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TOPOLOGICAL AND COMBINATORIAL PROPERTIES OF NEIGHBORHOOD AND CHESSBOARD COMPLEXES

Matthew Zeckner
University of Kentucky, matthew.zeckner@gmail.com

ABSTRACT OF DISSERTATION

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Matthew Zeckner
Lexington, Kentucky

Director: Dr. Benjamin Braun, Professor of Mathematics
Co-Director: Dr. Carl Lee, Professor of Mathematics
Lexington, Kentucky 2011

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TOPOLOGICAL AND COMBINATORIAL PROPERTIES OF NEIGHBORHOOD AND CHESSBOARD COMPLEXES

This dissertation examines the topological properties of simplicial complexes that arise from two distinct combinatorial objects. In 2003, A. Björner and M. de Longueville proved that the neighborhood complex of the stable Kneser graph $SG_{n,k}$ is homotopy equivalent to a k-sphere. Further, for $n = 2$ they showed that the neighborhood complex deformation retracts to a subcomplex isomorphic to the associahedron. They went on to ask whether or not, for all n and k, the neighborhood complex of $SG_{n,k}$ contains as a deformation retract the boundary complex of a simplicial polytope. Part one of this dissertation provides a positive answer to this question in the case $k = 2$. In this case it is also shown that, after partially subdividing the neighborhood complex, the resulting complex deformation retracts onto a subcomplex arising as a polyhedral boundary sphere that is invariant under the action induced by the automorphism group of $SG_{n,2}$. Part two of this dissertation studies simplicial complexes that arise from non-attacking rook placements on a subclass of Ferrers boards that have a_i rows of length i where $a_i > 0$ and $i \leq n$ for some positive integer n. In particular, enumerative properties of their facets, homotopy type, and homology are investigated.

KEYWORDS: Combinatorics, Topology, Simplicial Complex, Neighborhood Complex, Chessboard Complex

Author's signature: Matthew Zeckner

Date: May 3, 2011

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Matthew Zeckner

Director of Dissertation: Benjamin Braun

Co-Director of Dissertation: Carl Lee

Director of Graduate Studies: Qiang Ye

Date: May 3, 2011

RULES FOR THE USE OF DISSERTATIONS

DISSERTATION

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This dissertation is dedicated to my wife and sons who kept me sane throughout this entire ordeal.

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Chapter 1 Introduction

1.1 Overture

Basic Definitions

This dissertation is the study of simplicial complexes that arise from combinatorial objects. In Chapter 2 a collection of definitions and tools are provided that will be used throughout the remainder of the dissertation. Chapter 3 contains work pertaining to neighborhood complexes of stable Kneser graphs. This dissertation concludes with a study of chessboard complexes, including joint work with Eric Clark.

The remainder of this chapter is meant as a gentle introduction and overview to the topics and results that will be found throughout this dissertation We begin with the definition of one of the most basic combinatorial objects: a graph.

Definition 1.1.1. A *graph* $G = (V, E)$ is an ordered pair of sets of vertices V and edges E which are two-element subsets of the vertices. We say two vertices are *adjacent* if they share a common edge.

A classical way of studying graphs is to study graph colorings.

Definition 1.1.2. A *coloring* of the vertices of a graph G is a labeling of the vertices such that no two vertices that share an edge are labeled the same. A *k-coloring* of a graph G is a coloring of the vertices of G using at most k labels. The fewest number of colors needed to color a graph G is called the *chromatic number* of G and is denoted by $\chi(G)$.

Often times, it is of interest to know whether a graph is minimal with respect to colorings, i.e., whether or not the deletion of a vertex affects the chromatic number of a graph.

Definition 1.1.3. A graph G is called *vertex-critical* if the deletion of any vertex of G causes the chromatic number of G to decrease.

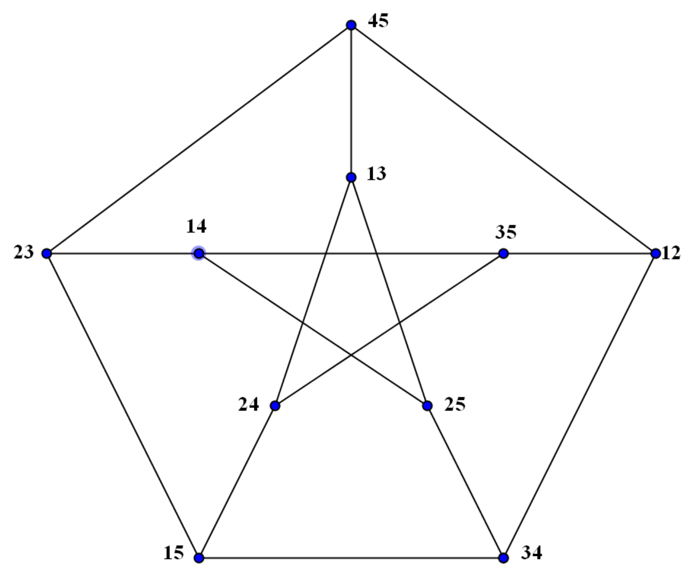

Figure 1.1: The graph of $KG_{2,1}$

Extending the basic structure of a graph to higher dimensions, we obtain a simplicial complex.

Definition 1.1.4. An *(abstract) simplicial complex* Δ on a finite set X is a family of subsets of X closed under the deletion of elements. We refer to singleton sets $\{x\}$ as vertices. The *dimension* of a simplex σ is $|\sigma| - 1$, while the *dimension* of a simplicial complex is the dimension of its largest simplex. We call a simplicial complex *pure* if all its facets (i.e., maximal faces) are of the same dimension.

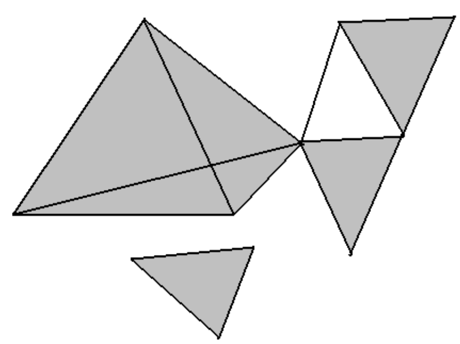

Figure 1.2: A sample simplicial complex. We note this complex is not pure.

Simplicial complexes arise from many combinatorial objects. We will focus our attention on two such occasions.

Definition 1.1.5. Given a graph $G = (V, E)$, the *neighborhood complex of* G is the simplicial complex $\mathcal{N}(G)$ with vertex set V and faces given by subsets of V sharing a common neighbor in G, i.e.,

$$\{F \subset V : \exists v \in V \ s.t. \ \forall u \in F, \{u, v\} \in E\}.$$

Neighborhood complexes are related to graph chromatic numbers and will be of particular interest in Chapter 3, as we study the neighborhood complex of stable Kneser graphs.

Definition 1.1.6. A *chessboard complex* is the collection of all non-attacking rook positions on an $m \times n$ chessboard.

It is clear that this is a simplicial complex, as the removal of one rook from an admissible rook placement yields another admissible rook placement. In general, we need not limit ourselves to rectangular chessboards, but may broaden our boards to include non-attacking rook positions on a board of any shape. Notice that a placement of $i+1$ rooks corresponds to a simplex of dimension i. We also notice that a chessboard complex can be reformulated as the independence complex of a graph G. The independence complex of G, $\mathrm{Ind}(G)$, is the simplicial complex whose faces are the independent (pairwise non-adjacent) sets of vertices of the graph. The graph G in question here is the graph whose vertices are the squares of the chessboard where two vertices are adjacent if their respective squares lie on the same row or column of the chessboard. We will examine chessboard complexes in further detail in Chapter 4.

Goal of Topological Combinatorics

As detailed in Jonsson's book [17], topological combinatorics is the study of the topology of spaces formed by combinatorial objects. We are particular interested in the (1) homology, (2) homotopy type, (3) connectivity degree, and (4) Euler characteristic of a space. Typically, a topological combinatorics problem will begin with a purely combinatorial problem. From this, we will create a CW complex. Often times this CW complex will be a simplicial complex. At this point, we shift our tool set to that of topology and begin examining the complex in order to understand any of the aforementioned categories, whichever is relevant to our problem. Once the topology of our complex is understood, we retract the problem back to its original combinatorial form and use our new knowledge as an aid in solving our combinatorial problem.

Frequently our mode of attack will involve discrete Morse theory, discussed in Chapter 2, a tool developed by Robin Forman as a combinatorial adaptation of traditional Morse Theory. Discrete Morse theory is a powerful tool that, at its most rudimentary level, is a method of pairing cells in a CW complex in an appropriate manner to study the homology and/or homotopy type of the complex.

1.2 Stable Kneser Graphs

For any natural number m define $[m] := \{1, \ldots, m\}$. Let n and k be any two natural numbers and consider all the subsets of size n of the set $[2n + k]$. We may form a graph called the Kneser graph, $KG_{n,k}$, whose vertices are the aforementioned n-sets. We connect two such vertices with an edge if they are disjoint as sets. In 1978 Lovász [20] proved that the chromatic number of $KG_{n,k}$ is $k + 2$, thus proving a conjecture by Kneser and beginning the field of topological combinatorics.

We call a subset S of $[2n + k]$ *stable* if:

1. $\{i, i + 1\} \not\subset S$ for $i = 1, \ldots, (2n + k - 1)$ and
2. $\{1, 2n + k\} \not\subset S$

As with the Kneser graph, we may form a graph whose vertices are the stable n-sets of $[2n + k]$ where two vertices are adjacent if they are disjoint as sets. This graph is called the stable Kneser graph and denoted $SG_{n,k}$.

Schrijver [21] showed that the stable Kneser graph $SG_{n,k}$ is a vertex-critical subgraph of $KG_{n,k}$, and Björner and de Longueville [5] showed that the neighborhood complex of $SG_{n,k}$, $\mathcal{N}(SG_{2,k})$, is a k-sphere. In addition, in the special case that

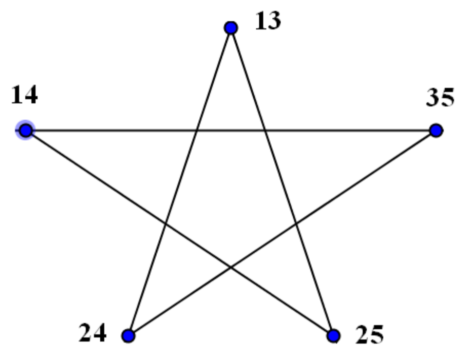

Figure 1.3: The induced subgraph of $KG_{2,1}$, the stable Kneser graph $SG_{2,1}$

$n = 2$ and $k \geq 0$, Björner and de Longueville also showed that $\mathcal{N}(SG_{2,k})$ contained the boundary of the $k+1$-dimensional associahedron as a deformation retract. With such a surprising result, they asked whether $\mathcal{N}(SG_{2,k})$ contained a simplicial polytope as a deformation retract for all n and k. We give in this dissertation a positive answer when $k = 2$. The theorem follows from two propositions.

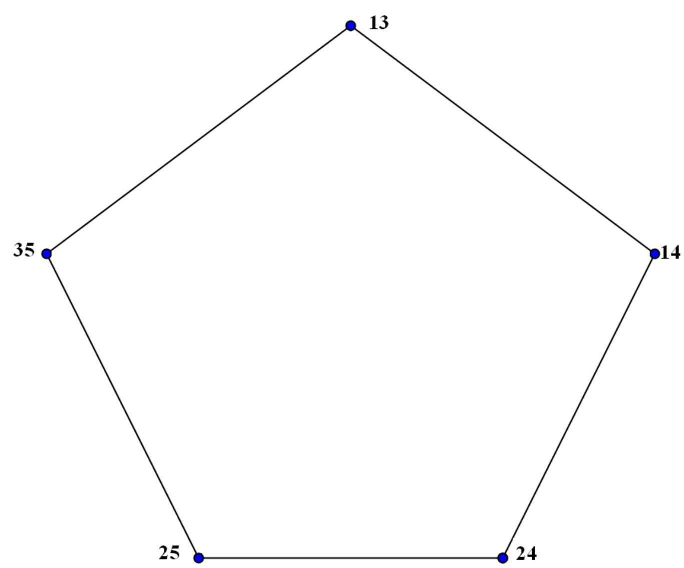

Figure 1.4: The neighborhood complex of $SG_{2,1}$. This is the boundary of a pentagon, the 2-dimensional associahedron.

Proposition 1.2.1. *The neighborhood complex of $SG_{n,2}$ collapses onto a pure subcomplex $\widetilde{\mathcal{N}}(SG_{n,2})$ of dimension 2.*

The method of proof here involves dividing the stable n-sets of $[2n+2]$ into two disjoint families. Each stable set from a given family yields a simplex of the same dimension as every other stable set in that family. To prove Proposition 1.2.1 we examine the face poset of $\mathcal{N}(SG_{n,2})$ and carefully use discrete Morse theory.

Proposition 1.2.2. *The simplicial complex $\widetilde{\mathcal{N}}(SG_{n,2})$ is the boundary complex of a 3-dimensional polytope.*

To prove Proposition 1.2.2, we verify that $\widetilde{\mathcal{N}}(SG_{n,2})$ meets the requirements of Steinitz's theorem [26]. More precisely, we show that the 1-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$ is a 3-connected, simple, planar graph. We begin by constructing a pure simplicial complex of dimension 1 (a graph) N and prove this is the 1-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$. Our new construction instantly verifies that the 1-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$ is both simple and planar. By finding three unique paths from any two given vertices on N, we verify the 1-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$ is 3-connected

These two propositions allow us to conclude the following.

Theorem 1.2.3. *The neighborhood complex of $SG_{n,2}$ collapses onto a subcomplex that is the boundary of a polytope.*

In a paper by Braun [7] it was shown that the dihedral group of order $2(2n+2)$, D_{2n+2}, acts on the stable sets of $[2n+2]$ and by extension on $\mathcal{N}(SG_{n,2})$. We wish this result to hold for $\widetilde{\mathcal{N}}(SG_{n,2})$. Unfortunately, our construction fails to maintain the dihedral group action on one of the families of simplices. In particular, consider the inner triangulated square in Figure 1.5. By comparing the vertices 135 and 137 it is clear that the dihedral group will not act on this triangulation; however, by passing these simplices through a barycentric subdivision we are able to say the following:

Theorem 1.2.4. *The dihedral group D_{2n+2} acts on a triangulated 2-sphere contained as a deformation retract of a partial subdivision of $\mathcal{N}(SG_{n,2})$.*

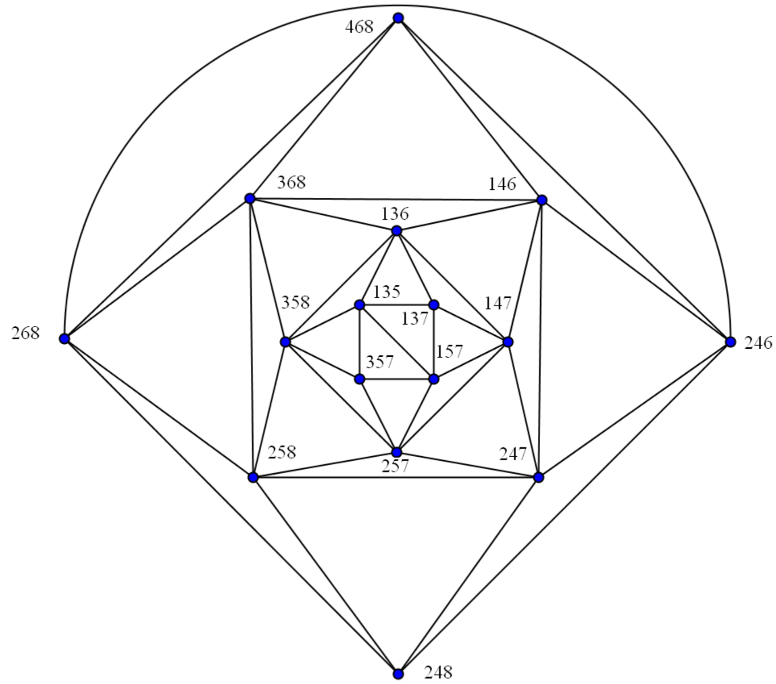

Figure 1.5: The collapsed simplicial complex of $\mathcal{N}(SG_{3,2})$

1.3 Ferrers Boards

Another area of recent interest is chessboard complexes. Chessboard complexes first appeared in the 1979 thesis of Garst [16] concerning Tits coset complexes. Chessboard complexes later appeared in a paper by Björner, Lovász, Vrećica, and Živaljević [4] where they gave a bound on the connectivity of the chessboard complex and conjectured that their bound was sharp. This conjecture was shown to be true by Shareshian and Wachs [24]. In that same paper, Shareshian and Wachs also showed that if the chessboard met certain criteria, then it contained torsion in its homology.

The triangular chessboard $\Psi_{a_n,\ldots,a_1}$ is a left justified board with $a_i > 0$ rows of length i for $1 \leq i \leq n$. In other words, given a positive integer n, the triangular board $\Psi_{a_n,\ldots,a_1}$ is the Ferrers board associated with the partition $\pi = (n^{a_n}, \ldots, 1^{a_1})$ with $a_i > 0$; see Figure 4.1.

We begin with a few definitions.

Definition 1.3.1. For a family Δ of sets and a set σ of Δ, the *link* of σ in Δ, $\mathrm{lk}_\Delta(\sigma)$, is the family of all $\tau \in \Delta$ such that $\tau \cap \sigma = \emptyset$, and $\tau \cup \sigma \in \Delta$. The *deletion* of σ in Δ, $\mathrm{del}_\Delta(\sigma)$, is the family of all $\tau \in \Delta$ such that $\sigma \not\subseteq \tau$.

Definition 1.3.2. A simplicial complex Δ is *vertex decomposable* if

1. Every simplex (including $\emptyset$ and $\{\emptyset\}$) is vertex decomposable.

2. Δ is pure and contains a 0-cell x — a *shedding vertex* — such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are both vertex decomposable.

Let $\Sigma(\Psi_{a_n,\ldots,a_1})$ denote the simplicial complex formed by all non-attacking rook placements on the triangular board. We call a rook placement *maximal* if no other non-attacking rook can be added to the placement, that is, every square on the board is attacked. In particular I am interested in two extreme cases of these simplicial complexes.

1. $\Sigma(\Psi_{a_n,\ldots,a_1})$ for $a_1 \geq 1$, $a_n \geq n$, and $a_i \geq i-1$ for all $i = 2, \ldots, n-1$, and

2. $\Sigma(\Psi_{a_n,\ldots,a_1})$ for $a_i = 1$ for all $i = 1, \ldots, n-1$

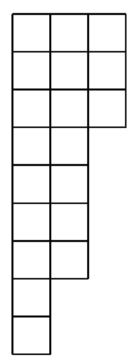

Figure 1.6: The Ferrers board $\Psi_{3,4,2}$.

In the former of the two cases, we can classify its homotopy type.

Theorem 1.3.3. *If $a_1 \geq 1$, $a_n \geq n$, and $a_i \geq i-1$ for all $i = 2, \ldots, n-1$, then the simplicial complex $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable.*

This result mirrors Ziegler's result [26] that the (rectangular) chessboard complex $M_{n,m}$ is vertex decomposable if $n \geq 2m-1$. That is, extending a triangular board, like extending a rectangular board, far enough allows one to conclude that the associated complex is vertex decomposable.

As the aforementioned complex $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable, we know it is homotopic to a wedge of spheres of the same dimension. The number of spheres can be computed by finding the reduced Euler characteristic, which is the alternating sum of the f-vector. The *f-vector* of a simplicial complex is the vector $(f_0, \ldots, f_n)$ where f_i represents the number of faces of dimension i. Let $\ell(i)$ denote the length of column i in $\Psi_{a_n,\ldots,a_1}$ with $\ell(i) = \sum_{j=i}^{n} a_j$.

Theorem 1.3.4. *The coefficients of the f-vector of $\Sigma(\Psi_{a_n,\ldots,a_1})$ are given by*

$$f_i = \sum_{S \in \binom{[n]}{i+1}} \prod_{j=0}^{i} (\ell(s_j) - j),$$

where $S = \{s_0 > s_1 > \cdots > s_i\}$.

The other extreme case, $\Sigma(\Psi_{1,\ldots,1})$, is called the Stirling complex. I studied this complex with Eric Clark, leading to our joint theorems found in Chapter 4, sections 4.3 4.4. There is a well-known bijection by Stanley, see [25, Corollary 2.4.2], that the f-vector of the board of size n, $\Sigma(\Psi_{1,\ldots,1})$, is given by the Stirling numbers of the second kind $S(n, k)$. Let B and C be two disjoint nonempty subsets of $[n]$ such that $\min(B \cup C) \in B$. Then B and C are *intertwined* if $\max(B) > \min(C)$. We say a partition P is *intertwined* if every pair of blocks in P is intertwined.

Theorem 1.3.5. *The set of maximal rook placements with k rooks on a triangular board $\Psi_{1,\ldots,1}$ of size n is in bijection with intertwined partitions of $n+1$ into $n+1-k$ blocks.*

Any simplicial complex can be entirely characterized by its facets. Thus, it is of interest to know how many facets are there. Moreover, if the complex is not pure, that is, if the facets are not all of the same dimension, then we would like to know how many facets there are of a given dimension. Hence, our bijection leads us to following theorem.

Theorem 1.3.6. *The number of partitions of* $[n]$ *into* k *intertwined blocks is given by*

$$I(n,k) = (k-1)! \sum_{i=k-1}^{n-k} S(i,k-1) \cdot S(n-i,k).$$

The generating function for $I(n,k)$ *is given by*

$$\sum_{n \geq 0} I(n,k)x^n = \frac{k! \cdot x^{2k}}{\left(\prod_{i=1}^{k}(1-ix)\right)^2 \cdot (1-(k+1)x)}.$$

The idea of intertwined partitions first appeared with the use of the *intertwining number* of a partition in [13] where they were used to provide a formula for q-Stirling numbers of the second kind.

We now turn our attention to questions of the homotopy type of the Stirling complex. The Stirling complex is clearly not vertex-decomposable as $\mathrm{Stir}(n)$ is not pure. Moreover, $\mathrm{Stir}(n)$ is not even non-pure shellable, making the homotopy type of $\mathrm{Stir}(n)$ all the more interesting. We can, however, prove the following.

Theorem 1.3.7. *The Stirling complex* $\mathrm{Stir}(n)$ *is homotopy equivalent to a CW complex with no cells of dimension* k *for* $k < \lceil n/2 \rceil - 1$ *and for* $k \geq n-1$*. In addition, the Stirling complex* $\mathrm{Stir}(2n)$ *is homotopy equivalent to a wedge of* $n!$ *spheres of dimension* $n-1$ *with a space* X *where* X *is* $(n-1)$*-connected.*

The proof of this theorem involves discrete Morse theory.

Chapter 2 Topological Tools

2.1 Topological Tools

For an introduction to combinatorial topology, basic definitions, and results, we refer the reader to the books by Jonsson [17] and Kozlov [18].

Definition 2.1.1. For a family Δ of sets and a set σ of Δ, the *link* $\mathrm{lk}_\Delta(\sigma)$ is the family of all $\tau \in \Delta$ such that $\tau \cap \sigma = \emptyset$, and $\tau \cup \sigma \in \Delta$. The *deletion* $\mathrm{del}_\Delta(\sigma)$ is the family of all $\tau \in \Delta$ such that $\sigma \not\subseteq \tau$.

Definition 2.1.2. A simplicial complex Δ is *vertex decomposable* if

1. Every simplex (including $\emptyset$ and $\{\emptyset\}$) is vertex decomposable.

2. Δ is pure and contains a 0-cell x — a *shedding vertex* — such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are both vertex decomposable.

Showing that a simplicial complex is vertex decomposable is useful in determining the topology of complex as can be seen in the following theorem.

Theorem 2.1.3. [17, Theorems 3.33 and 3.35] *Let Δ be a simplicial complex of dimension d. If the complex Δ is vertex decomposable, then Δ is homotopy equivalent to a wedge of spheres of dimension d. Moreover, we have the following implications Vertex Decomposable $\Longrightarrow$ Shellable $\Longrightarrow$ Constructible $\Longrightarrow$ Homotopically Cohen-Macaulay*

As the above theorem shows, being vertex decomposable is a very strong property.

2.2 Discrete Morse Theory

We recall the following definitions and theorems from discrete Morse theory. See [14, 15, 18] for further details.

Definition 2.2.1. A *partial matching* in a poset P is a partial matching in the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ such that $(x, y) \in M$ implies $x \prec y$ and each $x \in P$ belongs to at most one element of M. For $(x, y) \in M$ we write $x = d(y)$ and $y = u(x)$, where d and u stand for down and up, respectively.

Definition 2.2.2. A partial matching M on P is *acyclic* if there does not exist a cycle

$$z_1 \succ d(z_1) \prec z_2 \succ d(z_2) \prec \cdots \prec z_n \succ d(z_n) \prec z_1,$$

in P with $n \geq 2$, and all $z_i \in P$ distinct. Given a partial matching, the unmatched elements are called *critical*. If there are no critical elements, the acyclic matching is *perfect*.

We now state the main result from discrete Morse theory. For a simplicial complex Δ, let $\mathcal{F}(\Delta)$ denote the poset of faces of Δ ordered by inclusion.

Theorem 2.2.3. *Let Δ be a simplicial complex and let M be an acyclic matching on the face poset of Δ. Let c_i denote the number of critical i-dimensional cells of Δ. The space Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single 0-dimensional cell in the case where the empty set is paired in the matching.*

Remark 2.2.4. *If the critical cells of an acyclic matching on Δ form a subcomplex Γ of Δ, then Δ simplicially collapses to Γ, implying that Γ is a deformation retract of Δ.*

In Chapter 4 it will be convenient for us to make use of reduced discrete Morse theory; that is, we will include the empty set. In particular, if the matching in Theorem 2.2.3 is perfect, then Δ_c is contractible. Also, if the matching has exactly one critical cell then Δ_c is a d-sphere where d is the dimension of this cell.

Given a set of critical cells of differing dimension, in general it is difficult to conclude that the CW complex Δ_c is homotopy equivalent to a wedge of spheres. See

Kozlov [19] for a non-trivial example. However, when some critical cells are facets, it may be possible to say more as seen in the following theorem.

Theorem 2.2.5. *Let M be a Morse matching on $\mathcal{F}(\Delta)$ with k_i critical cells of dimension i. Assume that there are no critical cells of dimension less than j and that all critical cells of dimension j are facets. Then the complex Δ is homotopy equivalent to the wedge*

$$X \vee \bigvee_{k_j} \mathbb{S}^j,$$

where X is a CW complex consisting of one point and k_i i-dimensional cells for $i > j$.

Proof. The complex Δ without the critical cells of dimension j is homotopy equivalent to a CW complex X consisting of one point and k_i i-dimensional cells for $i > j$. Since every face of dimension less than j contracts to a point, the boundary of each j-dimensional critical cell contracts to a point. As all of these critical cells are maximal, they can be independently added back into the complex. $\square$

Kozlov [19] gives a more general sufficient condition on an acyclic Morse matching for the complex to be homotopy equivalent to a wedge of spheres enumerated by the critical cells.

It is often useful to create acyclic partial matchings on several different sections of the face poset of a simplicial complex and then combine them to form a larger acyclic partial matching on the entire poset. This process is detailed in the following theorem known as the *Cluster lemma* in [17] and as the *Patchwork theorem* in [18].

Theorem 2.2.6. *Assume that $\varphi : P \to Q$ is an order-preserving map. For any collection of acyclic matchings on the subposets $\varphi^{-1}(q)$ for $q \in Q$, the union of these matchings is itself an acyclic matching on P.*

Chapter 3 Stable Kneser Graphs

3.1 Introduction and Main Result

In 1978, Lovász proved in [20] Kneser's conjecture that if one partitions all the subsets of size n of a $(2n+k)$-element set into $(k+1)$ classes, then one of the classes must contain two disjoint subsets. Lovász proved this conjecture by modeling the problem as a graph coloring problem: see Section 3.2 for definitions of the following objects. For the Kneser graphs $KG_{n,k}$, Kneser's conjecture is equivalent to the statement that the chromatic number of $KG_{n,k}$ is equal to $k+2$. Lovász's proof methods actually provided a general lower bound on the chromatic number of any graph G as a function of the topological connectivity of an associated simplicial complex called the neighborhood complex of G. Of particular interest in his proof was the critical role played by the Borsuk-Ulam theorem. Later that year, Schrijver identified in [21] a vertex-critical family of subgraphs of the Kneser graphs called the stable Kneser graphs $SG_{n,k}$, or Schrijver graphs, and determined that the chromatic number of $SG_{n,k}$ is equal to $k+2$.

In 2003, Björner and de Longueville gave in [5] a new proof of Schrijver's result by applying Lovász's method to the stable Kneser graphs; in particular, they proved that the neighborhood complex of $SG_{n,k}$ is homotopy equivalent to a k-sphere. In the final section of their paper, Björner and De Longueville showed that the neighborhood complex of $SG_{2,k}$ contains the boundary complex of a $(k+1)$-dimensional associahedron as a deformation retract. Their paper concluded with the following:

Question 3.1.1. (Björner and De Longueville, [5]) *For all n and k, does the neighborhood complex of $SG_{n,k}$ contain as a deformation retract the boundary complex of a simplicial polytope?*

Our main contribution in this chapter is to provide a positive answer to Question 3.1.1 in the case $k=2$. Specifically, we show the following:

Theorem 3.1.2. *For every $n \geq 1$, the neighborhood complex of $SG_{n,2}$ simplicially collapses onto a subcomplex arising as the boundary of a three-dimensional simplicial polytope.*

The subcomplex for $\mathcal{N}(SG_{3,2})$ is shown in Figure 3.1.

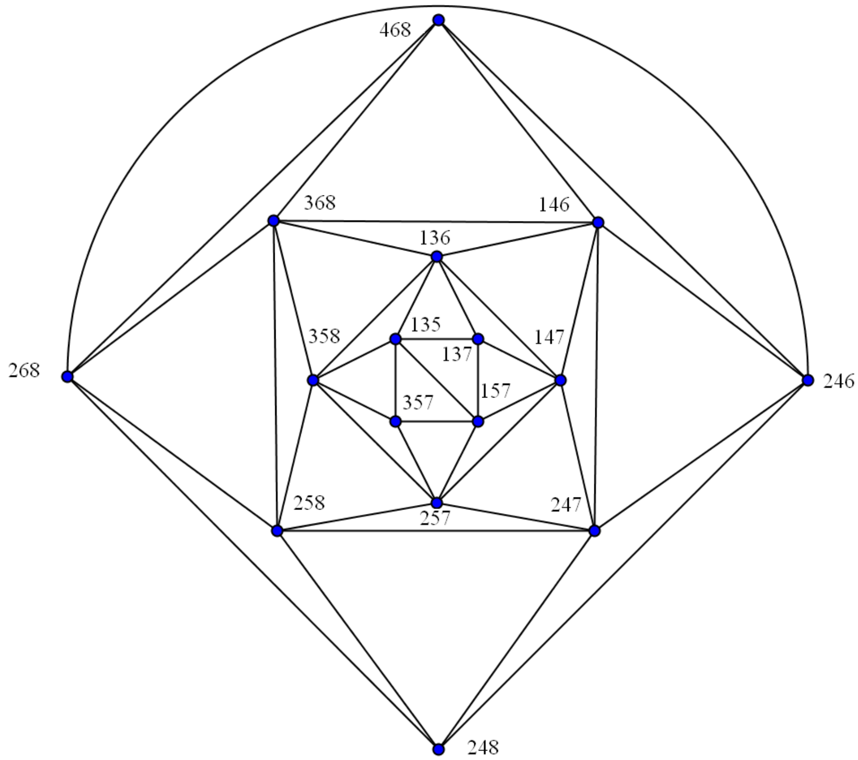

Figure 3.1: The subcomplex for $\mathcal{N}(SG_{3,2})$

In [7], Braun proved that for $k \geq 1$ and $n \geq 1$ the automorphism group of $SG_{n,k}$ is isomorphic to the dihedral group of order $2(2n+k)$. It is natural to ask if there exist spherical subcomplexes of the neighborhood complex of $SG_{n,k}$ that are invariant under the induced action of this group. While our spheres arising in Theorem 3.1.2 are not invariant, we are able to show the following:

Theorem 3.1.3. *For every $n \geq 1$, there exists a partial subdivision of the neighborhood complex of $SG_{n,2}$ that simplicially collapses onto a subcomplex invariant under the action induced by the automorphism group of $SG_{n,2}$ arising as the boundary of a three-dimensional simplicial polytope.*

In addition to its aesthetic attraction, there are two primary reasons we are interested in Question 3.1.1. First, any polytopes found in response to Question 3.1.1 will be common generalizations of simplices, associahedra, and 1-spheres given as odd cycles, due to the following observations: for $SG_{1,k} = K_{k+2}$, the neighborhood complex is a simplex boundary; for $SG_{n,1}$, the neighborhood complex is an odd cycle, hence a one-dimensional sphere; for $SG_{2,k}$, the neighborhood complex deformation retracts to an associahedron. A family of polytopes generalizing these objects would be interesting to identify. Second, a broad extension of the neighborhood complex construction is the graph homomorphism complex $HOM(H, G)$ studied in [1, 2, 10, 11, 22, 23]. The complex $HOM(K_2, G)$ is known to be homotopy equivalent to the neighborhood complex of G. The homomorphism complex construction leads to interesting phenomena, yet at present the lower bounds on graph chromatic numbers obtained by these are no better than those provided by the neighborhood complex. We believe it is appropriate to continue to focus attention on the neighborhood complex construction along with the HOM construction.

The rest of this chapter is as follows. In Section 3.2, we introduce the necessary background and notation regarding neighborhood complexes and stable Kneser graphs. In Sections 3.3 and 3.4, we provide a proof of Theorem 3.1.2. In Section 3.5 we provide a proof of Theorem 3.1.3.

3.2 Definitions and Background

Let $[n] := \{1, 2, \ldots, n\}$. The material in this section is adapted from the texts [17] and [18], where more details may be found.

Neighborhood Complexes and Stable Kneser Graphs

The following definition is due to Lovász.

Definition 3.2.1. Given a graph $G = (V, E)$, the *neighborhood complex of G* is the simplicial complex $\mathcal{N}(G)$ with vertex set V and faces given by subsets of V sharing a common neighbor in G, i.e. $\mathcal{N}(G) := \{F \subset V : \exists v \in V \text{ s.t. } \forall u \in F, \{u, v\} \in E\}$.

The graphs we are interested in are the following.

Definition 3.2.2. For $n \geq 1$ and $k \geq 0$ the *Kneser graph*, denoted $KG_{n,k}$, is the graph whose vertices are the subsets of $[2n+k]$ of size n. We connect two such vertices with an edge when they are disjoint as sets.

We call an n-set α of $[2n+k]$ *stable* if α does not contain the subset $\{1, 2n+k\}$ or any of the subsets $\{i, i+1\}$ for $i = 1, \ldots, 2n+k-1$. The *stable Kneser graph*, denoted $SG_{n,k}$, is the induced subgraph of $KG_{n,k}$ whose vertices are the stable subsets of $[2n+k]$.

Our focus in this dissertation is on the case $k = 2$; we will assume through the rest of the paper that this holds. In order to handle different stable n-sets, we distinguish between them as follows, with all addition on elements being modulo $2n+2$.

Definition 3.2.3. We call a stable n-set α *tight* if $\alpha = \{i, i+2, i+4, \ldots, i+2(n-1)\}$ for some $i \in [2n+2]$. Otherwise, we call α a *loose* stable n-set.

For $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and β stable n-sets, we call α and β *immediate neighbors* if $\alpha \oplus 1 = \beta$ or $\alpha \ominus 1 = \beta$, where $\alpha \oplus j := \{\alpha_1 + j, \ldots, \alpha_n + j\}$ and $\alpha \ominus j$ is defined similarly.

We call α and β *outer neighbors* if there is an ordering of the elements of α such that $\beta = (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_{i-1} + 1, \alpha_i + 2, \alpha_{i+1} + 1, \ldots, \alpha_n + 1)$ and α and β are neighbors in $SG_{n,2}$.

The following remarks provide some insight into the structure of these graphs; further discussion, including proofs of these remarks, can be found in [6, 7].

- A cycle is formed in $SG_{n,2}$ with vertices a stable n-set α and the stable n-sets $\alpha \oplus 1$, $\alpha \oplus 2$, etc, with the edges $\{\alpha \oplus i, \alpha \oplus (i+1)\}$. Thus, α and β are immediate neighbors if they are neighbors on such a cycle in $SG_{n,2}$.

- Stable n-sets α and β are outer neighbors in $SG_{n,2}$ if they are neighbors and lie on two different cycles created via the immediate neighbor process.

- A loose stable n-set has degree 4 in $SG_{n,2}$. Two of its neighbors are immediate neighbors while the other two are outer neighbors.

- The tight stable n-sets correspond to vertices that together induce a complete bipartite subgraph in $SG_{n,2}$.

3.3 Construction of the Acyclic Matching

In this section we use discrete Morse theory to describe a simplicial collapsing of $\mathcal{N}(SG_{n,2})$. Section 3.4 contains an analysis of the complex of critical cells of our discrete Morse matching. Our approach will be to produce poset maps from subposets of the face poset of $\mathcal{N}(SG_{n,2})$ to various target posets, construct acyclic matchings on inverse images of these poset maps, and apply Theorem 2.2.6 to obtain an acyclic matching on the entire face poset. In our construction of these poset maps, we will consider facets of $\mathcal{N}(SG_{n,2})$, which by definition arise in the following way.

Definition 3.3.1. For γ a vertex of $SG_{n,2}$, let Σ_γ be the facet in $\mathcal{N}(SG_{n,2})$ formed by the neighbors of γ.

A key role in our simplicial collapsing is played by the two simplices formed by the collections of all vertices of $SG_{n,2}$ of the form $\{\alpha_1, \ldots, \alpha_n\}$, where in each simplex the α_i have all even or all odd entries, respectively. These all even and all odd simplices may be viewed as North and South poles for the complex. As these pole simplices are not two-dimensional, we must collapse them to smaller dimension. The facets Σ_γ where γ is loose then collapse to pairs of triangles that interpolate between these two poles, forming our sphere.

Collapsing in facets of loose stable n-sets

For any loose stable n-set α, Σ_α is a 3-dimensional simplex in $\mathcal{N}(SG_{n,2})$ formed by the outer and immediate neighbors of α. A routine check reveals that the edge consisting of α's outer neighbors is free in $\mathcal{N}(SG_{n,2})$. Thus, for each such facet we may perform the following collapse.

Label the vertices of Σ_α by a, b, c, d with the outer neighbors of α labeled b and d. Let P_α be the face poset of Σ_α and $Q_\alpha := A < B_\alpha$ a chain of length 2. Let

$\theta_\alpha : P_\alpha \to Q_\alpha$ be defined by

$$\theta_\alpha(x) = \begin{cases} A & \text{if } \{b,d\} \nsubseteq x \\ B_\alpha & \text{if } \{b,d\} \subseteq x \end{cases}$$

It is immediate that θ_α is a poset map and that $\theta^{-1}(B_\alpha)$ yields a perfect acyclic matching when we match an element x in the inverse not containing a with $x \cup \{a\}$. This matching collapses each facet given by a loose stable n-set to two triangles that share a common edge.

Collapsing in facets of tight stable n-sets

Consider a tight stable n-set α in $[2n+2]$, and observe that all elements of α are of the same parity.

Lemma 3.3.2. *α has a unique outer neighbor.*

Proof. Observe that $[2n+2] \setminus \alpha$ consists of the $n+1$ elements of the opposite parity of the elements of α and the one remaining element of the same parity as the elements of α. An outer neighbor of α must contain the one element of the same parity as the elements of α, which we denote p. As the outer neighbor is a stable n-set, it cannot contain $p \pm 1$. Since there are only $n-1$ viable elements left in $[2n+2] \setminus \alpha$, an outer neighbor of α must contain them all. Hence, α has a unique outer neighbor. □

To simplify our presentation we introduce additional notation. Lexicographically assign the neighbors of α the labels $v^1, v^2, \ldots, v^{n+1}$ and η_α where the v^i's are all tight and of the opposite parity of α. The remaining vertex, η_α, denotes α's unique outer neighbor.

Let Σ_α denote the $(n+1)$-simplex formed by the neighbors of α and let P_α denote the face poset of Σ_α. Given α and its unique outer neighbor η_α, let p denote the element in the outer neighbor η_α of identical parity to the elements of α. For some j, we obtain v^j and v^{j+1} from η_α by replacing p with $p-1$ or $p+1$, respectively.

Lemma 3.3.3. *Σ_α collapses to the simplicial complex N_α where N_α consists of the following facets and their subsets:*

$$\{v^1, v^2, v^3\}, \{v^1, v^3, v^4\}, \{v^1, v^4, v^5\}, \ldots, \{v^1, v^n, v^{n+1}\}, \{v^j, v^{j+1}, \eta_\alpha\}$$

where if $j = n + 1$ then the last set listed above is replaced by $\{v^1, v^{n+1}, \eta_\alpha\}$. In other words, Σ_α collapses to a triangulated $(n + 1)$-gon where all diagonals in the triangulation emanate from the vertex labeled v^1 and the triangle $\{v^j, v^{j+1}, \eta_\alpha\}$ is attached to the $(n + 1)$-gon.

The idea behind our matching in the following proof is that the intersection of any two facets corresponding to tight sets is the simplex $\{v^1, v^2, v^3, \ldots, v^{n+1}\}$. To collapse Σ_α to N_α, we will pair unwanted faces contained in $\{v^1, v^2, v^3, \ldots, v^{n+1}\}$ with v^1, and pair unwanted faces containing η_α with v^j. Separating these matchings allows us to patch the relevant poset maps together in a coherent way in the following subsection.

Proof. Fix a tight stable n-set α, with outer neighbor η_α and associated v^j. Let $Q_\alpha := A < B < C_\alpha$ be a three element chain. Consider the map $\varphi_\alpha : P_\alpha \to Q_\alpha$ defined by

$$\varphi_\alpha(x) = \begin{cases} A & \text{if } |x| = 1,\ x = \{v^r, v^s\}, x = \{v^1, v^r, v^s\},\ \text{or } x \subseteq \{v^j, v^{j+1}, \eta_\alpha\} \\ B & \text{for all other } x \text{ such that } \eta_\alpha \notin x \\ C_\alpha & \text{otherwise} \end{cases}$$

where either $r = 1$ and $s \in [n + 1] \setminus \{1\}$ or $s = r + 1$ for $r \in [n] \setminus \{1\}$. Observe that $\varphi_\alpha^{-1}(A)$ is exactly the complex N_α defined above.

We now construct acyclic matchings on the posets $\varphi_\alpha^{-1}(B)$ and $\varphi_\alpha^{-1}(C_\alpha)$. We claim that matching each $x \in \varphi_\alpha^{-1}(B)$ not containing v^1 with $x \cup \{v^1\}$ yields a perfect acyclic matching. One first needs to check that no element is paired with an element of $\varphi_\alpha^{-1}(A)$ or $\varphi_\alpha^{-1}(C_\alpha)$, which is clear from the definitions. That every face is matched is similarly clear. To verify acyclicity, suppose a cycle exists, say $x_1 \prec u(x_1) \succ x_2 \prec u(x_2) \succ \cdots \prec u(x_m) \succ x_1$, for m minimal. Then, both $u(x_1)$ and $u(x_m)$ contain x_1 (as sets). However, our matching dictates that we match x_1 and $u(x_1)$ if and only if they are in $\varphi_\alpha^{-1}(B)$ and $u(x_1) = x_1 \cup \{v^1\}$. If $v^1 \in u(x_m)$, then

$u(x_m) = u(x_1)$ implying $x_m = x_1$, a contradiction. Otherwise, $v^1 \notin u(x_m)$ implies $u(x_m)$ is also matched with $u(x_m) \cup \{v^1\}$, a contradiction.

We claim that matching each $x \in \varphi_\alpha^{-1}(C_\alpha)$ such that $v^j \notin x$ with $x \cup \{v^j\}$ yields a perfect acyclic matching in $\varphi_\alpha^{-1}(C_\alpha)$. It is clear from the definitions that no element is paired with something outside $\varphi_\alpha^{-1}(C_\alpha)$, keeping in mind the observation that the pairs $(\eta_\alpha, \{v^j, \eta_\alpha\})$ and $(\{v^{j+1}, \eta_\alpha\}, \{v^j, v^{j+1}, \eta_\alpha\})$ are not included in this preimage; they are included in the preimage $\varphi_\alpha^{-1}(A)$. Verifying that this is a perfect acyclic matching is similar to the previous case. $\square$

Combining the loose and tight cases to form a single poset map

Our matchings were all defined by studying poset maps with domains the facets of $\mathcal{N}(SG_{n,2})$. To apply Theorem 2.2.6, we need to show that these maps may be combined into a single poset map in a coherent manner. Consider the poset $Q(n,2)$ formed by identifying along commonly named elements the posets Q_α from the constructions of our matchings. In other words, $Q(n,2)$ has a unique minimal element A, a maximal chain on two vertices labeled $A < B_\alpha$ for each loose n-set α, and a maximal chain of length three labeled $A < B < C_\alpha$ for each tight n-set α that all share the common subchain $A < B$. For each of the poset maps θ_α and φ_α defined in the previous subsection, we view them as a map from P_α to $Q(n,2)$.

Let $P(n,2)$ denote the face poset of $\mathcal{N}(SG_{n,2})$. We define a map Φ from $P(n,2)$ to $Q(n,2)$ by mapping a face $x \in \Sigma_\alpha$ to

$$\Phi(x) = \begin{cases} \theta_\alpha(x) & \text{if } \alpha \text{ is loose} \\ \varphi_\alpha(x) & \text{if } \alpha \text{ is tight} \end{cases}$$

Lemma 3.3.4. *Φ is a well-defined poset map.*

Proof. Assuming that Φ is well-defined, that it is a poset map is immediate since θ and φ are poset maps. To verify Φ is well-defined, we need to check that faces contained in more than one facet are mapped coherently by Φ. Let α^1 and α^2 be two stable sets that yield the facets Σ_{α^1} and Σ_{α^2} in $\mathcal{N}(SG_{n,2})$.

Case 1: Suppose α^1 and α^2 are both loose sets. We consider the size of the intersection of their respective facets. If $|\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}| = 4$, then $\alpha^1 = \alpha^2$ and we are done. Suppose $|\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}| = 3$. Say $\{v^1, v^2, v^3\} \subset \Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}$ along with all their subsets for some vertices v^1, v^2, and v^3. Consider the support of these vertices, $supp(v^1, v^2, v^3)$, where $supp(v^1, \ldots, v^k) := \cup_i v^i$ as sets. We know each of these vertices avoids the stable n-sets α^1 and α^2, thus there are at most $n+1$ viable elements remaining in $[2n+2]$. However, $|supp(v^1, v^2, v^3)| \geq n+2$, since the intersection of any two of the vertices can have at most $n-1$ elements in common. Thus, this case does not occur. Suppose $|\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}| \leq 2$. In this case, $\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}$ is either a single vertex or an edge between an inner and an outer neighbor. As any such face is sent to A by both θ_{α^1} and θ_{α^2}, we see that Φ is well-defined on the intersection of pairs of loose sets.

Case 2: If α^1 and α^2 are both tight sets with $\alpha^1 \neq \alpha^2$, then $|\alpha^1 \cap \alpha^2| = n-1$, implying that $\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2}$ is an n-dimensional simplex Σ. Using our previous notation, $\Sigma = \{v^1, v^2, v^3, \ldots, v^{n+1}\}$. For a given face $x \in \Sigma$, every map φ_α maps x to either A or B in a coherent manner, as the definitions of the φ-maps are the same on Σ. Thus, Φ is well-defined on the intersection of pairs of tight sets.

Case 3: Suppose α^1 is a tight set and α^2 is a loose set. If $|\alpha^1 \cap \alpha^2| \leq n-2$, then $|supp(\alpha^1, \alpha^2)| \geq n+2$. Hence, $|[2n+2] \setminus supp(\alpha^1, \alpha^2)| \leq n$. Thus, Σ_{α^1} and Σ_{α^2} intersect in a vertex x, and $\Phi(x) = A$ is well-defined.

If $|\alpha^1 \cap \alpha^2| = n-1$ consider $F = [2n+2] \setminus supp(\alpha^1, \alpha^2)$. We know $|F| = n+1$ as $|supp(\alpha^1, \alpha^2)| = n+1$. Moreover, F consists of n elements of the opposite parity of α^1 and one element, say p, of the same parity. From this we know that $p \pm 1$ is in F, but not both. Hence, F contains only two stable n-sets, one set β which is tight and whose elements are of opposite parity of α^1 and another set γ, which consists of p and $n-1$ elements of opposite parity of α^1 not including $p \pm 1$. Thus, $\Sigma_{\alpha^1} \cap \Sigma_{\alpha^2} = \{\beta, \gamma\}$, all faces of which are mapped to A by φ_{α^1}.

We next show that all faces of F are mapped to A by θ_{α^2} as well. As α^2 is loose and β is a tight neighbor, we know that β is an outer neighbor of α^2. In addition, γ is an immediate neighbor of α^2 by construction. The only edge sent to B_{α^2} by θ_{α^2} is the one formed by both outer neighbors of α^2, which this edge is not, thus it is sent to A. Hence, Φ is well-defined. $\square$

To use Theorem 2.2.6, we now need to verify that our previous matchings are valid. Recall that

$$\Phi^{-1}(A) = \cup_\alpha \varphi_\alpha^{-1}(A) \bigcup \cup_\alpha \theta_\alpha^{-1}(A),$$

and that none of these preimages carried matchings. The other preimages of Φ correspond to the preimages of θ_α or φ_α depending whether α is loose or tight. Our previous matchings may therefore be applied, after noting that on $\Phi^{-1}(B)$ the matching is independent of choice of α. Hence by Theorem 2.2.6 and the remark following it we have that $\mathcal{N}(SG_{n,2})$ simplicially collapses onto the complex whose face poset is $\Phi^{-1}(A)$.

3.4 Analysis of the Complex of Critical Faces

Throughout this section it will be useful to refer to Figure 3.1, illustrating the case $n = 3$. Denote by $\widetilde{\mathcal{N}}(SG_{n,2})$ the complex of critical faces given by $\Phi^{-1}(A)$. By construction, $\widetilde{\mathcal{N}}(SG_{n,2})$ is two-dimensional and pure; in this section we prove that it is the boundary of a three-dimensional simplicial polytope. Our approach is to first construct a planar graph inducing a triangulation of S^2 that realizes $\widetilde{\mathcal{N}}(SG_{n,2})$, then to apply the following theorem. Recall that a graph G is 3-*connected* if for any pair of vertices v and w in G, there exist three disjoint paths from v to w.

Theorem 3.4.1. (Steinitz' theorem, see [27]) *A simple graph G is the one-skeleton of a three-dimensional polytope if and only if it is planar and* 3*-connected.*

Construction of $\widetilde{\mathcal{N}}(SG_{n,2})$

We want to realize $\widetilde{\mathcal{N}}(SG_{n,2})$ as a triangulation of S^2; we will do so by constructing its one-skeleton in the plane. We begin with notation and several lemmas. For a stable n-set α, let α_{odd} be the set of all odd elements of α and let α_{even} be the set of all even elements of α. Throughout this subsection, unless otherwise indicated, we assume for a stable n-set $\alpha = \{\alpha_1, \dots, \alpha_n\}$ that $\alpha_{\text{odd}} = \{\alpha_1, \dots, \alpha_i\}$ and $\alpha_{\text{even}} = \{\alpha_{i+1}, \dots, \alpha_n\}$.

Let P_i denote the set of stable n-sets consisting of i even elements and $n-i$ odd elements.

Lemma 3.4.2. *$P_i = \{\alpha^0, \ldots, \alpha^n\}$ is lexicographically ordered by setting*

$$\alpha^0 := \{1, 3, \ldots, 2(n-i)-1, 2(n-i)+2, \ldots, 2n\}$$

and $\alpha^j := \alpha^0 \ominus 2j$. Also, $\alpha^n \ominus 2 = \alpha^0$.

Proof. For P_0, it is immediate that $\{1,3,5,\ldots,2n-1\} \leq \{1,3,5,\ldots,2n-3,2n+1\} \leq \{1,3,5,\ldots,2n-5,2n-1,2n+1\} \leq \cdots$ orders P_0 lexicographically. Given an $\alpha \in P_0$, it follows by inspection that $\alpha \ominus 2$ is the next term in the sequence. The set P_n is handled similarly.

For the case of P_i, $0 < i < n$, it is immediate that $\{1,3,5,\ldots,2(n-i)-1,2(n-i)+2,\ldots,2n\} \leq \{1,3,5,\ldots,2(n-i)-3,2(n-i),\ldots,2n,2n+1\} \leq \cdots \leq \{3,5,7,\ldots,2(n-i)+1,2(n-i)+4,\ldots,2n+2\}$ orders P_i lexicographically. Given an $\alpha \in P_i$, it follows by inspection that $\alpha \ominus 2$ is the next term in the sequence. $\square$

Lemma 3.4.3. *If $\alpha \in P_i$ and $\beta, \gamma \in P_{i+1}$ such that $|\alpha \cap \beta| = |(\alpha \ominus 2) \cap \beta| = n-1$, $|(\alpha \ominus 2) \cap \beta| = |(\alpha \ominus 2) \cap \gamma| = n-1$, and $|(\alpha \ominus 2) \cap (\beta \ominus 2)| = |(\alpha \ominus 4) \cap \beta \ominus 2| = n-1$, then $\gamma = \beta \ominus 2$.*

Proof. We maintain our ordering of the elements for a stable n-set $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ as $\alpha_{\text{odd}} = \{\alpha_1, \ldots, \alpha_i\}$ and $\alpha_{\text{even}} = \{\alpha_{i+1}, \ldots, \alpha_n\}$. As $\alpha \in P_i$ we have

$$\begin{aligned}
\alpha_{\text{odd}} &= \{\alpha_1, \ldots, \alpha_i\} \\
(\alpha \ominus 2)_{\text{odd}} &= \{\alpha_1 - 2, \ldots, \alpha_i - 2\} \\
&= \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_n + 1\} \\
(\alpha \ominus 4)_{\text{odd}} &= \{\alpha_1 - 4, \ldots, \alpha_i - 4\} \\
&= \{\alpha_1, \ldots, \alpha_{i-2}, \alpha_n - 1, \alpha_n + 1\}
\end{aligned}$$

By our assumptions about β and γ we have

$$\begin{aligned}
\beta_{\text{odd}} &= \alpha_{\text{odd}} \cap (\alpha \ominus 2)_{\text{odd}} = \{\alpha_1, \ldots, \alpha_{i-1}\} \\
\gamma_{\text{odd}} &= (\alpha \ominus 2)_{\text{odd}} \cap (\alpha \ominus 4)_{\text{odd}} = \{\alpha_1, \ldots, \alpha_{i-2}, \alpha_n + 1\}
\end{aligned}$$

Thus, $\beta_{\text{odd}} \ominus 2 = \gamma_{\text{odd}}$. By a similar argument we see that $\beta_{\text{even}} \ominus 2 = \gamma_{\text{even}}$ and hence $\beta \ominus 2 = \gamma$. So β and γ are neighbors in P_{i+1}. $\square$

To construct our planar graph, order the elements of P_0 lexicographically and denote them $v^0, \ldots, v^n$. Draw a regular $(n+1)$-gon, which we will also refer to as P_0, and cyclically label its vertices by $v^0, \ldots, v^n$. Triangulate P_0 so that each diagonal in the triangulation has the vertex v^0 as an endpoint. Next, we draw a second regular $(n+1)$-gon, denoted P_1, around P_0, satisfying two conditions:

- The vertices of P_1 lie outside P_0 on lines through the center point of P_0 and the midpoints of the edges of P_0, and
- The edges of P_1 do not intersect P_0.

Label the vertices of the polygon P_1 by the elements of the set P_1, where the labels are placed cyclically about the circle in the lexicographic order; the lexicographically first label for P_1 is placed on the ray between the center of P_0 and the edge between v^0 and v^1. Connect a vertex v of P_0 to a vertex w of P_1 if $|v \cap w| = n-1$; i.e., connect w to the endpoints of the edge of P_0 that it is nearest to.

We inductively continue this process for $i \leq n$ by drawing an $(n+1)$-gon denoted P_i around P_{i-1}. Label the vertices of the polygon P_i with the elements of the set P_i in such a way that one may connect a vertex v of P_{i-1} to a vertex w of P_i exactly when $|v \cap w| = n-1$. This results in the vertices of P_i being labeled cyclically with respect to lexicographic order, with the requirement that the lex-first label for P_i is placed on the ray between the center of P_0 and the edge between the lexicographically first and second vertices of P_{i-1}. To complete the construction, once P_n has been drawn and connected to P_{n-1}, draw arcs representing the edges $\{\{2,4,6,\ldots,2n\}, e\}$ for all even, tight stable n-sets e. It is immediate from our lemmas that this construction is legitimate, and also it is clear that it yields a triangulation of the sphere.

To finish our proof, we must show that the facets of $\widetilde{\mathcal{N}}(SG_{n,2})$ are the same as the facets of this triangulation, i.e. that this triangulation is actually a realization of $\widetilde{\mathcal{N}}(SG_{n,2})$. Observe that both P_0 and P_n bound triangulated $(n+1)$-gons where the vertices of P_0 are the odd tight sets while the vertices of P_n are the even tight sets and the diagonals in the triangulations all emanate from the lexicographically smallest tight stable set in each of P_0 and P_n. These triangulated polygons correspond exactly to the triangulated polygons contained in the N_α complexes defined with respect to facets of tight n-sets. What remains is to show that every other facet of our triangulation corresponds to a two-dimensional simplex in $\widetilde{\mathcal{N}}(SG_{n,2})$ and vice versa.

Let Σ be a simplex in $\widetilde{\mathcal{N}}(SG_{n,2})$. We will show that $\partial\Sigma$ exists in our constructed graph. We consider three cases.

<u>Case 1:</u> Suppose Σ consists of only tight vertices. Then $\Sigma = \{v^1, v^j, v^{j+1}\}$ for some $j = 2, \ldots, n$. As v^j and v^{j+1} are lexicographically ordered, we have $v^j = v^{j+1} \ominus 2$. In the construction of our graph we cyclically connected vertices ordered lexicographically, hence the edge $\{v^j, v^{j+1}\}$ exists in our graph. Moreover, in the construction of our graph we connected all vertices of P_0 to the vertex $\{1, 3, 5, \ldots, 2n-1\}$ and all vertices of P_n to the vertex $\{2, 4, 6, \ldots, 2n\}$. These are precisely the edges $\{v^1, v^j\}$ and $\{v^1, v^{j+1}\}$.

<u>Case 2:</u> Suppose Σ consists of tight and loose vertices. This case follows easily from the following lemma.

Lemma 3.4.4. *For $\alpha \in P_i$, there exists a unique vertex $\pi \in P_{i+1}$ such that $|\alpha \cap \pi| = |\alpha \ominus 2 \cap \pi| = n - 1$.*

Proof. Let α, $\alpha \ominus 2 \in P_i$ be two neighboring vertices. We consider two cases.

Suppose α and $\alpha\ominus 2$ are both tight sets. Without loss of generality we may assume $\alpha, (\alpha \ominus 2) \in P_0$ and we have $|\alpha \cap (\alpha \ominus 2)| = n - 1$. If π is a common neighbor to both α and $\alpha \ominus 2$ in $\widetilde{\mathcal{N}}(SG_{n,2})$ then, by construction, $\pi = \alpha \cap (\alpha \ominus 2) \cup \{p\}$ for some $p \in [2n + 2]$. We claim p must be $\alpha_n - 1$. By definition, p cannot be any element in $\alpha \cup (\alpha \ominus 2)$. There are $n + 1$ such elements. Additionally, p cannot be any of the n elements adjacent to an element in $\alpha \cap (\alpha \ominus 2)$. Thus, we are left with only one choice for p as claimed.

Suppose α and $\alpha \ominus 2$ are both loose sets. Set $\pi = (\alpha_{\text{odd}} \cap (\alpha_{\text{odd}} \ominus 2)) \cup (\alpha_{\text{even}} \cup (\alpha_{\text{even}} \ominus 2))$. From our definition of π it is immediate that π is a stable n-set.

Moreover, the definitions of $\alpha, (\alpha \ominus 2)$, and π we have $|\alpha \cap \pi| = |(\alpha \ominus 2) \cap \pi| = n-1$. Finally, as $|\alpha_{\text{odd}} \cap (\alpha_{\text{odd}} \ominus 2)| = i-1$ and $|\alpha_{\text{even}} \cup (\alpha_{\text{even}} \ominus 2)| = n-i+1$ we have that $\pi \in P_{i+1}$. The uniqueness of π follows from the definitions of α and $\alpha \ominus 2$. Thus our claim holds. $\square$

A similar argument shows that if $\alpha, (\alpha \ominus 2) \in P_i$, then there exists a unique vertex $\pi \in P_{i-1}$ that is a neighbor to both α and $(\alpha \ominus 2)$ for $i = 1, \ldots, n+1$, where $\pi = (\alpha_{\text{even}} \cap (\alpha_{\text{even}} \ominus 2)) \cup (\alpha_{\text{odd}} \cup (\alpha_{\text{odd}} \ominus 2))$.

Case 3: Suppose Σ consists of only loose vertices. By construction of our poset map, we know that two of these vertices, v^r and v^s, are immediate neighbors to some vertex α and the other vertex, v^t is an outer neighbor of α. Set $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ where α is a concatenation of α_{odd} and α_{even}. Then, without loss of generality, $\alpha \ominus 1 = v^r$ and $\alpha \oplus 1 = v^s$. This implies that $v^s \ominus 2 = v^r$ which is an edge in our graph. By definition of an outer neighbor, $v^t = \{(\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_{j-1} + 1, \alpha_j \pm 2, \alpha_{j+1} + 1, \ldots, \alpha_n + 1)$ where α_i is odd for $i = 1, \ldots, j-1$ and is even for $i = j+1, \ldots, n$. The parity of α_j is unknown. If α_j is odd, then $v^t = \{(\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_{j-1} + 1, \alpha_j + 2, \alpha_{j+1} + 1, \ldots, \alpha_n + 1)$. From this we immediately see that $v^t \setminus v^s = \{\alpha_j + 2\}$, so $|v^s \cap v^t| = n-1$. Consider $v^t \setminus v^r$. We claim that $v^t \setminus v^r = \{\alpha_n + 1\}$ implying $|v^r \cap v^t| = n-1$ so that the edges $\{v^r, v^t\}$ and $\{v^s, v^t\}$ exist in our graph by claim 2 of Lemma 3.4.4.

It is enough to show that $\alpha_j + 2 \in v^r$ as we know $\alpha_n + 1 \notin v^r$ and the remaining elements of v^t are in v^r by the definitions of v^t and v^r. Since, by assumption, α_j is odd, we know that there is a gap of size two between α_j and α_{j+1}. Now, $\alpha_{j+1} - 1 \in v^r$ by definition and is odd. As α_j is odd, it is also the case that $\alpha_j + 2$ is odd. Moreover, there is only one odd number between α_j and α_{j+1}. Thus $\alpha_j + 2 = \alpha_{j+1} - 1$. The case when α_j is even follows similarly.

Now consider a simplex σ in our constructed complex. If σ consists of only tight vertices, then it is immediate from Case 1 that $\tau \in \widetilde{\mathcal{N}}(SG_{n,2})$, as we constructed it to be so. If τ consists of any loose vertices then the fact that it is also in $\widetilde{\mathcal{N}}(SG_{n,2})$ is immediate from Lemma 3.4.4 or Case 3 above.

Proof that $\widetilde{\mathcal{N}}(SG_{n,2})$ is a simplicial polytope

Let G_n be the 1-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$. By definition, G_n is simple; the planarity of G_n is shown by the our construction. To apply Theorem 3.4.1 and complete our proof, we must show that G_n is 3-connected.

Let x and y be any two vertices of G_n. We will show that there exist (at least) three disjoint paths from x to y. The above construction shows us that G_n is built from $n+1$ concentric $(n+1)$-cycles, labeled from inside out $P_0, \ldots P_n$. Recall that each vertex v on a given cycle P_i, with the exception of the two cycles formed by tight vertices, is connected to two pairs of adjacent vertices off P_i, one pair on each of P_{i-1} and P_{i+1}. Each vertex v on either P_0 or P_n is connected to only one vertex on an adjacent cycle, either P_1 or P_{n-1}, respectively.

Suppose first that x and y lie on the same cycle P_i. Traverse P_i from x to y in opposite directions to obtain two edge-independent paths. The third path can be found by first moving from x to an adjacent cycle, P_{i+1} or P_{i-1}, then traveling around this cycle in either direction until a neighbor of y is reached.

Next, suppose x and y lie on different cycles, say x on P_j and y on P_k with $j < k$; we begin by finding a pair of disjoint paths from x to y. We first construct a pair of disjoint paths from x to P_k. Let v^1 and w^1 be the neighbors of x that lie on P_{j+1}. If $j+1 = k$, stop at this point having constructed paths x, v^1 and x, w^1, otherwise proceed. Let r^2 and v^2 be the neighbors of v^1 on P_{j+2} and v^2 and w^2 be the neighbors of w^1 on P_{j+2}, noting that v^2 is a common neighbor of v^1 and w^1. If $j+2 = k$, stop at this point having constructed paths x, v^1, v^2 and x, w^1, w^2, otherwise proceed.

Now we are in the same situation with v^2 and w^2 as we were in with v^1 and w^1, in that we may denote the neighbors of v^2 on P_{j+4} by r^3 and v^3 and the neighbors of w^3 by v^3 and w^3, which allows us to construct paths x, v^1, v^2, v^3 and x, w^1, w^2, w^3 from x to P_{j+3}. If y is not on P_{j+3}, then as in the previous cases, we may extend these two paths by setting v^4 equal to the unique common neighbor of v^3 and w^3 on P_{j+4} and setting w^4 equal to the other neighbor of w^3 on P_{j+4}. We continue in this fashion, creating two paths that curve side-by-side through the graph G_n, until we reach P_k with paths $x, v^1, \ldots, v^{k-j}$ and $x, w^1, \ldots, w^{k-j}$. Note that v^{k-j} and w^{k-j} are neighbors on P_k by construction. If v^{k-j} and w^{k-j} are both on P_k and neither is y, then we may extend these two paths along P_k in opposite directions until we meet y. If either v^{k-j} or w^{k-j} is y, then we may complete the other path by connecting via one edge.

Having completed two disjoint paths from x to y, we now need to find a third path disjoint from the first two. If $j+1=k$, let z be a neighbor of y on P_j; we may create a third path by considering the path in P_j from x to z followed by the edge from z to y. If $k > j+1$, then let z^0 be a neighbor of x on P_j not connected by a diagonal. There exists a common neighbor t of z^0 and x on P_1; let z^1 be the other neighbor of z^0 on P_1. We may choose z^2 to be the common neighbor of z^1 and v^1 on P_2. Continue in this fashion, choosing z^m to be the common neighbor of z^{m-1} and v^{m-1} on P_m, until one reaches z^{k-1} on P_{k-1}. If neither v^{k-j} nor w^{k-j} are equal to y, choose a neighbor s of y on P_{k-1} such that s is not v^{k-1} or w^{k-1}. Extend the path $x, z^1, z^2, \ldots, z^{k-1}$ to s by traversing P_{k-1}, then connect to y. If one of v^{k-j} or w^{k-j} is equal to y, then extend z^{k-1} to z^k on P_k and connect z^k to y on P_k to complete the path. Our result is a third path that is disjoint from the first two, connecting x to y. Thus, G_n is 3-connected and planar, hence the one-skeleton of a 3-dimensional polytope.

The f-vector and h-vector of $\widetilde{\mathcal{N}}(SG_{n,2})$

As we have shown, $\widetilde{\mathcal{N}}(SG_{n,2})$ is a simplicial polytope that is combinatorial in nature. It is natural to ask about its f-vector and, by extension, its g-vector. We begin with the following lemma which follows directly from Lemma 3.4.2.

Lemma 3.4.5. *Let $\alpha \subset [2n+2]$ be a stable Kneser set. Let i be the number of odd elements in α. Then there exists some element $\alpha_j \in \alpha$ such that*

$$\alpha_{odd} = \{\alpha_j, \alpha_j+2, \ldots, \alpha_j+2(i-1)\}$$

for $i = 0, 1, \ldots, n$.

Moreover, by Lemma 3.4.2, the same result holds for α_{even}. From this we are able to provide a direct count of the number of vertices in $\widetilde{\mathcal{N}}(SG_{n,2})$. In particular, the number of vertices is simply the number of stable Kneser sets.

Lemma 3.4.6. *For $n > 1$, there are $(n+1)^2$ unique stable Kneser sets in $[2n+2]$.*

Proof. Let i be the number of odd elements in a stable Kneser set α. From Lemma 3.4.5 we know that $\alpha_{\text{odd}} = \{\alpha_j, \alpha_j + 2, \ldots, \alpha_j + 2(i-1)\}$ for some $\alpha_j \in \alpha$. Hence, given an i, we need only concern ourselves with which odd element α_j is. As α_j has $n+1$ possible starting values for each of the $n+1$ possible values of i, our result follows immediately. □

With the above lemma, we are now able to produce our f-vector.

Theorem 3.4.7. *For and $n > 1$, the f-vector of $\widetilde{\mathcal{N}}(SG_{n,2})$ is given by*

$$\langle f_{-1}, f_0, f_1, f_2 \rangle = \left\langle 1, (n+1)^2, 3((n+1)^2 - 2), 2((n+1)^2 - 2) \right\rangle$$

Proof. We will show that $f_2 = 2((n+1)^2 - 2)$. The fact that $f_1 = 3((n+1)^2 - 2)$ will be immediate from the Euler characteristic of $\widetilde{\mathcal{N}}(SG_{n,2})$ using our result from 3.4.6.

We first notice that there are $n+1$, $(n+1)$-gons, the P_i's from our construction, in the one-skeleton of $\widetilde{\mathcal{N}}(SG_{n,2})$, G, yielding n adjacent pairs of $(n+1)$-gons. For each adjacent pair of $(n+1)$-gons there is an entrapped region. A simple count shows that in each region we have $2(n+1)$ 1-skeletons of triangles. Additionally, we recall that we have two triangulated $(n+1)$-gons created by the tight vertices. So, in total we have $2(n+1) \cdot n + 2(n-1) = 2((n+1)^2 - 2)$ 1-skeletons of triangles, each of which yields a two-dimensional face in $\widetilde{\mathcal{N}}(SG_{n,2})$. □

From this theorem and using the definition $h_j = \displaystyle\sum_{i=0}^{d} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$, where $d-1$ is the dimension of $\widetilde{\mathcal{N}}(SG_{n,2})$, we obtain the following corollary.

Corollary 3.4.8. *For $n > 1$, the h-vector of $\widetilde{\mathcal{N}}(SG_{n,2})$ is given by*

$$\langle h_0, h_1, h_2, h_3 \rangle = \left\langle 1, (n+1)^2 - 3, (n+1)^2 - 3, 1 \right\rangle$$

3.5 Invariant Subcomplexes

In [7], Braun proved that for $k \geq 1$ and $n \geq 1$ the automorphism group of $SG_{n,k}$ is isomorphic to the dihedral group of order $2(2n+k)$, which we denote D_{2n+k}. This action arises naturally, as D_{2n+k} acts on $[2n+k]$ thought of as a regular $(2n+k)$-gon with vertices labeled cyclically; this action preserves stable n-sets and disjointness, hence induces an action on $SG_{n,k}$. It is clear from the example in Figure 3.1 that this action does not restrict to simplicial automorphisms of $\widetilde{\mathcal{N}}(SG_{n,2})$, because the vertices $\{1,3,5\}$ and $\{3,5,7\}$ are in the same D_{2n+2}-orbit but do not have simplicially isomorphic neighborhoods. In general, the vertices $\{1,3,5,\ldots,2n-1\}$ and $\{3,5,7,\ldots,2n+1\}$ share this behavior. It is interesting to search for a polytopal boundary sphere contained in $\mathcal{N}(SG_{n,k})$ that is invariant under this group action. In the case $k=2$, we can find such a sphere after passing to a partial subdivision.

Subdividing and collapsing $\mathcal{N}(SG_{n,2})$

We subdivide $\mathcal{N}(SG_{n,2})$ into a complex we call $\overline{\mathcal{N}}(SG_{n,2})$ by leaving the facets of loose vertices unchanged and subdividing only the facets of tight vertices. We shall consider the case where α is a tight vertex consisting of even elements. For any such Σ_α, $n+1$ of its vertices are the even, tight vertices and the remaining vertex is a loose vertex consisting of $n-1$ even elements and one odd element. Order the tight even sets in $\mathcal{N}(SG_{n,2})$ lexicographically, denoted by $\alpha^1,\ldots,\alpha^{n+1}$, and label the loose set in Σ_{α^i} by η_{α^i}.

For the facet Σ_{α^i}, using the notation of Subsection 3.3, we have distinguished vertices v^j and v^{j+1}. Note that $(v^j \cap v^{j+1}) \subset \eta_{\alpha^i}$. Recall that since each of these facets Σ_{α^i} contain all the odd, tight vertices, they intersect in a common n-dimensional face which we will denote by F_o. Barycentrically subdivide F_o, and subdivide Σ_{α^i} by coning over the subdivision of F_o with η_{α^i}. To form $\overline{\mathcal{N}}(SG_{n,2})$, apply this subdivision and an identical procedure to the odd tight vertices; denote by F_e the n-dimensional face given by the even, tight vertices.

The complex we collapse onto will arise as a subcomplex of $\overline{\mathcal{N}}(SG_{n,2})$, which we denote $M(SG_{n,2})$. We will first produce a simplicial collapsing on $\mathcal{N}(SG_{n,2})$ that preserves F_o and F_e, then subdivide the F's and some adjacent cells, and finally complete the collapsing on this subdivided complex. Our strategy is very similar to the one used to create $\widetilde{\mathcal{N}}(SG_{n,2})$, and consists of the following steps:

1. In facets of loose vertices we collapse on the free edge formed by the outer neighbors of the vertex.

2. In each Σ_{α^i}, we collapse the faces of Σ_{α^i} containing η_{α^i}, except for the triangle $\{v^j, v^{j+1}, \eta_{\alpha^i}\}$.

3. On the F's, we barycentrically subdivide and then collapse all faces except the triangles $\{\{v^i, v^{i+1}\}, v^i, b\}$ and $\{\{v^i, v^{i+1}\}, v^{i+1}, b\}$, where b is the barycenter of F and the v^i's are the same notation introduced in Subsection 3.3. We also subdivide the triangles $\{v^j, v^{j+1}, \eta_{\alpha^i}\}$ by subdividing the edge $\{v^j, v^{j+1}\}$.

Via these collapses, the facets of loose simplices will collapse to our previous pairs of triangles sharing an edge, while the union of the subdivided Σ_{α^i}'s will deformation retract to complexes given as a barycentrically subdivided polygon with a triangle glued to each boundary edge.

For the first two steps of our process, we use the poset map Φ from Subsection 3.3, and apply the matchings used there on the preimages $\Phi^{-1}(B_\alpha)$ and $\Phi^{-1}(C_\alpha)$ ranging over all stable n-sets α. The resulting matching induces a simplicial collapse onto a subcomplex of $\mathcal{N}(SG_{n,2})$ consisting of a pair of triangles for each loose vertex, the simplices F_o and F_e, and a triangle of the form $\{v^j, v^{j+1}, \eta_{\alpha^i}\}$ for each tight set α^i.

For the third step in our process, we will subdivide and collapse F_o and F_e, along with the $\{v^j, v^{j+1}, \eta_{\alpha^i}\}$ triangles. We illustrate this only for F_o; F_e is handled identically. Label the odd, tight stable n-sets $v^1, \ldots, v^{n+1}$ as before. Apply this labeling to F_o; barycentrically subdivide F_o, relabeling the remaining vertices in the standard way except we use the label of b for the barycenter. To ensure that our subdivision remains a simplicial complex, we must also subdivide each $\{v^j, v^{j+1}, \eta_{\alpha^i}\}$ into two triangles, $\{\{v^j, v^{j+1}\}, v^j, \eta_{\alpha^i}\}$ and $\{\{v^j, v^{j+1}\}, v^{j+1}, \eta_{\alpha^i}\}$.

Let Ψ be the poset map from the face poset of F_o to the 2-chain $Q := 0 < 1$ such that

$$\Psi(x) = \begin{cases} 0 & \text{if } x \subseteq \{\{v^m, v^{m+1}\}, m, b\} \text{ or } x \subseteq \{\{v^m, v^{m+1}\}, v^{m+1}, b\} \\ 1_o & \text{if otherwise} \end{cases}$$

where $m \in [n+1]$ and $m+1 = 1$ if $m = n+1$. For $x \in \Psi^{-1}(1)$, we match x with $x \cup b$ if $b \notin x$. This matching is clearly acyclic, and it is perfect since if $w \in \Psi^{-1}(0)$ and $b \notin w$, then $b \cup w$ is contained in $\Psi^{-1}(0)$.

It is straightforward to paste the Ψ-maps for F_o and F_e together into a single poset map into the poset consisting of two 2-element chains sharing a common minimal element, i.e., $\{0, 1_o, 1_e\}$ such that $0 < 1_o$ and $0 < 1_e$. If a face of our collapsed, then subdivided, complex from the first two steps is not mapped by Ψ for F_o or F_e, then map it to 0. The resulting poset map allows the application of Theorem 2.2.6.

The proof that $M(SG_{n,2})$ is the boundary of a 3-dimensional polytope is almost identical to the proof in Subsection 3.4. One only needs to observe that the complex resulting from the current analysis is obtained from our previous case by removing the edges inside P_0 and P_n, barycentrically subdividing P_0 and P_n, and then subdividing the remaining triangles sharing an edge with P_0 or P_n. The proof that the one-skeleton is 3-connected is the same aside from handling the situation where vertices arising from the subdivision are involved, which is a straightforward modification of the argument given in the previous case.

Action of D_{2n+2} on $M(SG_{n,2})$

Our goal in this subsection is to show that D_{2n+2} acts simplicially on $M(SG_{n,2})$. Consider $[2n+2]$ as the set of vertices of a regular $(2n+2)$-gon on which D_{2n+2} acts in the usual way.

Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a loose stable set with $\alpha_{\text{odd}} = \{\alpha_1, \ldots, \alpha_i\}$ and $\alpha_{\text{even}} = \{\alpha_{i+1}, \ldots, \alpha_n\}$ Let the immediate neighbors of α be denoted $i(1) := \alpha \oplus 1$ and $i(2) :=$

$\alpha \ominus 1$. Let the outer neighbors of α be denoted

$$\begin{aligned} o(1) &:= \{\alpha_1 + 1, \ldots, \alpha_{i-1} + 1, \alpha_i + 2, \alpha_{i+1} + 1, \ldots, \alpha_n + 1\} \\ o(2) &:= \{\alpha_1 + 1, \ldots, \alpha_{n-1} + 1, \alpha_n + 2\}. \end{aligned}$$

There are two simplices associated to α in $M(SG_{n,2})$, given by $\{i(1), i(2), o(j)\}$ for $j = 1, 2$. As D_{2n+2} is the automorphism group of $SG_{n,2}$, the neighbors of α are mapped by an element $g \in D_{2n+2}$ to neighbors of $g(\alpha)$. Since α is a loose set and D_{2n+2} clearly preserves the loose and tight conditions, $g(\alpha)$ is also a loose set. We will show that the outer neighbors of α are carried by g to the outer neighbors of $g(\alpha)$, hence each of these two simplices associated to α are taken to one of the simplices associated to $g(\alpha)$.

For an element $g \in D_{2n+2}$, g is either a rotation or a flip of $[2n+2]$. If g is a rotation, then

$$\begin{aligned} g(o(1)) &= \{g(\alpha_1 + 1), \ldots, g(\alpha_{i-1} + 1), g(\alpha_i + 2), g(\alpha_{i+1} + 1), \ldots, g(\alpha_n + 1)\}, \\ &= \{g(\alpha_1) + 1, \ldots, g(\alpha_{i-1}) + 1, g(\alpha_i) + 2, g(\alpha_{i+1}) + 1, \ldots, g(\alpha_n) + 1\}. \end{aligned}$$

Otherwise, g is a flip and

$$\begin{aligned} g(o(1)) &= \{g(\alpha_1 + 1), \ldots, g(\alpha_{i-1} + 1), g(\alpha_i + 2), g(\alpha_{i+1} + 1), \ldots, g(\alpha_n + 1)\} \\ &= \{g(\alpha_1) - 1, \ldots, g(\alpha_{i-1}) - 1, g(\alpha_i) - 2, g(\alpha_{i+1}) - 1, \ldots, g(\alpha_n) - 1\} \\ &= \{g(\alpha_1) + 1, \ldots, g(\alpha_{i-1}) + 1, g(\alpha_i) + 1, g(\alpha_{i+1}) + 1, \ldots, g(\alpha_n) + 2\}. \end{aligned}$$

In either case, $g(o(1))$ is an outer neighbor of α and by a similar argument, $g(o(2))$ is an outer neighbor of α. Thus D_{2n+2} sends the associated simplices in $M(SG_{n,2})$ of a loose set α to the associated simplices of $g(\alpha)$.

Let α be a tight stable set and consider the four triangles $T_1 := \{v^j, \{v^j, v^{j+1}\}, b\}$, $T_2 := \{v^{j+1}, \{v^j, v^{j+1}\}, b\}$, $T_3 := \{v^j, \{v^j, v^{j+1}\}, \eta_\alpha\}$, and $T_4 := \{v^{j+1}, \{v^j, v^{j+1}\}, b\}$ in $M(SG_{n,2})$ where v^{j+1} is a neighboring tight vertex, $\{v^j, v^{j+1}\}$ is the barycenter of the edge $v^j v^{j+1}$, η_α is the unique vertex that is both a stable set and a neighbor of v^j and v^{j+1} in $M(SG_{n,2})$, and b is the barycenter of the $(n+1)$-gon. Every remaining facet in $M(SG_{n,2})$ can be associated to a tight stable set α in this way, e.g. every remaining facet contains some tight stable set as a vertex. We now apply g to these triangles and show that their image is contained in $M(SG_{n,2})$.

For T_1, we have $g(T_1) = \{\{g(v^j), g(\{v^j, v^{j+1}\}), g(b)\}$. As g is either parity preserving or reversing for all elements of $[2n+2]$, we know that $g(b)$ is either b or the corresponding element of opposite parity. In either case, $g(v^j)$ and $g(b)$ are neighbors in our complex as well as $g(\{v^j, v^{j+1}\})$ and $g(b)$. Finally, $g(\{v^j, v^{j+1}\})$ and $g(v^j)$ are neighbors in our complex as $g(\{v^j, v^{j+1}\}) = \{g(v^j), g(v^{j+1})\}$. So T_1 (as well as T_2 by symmetry) maps to a corresponding triangle in our complex for any $g \in D_{2n+2}$.

To see that T_3 and T_4 map to appropriate triangles, we need only check that $g(\eta_\alpha)$ is a neighbor of $g(v^j)$ and $g(v^{j+1})$ in our complex. By definition and construction the set $\eta_\alpha = \{v^j\} \cap \{v^{j+1}\} \cup \{p\}$ where p is the unique element of opposite parity of the elements of v^j and v^{j+1} that allows η_α to remain stable. Then $g(\eta_\alpha) = g(v^j) \cap g(v^{j+1}) \cup g(p)$. As $g(v^j)$ and $g(v^{j+1})$ are connected via $g(\{v^j, v^{j+1}\})$, we know that they have exactly $n-1$ elements in common. Moreover, $g(p)$ is of opposite parity of the elements of $g(v^j)$. Hence, $g(\eta_\alpha)$ is stable and a neighbor of both $g(v^j)$ and $g(v^{j+1})$ in $M(SG_{n,2})$, thus D_{2n+2} preserves triangles of this form as well.

3.6 Further Research

The question as to whether the neighborhood complex of $SG_{n,k}$ contains a simplicial polytope as a deformation retract for $n, k \geq 3$ remains unknown. In these cases the stable sets do not behave as nicely as they produce many simplices of several varying dimensions; however, it may be possible to apply certain constraints upon n and k to find some, not necessarily simplicial, polytope.

In related work, a paper by Chowdhury, Godsil, and Royle [8] has extended the idea of Kneser graphs to vector spaces over finite field of order q. These invoke q-binomial coefficients such that when $q = 1$, all of their results agree with what was previously known. In their paper they are able to find the chromatic number of some particular q-Kneser graphs, but many broad classes remain unsolved. I am interested in examining these q-Kneser graphs to see whether or not there is an analogous stable q-Kneser graph which behaves similarly to the traditional stable Kneser graphs.

Chapter 4 Triangular Chessboards

4.1 Introduction

A chessboard complex is the collection of all non-attacking rook positions on an $m \times n$ chessboard. It is clear that this is a simplicial complex as the removal of one rook from an admissible rook placement yields another admissible rook placement. Notice that a placement of $i+1$ rooks corresponds to a simplex of dimension i.

In this chapter, we will be studying the topology of the simplicial complex that arises from non-attacking rook placements on triangular boards. Subsections 3 and 4 of this chapter are joint with Eric Clark. The triangular board $\Psi_{a_n,\ldots,a_1}$ is a left justified board with $a_i > 0$ rows of length i for $1 \leq i \leq n$. In other words, given a positive integer n, the triangular board $\Psi_{a_n,\ldots,a_1}$ is the Ferrers board associated with the partition $\pi = (n^{a_n}, \ldots, 1^{a_1})$ with $a_i > 0$; see Figure 4.1. The squares of the triangular board will be labeled (i, j) for $i \leq j$ where i represents the columns (numbered left to right) while j represents the rows (labeled bottom to top). Motivated by results obtained using the Macaulay2 software package found in [9], we begin by showing that for a_i large enough, $\Sigma(\Psi_{a_n,\ldots,a_1})$, the simplicial complex associated with rook placements on $\Psi_{a_n,\ldots,a_1}$, is a pure complex that is vertex decomposable.

Next, we study the other extremal case. The Stirling complex $\mathrm{Stir}(n)$, originally defined by Ehrenborg and Hetyei [12], is the simplicial complex formed by rook placements on the board $\Psi_{1,1,\ldots,1}$ with n rows; see Figure 4.2. It is known that the f-vector of $\mathrm{Stir}(n)$ is given by $f_i = S(n+1, n+1-i)$ for $i = 1, \ldots, (n-1)$ where $S(n, i)$ denotes the Stirling number of the second kind; see [25]. However, this complex is not pure. We begin the study of $\mathrm{Stir}(n)$ by enumerating its facets via generating functions and then use discrete Morse theory to study its topology.

Chessboard complexes first appeared in the 1979 thesis of Garst [16] concerning Tits coset complexes. By setting $G = \mathfrak{S}_n$ and $G_i = \{\sigma | \sigma(i) = i\}$ for $i = 1, \ldots, m \leq n$, Garst obtained the chessboard complex $M_{m,n} = \Delta(G; G_1, \ldots, G_m)$.

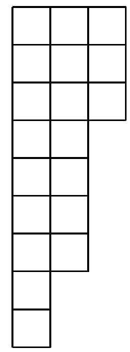

Figure 4.1: The Ferrers board $\Psi_{3,4,2}$.

Here, $\Delta(G; G_1, \ldots, G_m)$ is the simplicial complex whose vertices are the cosets of the subgroups and whose facets have the form $\{gG_1, \ldots, gG_m\}$, for $g \in G$.

The chessboard complex later appeared in a paper by Björner, Lovász, Vrećica, and Živaljević [4] where they gave a bound on the connectivity of the chessboard complex and conjectured that their bound was sharp. This conjecture was shown to be true by Shareshian and Wachs [24]. In that same paper, Shareshian and Wachs also showed that if the chessboard met certain criteria, then it contained torsion in its homology.

4.2 Triangular Boards

Let $\Sigma(\Psi_{a_n,\ldots,a_1})$ denote the simplicial complex formed by all non-attacking rook placements on the triangular board. We call a rook placement *maximal* if no other rook can be added to the placement, that is, every square on the board is attacked.

Theorem 4.2.1. *Let $a_1 \geq 1$, $a_n \geq n$, and $a_i \geq i-1$ for all $i = 2, \ldots, n-1$. Then the simplicial complex $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable.*

This theorem does not extend further in general. From a cursory glance at complexes $\Sigma(\Psi_{a_n,\ldots,a_1})$, where $a_n = n-1$ and $a_n = 0$, (for example, $\Sigma(\Psi_{2,1,0})$) we see we have a non-pure complex which, in general, is not even non-pure vertex decomposable. In addition, loosening our conditions to allow $a_i \geq i-2$ for $i = 1, \ldots, n-1$ and $a_n \geq n$ allows complexes such as $\Sigma(\Psi_{4,0,0})$ which is a torus [4].

In order to prove Theorem 4.2.1, we need the following lemmas. Recall that the squares of the first column of $\Psi_{a_n,\ldots,a_1}$ are labeled $(1,1),\ldots,(1,p)$ where $p = \sum_{i=1}^{n} a_i$. Let V_j denote a collection of the top j elements of the first column that is, $V_j = \{(1,p-j+1),\ldots,(1,p)\}$ for $j = 1,\ldots,p$ and let $V_0 = \emptyset$.

Lemma 4.2.2. *Consider $\Psi_{a_n,\ldots,a_1}$ with $a_1 \geq 1$, $a_n \geq n$, and $a_i \geq i-1$ for all $i = 2,\ldots,n-1$ Then for $j = 0,\ldots,p-1$ the simplicial complex $\mathrm{del}_{\Sigma(\Psi_{a_n,\ldots,a_1})}(V_j)$ is pure of dimension $n-1$.*

Proof. We have two cases to consider.

Let $j = 0$. Then $\mathrm{del}_{\Sigma(\Psi_{a_n,\ldots,a_1})}(V_0) = \Sigma(\Psi_{a_n,\ldots,a_1})$. Any facet of $\Sigma(\Psi_{a_n,\ldots,a_1})$ comes from some maximal rook placement on $\Psi_{a_n,\ldots,a_1}$. Any maximal rook placement must cover the rectangular board $n \times a_n$. Since $a_n \geq n$, this requires exactly n rooks, one in each of the n columns. Since every column contains a rook, the entire board $\Psi_{a_n,\ldots,a_1}$ is covered.

Let $1 \leq j \leq p-1$. Here, $\mathrm{del}_{\Sigma(\Psi_{a_n,\ldots,a_1})}(V_j)$ is the simplicial complex formed by all non-attacking rook placements on the Ferrers board $\Psi_{a_n,\ldots,a_1}$ where the top j entries in the first column have been deleted. Any maximal rook placement on this board must cover the $a_n \times n-1$ rectangular sub-board created by rows $p-a_n+1,\ldots,p$ and columns $2,3,\ldots,n$. Since $a_n \geq n-1$, this requires exactly $n-1$ rooks to cover, one in each of the columns $2,3,\ldots,n$. The first column will contain at least one square (namely $(1,1)$) that is not covered by any of these $n-1$ rooks. Thus by placing a rook in the first column, we see that any facet of $\mathrm{del}_{\Sigma(\Psi_{a_n,\ldots,a_1})}(V_j)$ comes from a maximal rook placement utilizing n rooks. $\square$

Since this simplicial complex is pure, it is natural to ask about its topology.

Lemma 4.2.3. *If $a_i \geq i$ for all $i = 1,\ldots,n$, then $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable.*

Proof. We will proceed by induction on n, the length of the largest row.

For $n = 1$, we have a $1 \times a_1$ chessboard which yields a simplicial complex that is clearly vertex decomposable. Now assume $\Sigma(\Psi_{a_k,\ldots,a_1})$ is vertex decomposable and consider $\Sigma(\Psi_{a_{k+1},\ldots,a_1})$. We maintain our labeling of the vertices using $p = \sum_{i=1}^{k+1} a_i$. We note that $\Sigma(\Psi_{a_{k+1},\ldots,a_1})$ is pure by Lemma 4.2.2.

We claim that the vertex corresponding to the square $(1, p)$ is a shedding vertex of $\Sigma(\Psi_{a_{k+1},\ldots,a_1})$. First, $\mathrm{lk}_{\Sigma(\Psi_{a_{k+1},\ldots,a_1})}(1, p)$ is the set of all faces in bijection with maximal rook placements on the Ferrers board $\Psi_{a_{k+1},\ldots,a_1}$ where the pth row (i.e., top row) and the first column of $\Psi_{a_{k+1},\ldots,a_1}$ have been deleted. That is,

$$\mathrm{lk}_{\Sigma(\Psi_{a_{k+1},\ldots,a_1})}(1, p) = \Sigma(\Psi_{a_{k+1}-1,a_k,\ldots,a_2}).$$

Note that the largest row is now length k. Since $a_i \geq i-1$ and $a_{k+1} - 1 \geq k$ the link of $(1, p)$ is vertex decomposable by our induction hypothesis.

We now must show that $\mathrm{del}_{\Sigma(\Psi_{a_{k+1},\ldots,a_1})}(1, p)$ is vertex decomposable by showing that the vertex corresponding to the square $(1, p-1)$ is a shedding vertex which begins a recursive process.

At the jth iteration, we need to show that $\Delta_j = \mathrm{del}_{\Sigma(\Psi_{a_{k+1},\ldots,a_1})}(V_j)$ is vertex decomposable by showing that the vertex corresponding to $(1, p-j)$ is a shedding vertex. Suppose row $p-j$ has length ℓ. First, $\mathrm{lk}_{\Delta_j}(1, p-j)$ corresponds to the set of all rook placements on the Ferrers board $\Psi_{a_{k+1},\ldots,a_1}$ with row $p-j$ and the first column deleted. This board has a_i rows of length $i-1$ for $i = 2, \ldots, \widehat{\ell+1}, \ldots, k+1$ and $a_{\ell+1} - 1$ rows of length ℓ. From this we see that

$$\mathrm{lk}_{\Delta_j}(1, p-j) = \Sigma(\Psi_{a_{k+1},\ldots,a_{\ell+1}-1,\ldots,a_2}).$$

Once again as $a_i \geq i-1$ and $a_{\ell+1} - 1 \geq \ell$, $\mathrm{lk}_{\Delta_j}(1, p-j)$ is vertex decomposable by our induction hypothesis.

Similarly, the vertex decomposability of $\mathrm{del}_{\Delta_j}(1, p-j)$ remains undetermined and we proceed to another iteration of this process.

At the pth and final step of this process we test the link and deletion of the vertex corresponding to the square $(1, 1)$ on the board $\Psi_{a_{k+1},\ldots,a_2} \cup \{(1, 1)\}$, where $(1, 1)$ forms its own row and column. This complex remains pure by Lemma 4.2.2. Moreover, $\mathrm{lk}_{\Delta_p}(1, 1) = \mathrm{del}_{\Delta_p}(1, 1) = \Sigma(\Psi_{a_{k+1},\ldots,a_2})$ which is vertex decomposable by our induction hypothesis, verifying vertex decomposability by moving backwards through our deletions. Thus $\Sigma(\Psi_{a_{k+1},\ldots,a_1})$ is vertex decomposable. $\square$

Proof of Theorem 4.2.1. Proceed by induction on n with base case $n = 2$. Apply the above proof and Lemma 4.2.3. $\square$

The key observation to make in the proof of Theorem 4.2.1 is that the link of any vertex corresponding to a vertex in the first column yields a simplicial complex that satisfies the conditions of Lemma 4.2.3. Once again, it is the deletion of these vertices which remains in question and this is answered through the same recursive process.

Since $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable for $a_i \geq i-1$, $i = 2, \ldots, n-1$, $a_i \geq 1$, and $a_n \geq n$, we know by Theorem 2.1.3 that it will be homotopy equivalent to a wedge of spheres of dimension $n-1$, or contractible. The number of spheres can be computed by finding the reduced Euler characteristic which is the alternating sum of the coefficients of the f-vector. Let $\ell(i)$ denote the length of column i in $\Psi_{a_n,\ldots,a_1}$ with $\ell(i) = \sum_{j=i}^{n} a_j$.

Theorem 4.2.4. *The coefficients of the f-vector of $\Sigma(\Psi_{a_n,\ldots,a_1})$ are given by*

$$f_i = \sum_{S \in \binom{[n]}{i+1}} \prod_{j=0}^{i} (\ell(s_j) - j),$$

where $S = \{s_0 > s_1 > \cdots > s_i\}$.

Proof. We will compute the coefficients of the f-vector by considering all rook placements on the board $\Psi_{a_n,\ldots,a_1}$. Let S be the collection of $i+1$ columns where the rooks are placed. Notice that $\ell(s_k) \leq \ell(s_{k+1})$. Therefore, placing a rook in column s_k removes a possible location to place a rook in columns $s_{k+1}, \ldots, s_i$. Thus, there are $\prod_{j=0}^{i}(\ell(s_j) - j)$ ways to place $i+1$ rooks on these $i+1$ columns. The result follows by summing over all subsets of $i+1$ columns. $\square$

Theorem 4.2.1 mirrors the result of Ziegler [26] which says that the (rectangular) chessboard complex $M_{n,m}$ is vertex decomposable if $n \geq 2m-1$. That is, extending

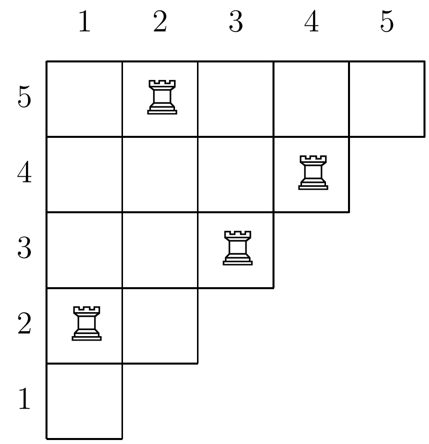

Figure 4.2: A rook placement in bijective correspondence with the partition {{1,3,4,5},{2,6}}.

a triangular board, like extending a rectangular board, far enough allows one to conclude that the associated complex is vertex decomposable.

4.3 Facet Enumeration

We now turn our attention to the Stirling complex. Recall the Stirling complex $\mathrm{Stir}(n)$ is equal to the simplicial complex associated to valid rook placements on the triangular board of size n, $\Psi_{1,\ldots,1}$. It is clear that the Stirling complex is not pure. In this section, we will enumerate the facets of the Stirling complex in each dimension.

The f-vector of the Stirling complex is given by Stirling numbers of the second kind; that is, faces of the Stirling complex are in bijection with partitions. This is done using the map $\mathcal{R}$ where any placement of k non-attacking rooks gets mapped to the partition A where if a rook occupies the square (i, j) then i and $j + 1$ are in the same block of the partition A, see [25, Corollary 2.4.2]. In what follows, we show that facets of the Stirling complex are in bijection with a particular subset of partitions.

Definition 4.3.1. Let B and C be two disjoint nonempty subsets of $[n]$. Then B and C are *intertwined* if $\max(B) > \min(C)$ and $\max(C) > \min(B)$. We say a partition P is *intertwined* if every pair of blocks in P is intertwined.

The idea of intertwined partitions first appeared with the use of the *intertwining number* of a partition in [13] where they were used to provide a combinatorial interpretation for q-Stirling numbers of the second kind. The following definitions are from [13]. For two integers i and j, define the interval $\mathrm{int}(i,j)$ to be the set

$$\mathrm{int}(i,j) = \{n \in \mathbb{Z} : \min(i,j) < n < \max(i,j)\}.$$

Definition 4.3.2. For two disjoint nonempty subsets B and C of $[n]$, define the *intertwining number* $\iota(B,C)$ to be

$$\iota(B,C) = |\{(b,c) \in B \times C : \mathrm{int}(b,c) \cap (B \cup C) = \emptyset\}|.$$

These two ideas are connected as can be seen in the following proposition.

Proposition 4.3.3. *Let B and C be two disjoint nonempty subsets of $[n]$. Then B and C are intertwined if and only if $\iota(B,C) \geq 2$.*

Proof. ($\Rightarrow$) Suppose $\min(B) < \min(C) < \max(B)$. Let b_0 be the maximum element of B such that $\min(B) \leq b_0 < \min(C)$. Then $\mathrm{int}(b_0, \min(C)) \cap (B \cup C) = \emptyset$. Let c_1 be the maximum element of C such that $c_1 < \max(B)$ and b_1 be the minimum element of B such that $c_1 < b_1$. Then $\mathrm{int}(b_1, c_1) \cap (B \cup C) = \emptyset$. Therefore, the intertwining number is at least 2.

($\Leftarrow$) Suppose $\max(B) < \min(C)$. Clearly, for $b \in B$ and $c \in C$, $\mathrm{int}(b,c) \cap (B \cup C) \neq \emptyset$ unless $b = \max(B)$ and $c = \min(C)$. Thus, the intertwining number is 1. $\square$

The bijection between facets of the Stirling complex and intertwined partitions can now be verified.

Theorem 4.3.4. *The set of maximal rook placements with k rooks on a triangular board $\Psi_{1,\ldots,1}$ of size n is in bijection with intertwined partitions of $[n+1]$ into $n+1-k$ blocks.*

Proof. We first show that a maximal rook placement gives rise to an intertwined partition. Let $P = \{P_1, P_2, \ldots, P_{n+1-k}\}$, be a partition of $n+1$ into $n+1-k$ blocks such that there exists two blocks, P_i and P_j, that are not intertwined. Then, without loss of generality, $M_i < m_j$ where M_i is the maximal element of P_i and m_j is the minimal element of P_j. We show that $\mathcal{R}^{-1}(P)$ is not a maximal placement of non-attacking rooks.

As P_i and P_j are not intertwined, we may form a new partition P' where $P' = (P \setminus (P_i, P_j)) \cup \{P_i \cup P_j\}$. That is, P' is the partition P where we have joined P_i and P_j into a single block. Then it is clear that $\mathcal{R}^{-1}(P')$ contains the rook placement $\mathcal{R}^{-1}(P)$ along with an additional rook. As $\mathcal{R}$ is a bijection, $\mathcal{R}^{-1}(P)$ is not maximal.

We now show that an intertwined partition gives rise to a maximal rook placement. Suppose R_k is not a maximal rook placement. We consider $\mathcal{R}(R_k) = Q$. As R_k not maximal, there exists a square (i, j) where we may place a rook. As there is no rook in column i, this implies that i is the maximal element of some block Q_i in Q. Similarly, no rook in row j implies that $j + 1$ is the minimal element of some block Q_j in Q. Hence, Q contains two blocks that are not intertwined. □

We now count the number of partitions with intertwined blocks.

Theorem 4.3.5. *The number of partitions of* $[n]$ *into* k *intertwined blocks is given by*

$$I(n, k) = (k-1)! \sum_{i=k-1}^{n-k} S(i, k-1) \cdot S(n-i, k).$$

Proof. Let $1 \leq i \leq n$. Let P be a partition of $[i]$ into $k-1$ blocks. This can be done in $S(i, k-1)$ ways. Next, let Q be a partition of the remaining $n-i$ elements into k blocks which can be done in $S(n-i, k)$ ways. In order to combine these two partitions into a single partition of $[n]$ into k intertwined blocks, ignore the block containing $i+1$ in Q and pair each block of P to exactly one unique block in Q. This can be done in $(k-1)!$ ways. Clearly any partition obtained with this construction is intertwined.

We now show that every partition with intertwined blocks can be obtained uniquely in this way. When summing over i, notice that $i+1$ is the largest minimal element of the blocks of the partition. Thus, as i varies, so do the intertwined partitions gen-

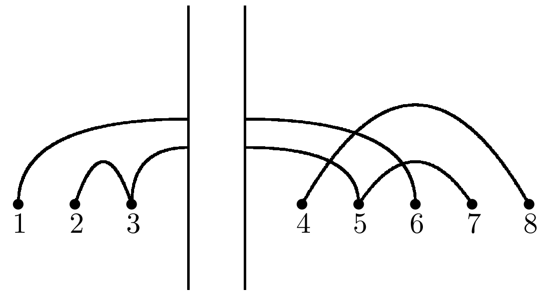

Figure 4.3: This diagram denotes the intertwined partition $\{\{1,6\},\{2,3,5,7\},\{4,8\}\}$ or the intertwined partition $\{\{1,5,7\},\{2,3,6\},\{4,8\}\}$ depending on how the edges from the first part are connected to the edges of the second part.

erated. For a fixed i, it is clear that any intertwined partition can only be obtained in at most one way.

□

Corollary 4.3.6. *The number of maximal rook placements of size $n-k$ on a triangular board $\Psi_{1,\ldots,1}$ of size n is given by*

$$F^n_{n-k} = k! \sum_{i=k}^{n-k} S(i,k) \cdot S(n+1-i,k+1).$$

Using this corollary, we can write the generating function for the facets of the Stirling complex. It is interesting to note that the generating function obtained is the product of the generating function for $S(n,k)$, a shift of the generating function for $S(n,k+1)$, and $k!$.

Corollary 4.3.7. *The generating function for F^n_{n-k} is given by*

$$\sum_{n\geq 0} F^n_{n-k} x^n = \frac{k! \cdot x^{2k}}{\left(\prod_{i=1}^{k}(1-ix)\right)^2 \cdot (1-(k+1)x)}.$$

Proof. We have

$$\begin{aligned}
\sum_{n\geq 0} F^n_{n-k} x^n &= k! \sum_{n\geq 0}\sum_{i=0}^{n} S(i,k)\cdot S(n+1-i,k+1)x^n \\
&= k!\left(\sum_{n\geq 0} S(n,k)x^n\right)\cdot\left(\sum_{n\geq 0} S(n+1,k+1)x^n\right) \\
&= k!\left(\frac{x^k}{\prod_{i=1}^{k}(1-ix)}\right)\cdot\left(\frac{x^k}{\prod_{i=1}^{k+1}(1-ix)}\right) \\
&= \frac{k!\cdot x^{2k}}{\left(\prod_{i=1}^{k}(1-ix)\right)^2\cdot(1-(k+1)x)}.
\end{aligned}$$

□

4.4 Homology

The work in this section is joint with Eric Clark. In this section we examine the topology of the Stirling complex. Work on this has been done by Barmak [3] where he showed that the Stirling complex $\mathrm{Stir}(n)$ is $\lfloor \frac{n-3}{2} \rfloor$-connected. Our technique uses discrete Morse theory by defining poset maps and creating a Morse matching using the Patchwork theorem. We provide an alternate proof of Barmak's connectivity bound and further give a partial description of its homotopy type when n is even.

For a positive integer n, let P be the following poset on the set $\{2, 3, 4, \ldots, 2n\}$. The even integers have the order

$$\begin{aligned}
2n <_P 2(1) <_P 2(n-1) <_P 2(2) &<_P \cdots <_P 2(n-k) <_P 2(k+1) \\
&<_P 2(n-k-1) <_P \cdots <_P 2\lceil n/2 \rceil,
\end{aligned}$$

while the odd integers have the cover relations $k_i <_P k_{i+1}$ where $k_{i+1} = k_i + 2i \cdot (-1)^{n+i+1}$ and $k_1 = 2\lceil n/2 \rceil + 1$. The evens and odds are not comparable, see Figure 4.4.

Using P, we define a total order Q on the squares of the triangular board $\Psi_{1,\ldots,1}$. For $(i,j) \in [n]\times[n]$ with $i \leq j$, $(i,j) <_Q (k,\ell)$ if $j-i < \ell-k$ in the standard order. If $j-i = \ell-k$ then $(i,j) <_Q (k,\ell)$ if $i+j <_P k+\ell$.

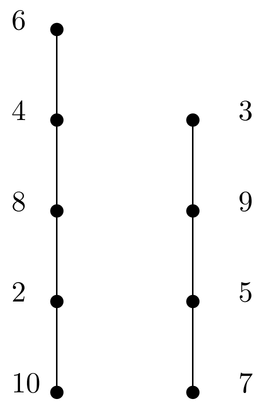

Figure 4.4: The order P for $n = 5$.

	1	2	3	4	5
5	15	13	10	8	1
4	14	12	6	3	
3	11	7	5		
2	9	4			
1	2				

Figure 4.5: The order Q of the squares of a triangular board of size five.

Informally, this order is obtained by starting on the largest diagonal and alternating upper-right to lower-left from the outside to the middle. We continue on the next diagonal alternating from the middle to the outside. The next diagonal moves again from the outside to the middle, etc., see Figure 4.5. It so happens that this order pairs more faces in $\mathrm{Stir}(n)$ than any more traditional ordering.

Let Q_1 be the sub-chain of Q consisting of the lowest elements $(n, n) <_Q (1, 1) <_Q (n-1, n-1) <_Q \cdots <_Q \left(\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right)$ adjoined with a maximal element $\widehat{1}_{Q_1}$. We define a map φ from the face poset of $\mathrm{Stir}(n)$ to the poset Q_1. For $x \in \mathrm{Stir}(n)$, let

$$\varphi(x) = \begin{cases} (i, i), & \text{if } (i, i) \text{ is the smallest in } Q_1 \text{ such that} \\ & x \cup \{(i, i)\} \in \mathrm{Stir}(n) \\ \widehat{1}_{Q_1}, & x \cup \{(i, i)\} \notin \mathrm{Stir}(n) \end{cases}$$

Lemma 4.4.1. *The map $\varphi : \mathcal{F}(\mathrm{Stir}(n)) \longrightarrow Q_1$ is an order-preserving poset map.*

Proof. Let $x, y \in \mathrm{Stir}(n)$ with $x \subset y$. Suppose $\varphi(x) = (i,i)$. That is, (i,i) is the smallest ordered pair such that $(1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n)$ are not elements of x. Since y contains x, $\varphi(y)$ can be no smaller than (i,i). Therefore, $\varphi(x) \leq \varphi(y)$. Suppose $\varphi(x) = \widehat{1}_{Q_1}$. Again, since y contains x, we have $\varphi(y) = \widehat{1}_{Q_1}$ also. $\square$

Lemma 4.4.2. *For $(i,i) <_{Q_1} \widehat{1}_{Q_1}$, the collection $\{(x, x \cup \{(i,i)\}) \ : \ (i,i) \notin x \in \varphi^{-1}(i,i)\}$ is a perfect acyclic matching on the fiber $\varphi^{-1}(i,i)$.*

Proof. Suppose $\varphi(x) = (i,i)$ and $(i,i) \notin x$. Since $(1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n)$ are all not in x, $u(x) = x \cup \{(i,i)\}$ is a valid rook placement in $\mathrm{Stir}(n)$. Also, $\varphi(u(x)) = (i,i)$. Suppose $\varphi(x) = (i,i)$ and $(i,i) \in x$. It is clear that $d(x) = x - \{(i,i)\}$ is a valid rook placement. Also, removing the element (i,i) will not affect the mapping under φ. Therefore, $\varphi(d(x)) = (i,i)$. Finally, this matching is clearly acyclic since the same element is either added to or removed from a placement. $\square$

Using the Patchwork theorem, we have an acyclic matching on $\mathcal{F}(\mathrm{Stir}(n))$ whose only critical cells are the elements of the fiber $\Gamma = \varphi^{-1}(\widehat{1}_{Q_1})$. From the definition of the function φ, the following is clear.

Lemma 4.4.3. *The rook placement x is an element of Γ if for each $i \in [n]$, there is a $j \neq i$ such that $(i,j) \in x$ or $(j,i) \in x$.*

We will exhibit an acyclic matching on the fiber Γ.

Given $\mathrm{Stir}(2n)$, the elements of $\Gamma = \varphi^{-1}(\widehat{1}_{Q_1})$ can be viewed as rook placements on a $(2n-1)$ board. Moreover, these rook placements correspond to partitions of $2n$ $(n > 1)$ with block size greater than 1. This can be easily seen when one considers that a partition with a block of size one implies that the corresponding rook placement leaves an entire row and column unattacked. If such were the case, then a rook placed on the appropriate diagonal would create a valid rook placement. This indicates that the rook placement would have been previously matched.

Continuing our analysis we note that there are no rook placements with fewer than n rooks in Γ. Furthermore, we also note that any placement of n rooks contained entirely within the Durfee square are maximal and cannot have any rooks added, where the *Durfee square* of a Ferrers board is the largest contiguous square sub-board. For example, the Durfee square of the board in Figure 4.2 is of size 3. Thus, these rook placements cannot be matched.

Let us turn our attention to the non-facets with exactly n rooks in Γ. Let x be such a face. By Theorem 4.3.4, as x is not a facet it does not correspond to an intertwined partition of $[2n+1]$. Furthermore, the removal of the element $2n+1$ from the corresponding partition of x does not yield an intertwined partition of $[2n]$. Hence, x corresponds to a partition $P = \{B_1, \ldots, B_k\}$ where some pair of blocks B_*, and $B_{*'}$ are not intertwined. If more than one of such pairs of blocks exist in P, choose B_*, and $B_{*'}$ such that $(\max(B_*), \min(B_{*'}))$ is the smallest in our ordering Q.

We note that $|B_i| = 2$ for all blocks in P and we consider

$$P' = \{B_1, \ldots, \hat{B}_*, \ldots, \hat{B}_{*'}, \ldots, B_k\} \cup \{B_* \cup B_{*'}\}.$$

This corresponds to the rook placement $x' = x \cup (\max B_*, \min B_{*'})$. We define a poset Q_2 that consists of a $\widehat{1}_{Q_2}$ along with an atom P_x for each non-facet x in Γ of dimension $(n-1)$, each of which forms a 2-chain with $\widehat{1}_{Q_2}$.

We define a map $\psi : \Gamma \longrightarrow Q_2$ where we map x and x' to P_x for all x and x' defined as above and all other elements are sent to $\widehat{1}_{Q_2}$.

Lemma 4.4.4. *The map ψ is an order-preserving poset map.*

Proof. The fact that ψ is order-preserving is clear. We need to show that ψ is well-defined. In particular, it is enough to show that each x' is mapped uniquely. For this, we consider the corresponding partition P' of x'. The partition P' consists of $k-1$ blocks of size 2 and one block B of size 4. The block B was formed by merging two non-intertwined blocks of size two into one block of size four. We claim that there is only one way to break B into two non-intertwined blocks of size two. This, however, is immediate when one considers the block $\{1, 2, 3, 4\}$ which can only be broken into the blocks $\{1, 2\}$,$\{3, 4\}$ and be both non-intertwined and of size two. $\square$

We define a matching on the fiber $\psi^{-1}(P_x)$ in the obvious way as each fiber only consists of two elements. The matching is clearly acyclic and perfect.

We are now able to say something about the topology of the Stirling complex.

Theorem 4.4.5. *The Stirling complex* $\mathrm{Stir}(n)$ *is homotopy equivalent to a CW complex with no cells of dimension* k *for* $k < \lceil n/2 \rceil - 1$ *and for* $k \geq n-1$*. Moreover, the Stirling complex* $\mathrm{Stir}(2n)$ *is homotopy equivalent to a wedge of* $n!$ *spheres of dimension* $n-1$ *with a space* X *where* X *is* $(n-1)$*-connected.*

Proof. For the first part of the theorem we see from Lemma 4.4.3 that without at least $\lceil n/2 \rceil$ rooks, a placement x cannot get mapped by φ to $\widehat{1}_{Q_1}$ and will therefore not be critical. Also, it is clear that there cannot be a rook placement with greater than n rooks. The single placement with n rooks will necessarily get sent by φ to (n, n). Thus, any possible critical rook placement will have no more than $n-1$ rooks.

For the second part of the theorem we see from Corollary 4.3.6 that for a triangular board $\Psi_{1,\ldots,1}$ of size $2n$, there are $n!$ facets using n rooks. In fact, these are precisely the placements that fit inside the Durfee square of the triangular board. These will all clearly be mapped to $\widehat{1}_{Q_1}$ by φ. Also, since every position in the Durfee square has coordinates (i, j) for $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$, (ℓ, i) and (j, k) cannot be elements of the placement for $1 \leq \ell, k \leq 2n$. Thus, these will all in turn be mapped to $\widehat{1}_{Q_2}$ by ψ and are therefore critical.

We now must show that no other placement with n rooks will be critical from matchings. Let x be a rook placement with n rooks that are not all contained in the Durfee square and suppose $\varphi(x) = \widehat{1}_{Q_1}$. We show that $\psi(x) \neq \widehat{1}_{Q_2}$. From our observations above, we know that x is not a facet and so there exists a position (i, j) that is not attacked. This means that column i and row j do not contain any rooks. However, since $\varphi(x) = \widehat{1}_{Q_1}$, the positions (i, i) and (j, j) must be attacked. This implies that there exist ℓ and k such that (ℓ, i) and (j, k) are elements of x. Also, since (i, j) was previously not attacked, $x' = x \cup \{(i, j)\}$ is a face of $\mathrm{Stir}(n)$, and hence $\psi(x) = P_x$.

To see that the $n!$ critical cells of size n form the claimed wedge of spheres, let x be

n	$\widetilde{\beta}_0$	$\widetilde{\beta}_1$	$\widetilde{\beta}_2$	$\widetilde{\beta}_3$	$\widetilde{\beta}_4$	$\widetilde{\beta}_5$
1	0					
2	1					
3	0	1				
4	0	2				
5	0	0	9			
6	0	0	6	15		
7	0	0	0	58	8	
8	0	0	0	24	292	1

Table 4.1: The reduced Betti numbers of the Stirling complex through Stir(8).

a rook placement contained within the Durfee square. From Theorem 4.4.5, we know that for $w \subset x$, w will be matched. Also, since x is a facet, there is no placement above it. Therefore, using Theorem 2.2.5, we can conclude that these critical cells form a wedge of $n!$ spheres of dimension $n - 1$. □

We now discuss two corollaries.

Corollary 4.4.6. *The Stirling complex* $\mathrm{Stir}(n)$ *is exactly* $\lfloor \frac{n-3}{2} \rfloor$*-connected.*

Proof. We note that in the case that n is even, this corollary can be obtained from work done by Barmak [3] by observing that the star cluster of the diagonal, K, is the collection of faces not in $\Gamma = \varphi^{-1}(\widehat{1}_{Q_1})$ and $\mathrm{Stir}(n) = Stir_{n+1}$. In the case that n is odd we note the facet from the placement $\{(1, \frac{n+1}{2}), (2, \frac{n+1}{2} + 1), \ldots, (\frac{n+1}{2}, n)\}$ is critical and its boundary is contractible. Therefore, $\mathrm{Stir}(n)$ contains a sphere of dimension $\frac{n-1}{2}$. □

Moreover, we obtain the following corollary from Theorem 4.4.5.

Corollary 4.4.7. *The* $(n-1)$*st reduced Betti number of the Stirling complex* $\mathrm{Stir}(2n)$ *is* $n!$*.*

4.5 Further Research

With the Stirling complex I would like to continue to try and classify its homotopy type or its homology. When using Macaulay 2 to find the homology of $\mathrm{Stir}(n)$, the program failed to produce a result for $n > 8$. At $n = 9$, the Durfee square (the largest contiguous square sub-board) of $\Psi_{1,\ldots,1}$ is a 5×5 square chessboard which is known to contain torsion in its homology. I am interested to see if $\mathrm{Stir}(n)$ contains torsion in its homology at this step or if it contains torsion anywhere. Furthermore, I would like to see if we can extend our matching to further classify the homology and by doing so, determine whether our matching is maximal or not.

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Vita

- **Personal Information**
  - Matthew David Zeckner
  - Born February 26, 1983 in Louisville, Kentucky
- **Education**
  - 2011 Ph.D.(expected), University of Kentucky
  - 2008 M.A., University of Kentucky
  - 2005 B.M., University of Cincinnati
- **Scholastic and Professional Awards**
  - Max Steckler Fellowship, Fall 2010 – Spring 2011.
  - Graduate Fellowship, Funding from NSF award DMS-0758321, Summer 2009, '10.
- **Publications**
  - "Deformation retracts of neighborhood complexes of stable Kneser graphs" with Benjamin Braun. http://arxiv.org/abs/1102.1984
  - "Simplicial complexes of triangular Ferrers boards" with Eric Clark. http://arxiv.org/abs/1008.4525