0, \qquad \frac{1}{z_0} < \omega \ll \mathcal{T}_\infty \ll \Phi_S, \qquad 0 < v_{\,\mathrm{min}} < v_{\,\mathrm{max}} < 1. \tag{4.90}$$

Chapter 5 Small Amplitude Forced Fluid Dynamics from Gravity at $T = 0$

The AdS/CFT correspondence has provided useful insight into the dynamics of strongly coupled quantum field theories, particularly nonabelian gauge theories. Recently this has led to a fluid-gravity correspondence which provides a study of conformal fluid dynamics, an effective description of strongly coupled conformal field theory at long wavelengths in local equilibrium. In fact there is a precise mathematical connection between a long distance limit of the Einstein gravity and its holographic dual fluid dynamics [25, 44, 45, 60, 61, 68, 69]. Explicitly, it has been demonstrated that certain deformations of asymptotically AdS_{d+1} black branes which are slowly varying along the boundary directions (but can have $O(1)$ amplitudes) provide dual descriptions of solutions of the equations of fluid dynamics.

In fluid dynamics, there are local thermodynamic quantities such as local temperature, chemical potentials, R-charges or d-velocity, which are slowly varying along the boundary directions compared to effective equilibrium length scale of the fluid, the mean free path l_{mfp}. Bulk dual solutions of this inhomogeneous fluids are constructed order by order in derivatives respect to boundary coordinate.

More precisely, the asymptotically AdS space is foliated into a collection of tubes, each characterized by a value of the boundary coordinate. Each tube is centered about a radial ingoing null geodesic starting from AdS boundary. The width of each tube in the boundary direction is smaller than the scale of dual fluid dynamics. The gravity solution is developed locally in each tube, which turns out to be black brane with local thermodynamic quantities of the boundary fluid. The global geometry is constructed by gluing the tubes, in which the local thermodynamic quantities change along the boundary direction.

One crucial aspect of this construction is ultra-locality. The local thermodynamic quantities and metric corrections are expanded around a point on the boundary, which may be chosen to be x^μ=0. For some local thermodynamic quantity $q_i(x^\mu)$,

$$q_i(x^\mu) = q_i(0) + x^\mu \partial_\mu q_i(0) + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu q_i(0) + ... \tag{5.1}$$

In equilibrium these quantities are independent of x^μ so that all the higher terms vanish. The bulk then corresponds to some static black brane. These quantities appear in the bulk metric and other fields, generically denoted by $g(r, x^\mu)$, where r is the AdS radial coordinate. One then expands

$$g(r, x^\mu) = g^{(0)}(r, x^\mu) + g^{(1)}(r, x^\mu) + g^{(2)}(r, x^\mu) + ..., \tag{5.2}$$

where $g^{(0)}(r, x^\mu)$ is the bulk field obtained by replacing q_i in the equilibrium solution by $q_i(x^\mu)$. $g^{(0)}(r, x^\mu)$ is clearly not a solution of the bulk equations of motion. The higher order terms $g^{(n)}(r, x^\mu)$ are then determined by requiring that the full $g(r, x^\mu)$ solves the bulk equations of motion. The expansion above then constitutes a derivative expansion of a bulk solution. The equation satisfied by $g^{(n)}$ is of the form

$$H\left(g^{(0)}(q_i(0))\right) g^{(n)}(r, x^\mu) = s_n. \tag{5.3}$$

H is a linear differential operator in the second order in the radial variable. s_n is a source term from mth order corrections where $m < n$. The operator does not contain any derivative in the boundary directions, since this would produce terms of higher order. In other words, in this derivative expansion, H becomes ultra-local operator. The corrections $g^{(n)}(r, x^\mu)$ are required to be regular everywhere outside the black brane horizon. They are also required to fall of sufficiently fast at the boundary $r = \infty$, so that they do not lead to additional sources in the boundary theory.

In [60, 61, 68], this program has been carried out for conformal fluids carrying a global R charge. However, it was found that in the tubes where the local temperature is zero, i.e. the black hole locally looks like an extremal black hole, the solutions are singular at the horizon. These singularities seem to be genuine in that those possibly cause singularities in the curvature invariants at black brane horizon.

The failure of the derivative expansion method of deriving the bulk solution in these tubes does not agree with expectations from the dual fluid dynamics. In the dual fluid dynamics, the effective equilibrium length scale is the mean free path, which is given by $l_{mfp} \sim \frac{\eta}{\epsilon}$ [42, 57]. η is shear viscosity and ϵ is energy density. For the fluids described by the gravity solutions in [25, 43, 44, 45, 60, 61, 68, 70], $\eta = \frac{s}{4\pi}$, where s is entropy density. Consequently, $l_{mfp} \sim \frac{s}{4\pi\epsilon}$. In a tube where the local geometry becomes extremal, the zeroth order gravity solutions in the derivative expansion in these papers have finite entropy and energy densities. This implies that there should be a reasonable fluid dynamics even at zero temperature.

These divergences appear even in the linearized regime, i.e small amplitude *and* small frequency perturbations around an extremal black brane. Consider for example a linearized scalar field in the extremal background. It has been shown in [64] that in this case performing a small frequency expansion in the scalar field equation is not straight-forward. Technically, this is because the equation contains terms which have powers of the frequency multiplied by functions which blow up at the horizon. This may be traced to the fact that the components of the background metric have double zeroes or poles at the horizon as opposed to single zeroes or poles for a geometry at finite temperature.

A clue to resolve this problem is obtained by observing that the near horizon geometry of the charged black brane becomes $AdS_2 \times R^{d-1}$, where $d+1$ becomes bulk space-time dimension. For the near horizon region, it is natural to introduce a new radial coordinate $\zeta \sim \frac{\omega}{r-r_0}$, where ω is frequency of the scalar field, r is radial coordinate of the black brane, and r_0 is the horizon of the black brane. Motivated by the fact that in the near horizon region, the equations involve ζ with no further ω dependence, the small frequency expansion is then obtained by writing everything in terms of ζ and then expanding in powers of ω. To lowest order in this modified small frequency expansion, the equations are then solved in two regions : (i) the inner region close to the horizon, which is defined as $\zeta \to \infty$ with $\frac{\omega}{\zeta} \to 0$, and (ii) the outer region which corresponds to small ζ or $r \gg r_0$. It is essential to use the coordinate ζ in the inner region rather than r. The solution in the full space is then obtained by matching the inner and outer region solutions. By construction, a solution which is regular at the horizon exists. This technique has been used to obtained various response functions by solving the linearized gravity equations and gauge field equations [65, 66].

As will be clarified below, this modified low frequency expansion implies that the tubewise approximation used in obtaining gravity duals of boundary fluid flows breaks down. This is because the change of radial coordinates from r to ζ involves the frequency, so that in position space this implies a non-local (on the boundary) redefinition of fields. This reorganizes the low frequency expansion in which differential operators in the equations are not ultra-local any longer. The implications of this fact for fluid mechanics are not clear at the moment.

In this paper we consider a related problem where similar divergences appear and show how these get resolved. We consider four dimensional Einstein-Maxwell-dilaton theory with a negative cosmological constant. All bulk fields are spatially homogeneous but time dependent. The dilaton has a nonzero boundary value and the velocity of the boundary fluid is zero. First we consider slowly varying deformations of a charged black hole in this geometry in the presence of the dilaton source. These deformations are *entirely* due to the dilaton source. The setup is similar to that of [72] where a time dependent boundary value of the dilaton evolves an initially pure AdS geometry into that with a space-like region of large curvature. In the sense of fluid dynamics, our set up is a natural extension of [25] to R-charged fluid with vanishing velocities. The authors in [25] solve the Einstein-dilaton system with negative cosmological constant in which the boundary value of the dilaton field is slowly varying with arbitrary large amplitude. The field theory dual of the gravity system becomes a certain fluid dynamics satisfying Navier Stokes equations with dilaton dependent forcing term. We first treat the problem in a naive derivative expansion. In this expansion, only the dilaton can have $O(1)$ changes, as in [72]. The changes of the metric and gauge fields are due to the backreaction of the dilaton, and therefore suppressed by powers of the frequency. In [72] (as in [25, 45, 68]), the equations which determine the corrections to the fields at any given order n are linear and contain fields of order $(n-1)$. As expected, we find that the derivative expansion breaks down and solutions are singular at the horizon.

We then explore if the modified low energy limit can tame these divergences, keeping the variation of the dilaton field $O(1)$. We find that the effect of backreaction cannot be ignored in solving for the corrections to the dilaton and other fields even in the lowest order, even though the backreaction is small. This is because the nonlocal change of the radial coordinate typically implies that radial derivatives are *large* (in terms of the parameter of the derivative expansion) rather than $O(1)$, thus making the effect of the backreaction large.

This motivates us to consider the problem for the case where the dilaton has both small frequency as well as a small amplitude. We calculate the bulk dilaton and its backreaction to the metric and gauge fields to the leading order in amplitude. We find that in this case the modified low frequency expansion of [64] and the matching procedure indeed leads to bulk solutions which are smooth everywhere. This setup is similar to [64, 65, 66]. In these papers, the linearized problem was solved for fields which also depend on the spatial coordinates on the boundary, but the backreaction of the fields were not calculated. In our problem there is no such spatial dependence; the change of the metric and the gauge fields are entirely due to the backreaction of the dilaton. A consistent treatment of the backreaction however require us to go to higher orders in the frequency(in some case up to the fourth order). At the same time, we need to go to the second order in the amplitude of the boundary dilaton. The fact that this scheme works to lowest non-trivial order in the presence of back-reaction indicates that there is a systematic double expansion in the frequency and the amplitude which leads to non-singular solutions.

5.1 Charged Black Brane with Dilaton Field

In this section, we define our model to address the problem. We consider 4-dimensional Maxwell-Gravity theory with time dependent dilaton. The solutions are constructed order by order in small frequency in the perturbation theory. It turns out that leading corrections in the perturbation theory are divergent as they approach the black brane horizon in the extremal limit.

Derivative Expansion

A consistently truncated theory from M-theory with S^7 compactification [71] motivates us to consider Einstein-Maxwell-dilaton theory with negative cosmological constant,

$$S = \frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left(\frac{1}{4}R - \frac{1}{4}F_{MN}F^{MN} + \frac{3}{2L^2} - \frac{1}{8}\partial_M\phi\partial^M\phi\right), \tag{5.4}$$

where κ_4 is the gravitational constant. Indices M,N.. run from 0 to 3, and F_{MN} is field strength from $U(1)$ gauge field A_M. We choose units with $L = 1$. The equations of motion are

$$\begin{aligned} W_{MN} &\equiv R_{MN} + 3g_{MN} - 2F_{MP}F^P_N + \frac{1}{2}g_{MN}F_{PQ}F^{PQ} \qquad (5.5)\\ &- \frac{1}{2}\partial_M\phi\partial_N\phi = 0,\\ Y^N &\equiv \nabla_M F^{MN} = 0, \qquad (5.6)\\ X &\equiv \nabla^2\phi = 0, \qquad (5.7)\end{aligned}$$

where ∇ denotes a covariant derivative with bulk metric g_{MN}. A charged black brane solution of these equations of motion in Eddington-Finkelstein coordinates with constant dilaton is

$$\begin{aligned} ds^2 &= 2dvdr - U_0(r)dv^2 + r^2dx^idx^i, \qquad (5.8)\\ A &= \rho\left(\frac{1}{r_0} - \frac{1}{r}\right)dv, \qquad (5.9)\\ \phi &= \text{const}, \qquad (5.10)\end{aligned}$$

where $U_0(r) = r^2 + \frac{\rho^2}{r^2} - \frac{2\epsilon}{r}$ and r_0 is the outer horizon of the black brane, the largest root of $U_0(r_0) = 0$. v is the ingoing null coordinate which is time coordinate in the AdS_4 boundary. ρ and ϵ are charge density and energy density of the black brane respectively.

Let us consider time dependent dilaton which is slowly varying compared to r_0, $\phi = \phi_{(0)}(v)$. More precisely, the dialton has a form of

$$\phi_{(0)}(v) = f(\frac{\varepsilon v}{r_0}), \tag{5.11}$$

where ε is dimensionless small parameter. The function f satisfies

$$f'(\frac{\varepsilon v}{r_0}) \sim O(1), \tag{5.12}$$

where prime denotes derivative with respect to its argument. The derivative of the dilaton with time is suppressed by ε.

$$\frac{d\phi_0(v)}{dv} = \frac{\varepsilon}{r_0} f'(\frac{\varepsilon v}{r_0}) \sim \frac{\varepsilon}{r_0} \sim O(\varepsilon). \tag{5.13}$$

$\phi_{(0)}(v)$ is obviously not a solution of the equations of motions. To solve the dilaton equation perturbatively, we add correction terms. The dilaton is expanded as

$$\phi(r, v) = \phi_{(0)}(v) + \phi_{(1)}(r, v) + \phi_{(2)}(r, v)..., \tag{5.14}$$

where $\phi_{(0)}$ is zeroth order in ε, and $\phi_{(1)}$ is 1st order in ε and so on. The dilaton solution can be calculated perturbatively order by order in ε, which becomes the expansion parameter of the perturbation theory. We promote the energy density and the charge density to be functions of time as

$$\rho(v) = \rho_0 + r_0^2 C(v), \tag{5.15}$$
$$\epsilon(v) = \epsilon_0 + r_0^3 E(v), \tag{5.16}$$

where ρ_0 and ϵ_0 are constants. For further convenience, we set $C(-\infty) = E(-\infty) = 0$ as initial conditions. We expand $C(v)$ and $E(v)$ as

$$C(v) = C^{(0)}(v) + C^{(1)}(v) + C^{(2)}(v)... \tag{5.17}$$
$$E(v) = E^{(0)}(v) + E^{(1)}(v) + E^{(2)}(v)... \tag{5.18}$$

The dilaton equation up to first order in ε becomes

$$0 = \partial_r \left(r^2 U(r, v)\partial_r \phi_{(1)}(r, v)\right) + 2r\partial_v \phi_{(0)}(v), \tag{5.19}$$

where $U(r, v) = r^2 + \frac{(\rho_0 + r_0^2 C(v))^2}{r^2} - \frac{2(\epsilon_0 + r_0^3 E(v))}{r}$. As we will show below (See Eq(5.35) and Eq(5.36)), $C(v)$ and $E(v)$ are higher order in ε. The first order correction to the dilaton is given by

$$\phi_{(1)}(r, v) = \int^r \frac{r_0^2 \Lambda_1(v) - r^2}{r^2 U_0(r)} (\partial_v \phi_{(0)}(v)) + \Lambda_2(v), \tag{5.20}$$

where $U_0(r) = r^2 + \frac{\rho_0^2}{r^2} - \frac{2\epsilon_0}{r}$ and $\Lambda_1(v)$ and $\Lambda_2(v)$ are integration constants which are to be determined by boundary conditions that we demand. The regularity condition at the black brane horizon requires $\Lambda_1(v) = 1$. Moreover, we want a specific boundary condition that as $r \to \infty$, $\phi(r,v) = \phi_{(0)}(v)$. $\Lambda_2(v)$ is determined by this boundary condition.

We pause here to demonstrate the relationship of the tube-wise solution with derivative expansion. Consider the congruence of null geodesics emanating from AdS_4 boundary and tubes which are centered along the null geodesics. The set up is spatially homogeneous, so we classify the tubes by v. Without loss of generality, we set $v = 0$ for every individual tube. We expand $\phi_{(0)}(v)$ in the neighborhoods of $v = 0$ as

$$\phi_{(0)}(v) = \phi_{(0)}(0) + \varepsilon v \partial_v \phi_{(0)}(0) + \frac{1}{2}\varepsilon^2 v^2 \partial_v^2 \phi_{(0)}(0) + ... \tag{5.21}$$

The charge density and energy density can be expanded as

$$\rho(v) = \rho(0) + \varepsilon v \partial_v \rho(0) + \frac{1}{2}\varepsilon^2 v^2 \partial_v^2 \rho(0)..., \tag{5.22}$$

$$\epsilon(v) = \epsilon(0) + \varepsilon v \partial_v \epsilon(0) + \frac{1}{2}\varepsilon^2 v^2 \partial_v^2 \epsilon(0)... \tag{5.23}$$

We add correction terms to the dialton field as

$$\phi(r) = \phi_{(0)} + \varepsilon \phi_{(1)}(r) + \varepsilon^2 \phi_{(2)}(r)... \tag{5.24}$$

We omit v-dependence in the correction terms because the differential operator acting on these becomes ultra-local as argued below Eq(5.3). Plugging these into the dialton equation and evaluating it up to first order in ε at $v = 0$, we obtain

$$0 = \partial_r \left(r^2 U(r) \partial_r \phi_{(1)}(r)\right) + 2r \partial_v \phi_{(0)}(0), \tag{5.25}$$

where $U(r) = r^2 + \frac{\rho^2(0)}{r^2} - \frac{2\epsilon(0)}{r}$. The solution of this equation becomes

$$\phi_{(1)}(r) = \int^r \frac{r_0^2 \Lambda_1(0) - r^2}{r^2 U(r)} (\partial_v \phi_{(0)}(0)) + \Lambda_2(0), \tag{5.26}$$

where $\Lambda_1(0) = 1$ for the regularity of the solution $\phi_{(1)}(r)$ at the horizon and $\Lambda_2(0)$ is determined by the same boundary condition of our solution at AdS_4 boundary. To get global solution, we should patch every local solution. Even if the way of getting solution is different, Eq(5.20) and Eq(5.26) have the same form at least in the first order in small frequency expansion. This is also true for its back reactions.

Back reactions are obtained perturbatively order by order in ε with gauge field and metric being expanded as

$$g_{MN} = g_{MN}^{(0)} + g_{MN}^{(1)} + g_{MN}^{(2)}... \tag{5.27}$$

$$A_M = A_M^{(0)} + A_M^{(1)} + A_M^{(2)}... \tag{5.28}$$

For leading order correction, we try following form of metric and gauge field solutions:

$$
\begin{aligned}
ds^2 &= 2dvdr - U(r,v)dv^2 + r^2 dx^i dx^i + \frac{k(r,v)}{r^2}dv^2 - 2h(r,v)drdv, \quad (5.29)\\
A &= \rho(v)(\frac{1}{r_0(v)} - \frac{1}{r})dv + a(r,v)dv,
\end{aligned}
$$

where $k(r,v)$, $h(r,v)$, and $a(r,v)$ are leading order corrections in the perturbation theory. Details of equations of motions and calculations of the solutions are in Appendix B.1. We briefly list the leading back reactions. The leading corrections to gauge field and metric are

$$
\begin{aligned}
g^{(1)}_{MN} &= A^{(1)}_M = 0, \quad (5.30)\\
h(r,v) &\equiv -g^{(2)}_{rv} = -\frac{1}{4}(\partial_v\phi_{(0)}(v))^2 \int^r r'\left(\frac{r_0^2 - r'^2}{r'^2 U_0(r')}\right)^2 dr' + \bar{h}_1(v), \quad (5.31)\\
k(r,v) &\equiv r^2 g^{(2)}_{vv} = -\frac{r^2}{2}U_0(r)(\partial_v\phi_{(0)})^2 \int^r r'\left(\frac{r_0^2 - r'^2}{r'^2 U_0(r')}\right)^2 dr' \quad (5.32)\\
&+ \frac{r}{4}(\partial_v\phi_{(0)}(v))^2 \int^r \frac{(r_0^2 - r'^2)^2}{r'^2 U_0(r')}dr' + r\bar{k}_1(v) - 2\rho_0\bar{a}_1(v) + 2r^2 U_0(r)\bar{h}_1(v),\\
a(r,v) &\equiv A^{(2)}_v = \frac{\rho_0}{4}(\partial_v\phi_{(0)}(v))^2 \int^r \frac{dr'}{r'^2}\int^{r'} r''\left(\frac{r_0^2 - r''^2}{r''^2 U_0(r'')}\right)^2 dr'' \quad (5.33)\\
&- \frac{\bar{a}_1(v) - \rho_0\bar{h}_1(v)}{r} + \bar{a}_2(v).
\end{aligned}
$$

$\bar{h}_1(v)$, $\bar{a}_1(v)$, $\bar{a}_2(v)$ and $\bar{k}_1(v)$ are integration constants. They are determined by specific boundary conditions that we demand. At the black brane horizon, these solutions are already regular by choosing $\Lambda_1(v) = 1$. For the boundary condition at $r = \infty$ we demand that each leading correction of the perturbation theory behaves as

$$
\begin{aligned}
h(r,v) &\sim O(r^0), \quad (5.34)\\
k(r,v) &\sim O(r^3),\\
a(r,v) &\sim O(r^{-2}).
\end{aligned}
$$

The motivation behind these boundary conditions is that there are no non-normalizable modes which deform the boundary metric, chemical potential or charge density [45, 25, 68, 60, 61].

There are the constraint equations which are certain combinations of the equations of back reactions. They are in fact the equations of dual fluid dynamics [25].

$$
\begin{aligned}
C^{(0)}(v) &= C^{(1)}(v) = E^{(0)}(v) = 0, \quad (5.35)\\
\dot{E}^{(1)}(v) &= \frac{1}{4r_0}(\partial_v\phi_{(0)}(v))^2. \quad (5.36)
\end{aligned}
$$

The other components of gauge field and metric are trivial.

Divergences of the Leading Order Corrections in the Extremal Limit

The regularity of the solution that we impose for each perturbative correction in the previous section breaks down in the background of extremal black brane. In Appendix B.2, we derive the near horizon behavior of leading corrections of the dilaton and its back reactions in the extremal limit by expansion in $u-1$, where u is a rescaled radial coordinate, $u \equiv \frac{r}{r_0}$. To be more general, we keep $\Lambda_1(v)$ to be arbitrary in Eq(B.9). As shown above, $\Lambda_1(v) = 1$ ensures regularity at the horizon for non-extremal black brane. However in the extremal limit, all the corrections, Eq(5.20),Eq(5.31),Eq(5.32) and Eq(5.33) have singularities at the horizon. For example,

$$\phi(u,v) = \phi_{(0)} - \frac{\partial_v \phi_{(0)}(v)}{3r_0} ln(u-1) + O(1), \tag{5.37}$$

as $u \to 1$. The near horizon expansion of the back reactions are also leading to divergences in physical quantities like curvature invariants and field strengths. The $u \to 1$ behavior of the leading correction to the gauge field is given by

$$a(u,v) = -\frac{\sqrt{3}}{108}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0}\left(3ln(u-1) - 2(u-1)ln(u-1) + O(u-1)^2\right) + O(1). \tag{5.38}$$

The first two terms can cause singularity in the field strength, F_{rv}. The divergences of metric corrections are

$$h(u,v) = -\frac{1}{4}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0^2}\left(-\frac{1}{9(u-1)} + \frac{2}{27}ln(u-1) + O(1)\right), \tag{5.39}$$

$$k(u,v) = -\frac{1}{4}r_0^2(\partial_v\phi_{(0)}(v))^2\left(\frac{8}{9}(u-1)^2 ln(u-1) + O(1)\right). \tag{5.40}$$

In particular the term multiplying $(u-1)^2 ln(u-1)$ in $k(u,v)$ can possibly cause singularities in curvature invariants.

Divergence Resolution (The Main Result)

In this subsection we briefly discuss the main result of this paper without much technical details. To deal with divergences discussed in Sec.5.1, we follow Ref.[64] and divide the radial coordinate into two regions.

$$\text{Inner Region} \;:\; u-1 = \frac{\nu}{\xi} \;\; \text{for} \;\; \delta < \xi < \infty, \tag{5.41}$$

$$\text{Outer Region} \;:\; \frac{\nu}{\delta} < u-1,$$

with a certain scaling limit,

$$\nu \to 0, \;\; \xi = \text{ finite}, \;\; \delta \to 0, \;\; \text{and} \;\; \frac{\nu}{\delta} \to 0. \tag{5.42}$$

where ν is frequency of the fields in the perturbation theory. Note that switching the radial variable u to ξ is a non-local transformation. For this transformation, we need to evaluate our equations in the frequency space. We use ξ as a radial coordinate for the inner region and u as that for the outer region. We also define overlapping region (or matching region)as a region that $\xi \sim \delta$. The black brane horizon is located at $\xi = \infty$. As $\xi \to 0$, we approach the overlapping region by the scaling limit(5.42). The solutions listed in Sec.5.1 are the outer region solutions. The expansions of the dilaton and its back reactions as $u \to 1$ in Sec.5.1 are therefore the fields in the overlapping region. With the scaling limit we define a new perturbation theory in small frequency in the inner region. If the inner region solutions are

- Regular at the black brane horizon,
- Smoothly connected to the outer region solutions in the overlapping region,

the solutions are regular everywhere.

The dilaton equation in the extremal background is

$$\nabla^2\phi(u,v) = \frac{1}{u^2}\left(\partial_u(u^2 U(u,v)\partial_u\phi(u,v)) + \frac{1}{r_0}\partial_u(u^2\partial_v\phi(u,v)) + \frac{u^2}{r_0}\partial_u\partial_v\phi(u,v)\right) = 0, \tag{5.43}$$

where again we define dimensionless radial coordinate, $u \equiv \frac{r}{r_0}$. The metric factor $U(u,v) = \frac{(u-1)^2}{u^2}(u^2+2u+3) + \frac{2\sqrt{3}C(v)+C^2(v)}{u^2} - \frac{2E(v)}{u}$. More explicitly, the equation has a form of

$$\begin{aligned} 0 &= \partial_u\left((u-1)^2(u^2+2u+3)\partial_u\phi(u,v)\right) + \frac{1}{r_0}\partial_u(u^2\partial_v\phi(u,v)) \\ &+ \frac{u^2}{r_0}\partial_u\partial_v\phi(u,v) - 2E(v)\partial_u\left(u\partial_u\phi(u,v)\right). \end{aligned} \tag{5.44}$$

In this equation, we see that the term multiplying the energy density $E(v)$ is of $O(\varepsilon^2)$ whereas the terms proportional to $\partial_u^2\phi(u,v)$ is of $O(\varepsilon)$, using the constraint equation(5.36)(which shows $E(v) \sim O(\varepsilon)$) and the dilaton solution(5.20). Then, it may appear that the last term in Eq(5.44) can be ignored for obtaining the first order solution in ε. However, this is no longer true when we switch the radial variable u to ξ. The momentum ν which appears in Eq(5.41) is effectively proportional to ε. This is because the dilaton field is localized around $\nu \sim \varepsilon$ in the momentum space(See the discussion in the beginning of Sec.5.2). In ξ coordinate, each u-derivative in Eq(5.44) produces extra factor of $\frac{1}{\varepsilon}$. Therefore, the first term in Eq(5.44) becomes the higher order in ε than the term multiplying the energy density in ξ coordinate. This means that this later term cannot be ignored any longer. We have not yet been able to solve this type of equation. In the following we will choose a regime where the amplitude of the dilaton field is small. This will allow us to ignore the last term in Eq(5.44), but retain the essential feature of the problem.

The inner region solutions that we solve are completely agree with above two conditions for the entire solutions to be regular everywhere. To be precise, we briefly list our inner region solutions. The inner region solution of the dilaton has a form of

$$\begin{aligned}\phi_{(in)\nu}(\xi) &= \phi_\nu^{(0)} + \nu\left(A_\nu^{(1)} - \frac{i\phi_\nu^{(0)}}{3r_0}e^{\frac{i}{3r_0}\xi}E_1(\frac{i}{3r_0}\xi)\right) + \nu^2 A_\nu^{(2)} \\ &- \frac{ir_0\nu^2}{2}\left(\int_\infty^\xi \frac{g_{(2)\nu}(\xi')}{\xi'^2}d\xi' - e^{\frac{i}{3r_0}\xi}\int_\infty^\xi \frac{g_{(2)\nu}(\xi')}{\xi'^2}e^{-\frac{i}{3r_0}\xi'}d\xi'\right) + O(\nu)^3,\end{aligned} \tag{5.45}$$

where

$$g_{(2)\nu}(\xi) = \frac{2iA_\nu^{(1)}}{r_0} - \frac{4i\phi_\nu^{(0)}}{3r_0\xi} + \frac{2i\phi_\nu^{(0)}}{3r_0\xi}\left(1 - \frac{4i\xi}{3r_0}\right)e^{\frac{i\xi}{3r_0}}E_2(\frac{i\xi}{3r_0}). \tag{5.46}$$

$E_k(x) \equiv \int_1^\infty \frac{e^{-xt}}{t^k}dt$ is The Integral Exponential Function, where k is an integer. The numerical coefficients, $A_\nu^{(1)}$, $A_\nu^{(2)}$ and so on, are determined by matching with the outer region dilaton solution(5.37) in the overlapping region, more explicitly with Eq(B.9). The inner region solutions are developed in the frequency space. Therefore, for matching we need to take the outer region solution to momentum space by Fourier transformation defined in Eq(5.59). The precise expression of the outer region solution of the dilaton in frequency space is in Sec.5.2. We have a precise form of $A_\nu^{(1)}$ as

$$A_\nu^{(1)} = \frac{i\phi_\nu^{(0)}}{36r_0}\left(6\sqrt{2}tan^{-1}(\sqrt{2}) - 6ln(6) - \frac{6\pi}{\sqrt{2}}\right) + \frac{i\gamma}{3r_0}\phi_\nu^{(0)} + \frac{i\phi_\nu^{(0)}}{3r_0}ln\left(\frac{i\nu}{3r_0}\right), \tag{5.47}$$

where $\gamma = 0.57721..$ is Euler's constant. For the regularity at the horizon and the matching with the outer region dilaton solution, only ingoing waves at the black brane horizon are allowed in Eq(5.45). As a homogeneous solution of the dilaton equation we could add outgoing waves as $B_\nu^{(n)}e^{\frac{i\xi}{3r_0}}$ to Eq(5.45), where $B_\nu^{(n)}$ is a constant of order n in ε. The matching condition forces $B_\nu^{(0)} = 0$. For $n > 1$, it turns out that $B_\nu^{(n)}$ spoils the regularity of the dilaton solution in $n+1$th order in ϵ. Thus, we naturally impose ingoing boundary condition at the horizon for the smooth dilaton field. The other coefficients are obtained by matching with higher order solutions in the outer region. Near horizon, the inner solutions behave as

$$\begin{aligned}\phi_{(in)\nu}(\xi) &\sim \phi_\nu^{(0)} + \nu\left(A_\nu^{(1)} - \frac{\phi_\nu^{(0)}}{\xi} + ...\right) + \nu^2\left(A_\nu^{(2)} - \frac{A_\nu^{(1)}}{\xi}\right. \\ &+ \left.\frac{\phi_\nu^{(0)} - 3ir_0A_\nu^{(1)}}{\xi^2} + ...\right) + ...,\end{aligned} \tag{5.48}$$

which is manifestly regular as $\xi \to \infty$.

We evaluate back reactions from the dilaton up to the second order in small frequency as

$$
\begin{aligned}
h_{(in)\nu}(\xi) &= h^{(1)}_{(in)\nu}(\xi) + \nu^2 \bar{H}^{(2)}_\nu + \nu h^{(2)}_{(in)\nu}(\xi) ..., && (5.49)\\
a_{(in)\nu}(\xi) &= \nu^2 \bar{A}^{(2)}_\nu + \nu^2 \tilde{A}^{(2)}_\nu + \sqrt{3} r_0 \nu \int_\infty^\xi \frac{dy}{y^2} h^{(1)}_{(in)\nu}(y) - \sqrt{3} r_0 \nu^3 \frac{\bar{H}^{(2)}_\nu}{\xi} + \nu^3 \bar{A}^{(3)}_\nu && (5.50)\\
&+ \nu^3 \tilde{A}^{(3)}_\nu + \sqrt{3} r_0 \nu^2 \int_\infty^\xi \frac{dy}{y^2} \left(h^{(2)}_{(in)\nu}(y) - \frac{2}{y} h^{(1)}_{(in)\nu}(y) \right) ...,\\
k_{(in)\nu}(\xi) &= \nu^2 \bar{K}^{(2)}_\nu + \nu^3 \tilde{K}^{(3)}_\nu + 6 r_0^4 \nu^2 \left(\frac{h^{(1)}_{(in)\nu}(\xi)}{\xi^2} - 2 \int_\infty^\xi \frac{dy}{y^3} h^{(1)}_{(in)\nu}(y) \right) && (5.51)\\
&+ \nu^3 \left(\bar{K}^{(3)}_\nu + \frac{\bar{K}^{(2)}_\nu}{\xi} \right) + \nu^4 \left(\bar{K}^{(4)}_\nu + \frac{\bar{K}^{(3)}_\nu}{\xi} \right) + \nu^4 \left(\tilde{K}^{(4)}_\nu + \frac{\tilde{K}^{(3)}_\nu}{\xi} \right)\\
&- 2 r_0^4 \nu^3 \left(6 \int_\infty^\xi \frac{dy}{y^3} h^{(2)}_{(in)\nu}(y) + 3 \int_\infty^\xi \frac{dy}{y^4} h^{(1)}_{(in)\nu}(y) - 3 \frac{h^{(2)}_{(in)\nu}(\xi)}{\xi^2} - 2 \frac{h^{(1)}_{(in)\nu}(\xi)}{\xi^3} \right)\\
&+ 12 r_0^4 \nu^3 \int_\infty^\xi \frac{dy}{y^2} \int_\infty^y \frac{dz}{z^3} h^{(1)}_{(in)\nu}(z) + 12 \nu^4 r_0^4 \frac{\bar{H}^{(2)}_\nu}{\xi^2} ...,
\end{aligned}
$$

where

$$
\begin{aligned}
h^{(1)}_{(in)\nu}(\xi) &= \nu \tilde{H}^{(1)}_\nu - \frac{1}{36 r_0^2} \int_{-\infty}^{\infty} d\omega \phi^{(0)}_\omega \phi^{(0)}_{\nu-\omega} \frac{\omega(\nu-\omega)}{\nu} \int_\infty^\xi dy && (5.52)\\
&\times \left(1 - \frac{2 i \omega y}{3 r_0 \nu} e^{\frac{i\omega y}{3 r_0 \nu}} E_1(\frac{i \omega y}{3 r_0 \nu}) - \frac{\omega(\nu-\omega) y^2}{9 r_0^2 \nu^2} e^{\frac{i}{3 r_0} y} E_1(\frac{i\omega y}{3 r_0 \nu}) E_1(\frac{i(\nu-\omega) y}{3 r_0 \nu}) \right),\\
h^{(2)}_{(in)\nu}(\xi) &= \nu \tilde{H}^{(2)}_\nu - \frac{1}{36 r_0^2} \int_{-\infty}^{\infty} d\omega \phi^{(0)}_\omega \phi^{(0)}_{\nu-\omega} \frac{\omega(\nu-\omega)}{\nu} \int_\infty^\xi \frac{dy}{y} && (5.53)\\
&\times \left(1 - \frac{2 i \omega y}{3 r_0 \nu} e^{\frac{i\omega y}{3 r_0 \nu}} E_1(\frac{i \omega y}{3 r_0 \nu}) - \frac{\omega(\nu-\omega) y^2}{9 r_0^2 \nu^2} e^{\frac{i}{3 r_0} y} E_1(\frac{i\omega y}{3 r_0 \nu}) E_1(\frac{i(\nu-\omega) y}{3 r_0 \nu}) \right)\\
&- \frac{i}{36 r_0} \int_{-\infty}^{\infty} d\omega \phi^{(0)}_{\nu-\omega} \frac{\omega^3 (\nu-\omega)^2}{\nu^4} \int_\infty^\xi y^2 dy \left(\frac{\nu}{(\nu-\omega) y} e^{\frac{i\omega y}{3 r_0 \nu}} \right.\\
&\left. - \frac{i}{3 r_0} e^{\frac{i}{3 r_0} y} E_1(\frac{i(\nu-\omega) y}{3 r_0 \nu}) \right) \int_\infty^{\frac{\omega}{\nu} y} dz \frac{g_{(2)\omega}(z)}{z^2} e^{-\frac{i}{3 r_0} z}.
\end{aligned}
$$

$\bar{H}^{(2)}_\nu$, $\bar{A}^{(2)}_\nu$, $\bar{A}^{(3)}_\nu$, $\bar{K}^{(2)}_\nu$, $\bar{K}^{(3)}_\nu$ and $\bar{K}^{(4)}_\nu$ are integration constants which are determined by matching with the outer region solutions. Eq(5.49), Eq(5.50) and Eq(5.51) are smoothly connected to frequency space expressions of the outer region solutions of Eq(5.39), Eq(5.38) and Eq(5.40) respectively. Details are discussed in Sec.5.3. We determine some of the integration constants in the inner region solutions with this as

$$\begin{aligned}
\bar{H}^{(2)}_{\nu} &= \frac{1}{864r_0^2}\int_{-\infty}^{\infty} d\omega \frac{\omega(\nu-\omega)}{\nu^2}\phi^{(0)}_{\omega}\phi^{(0)}_{\nu-\omega}\left(8+\frac{13\sqrt{2}\pi}{2} - 13\sqrt{2}tan^{-1}(\sqrt{2}) \right. &(5.54)\\
&+ \left. 16ln(\nu) - 8ln(6)\right),\\
\bar{A}^{(2)}_{\nu} &= -\frac{\sqrt{3}}{288}\int_{-\infty}^{\infty} d\omega \frac{\omega(\nu-\omega)}{\nu^2}\phi^{(0)}_{\omega}\phi^{(0)}_{\nu-\omega}\left(-10+2\sqrt{2}tan^{-1}(\sqrt{2}) - \sqrt{2}\pi \right. &(5.55)\\
&- \left. 8ln(\nu) + 4ln(6)\right),\\
\bar{K}^{(2)}_{\nu} &= \frac{1}{4}r_0^2\int_{-\infty}^{\infty} d\omega \frac{\omega(\nu-\omega)}{\nu^2}\phi^{(0)}_{\omega}\phi^{(0)}_{\nu-\omega}\left(\sqrt{2}tan^{-1}(\sqrt{2}) - 1\right) + \frac{r_0\bar{k}^{\nu}_1}{\nu^2}. &(5.56)
\end{aligned}$$

$\bar{A}^{(3)}_{\nu}$, $\bar{K}^{(3)}_{\nu}$ and $\bar{K}^{(4)}_{\nu}$ are determined by higher orders in the outer region solutions. $\bar{k}^{\nu}_1$ is the Fourier transform of $\bar{k}_1(v)$. $\tilde{H}^{(1)}_{\nu}$, $\tilde{H}^{(2)}_{\nu}$, $\tilde{A}^{(2)}_{\nu}$, $\tilde{A}^{(3)}_{\nu}$, $\tilde{K}^{(3)}_{\nu}$ and $\tilde{K}^{(4)}_{\nu}$ are finite numerical constants, which cannot be obtained analytically(In principle we can. See Sec.5.3 for the discussion about these constants). The near horizon behaviors of the back reactions are given by

$$\begin{aligned}
h_{(in)\nu}(\xi) &\sim \nu\tilde{H}^{(1)}_{\nu} + \nu^2\bar{H}^{(2)}_{\nu} + \nu^2\tilde{H}^{(2)}_{\nu}... + O(\frac{1}{\xi}), &(5.57)\\
a_{(in)\nu}(\xi) &\sim \nu^2\bar{A}^{(2)}_{\nu} + \nu^2\tilde{A}^{(2)}_{\nu} + \nu^3\bar{A}^{(3)}_{\nu} + \nu^3\tilde{A}^{(3)}_{\nu}... + O(\frac{1}{\xi}),\\
\text{and}\quad k_{(in)\nu}(\xi) &\sim \nu^2\bar{K}^{(2)}_{\nu} + \nu^3\bar{K}^{(3)}_{\nu} + \nu^3\tilde{K}^{(3)}_{\nu} + \nu^4\bar{K}^{(4)}_{\nu} + \nu^4\tilde{K}^{(4)}_{\nu}... + O(\frac{1}{\xi}).
\end{aligned}$$

Thus the solutions are regular everywhere.

Finally, the constraint equations in the inner region turns out to be the same with those in the outer region(5.35), which have forms of

$$\begin{aligned}
C^{(0)}_{\nu} &= C^{(1)}_{\nu} = E^{(0)}_{\nu} = 0, &(5.58)\\
\nu^2E^{(1)}_{\nu} &= -\frac{1}{4ir_0}\int_{-\infty}^{\infty} d\omega\omega(\nu-\omega)\phi^{(0)}_{\omega}\phi^{(0)}_{\nu-\omega} \quad \text{and so on...},
\end{aligned}$$

in the frequency space.

5.2 Divergence Resolution of Dilaton Field

In this section, we solve the dialton equation in the inner region. The scaling limit(5.42) cannot be applied to Eq(5.44) because switching radial variable u to ξ is non-local transformation. To deal with this, we need to rewrite the equation in the frequency space by Fourier transformation as

$$\phi(u,v) = \int_{-\infty}^{\infty} e^{i\omega v}\phi_{\omega}(u)d\omega, \tag{5.59}$$

where $\phi_{\omega}(u)$ is a localized and normalizable function in frequency space. For example, we can choose $\phi_{\omega}(u)$ to be

$$\phi_{\omega}(u) \sim e^{-\frac{\omega^2}{\varepsilon^2}}e^{-\frac{\varepsilon^2}{\omega^2}}f(\frac{\omega}{\varepsilon})g_{\omega}(u), \tag{5.60}$$

where $g_\omega(u)$ is a function carrying radial variable u and $f(\frac{\omega}{\varepsilon})$ is arbitrary $O(1)$ function of in frequency space. Then, $\phi_\omega(u)$ is suppressed as ω approach either zero or ∞ by the exponential factors in it. Consequently, this function is extremely localized around $\omega = \pm\varepsilon$ and normalizable. Using the argument in Appendix B.4, one can show that Fourier transformation of Eq(5.60) becomes a form as Eq(5.11) in the frequency space. This shows that the properties of Eq(5.60) is consistent with those of $\phi_{(0)}(v)$ introduced in Sec.5.1. The dilaton equation in momentum space can be read off by acting an integral operator $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\nu v}dv$ on Eq(5.44). The equation in the momentum space has a form of

$$\begin{aligned} 0 &= \partial_u\left((u-1)^2(u^2+2u+3)\partial_u\phi_\nu(u)\right) + \frac{i\nu}{r_0}\partial_u(u^2\phi_\nu(u)) + \frac{i\nu u^2}{r_0}\partial_u\phi_\nu(u) \quad (5.61)\\ &- 2\int_{-\infty}^{\infty}\partial_u\left(u\partial_u\phi_\omega(u)\right)E_{\nu-\omega}d\omega, \end{aligned}$$

where $E_{\nu-\omega}$ is Fourier transfor of $E(v)$ as defined in Eq(B.25). In the inner and outer region, we expand the dilaton field as

$$\text{Inner Region}: \ \phi_{(in)\nu}(\xi) = \phi^{(0)}_{(in)\nu}(\xi) + \nu\phi^{(1)}_{(in)\nu}(\xi) + \nu^2\phi^{(2)}_{(in)\nu}(\xi) + ..., \quad (5.62)$$

$$\text{Outer Region}: \ \phi_\nu(u) = \phi^{(0)}_\nu + \nu\phi^{(1)}_\nu(u) + \nu^2\phi^{(2)}_\nu(u) + ..., \quad (5.63)$$

respectively. To avoid confusion, we do not tag outer region solutions with "(out)".

Inner Solution

As discussed in Sec.5.1, we solve linear dilaton equation by ignoring the last term in Eq(5.61). Switching radial variable u to ξ, the dilaton equation in the inner region becomes

$$\xi^2\partial_\xi\left((6+\frac{4\nu}{\xi}+\frac{\nu^2}{\xi^2})\partial_\xi\phi_{(in)\nu}(\xi)\right) - \frac{2i\xi^2}{r_0}(1+\frac{\nu}{\xi})^2\partial_\xi\phi_{(in)\nu}(\xi) + \frac{2i\xi^2}{r_0}(\frac{\nu}{\xi^2}+\frac{\nu^2}{\xi^3})\phi_{(in)\nu}(\xi) = 0. \quad (5.64)$$

The zeroth order equation in ν is

$$6\xi^2\partial_\xi^2\phi^{(0)}_{(in)\nu}(\xi) - \frac{2i\xi^2}{r_0}\partial_\xi\phi^{(0)}_{(in)\nu}(\xi) = 0. \quad (5.65)$$

This equation gives two linearly independent solutions,

$$\phi^{(0)}_{(in)\nu} = A^{(0)}_\nu + B^{(0)}_\nu e^{\frac{i}{3r_0}\xi}, \quad (5.66)$$

where $A^{(0)}_\nu$ is purely incoming wave and the term multiplying $B^{(0)}_\nu$ is purely outgoing wave at the extremal black brane horizon: $\xi = \infty$. The subscript "ν" denotes that the integration constants depend on frequency. These two independent solutions are regular at the horizon up to the zeroth order in ν. First order equation in ν is

$$6\xi^2\partial_\xi^2\phi^{(1)}_{(in)\nu}(\xi) - \frac{2i\xi^2}{r_0}\partial_\xi\phi^{(1)}_{(in)\nu}(\xi) + \frac{2i}{r_0}\left(A^{(0)}_\nu + B^{(0)}_\nu e^{\frac{i}{3r_0}\xi}\right) = 0. \quad (5.67)$$

The solution of this equation is

$$\phi^{(1)}_{(in)\nu} = -\frac{iA^{(0)}_\nu}{3r_0}\int_\infty^\xi e^{\frac{i}{3r_0}\xi'}d\xi'\left(\int_\infty^{\xi'}\frac{e^{-\frac{i}{3r_0}\xi''}}{\xi''^2}d\xi'' + B^{(1)}_\nu\right) + A^{(1)}_\nu + \frac{iB^{(0)}_\nu}{3r_0}\int_\infty^\xi \frac{e^{\frac{i}{3r_0}\xi'}}{\xi'}d\xi', \tag{5.68}$$

where $A^{(1)}_\nu$ and $B^{(1)}_\nu$ are integration constants, which are corresponding to incoming and outgoing waves at the horizon respectively. Using

$$\int^x e^{i\alpha x'}dx'\int^{x'}\frac{e^{-i\alpha x''}}{x''^2}dx'' = e^{i\alpha x}E_1(i\alpha x), \tag{5.69}$$

Eq(5.68) becomes

$$\phi^{(1)}_{(in)\nu}(\xi) = A^{(1)}_\nu - A^{(0)}_\nu B^{(1)}_\nu e^{\frac{i\xi}{3r_0}} - \frac{iA^{(0)}_\nu}{3r_0}e^{\frac{i}{3r_0}\xi}E_1(\frac{i}{3r_0}\xi) - \frac{iB^{(0)}_\nu}{3r_0}E_1(-\frac{i}{3r_0}\xi) \tag{5.70}$$

The first and the third terms are incoming waves at the horizon, whereas the others are outgoing waves. Let us discuss the asymptotic form of the solution. For large y, $E_1(y)$ is expanded as

$$E_1(y) = \frac{e^{-y}}{y}\sum_{n=0}^{\infty}\frac{(-1)^n(n)!}{y^n}, \tag{5.71}$$

where y is pure imaginary number, so e^{-y} term is bounded. It is manifest that $E_1(y)$ is regular as y approaches infinity. In the case that y goes to zero, the function $E_1(y)$ becomes divergent. Let us argue what the leading divergence is. To see the leading divergence, we calculate following object:

$$\lim_{y\to 0}\frac{E_1(y)}{ln(y)} = \lim_{y\to 0}\frac{\frac{dE_1(y)}{dy}}{\frac{dln(y)}{dy}} = -1, \tag{5.72}$$

where we have used Hospital's theorem for the first equality. Then, the leading divergent term is logarithmic. Consequently, this solution is regular at the horizon and divergent logarithmically near the matching region.

Outer Solution

In Appendix B.2, we evaluate the dilaton in outer region as $u \to 1$ in the extremal limit. The solution in momentum space is given by

$$\phi_\nu(u) \equiv (\phi^{(0)}_\nu + \nu\phi^{(1)}_\nu(u)) = \phi^{(0)}_\nu + \frac{i\nu\phi^{(0)}_\nu}{r_0}\int^u\frac{\Lambda_1 - u^2}{u^2U_0(u)}du + \Lambda_2, \tag{5.73}$$

where Λ_1 and Λ_2 come from the Fourier transforms of the integration constants $\Lambda_1(v)$ and $\Lambda_2(v)$. They depend on ν. Near matching region expansion of Eq(5.73) is

$$\begin{aligned}\phi_\nu(u) &= \phi_\nu^{(0)} + \frac{i\nu\phi_\nu^{(0)}}{6(u-1)r_0}(1-\Lambda_1) - \frac{i\nu\phi_\nu^{(0)}}{9r_0} ln(u-1)\,(2+\Lambda_1) \\ &+ \frac{i\nu\phi_\nu^{(0)}}{36r_0}\left((\Lambda_1-7)\sqrt{2}tan^{-1}(\sqrt{2}) + 2(\Lambda_1+2)ln(6) + \frac{\pi}{\sqrt{2}}(7-\Lambda_1)\right) \\ &+ \frac{i\nu\phi_\nu^{(0)}}{18r_0}(u-1) + O(u-1)^2.\end{aligned} \tag{5.74}$$

Matching

For the inner solution to match the outer one, we need to switch the radial coordinate ξ to u near matching region. The matching region is defined as a region with $\frac{\nu}{u-1} \sim \delta$. The scaling limit(5.42) shows that $\frac{\nu}{u-1}$ can become a small expansion parameter near matching region. The outer region solution is perturbative solution order by order in ν. It is justified that one can do series expansion of $\phi_{(in)\nu}$ in ν for matching the outer solution. The inner region solution up to the leading order correction in ν, Eq(5.70), in the radial variable u is given by

$$\begin{aligned}\phi_{(in)\nu}(u) &= A_\nu^{(0)} + B_\nu^{(0)} e^{\frac{i\nu}{3r_0(u-1)}} + \nu\left(A_\nu^{(1)} - A_\nu^{(0)}B_\nu^{(1)}e^{\frac{i\nu}{3r_0(u-1)}}\right. \\ &- \left.\frac{iA_\nu^{(0)}}{3r_0}e^{\frac{i\nu}{3r_0(u-1)}}E_1\left(\frac{i\nu}{3r_0(u-1)}\right) - \frac{iB_\nu^{(0)}}{3r_0}E_1\left(\frac{-i\nu}{3r_0(u-1)}\right)\right).\end{aligned} \tag{5.75}$$

Using asymptotic expansion of $E_1(y)$ for small y as

$$E_1(y) = -\gamma - ln(y) - \sum_{n=1}^{\infty}\frac{(-1)^n y^n}{nn!}, \tag{5.76}$$

we expand the inner region solution in terms of $\frac{\nu}{u-1}$. The expansion has a form of

$$\begin{aligned}\phi_{(in)\nu}(u) &= A_\nu^{(0)} + B_\nu^{(0)} + \nu\left(\frac{iB_\nu^{(0)}}{3r_0(u-1)} - \frac{i}{3r_0}(A_\nu^{(0)}+B_\nu^{(0)})ln(u-1) + A_\nu^{(1)}\right. \\ &- \left.A_\nu^{(0)}B_\nu^{(1)} + \frac{i\gamma}{3r_0}(A_\nu^{(0)}+B_\nu^{(0)}) + \frac{iA_\nu^{(0)}}{3r_0}ln\left(\frac{i\nu}{3r_0}\right) + \frac{iB_\nu^{(0)}}{3r_0}ln\left(-\frac{i\nu}{3r_0}\right)\right) \\ &+ O\left(\frac{\nu}{u-1}\right)^2\end{aligned} \tag{5.77}$$

We compare the asymptotes of $\phi_{(in)\nu}(u)$ with Eq(5.74) to determine each coefficient in it. At the zeroth order in ν,

$$A_\nu^{(0)} + B_\nu^{(0)} = \phi_\nu^{(0)}. \tag{5.78}$$

At the first order,

$$A_\nu^{(0)} + B_\nu^{(0)} = \frac{2+\Lambda_1}{3}\phi_\nu^{(0)}, \tag{5.79}$$

$$B_\nu^{(0)} = \frac{1-\Lambda_1}{2}\phi_\nu^{(0)}, \tag{5.80}$$

$$\begin{aligned} A_\nu^{(0)}B_\nu^{(1)} - A_\nu^{(1)} &= -\frac{i\gamma}{3r_0}(A_\nu^{(0)} + B_\nu^{(0)}) - \frac{iA_\nu^{(0)}}{3r_0}ln\left(\frac{i\nu}{3r_0}\right) - \frac{iB_\nu^{(0)}}{3r_0}ln\left(-\frac{i\nu}{3r_0}\right) \\ &+ \frac{i\phi_\nu^{(0)}}{36r_0}\left((\Lambda_1-7)\sqrt{2}tan^{-1}(\sqrt{2}) + 2(\Lambda_1+2)ln(6) + \frac{\pi}{\sqrt{2}}(7-\Lambda_1)\right) \end{aligned} \tag{5.81}$$

Eq(5.78), Eq(5.79), Eq(5.80) and Eq(5.81) provide

$$\Lambda_1 = 1, \quad A_\nu^{(0)} = \phi_\nu^{(0)}, \quad B_\nu^{(0)} = 0, \tag{5.82}$$

and

$$A_\nu^{(1)} - A_\nu^{(0)}B_\nu^{(1)} = \frac{i\phi_\nu^{(0)}}{6r_0}\left(\sqrt{2}tan^{-1}(\sqrt{2}) - ln(6) - \frac{\pi}{\sqrt{2}}\right) + \frac{i\gamma\phi_\nu^{(0)}}{3r_0} + \frac{i\phi_\nu^{(0)}}{3r_0}ln\left(\frac{i\nu}{3r_0}\right). \tag{5.83}$$

More on Dilaton Solution in Inner Region

As is clear from Eq(5.77), the lowest order solution in the inner region, expressed in terms of coordinate u contains all power of ν. This is true for all higher order corrections to the inner region solution as well. More precisely, $\nu^n\phi^{(n)}_{(in)\nu}(\xi)$, expressed in terms of u will have terms of $O(\nu^m)$ with $m \geqslant 1$. These terms are crucial in ensuring a smooth matching with the outer region solution. For example, in Eq(5.74) there is a term $\sim \nu\phi_\nu^{(0)}(u-1)$- but such a term is not present in $\nu\phi^{(1)}_{(in)\nu}(u)$. We now show that such a term is actually present in $\nu^2\phi^{(2)}_{(in)\nu}(u)$ with precisely the correct coefficient. To see this, let us evaluate the second order correction of dialton field. The second order equation is

$$6\xi^2\partial_\xi^2\phi^{(2)}_{(in)\nu}(\xi) - \frac{2i\xi^2}{r_0}\partial_\xi\phi^{(2)}_{(in)\nu}(\xi) + g_{(2)\nu}(\xi) = 0, \tag{5.84}$$

where $g_{(2)\nu}(\xi)$ is

$$g_{(2)\nu}(\xi) = 4\xi\phi^{(1)\prime\prime}_{(in)\nu}(\xi) - 4(1+\frac{i\xi}{r_0})\phi^{(1)\prime}_{(in)\nu}(\xi) + \frac{2i}{r_0}\phi^{(1)}_{(in)\nu}(\xi) + \frac{2i}{\xi r_0}\phi^{(0)}_{(in)\nu}. \tag{5.85}$$

The prime indicates derivative respect to ξ. The solution of this equation is

$$\phi^{(2)}_{(in)\nu}(\xi) = A_\nu^{(2)} + \frac{iB_\nu^{(2)}r_0}{2}e^{\frac{i}{3r_0}\xi} - \frac{ir_0}{2}\left(\int_\infty^\xi \frac{g_{(2)\nu}(\xi')}{\xi'^2}d\xi' - e^{\frac{i}{3r_0}\xi}\int_\infty^\xi \frac{g_{(2)\nu}(\xi')}{\xi'^2}e^{-\frac{i}{3r_0}\xi'}d\xi'\right), \tag{5.86}$$

where we set $B_\nu^{(2)}=0$ because it cause logarithmic divergence at the horizon in the third order in ν. Near horizon, $\phi^{(2)}_{(in)\nu}(\xi)$ is expanded as

$$\phi^{(2)}_{(in)\nu}(\xi)=\sum_{n=0}^{\infty}\frac{\alpha_\nu^n}{\xi^n}+\phi_\nu^{(0)}B_\nu^{(1)}\left(-\frac{4i}{9r_0}ln\xi+\frac{1}{\xi}+...\right), \tag{5.87}$$

where $\alpha_\nu^0=A_\nu^{(2)}$, $\alpha_\nu^1=-A_\nu^{(1)}$, $\alpha_\nu^2=\phi_\nu^{(0)}-3ir_0A_\nu^{(1)}$ and so on. Regularity condition of $\phi^{(2)}_{(in)\nu}(\xi)$ at the horizon forces $B_\nu^{(1)}=0$. Only incoming wave is allowed in $\phi^{(1)}_{(in)\nu}(\xi)$ too. Near the matching region, we switch the radial variable ξ to u for matching with the outer solution. Defining a new integral variable y as $\xi'\equiv\frac{\nu}{y-1}$, Eq(5.86) becomes

$$\phi^{(2)}_{(in)\nu}(u)=A_\nu^{(2)}+\frac{ir_0}{2\nu}\left(\int^u g_{(2)\nu}(\frac{\nu}{y-1})dy-e^{\frac{i\nu}{3r_0(u-1)}}\int^u g_{(2)\nu}(\frac{\nu}{y-1})e^{-\frac{i\nu}{3r_0(y-1)}}dy\right). \tag{5.88}$$

For matching, we expand $\phi^{(2)}_{(in)\nu}(u)$ in ν as

$$\phi^{(2)}_{(in)\nu}(u)=\frac{i\phi_\nu^{(0)}(u-1)}{18r_0\nu}+\sum_{j=0}^{\infty}\frac{\nu^j}{(u-1)^j}\left(\beta_{j\nu}^{(2)}+\beta_{j\nu}^{(2)\prime}ln(u-1)+\beta_{j\nu}^{(2)\prime\prime}\left(ln(u-1)\right)^2\right), \tag{5.89}$$

where $\beta_{j\nu}^{(2)}$,$\beta_{j\nu}^{(2)\prime}$ and $\beta_{j\nu}^{(2)\prime\prime}$ are $O(1)$ constants, some of which are given by $\beta_0^{(2)\prime}=\frac{6iA_\nu^{(1)}r_0+\phi_\nu^{(0)}}{18r_0^2}$, $\beta_0^{(2)\prime\prime}=-\frac{\phi_\nu^{(0)}}{54r_0^2}$ and so on. The first term in Eq(5.89) is proportional to ν, which matches the last term in Eq(5.74) precisely. We expect that similar mechanism ensures matching of the higher order terms.

5.3 Divergence Resolution of Back Reacted Metric and Gauge Field

In this section, we extend our discussion into back reactions. In Appendix.B.3, we obtain the equations of the back reactions without ignoring time derivatives. These equations are the starting point for our discussion.

Inner Solution

We begin with Eq(B.24). To solve this equation in the inner region, we need to substitute the inner region solution of the dilaton field into it. To do this, we take the inner region dilaton solution back to the outer region: $\phi_{(in)\omega}(\xi)\to\phi_{(in)\omega}(\frac{\omega}{u-1})$ and plug it into Eq(B.24). With this, Eq(B.24) becomes

$$\partial_u h_\nu(u)=-\frac{u}{4}\int_{-\infty}^{\infty}d\omega\partial_u\phi_{(in)\nu-\omega}(\frac{\nu-\omega}{u-1})\partial_u\phi_{(in)\omega}(\frac{\omega}{u-1}). \tag{5.90}$$

The u-derivative acting on the dilaton can be switched to derivative with respect to its argument as

$$\partial_u\phi_{(in)\omega}(\frac{\omega}{u-1})=-\frac{\omega}{(u-1)^2}\phi'_{(in)\omega}(\frac{\omega}{u-1}). \tag{5.91}$$

After this, we replace the radial coordinate u with ξ. Then, Eq(B.24) becomes

$$h'_{(in)\nu}(\xi) = \frac{\xi^2(1+\frac{\nu}{\xi})}{4\nu^3}\left(\int_{-\infty}^{\infty} d\omega\omega(\nu-\omega)\phi'_{(in)\omega}(\frac{\omega}{\nu}\xi)\phi'_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi)\right), \qquad (5.92)$$

where the prime on the dilaton denotes derivative with respect to its argument, whereas the prime on $h_{(in)\nu}(\xi)$ does derivative with respect to ξ. Plugging the dilaton expansion(5.62) into Eq(5.92), it becomes

$$\begin{aligned} h'_{(in)\nu}(\xi) &= \frac{\xi^2}{4\nu^3}(1+\frac{\nu}{\xi})\int_{-\infty}^{\infty} d\omega\omega^2(\nu-\omega)^2\left(\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi)\phi^{(1)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi)\right. \qquad (5.93)\\ &+ \omega\phi^{(2)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi)\phi^{(1)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi)+(\nu-\omega)\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi)\phi^{(2)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi)\\ &+ \left. ...\right). \end{aligned}$$

$h_{(in)\nu}(\xi)$ is also localized function around $\nu \sim \pm\varepsilon$ in frequency space as the dilaton field (See the discussion below Eq(5.60) in Sec.5.2). Then, we expand $h_{(in)\nu}(\xi)$ as

$$h_{(in)\nu}(\xi) = \bar{H}_\nu + h^{(1)}_{(in)\nu}(\xi) + \nu h^{(2)}_{(in)\nu}(\xi) + ..., \qquad (5.94)$$

where $h^{(1)}_{(in)\nu}(\xi)$ is in the first order in small frequency and $\nu h^{(2)}_{(in)\nu}(\xi)$ is in the second order and so on. We expand the other corrections of the back reactions in the same way. $\bar{H}_\nu$ is an integration constant which can be expanded as $\bar{H}_\nu = \nu\bar{H}^{(1)}_\nu + \nu^2\bar{H}^{(2)}_\nu ...$, where $\bar{H}^{(1)}_\nu$ and $\bar{H}^{(2)}_\nu ...$ are $O(1)$ constants. We note that counting power of ε of the solutions to show that each solution is in the correct order in small frequency expansion is not manifest in the frequency space. In Appendix B.4, we discuss details about this power counting by scaling all the frequencies appeared in the solutions with ε. The solutions up to the second order in small frequency are

$$\begin{aligned} h^{(1)}_{(in)\nu}(\xi) &= \nu\tilde{H}^{(1)}_\nu + \int_{\infty}^{\xi}\frac{\xi'^2 d\xi'}{4\nu^3}\int_{-\infty}^{\infty} d\omega\omega^2(\nu-\omega)^2\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi')\phi^{(1)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi'), (5.95)\\ h^{(2)}_{(in)\nu}(\xi) &= \nu\tilde{H}^{(2)}_\nu \qquad (5.96)\\ &+ \int_{\infty}^{\xi}\frac{\xi'^2 d\xi'}{4\nu^4}\int_{-\infty}^{\infty} d\omega\omega^2(\nu-\omega)^2\left(\frac{\nu}{\xi'}\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi')\phi^{(1)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi')\right.\\ &+ \left.\omega\phi^{(2)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi')\phi^{(1)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi')+(\nu-\omega)\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi')\phi^{(2)\prime}_{(in)\nu-\omega}(\frac{\nu-\omega}{\nu}\xi')\right). \end{aligned}$$

As will be shown below, the terms containing ξ'-integrations appearing in Eq(5.95) and Eq(5.96) vanish as $\xi \to \infty$. However, the near matching region expansions of those terms present additive constant terms. $\tilde{H}^{(1)}_\nu$ and $\tilde{H}^{(2)}_\nu$ are $O(1)$ numerical constants which are designed so that they precisely cancel those additive constant terms. With such a choice of $\tilde{H}^{(1)}_\nu$ and $\tilde{H}^{(2)}_\nu$, the only constant term in the overlapping region expansion of $h_{(in)\nu}(\xi)$ to match that in the outer region solution $h_\nu(u)$ become $\bar{H}_\nu$ (See Eq(5.94) and Eq(5.124)). There will be numerical constants as $\tilde{A}^{(2)}_\nu$, $\tilde{A}^{(3)}_\nu$, $\tilde{K}^{(3)}_\nu$

and $\tilde{K}^{(4)}_\nu$ appearing in $a_{(in)\nu}(\xi)$ and $k_{(in)\nu}(\xi)$ to be determined by the same manner. They provide the near matching region expansions of $a_{(in)\nu}(\xi)$ and $k_{(in)\nu}(\xi)$ to be exactly given by Eq(5.127) and Eq(5.128). We cannot determine these numerical constants analytically but in principle one can obtain the precise values.

Let us discuss regularity of the solutions. $\bar{H}_\nu$ is regular everywhere. To see the behaviors of $h^{(1)}_{(in)\nu}(\xi)$ and $h^{(2)}_{(in)\nu}(\xi)$ we expand $\phi^{(1)\prime}_{(in)\omega}(x)$ and $\phi^{(2)\prime}_{(in)\omega}(x)$ in Eq(5.95) and Eq(5.96) in the limit of large value of their argument x using that Eq(5.70), Eq(5.71), Eq(5.87) and $B^{(0)}_\nu = B^{(1)}_\nu = 0$. They are given by

$$\phi^{(1)\prime}_{(in)\omega}(x) = -\frac{i\phi^{(0)}_\omega}{3r_0 x}\sum_{n=1}^{\infty}\frac{(-1)^n n!}{(\frac{ix}{3r_0})^n}, \tag{5.97}$$

$$\phi^{(2)\prime}_{(in)\omega}(x) = -\sum_{n=1}^{\infty}\frac{n\alpha^n_\omega}{x^{n+1}}. \tag{5.98}$$

Substitution of Eq(5.97) to Eq(5.95) provides near horizon expansion of $h^{(1)}_{(in)\nu}(\xi)$ as

$$h^{(1)}_{(in)\nu}(\xi) = \nu\tilde{H}^{(1)}_\nu + \sum_{n,m=1}^{\infty}\frac{A_{mn}}{\xi^{m+n-1}}, \tag{5.99}$$

where

$$A_{mn} = \frac{1}{4}\frac{m!n!(-1)^{m+n+1}}{m+n-1}\nu^{m+n-1}\left(\frac{i}{3r_0}\right)^{2-m-n}\int_{-\infty}^{\infty}\phi^{(0)}_\omega\phi^{(0)}_{\nu-\omega}\omega^{1-n}(\nu-\omega)^{1-m}d\omega, \tag{5.100}$$

where m and n are integers. We note that the ω-integration in Eq(5.100) seems to have poles at $\omega = 0$ and $\omega = \nu$ for $m+n > 2$ and could lead to an infinite integrand. However, we set $\phi^{(0)}_\omega \sim e^{-\frac{\varepsilon^2}{\omega^2}}$ as $\omega \to 0$ as discussed in the beginning of Sec.5.2 (See Eq(5.60)). This ensures that the integration is finite. Then, near horizon behavior of $h^{(1)}_{(in)\nu}(\xi)$ is given by $h^{(1)}_{(in)\nu}(\xi) \sim \nu\tilde{H}^{(1)}_\nu + O(\frac{1}{\xi})$.

We obtain near horizon behavior of $h^{(2)}_{(in)\nu}(\xi)$ by plugging Eq(5.97) and Eq(5.98) into Eq(5.96), which is given by

$$h^{(2)}_{(in)\nu}(\xi) = \nu\tilde{H}^{(2)}_\nu + \sum_{n,m=1}^{\infty}\left(\frac{m+n-1}{m+n}\frac{A_{mn}}{\xi^{m+n}} + \frac{B_{mn}}{\xi^{m+n-1}}\right), \tag{5.101}$$

where

$$B_{mn} = \frac{m!(n)(-1)^{m+1}}{4(m+n-1)}\nu^{m+n-2}\left(\frac{i}{3r_0}\right)^{1-m}\int_{-\infty}^{\infty}\Big(\phi^{(0)}_{\nu-\omega}\alpha^n_\omega\omega^{2-n}(\nu-\omega)^{1-m} + \phi^{(0)}_\omega\alpha^n_{\nu-\omega}\omega^{1-m}(\nu-\omega)^{2-n}\Big)d\omega, \tag{5.102}$$

The α^n_ω in Eq(5.102) are proportional to $\phi^{(0)}_\omega$ (See Eq(5.82), Eq(5.83), Eq(5.87) and discussion below it). This ensures that the integration in Eq(5.102) is also finite. Consequently, $h^{(2)}_{(in)\nu}(\xi) \sim \nu\tilde{H}^{(2)}_\nu + O(\frac{1}{\xi})$ near horizon.

Secondly we solve Eq(B.29) which provides solutions of gauge field corrections in the inner region. Switching radial variable u to ξ, Eq(B.29) becomes

$$a'_{(in)\nu}(\xi) = \frac{\nu r_0}{\xi^2(1+\frac{\nu}{\xi})^2}\left(C_\nu + \sqrt{3}h_{(in)\nu}(\xi)\right). \tag{5.103}$$

We expand the charge density as $C_\nu = C_\nu^{(0)} + \nu C_\nu^{(1)} + \nu^2 C_\nu^{(2)}....$ $a_{(in)\nu}(\xi)$ can be expanded as $a_{(in)\nu}(\xi) = \bar{A}_\nu + a^{(1)}_{(in)\nu}(\xi) + \nu a^{(2)}_{(in)\nu}(\xi) + \nu^2 a^{(3)}_{(in)\nu}(\xi)...$, where again $\bar{A}_\nu$ is an integration constant which is also expanded as $\bar{A}_\nu = \nu\bar{A}_\nu^{(1)} + \nu^2\bar{A}_\nu^{(2)}...$ The solutions up to the third order expansion are given by

$$\begin{aligned}
a^{(1)}_{(in)\nu}(\xi) &= -r_0\nu\frac{C_\nu^{(0)}}{\xi}, && (5.104)\\
a^{(2)}_{(in)\nu}(\xi) &= \nu\tilde{A}_\nu^{(2)} + \sqrt{3}r_0\int_\infty^\xi \frac{d\xi'}{\xi'^2}h^{(1)}_{(in)\nu}(\xi') - \frac{\sqrt{3}r_0\nu}{\xi}\bar{H}_\nu^{(1)} && (5.105)\\
&+ r_0\nu\left(\frac{C_\nu^{(0)}}{\xi^2} - \frac{C_\nu^{(1)}}{\xi}\right),\\
a^{(3)}_{(in)\nu}(\xi) &= \nu\tilde{A}_\nu^{(3)} + \sqrt{3}r_0\int_\infty^\xi \frac{d\xi'}{\xi'^2}\left(h^{(2)}_{(in)\nu}(\xi') - \frac{2}{\xi'}h^{(1)}_{(in)\nu}(\xi')\right) && (5.106)\\
&+ \sqrt{3}r_0\nu\left(\frac{\bar{H}_\nu^{(1)}}{\xi^2} - \frac{\bar{H}_\nu^{(2)}}{\xi}\right) - r_0\nu\left(\frac{C_\nu^{(0)}}{\xi^3} - \frac{C_\nu^{(1)}}{\xi^2} + \frac{C_\nu^{(2)}}{\xi}\right).
\end{aligned}$$

The reason why we need to obtain $a_{(in)\nu}(\xi)$ up to the third order in the small frequency is that when we plug the nth order solution of $h_{(in)\nu}(\xi)$ into Eq(5.103), we get $n+1$th order solution of $a_{(in)\nu}(\xi)$. For the same reason, we need to get the inner solution of $k_{(in)\nu}(\xi)$ up to the fourth order in small frequency.

Let us explore near horizon limit of the solutions. $a^{(1)}_{(in)\nu}(\xi)$ is manifestly regular at the horizon. Using Eq(5.99) and Eq(5.101), we evaluate near horizon expansions of $a^{(2)}_{(in)\nu}$ and $a^{(3)}_{(in)\nu}$ which have forms of

$$\begin{aligned}
a^{(2)}_{(in)\nu} &= \nu\tilde{A}_\nu^{(2)} - \sqrt{3}r_0\sum_{m,n=1}^{\infty}\frac{A_{mn}}{m+n}\frac{1}{\xi^{m+n}} - \frac{\sqrt{3}r_0\nu}{\xi}\bar{H}_\nu^{(1)} && (5.107)\\
&+ r_0\nu\left(\frac{C_\nu^{(0)}}{\xi^2} - \frac{C_\nu^{(1)}}{\xi}\right) - \frac{\sqrt{3}r_0\nu}{\xi}\tilde{H}_\nu^{(1)},\\
a^{(3)}_{(in)\nu} &= \nu\tilde{A}_\nu^{(3)} + \sqrt{3}r_0\sum_{m,n=1}^{\infty}\left(\frac{A_{mn}}{m+n}\frac{1}{\xi^{m+n+1}} - \frac{B_{mn}}{m+n}\frac{1}{\xi^{m+n}}\right) && (5.108)\\
&+ \sqrt{3}r_0\nu\frac{\bar{H}_\nu^{(1)}}{\xi^2} - \sqrt{3}r_0\nu\frac{\bar{H}_\nu^{(2)}}{\xi} - r_0\nu\left(\frac{C^{(0)}}{\xi^3} - \frac{C^{(1)}}{\xi^2} + \frac{C^{(2)}}{\xi}\right)\\
&+ \sqrt{3}r_0\nu\left(\frac{\tilde{H}_\nu^{(1)}}{\xi^2} - \frac{\tilde{H}_\nu^{(2)}}{\xi}\right).
\end{aligned}$$

Then, $a_{(in)\nu}(\xi)$ is finite at the horizon.

Combining Eq(B.28) and Eq(B.30), we get

$$0 = -6(u^4-1)h_\nu(u) - u(u^4-4u+3)h'_\nu(u) + \frac{1}{r_0^4}\left(uk'_\nu(u) - k_\nu(u)\right) + 2\sqrt{3}C_\nu, \quad (5.109)$$

which gives solutions of $k_{(in)\nu}(\xi)$. Changing radial coordinate u into ξ, this equation becomes

$$\begin{aligned} 0 &= 2\sqrt{3}\nu C_\nu - \frac{1}{r_0^4}\left(\nu k_{(in)\nu}(\xi) + (\xi^2+\nu\xi)k'_{(in)\nu}(\xi)\right) \quad (5.110)\\ &+ \nu^2\left(6 + \frac{10\nu}{\xi} + \frac{5\nu^2}{\xi^2} + \frac{\nu^3}{\xi^3}\right)h'_{(in)\nu}(\xi) - 6\nu\left(\frac{4\nu}{\xi} + \frac{6\nu^2}{\xi^2} + \frac{4\nu^3}{\xi^3} + \frac{\nu^4}{\xi^4}\right)h_{(in)\nu}(\xi). \end{aligned}$$

We expand $k_{(in)\nu}(\xi)$ as $k_{(in)\nu}(\xi) = \bar{K}_\nu(\xi) + k^{(1)}_{(in)\nu}(\xi) + \nu k^{(2)}_{(in)\nu}(\xi) + \nu^2 k^{(3)}_{(in)\nu}(\xi)...$, where $\bar{K}_\nu(\xi)$ is a homogeneous solution of Eq(5.110) which satisfies

$$\nu\bar{K}_\nu(\xi) + (\xi^2+\nu\xi)\bar{K}'_\nu(\xi) = 0. \quad (5.111)$$

The solution of Eq(5.111) is given by

$$\bar{K}_\nu(\xi) = \nu\bar{K}^{(1)}_\nu + \nu^2\left(\bar{K}^{(2)}_\nu + \frac{\bar{K}^{(1)}_\nu}{\xi}\right) + \nu^3\left(\bar{K}^{(3)}_\nu + \frac{\bar{K}^{(2)}_\nu}{\xi}\right)..., \quad (5.112)$$

where $\bar{K}^{(1)}_\nu$, $\bar{K}^{(2)}_\nu$ and $\bar{K}^{(3)}_\nu$... are arbitrary $O(1)$ constants. We solve Eq(5.110) up to fourth order in small frequency which are given by

$$\begin{aligned} k^{(1)}_{(in)\nu}(\xi) &= -2\sqrt{3}r_0^4\nu\frac{C^{(0)}_\nu}{\xi} \quad (5.113)\\ k^{(2)}_{(in)\nu}(\xi) &= -2\sqrt{3}r_0^4\nu\frac{C^{(1)}_\nu}{\xi} \quad (5.114)\\ k^{(3)}_{(in)\nu}(\xi) &= \nu\tilde{K}^{(3)}_\nu + 6r_0^4\int_\infty^\xi\frac{d\xi'}{\xi'^2}\left(h^{(1)\prime}_{(in)\nu}(\xi') - \frac{4}{\xi'}h^{(1)}_{(in)\nu}(\xi')\right) + 12r_0^4\nu\frac{\bar{H}^{(1)}_\nu}{\xi^2} \quad (5.115)\\ &- 2\sqrt{3}r_0^4\nu\frac{C^{(2)}_\nu}{\xi},\\ k^{(4)}_{(in)\nu}(\xi) &= \nu\tilde{K}^{(4)}_\nu - 6r_0^4\int_\infty^\xi\frac{d\xi'}{\xi'^2}\int_\infty^{\xi'}\frac{d\xi''}{\xi''^2}\left(h^{(1)\prime}_{(in)\nu}(\xi'') - \frac{4}{\xi''}h^{(1)}_{(in)\nu}(\xi'')\right) \quad (5.116)\\ &- 2r_0^4\int_\infty^\xi\frac{d\xi'}{\xi'^2}\left(\frac{12}{\xi'}h^{(2)}_{(in)\nu}(\xi') + \frac{6}{\xi'^2}h^{(1)}_{(in)\nu}(\xi') - 3h^{(2)\prime}_{(in)\nu}(\xi') - \frac{2}{\xi'}h^{(1)\prime}_{(in)\nu}(\xi')\right)\\ &+ 12r_0^4\nu\frac{\bar{H}^{(2)}_\nu}{\xi^2} + 8r_0^4\nu\frac{\bar{H}^{(1)}_\nu}{\xi^3} - 2\sqrt{3}r_0^4\nu\frac{C^{(3)}_\nu}{\xi} + \nu\frac{\tilde{K}^{(3)}_\nu}{\xi}. \end{aligned}$$

$\bar{K}_\nu(\xi)$, $k^{(1)}_{(in)\nu}(\xi)$ and $k^{(2)}_{(in)\nu}(\xi)$ are manifestly regular at the horizon. We list behaviors of $k^{(3)}_{(in)\nu}(\xi)$ and $k^{(4)}_{(in)\nu}(\xi)$ near horizon:

$$\begin{aligned} k^{(3)}_{(in)\nu}(\xi) &= 12r_0^4\nu\frac{\bar{H}^{(1)}_\nu+\tilde{H}^{(1)}_\nu}{\xi^2}-2\sqrt{3}r_0^4\nu\frac{C^{(2)}_\nu}{\xi}+6r_0^4\sum_{m,n=1}^{\infty}\frac{m+n+3}{m+n+1}\frac{A_{mn}}{\xi^{m+n+1}} \quad (5.117)\\ &+ \nu\tilde{K}^{(3)}_\nu, \\ k^{(4)}_{(in)\nu}(\xi) &= 12r_0^4\nu\frac{\bar{H}^{(2)}_\nu+\tilde{H}^{(2)}_\nu}{\xi^2}+8r_0^4\nu\frac{\bar{H}^{(1)}_\nu+\tilde{H}^{(1)}_\nu}{\xi^3}-2\sqrt{3}r_0^4\nu\frac{C^{(3)}_\nu}{\xi}+\nu\frac{\tilde{K}^{(3)}_\nu}{\xi} \quad (5.118)\\ &+ r_0^4\sum_{m,n=1}^{\infty}\frac{6(m+n+3)}{m+n+1}\frac{B_{mn}}{\xi^{m+n+1}}+r_0^4\sum_{m,n=1}^{\infty}\frac{A_{mn}}{\xi^{m+n+2}}\left(\frac{10m+10n+2}{m+n+2}\right.\\ &+ \left.\frac{6(m+n+3)}{(m+n+1)(m+n+2)}+\frac{24(m+n-1)}{(m+n)(m+n+2)}\right)+\nu\tilde{K}^{(4)}_\nu. \end{aligned}$$

As $\xi\to\infty$, $k^{(3)}_{(in)\nu}(\xi)\sim\nu\tilde{K}^{(3)}_\nu+O(\frac{1}{\xi})$ and $k^{(4)}_{(in)\nu}(\xi)\sim\nu\tilde{K}^{(4)}_\nu+O(\frac{1}{\xi})$. Therefore, the inner region solutions are regular solutions.

Finally, we solve Eq(B.19) to obtain the constraint equations in the inner region, which are given by

$$0 = E^{(0)}_\nu-\sqrt{3}C^{(0)}_\nu, \quad (5.119)$$

$$0 = i\nu^2 r_0\left(-2\sqrt{3}C^{(1)}_\nu+2E^{(1)}_\nu+\frac{\bar{K}^{(1)}_\nu}{r_0^4}\right)+\frac{1}{2}\int_{-\infty}^{\infty}d\omega(\nu-\omega)\omega\phi^{(0)}_{\nu-\omega}\phi^{(0)}_\omega, \quad (5.120)$$

$$0 = i\nu^3 r_0\left(-2\sqrt{3}C^{(2)}_\nu+2E^{(2)}_\nu+\frac{\bar{K}^{(2)}_\nu}{r_0^4}\right)-\int_{-\infty}^{\infty}d\omega(\nu-\omega)\omega^2\phi^{(0)}_{\nu-\omega}A^{(1)}_\omega. \quad (5.121)$$

Matching

In this subsection, we show that the inner region solutions solved in the previous subsection are smoothly connected to the outer region solutions in Sec.5.1. Let us start with $h_{(in)\nu}(\xi)$. Near matching region, we expand Eq(5.95) with small ξ using Eq(5.70) and Eq(5.76). Integrating by ξ, Eq(5.95) becomes

$$h^{(1)}_{(in)\nu}(\frac{\nu}{u-1})=\frac{1}{4}\left(\frac{i}{3r_0}\right)^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi^{(0)}_\omega\phi^{(0)}_{\nu-\omega}\frac{1}{u-1}+\text{ subleading terms...}, \quad (5.122)$$

where the "subleading terms" denote terms which are higher order in ν when $h^{(1)}_{(in)\nu}(\xi)$ is expressed in terms of u (We obtain leading corrections only in the outer region). The same procedure is applied to Eq(5.96). Near the overlapping region, the expansion of $h^{(2)}_{(in)\nu}(u)$ becomes

$$\nu h^{(2)}_{(in)\nu}(\frac{\nu}{u-1})=-\frac{1}{6}\left(\frac{i}{3r_0}\right)^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi^{(0)}_\omega\phi^{(0)}_{\nu-\omega}ln\left(\frac{u-1}{\nu}\right)+\text{ subleading terms...} \quad (5.123)$$

Combining these, we get near matching region expansion of $h_{(in)\nu}(u)$ which is a form of

$$h_{(in)\nu}(u) = \nu\bar{H}_\nu^{(1)}+\nu^2\bar{H}_\nu^{(2)}+\frac{1}{4}\left(\frac{i}{3r_0}\right)^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(\frac{1}{u-1}-\frac{2}{3}ln(\frac{u-1}{\nu})\right)... \tag{5.124}$$

As $u \to 1$, the outer region solution of h_ν in momentum space is expanded as

$$\begin{aligned} h_\nu(u) &= \frac{1}{4}(\frac{i}{3r_0})^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(\frac{1}{u-1}-\frac{2}{3}ln(u-1)\right. \qquad (5.125)\\ &- \left.\frac{1}{24}\left(8+\frac{13\sqrt{2}\pi}{2}-13\sqrt{2}tan^{-1}(\sqrt{2})-8ln(6)\right)+...\right) \end{aligned}$$

(See Appendix B.2). Consequently, Eq(5.124) completely matches Eq(5.125) requesting that

$$\begin{aligned} \bar{H}_\nu^{(1)} &= 0, \qquad (5.126)\\ \nu^2\bar{H}_\nu^{(2)} &= \frac{1}{864r_0^2}\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(8+\frac{13\sqrt{2}\pi}{2}-13\sqrt{2}tan^{-1}(\sqrt{2})\right.\\ &- \left.8ln(6)+16ln(\nu)\right), \end{aligned}$$

and so on.

In the overlapping region, $a_{(in)\nu}(u)$ and $k_{(in)\nu}(u)$ are obtained in the same way. As ξ approaches zero, $a_{(in)\nu}(u)$ and $k_{(in)\nu}(u)$ are expanded as

$$\begin{aligned} a_{(in)\nu}(u) &= -r_0C_\nu^{(0)}\left((u-1)-(u-1)^2+(u-1)^3...\right) \qquad (5.127)\\ &- r_0\nu C_\nu^{(1)}\left((u-1)-(u-1)^2...\right)\\ &- r_0\nu^2C_\nu^{(2)}(u-1)...+\nu\bar{A}_\nu^{(1)}+\nu^2\bar{A}_\nu^{(2)}+\nu^2\bar{A}_\nu^{(3)}...\\ &- \sqrt{3}r_0\nu\bar{H}_\nu^{(1)}\left((u-1)-(u-1)^2...\right)-\sqrt{3}r_0\nu^2\bar{H}_\nu^{(2)}(u-1)...\\ &+ \frac{\sqrt{3}r_0}{4}(\frac{i}{3r_0})^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(-ln\left(\frac{u-1}{\nu}\right)\right.\\ &+ \left.\frac{2}{3}(u-1)ln\left(\frac{u-1}{\nu}\right)+\frac{4}{3}(u-1)...\right), \end{aligned}$$

$$\begin{aligned} k_{(in)\nu}(u) &= \nu\bar{K}_\nu^{(1)}+\nu(u-1)\bar{K}_\nu^{(1)}+\nu^2\bar{K}_\nu^{(2)}+\nu^2(u-1)\bar{K}_\nu^{(2)}+\nu^3\bar{K}_\nu^{(3)} \qquad (5.128)\\ &+ \nu^3(u-1)\bar{K}_\nu^{(3)}+\nu^4\bar{K}_\nu^{(4)}...\\ &- 2\sqrt{3}r_0^4(u-1)\left(C_\nu^{(0)}+\nu C_\nu^{(1)}+\nu^2C_\nu^{(2)}+\nu^3C_\nu^{(3)}...\right)\\ &+ 4r_0^4\nu\bar{H}_\nu^{(1)}\left(3(u-1)^2+2(u-1)^3...\right)\\ &+ 12r_0^4\nu^2(u-1)^2\bar{H}_\nu^{(2)}...+\nu^3\tilde{K}_\nu^3(u-1)...\\ &+ \frac{1}{4}r_0^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(\frac{8}{9}(u-1)^2ln\left(\frac{u-1}{\nu}\right)-2(u-1)\right.\\ &- \left.\frac{5}{3}(u-1)^2...\right). \end{aligned}$$

In the overlapping region, the outer region solutions $a_\nu(u)$ and $k_\nu(u)$ in momentum space are given by

$$
\begin{aligned}
a_\nu(u) &= \frac{\sqrt{3}r_0}{4}(\frac{i}{3r_0})^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}(-ln(u-1) \qquad (5.129)\\
&+ \frac{2}{3}(u-1)ln(u-1)+\frac{1}{8}\left(-10+2\sqrt{2}tan^{-1}(\sqrt{2})-\sqrt{2}\pi+4ln(6)\right)\\
&+ \frac{1}{24}\left(40-13\sqrt{2}tan^{-1}(\sqrt{2})-8ln(6)+\frac{13\sqrt{2}\pi}{2}\right)(u-1)+...)+O(u-1)^2,\\
k_\nu(u) &= \frac{1}{4}r_0^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(\frac{8}{9}(u-1)^2ln(u-1)-1\right. \qquad (5.130)\\
&+ \sqrt{2}tan^{-1}(\sqrt{2})+(\sqrt{2}tan^{-1}(\sqrt{2})-3)(u-1)+\left(\frac{13\sqrt{2}\pi}{36}-\frac{11}{9}-\frac{4ln(6)}{9}\right.\\
&- \left.\left.\frac{13tan^{-1}(\sqrt{2})}{9\sqrt{2}}\right)(u-1)^2\right)+r_0\bar{k}_1^\nu+r_0\bar{k}_1^\nu(u-1)+O(u-1)^3,
\end{aligned}
$$

where again $\bar{k}_1^\nu$ is the Fourier transform of $\bar{k}_1(v)$. We compare Eq(5.127), Eq(5.128) with Eq(5.129), Eq(5.130) respectively to decide that $C_\nu^{(0)}=C_\nu^{(1)}=C_\nu^{(2)}=\bar{A}_\nu^{(1)}=\bar{K}_\nu^{(1)}=0$,

$$
\begin{aligned}
\nu^2\bar{A}_\nu^{(2)} &= -\frac{\sqrt{3}}{288}\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(-10+2\sqrt{2}tan^{-1}(\sqrt{2})-\sqrt{2}\pi\right.\\
&+ 4ln(6)-8ln(\nu))\\
\text{and } \nu^2\bar{K}^{(2)} &= \frac{1}{4}r_0^2\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}\left(\sqrt{2}tan^{-1}(\sqrt{2})-1\right)+r_0\bar{k}_1^\nu.
\end{aligned}
$$

$\bar{A}_\nu^{(3)}$, $\bar{K}_\nu^{(3)}$ and $\bar{K}_\nu^{(4)}$ are determined by matching with higher orders in the outer region solutions. With the coefficients determined in this fashion, two solutions are connected smoothly in the matching region.

The constraint equations in the inner region are the same with those in the outer region. Plugging $C_\nu^{(0)}=0$ into Eq(5.119), we obtain $E_\nu^{(0)}=0$. By using $C_\nu^{(1)}=\bar{K}_\nu^{(1)}=0$, Eq(5.120) becomes

$$\nu^2E_\nu^{(1)}=-\frac{1}{4ir_0}\int_{-\infty}^{\infty}d\omega\omega(\nu-\omega)\phi_\omega^{(0)}\phi_{\nu-\omega}^{(0)}, \qquad (5.131)$$

They are the same with the momentum space expression of Eq(5.35) and Eq(5.36).

Chapter A Appendices of Chapter3

A.1 Comments on Metric to $O(\epsilon^2)$

We are interested in calculating the back reaction on the metric to $O(\epsilon^2)$ that arises due to the dilaton Φ_0. Without loss of generality we can assume that the metric is S^3 symmetric and therefore of form,

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + R^2 d\Omega^2 \tag{A.1}$$

where the metric coefficients are functions of r, t. The zeroth order metric is that of AdS_5, Eq(3.7). We argued above that the backreaction to the dilaton source arises at order ϵ^2. Thus g_{tr} in Eq(A.1) is of order ϵ^2.

We now show that by doing a suitable coordinate transformation, the mixed component g_{tr} can be set to vanish up to order ϵ^2. The coordinate transformation is, from (t, r) to $(t, \tilde{r})$, where,

$$r = \tilde{r} - \frac{g_{tr}}{g_{rr}}t, \tag{A.2}$$

which leads to

$$dr = d\tilde{r} - (\frac{g_{tr}}{g_{rr}})' t d\tilde{r} - \frac{g_{tr}}{g_{rr}}dt + O(\epsilon^3). \tag{A.3}$$

Prime above indicates derivatives with respect to r, We can drop the ϵ^3 terms for our purpose, these originate from additional time derivatives on the metric components. Substituting in Eq(A.1) we see that in the new coordinates the $g_{t\tilde{r}}$ components of the metric vanish up to $O(\epsilon^3)$ corrections which we are neglecting anyways. To avoid clutter we will henceforth drop the tilde on the r coordinate and write the metric as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + R^2 d\Omega^2 \tag{A.4}$$

Next we show that up to $O(\epsilon^2)$ we can set R equal to the coordinate r without reintroducing the mixed components. First define,

$$\bar{r} = R \tag{A.5}$$

leading to,

$$d\bar{r} = R' dr + \dot{R}dt \tag{A.6}$$

where dot indicates a time derivative. Now any time dependence in R arises only due to the dilaton and therefore is of order ϵ^2. This means that $\dot{R}$ is $O(\epsilon^3)$ and can be neglected. So up to $O(\epsilon^2)$ no mixed components arise in the metric due to this coordinate transformation. We now drop the bar on the radial coordinate and write the final metric as,

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2. \tag{A.7}$$

A.2 More on the Driven Harmonic Oscillator

In this appendix we provide the steps leading to (3.98) and (3.99). The time derivative of the state vector $|\psi(t)>$ in (3.97) is

$$i\frac{\partial}{\partial t}|\psi(t)>= i(\dot{\lambda}+\dot{\alpha})a^\dagger|\psi(t)> +i(\frac{\dot{N}}{N}+\frac{\dot{N}_\alpha}{N_\alpha})|\psi(t)> \tag{A.8}$$

where we have used the expression for $|\phi_0>$ in (3.90). The action of the hamiltonian H on the state is easily obtained by noting that

$$[H, e^{\lambda a^\dagger}] = \left(\omega_0\lambda a^\dagger + \frac{J\lambda}{\sqrt{2\omega_0}}\right)e^{\lambda a^\dagger}. \tag{A.9}$$

This leads to

$$H|\psi(t)>= \left(\omega_0\lambda a^\dagger + \frac{J\lambda}{\sqrt{2\omega_0}}\right)|\psi(t)> +\frac{\omega_0}{2}|\psi(t)> . \tag{A.10}$$

It may easily be checked that the states $|\psi(t)>$ and $a^\dagger|\psi(t)>$ are linearly independent. Equating the coefficients of $a^\dagger|\psi(t)>$ in Eq(A.8) and (A.10) and using Eq.(3.92) then leads to Eq.(3.98). Equating the coefficients of $|\psi(t)>$ in Eq.(A.8) and (A.10) gives an equation that determines $N(t)$. Note that $|N(t)|$ is determined directly from the requirement that $<\psi|\psi>= 1$.

A.3 The normalization factor $F(2n)$

In computing the normalization $F(2n)$ in (3.115) it is best to first continue to euclidean signature and then perform a conformal transformation from $R\times S^3$ to R^4. The radial coordinate on the R^4 is given by $r = e^\tau$. where τ is the euclidean time in $R\times S^3$. Then the Heisenberg picture operator on R^4 is given by

$$\hat{\mathcal{O}}_{l=0} = \sum_{m=-\infty}^{\infty}\frac{\mathcal{O}_m}{r^{m+4}} \tag{A.11}$$

The factor of r^{m+4} in the denominator reflects the fact that the operator $\hat{\mathcal{O}}_{l=0}$ has dimension 4. The conformally invariant vacuum satisfies

$$\begin{aligned} \mathcal{O}_m|0> &= 0 \qquad m\geq -3 \\ <0|\mathcal{O}_m &= 0 \qquad m\leq 3 \end{aligned} \tag{A.12}$$

Then the radial time ordered 2 point function is given by

$$<\hat{\mathcal{O}}_{l=0}(r)\hat{\mathcal{O}}_{l=0}(r')>= \sum_{m=4}^{\infty}\sum_{n=-\infty}^{-4}\frac{<0|\mathcal{O}_m\mathcal{O}_n|0>}{r^{m+4}(r')^{n+4}} \tag{A.13}$$

The 2 point function only involves the central term in the operator algebra. This means we can write

$$\begin{aligned} \mathcal{O}_m &= NF(m)A_m \qquad (m>0) \\ \mathcal{O}_{-m} &= NF^\star(m)A_m^\dagger \qquad (m>0) \end{aligned} \tag{A.14}$$

where the operators $A_m, A^\dagger$ satisfies an operator algebra and $F(m)$ is a normalization

$$[A_m, A_n] = [A_m^\dagger, A_n^\dagger] = 0 \qquad [A_m, A_n^\dagger] = \delta_{mn} \tag{A.15}$$

Note that because of (A.13) only terms for $n \geq 4$ contribute to the sum. This leads to the result

$$< \hat{\mathcal{O}}_{l=0}(r)\hat{\mathcal{O}}_{l=0}(r') >= \frac{N^2}{r^8}\sum_{m=4}^{\infty}|F(m)|^2\left(\frac{r'}{r}\right)^{m-4} \tag{A.16}$$

On the other hand since the dimension of the operator $\hat{\mathcal{O}}^\Phi(r,\Omega_3)$ is 4 we know the 2 point function on R^4. This is given by

$$< \hat{\mathcal{O}}(r,\Omega_3)\hat{\mathcal{O}}(r',\Omega_3') >= \frac{AN^2}{|\vec{r}-\vec{r'}|^8} \tag{A.17}$$

where A is a order one numerical constant. Here $\vec{r} = (r,\Omega)$ etc., is the location of the operator on R^4. Integrating over Ω_3, Ω_3' we get

$$\int d\Omega_3 \int d\Omega_3' < \hat{\mathcal{O}}(r,\Omega_3)\hat{\mathcal{O}}(r',\Omega_3') >= AN^2(8\pi^3)\int_0^\pi \frac{\sin^2\theta\ d\theta}{(r^2+(r')^2-2rr'\cos\theta)^4} \tag{A.18}$$

The integral can be performed. The result is, for $r > r'$

$$\int d\Omega_3 \int d\Omega_3' < \hat{\mathcal{O}}(r,\Omega_3)\hat{\mathcal{O}}(r',\Omega_3') >= N^2\frac{4A\pi^4}{r^8}\frac{\left(\frac{r'}{r}\right)^2+1}{(1-\left(\frac{r'}{r}\right)^2)^5} \tag{A.19}$$

Using the power series expansion

$$\frac{1+x}{(1-x)^5} = \sum_{m=0}^{\infty}\frac{1}{12}(m+1)(m+2)^2(m+3)x^m \tag{A.20}$$

we finally get

$$\int d\Omega_3 \int d\Omega_3' < \hat{\mathcal{O}}(r,\Omega_3)\hat{\mathcal{O}}(r',\Omega_3') >= N^2\frac{A\pi^4}{3}\frac{1}{r^8}\sum_{m=0}^{\infty}(m+1)(m+2)^2(m+3)\left(\frac{r'}{r}\right)^{2m} \tag{A.21}$$

The result clearly shows that only operators with even mode numbers are present in the expansion (A.11). Comparing (A.21) and (A.16) we get

$$F(2m+1) = 0 \qquad |F(2m)|^2 = \frac{A\pi^4}{3}m^2(m^2-1) \tag{A.22}$$

which is the result in equation (3.115).

Chapter B Appendices of Chapter5

B.1 Leading Corrections of the Toy-Model

Equations of motions

With Eq(5.29), the leading order Einstein equations for $h(r,v)$, $a(r,v)$ and $k(r,v)$ become

$$\begin{aligned}
W_{rr} &= -\frac{2h'(r,v)}{r} - \frac{1}{2}(\partial_v\phi_0)^2\left(\frac{r_0^2-r^2}{r^2U_0(r)}\right)^2 = 0 && \text{(B.1)}\\
W_{rv} &= \frac{1}{2r^4}\big(-12r^4h(r,v) - 2r(r^4+r\epsilon_0-\rho_0^2)h'(r,v) && \text{(B.2)}\\
&- 2rk'(r,v) + 4r^2\rho_0 a'(r,v) + 2k(r,v) + r^2k''(r,v)\big)\\
&- \frac{1}{2}(\partial_v\phi_0)^2\left(\frac{r_0^2-r^2}{r^2U_0(r)}\right) = 0\\
W_{vv} &= -\frac{U_0(r)}{2r^4}\big(-12r^4h(r,v) - 2r(r^4+r\epsilon_0-\rho_0^2)h'(r,v) && \text{(B.3)}\\
&- 2rk'(r,v) + 4r^2\rho_0 a'(r,v) + 2k(r,v) + r^2k''(r,v)\big)\\
&- \frac{1}{2}(\partial_v\phi_0)^2 + \frac{1}{r^3}(2rr_0^3\dot{E}(v) - 2\rho_0 r_0^2\dot{C}(v)) = 0\\
W_{ii} &= -\frac{1}{r^2}(6r^4h(r,v) + k(r,v) + r^3U_0(r)h'(r,v) && \text{(B.4)}\\
&- rk'(r,v) + 2r^2\rho_0 a'(r,v))) = 0
\end{aligned}$$

,where $W_{ii} \equiv W_{xx} = W_{yy}$ and dots and primes indicate derivatives with respect to v and r respectively. The gauge field equations are

$$\begin{aligned}
Y^v &= -\frac{1}{r^2}\left(\rho_0 h'(r,v) + 2ra'(r,v) + r^2a''(r,v)\right) = 0, && \text{(B.5)}\\
Y^r &= \frac{r_0^2}{r^2}\dot{C}(v) = 0. && \text{(B.6)}
\end{aligned}$$

The other components of the Einstein equations and gauge field equations are zero. These are the leading order equations in the naive derivative expansion. This means that v-derivatives on $h(r,v)$, $a(r,v)$ and $k(r,v)$ are ignored.

Eq(B.6) shows that there is no dynamics for the charge density. By the initial conditions mentioned in Sec.5.1, $C(v) = 0$. A particular combination of Einstein equations, $W_{rv}U_0(r,v) + W_{vv} = 0$, gives

$$\dot{E}(v) = \frac{1}{4r_0}(\partial_v\phi_{(0)}(v))^2 \tag{B.7}$$

This equation indicates that $E(v) \sim O(\varepsilon)$. This justifies that $E(v)$ in the metric factor $U(r,v)$ is suppressed by ε to produce the second order terms in Eq(5.19). We solve Eq(B.1), Eq(B.5), Eq(B.4) to get Eq(5.31), Eq(5.33),Eq(5.32) respectively. The other Einstein equations are satisfied with the solutions.

B.2 Outer Solution in Extremal Limit

Dilaton solution

We start with Eq(5.20). In the case of extremality, Eq(5.20) can have a form of

$$\begin{aligned}\phi(u,v) &= \phi_{(0)}(v) + \frac{\partial_v\phi_{(0)}(v)}{6(u-1)r_0}(1-\Lambda_1(v)) - \frac{\partial_v\phi_{(0)}(v)}{9r_0}ln(u-1)\,(2+\Lambda_1(v)) \\ &+ \frac{\partial_v\phi_{(0)}(v)}{36r_0}\left((\Lambda_1(v)-7)\sqrt{2}tan^{-1}(\frac{u+1}{\sqrt{2}}) + 2(\Lambda_1(v)+2)ln(u^2+2u+3)\right) + \Lambda_2(v).\end{aligned} \quad (B.8)$$

As $u \to \infty$, a boundary condition that we demand for the dialton is $\phi(u,v)|^{u=\infty} = \phi_{(0)}(v)$. This boundary condition yields $\Lambda_2(v) = -\frac{i\pi\phi_0}{36\sqrt{2}r_0}(\Lambda_1(v)-7)$. With this, $\phi(r,v)$ has an asymptotic behavior of

$$\begin{aligned}\phi(u,v) &= \phi_{(0)}(v) + \frac{\partial_v\phi_{(0)}(v)}{6(u-1)r_0}(1-\Lambda_1(v)) - \frac{\partial_v\phi_{(0)}(v)}{9r_0}ln(u-1)\,(2+\Lambda_1(v)) \\ &+ \frac{\partial_v\phi_{(0)}(v)}{36r_0}\left((\Lambda_1(v)-7)\sqrt{2}tan^{-1}(\sqrt{2}) + 2(\Lambda_1(v)+2)ln(6) + \frac{\pi}{\sqrt{2}}(7-\Lambda_1(v))\right) \\ &+ O(u-1),\end{aligned} \quad (B.9)$$

as $u \to 1$, near the black brane horizon.

Metric and Gauge Field Solution

As we discussed in Sec.5.2, the regularity condition of the dilaton field forces $\Lambda_1(v) = 1$. In this subsection, we follow this. For the extremal limit, we set $\epsilon_0 = 2r_0^3$ and $\rho_0 = \sqrt{3}r_0^2$. Eq(5.31) becomes

$$\begin{aligned}h(u,v) &= \bar{h}_1(v) - \frac{1}{864}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0^2}\left(\frac{6(u^2-10u-15)}{(u-1)(u^2+2u+3)} - 13\sqrt{2}tan^{-1}(\frac{1+u}{\sqrt{2}})\right. \\ &+ \left. 16ln(u-1) - 8ln(u^2+2u+3)\right).\end{aligned} \quad (B.10)$$

As $u \to 1$, this can be expanded as

$$\begin{aligned}h(u,v) &= \bar{h}_1(v) - \frac{1}{4}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0^2}\left(-\frac{1}{9(u-1)} + \frac{2}{27}ln(u-1)\right. \\ &+ \left.\frac{1}{216}(8 - 13\sqrt{2}tan^{-1}(\sqrt{2}) - 8ln(6)) + O(u-1)\right).\end{aligned} \quad (B.11)$$

The near horizon expansions of Eq(5.33) and Eq(5.32) are also given by

$$a(u,v) = \frac{\sqrt{3}}{864}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0}\Bigg(-8(\frac{2}{u}+1)ln(u-1) - \frac{30}{u} + \sqrt{2}(\frac{13}{u}-7)tan^{-1}(\frac{1+u}{\sqrt{2}}) \quad \text{(B.12)}$$
$$+ \; 4(\frac{2}{u}+1)ln(u^2+2u+3)\Bigg) + \bar{a}_2(v) - \frac{\bar{a}_1(v)-\sqrt{3}r_0^2\bar{h}_1(v)}{r_0 u},$$
$$k(u,v) = -\frac{1}{4}r_0^2(\partial_v\phi_{(0)}(v))^2\Bigg(\frac{4}{27}(u-1)^2(u^2+2u+3)ln(u-1) - u^2 \quad \text{(B.13)}$$
$$- \; \sqrt{2}tan^{-1}(\frac{u+1}{\sqrt{2}})\Bigg(-u + \frac{13}{108}(u-1)^2(u^2+2u+3)\Bigg)$$
$$- \; \frac{2(u-1)^2(u^2+2u+3)}{27}ln(u^2+2u+3) + \frac{1}{18}(u^2-10u-15)(u-1)\Bigg)$$
$$+ \; r_0 u\bar{k}_1(v) - 2\sqrt{3}r_0^2\bar{a}_1(v) + 2r_0^4(u-1)^2(u^2+2u+3)\bar{h}_1(v),$$

The integration constants,$\bar{h}_1(v)$, $\bar{a}_1(v)$, $\bar{a}_2(v)$ and $\bar{k}_1(v)$ are determined by the boundary condition (5.34). As $u\to\infty$, the asymptotic expansion of $k(u,v)$ can have terms of $O(u^4)$. These are non-normalizable modes which give deformation of the boundary metric. The terms are removed by imposing $\bar{h}_1(v) = -\frac{13\sqrt{2}\pi}{1728r_0^2}(\partial_v\phi_{(0)}(v))^2$. Near AdS_4 boundary, $a(u,v)$ presents $O(1)$ and $O(\frac{1}{u})$ terms. The former corrects the chemical potential and the later does the charge density. To eliminate these terms, $\bar{a}_1(v)$ and $\bar{a}_2(v)$ should be properly chosen as $\bar{a}_1(v)=0$ and $\bar{a}_2(v) = \frac{7\sqrt{6}\pi}{1728r_0}(\partial_v\phi_{(0)}(v))^2$.

Near the black brane horizon, the behavior of the leading back reactions are given by

$$h(u,v) = -\frac{1}{4}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0^2}\Bigg(-\frac{1}{9(u-1)} + \frac{2}{27}ln(u-1) \quad \text{(B.14)}$$
$$+ \; \frac{1}{216}(8+\frac{13\sqrt{2}\pi}{2} - 13\sqrt{2}tan^{-1}(\sqrt{2}) - 8ln(6)) + O(u-1)\Bigg),$$
$$a(u,v) = \frac{\sqrt{3}}{864}\frac{(\partial_v\phi_{(0)}(v))^2}{r_0}\big(-8ln(u-1)(3-2(u-1)+O(u-1)^2) - 30 \quad \text{(B.15)}$$
$$+ \; 6\sqrt{2}tan^{-1}(\sqrt{2}) - 3\sqrt{2}\pi + 12ln(6)$$
$$+ \; \Bigg(40 - 13\sqrt{2}tan^{-1}(\sqrt{2}) - 8ln(6) + \frac{13\sqrt{2}\pi}{2}\Bigg)(u-1) + O(u-1)^2\Bigg),$$
$$k(u,v) = -\frac{1}{4}r_0^2(\partial_v\phi_{(0)}(v))^2\Bigg(\frac{8}{9}(u-1)^2ln(u-1) - 1 + \sqrt{2}tan^{-1}(\sqrt{2}) \quad \text{(B.16)}$$
$$+ \; (\sqrt{2}tan^{-1}(\sqrt{2})-3)(u-1) + \Bigg(\frac{13\sqrt{2}\pi}{36} - \frac{11}{9} - \frac{4ln(6)}{9}$$
$$- \; \frac{13tan^{-1}(\sqrt{2})}{9\sqrt{2}}\Bigg)(u-1)^2\Bigg) + r_0\bar{k}_1(v) + r_0\bar{k}_1(v)(u-1) + O(u-1)^3.$$

B.3 Equations in Extremal Backgrounds with v-derivative Retained

In this section, we develop the Einstein equations(5.5) and the gauge field equations(5.6) without ignoring v-derivatives. We start with Eq(5.29). The only assumption in this section is that the equations are linear in $h(u,v)$, $a(u,v)$ and $k(u,v)$. The Einstein equations are

$$
\begin{aligned}
r_0^2 W_{rr} &= -\frac{2h'(u,v)}{u} - \frac{1}{2}\partial_u\phi(u,v)\partial_u\phi(u,v) = 0, && \text{(B.17)}\\
W_{rv} &= \frac{1}{2u^4}\Bigg(-12u^4 h(u,v) + 2u(3-u^4-2u)h'(u,v) + \frac{2u^4}{r_0}\dot{h}'(u,v) && \text{(B.18)}\\
&+ \frac{4\sqrt{3}u^2}{r_0}a'(u,v) + \frac{1}{r_0^4}\left(2k(u,v) - 2uk'(u,v) + u^2k''(u,v)\right),\\
&- \frac{1}{2r_0}\partial_v\phi(u,v)\partial_u\phi(u,v) = 0,\\
\bar{W} &\equiv W_{vv} + \left(\frac{r^4 - 4rr_0^3 + 3r_0^4}{r^2}\right)W_{rv} && \text{(B.19)}\\
&= \frac{r_0}{u^3}\left(-2\sqrt{3}\dot{C}(v) + 2u\dot{E}(v) - 2(u^4-4u+3)\dot{h}(u,v) + \frac{\dot{k}(u,v)}{r_0^4}\right)\\
&- \frac{1}{2}\partial_v\phi(u,v)\partial_v\phi(u,v) - r_0\left(\frac{u^4-4u+3}{2u^2}\right)\partial_u\phi(u,v)\partial_v\phi(u,v) = 0,\\
\frac{r^2}{r_0^2}W_{ii} &= -6u^4h(u,v) - u(u^4-4u+3)h'(u,v) - \frac{2\sqrt{3}u^2}{r_0}a'(u,v) && \text{(B.20)}\\
&+ \frac{1}{r_0^4}\left(uk'(u,v) - k(u,v)\right) = 0,
\end{aligned}
$$

where $u = \frac{r}{r_0}$, the rescaled radial coordinate again. The primes and dots denote derivatives with respect to u and v respectively. Gauge field equations are

$$
\begin{aligned}
\frac{r^2}{r_0^2}Y^r &= \dot{C}(v) + \sqrt{3}\dot{h}(u,v) + \frac{u^2}{r_0}\dot{a}'(u,v) = 0, && \text{(B.21)}\\
r^2 Y^v &= -2ua'(u,v) - \sqrt{3}r_0h'(u,v) - u^2a''(u,v) = 0. && \text{(B.22)}
\end{aligned}
$$

As discussed in the beginning of Sec.5.2, for introducing the scaling(5.42) the equations in position space of v should be transformed to the momentum space. It is worth showing how one gets an expression Eq(B.17) in momentum space, for example. We define the Fourier transform of $h(u,v)$ by

$$h(u,v) = \int_{-\infty}^{\infty} h_\omega(u)e^{i\omega v}d\omega. \tag{B.23}$$

Substituting of Eq(5.59) and Eq(B.23) into Eq(B.17) and acting an integral operator $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\nu v}dv$ on it, we get

$$r_0^2 W_{rr} = -\frac{2h'_\nu(u)}{u} - \frac{1}{2}\int_{-\infty}^{\infty} d\omega\partial_u\phi_{\nu-\omega}(u)\partial_u\phi_\omega(u) = 0. \tag{B.24}$$

To deal with other equations, we define Fourier transforms of $a(u,v)$ and $k(u,v)$ as that of $h(u,v)$. For $E(v)$ and $C(v)$,

$$
\begin{aligned}
E(v) &= \int_{-\infty}^{\infty} e^{i\omega v} E_\omega d\omega, &\text{(B.25)}\\
C(v) &= \int_{-\infty}^{\infty} e^{i\omega v} C_\omega d\omega.
\end{aligned}
$$

With these, the other Einstein equations in momentum space are given by

$$
\begin{aligned}
W_{rv} &= \frac{1}{2u^4}\Bigg(-12u^4 h_\nu(u) + 2u(3-u^4-2u)h'_\nu(u) + i\nu\frac{2u^4}{r_0}h'_\nu(u) &\text{(B.26)}\\
&+ \frac{4\sqrt{3}u^2}{r_0}a'_\nu(u) + \frac{1}{r_0^4}\left(2k_\nu(u) - 2uk'_\nu(u) + u^2k''_\nu(u)\right)\\
&- \frac{1}{2r_0}\int_{-\infty}^{\infty} d\omega i(\nu-\omega)\phi_{\nu-\omega}(u)\partial_u\phi_\omega(u) = 0,\\
\bar{W} &\equiv W_{vv} + \left(\frac{r^4 - 4rr_0^3 + 3r_0^4}{r^2}\right)W_{rv} &\text{(B.27)}\\
&= \frac{i\nu r_0}{u^3}\left(-2\sqrt{3}C_\nu + 2uE_\nu - 2(u^4-4u+3)h_\nu(u) + \frac{k_\nu(u)}{r_0^4}\right)\\
&- r_0\left(\frac{u^4-4u+3}{2u^2}\right)\int_{-\infty}^{\infty} d\omega i(\nu-\omega)\phi_{\nu-\omega}(u)\partial_u\phi_\omega(u)\\
&+ \frac{1}{2}\int_{-\infty}^{\infty} d\omega(\nu-\omega)\omega\phi_{\nu-\omega}(u)\phi_\omega(u) = 0,\\
\frac{r^2}{r_0^2}W_{ii} &= -6u^4h_\nu(u) - u(u^4-4u+3)h'_\nu(u) - \frac{2\sqrt{3}u^2}{r_0}a'_\nu(u) &\text{(B.28)}\\
&+ \frac{1}{r_0^4}\left(uk'_\nu(u) - k_\nu(u)\right) = 0.
\end{aligned}
$$

The gauge field equations become

$$
\begin{aligned}
\frac{r^2}{r_0^2}Y^r &= i\nu\left(C_\nu + \sqrt{3}h_\nu(u) + \frac{u^2}{r_0}a'_\nu(u)\right) = 0, &\text{(B.29)}\\
r^2Y^v &= -2ua'_\nu(u) - \sqrt{3}r_0h'_\nu(u) - u^2a''_\nu(u) = 0. &\text{(B.30)}
\end{aligned}
$$

B.4 Counting Power of ε

In this section, we argue the parametric order of the inner region solutions in ε. The basic idea is that we transfrom the inner region solutions to the position space and check their powers of ε. For a simple example, we discuss the dilaton solution. We design the zeroth order dilaton solution in position space as in Eq(5.11). By Fourier transformation, we obtain its expression in frequency space as

$$
\phi_\omega^{(0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega v} f\left(\frac{\varepsilon v}{r_0}\right)dv. \qquad \text{(B.31)}
$$

Scaling of the integration variable v as $\tau = \varepsilon v$ takes this expression to

$$\phi^{(0)}_{\omega} = \frac{1}{2\pi\varepsilon}\int_{-\infty}^{\infty} e^{-i\frac{\omega}{\varepsilon}\tau} f(\frac{\tau}{r_0}) d\tau \equiv \frac{1}{\varepsilon} g(\frac{\omega}{\varepsilon}), \tag{B.32}$$

where $g(\frac{\omega}{\varepsilon})$ becomes an $O(1)$ function. By observing Eq(B.32) and dilaton solution in the inner region(5.45), one can recognize that each subleading correction to the dialton field in momentum space can be written as

$$\phi^{(i)}_{(in)\omega} \equiv \frac{1}{\varepsilon} g(\frac{\omega}{\varepsilon}) \left(h_{(i)}(\xi) + a_{(i)}(\xi) ln(\nu) \right), \tag{B.33}$$

where $\phi^{(i)}_{(in)\omega}$ denotes ith order correction in small frequency to the dilaton field. $h_{(i)}(\xi)$ and $a_{(i)}(\xi)$ are functions of ξ only. It turns out that the terms multiplying $a_{(i)}(\xi)$ produce terms which are proportional to $\varepsilon^i ln(\varepsilon)$ in the position space. It is obscure to count power of ε of these terms. In this discussion, we exclude these. The perturbation expansion of the dilaton solution becomes a form of

$$\phi_{(in)\omega} = \frac{1}{\varepsilon} g(\frac{\omega}{\varepsilon}) \left(1 + \nu h_{(1)}(\xi) + \nu^2 h_{(2)}(\xi) ... \right), \tag{B.34}$$

up to the logarithmic terms. Fourier transformation defined in Eq(5.59) takes this expression to position space, which is given by

$$\phi(u,v) = \int_{-\infty}^{\infty} e^{i\omega v} d\omega \frac{1}{\varepsilon} g(\frac{\omega}{\varepsilon}) \left(1 + \omega h_{(1)}(\xi) + \omega^2 h_{(2)}(\xi) ... \right). \tag{B.35}$$

Again we rescale the integration variable ω as $\omega = \varepsilon\bar{\omega}$, then the expression becomes

$$\begin{aligned} \phi(u,v) &= \int_{-\infty}^{\infty} e^{i\varepsilon\bar{\omega}v} d\bar{\omega} g(\bar{\omega}) \left(1 + \varepsilon\bar{\omega} h_{(1)}(\xi) + \varepsilon^2\bar{\omega}^2 h_{(2)}(\xi) ... \right) \\ &\equiv F^{(0)}(\varepsilon v) + (-i)\varepsilon F^{(0)\prime}(\varepsilon v) h_{(1)}(\xi) + (-i)^2 \varepsilon^2 F^{(0)\prime\prime}(\varepsilon v) h_{(2)}(\xi), \end{aligned} \tag{B.36}$$

where $F^{(0)}(\varepsilon v) = \int_{-\infty}^{\infty} e^{i\varepsilon\bar{\omega}v} d\bar{\omega} g(\bar{\omega})$ and the prime indicates derivative with respect to its argument. The property of $F^{(0)}(\varepsilon v)$ as noted in Eq (5.12) shows that $F^{(0)}(\varepsilon v)$ and its derivatives with its argument are $O(1)$ functions. Compare Eq(B.34) to Eq(B.36). This shows that counting power of ν in the momentum space is the same as the counting power of ε in the position space.

We apply this argument to the back reactions in the inner region. For the simplest case, let us check $h^{(1)}_{(in)\nu}(\xi)$ with Eq(5.95). This contains derivative of the dilaton field, which can be expressed as $\phi^{(1)\prime}_{(in)\omega}(\frac{\omega}{\nu}\xi) = \frac{1}{\varepsilon} f(\frac{\omega}{\varepsilon}) k(\frac{\omega}{\nu}\xi)$. We switch Eq(5.95) to position space with Fourier transformation as defined in Eq(B.23) and rescale integration variables as in the discussion of the dilaton. This time, we scale ω as well as ν in Eq(5.95). Then, we obtain following expression:

$$\begin{aligned} h^{(1)}_{(in)}(v,\xi) &= \varepsilon \int^{\xi} \frac{\xi'^2 d\xi'}{4\nu^3} \int_{-\infty}^{\infty} e^{i\bar{\nu}\varepsilon v} d\bar{\nu} d\bar{\omega} \bar{\omega}^2 (\bar{\nu}-\bar{\omega})^2 f(\bar{\omega}) f(\bar{\nu}-\bar{\omega}) g(\frac{\bar{\omega}}{\bar{\nu}}\xi) g(\frac{\bar{\nu}-\bar{\omega}}{\bar{\nu}}\xi) \\ &\equiv \varepsilon G(\varepsilon v), \end{aligned} \tag{B.37}$$

where $G(\varepsilon v)$ is an $O(1)$ function. Consequently, it turns out that $h^{(1)}_{(in)}(v,\xi)$ is in the first order in ε in position space. Comparing Eq(B.37) to Eq(5.95), one can recognize that the correct power counting of the small frequency in the momentum space is to count not only ν but also ω and $\nu-\omega$ in the integrand of Eq(5.95). By the the same argument, we get

$$\begin{aligned}
a^{(1)}_{(in)}(v,\xi) &\sim k^{(1)}_{(in)}(v,\xi) \sim O(\varepsilon), \\
h^{(2)}_{(in)}(v,\xi) &\sim a^{(2)}_{(in)}(v,\xi) \sim k^{(2)}_{(in)}(v,\xi) \sim O(\varepsilon^2), \\
a^{(3)}_{(in)}(v,\xi) &\sim k^{(3)}_{(in)}(v,\xi) \sim O(\varepsilon^3) \;\text{ and }\; k^{(4)}_{(in)}(v,\xi) \sim O(\varepsilon^4).
\end{aligned} \tag{B.38}$$

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