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# Convergence of Eigenvalues for Elliptic Systems on Domains with Thin Tubes and the Green Function for the Mixed Problem

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ABSTRACT OF DISSERTATION

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The Graduate School
University of Kentucky
2011

Convergence of Eigenvalues for Elliptic Systems on Domains with Thin Tubes and the Green Function for the Mixed Problem

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Justin L. Taylor
Lexington, Kentucky

Director: Dr. Russell M. Brown, Professor of Mathematics
Lexington, Kentucky 2011

ABSTRACT OF DISSERTATION

Convergence of Eigenvalues for Elliptic Systems on Domains with Thin Tubes and the Green Function for the Mixed Problem

I consider Dirichlet eigenvalues for an elliptic system in a region that consists of two domains joined by a thin tube. Under quite general conditions, I am able to give a rate on the convergence of the eigenvalues as the tube shrinks away. I make no assumption on the smoothness of the coefficients and only mild assumptions on the boundary of the domain.

Also, I consider the Green function associated with the mixed problem on a Lipschitz domain with a general decomposition of the boundary. I show that the Green function is Hölder continuous, which shows how a solution to the mixed problem behaves.

KEYWORDS: eigenvalues, elliptic systems, thin tubes, Green function, mixed problem.

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This is dedicated to all the men and women who have died or fought for the United States of America.

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# Chapter 1 Convergence of Eigenvalues for Elliptic Systems on Domains with Thin Tubes

## 1.1 Introduction

In this chapter, we consider the behavior of eigenvalues for elliptic systems in singularly perturbed domains. We give a simple characterization of the family of domains that we can study and it is easy to see that this class includes dumbbell domains formed by connecting two domains by a thin tube. We are able to give a rate on the convergence of the eigenvalues as the tube shrinks away. We make no assumption on the smoothness of the coefficients and only mild assumptions on the boundary of the domain. There does not seem to be much work on eigenvalues for elliptic systems. The work of Rauch and Taylor [32] gives limiting values of eigenvalues in domains with low regularity, but only treats elliptic equations and does not give a rate of convergence. Also, the work of Brown, Hislop, and Martinez [5] provides upper and lower bounds on the splitting between the first two Dirichlet eigenvalues in a symmetric dumbbell region with a straight tube. Furthermore, the work of Anné [1] examines the behavior of eigenfunctions of the Laplace operator under a singular perturbation obtained by adding a thin handle to a compact manifold, but requires more regularity than we use.

There is also a great deal of research on eigenvalues for the Neumann Laplacian in domains with thin tubes. Courant and Hilbert [7] point this out by taking the unit square in $\mathbb{R}^2$ and attaching a thin handle with a proportional square attached to the other end. They show that if $\{\lambda_n^\varepsilon\}$ and $\{\lambda_n^0\}$ are the Neumann eigenvalues of $-\Delta$ in increasing order including multiplicities with respect to the unit square and the perturbed square, then $\lambda_2^\varepsilon \to 0$ as $\varepsilon \to 0$, but $\lambda_2^0 > 0$. Furthermore, Arrieta, Hale, and Han [3] show that for this type of domain, $\lambda_m^\varepsilon \to \lambda_{m-1}^0$, as $\varepsilon \to 0$ for $m \geq 3$.

Jimbo and Morita [22] show that for $N$ disjoint domains connected by thin tubes whose axes are straight lines, the Neumann eigenvalues of $-\Delta$ converge at a rate of order $\varepsilon^{n-1}$, where $\varepsilon$ is the tube width. Jimbo [21] also shows that if $\{\mu_l\}$ are the Neumann eigenvalues of $-\Delta$ in $D = D_1 \cup D_2$ and $\{\lambda_j\}$ are the Dirichlet eigenvalues of $\frac{d^2}{dx^2}$ in $(0,1)$, then for $\{\sigma_k\} = \{\mu_l\} \cup \{\lambda_j\}$ and the eigenvalues of $D_1 \cup D_2 \cup T_\varepsilon$ being $\{\sigma_k^\varepsilon\}$, where $T_\varepsilon$ is a tube with axis $(0,1)$, it is the case that $\sigma_k^\varepsilon \to \sigma_k$ as $\varepsilon \to 0$. Also, Brown, Hislop, and Martinez [4] show that if $\sigma_k \in \{\mu_l\} \backslash \{\lambda_j\}$ then

$$|\sigma_k - \sigma_k^\varepsilon| \leq C \left[\log\left(\frac{1}{\varepsilon}\right)\right]^{\frac{-1}{2}} \qquad n = 2$$

$$|\sigma_k - \sigma_k^\varepsilon| \leq C \varepsilon^{\frac{n-2}{2}} \qquad n \geq 3.$$

Our technique relies on a reverse-Hölder inequality for eigenfunctions that uses a technique introduced by Gehring [12]. This gives $L^p$-integrability of the gradient of eigenfunctions for $p > 2$, which implies that they are not concentrated in the tube. From this inequality, we are able to prove several estimates on eigenfunctions that lead to the result. As a by-product of our research, we give a simple proof of Shi and Wright's [35] $L^p$-estimates for the gradient of the Lamé system as well as other elliptic systems.

### 1.2 Preliminaries

We now define the family of domains $\Omega_\varepsilon$. We let $\Omega$ and $\widetilde{\Omega}$ in $\mathbb{R}^n$ be two non-empty, open, disjoint, and bounded sets. We fix $\varepsilon_0 > 0$, and then let $\{T_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be a family of sets such that if $|T_\varepsilon|$ denotes the Lebesgue measure of $T_\varepsilon$, then

$$|T_\varepsilon| \leq C\varepsilon^d \tag{1.1}$$

where $C$ and $d > 0$ are independent of $\varepsilon$. The connections from $T_\varepsilon$ to $\Omega$ and $\widetilde{\Omega}$ will be contained in $B_\varepsilon$ and $\widetilde{B}_\varepsilon$, which will be balls of radius $\varepsilon$ in $\mathbb{R}^n$ so that $T_\varepsilon \cap \Omega = \emptyset$ and $\overline{T_\varepsilon} \cap \overline{\Omega} \subset B_{\frac{\varepsilon}{2}}$ where $B_{\frac{\varepsilon}{2}}$ is the concentric ball to $B_\varepsilon$ of radius $\frac{\varepsilon}{2}$. Also, suppose a similar condition for $\widetilde{\Omega}$ and $\widetilde{B}_\varepsilon$. Then for any $\varepsilon$, define $\Omega_\varepsilon$ to be the set $\Omega \cup \widetilde{\Omega} \cup T_\varepsilon$, which we assume to be open, and $\Omega_0 = \Omega \cup \widetilde{\Omega}$. So, you may think of $T_\varepsilon$ as a "tube" connecting the two domains. We now have the family of domains $\{\Omega_\varepsilon\}_{0 \leq \varepsilon < \varepsilon_0}$.

Next, we give a condition on the boundary of $\Omega_\varepsilon$. If $B_r$ is any ball of radius $r$ satisfying $B_r \cap \Omega_\varepsilon^c \neq \emptyset$, then

$$|B_{2r} \cap \Omega_\varepsilon^c| \geq C_0 r^n \tag{1.2}$$

where $C_0$ is a constant independent of $r$ and $\varepsilon$. This eliminates domains with "cracks."

Throughout this paper we use the convention of summing over repeated indices, where $i$ and $j$ will run from 1 to $n$ and $\alpha$, $\beta$, and $\gamma$ will run from 1 to $m$. We let $a_{ij}^{\alpha\beta}(x)$ be bounded, measurable, real-valued functions on $\mathbb{R}^n$ which satisfy the symmetry condition

$$a_{ij}^{\alpha\beta}(x) = a_{ji}^{\beta\alpha}(x), \quad i, j = 1, 2, ..., n, \quad \alpha, \beta = 1, 2, ..., m. \tag{1.3}$$

We let $L^2(\Omega_\varepsilon)$ denote the space of square integrable functions taking values in $\mathbb{R}^m$ and $H_0^1(\Omega_\varepsilon)$ denotes the Sobolev space of vector valued functions having one derivative in $L^2(\Omega_\varepsilon)$ and which vanish on the boundary. We use $u_j^\alpha$ to denote the partial derivative $\frac{\partial u^\alpha}{\partial x_j}$.

Let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be a cutoff function so that $\eta_\varepsilon = 0$ in $T_\varepsilon$, $\eta_\varepsilon = 1$ in $\Omega_0 \backslash (B_\varepsilon \cup \widetilde{B_\varepsilon})$, $|\nabla \eta_\varepsilon| \leq \frac{C_n}{\varepsilon}$, and $0 \leq \eta_\varepsilon \leq 1$, where $C_n$ only depends on $n$. We emphasize that $B_\varepsilon$, $\widetilde{B_\varepsilon}$, and $\eta_\varepsilon$ depend on the parameter $\varepsilon$. With these assumptions and definitions, we have that for any $u \in H_0^1(\Omega_\varepsilon)$, $\eta_\varepsilon u$ will be in $H_0^1(\Omega_0)$.

We now introduce the notion of a weak eigenvalue and corresponding weak eigenvector. We say that the number $\sigma$ is a weak Dirichlet eigenvalue of $L$ with weak

Dirichlet eigenfunction $u \in H_0^1(\Omega)$, if $u \neq 0$ and

$$\int_\Omega a_{ij}^{\alpha\beta}(x) u_i^\alpha(x) \phi_j^\beta(x)\ dx = \sigma \int_\Omega u^\gamma(x)\phi^\gamma(x)\ dx \qquad \textit{for any } \phi \in H_0^1(\Omega). \tag{1.4}$$

As we will see in a later section, the eigenvalues for the elliptic systems we consider form an increasing sequence. The lower bound on the smallest eigenvalue, however, depends on which ellipticity condition we use.

## 1.3 Ellipticity Conditions

If we define a norm on matrices $A = A_j^i \in \mathbb{R}^{m\times n}$ as $|A|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|A_j^i|^2$, then we say that $L$ satisfies a *strong Legendre condition* or a *strong ellipticity condition* if there exists $\theta > 0$ so that

$$a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \geq \theta|\xi|^2, \quad \xi \in \mathbb{R}^{m\times n}, \quad a.e.\ x \in \Omega_\varepsilon. \tag{1.5}$$

We introduce the Lamé system as $Lu = -\text{div}\zeta(u)$, where $\zeta(u)$ denotes the stress tensor defined by

$$\zeta_j^\beta(u) := a_{ij}^{\alpha\beta} u_i^\alpha \tag{1.6}$$

which is defined in terms of the Lamé moduli $\upsilon$ and $\mu$ by

$$a_{ij}^{\alpha\beta} = \upsilon\delta_{i\alpha}\delta_{j\beta} + \mu\delta_{ij}\delta_{\alpha\beta} + \mu\delta_{i\beta}\delta_{j\alpha}. \tag{1.7}$$

Also, define the strain tensor $\kappa(u)$ as

$$\kappa_{ij}(u) := \frac{1}{2}\left(u_j^i + u_i^j\right). \tag{1.8}$$

Note that for the Lamé system, $m = n$ and the Lamé parameters $\upsilon$ and $\mu$ given in (1.7) are bounded, measurable, and satisfy the conditions

$$\upsilon(x) > 0 \qquad \mu(x) \geq \delta > 0. \tag{1.9}$$

The Lamé system does not satisfy the strong ellipticity condition, but does satisfy the ellipticity condition

$$a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \geq \tau \left|\kappa(u)\right|^2, \quad u \in H_0^1(\Omega_\varepsilon) \tag{1.10}$$

where $\tau = 2\delta$. Next, consider a well-known inequality from Oleinik [30, p. 13].

**Theorem 1.3.1.** ***Korn's Inequality*** *Let $\Omega$ be a bounded domain. If $u \in H_0^1(\Omega)$, then*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq 2\|\kappa(u)\|^2_{L^2(\Omega)} \tag{1.11}$$

*where $\kappa(u)$ is from (1.8) and $C$ only depends on $n$.*

With Korn's Inequality (1.11), it is easy to see that for the Lamé system, we have

$$\frac{\tau}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 \, dy \leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy, \qquad u \in H_0^1(\Omega_\varepsilon).$$

Furthermore, we say that $L$ satisfies the *Legendre-Hadamard condition* if there exists $\theta > 0$ so that

$$a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\psi_i\psi_j \geq \theta|\xi|^2|\psi|^2, \quad \xi \in \mathbb{R}^m, \quad \psi \in \mathbb{R}^n, \quad a.e.\ x \in \Omega_\varepsilon. \tag{1.12}$$

For scalar equations, the Legendre-Hadamard condition is equivalent to the strong Legendre condition. However, for systems, this is not the case, as illustrated in this example taken from Chen [6, p. 133]. Let $m = n = 2$ and

$$a_{ij}^{\alpha\beta} = s\delta_{\alpha\beta}\delta_{ij} + b_{ij}^{\alpha\beta}, \qquad 0 < s < \frac{1}{2}$$

where $b_{21}^{21} = 1$, $b_{21}^{12} = -1$, and $b_{ij}^{\alpha\beta} = 0$ otherwise. Then,

$$\begin{aligned} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\psi_i\psi_j &= s\xi_1^2(\psi_1^2+\psi_2^2) + s\xi_2^2(\psi_1^2+\psi_2^2) \\ &= s(\xi_1^2+\xi_2^2)(\psi_1^2+\psi_2^2) \\ &= s|\xi|^2|\psi|^2, \end{aligned}$$

which means that this system satisfies the Legendre-Hadamard condition. But, if $\xi = (0, 1, 2s, 0)^t$, we obtain

$$\begin{aligned} a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta &= s|\xi|^2 + (\xi_2^2\xi_1^1 - \xi_1^2\xi_2^1) \\ &= s(1 + 4s^2) - 2s \\ &= s(4s^2 - 1). \end{aligned}$$

Hence, this system does not satisfy the strong Legendre condition.

Even in the case of the coefficients satisfying a symmetry condition, the Legendre-Hadamard condition is still a weaker condition. As stated earlier, the Lamé system does not satisfy the strong ellipticity condition. This can be observed by noting that for any $\xi \in \mathbb{R}^{n^2}$, we have

$$a_{ij}^{\alpha\beta}\xi_i^\alpha \xi_j^\beta = \upsilon\xi_i^i\xi_j^j + \mu|\xi_i^\alpha|^2 + \mu\xi_i^j\xi_j^i$$

so that by choosing $n = 2$, $\xi_2^1 = -1$, $\xi_1^2 = 1$, and $\xi_1^1 = \xi_2^2 = 0$, we have

$$\begin{aligned} a_{ij}^{\alpha\beta}\xi_i^\alpha \xi_j^\beta &= 2\mu - 2\mu \\ &= 0 \end{aligned}$$

which implies that the Lamé system does not satisfy the strong ellipticity condition. However, note that for $\xi,\ \eta \in \mathbb{R}^n$, we have

$$\begin{aligned} a_{ij}^{\alpha\beta}\xi_i\xi_j\eta_\alpha\eta_\beta &= \upsilon\xi_i\xi_j\eta_i\eta_j + \mu\xi_i\xi_i\eta_\alpha\eta_\alpha + \mu\xi_i\xi_j\eta_j\eta_i \\ &= (\upsilon + \mu)\left(\xi_i\eta_i\right)^2 + \mu|\xi|^2|\eta|^2 \\ &\geq \delta|\xi|^2|\eta|^2 \end{aligned}$$

so that the Lamé system satisfies the Legendre-Hadamard ellipticity condition. In general, systems with continuous coefficients satisfying the Legendre-Hadamard ellipticity condition also satisfy the following inequality taken from Treves [37, p. 347].

**Proposition 1.3.2.** ***Gårding's Inequality*** *If $L$ satisfies the Legendre-Hadamard condition (1.12) with continuous coefficients in $\overline{\Omega}_\varepsilon$, then for any $u \in H_0^1(\Omega_\varepsilon)$,*

$$C_1 \int_{\Omega_\varepsilon} |\nabla u|^2 \, dy \leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy + C_2 \int_{\Omega_\varepsilon} |u|^2 \, dy \tag{1.13}$$

*where both $C_1$ and $C_2$ depend on the ellipticity constant in (1.12) and the coefficients $a_{ij}^{\alpha\beta}$.*

*Proof.* We first restrict to when the domain is a small ball, $B_r$, and consider the case when the coefficients are constant. It suffices to consider $u \in C_c^\infty(B_r)$. We define the Fourier transform for scalar-valued functions $f \in L^2(\mathbb{R}^n)$ as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx,$$

and set

$$(u)\widehat{\ } = ((u^1)\widehat{\ }, ..., (u^m)\widehat{\ })^t.$$

Parseval's identity and properties of the Fourier transform then yield

$$\begin{aligned}
\int_{B_r} a_{ij}^{\alpha\beta} u_i^\alpha(y) u_j^\beta(y) \, dy &= \int_{B_r} a_{ij}^{\alpha\beta} (u_i^\alpha)\widehat{\ }(\xi) (\overline{u_j^\beta})\widehat{\ }(\xi) \, d\xi \\
&= \int_{B_r} a_{ij}^{\alpha\beta} (2\pi i \xi_i)(2\pi i \xi_j)(u^\alpha)\widehat{\ }(\xi)(u^\beta)\widehat{\ }(\xi) \, d\xi \\
&\geq \int_{B_r} \theta |2\pi i \xi|^2 |(u)\widehat{\ }(\xi)|^2 \, d\xi
\end{aligned}$$

where the ellipticity condition (1.12) was used on the last line. Thus, since

$$\begin{aligned}
\int_{B_r} \theta |2\pi i \xi|^2 |(u)\widehat{\ }(\xi)|^2 \, d\xi &= \sum_{j=1}^n \sum_{\alpha=1}^m \int_{B_r} \theta |2\pi i \xi_j (u^\alpha)\widehat{\ }(\xi)|^2 \, d\xi \\
&= \sum_{j=1}^n \sum_{\alpha=1}^m \int_{B_r} \theta |(u_j^\alpha)\widehat{\ }(\xi)|^2 \, d\xi \\
&= \int_{B_r} \theta |\nabla u(y)|^2 \, dy
\end{aligned}$$

we thus obtain

$$\theta \int_{B_r} |\nabla u|^2 \, dy \leq \int_{B_r} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy. \tag{1.14}$$

Next, define the modulus of continuity to be

$$M(x_0, R) = \max_{\substack{y \in \overline{B_R(x_0)} \\ i,j,\alpha,\beta}} |a_{ij}^{\alpha\beta}(y) - a_{ij}^{\alpha\beta}(x_0)|. \tag{1.15}$$

We have

$$\left| \int_{B_r(x_0)} [a_{ij}^{\alpha\beta}(x_0) - a_{ij}^{\alpha\beta}] u_i^\alpha u_j^\beta \, dy \right| \le M(x_0, r) \int_{B_r(x_0)} |\nabla u|^2 \, dy$$

so that by freezing the coefficients at $x_0$,

$$\int_{B_r(x_0)} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy = \int_{B_r(x_0)} a_{ij}^{\alpha\beta}(x_0) u_i^\alpha u_j^\beta \, dy + \int_{B_r(x_0)} [a_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x_0)] u_i^\alpha u_j^\beta \, dy$$

and using the constant coefficient case (1.14), we obtain

$$(\theta - M(x_0, r)) \int_{B_r(x_0)} |\nabla u|^2 \, dy \le \int_{B_r(x_0)} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy. \tag{1.16}$$

Now, for the global estimate, since the coefficients are uniformly continuous in $\overline{\Omega}_\varepsilon$, we may fix $r_0$ small enough so that

$$\theta - M(y, r_0) > \frac{\theta}{2}, \qquad y \in \Omega_\varepsilon. \tag{1.17}$$

Cover $\Omega_\varepsilon$ with a finite number of balls $\{B_{r_0}(x_k)\}_{k=1}^N$. There exists a smooth partition of unity $\{\rho_k\}_{k=1}^N$ subordinate to the cover $\{B_{r_0}(x_k)\}_{k=1}^N$ so that

$$\begin{cases} 0 \le \rho_k \le 1 \quad & k = 1, ..., N \\ \displaystyle\sum_{k=1}^N \rho_k^2(x) = 1 \quad & \text{for each } x \in \Omega_\varepsilon \\ |\nabla \rho_k| \le \frac{C}{r_0} \quad & k = 1, ..., N. \end{cases}$$

We may write

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \ dy = \sum_{k=1}^N \int_{\Omega_\varepsilon} \rho_k^2 a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \ dy \\
&= \sum_{k=1}^N \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} (\rho_k u)_i^\alpha (\rho_k u)_j^\beta \ dy \\
&\qquad - \sum_{k=1}^N \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} [(\rho_k)_i (\rho_k)_j u^\alpha u^\beta + \rho_k (\rho_k)_j u_i^\alpha u^\beta + (\rho_k)_i \rho_k u^\alpha u_j^\beta] \ dy \\
&= I - II.
\end{aligned}
$$

We have that

$$
II \leq \left( \frac{CN}{r_0^2} + \frac{CN^2}{r_0^2 \omega} \right) \int_{\Omega_\varepsilon} |u|^2 \ dy + \omega \int_{\Omega_\varepsilon} |\nabla u|^2 \ dy \tag{1.18}
$$

for any $\omega > 0$.

Also, since $\rho_k u$ has compact support in $B_{r_0}(x_k)$, we may apply (1.16) and (1.17) to obtain

$$
\begin{aligned}
I &\geq \frac{\theta}{2} \sum_{k=1}^N \int_{\Omega_\varepsilon} |\nabla(\rho_k u)|^2 \\
&\geq \frac{\theta}{2} \sum_{k=1}^N \int_{\Omega_\varepsilon} (\rho_k^2 |\nabla u|^2 - |\nabla \rho_k|^2 |u|^2) \ dy \\
&\geq \frac{\theta}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 \ dy - \frac{C\theta}{r_0^2} \int_{\Omega_\varepsilon} |u|^2 \ dy.
\end{aligned} \tag{1.19}
$$

So, now using (1.19) and choosing $\omega = \frac{\theta}{4}$ in (1.18), we obtain

$$
\frac{\theta}{4} \int_{\Omega_\varepsilon} |\nabla u|^2 \ dy \leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \ dy + \frac{C}{r_0^2} \left( N + \frac{N}{\theta} + \theta \right) \int_{\Omega_\varepsilon} |u|^2 \ dy.
$$

$\square$

## 1.4 Construction of Eigenvalues

The construction of eigenvalues and eigenfunctions is taken from Gilbarg and Trudinger [15, p. 212] and is well-known. We will construct eigenvalues assuming that $u \in H_0^1(\Omega_\varepsilon)$ satisfies (1.13). We note that if $L$ satisfies the strong Legendre ellipticity condition (1.5) or the ellipticity condition (1.10), then the construction is a special case of this construction. Define the bilinear form $B_\varepsilon$ on $H_0^1(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)$ as

$$B_\varepsilon(u, v) = \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha v_j^\beta \, dy \tag{1.20}$$

and define the Rayleigh quotient $R_\varepsilon$ as

$$R_\varepsilon(u) = \frac{B_\varepsilon(u, u)}{\|u\|^2_{L^2(\Omega_\varepsilon)}} \tag{1.21}$$

for $u \neq 0$. From Gårding's inequality (1.13),

$$R_\varepsilon(u) \geq \frac{C_1 \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} - C_2 \|u\|^2_{L^2(\Omega_\varepsilon)}}{\|u\|^2_{L^2(\Omega_\varepsilon)}} \geq -C_2. \tag{1.22}$$

So, $\sigma = \inf\limits_{0 \neq w \in H_0^1(\Omega_\varepsilon)} R_\varepsilon(w)$ exists and is finite.

**Claim 1.4.1.** *There exists $u \in H_0^1(\Omega_\varepsilon)$ such that $\sigma = R_\varepsilon(u)$.*

*Proof.* Choose a sequence $\{w_p\} \in H_0^1(\Omega_\varepsilon)$ so that $R_\varepsilon(w_p) \to \sigma$. Then set

$$u_p = \frac{w_p}{\|w_p\|_{L^2(\Omega_\varepsilon)}}$$

so that $\|u_p\|_{L^2(\Omega_\varepsilon)} = 1$ and $R_\varepsilon(u_p) \to \sigma$. By Gårding's inequality (1.13),

$$\begin{aligned} C_1 \|\nabla u_p\|^2_{L^2(\Omega_\varepsilon)} &\leq \int_\Omega a_{ij}^{\alpha\beta} (u_p)_i^\alpha (u_p)_j^\beta \, dy + C_2 \|u_p\|^2_{L^2(\Omega_\varepsilon)} \\ &= R_\varepsilon(u_p) + C_2 \\ &\leq C \end{aligned}$$

the last line owing to the fact that $\{R_\varepsilon(u_p)\}$ converges. Thus, by the compact imbedding of $H_0^1(\Omega_\varepsilon)$ into $L^2(\Omega_\varepsilon)$, there exists $u \in L^2(\Omega_\varepsilon)$ so that by passing to a subsequence of $\{u_p\}$, and renaming it $\{u_p\}$, we have $\|u_p - u\|_{L^2(\Omega_\varepsilon)} \to 0$ and $\|u\|_{L^2(\Omega_\varepsilon)} = 1$.

We will next show $\|u_p - u\|_{H_0^1(\Omega_\varepsilon)} \to 0$. Define $Q(w) = B_\varepsilon(w,w)$. Then, for any $l$ and $k$, we have

$$
\begin{aligned}
&Q\left(\frac{u_l+u_p}{2}\right) + Q\left(\frac{u_l-u_p}{2}\right)\\
&\qquad = \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}\left(\frac{u_l+u_p}{2}\right)_i^\alpha \left(\frac{u_l+u_p}{2}\right)_j^\beta \, dy + \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}\left(\frac{u_l-u_p}{2}\right)_i^\alpha \left(\frac{u_l-u_p}{2}\right)_j^\beta \, dy\\
&\qquad = \frac{1}{2}\left(\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(u_l)_i^\alpha (u_l)_j^\beta \, dy + \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(u_p)_i^\alpha (u_p)_j^\beta \, dy\right)\\
&\qquad = \frac{1}{2}\left(Q(u_l) + Q(u_p)\right).
\end{aligned}
$$

Thus, since $\sigma = \inf\limits_{0 \neq w \in H_0^1(\Omega_\varepsilon)} R_\varepsilon(w)$, we have

$$
\begin{aligned}
Q\left(\frac{u_l-u_p}{2}\right) &\le \frac{1}{2}\left(Q(u_l)+Q(u_p)\right) - \sigma \int_{\Omega_\varepsilon} \left|\frac{u_l+u_p}{2}\right|^2 \, dy\\
&= \frac{1}{2}\left(Q(u_l)+Q(u_p)\right) - \frac{\sigma}{4}\int_{\Omega_\varepsilon} |u_l|^2 + |u_p|^2 + 2(u_l)^\alpha (u_p)^\alpha \, dy\\
&\to \frac{1}{2}(\sigma+\sigma) - \frac{\sigma}{4}(4) \qquad (\text{as } p, l \to \infty)\\
&= 0.
\end{aligned}
$$

Therefore, using Gårding's inequality (1.13) and since $\{u_p\}$ converges in $L^2(\Omega_\varepsilon)$,

$$
\begin{aligned}
C_1\|\nabla(u_l-u_p)\|_{L^2(\Omega_\varepsilon)}^2 &\le \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(u_l-u_p)_i^\alpha (u_l-u_p)_j^\beta \, dy + C_2\|u_l-u_p\|_{L^2(\Omega_\varepsilon)}^2\\
&= 4Q\left(\frac{u_l-u_p}{2}\right) + C_2\|u_l-u_p\|_{L^2(\Omega_\varepsilon)}^2\\
&\to 0 \qquad (\text{as } p, l \to \infty)
\end{aligned}
$$

so that $\{u_p\}$ is a Cauchy sequence in $H_0^1(\Omega_\varepsilon)$. It now follows that $u_p \to u$ in $H_0^1(\Omega_\varepsilon)$.

To finish up the proof of the claim, we will now show $Q(u) = R_\varepsilon(u) = \sigma$. We have

$$
\begin{aligned}
&\left|\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(u_p)_i^\alpha (u_p)_j^\beta \, dy - \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \, dy\right|\\
&\qquad \le C\int_{\Omega_\varepsilon} \left|(u_p)_i^\alpha (u_p)_j^\beta - u_i^\alpha (u_p)_j^\beta + u_i^\alpha (u_p)_j^\beta - u_i^\alpha u_j^\beta\right|\\
&\qquad \le C\int_{\Omega_\varepsilon} |(u_p)_j^\beta|\,|(u_p)_i^\alpha - u_i^\alpha| + |u_i^\alpha|\left|(u_p)_j^\beta - u_j^\beta\right|.
\end{aligned}
$$

So, since $u_p \to u$ in $H_0^1(\Omega_\varepsilon)$, we may apply Hölder's inequality to obtain

$$\begin{aligned}
&\left| \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} (u_p)_i^\alpha (u_p)_j^\beta \; dy - \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \; dy \right| \\
&\qquad \le C \|u_p\|_{H_0^1(\Omega_\varepsilon)} \|u_p - u\|_{H_0^1(\Omega_\varepsilon)} + \|u\|_{H_0^1(\Omega_\varepsilon)} \|u_p - u\|_{H_0^1(\Omega_\varepsilon)} \\
&\qquad \to 0 \qquad (\text{as } p \to \infty)
\end{aligned}$$

so that $\sigma = \lim_{p\to\infty} R_\varepsilon(u_p) = R_\varepsilon(u)$ and the proof of the claim is complete. □

**Claim 1.4.2.** *$\sigma = R_\varepsilon(u)$ from Claim 1.4.1 is the minimum eigenvalue with eigenfunction $u$.*

*Proof.* Fix $v \in H_0^1(\Omega_\varepsilon)$ and define $f(t) = R_\varepsilon(u + tv)$ where $t \in \mathbb{R}$. Then, by the symmetry of the coefficients (1.3) and the normalization of $u$,

$$\begin{aligned}
f'(0) &= \frac{\left(\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} \left( v_i^\alpha u_j^\beta + u_i^\alpha v_j^\beta \right) \; dy\right) \left(\int_{\Omega_\varepsilon} |u|^2 \; dy\right) - \left(\int_{\Omega_\varepsilon} 2u^\alpha v^\alpha \; dy\right) \left(\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta\right)}{\left(\int_{\Omega_\varepsilon} |u|^2\right)^2} \\
&= 2B_\varepsilon(u, v) - 2\sigma \int_{\Omega_\varepsilon} u^\alpha v^\alpha.
\end{aligned}$$

So, since $R_\varepsilon$ achieves a minimum at $u$, we have $2B_\varepsilon(u,v) - 2\sigma \int_{\Omega_\varepsilon} u^\alpha v^\alpha = 0$ or $B_\varepsilon(u,v) = \sigma \int_{\Omega_\varepsilon} u^\alpha v^\alpha$ which implies $u$ is an eigenfunction of $L$ with eigenvalue $\sigma$.

Also, if $\lambda < \sigma$ is another eigenvalue with eigenfunction $w$, then $B_\varepsilon(w, w) = \lambda \int_{\Omega_\varepsilon} w^\alpha w^\alpha$ which implies

$$\frac{B_\varepsilon(w,w)}{\|w\|_{L^2(\Omega_\varepsilon)}^2} = \lambda < \sigma$$

which contradicts $\sigma = \inf_{0 \ne w \in H_0^1(\Omega_\varepsilon)} R_\varepsilon(w)$. The proof of the claim is now complete. □

To construct the remaining eigenvalues, we need to make sure the eigenspaces are all finite-dimensional.

**Claim 1.4.3.** *We have $E_N = span\{u_k : \sigma_k \le N\} \subset L^2(\Omega_\varepsilon)$ is finite-dimensional for every $N$.*

*Proof.* We prove by contradiction. So, suppose there is an infinite orthonormal sequence $\{u_k\}$ in $E_N$. Then by the ellipticity condition (1.13), for each $k$, we have

$$\begin{aligned} C_1 \int_{\Omega_\varepsilon} |\nabla u_k|^2 \, dy &\leq B_\varepsilon(u_k, u_k) + C_2 \int_{\Omega_\varepsilon} |u_k|^2 \, dy \\ &\leq \sigma_k + C_2 \\ &\leq N + C_2. \end{aligned} \tag{1.23}$$

So, we have that the sequence $\{u_k\}$ is bounded in $H_0^1(\Omega_\varepsilon)$. So, again using the compact imbedding of $H_0^1(\Omega_\varepsilon)$ into $L^2(\Omega_\varepsilon)$, there exists a convergent subsequence in $L^2(\Omega_\varepsilon)$. Renaming this subsequence $\{u_k\}$ and using that this subsequence is orthonormal, we have

$$\begin{aligned} \|u_l - u_p\|^2_{L^2(\Omega_\varepsilon)} &= \langle u_l - u_p, u_l - u_p \rangle_{L^2(\Omega_\varepsilon)} \\ &= \|u_l\|^2_{L^2(\Omega_\varepsilon)} + \|u_p\|^2_{L^2(\Omega_\varepsilon)} - 2\langle u_l, u_p \rangle_{L^2(\Omega_\varepsilon)} \\ &= 2 \quad (l \neq p) \end{aligned}$$

which implies this subsequence is not Cauchy in $L^2(\Omega_\varepsilon)$. This contradicts that this subsequence converges. So, there cannot be an infinite orthonormal sequence. □

Now that we have that each eigenspace is finite-dimensional, we may continue the construction of subsequent eigenvalues. Given the $(k-1)th$ eigenfunction $u_{k-1}$, set

$$\sigma_k = \inf_{\substack{0 \neq w \in H_0^1(\Omega_\varepsilon) \\ w \in \{u_1, u_2, ..., u_{k-1}\}^\perp}} R_\varepsilon(w) \tag{1.24}$$

where the orthogonal complement is taken in $L^2(\Omega_\varepsilon)$. We note that $\sigma_k$ exists since $R_\varepsilon(w)$ is bounded below. Furthermore, following the same arguments from Claim 1.4.1, there exists $u_k \in H_0^1(\Omega_\varepsilon)$ such that $R_\varepsilon(u_k) = \sigma_k$ and $\|u_k\|^2_{L^2(\Omega_\varepsilon)} = 1$. To show that $u_k$ is an eigenfunction of $L$ with eigenvalue $\sigma_k$, we decompose

$$L^2(\Omega_\varepsilon) = \operatorname{span}\{u_1, u_2, ..., u_{k-1}\} \oplus \{u_1, u_2, ..., u_{k-1}\}^\perp.$$

If $v \in H_0^1(\Omega_\varepsilon) \cap \{u_1, u_2, ..., u_{k-1}\}^\perp$, then by construction of the eigenvalues, we may set $f(t) = R_\varepsilon(u_k + tv)$ and follow the same argument from Claim 1.4.2 to get that $B_\varepsilon(u_k, v) = \sigma_k \int_{\Omega_\varepsilon} u_k^\alpha v^\alpha$. If $v \in H_0^1(\Omega_\varepsilon) \cap \text{span}\{u_1, u_2, ..., u_{k-1}\}$, then write $v = \sum_{l=1}^{k-1} c_l u_l$. We have $B_\varepsilon(v, w) = \displaystyle\sum_{l=1}^{k-1} c_l \sigma_l \int_{\Omega_\varepsilon} u_l^\alpha w^\alpha$ for any $w \in H_0^1(\Omega_\varepsilon)$. Consequently, by the symmetry condition (1.3) and since $u_k \in \{u_1, u_2, ..., u_{k-1}\}^\perp$, we have

$$\begin{aligned} B_\varepsilon(u_k, v) &= B_\varepsilon(v, u_k) \\ &= \sum_{l=1}^{k-1} c_l \sigma_l \int_{\Omega_\varepsilon} u_l^\alpha u_k^\alpha \\ &= 0 \\ &= \sigma_k \int_{\Omega_\varepsilon} u_k^\alpha v^\alpha. \end{aligned}$$

We now have that $u_k$ is an eigenfunction of $L$ with eigenvalue $\sigma_k$. We also note that by construction, $\sigma_l \le \sigma_k$ if $l \le k$. We thus have a non-decreasing sequence of eigenvalues, listed according to multiplicity such that

$$\min_{\substack{0 \ne w \in H_0^1(\Omega_\varepsilon) \\ w \in \{u_1, u_2, ..., u_{k-1}\}^\perp}} R_\varepsilon(w) = R_\varepsilon(u_k) = \sigma_k \tag{1.25}$$

and

$$\|u_k\|_{L^2(\Omega_\varepsilon)} = 1 \tag{1.26}$$

for any $k$.

**Claim 1.4.4.** *The constructed sequence of eigenvalues $\{\sigma_k\}_{k=1}^\infty$ is increasing and satisfies $\sigma_k \to \infty$ as $k \to \infty$.*

*Proof.* We show $\sigma_k \to \infty$ by contradiction. Suppose $\sigma_k \le C$ uniformly in $k$. Then, by construction of the eigenvalues, $E_C$ is infinite-dimensional, but Claim 1.4.3 guarantees that $E_C$ is finite-dimensional. We thus have

$$\sigma_k \to \infty \quad \text{as} \quad k \to \infty. \tag{1.27}$$

□

## 1.5 Theorem for Convergence of Eigenvalues

We now state the main result for this chapter.

**Theorem 1.5.1.** *Let*

$$(Lu)^{\beta} = -\frac{\partial}{\partial x_j}\left(a_{ij}^{\alpha\beta}\frac{\partial u^{\alpha}}{\partial x_i}\right) \qquad \beta = 1, ..., m$$

*satisfy one of the following:*

1. *L has uniformly bounded coefficients and satisfies either the ellipticity condition (1.5) or the ellipticity condition (1.10).*

2. *L has continuous coefficients and satisfies the ellipticity condition (1.12).*

*Also assume* $\{\sigma_k^0\}_{k=1}^{\infty}$ *and* $\{\sigma_k^{\varepsilon}\}_{k=1}^{\infty}$ *are the Dirichlet eigenvalues of* $L$ *with respect to* $\Omega_0$ *and* $\Omega_{\varepsilon}$ *in increasing order numbered according to multiplicity. Then for each* $J \in \mathbb{N}$*, we have the following estimate:*

$$|\sigma_J^{\varepsilon} - \sigma_J^0| \leq C\varepsilon^a$$

*where* $a > 0$ *is independent of any eigenvalue and* $C$ *only depends on* $\sigma_J^0$ *and the distance from* $\sigma_J^0$ *to nearby eigenvalues.*

The proof relies on the reverse-Hölder inequality for the gradient of solutions of elliptic equations that is established by a technique introduced by Gehring [12]. This gives $L^p$-integrability of the gradient of eigenfunctions for $p > 2$, which implies that they are not concentrated in the tube.

### A Reverse-Hölder Inequality

If $\fint_E |f(y)|\, dy$ is defined to be the average of $f$ on $E$, then recall that the maximal function is defined for $f \in L^1_{loc}(\mathbb{R}^n)$ to be

$$M(f)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy$$

where $B_r(x)$ is a ball of radius $r$ centered at $x$. Also, define $M_R(f)(x)$ to be

$$M_R(f)(x) = \sup_{r<R} \fint_{B_r(x)} |f(y)| \, dy.$$

We will need the following theorem, which uses a technique introduced by Gehring [12] and was refined by Giaquinta and Modica [14].

**Theorem 1.5.2.** *Let $r > q > 1$, and $Q = Q_R$ be a cube in $\mathbb{R}^n$ with sidelength $R$ centered at 0. Also, define $d(x) = dist(x, \partial Q)$. If $f$ and $g$ are non-negative measurable functions such that $f \in L^r(Q)$, $g \in L^q(Q)$, $f = g = 0$ outside $Q$, and with the added condition that*

$$M_{\frac{d(x)}{2}}(g^q)(x) \leq bM^q(g)(x) + M(f^q) + aM(g^q)(x)$$

*for almost every $x$ in $Q$ where $b \geq 0$ and $0 \leq a < 1$, then $g \in L^p(Q_{\frac{R}{2}}(0))$, for $p \in [q, q+\epsilon)$ and*

$$\left( \fint_{Q_{R/2}} g^p(y) \, dy \right)^{\frac{1}{p}} \leq C \left[ \left( \fint_{Q_R} g^q(y) \, dy \right)^{\frac{1}{q}} + \left( \fint_{Q_R} f^p(y) \, dy \right)^{\frac{1}{p}} \right] \tag{1.28}$$

*where $\epsilon$ and $C$ depend on $b, q, n, a$ and $r$.*

The conclusion of this theorem is known as a reverse-Hölder inequality. To show that the gradient of eigenfunctions satisfy this inequality, we will need to prove a Caccioppoli inequality. However, to show this Caccioppoli inequality, we first need the following two well-known inequalities taken from Hebey [19, p. 44] and Oleinik [30, p. 27].

**Theorem 1.5.3.** ***Sobolev-Poincaré Inequality*** *Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Also, let $B_r$ be any ball of radius $r$ with $u \in W^{1,p}(B_r)$. Then, for $S$ contained in $B_r$ with $|S| \geq c_0 r^n$,*

$$\int_{B_r} |u(x) - u_S|^q \, dx \leq C \left( \int_{B_r} |\nabla u|^p(x) \, dx \right)^{\frac{q}{p}} \tag{1.29}$$

*where $u_S = \fint_S u \, dy$ and for some constant $C(n, p, c_0)$, independent of $u$.*

**Theorem 1.5.4.** ***Korn's Inequality on Balls*** *If $u \in H^1(B_r)$ then*

$$\|\nabla u\|^2_{L^2(B_r)} \leq C\left(\|\kappa(u)\|^2_{L^2(B_r)} + \frac{1}{r^2}\|u\|^2_{L^2(B_r)}\right) \tag{1.30}$$

*where $C$ only depends on $n$.*

We now state and prove a Caccioppoli inequality for eigenfunctions.

**Theorem 1.5.5.** *Let $u$ be an eigenfunction with eigenvalue $\sigma$ associated to the operator $L$ satisfying either (1.5) or (1.10) with uniformly bounded coefficients or associated to (1.12) with continuous coefficients. Extending $u$ to be 0 outside $\Omega_\varepsilon$, there exists $r_0 > 0$ so that if $r_0 \geq r > 0$, $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} \fint_{B_r} |\nabla u|^2 \, dy \leq C_1 \left(\fint_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} \, dy\right)^{\frac{n+2}{n}} \\ + C_2|\sigma| \fint_{B_{2r}} |u|^2 \, dy + C_3 \fint_{B_{2r}} |\nabla u|^2 \, dy \end{aligned} \tag{1.31}$$

*where $B_r$ is a ball with radius $r$ centered at $x$, $C_3 < 1$, and $C_l > 0$ only depends on $M = \max_{i,j,\alpha,\beta} \|a^{\alpha\beta}_{ij}\|_{L^\infty(\Omega_\varepsilon)}$, $n$, $m$, $\theta$, $\tau$, and $C_0$. Furthermore, if $L$ satisfies either (1.5) or (1.10) with uniformly bounded coefficients, then the inequality (1.31) holds for any $r > 0$.*

*Proof.* First, choose a ball $B_r$ and define a cutoff function $\nu \in C^\infty_c(\mathbb{R}^n)$ to be so that $\nu = 1$ in $B_r$, $\nu = 0$ outside $B_{2r}$, $|\nabla \nu| \leq \frac{C_n}{r}$, and $0 \leq \nu \leq 1$, where $C_n$ only depends on $n$. Below, we will find an appropriate constant vector $\rho \in \mathbb{R}^m$, so that $\nu^2(u - \rho) \in H^1_0(\Omega_\varepsilon)$. By the weak formulation (1.4), we have

$$\int_{\Omega_\varepsilon} a^{\alpha\beta}_{ij} u^\alpha_i [\nu^2(u-\rho)]^\beta_j \, dy = \sigma \int_{\Omega_\varepsilon} u^\gamma [\nu^2(u-\rho)]^\gamma \, dy.$$

By performing the differentiations, we then get

$$\int_{\Omega_\varepsilon} a^{\alpha\beta}_{ij} u^\alpha_i [2\nu\nu_j(u-\rho)^\beta + \nu^2 u^\beta_j] \, dy = \sigma \int_{\Omega_\varepsilon} u^\gamma \nu^2 (u-\rho)^\gamma \, dy. \tag{1.32}$$

From this point, the argument depends on which ellipticity condition $L$ satisfies. We have 3 cases.

*case 1: L satisfies the strong ellipticity condition (1.5).*

Using (1.5) and properties of $\nu$, we obtain the inequality

$$\int_{B_{2r}} \nu^2 a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \; dy \leq \int_{B_{2r}} 2M\frac{C_n}{r}\nu|\nabla u||u-\rho| \; dy + \int_{B_{2r}} |\sigma||u||u-\rho| \; dy$$

which, for any constant $\omega > 0$, then leads to

$$\begin{aligned} \int_{B_{2r}} \nu^2 a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \; dy \leq \int_{B_{2r}} \frac{\omega\nu^2|\nabla u|^2}{2} \; dy + \frac{C}{\omega r^2}\int_{B_{2r}} |u-\rho|^2 \; dy \\ + C|\sigma|\int_{B_{2r}} |u|^2 \; dy \end{aligned} \tag{1.33}$$

where $C$ depends on $M$ and $C_n$. Then choosing $\omega = \theta$ in (1.33) gives

$$\frac{\theta}{2}\int_{B_{2r}} \nu^2|\nabla u|^2 \; dy \leq \frac{C}{\theta r^2}\int_{B_{2r}} |u-\rho|^2 \; dy + C|\sigma|\int_{B_{2r}} |u|^2 \; dy.$$

Then, multiplying both sides by $\frac{2}{\theta}$ and using that $\nu = 1$ on $B_r$ gives

$$\int_{B_r} |\nabla u|^2 \; dy \leq \frac{2C}{\theta^2 r^2}\int_{B_{2r}} |u-\rho|^2 \; dy + \frac{2C|\sigma|}{\theta}\int_{B_{2r}} |u|^2 \; dy. \tag{1.34}$$

Now, for the term $\displaystyle\int_{B_{2r}} |u-\rho|^2 \; dy$, we must consider two subcases.

*subcase A*

If $B_{2r} \subset \Omega_\varepsilon$, then let $\rho^\alpha = \displaystyle⨍_{B_{2r}} u^\alpha \; dy$. Our condition on the support of $\nu$ implies $\nu^2(u-\rho) \in H_0^1(\Omega_\varepsilon)$. So, setting $q = 2$ and $S = B_{2r}$ in the Sobolev-Poincaré Inequality (1.29), we obtain

$$\int_{B_{2r}} |u-\rho|^2 \; dy \leq C\left(\int_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} \; dy\right)^{\frac{n+2}{n}}.$$

Using this estimate with (1.34) gives

$$\int_{B_r} |\nabla u|^2 \; dy \leq \frac{C}{r^2}\left(\int_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} \; dy\right)^{\frac{n+2}{n}} + C|\sigma|\int_{B_{2r}} |u|^2 \; dy.$$

Now, dividing through by $r^n$ gives the desired result with $C_3 = 0$.

*subcase B*

If $B_{2r} \cap \Omega_\varepsilon^c \neq \emptyset$, then set $\rho = 0$, which, again, guarantees that $\nu^2(u-\rho) \in H_0^1(\Omega_\varepsilon)$. So setting $q = 2$ and $S = B_{4r} \cap \Omega_\varepsilon$ in the Sobolev-Poincaré Inequality (1.29), we have by our assumption on $\Omega_\varepsilon^c$ (1.2) that

$$\int_{B_{4r}} |u-\rho|^2 \, dy \leq C \left( \int_{B_{4r}} |\nabla u|^{\frac{2n}{n+2}} \, dy \right)^{\frac{n+2}{n}}.$$

From (1.34), we obtain

$$\int_{B_r} |\nabla u|^2 \, dy \leq \frac{C}{r^2} \left( \int_{B_{4r}} |\nabla u|^{\frac{2n}{n+2}} \, dy \right)^{\frac{n+2}{n}} + C|\sigma| \int_{B_{4r}} |u|^2 \, dy.$$

A simple covering argument gives the estimate with $B_{4r}$ replaced with $B_{2r}$.

*case 2: L satisfies the ellipticity condition (1.10).*

From (1.10) and (1.33), we have

$$\int_{B_r} \tau |\kappa(u)|^2 \, dy \leq \int_{B_{2r}} \frac{\omega \nu^2 |\nabla u|^2}{2} \, dy + \frac{C}{\omega r^2} \int_{B_{2r}} |u-\rho|^2 \, dy + C|\sigma| \int_{B_{2r}} |u|^2 \, dy.$$

Also, by Korn's inequality (1.30), we have

$$\frac{\tau}{C} \int_{B_r} |\nabla u|^2 \, dy - \frac{\tau}{r^2} \int_{B_r} |u-\rho|^2 \, dy \leq \int_{B_r} \tau |\kappa(u)|^2 \, dy.$$

This implies

$$\int_{B_r} |\nabla u|^2 \, dy \leq \frac{C\omega}{2\tau} \int_{B_{2r}} |\nabla u|^2 \, dy + C \left( \frac{1}{\omega \tau r^2} + \frac{1}{r^2} \right) \int_{B_{2r}} |u-\rho|^2 \, dy + \frac{C|\sigma|}{\tau} \int_{B_{2r}} |u|^2 \, dy.$$

This again leads to two subcases as in case 1. We must choose $\rho$ appropriately and use the Sobolev-Poincaré inequality (1.29) as in case 1. Then, by taking $\omega$ sufficiently small, we obtain the desired result.

*case 3: L satisfies the Legendre-Hadamard condition* (1.12) *with continuous coefficients in* $\overline{\Omega}_\varepsilon$.

We note that it suffices to study $u \in C_c^\infty(\Omega_\varepsilon)$ and first consider when the coefficients are constant. We rewrite the left side of (1.32) as

$$\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}((u-\rho)^\alpha \nu)_i((u-\rho)^\beta \nu)_j \, dy$$
$$+ \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}[\nu\nu_j u_i^\alpha (u-\rho)^\beta - \nu_i \nu (u-\rho)^\alpha u_j^\beta - \nu_i\nu_j (u-\rho)^\alpha (u-\rho)^\beta] \, dy.$$

This, then implies that

$$\begin{aligned} \int_{B_{2r}} & a_{ij}^{\alpha\beta}((u-\rho)^\alpha \nu)_i((u-\rho)^\beta \nu)_j \, dy \\ & \leq C \int_{B_{2r}} |\nabla \nu||\nabla((u-\rho)\nu)||u-\rho| + |u-\rho|^2|\nabla\nu|^2 + |\sigma||u||u-\rho| \, dy. \end{aligned} \tag{1.35}$$

We note that we may use the Fourier transform to get a lower bound of

$$\int_{B_{2r}} \theta |\nabla((u-\rho)\nu)|^2 \, dy$$

on the left side of (1.35) as in the derivation of (1.14). This leads to the estimate

$$\int_{B_r} |\nabla u|^2 \, dy \leq \int_{B_{2r}} |\nabla((u-\rho)\nu)|^2 \, dy \leq \frac{C}{r^2} \int_{B_{2r}} |u-\rho|^2 \, dy + C|\sigma| \int_{B_{2r}} |u|^2 \, dy. \tag{1.36}$$

So, again, if we employ the Sobolev-Poincaré inequality (1.29), we get the desired result in the case of constant coefficients.

If the coefficients are continuous and non-constant, then we freeze the coefficients at $x$. That is, from the weak formulation (1.4), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(x) u_i^\alpha ((u-\rho)\nu^2)_j^\beta \, dy + \int_{\Omega_\varepsilon} (a_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x)) u_i^\alpha ((u-\rho)\nu^2)_j^\beta \, dy \\ = \sigma \int_{\Omega_\varepsilon} u^\gamma ((u-\rho)\nu^2)^\gamma \, dy. \end{aligned} \tag{1.37}$$

So, recalling the definition of the modulus of continuity from (1.15), we have that

$$\begin{aligned} \int_{B_{2r}} & (a_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x)) u_i^\alpha ((u-\rho)\nu^2)_j^\beta \, dy \\ & \leq M(x,2r) \int_{B_{2r}} \nu^2 |\nabla u|^2 \, dy + 2M(x,2r) \int_{B_{2r}} \nu |\nabla \nu||\nabla u||u-\rho| \, dy \\ & \leq C(M(x,2r) + M(x,2r)^2) \int_{B_{2r}} |\nabla u|^2 \, dy + \frac{C}{r^2} \int_{B_{2r}} |u-\rho|^2 \, dy. \end{aligned}$$

Also, by the uniform continuity of the coefficients on $\overline{\Omega}_\varepsilon$, for any $c < 1$, there exists $r_0$ depending on $c$, so that if $C(x_0, R) = C(M(x_0, 2R) + M(x_0, 2R)^2)$ and $r \leq r_0$, then

$$C(x_0, r) \leq c$$

for all $x_0 \in \overline{\Omega}_\varepsilon$. So, now moving the second term on the left side of (1.37) to the right and using the constant coefficient case (1.36), we obtain that for any $c < 1$, there exists $r_0$ so that if $r \leq r_0$,

$$\int_{B_r} |\nabla u|^2 \, dy \leq \frac{C}{r^2} \int_{B_{2r}} |u - \rho|^2 \, dy + C|\sigma| \int_{B_{2r}} |u|^2 \, dy + c \int_{B_{2r}} |\nabla u|^2 \, dy.$$

We again note that here, we must choose $\rho$ appropriately and apply the Sobolev-Poincaré inequality (1.29) to get the desired result. $\square$

As stated earlier, our proof of Theorem 1.5.1 relies on the gradient of an eigenfunction satisfying the reverse-Hölder inequality, as in our next theorem.

**Theorem 1.5.6.** *There exists $\epsilon_1 > 0$ so that if $u$ is an eigenfunction with eigenvalue $\sigma$, then*

$$\fint_{\Omega_\varepsilon} |\nabla u|^{\widetilde{p}} \, dy \leq C \left[ \left( \fint_{\Omega_\varepsilon} |\nabla u|^2 \, dy \right)^{\frac{\widetilde{p}}{2}} + |\sigma|^{\frac{\widetilde{p}}{2}} \fint_{\Omega_\varepsilon} |u|^{\widetilde{p}} \, dy \right] \tag{1.38}$$

*where $2 \leq \widetilde{p} < 2 + \epsilon_1$, and $\epsilon_1$ and $C$ are independent of $\varepsilon$ and any eigenvalue.*

*Proof.* Now if $u$ is an eigenfunction with eigenvalue $\sigma$, we have $u \in H_0^1(\Omega_\varepsilon)$, and thus we may employ the Sobolev inequality to get that $|u| \in L^r(\Omega_\varepsilon)$ for some $r > 2$. If $L$ satisfies either (1.5) or (1.10) with uniformly bounded coefficients, then we may choose a cube $Q_R$, centered at 0, with radius $R$ such that $\Omega_\varepsilon \subset Q_{\frac{R}{2}}$, uniformly in $\varepsilon$, and set $g = |\nabla u|^{\frac{2n}{n+2}}$, $f = (C_3|\sigma|)^{\frac{n}{n+2}} |u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$, and $u = 0$ outside $\Omega_\varepsilon$, we may conclude by (1.31) and (1.28) that

$$\left( \fint_{\Omega_\varepsilon} |\nabla u|^{\frac{2np}{n+2}} \, dy \right)^{\frac{1}{p}} \leq C \left[ \left( \fint_{\Omega_\varepsilon} |\nabla u|^2 \, dy \right)^{\frac{n}{n+2}} + \sigma^{\frac{n}{n+2}} \left( \fint_{\Omega_\varepsilon} |u|^{\frac{2np}{n+2}} \, dy \right)^{\frac{1}{p}} \right]$$

where $\frac{n+2}{n} \leq p \leq \frac{n+2}{n} + \epsilon$, which is independent of $\varepsilon$ and any eigenvalue. So, setting $\widetilde{p} = \frac{2np}{n+2}$, we have the result. If $L$ satisfies (1.12) with continuous coefficients, then since we only have Theorem 1.5.5 true for small $r$, we must cover $\Omega_\varepsilon$ with a fixed number of cubes and apply (1.28) to each cube to obtain the result. $\square$

### Eigenvalue Estimates

From this point, let $\sigma_k^\varepsilon$ be the *kth* eigenvalue with respect to $\Omega_\varepsilon$, and $\phi_k^\varepsilon$ be its corresponding eigenfunction with $\phi_k^\varepsilon = 0$ outside $\Omega_\varepsilon$ for $\varepsilon \geq 0$. We also *fix* an eigenvalue $\sigma_J^0$ with multiplicity $m_J$ where $\sigma_{J-1}^0 < \sigma_J^0$ if $J \geq 2$. We will consider the family $\{\sigma_J^\varepsilon\}$ as $\varepsilon > 0$ tends to 0. We begin with the following proposition taken from Anné [2, p. 2595-2596]:

**Lemma 1.5.7.** *Let $(q, \boldsymbol{D})$ be a closed non-negative quadratic form in the Hilbert space $(\boldsymbol{H}, \langle , \rangle)$. Define the associated norm $\|f\|_1^2 = \|f\|_{\boldsymbol{H}}^2 + q(f)$, and the spectral projector $\Pi_I$ for any interval $I = (\alpha, \beta)$ for which the boundary does not meet the spectrum.*

1. *Suppose $f \in \boldsymbol{D}$ and $\lambda \in I$ satisfy*

$$|q(f, g) - \lambda\langle f, g\rangle| \leq \delta\|f\|\|g\|_1 \qquad g \in \boldsymbol{D}.$$

   *Then there exists a constant $C > 0$, which depends on $I$, such that if $a$ is less than the distance of $\alpha$ or $\beta$ to the spectrum of $q$,*

$$\|\Pi_I(f) - f\|_1 = \|\Pi_{I^c}(f)\|_1 \leq \frac{C\delta}{a}\|f\|.$$

2. *Suppose the spectral space $E(I)$ has dimension $m$ and $f_1, ..., f_m$ is an orthonormal family which satisfies*

$$\|\Pi_{I^c}(f_j)\|_1 \leq \delta \qquad j = 1, ..., m.$$

   *Also let $E$ be the space spanned by the $f_j$'s. Then,*

$$dist(E(I), E) \leq C\delta$$

   *where the distance is measured as the distance between the two orthogonal projectors.*

This lemma will give us the results we need for the convergence of eigenvalues. We will prove estimates on eigenfunctions using the reverse-Hölder inequality (1.38), which will allow us to use this lemma. We begin with the following well-known mini-max theorem for systems taken from Grubb and Sharma [16].

**Theorem 1.5.8.** *Let $S^k$ denote any subspace of $L^2(\Omega_\varepsilon)$, with dimension $k$. Then*

$$\sigma_k^\varepsilon = \min_{S^k} \max_{0 \neq u \in S^k} R_\varepsilon(u). \tag{1.39}$$

This leads to the following proposition.

**Proposition 1.5.9.** *We have for any $\varepsilon > 0$, and any $k \in \mathbb{N}$,*

$$\sigma_k^\varepsilon \leq \sigma_k^0. \tag{1.40}$$

*Proof.* Now, by (1.39),

$$\min_{S^k} \max_{0 \neq u \in S^k} R_\varepsilon(u) = \sigma_k^\varepsilon.$$

Set $T^k = \text{span}\{\phi_1^0, ..., \phi_k^0\}$. Then for $w \in T^k$, say $w = \sum_{l=1}^{k} c_l \phi_l^0$, and by the definition of $R_\varepsilon(w)$, we have

$$\begin{aligned} R_\varepsilon(w) &= \frac{\sum_{l,s=1}^k B_\varepsilon(c_l \phi_l^0, c_s \phi_s^0)}{\sum_{l,s=1}^k \langle c_l \phi_l^0, c_s \phi_s^0 \rangle_{L^2(\Omega_\varepsilon)}} \\ &= \frac{\sum_{l,s=1}^k c_l c_s B_\varepsilon(\phi_l^0, \phi_s^0)}{\sum_{l,s=1}^k c_l c_s \langle \phi_l^0, \phi_s^0 \rangle_{L^2(\Omega_\varepsilon)}} \\ &= \frac{\sum_{l,s=1}^k c_l c_s \sigma_l^0 \langle \phi_l^0, \phi_s^0 \rangle_{L^2(\Omega_\varepsilon)}}{\sum_{l,s=1}^k c_l c_s \langle \phi_l^0, \phi_s^0 \rangle_{L^2(\Omega_\varepsilon)}} \end{aligned}$$

where we have used the weak formulation of an eigenfunction (1.4) on the last line. So, by the orthogonality in $L^2$ of the eigenfunctions and since eigenvalues form an increasing sequence,

$$\begin{aligned} R_\varepsilon(w) &= \frac{\sum_{l=1}^k \sigma_l^0 c_l^2 \langle \phi_l^0, \phi_l^0 \rangle_{L^2(\Omega_\varepsilon)}}{\sum_{l=1}^k c_l^2 \langle \phi_l^0, \phi_l^0 \rangle_{L^2(\Omega_\varepsilon)}} \\ &\leq \sigma_k^0 \frac{\sum_{l=1}^k c_l^2 \langle \phi_l^0, \phi_l^0 \rangle_{L^2(\Omega_\varepsilon)}}{\sum_{l=1}^k c_l^2 \langle \phi_l^0, \phi_l^0 \rangle_{L^2(\Omega_\varepsilon)}} \\ &= \sigma_k^0 \end{aligned}$$

so that by the construction of eigenvalues,

$$\sigma_k^0 = R_\varepsilon(\phi_k^0) = \max_{u \in \mathrm{span}\{\phi_1^0, \dots, \phi_k^0\}} R_\varepsilon(u).$$

Thus, since span$\{\phi_1^0, ..., \phi_k^0\}$ is one of the $S^k$'s, we have the result. □

This proposition gives us the easy half of the inequality in our theorem. To prove the second half of the inequality, we will need a few items.

**Proposition 1.5.10.** *For any $\varepsilon > 0$, and $k \geq 1$, if $\phi = \phi_k^\varepsilon$, then we have*

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{\widetilde{p}} \, dy \leq C \tag{1.41}$$

*where $\widetilde{p} > 2$ is from (1.38) and $C$ depends on the domain $\Omega_0$ and $n$, and has order $(\sigma_k^0)^{\frac{2\widetilde{p}+n(\widetilde{p}-2)}{4}}$ for $n \geq 3$ or $(\sigma_k^0)^{\frac{q\widetilde{p}+2(\widetilde{p}-q)}{2q}}$ for $n = 2$, where $2 - \kappa < q < 2$ for small $\kappa$. Furthermore, $\widetilde{p}$ and $C$ are both independent of $\varepsilon$, and if $n = 2$, $C$ blows up as $q \to 2$.*

*Proof.* Now, from (1.38), we have

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{\widetilde{p}} \, dy \leq C \left[ |\Omega_\varepsilon|^{\frac{2-\widetilde{p}}{2}} \left( \int_{\Omega_\varepsilon} |\nabla \phi|^2 \, dy \right)^{\frac{\widetilde{p}}{2}} + (\sigma_k^\varepsilon)^{\frac{\widetilde{p}}{2}} \left( \int_{\Omega_\varepsilon} |\phi|^{\widetilde{p}} \, dy \right) \right] \tag{1.42}$$

where $\widetilde{p} > 2$ is from (1.38). Observe that by Gårding's inequality (1.13), we have

$$\begin{aligned} C \int_{\Omega_\varepsilon} |\nabla \phi|^2 \, dy &\leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} \phi_i^\alpha \phi_j^\beta \, dy + C \int_{\Omega_\varepsilon} |\phi|^2 \, dy \\ &\leq C(1 + |\sigma_k^\varepsilon|) \int_{\Omega_\varepsilon} |\phi|^2 \, dy \\ &\leq C(1 + |\sigma_k^\varepsilon|) \end{aligned} \tag{1.43}$$

the last line owing to the normalization of the eigenfunctions. Next, we will consider $n \geq 3$ and estimate

$$\int_{\Omega_\varepsilon} |\phi|^{\widetilde{p}} \, dy.$$

Using Sobolev's inequality and (1.43), we have

$$\left(\int_{\Omega_\varepsilon} |\phi|^{\frac{2n}{n-2}}\ dy\right)^{\frac{n-2}{2n}} \le C\left(\int_{\Omega_\varepsilon} |\nabla\phi|^2\ dy\right)^{\frac{1}{2}}$$
$$\le C(1+|\sigma_k^\varepsilon|^{\frac{1}{2}}). \tag{1.44}$$

Also, by Hölder's inequality, we have

$$\left(\int_{\Omega_\varepsilon} |\phi|^{\widetilde{p}}\ dy\right)^{\frac{1}{\widetilde{p}}} \le \left(\int_{\Omega_\varepsilon} |\phi|^2\ dy\right)^{\frac{1-t}{2}} \left(\int_{\Omega_\varepsilon} |\phi|^{\frac{2n}{n-2}}\ dy\right)^{\frac{t(n-2)}{2n}}$$

where $t$ satisfies

$$\frac{1}{\widetilde{p}} = \frac{1-t}{2} + \frac{t(n-2)}{2n}.$$

From this inequality and (1.44), it follows that

$$\left(\int_{\Omega_\varepsilon} |\phi|^{\widetilde{p}}\ dy\right)^{\frac{1}{\widetilde{p}}} \le C\left(1+|\sigma_k^\varepsilon|^{\frac{t}{2}}\right)$$
$$= C\left(1+|\sigma_k^\varepsilon|^{\frac{n(\widetilde{p}-2)}{4\widetilde{p}}}\right).$$

Now, using this inequality along with (1.42), (1.43), and (1.40), we obtain

$$\int_{\Omega_\varepsilon} |\nabla\phi|^{\widetilde{p}}\ dy \le C\left[\left(1+\sigma_k^0\right)^{\frac{\widetilde{p}}{2}} + (\sigma_k^0)^{\frac{\widetilde{p}}{2}}\left(1+|\sigma_k^0|^{\frac{n(\widetilde{p}-2)}{4}}\right)\right]$$
$$\le C\left[\left(\sigma_k^0\right)^{\frac{2\widetilde{p}+n(\widetilde{p}-2)}{4}} + \left(\sigma_k^0\right)^{\frac{\widetilde{p}}{2}} + 1\right].$$

This completes the proof for $n \ge 3$.

If $n = 2$, then from Gilbarg and Trudinger [15, p. 158], we use Sobolev's inequality, along with Hölder's inequality and (1.43) to obtain

$$\left(\int_{\Omega_\varepsilon} |\phi|^{q^*}\ dy\right)^{\frac{1}{q^*}} \le \frac{C}{(2-q)^{\frac{1}{2}}}\left(\int_{\Omega_\varepsilon} |\nabla\phi|^q\ dy\right)^{\frac{1}{q}}$$
$$\le \frac{C}{(2-q)^{\frac{1}{2}}}\left(\int_{\Omega_\varepsilon} |\nabla\phi|^2\ dy\right)^{\frac{1}{2}} |\Omega_\varepsilon|^{\frac{1}{q^*}}$$
$$\le \frac{C}{(2-q)^{\frac{1}{2}}}\left(1+|\sigma_k^\varepsilon|^{\frac{1}{2}}\right)$$

where $q^* = \frac{2q}{2-q}$ is the Sobolev conjugate of $q$. Then, applying Hölder's inequality, we obtain

$$\begin{aligned}\left(\int_{\Omega_\varepsilon} |\phi|^{\widetilde{p}}\, dy\right)^{\frac{1}{\widetilde{p}}} &\leq \frac{C}{(2-q)^{\frac{t}{2}}}\left(1 + |\sigma_k^\varepsilon|^{\frac{t}{2}}\right) \\ &= \frac{C}{(2-q)^{\frac{(\widetilde{p}-q)}{\widetilde{p}q}}}\left(1 + |\sigma_k^\varepsilon|^{\frac{(\widetilde{p}-q)}{\widetilde{p}q}}\right)\end{aligned}$$

and using (1.42), (1.43), and (1.40), we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla\phi|^{\widetilde{p}}\, dy &\leq \frac{C}{(2-q)^{\frac{(\widetilde{p}-q)}{q}}}\left[\left(1+\sigma_k^0\right)^{\frac{\widetilde{p}}{2}} + (\sigma_k^0)^{\frac{\widetilde{p}}{2}}\left(1 + |\sigma_k^0|^{\frac{(\widetilde{p}-q)}{q}}\right)\right] \\ &\leq \frac{C}{(2-q)^{\frac{(\widetilde{p}-q)}{q}}}\left[\left(\sigma_k^0\right)^{\frac{q\widetilde{p}+2(\widetilde{p}-q)}{2q}} + \left(\sigma_k^0\right)^{\frac{\widetilde{p}}{2}} + 1\right].\end{aligned}$$

□

**Lemma 1.5.11.** *For the eigenfunction $\phi_k^\varepsilon$, $J \leq k \leq J+m_J-1$, and any $w \in H_0^1(\Omega_0)$, we have the following estimate:*

$$\left|\int_{\Omega_0} a_{ij}^{\alpha\beta}(\eta_\varepsilon\phi_k^\varepsilon)_i^\alpha w_j^\beta\, dy - \sigma_k^\varepsilon\int_{\Omega_0}(\eta_\varepsilon\phi_k^\varepsilon)^\alpha w^\alpha\, dy\right| \leq C\varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}}\|w\|_1 \tag{1.45}$$

*where $\|w\|_1$ is from Lemma 1.5.7 and $C$ only depends on the domain $\Omega_0$, $n$, $\sigma_J^0$, and is independent of $\varepsilon$.*

*Proof.* First, we extend $w$ to be 0 outside $\Omega_0$ and $\phi_k^\varepsilon$ to be 0 in $(B_\varepsilon \cup \widetilde{B}_\varepsilon) \cap \Omega_\varepsilon^c$. Then we have

$$\begin{aligned}&\left|\int_{\Omega_0} a_{ij}^{\alpha\beta}(\eta_\varepsilon\phi_k^\varepsilon)_i^\alpha w_j^\beta\, dy - \sigma_k^\varepsilon\int_{\Omega_0}(\eta_\varepsilon\phi_k^\varepsilon)^\alpha w^\alpha\, dy\right| \\ &\qquad\leq \left|\int_{\Omega_0} a_{ij}^{\alpha\beta}[(\eta_\varepsilon)_i(\phi_k^\varepsilon)^\alpha w_j^\beta - (\eta_\varepsilon)_j(\phi_k^\varepsilon)^\alpha w^\beta]\, dy\right| \\ &\qquad\quad + \left|\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(\phi_k^\varepsilon)_i^\alpha(\eta_\varepsilon w)_j^\beta\, dy - \sigma_k^\varepsilon\int_{\Omega_\varepsilon}(\phi_k^\varepsilon)^\alpha(\eta_\varepsilon w)^\alpha\, dy\right| \\ &\qquad= |I + II| + |III + IV|.\end{aligned}$$

First, since $\phi_k^\varepsilon$ is an eigenfunction with eigenvalue $\sigma_k^\varepsilon$, we have that $III + IV = 0$. Also, by Hölder's inequality and Poincaré's inequality, we have

$$\begin{aligned} |I + II| &\leq \frac{C}{\varepsilon} \|\phi_k^\varepsilon\|_{L^2(B_\varepsilon \cup \widetilde{B}_\varepsilon)} \|\nabla w\|_{L^2(B_\varepsilon \cup \widetilde{B}_\varepsilon)} \\ &\leq C \|\nabla \phi_k^\varepsilon\|_{L^2(B_\varepsilon \cup \widetilde{B}_\varepsilon)} \|w\|_1 \end{aligned}$$

where we have used Gårding's inequality (1.13) on the last line for $w$. Thus, from Hölder's inequality and Proposition 1.5.10,

$$\begin{aligned} |I + II| &\leq C \varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}} \|\nabla \phi_k^\varepsilon\|_{L^{\widetilde{p}}(\Omega_\varepsilon)} \|w\|_1 \\ &\leq C \varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}} \|w\|_1. \end{aligned}$$

This concludes the proof of the lemma. □

If we choose an interval $I$ around $\sigma_k^0$ such that $\sigma_k^\varepsilon \in I$, then it is easy to see that for $q(f, g) = \displaystyle\int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta \, dy$ and $f = \eta_\varepsilon \phi_k^\varepsilon$, we have satisfied the hypotheses for part 1 of Lemma 1.5.7. To satisfy part 2, we start with the following well-known proposition.

**Proposition 1.5.12.** *If $A$ is an $N \times N$ matrix and $v$ is a $N \times 1$ vector such that $Av = 0$ and $\displaystyle\sum_{i \neq l}^{N} |A_{li}| < |A_{ll}| \quad$ for each $l = 1, ..., N$, then $v = 0$.*

The next proposition shows that the functions $\{\eta_\varepsilon \phi_k^\varepsilon\}_{k=J}^{J+m_J-1}$ are almost orthonormal.

**Proposition 1.5.13.** *For any $\varepsilon > 0$ and $l, k \in \mathbb{N}$, $(J \leq l, k \leq J+m_J-1)$, if $\phi_k = \phi_k^\varepsilon$, we have the following estimates:*

$$\int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\phi_k|^2 \, dy \geq 1 - C\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}} \tag{1.46}$$

$$\left| \int_{\Omega_\varepsilon} \eta_\varepsilon^2 \phi_k \cdot \phi_l \, dy \right| \leq C\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}} \quad \text{if } k \neq l \tag{1.47}$$

*where $C$ only depends on $|\Omega_0|$, $n$, and $\sigma_J^0$, and is independent of $\varepsilon$.*

*Proof.* We start by showing (1.46). Since the eigenfunctions are normalized, we obtain for each $k$,

$$
\begin{aligned}
1-\int_{\Omega_\varepsilon}\eta_\varepsilon^2|\phi_k|^2\ dy &= \int_{\Omega_\varepsilon}(1-\eta_\varepsilon^2)|\phi_k|^2\ dy \\
&= \int_{T_\varepsilon\cup B_\varepsilon\cup\widetilde{B}_\varepsilon}(1-\eta_\varepsilon^2)|\phi_k|^2\ dy \\
&\le \|\nabla\phi_k\|^2_{L^{\widetilde{p}}(\Omega_\varepsilon)}|T_\varepsilon\cup B_\varepsilon\cup\widetilde{B}_\varepsilon|^{\frac{\widetilde{p}-2}{\widetilde{p}}} \\
&\le C_k\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}}
\end{aligned}
$$

where, from (1.41), $C_k$ depends on $\sigma_k^0$. So, since $C_k = C_J$, we have (1.46).

Next, to show (1.47), we have

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon}\eta_\varepsilon^2\phi_k\cdot\phi_l\ dy\right| &\le \left|\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}\eta_\varepsilon^2\phi_k\cdot\phi_l\ dy\right| + \left|\int_{\Omega_0\setminus(B_\varepsilon\cup\widetilde{B}_\varepsilon)}\eta_\varepsilon^2\phi_k\cdot\phi_l\ dy\right| \\
&= \left|\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}\eta_\varepsilon^2\phi_k\cdot\phi_l\ dy\right| + \left|\int_{\Omega_0\setminus(B_\varepsilon\cup\widetilde{B}_\varepsilon)}\phi_k\cdot\phi_l\ dy - \int_{\Omega_\varepsilon}\phi_k\cdot\phi_l\ dy\right| \\
&\le \int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_k\cdot\phi_l|\ dy + \int_{T_\varepsilon\cup B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_k\cdot\phi_l|\ dy
\end{aligned}
$$

the second inequality following since the set of eigenfunctions form an orthogonal set in $L^2(\Omega_\varepsilon)$. So, next by Hölder's inequality, we get

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon}\eta_\varepsilon^2\phi_k\cdot\phi_l\ dy\right| &\le \left(\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_k|^2\ dy\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_l|^2\ dy\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{T_\varepsilon\cup B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_k|^2\ dy\right)^{\frac{1}{2}}\left(\int_{T_\varepsilon\cup B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_l|^2\ dy\right)^{\frac{1}{2}} \\
&= I + II.
\end{aligned}
$$

Now, from Poincaré's inequality and (1.41), we get

$$
\begin{aligned}
I &\le \left[\left(\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_k|^{\widetilde{p}}\ dy\right)^{\frac{2}{\widetilde{p}}}|B_\varepsilon\cup\widetilde{B}_\varepsilon|^{\frac{\widetilde{p}-2}{\widetilde{p}}}\right]^{\frac{1}{2}}\left[\left(\int_{B_\varepsilon\cup\widetilde{B}_\varepsilon}|\phi_l|^{\widetilde{p}}\ dy\right)^{\frac{2}{\widetilde{p}}}|B_\varepsilon\cup\widetilde{B}_\varepsilon|^{\frac{\widetilde{p}-2}{\widetilde{p}}}\right]^{\frac{1}{2}} \\
&\le \|\nabla\phi_k\|_{L^{\widetilde{p}}(\Omega_\varepsilon)}\varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}}\|\nabla\phi_l\|_{L^{\widetilde{p}}(\Omega_\varepsilon)}\varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}} \\
&\le C_k\varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}}C_l\varepsilon^{\frac{n(\widetilde{p}-2)}{2\widetilde{p}}}
\end{aligned}
$$

where $C_k$ again depends on $\sigma_k^\varepsilon$ and $C_l$ depends on $\sigma_l^\varepsilon$. Thus, we have

$$I \le C\varepsilon^{\frac{n(\widetilde{p}-2)}{\widetilde{p}}} \tag{1.48}$$

where $C$ depends only on $|\Omega_0|$, $n$, and $\sigma_J^0$. Similarly,

$$II \le C\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}} \tag{1.49}$$

so that the proposition is proved. □

To satisfy the hypotheses for part 2 of Lemma 1.5.7, we need an orthonormal basis. The next proposition shows that for small $\varepsilon$, we have a basis.

**Proposition 1.5.14.** *For $\varepsilon > 0$ small enough, $\{\eta_\varepsilon \phi_k^\varepsilon\}_{k=J}^N$ forms a linearly independent set for any $N \ge J$.*

*Proof.* Assume $C_J \eta_\varepsilon \phi_J^\varepsilon + ... + C_N \eta_\varepsilon \phi_N^\varepsilon = 0$. Then, multiplying this equation by $\eta_\varepsilon \phi_l^\varepsilon$, we achieve

$$\sum_{k=J}^{N} C_k \langle \eta_\varepsilon \phi_k^\varepsilon, \eta_\varepsilon \phi_l^\varepsilon \rangle_{L^2(\Omega_\varepsilon)} = 0, \quad l = J, ..., N.$$

So, if $A_{lk} = \langle \eta_\varepsilon \phi_k^\varepsilon, \eta_\varepsilon \phi_l^\varepsilon \rangle_{L^2(\Omega_\varepsilon)}$, we obtain by (1.46) and (1.47) that

$$\begin{aligned} |A_{kk}| &\ge 1 - C\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}} \\ &> C\varepsilon^{\frac{d(\widetilde{p}-2)}{\widetilde{p}}} \\ &\ge \sum_{\substack{k=J \\ i\ne k}}^{N} |A_{ki}| \end{aligned}$$

if $\varepsilon$ is small enough. Thus, if $C = (C_J, ..., C_N)^t$, since $AC = 0$, we have by Proposition 1.5.12 that $C = 0$ so that the proposition is proved. □

Now we define $I = \left(\sigma_J^0 - M\varepsilon^{\frac{n(\widetilde{p}-2)}{4\widetilde{p}}}, \frac{\sigma_J^0 + \sigma_{J+m_J}^0}{2}\right)$ for $M > 0$ to be chosen later. Also, let $\Pi$ be the projector onto the space spanned by the eigenfunctions corresponding to the eigenvalues, $\{\sigma_k^\varepsilon\}$, in $I$. We note that for fixed $\varepsilon$, we may choose $M$ so that $\sigma_k^\varepsilon$ is in $I$ for $J \le k \le N$ where $N \ge J + m_J - 1$. This is due to Proposition 1.5.9. We next define $J_0 : L^2(\Omega_\varepsilon) \to L^2(\Omega_0)$ to be given by $J_0 f = \eta_\varepsilon f$, and similarly, we define $J_\varepsilon : L^2(\Omega_0) \to L^2(\Omega_\varepsilon)$ to be such that

$$J_\varepsilon f(x) = \begin{cases} f(x), & \text{if} \quad x \in \Omega_0 \\ 0, & \text{if} \quad x \in \Omega_\varepsilon \backslash \Omega_0. \end{cases}$$

By Proposition 1.5.14, $\{\eta_\varepsilon \phi_k^\varepsilon\}_{k=J}^N$ is a basis for the range of $J_0 \Pi J_\varepsilon$. Thus, we may apply the Gram-Schmidt process to this basis. That is, define

$$\begin{aligned} &f_J = \eta_\varepsilon \phi_J^\varepsilon \\ &\vdots \\ &f_k = \eta_\varepsilon \phi_k^\varepsilon - \frac{\langle \eta_\varepsilon \phi_k^\varepsilon, f_J \rangle}{\|f_J\|^2} f_J - \ldots - \frac{\langle \eta_\varepsilon \phi_k^\varepsilon, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1} \\ &\vdots \end{aligned}$$

**Lemma 1.5.15.** *Let $I$ be as defined above. For each $k$, $J \leq k \leq J + m_J - 1$, we have $\|\Pi_{I^c}(f_k)\|_1 \leq \dfrac{C\varepsilon^{\frac{n(\tilde{p}-2)}{4\tilde{p}}}}{M}$, for $\varepsilon \leq 1$, and where $M$ only depends on $\sigma_J^0$ and $\sigma_{J-1}^0$.*

*Proof.* First let $\varepsilon = 1$. We note that from Proposition 1.5.9, for each $k$, $J \leq k \leq J + m_J - 1$, we may choose $M$ so that $\sigma_k^\varepsilon$ lies in $I$. So, defining $q(f, g) = \int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta \, dy$, we may apply Lemma 1.5.11 and then Lemma 1.5.7 (part 1) to obtain

$$\|\Pi_{I^c}(f_J)\|_1 \leq \frac{C\varepsilon^{\frac{n(\tilde{p}-2)}{4\tilde{p}}}}{M}$$

where $C$ depends on $\Omega_0$, $n$, $\sigma_J^0$, and $\sigma_{J+m_J}^0$. Then, from Proposition 1.5.13, Lemma 1.5.11, and properties of the norm, we get the result. We next note that if $\varepsilon \leq 1$, since $\sigma_k^1 \leq \sigma_k^\varepsilon$, $M$ will grow as $\varepsilon$ shrinks. This means that we obtain the same estimate. $\square$

**Corollary 1.5.16.** $\|\Pi_I - J_0 \Pi J_\varepsilon\|_{L^2(\Omega_0) \to L^2(\Omega_0)} \leq \dfrac{C\varepsilon^{\frac{n(\tilde{p}-2)}{4\tilde{p}}}}{M}$, *where $M$ only depends on $\sigma_J^0$ and $\sigma_{J-1}^0$.*

*Proof.* Normalize the $f_k$'s and observe that $\dfrac{1}{\|f_k\|} \leq \dfrac{1}{1 - C\varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}}}$. Then apply Lemma 1.5.7 (part 2) to the normalized functions. $\square$

We are now ready to prove Theorem 1.5.1.

*Proof.* When choosing $M$, we must be careful that no smaller eigenvalues for $\Omega_0$ are in $I$. So, we first prove for $J = 1$. Since every eigenvalue is bounded below, we can choose such an $M$. We have $\operatorname{rank}(J_0 \Pi J_\varepsilon) = \operatorname{rank}(\Pi) = N$ for $\varepsilon \leq \widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ is chosen small from Proposition 1.5.14. Then we use Corollary 1.5.16 to apply Lemma I-4.10 from Kato [23, p.34] to get that for $\varepsilon < \min\{1, \widetilde{\varepsilon}\}$, $m_1 = \operatorname{rank}(\Pi_I) = \operatorname{rank}(\Pi) = N$. This implies that $\sigma_k^\varepsilon \in I$ only for $k$, $1 \leq k \leq m_1$, and hence, the result for $J = 1$. The result for $J = 1$ implies that not only may we choose $M$ so that all eigenvalues $\{\sigma_k^\varepsilon\}_{k=m_1+1}^{m_1+m_2}$ are in the interval corresponding to the next highest eigenvalue $\sigma_{m_1+1}^0$, but also that $\sigma_1^0$ is not in this interval. Thus, we apply the same reasoning here to get the result for $\sigma_{m_1+1}^0$. Then, by an induction argument, we get the result for each $J \in \mathbb{N}$, satisfying $\sigma_J^0 > \sigma_{J-1}^0$. $\square$

### Future Work

We close this chapter with a list of questions.

- Is the rate of convergence optimal?
- Can the methods used for Dirichlet eigenvalues be extended to Neumann eigenvalues, if we have some additional regularity on the domain?
- For particular systems, can we determine if there is a lower bound for $|\sigma_J^\varepsilon - \sigma_J^0|$?

# Chapter 2 The Green Function for the Mixed Problem on Lipschitz Domains

## 2.1 Introduction

There has been much activity recently on the study of classical boundary value problems for the Laplacian on domains that are not smooth especially including Lipschitz domains as in Dahlberg [8], Dahlberg and Kenig [9], Jerison and Kenig [20], and Verchota [38]. This is of interest because it allows us to treat physically realistic problems in regions with corners and edges and it is interesting from a mathematical viewpoint because the conditions on the domain are scale invariant; thus, we are able to study something that is really new, rather than study problems that are really just perturbations of a boundary value problem in half-plane.

The study of the mixed problem in Lipschitz domains appears as problem 3.2.15 in Kenig's CBMS lecture notes [24]. The work of Brown and Sykes [36] establishes results for the mixed problem in Lipschitz graph domains. I. Mitrea and M. Mitrea [27] studied the mixed problem for the Laplacian with data taken from a large family of function spaces. More recently, Ott and Brown [31] studied the mixed problem when the boundary between the Dirichlet set $D$ and the Neumann set $N$ is a Lipschitz surface. It is well-known that an elliptic operator with bounded measurable coefficients [26] has a Green function in all of space, provided the dimension is at least three. Given this free space fundamental solution, if the boundary between $D$ and $N$ is Lipschitz, then by using a reflection argument as in Dahlberg and Kenig [9], there is a Green function $G$ such that the solution $u$ to the mixed problem with $f_D = 0$ and $f_N \in W_D^{-1/2,2}(\partial\Omega)$ may be represented as

$$u(x) = -\int_{\partial\Omega} f_N(y)G(x,y)\,dy$$

Then, from the methods of de Giorgi [11], Nash [29], and Moser [28], one may obtain regularity results of the Green function that show how the solution behaves. In Stampacchia [34], a study of Hölder continuity of solutions to elliptic equations is given with a more restrictive condition on the decomposition of the boundary. Also, Haller-Dintelmann et al. [18] show Hölder continuity for solutions to the mixed problem under a condition similar to Stampacchia's. Roughly speaking, Stampacchia's condition is that the Dirichlet set $D \subset \partial\Omega$ and Neumann set $N \subset \partial\Omega$ are separated by a Lipschitzian hypersurface of $\partial\Omega$. In this chapter, we consider properties of the Green function for the mixed problem where the decomposition of the boundary is more general.

### 2.2 Preliminaries

A bounded, connected open set $\Omega$ is called a Lipschitz domain with Lipschitz constant $M$ if the boundary is locally given by the graph of a Lipschitz function. To make this precise, use coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and define a coordinate cylinder $Z_r(x)$ to be a set of the form $Z_r(x) = \{y : |y'-x'| < r,\ |y_n - x_n| < (1+M)r\}$. We assume that this coordinate system is a translation and rotation of the standard coordinates. For each $x$ in the boundary, we assume that we may find a coordinate cylinder and a Lipschitz function $\phi$ with Lipschitz constant $M$ so that

$$\Omega \cap Z_r(x) = \{(y', y_n) : y_n > \phi(y')\} \cap Z_r(x)$$

$$\partial\Omega \cap Z_r(x) = \{(y', y_n) : y_n = \phi(y')\} \cap Z_r(x).$$

To describe the mixed problem, let $\Omega$ be a bounded, connected open Lipschitz domain in $\mathbb{R}^n$ and decompose the boundary $\partial\Omega = D \cup N$, where $D$ is an open subset in $\partial\Omega$ and $N = \partial\Omega \backslash D$. Also, let $\Lambda$ be the boundary between $D$ and $N$ relative to $\partial\Omega$. We define the space $W^{1,2}_D(\Omega)$ to be the closure in $W^{1,2}(\Omega)$ of $C^\infty(\overline{\Omega})$ functions which vanish on $D$. We note here that by definition, if $w \in W^{1,2}_D(\Omega)$, then $w = \lim_{n\to\infty} w_n$

where each $w_n \in C^\infty(\overline{\Omega})$ and $w_n = 0$ on $D$. The limit here is taken in $W^{1,2}(\Omega)$. Since we have a bounded Lipschitz domain, we define the trace map as $\text{trace}(w) = \lim_{n\to\infty} w_n$ where the limit is taken in $L^2(\partial\Omega)$. We let $W_D^{1/2,2}(\partial\Omega)$ be these restrictions to $\partial\Omega$ of $W_D^{1,2}(\Omega)$ and let $W_D^{-1/2,2}(\partial\Omega)$ be the dual of $W_D^{1/2,2}(\partial\Omega)$. Then the mixed problem is given as

$$\begin{cases} Lu = -(a_{ij}u_{x_i})_{x_j} = f & \text{in} \quad \Omega \\ u = f_D & \text{on} \quad D \\ a_{ij}u_{x_i}\nu_j = f_N & \text{on} \quad N \end{cases} \tag{2.1}$$

with the following:

1. We use the convention of summing over repeated indices, where $i$ and $j$ sum from 1 to $n$.
2. The coefficients $a_{ij}$ are bounded and measurable functions satisfying the ellipticity condition $\theta|\xi|^2 \leq a_{ij}\xi_i\xi_j$ for any $\xi \in \mathbb{R}^n$.
3. $f$ is taken from $L^{q/2}(\Omega)$, for $q > n$, and we have $\|f\|_{L^{q/2}(\Omega)} \leq M_f$.
4. $f_D$ is the trace of a function $\widetilde{f}_D$ from $W^{1,2}(\Omega)$.
5. $f_N$ is taken from $W_D^{-1/2,2}(\partial\Omega)$.

We will also assume 2 conditions on $\partial\Omega$. The first is a condition on $D$. There exists $C > 0$ such that

$$\text{for any } x \in \Lambda,\ \sigma(B_r(x) \cap D) \geq Cr^{n-1}, \quad 0 < r \leq r_0 \tag{2.2}$$

where $\sigma(E)$ is the $\mathbb{R}^{n-1}$ surface measure of a set $E$. The next condition is on $N$. There exists $c > 0$ such that

$$\text{for any } x \in N,\ \text{if } B_r(x) \cap D = \emptyset,\ \text{then } |B_r(x) \cap \Omega| \geq cr^n, \quad 0 < r \leq r_0. \tag{2.3}$$

Even though this is a restriction on $\partial\Omega$, it still allows for a quite general decomposition of the boundary. We will use (2.2) and (2.3) in order to apply Sobolev and Poincaré inequalities.

We say that $u \in W^{1,2}(\Omega)$ is a weak solution to the mixed problem (2.1) if $u - \widetilde{f}_D \in W_D^{1,2}(\Omega)$ and

$$\int_\Omega a_{ij} u_{x_i} w_{x_j}\ dx = \int_\Omega fw\ dx + \langle f_N, w\rangle_N \quad \text{for any } w \in W_D^{1,2}(\Omega) \tag{2.4}$$

where $\langle f_N, w\rangle_N$ is interpreted as the pairing of $f_N$ and $\operatorname{trace}(w) \in W_D^{1/2,2}(\partial\Omega)$.

## 2.3 Global Boundedness and Hölder Continuity for Solutions to the Mixed Problem

The next theorem is adapted from Gilbarg and Trudinger [15] and uses an iteration technique introduced by Moser [28]

**Theorem 2.3.1.** *Let $u$ solve the mixed problem (2.1) with $f_D = 0$ and $f_N = 0$. Then, $\sup_\Omega u \le C(\|u\|_{L^2(\Omega)} + 1)$ where $C$ depends on $|\Omega|$, $\|f\|_{L^{q/2}(\Omega)}$, $n$, $q$, and $\theta$.*

*Proof.* Set $k \ge 1$, $\beta \ge 1$, and define $H \in C^1[k, \infty)$ by

$$H(z) = \begin{cases} z^\beta - k^\beta, & z \in [k, N] \\ \frac{N^\beta - k^\beta}{N} z, & z \ge N \end{cases}$$

Next, set $w = u^+ + k$ where $u^+ = \sup(u, 0)$ is the positive part of $u$. Then, if $v = G(w) = \displaystyle\int_k^w |H'(s)|^2\ ds$, we have for $x \in D$ that

$$\begin{aligned} v(x) &= \int_k^{w(x)} |H'(s)|^2\ ds \\ &= \int_k^k |H'(s)|^2\ ds \\ &= 0 \end{aligned}$$

So, by the chain rule [15, p. 151], $v \in W_D^{1,2}(\Omega)$ is an acceptable test function in the weak formulation for $u$. So, from (2.4),

$$\int_\Omega a_{ij} u_{x_i} v_{x_j} \, dx = \int_\Omega f v \, dx$$

or

$$\int_\Omega a_{ij} u_{x_i} G'(w) w_{x_j} \, dx = \int_\Omega f G(w) \, dx$$

Since $w_{x_j} = u_{x_j}$ when $u \geq 0$ and $w_{x_j} = 0$ otherwise, and $G'(w) \geq 0$, we have by the ellipticity condition that

$$\begin{aligned} \int_\Omega |\nabla w|^2 G'(w) \, dx &\leq \frac{1}{\theta} \int_\Omega a_{ij} u_{x_i} w_{x_j} G'(w) \, dx \\ &\leq \frac{1}{\theta} \int_\Omega |f||G(w)| \, dx \end{aligned}$$

Also,

$$\begin{aligned} G(t) &= \int_k^t |H'(s)|^2 \, ds \\ &\leq \int_0^t |H'(t)|^2 \, ds \\ &= tG'(t) \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \int_\Omega |\nabla w|^2 G'(w) \, dx &\leq \frac{1}{\theta} \int_\Omega |f||w||G'(w)| \, dx \\ &\leq \frac{1}{\theta} \int_\Omega |f||w|^2 |G'(w)| \, dx \end{aligned}$$

the last line owing to $w \geq 1$. This is equivalent to

$$\int_\Omega |\nabla H(w)|^2 \, dx \leq C \int_\Omega |f||H'(w)w|^2 \, dx$$

so that by applying Sobolev's inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
\|H(w)\|_{L^{\frac{2\widehat{n}}{\widehat{n}-2}}(\Omega)} &\leq \left(C\int_\Omega |f||H'(w)w|^2\,dx\right)^{\frac{1}{2}} \\
&\leq C\|f\|^{\frac{1}{2}}_{L^{q/2}(\Omega)}\|H'(w)w\|_{L^{\frac{2q}{q-2}}(\Omega)} \\
&\leq C\|H'(w)w\|_{L^{\frac{2q}{q-2}}(\Omega)} \qquad (2.5)
\end{aligned}
$$

where $\widehat{n}=n$ if $n\geq 3$ and $2<\widehat{n}<q$ if $n=2$. So, now letting $N\to\infty$ in (2.5), we obtain the condition that if $w\in L^{\beta\frac{2q}{q-2}}(\Omega)$, then also $w\in L^{\beta\frac{2\widehat{n}}{\widehat{n}-2}}(\Omega)$. Furthermore, setting $q^*=\dfrac{2q}{q-2}$ and $\xi=\dfrac{\widehat{n}(q-2)}{q(\widehat{n}-2)}>1$, we have

$$
\|w\|_{L^{\beta\xi q^*}(\Omega)}\leq (C\beta)^{\frac{1}{\beta}}\|w\|_{L^{\beta q^*}(\Omega)} \qquad (2.6)
$$

By the Sobolev inequality, we may set $\beta q^*=\frac{2\widehat{n}}{\widehat{n}-2}$ which means $\beta=\frac{\widehat{n}q-2\widehat{n}}{\widehat{n}q-2q}>1$ to obtain $w\in L^{\xi\frac{2\widehat{n}}{\widehat{n}-2}}(\Omega)$ in (2.6). Then by an induction argument, we show $w\in\bigcap\limits_{1\leq p<\infty}L^p(\Omega)$. Moreover, setting $\beta=\xi^m$ for $m=0,1,2,\ldots$, and iterating (2.6), we obtain

$$
\begin{aligned}
\|w\|_{L^{\xi^N q^*}(\Omega)} &\leq \prod_{m=0}^{N-1}(C\xi^m)^{\xi^{-m}}\|w\|_{L^{q^*}(\Omega)} \\
&\leq C\|w\|_{L^{q^*}(\Omega)} \qquad (2.7)
\end{aligned}
$$

where $C$ depends on $n$, $q$, $\|f\|_{L^{q/2}(\Omega)}$, and $\theta$. Now let $N\to\infty$ in (2.7) to obtain

$$
\sup_\Omega w\leq C\|w\|_{L^{q^*}(\Omega)}
$$

Using a simple result from Hölder's inequality, we obtain

$$
\sup_\Omega w\leq C\|w\|_{L^2(\Omega)}
$$

Now repeating this argument with $u^+$ replaced with $u^-$, we get the desired result. $\square$

We aim to show Hölder continuity of solutions to the mixed problem with the general decomposition of the boundary described earlier. To achieve this, we will modify the well-known de Giorgi methods [11] from Ladyzhenskaya and Ural'tseva [25, p. 81]. We start with a definition. We say $u \in H^1(\Omega) = W^{1,2}(\Omega)$ belongs to $ß_m(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$ for $M, \gamma, \delta > 0$ and $q > n$ if $\|u\|_\infty \leq M$ and if both $u$ and $-u$ satisfy the following inequalities for an arbitrary region $B_r \subset \Omega$ or $\Omega_r = B_r \cap \Omega$ if $B_r$ is centered on $\partial\Omega$ and arbitrary $\sigma \in (0,1)$:

$$\int_{A_{k,r-\sigma r}} |\nabla u|^m \, dx \leq \gamma \left[ \frac{1}{\sigma^m r^{m-\frac{mn}{q}}} \sup_{A_{k,r}} (u(x)-k)^m + 1 \right] |A_{k,r}|^{1-\frac{m}{q}} \tag{2.8}$$

for $k$ satisfying both $k \geq 0$ and $k \geq \sup_{\Omega_r} u(x) - \delta$ if $B_r \cap D \neq \emptyset$ and for only $k \geq \sup_{\Omega_r} u(x) - \delta$ otherwise, where $A_{k,r} = \{x \in \Omega_r : u(x) > k\}$. Here $B_{r-\sigma r}$ is the concentric ball to $B_r$ and $r \leq r_0$ for some positive $r_0$.

With this definition, we can state

**Proposition 2.3.2.** *Let $u$ solve the mixed problem (2.1) with $f_D = 0$ and $f_N = 0$. Then $u \in ß_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$ where $\delta = \frac{1}{M_f}$ and $\gamma = \gamma(n, \theta)$.*

*Proof.* We note that by Theorem 2.3.1, $u$ is bounded. Next, fix $\Omega_r$ and define $\eta \in C_c^\infty(\mathbb{R}^n)$ to be so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{r-\sigma r}$, $\eta = 0$ outside $B_r$, and $|\nabla \eta| \leq \frac{C_n}{\sigma r}$. We aim to show that $\max\{u-k, 0\} \in W_D^{1,2}(\Omega)$, so that we may use $\phi = \eta^2 \max\{u - k, 0\} \in W_D^{1,2}(\Omega)$ as a test function. To do this let $F(x) = \max\{x, 0\}$. Then, $F$ is piecewise smooth on $\mathbb{R}$ and $\|F'\|_\infty \leq 1$. So, since $u - k \in W^{1,2}(\Omega)$, we may use Theorem 7.8 from Gilbarg and Trudinger [15, p. 153] to get that $F(u-k) \in W^{1,2}(\Omega)$. Furthermore, since $\operatorname{trace}(u-k) = -k$ on $D$, we have $\operatorname{trace}(F(u-k)) = \max\{-k, 0\} = 0$ on $D$, for $k \geq 0$.

We may set $\phi = \eta^2 \max\{u-k, 0\} \in W_D^{1,2}(\Omega)$. Since $\phi$ is non-zero only in $A_{k,r}$, we have by the weak formulation (2.4)

$$\int_{A_{k,r}} a_{ij} u_{x_i} \phi_{x_j} \, dx = \int_{A_{k,r}} f\phi \, dx \tag{2.9}$$

Performing the differentiations, and using ellipticity, we have

$$\int_{A_{k,r}} \theta|\nabla u|^2\eta^2 \, dx \le C\int_{A_{k,r}} |\nabla u|\eta|\nabla\eta||u-k| \, dx + \int_{A_{k,r}} |f|\eta^2|u-k| \, dx$$
$$= I + II$$

Using the Cauchy inequality, we obtain

$$I \le \int_{A_{k,r}} \varepsilon|\nabla u|^2\eta^2 \, dx + \frac{C}{\varepsilon}\int_{A_{k,r}} |\nabla\eta|^2|u-k|^2 \, dx$$

so that by choosing $\varepsilon = \frac{\theta}{2}$, we obtain

$$\frac{\theta}{2}\int_{A_{k,r}} |\nabla u|^2\eta^2 \, dx \le \int_{A_{k,r}} |f|\eta^2|u-k| \, dx + C\int_{A_{k,r}} |\nabla\eta|^2|u-k|^2 \, dx \qquad (2.10)$$
$$= II + III$$

Also, since $1 \le C\left(\frac{r^n}{|A_{k,r}|}\right)^{\frac{2}{q}}$, it follows that

$$\frac{|A_{k,r}|}{r^2} \le C\frac{|A_{k,r}|^{1-\frac{2}{q}}}{r^{2-\frac{2n}{q}}}$$

From this, we obtain that

$$III \le \frac{C}{\sigma^2 r^2}\sup_{A_{k,r}}|u-k|^2|A_{k,r}|$$
$$\le \frac{C}{\sigma^2 r^{2-\frac{2n}{q}}}\sup_{A_{k,r}}|u-k|^2|A_{k,r}|^{1-\frac{2}{q}}$$

Next, from Hölder's inequality,

$$II \le \|f\|_{L^{\frac{q}{2}}(\Omega)}\left(\int_{A_{k,r}} (|u-k|\eta^2)^{\frac{q}{q-2}} \, dx\right)^{1-\frac{2}{q}}$$
$$\le M_f\left(\frac{1}{M_f}\right)|A_{k,r}|^{1-\frac{2}{q}}$$
$$= |A_{k,r}|^{1-\frac{2}{q}}$$

It now follows from (2.10) that

$$\int_{A_{k,r-\sigma r}} |\nabla u|^2 \, dx \le \int_{A_{k,r}} |\nabla u|^2\eta^2 \, dx$$
$$\le C\left(\frac{1}{\sigma^2 r^{2-\frac{2n}{q}}}\sup_{A_{k,r}}|u-k|^2 + 1\right)|A_{k,r}|^{1-\frac{2}{q}}$$

Thus, noting that this inequality for $-u$ is true by a similar proof, we have that $u \in \text{ß}_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$. $\square$

Before stating a theorem for Hölder continuity in the interior of $\Omega$, we need several lemmas taken from Ladyzhenskaya and Ural'tseva [25]. The first is a consequence of Poincaré's inequality.

**Lemma 2.3.3.** *If $u \in W^{1,1}(B_r)$, then*

$$(l-k)|A_{l,r}|^{1-\frac{1}{n}} \leq \frac{\beta r^n}{|B_r \backslash A_{k,r}|} \int_{A_{k,r} \backslash A_{l,r}} |\nabla u| \; dx$$

*where $l \geq k$ and $\beta = \beta(n)$.*

**Lemma 2.3.4.** *Suppose a sequence $y_l$ satisfies*

$$0 \leq y_{l+1} \leq c b^l y_l^{1+\varepsilon}$$

*and*

$$y_0 \leq c^{\frac{-1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}},$$

*where $c, \varepsilon,$ and $b$ are positive constants with $b > 1$. Then,*

$$y_l \to 0 \quad as \quad l \to \infty.$$

The proof of Lemma 2.3.5 is presented, but can also be found in Ladyzhenskaya and Ural'tseva [25, p. 83].

**Lemma 2.3.5.** *There exists $\theta_1 > 0$ so that for any $u \in \text{ß}_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$ and for any $\Omega_r$ with $k \geq \sup_{\Omega_r} u(x) - \delta$, the inequalities*

1. $|A_{k,r}| \leq \theta_1 r^n$
2. $H = \sup_{\Omega_r} u(x) - k \geq r^{1-\frac{n}{q}}$

*imply*

$$|A_{k+\frac{H}{2},\frac{r}{2}}| = 0.$$

*Proof.* Fix $B_r$ and $u \in ß_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$. Define the sequences

- $r_h = \frac{r}{2} + \frac{r}{2^{h+1}}$
- $k_h = k + \frac{H}{2} - \frac{H}{2^{h+1}}$

for $h = 0, 1, 2, ...,$ and consider the balls $B_{r_h}$ that are concentric to $B_r$. Also, set $\sigma = \frac{r_h - r_{h+1}}{r_h}$ in (2.8) to obtain

$$\int_{A_{k_h, r_{h+1}}} |\nabla u|^2 \, dx \leq \gamma \left[ \frac{r_h^{\frac{2n}{q}}}{(r_h - r_{h+1})^2} \sup_{A_{k_h, r_h}} (u(x) - k_h)^2 + 1 \right] |A_{k_h, r_h}|^{1-\frac{2}{q}} \tag{2.11}$$

then use Lemma 2.3.3 with $k = k_h$ and $l = k_{h+1}$ to obtain

$$(k_{h+1} - k_h)|A_{k_{h+1}, r_{h+1}}|^{1-\frac{1}{n}} \leq \frac{\beta r_{h+1}^n}{|B_{r_{h+1}} \backslash A_{k_h, r_{h+1}}|} \int_{A_{k_h, r_{h+1}}} |\nabla u| \, dy \tag{2.12}$$

If we impose that $\theta_1 \leq \frac{w_n}{2^{n+1}}$, then by assumption, we have

$$|A_{k_h, r_{h+1}}| \leq |A_{k,r}| \leq \frac{|B_{r_{h+1}}|}{2}$$

Thus, since $\frac{H}{2^{h+2}} = k_{h+1} - k_h$, by (2.12), we have

$$\begin{aligned} \frac{H}{2^{h+2}} |A_{k_{h+1}, r_{h+1}}|^{1-\frac{1}{n}} &\leq \frac{2\beta}{\omega_n} \int_{A_{k_h, r_{h+1}}} |\nabla u| \, dx \\ &\leq \frac{2\beta}{\omega_n} \left( \int_{A_{k_h, r_{h+1}}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} |A_{k_h, r_{h+1}}|^{\frac{1}{2}} \\ &\leq \frac{2\beta}{\omega_n} \left( \int_{A_{k_h, r_{h+1}}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} |A_{k_h, r_h}|^{\frac{1}{2}} \end{aligned} \tag{2.13}$$

Then from (2.11) and (2.13), we arrive at

$$\begin{aligned} \left( \frac{H\omega_n}{2^{h+3}\beta} \right)^2 |A_{k_{h+1}, r_{h+1}}|^{2-\frac{2}{n}} |A_{k_h, r_h}|^{-1} &\leq \int_{A_{k_h, r_{h+1}}} |\nabla u|^2 \, dx \\ &\leq \gamma \left[ (2^{h+2})^2 H^2 r^{2(\frac{n}{q}-1)} + 1 \right] |A_{k_h, r_h}|^{1-\frac{2}{q}} \end{aligned}$$

which implies

$$|A_{k_{h+1}, r_{h+1}}|^{2-\frac{2}{n}} \leq \gamma \left( \frac{2^{h+3}\beta}{\omega_n} \right)^2 \left[ 2^{2h+4} r^{(\frac{2n}{q}-2)} + H^{-2} \right] |A_{k_h, r_h}|^{2-\frac{2}{q}}$$

so that, by the assumption $H \geq r^{(1-\frac{n}{q})}$, we have

$$\left(\frac{|A_{k_{h+1},r_{h+1}}|}{r^n}\right)^{1-\frac{1}{n}} \leq C2^{2h}\left(\frac{|A_{k_h,r_h}|}{r^n}\right)^{1-\frac{1}{q}} \tag{2.14}$$

where $C = C(\gamma, \beta, n)$. So, if we define $\mu_h = \frac{|A_{k_h,r_h}|}{r^n}$, we have the inequality

$$\mu_{h+1} \leq C^{\frac{n}{n-1}} 2^{\frac{2n}{n-1}h} \mu_h^{1+\varepsilon}$$

where $\varepsilon = \frac{q-n}{q(n-1)}$. Hence, in accordance with Lemma 2.3.4, if

$$\mu_0 \leq \frac{1}{C^{\frac{n}{\varepsilon(n-1)}} 2^{\frac{2n}{\varepsilon^2(n-1)}}} = C_0$$

then $\mu_h \to 0$ as $h \to \infty$. To satisfy this condition, we let $\theta_1 = \min\left\{\frac{\omega_n}{2^{n+1}}, C_0\right\}$. Finally, observing that $k_h \to k + \frac{H}{2}$ and $r_h \to \frac{r}{2}$ as $h \to \infty$, we get the desired result.

□

If we are able to satisfy the hypotheses of the next lemma taken from Ladyzhenskaya and Ural'tseva [25, p. 66], we will have the Hölder continuity we desire.

**Lemma 2.3.6.** *Suppose $u$ is bounded and measurable in some $\Omega_{r_0}$. Consider $B_r$ and $B_{br}$ for $b > 1$ which are concentric with $B_{r_0}$. Suppose for arbitrary $r \leq \frac{r_0}{b}$ at least one of the following holds:*

1. *$osc(u, \Omega_r) \leq c_1 r^{\varepsilon}$*
2. *$osc(u, \Omega_r) \leq \Theta osc(u, \Omega_{br})$*

*where $c_1, \varepsilon \leq 1$ and $\Theta < 1$. Then for $r \leq r_0$,*

$$osc(u, \Omega_r) \leq c r_0^{-\alpha} r^{\alpha}$$

*where $\alpha = \min\{-\log_b(\Theta), \varepsilon\}$, $c = b^{\alpha} \max\{\omega_0, c_1 r_0^{\varepsilon}\}$, and $\omega_0 = osc(u, \Omega_{r_0})$.*

The next lemma is taken from Ladyzhenskaya and Ural'tseva [25, p. 85], and we present the proof with more detail. This lemma will allow us to use Lemma 2.3.6, and hence obtain interior continuity for a solution to the mixed problem.

**Lemma 2.3.7.** *Let $u \in \mathfrak{B}_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$. There exists a positive integer $s = s(n, \theta, M, \delta)$ so that for any $B_r$, concentric with $B_{4r} \subset \Omega$, at least one of the following inequalities hold for $u$:*

1. $osc(u, B_r) \leq 2^s r^{1-\frac{n}{q}}$
2. $osc(u, B_r) \leq \left(1 - \frac{1}{2^{s-1}}\right) osc(u, B_{4r})$

*Proof.* We impose the condition

$$\frac{M}{2^{s-3}} \leq \delta \tag{2.15}$$

on $s$ and assume condition 1. is false. Define

- $M_r = \sup_{B_r} u$
- $m_r = \inf_{B_r} u$
- $\overline{M}_r = \dfrac{M_r + m_r}{2}$
- $\operatorname{osc}(u, B_r) = \omega_r = M_r - m_r$
- $D_t = \left(A_{M_{4r} - \frac{\omega_{4r}}{2^t}, 2r} \backslash A_{M_{4r} - \frac{\omega_{4r}}{2^{t+1}}, 2r}\right)$ $, t = 1, 2, ..., s$

where $A_{k,r} = \{x \in B_r : u(x) > k\}$. We may also assume that

$$\left|A_{\overline{M}_{4r}, 2r}\right| \leq \frac{|B_{2r}|}{2}, \tag{2.16}$$

for, if not, we replace $u$ with $-u$ and then prove the lemma for $-u$.

First, use Lemma 2.3.3 with $k = M_{4r} - \frac{\omega_{4r}}{2^t}$ and $l = M_{4r} - \frac{\omega_{4r}}{2^{t+1}}$ to obtain

$$\begin{aligned} \frac{\omega_{4r}}{2^{t+1}} \left|A_{M_{4r} - \frac{\omega_{4r}}{2^{t+1}}, 2r}\right|^{1-\frac{1}{n}} &\leq \frac{\beta(2r)^n}{\frac{1}{2}|B_{2r}|} \int_{D_t} |\nabla u| \, dx \\ &= \frac{2\beta}{\omega_n} \int_{D_t} |\nabla u| \, dx \end{aligned} \tag{2.17}$$

where we have also used (2.16) on the first line. So, by Hölder's inequality,

$$\left(\frac{\omega_{4r}}{2^{t+1}}\right)^2 \left|A_{M_{4r} - \frac{\omega_{4r}}{2^{t+1}}, 2r}\right|^{2-\frac{2}{n}} \leq \left(\frac{2\beta}{\omega_n}\right)^2 |D_t| \int_{D_t} |\nabla u|^2 \, dx \quad t = 1, 2, ..., s \tag{2.18}$$

Next, we aim to place conditions on $k$ so that we may use the inequality from (2.8). We need $k = M_{4r} - \frac{\omega_{4r}}{2^t} \geq M_{4r} - \delta$. This will mean that we need $t \geq \log_2(\frac{2M}{\delta}) = t_0$. With these values of $t$, we may use the inequality (2.8) with $\sigma = 1/2$ to obtain

$$\int_{A_{M_{4r}-\frac{\omega_{4r}}{2^t},2r}} |\nabla u|^2 \, dx \leq \gamma \left[ 4(4r)^{2n/q-2} \sup_{A_{M_{4r}-\frac{\omega_{4r}}{2^t},4r}} \left| u - (M_{4r} - \frac{\omega_{4r}}{2^t}) \right|^2 + 1 \right] |A_{M_{4r}-\frac{\omega_{4r}}{2^t},4r}|^{1-2/q}$$
$$\leq \gamma r^{2n/q-2} \left[ (4)^{2n/q-1} \left( \frac{\omega_{4r}}{2^t} \right)^2 + r^{2-2n/q} \right] |A_{M_{4r}-\frac{\omega_{4r}}{2^t},4r}|^{1-2/q}$$

Also, since we are assuming condition 1. is false and $1 \leq t \leq s$, we have

$$\int_{A_{M_{4r}-\frac{\omega_{4r}}{2^t},2r}} |\nabla u|^2 \, dx \leq \gamma r^{2n/q-2} \left( \frac{\omega_{4r}}{2^t} \right)^2 |A_{M_{4r}-\frac{\omega_{4r}}{2^t},4r}|^{1-2/q} \left[ (4)^{2n/q-1} + 1 \right]$$
$$\leq C \left( \frac{\omega_{4r}}{2^t} \right)^2 r^{n-2}$$

so that by (2.18),

$$\left( \frac{\omega_{4r}}{2^{t+1}} \right)^2 \left| A_{M_{4r}-\frac{\omega_{4r}}{2^{t+1}},2r} \right|^{2-\frac{2}{n}} \leq C \left( \frac{\omega_{4r}}{2^t} \right)^2 |D_t| r^{n-2}, \quad t = 1, 2, ..., s$$

or

$$\left| A_{M_{4r}-\frac{\omega_{4r}}{2^{s-2}},2r} \right|^{2-\frac{2}{n}} \leq C |D_t| r^{n-2}, \quad t = 1, 2, ..., s-3 \tag{2.19}$$

Then, summing (2.19) from $t = 1$ to $t = s - 3$, we obtain

$$(s-3) \left| A_{M_{4r}-\frac{\omega_{4r}}{2^{s-2}},2r} \right|^{2-\frac{2}{n}} \leq C r^{n-2} \sum_{t=1}^{s-3} |D_t|$$
$$\leq C r^{n-2} |B_{2r}|$$
$$= C r^{2n-2}$$

Thus, we obtain the inequality

$$\left| A_{M_{4r}-\frac{\omega_{4r}}{2^{s-2}},2r} \right| \leq \left( \frac{C \omega_n 2^n}{s-3} \right)^{\frac{n}{2n-2}} r^n \tag{2.20}$$

We now look at $H = M_{2r} - k = M_{2r} - M_{4r} + \frac{\omega_{4r}}{2^{s-2}}$, defined in accordance with Lemma 2.3.5. We have two cases, depending on $H$.

*(case 1)* $H < (2r)^{1-\frac{n}{q}}$

Again, since we are assuming $\omega_{4r} > 2^s r^{1-\frac{n}{q}}$, we have

$$\begin{aligned} M_{2r} &\leq M_{4r} - \frac{\omega_{4r}}{2^{s-2}} + (2r)^{1-\frac{n}{q}} \\ &< M_{4r} - \frac{\omega_{4r}}{2^{s-2}} + 2^{1-\frac{n}{q}} \left( \frac{\omega_{4r}}{2^s} \right) \\ &\leq M_{4r} - \frac{\omega_{4r}}{2^{s-1}} \end{aligned}$$

which implies

$$\begin{aligned} M_{2r} - m_{2r} &< M_{4r} - m_{2r} - \frac{\omega_{4r}}{2^{s-1}} \\ &< M_{4r} - m_{4r} - \frac{\omega_{4r}}{2^{s-1}} \\ &= \left( 1 - \frac{1}{2^{s-1}} \right) \omega_{4r} \end{aligned}$$

so that $\operatorname{osc}(u, B_r) < \left(1 - \frac{1}{2^{s-1}}\right) \operatorname{osc}(u, B_{4r})$. For case 1, the proof of the lemma is now complete.

*(case 2)* $H \geq (2r)^{1-\frac{n}{q}}$

For this case, from the condition (2.15), we have $H \leq \omega_{4r} 2^{2-s} \leq 2M 2^{2-s} \leq \delta$. Thus, we may apply Lemma 2.3.5 to get the existence of $\theta_1$ so that with our choice of $k$ and $H$, the inequalities

- $|A_{k,2r}| \leq \theta_1 (2r)^n$
- $H \geq (2r)^{1-\frac{n}{q}}$

imply

$$|A_{k+\frac{H}{2},r}| = 0.$$

This inequality implies

$$|A_{M_{4r}-\frac{\omega_{4r}}{2^{s-2}}+\frac{\omega_{4r}}{2^{s-1}},r}| = 0.$$

Hence,

$$\begin{aligned} M_r &\leq M_{4r} - \frac{\omega_{4r}}{2^{s-2}} + \frac{\omega_{4r}}{2^{s-1}} \\ &= \left(M_{4r} - \frac{\omega_{4r}}{2^{s-1}}\right) \end{aligned}$$

which again leads to $\operatorname{osc}(u, B_r) < \left(1 - \frac{1}{2^{s-1}}\right) \operatorname{osc}(u, B_{4r})$. □

Now that we have interior continuity for solutions to the mixed problem with zero Dirichlet data and zero Neumann data, we aim to extend this result up to the boundary. In order to do this, we will use a similar lemma to Lemma 2.3.7. The proof requires the use of Lemma 2.3.3, but the right side of the inequality in this lemma may blow up as we approach the Dirichlet set $D$. To compensate, we must replace $\dfrac{\beta r^n}{|B_r \backslash A_{k,r}|}$ from Lemma 2.3.3 with a constant which does not depend on $r$, as we approach $D$. To do this, we first need a well-known theorem taken from Ladyzhenskaya and Ural'tseva [25, p. 54]. Again, we present the proof with more detail.

**Theorem 2.3.8.** *Let $u \in W^{1,1}(B_r)$, $S \subset B_r$, and $S_0 = \{x \in B_r : u(x) = 0\}$. Then*

$$\int_S |u| \, dy \leq \frac{\beta r^n |S|^{1/n}}{|S_0|} \int_{B_r} |\nabla u| \, dy \tag{2.21}$$

*Proof.* It suffices to prove for smooth $u$. Fix $x \in B_r$ and $y \in S_0$. Then for $\omega = \frac{y-x}{|y-x|}$, we have

$$-u(x) = u(y) - u(x) = \int_0^{|y-x|} \frac{\partial u(x + r\omega)}{\partial r} \, dr$$

or

$$-u(x)|S_0| = \int_{S_0} \int_0^{|y-x|} \frac{\partial u(x + r\omega)}{\partial r} \, dr \, dy \tag{2.22}$$

Also,

$$\left|\int_{S_0}\int_0^{|y-x|}\frac{\partial u(x+r\omega)}{\partial r}\;dr\;dy\right| \leq \int_0^{2r}\rho^{n-1}\int_0^{|y-x|}\left|\frac{\partial u(x+r\omega)}{\partial r}\right|\;dr\;d\omega\;d\rho$$
$$\leq \frac{(2r)^n}{n}\int_{B_r}\frac{|\nabla u(z)|}{|x-z|^{n-1}}\;dz$$

So, from (2.22), we obtain

$$|u(x)||S_0| \leq \frac{(2r)^n}{n}\int_{B_r}\frac{|\nabla u(z)|}{|x-z|^{n-1}}\;dz$$

Integrating over $S$, we obtain

$$\begin{aligned}|S_0|\int_S |u(x)|\;dx &\leq \frac{(2r)^n}{n}\int_{B_r}|\nabla u(z)|\;\int_S\frac{dx}{|x-z|^{n-1}}\;dz\\ &= \frac{(2r)^n}{n}\int_{B_r}|\nabla u(z)|\;\left(\int_{S\cap\{x:|x-z|\leq\varepsilon\}}\frac{dx}{|x-z|^{n-1}}+\int_{S\cap\{x:|x-z|\geq\varepsilon\}}\frac{dx}{|x-z|^{n-1}}\right)dz\\ &= \frac{(2r)^n}{n}\int_{B_r}|\nabla u(z)|\;(I+II)\,dz\end{aligned}$$

for $\varepsilon > 0$. We have that

$$I \leq \varepsilon\sigma(\partial B(0,1))$$

and

$$II \leq \varepsilon^{1-n}|S|$$

So, choosing $\varepsilon = |S|^{1/n}$, we obtain the result with $\beta = \frac{2^n}{n}(\sigma(\partial B(0,1))+1)$. $\square$

Using the previous theorem, we are able to state and prove a version of Lemma 2.3.3, when we are on the Dirichlet set $D$.

**Lemma 2.3.9.** *Let $\Omega$ be a Lipschitz domain and $B_{r/2}(x)$ be a ball centered on $\partial\Omega$ such that $B_{\frac{r}{2}}(x)\cap D \neq \emptyset$. Also, recall the definition of $A_{k,r}$ from (2.8). Then for $r \leq r_0$ from (2.2) and (2.3), and $u \in W^{1,2}_D(\Omega)$, we have*

$$(l-k)|A_{l,r/2}|^{(n-1)/n} \leq \widetilde{C}\int_{A_{k,Cr}\setminus A_{l,Cr}}|\nabla u|\;dy \tag{2.23}$$

*for $l > k \geq 0$, where $\widetilde{C}$ is independent of $r$, and where $C$ depends on the Lipschitz constant $M$.*

*Proof.* Since $\partial\Omega$ is Lipschitz, there is a coordinate cylinder $Z_r$ so that we may extend $u$ by even reflection to $\widetilde{u}$ in $Z_r$. That is, for $x \in Z_r$, define

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq \phi(x') \\ u(Rx) & \text{if } x_n < \phi(x') \end{cases}$$

where $Rx = (x', 2\phi(x') - x_n)$. We note

$$\begin{aligned} \int_{B_r \setminus \Omega_r} |\widetilde{u}(x)| dx &= \int_{B_r \cap \{x_n < \phi(x')\}} |u(x', 2\phi(x') - x_n)| dx' dx_n \\ &\leq \int_{Z_r \cap \{w > \phi(x')\}} |u(x', w)| dw dx \\ &= \int_{Z_r \cap \Omega} |u(x)| dx \end{aligned}$$

and

$$\begin{aligned} \int_{B_r \setminus \Omega_r} \left| \frac{\partial}{\partial x_n} \widetilde{u}(x) \right| dx &\leq \int_{Z_r \cap \{w > \phi(x')\}} |-u_{x_n}(x', w)| dw dx \\ &= \int_{Z_r \cap \Omega} |u_{x_n}(x)| dx \end{aligned}$$

Thus,

$$\int_{B_r} |\widetilde{u}| \; dy \leq \int_{Z_r \cap \Omega} |u| \; dy \tag{2.24}$$

and

$$\int_{B_r} |\nabla \widetilde{u}| \; dy \leq C \int_{Z_r \cap \Omega} |\nabla u| \; dy \tag{2.25}$$

We let $\eta$ be so that $\eta = 1$ in $B_{r/2}$, $\eta = 0$ outside $B_{3r/4}$, and $|\nabla \eta| \leq C_n/r$. For any $S \subset B_r$, we use (2.21), (2.24), and (2.25) to obtain

$$\begin{aligned} \int_S \eta \widetilde{u} \; dy &\leq C|S|^{1/n} \int_{B_r} |\nabla(\eta \widetilde{u})| \; dy \\ &\leq C|S|^{1/n} \left( \int_{B_r} |\nabla \eta| |\widetilde{u}| \; dy + \int_{B_r} |\nabla \widetilde{u}| \; dy \right) \\ &\leq C|S|^{1/n} \left( \int_{Z_r \cap \Omega} \frac{1}{r} |u| \; dy + \int_{Z_r \cap \Omega} |\nabla u| \; dy \right) \\ &\leq C|S|^{1/n} \left( \int_{Z_r \cap \Omega} |\nabla u| \; dy \right) \end{aligned}$$

where we have used Poincaré's inequality on the last line since $\sigma(B_r \cap D) \geq Cr^{n-1}$ and $u \in W^{1,2}_D(\Omega)$. Consequently,

$$\int_S \eta \widetilde{u}\; dy \leq C|S|^{1/n} \int_{Z_r \cap \Omega} |\nabla u|\; dy$$

Now, define

$$\overline{u}(x) = \begin{cases} 0 & \text{if } \; u(x) \leq k \\ u(x) - k & \text{if } \; k \leq u(x) \leq l \\ l - k & \text{if } \; u(x) \geq l \end{cases}$$

and $S = A_{l,r/2}$. Since $k \geq 0$, we have $\overline{u} \in W^{1,2}_D(\Omega)$. So, we may replace $u$ by $\overline{u}$ in the previous inequality and choose $C$ so that $B_{Cr}$ is the smallest ball which contains $Z_r$ to obtain the result. □

We can now state a theorem for Hölder continuity up to the boundary.

**Lemma 2.3.10.** *Let $\Omega$ be a Lipschitz domain and $u \in ß_2(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q}) \cap W^{1,2}_D(\Omega)$. Fix $x \in \partial\Omega$ and assume $r \leq r_0/16C$ along with the boundary conditions (2.2) and (2.3). There exists a positive integer $s = s(n, \theta, M, \delta, C)$ so that for any $\Omega_r(x)$, concentric with $\Omega_{16Cr}(x)$, at least one of the following inequalities hold for $u$:*

1. $osc(u, \Omega_r) \leq 2^s r^{1-\frac{n}{q}}$
2. $osc(u, \Omega_r) \leq \left(1 - \frac{1}{2^{s-1}}\right) osc(u, \Omega_{16Cr})$

*Here, $C$ is from Lemma 2.3.9 and depends on the Lipschitz constant $M$.*

*Proof.* We will modify the proof of Lemma 2.3.7. If $\Omega_{4r} \cap D = \emptyset$, the proof is the same as the proof of Lemma 2.3.7. So, assume $\Omega_{4r} \cap D \neq \emptyset$. In this case, we do not impose a condition of the form (2.16). Instead, we assume $k = M_{16Cr} - \frac{\omega_{16Cr}}{2^t} \geq 0$, for, if not, we replace $u$ with $-u$ in the definitions of $M_r$ and $m_r$. Since $k \geq 0$, we use (2.23) with $r$ replaced with $8r$. This leads to the inequality

$$\frac{\omega_{16Cr}}{2^{t+1}} \left| A_{M_{16Cr} - \frac{\omega_{16Cr}}{2^{t+1}}, 4r} \right|^{1-\frac{1}{n}} \leq \widetilde{C} \int_{A_{k,8Cr} \setminus A_{l,8Cr}} |\nabla u|\; dx \tag{2.26}$$

We replace (2.17) with (2.26). We next redefine $D_t$ on balls of radius $8Cr$ and use (2.8) with $A_{k,16Cr}$ and $\sigma = 1/2$. This replaces (2.20) with

$$\left| A_{M_{16Cr} - \frac{\omega_{16Cr}}{2^{s-2}}, 2r} \right| \leq \left( \frac{C\omega_n 2^n}{s-3} \right)^{\frac{n}{2n-2}} r^n. \tag{2.27}$$

Then, from here, letting $H = M_{8Cr} - k = M_{8Cr} - M_{16Cr} + \frac{\omega_{16Cr}}{2^{s-2}}$, we obtain the result.

□

**Corollary 2.3.11.** *Let $\Omega$ be a Lipschitz domain and assume the boundary conditions (2.2) and (2.3). Let $u$ solve the mixed problem (2.1) with $f_D = 0$ and $f_N = 0$. Then $u$ is Hölder continuous in $\overline{\Omega}$. Moreover, for each $r \leq r_0$, if either $\Omega_r(x) \subset \Omega$ or $x \in \partial\Omega$, there exists $\alpha$ such that $u$ satisfies the estimate*

$$|u(z) - u(z')| \leq C \left( \frac{|z - z'|}{r} \right)^{\alpha} \left( 1 + \sup_{\Omega_r} |u(x)| \right), \qquad z, z' \in \Omega_r \tag{2.28}$$

*where $C$ and $\alpha$ both depend on $n$, $|\Omega|$, $\|f\|_{L^{q/2}(\Omega)}$, $q$, $\|u\|_{L^2(\Omega)}$, $\theta$, and the Lipschitz constant.*

*Proof.* The proof follows immediately from applying Proposition 2.3.2, Lemmas 2.3.7 and 2.3.10, and Lemma 2.3.6. □

## 2.4 The Green Function for the Mixed Problem

We introduce an approximation for the Green function for the mixed problem. We fix $y \in \Omega$ and $\rho > 0$. If we define the bilinear form $a(u, v)$ on $W^{1,2}_D(\Omega) \times W^{1,2}_D(\Omega)$ as

$$a(u, v) = \int_{\Omega} a_{ij} u_{x_i} v_{x_j} \, dx$$

then the Lax-Milgram theorem guarantees the existence of a unique function $G^\rho \in W^{1,2}_D(\Omega)$ so that

$$a(G^\rho, \phi) = \fint_{B_\rho(y)} \phi \, dx \quad \text{for any } \phi \in W^{1,2}_D(\Omega) \tag{2.29}$$

This function, $G^\rho$, is then a weak solution to the mixed problem (2.1) with $f_D = 0$, $f_N = 0$, and $f = \chi \frac{1}{|B_\rho(y)|}$, where $\chi$ is the characteristic function over $B_\rho(y)$. Before we list some properties of $G^\rho$, we have a definition.

We say the operator $L$ satisfies a symmetry condition if

$$a_{ij} = a_{ji} \quad \text{for each} \quad i \quad \text{and} \quad j. \tag{2.30}$$

From this point we assume (2.30) on the coefficients. Our first property of $G^\rho$ is the following:

**Lemma 2.4.1.** *For any $x \in \Omega$, $G^\rho(x) \geq 0$.*

*Proof.* We have that

$$a(G^\rho - |G^\rho|, G^\rho - |G^\rho|) = a(G^\rho, G^\rho) + a(|G^\rho|, |G^\rho|) - 2a(G^\rho, |G^\rho|).$$

Thus, noting that $|G^\rho|_{x_i} = \operatorname{sign}(G^\rho) G^\rho_{x_i}$, we obtain

$$\begin{aligned} a(G^\rho - |G^\rho|, G^\rho - |G^\rho|) &= 2(a(G^\rho, G^\rho) - a(G^\rho, |G^\rho|)) \\ &= 2 \left( \fint_{B_\rho(y)} G^\rho \, dx - \fint_{B_\rho(y)} |G^\rho| \, dx \right) \\ &\leq 0 \end{aligned}$$

so that, by ellipticity, $|\nabla(G^\rho - |G^\rho|)| = 0$. So, since $G^\rho$ vanishes on $D$, we obtain $G^\rho = |G^\rho|$. $\square$

The next estimate, due to Moser, gives a local estimate.

**Theorem 2.4.2.** *If $u \in W^{1,2}(\Omega)$ is a bounded weak solution to the mixed problem (2.1) with $f = 0$, then*

$$\sup_{\Omega_r(x_0)} |u| \leq C \fint_{\Omega_{2r}(x_0)} |u| \, dz$$

*for either*

*1.* $\Omega_{2r}(x_0) = B_{2r}(x_0) \subset \Omega$ *or*

*2.* $\Omega_{2r}(x_0) = B_{2r}(x_0) \cap \Omega$ *for* $x_0 \in \partial\Omega$ *and* $f_D = 0$*,* $f_N = 0$ *on* $\partial\Omega_{2r}(x_0) \cap \partial\Omega$

*Proof.* We first prove for $r = 1$. We also first assume $u \geq 0$. Define $r_1$ and $r_2$ to be such that $1 \leq r_1 < r_2 \leq 2$. Also, let $\eta \in C_c^\infty(\mathbb{R}^n)$ to be so that $\eta = 1$ in $B_{r_1}$, $\eta = 0$ outside $B_{r_2}$, and $|\nabla \eta| \leq \frac{C}{r_2 - r_1}$. Then, for $m \geq 1$, since $u$ is bounded, we have $v = \eta^2 u^m \in W_D^{1,2}(\Omega)$, we have

$$\begin{aligned}\int_\Omega a_{ij} u_{x_i} v_{x_j} \; dx &= \int_\Omega a_{ij} u_{x_i} (\eta^2 m u^{m-1} u_{x_j} + 2\eta\eta_{x_j} u^m) \; dx \\ &= 0\end{aligned}$$

This gives

$$\begin{aligned}\int_\Omega \theta m \eta^2 u^{m-1} |\nabla u|^2 \; dx &\leq \int_\Omega 2\eta |\nabla u| |\nabla \eta| u^m \\ &\leq \int_\Omega C\varepsilon \eta^2 |\nabla u|^2 u^{m-1} + \frac{C}{\varepsilon} |\nabla \eta|^2 u^{m+1}\end{aligned}$$

so that by choosing $\varepsilon = \frac{\theta m}{2C}$, we obtain

$$\int_\Omega \eta^2 u^{m-1} |\nabla u|^2 \; dx \leq \frac{C}{m^2} \int_\Omega |\nabla \eta|^2 u^{m+1} \; dx \tag{2.31}$$

Now, defining $w = u^{(m+1)/2}$, we may use Sobolev's inequality to obtain

$$\begin{aligned}\|\eta w\|^2_{\frac{2\widehat{n}}{\widehat{n}-2}} &\leq C \int_\Omega |\eta \nabla w|^2 + |w \nabla \eta|^2 \; dx \\ &\leq C \left(\frac{m+1}{m}\right)^2 \int_\Omega |w \nabla \eta|^2\end{aligned}$$

where $\widehat{n} = n$ for $n \geq 3$ and $2 < \widehat{n} < q$ when $n = 2$. So, now defining $\chi = \frac{\widehat{n}}{\widehat{n}-2}$, we obtain

$$\|w\|_{L^{2\chi}(\Omega_{r_1}(x_0))} \leq C \left(\frac{m+1}{m}\right) \frac{1}{r_2 - r_1} \|w\|_{L^2(\Omega_{r_2}(x_0))} \tag{2.32}$$

Now for $p \geq 2$, setting $m+1 = \chi^l p$ and $r_l = 1 + 2^{-l}$ for $l = 0, 1, 2, ..,$ we iterate (2.32) to get

$$\|u\|_{L^{p\chi^l}(\Omega_1(x_0))} \leq C \left( \prod_{j=1}^{l} (2^j)^{\frac{1}{\chi^{j-1}}} \right)^{2/p} \|u\|_{L^p(\Omega_2(x_0))}$$

$$\leq C\|u\|_{L^p(\Omega_2(x_0))} \tag{2.33}$$

Taking $l \to \infty$ in (2.33), we get

$$\sup_{\Omega_1(x_0)} |u| \leq C\|u\|_{L^p(\Omega_2(x_0))}, \qquad p \geq 2 \tag{2.34}$$

We note that by employing a technique from Fabes and Stroock [10], we obtain (2.34) for $p > 0$. Then we rescale to obtain the result for $u \geq 0$. Then for general $u$, write as $u = u^+ + u^-$ and apply (2.34) to each of $u^+$ and $u^-$. $\square$

**We now only consider $G^\rho$ for $n \geq 3$.**

We prove a weak $L^{\frac{n}{n-2}}$ estimate for $G^\rho$. Define $\Omega_\alpha = \{x \in \Omega : G^\rho(x) > e\alpha\}$.

**Lemma 2.4.3.** *We have $|\Omega_\alpha| \leq C\alpha^{\frac{-n}{n-2}}$ for any $\alpha > 0$, where $C = C(\theta, n)$.*

*Proof.* Set $\phi = \left[\frac{1}{\alpha} - \frac{1}{G^\rho}\right]^+ \in W_D^{1,2}(\Omega)$. Then, by Lemma 2.4.1,

$$\frac{1}{\alpha} = \fint_{B_\rho(y)} \frac{1}{\alpha}\, dx$$

$$\geq \fint_{B_\rho(y)} \phi\, dx$$

$$= a(G^\rho, \phi)$$

So, since $\phi$ is positive only in $E = \{x \in \Omega : G^\rho(x) > \alpha\}$, we have

$$\frac{1}{\alpha} \geq \int_E a_{ij} G^\rho_{x_i} \frac{G^\rho_{x_j}}{(G^\rho)^2}\, dx$$

$$\geq \theta \int_E \frac{|\nabla G^\rho|^2}{(G^\rho)^2}\, dx$$

$$= \theta \int_E \left| \nabla \left( \log \left( \frac{G^\rho}{\alpha} \right) \right)^+ \right|^2 dx \tag{2.35}$$

We thus obtain by Sobolev embedding that

$$
\begin{aligned}
\frac{1}{\alpha^{\frac{1}{2}}} &\geq \theta^{\frac{1}{2}} \left( \int_E \left| \nabla \left( \log \left( \frac{G^\rho}{\alpha} \right) \right)^+ \right|^2 \, dx \right)^{\frac{1}{2}} \\
&\geq C\theta^{\frac{1}{2}} \left( \int_E \left| \left( \log \left( \frac{G^\rho}{\alpha} \right) \right)^+ \right|^{2^*} \, dx \right)^{\frac{1}{2^*}}
\end{aligned} \tag{2.36}
$$

Also, by the Chebyshev inequality,

$$
\int_{\Omega_\alpha} \left| \log \left( \frac{G^\rho}{\alpha} \right) \right|^{2^*} \, dx \geq C|\Omega_\alpha|
$$

Hence, putting this with (2.36), we obtain

$$
\theta^{\frac{1}{2}} |\Omega_\alpha|^{\frac{1}{2^*}} \leq \frac{C}{\alpha^{\frac{1}{2}}}
$$

or

$$
\begin{aligned}
|\Omega_\alpha| &\leq \frac{C}{\alpha^{\frac{2^*}{2}}} \\
&= C\alpha^{\frac{-n}{n-2}}
\end{aligned}
$$

$\square$

We now state and prove a pointwise estimate for $G^\rho$.

**Theorem 2.4.4.** *Let $x \in \Omega$ be so that $|x - y| \geq 2\rho$. We have the estimate*

$$
G^\rho(x) \leq C|x - y|^{2-n}.
$$

*Proof.* We start by showing

$$
⨍_{B_r(x)} G^\rho \, dz \leq Cr^{2-n} \tag{2.37}
$$

for any $r$ and $x$ such that $B_r(x) \subset \Omega$. We have

$$
\begin{aligned}
⨍_{B_r(x)} G^\rho \, dz &\leq Cr^{-n} \int_0^\infty |\Omega_\alpha \cap B_r(x)| \, d\alpha \\
&= Cr^{-n} \left( \int_0^s + \int_s^\infty \right) |\Omega_\alpha \cap B_r(x)| \, d\alpha \\
&\leq Cr^{-n} \left( \int_0^s |B_r(x)| \, d\alpha + \int_s^\infty \alpha^{\frac{-n}{n-2}} \, d\alpha \right)
\end{aligned}
$$

for any $s > 0$, where we have used Lemma 2.4.3 in the last line. Choosing $s = r^{2-n}$, we obtain

$$\begin{aligned} \fint_{B_r(x)} G^\rho \, dz &\leq C\left(r^{2-n} + r^{-n} r^{(2-n)\left(\frac{-2}{n-2}\right)}\right) \\ &= C r^{2-n} \end{aligned}$$

so that (2.37) is true. So, if we set $r = \frac{|x-y|}{2}$ in (2.37), since we have $LG^\rho = 0$ in $\Omega \backslash B_\rho(y)$ for the mixed problem (2.1), we have by Theorem 2.4.2 that

$$\begin{aligned} G^\rho(x) &\leq C \fint_{B_r(x)} G^\rho \, dz \\ &\leq C r^{2-n} \\ &= C|x-y|^{2-n} \end{aligned}$$

as required. The proof for $\Omega_r(x) = B_r(x) \cap \Omega$ for $x \in \partial\Omega$ is similar. □

We now discuss Holder continuity for the Green function. We first state a definition, which is a slight modification of the definition of $\text{ß}_m(\overline{\Omega}, M, \gamma, \delta, \frac{1}{q})$ as in (2.8).

We say $u \in H^1(\Omega)$ belongs to $\widetilde{\text{ß}}_m(\overline{\Omega}_R, M, \gamma, \delta, \frac{1}{q})$ for $M, \gamma, \delta > 0$ and $q > n$ if $\|u\|_{L^\infty(\Omega_R)} \leq M$ and if both $u$ and $-u$ satisfy the following inequalities for an arbitrary concentric $\Omega_r \subset \Omega_R$ and arbitrary $\sigma \in (0, 1)$:

$$\int_{A_{k,r-\sigma r}} |\nabla u|^m \, dx \leq \gamma \left[\frac{1}{\sigma^m r^{m-\frac{mn}{q}}} \sup_{A_{k,r}} (u(x) - k)^m + 1\right] |A_{k,r}|^{1-\frac{m}{q}} \tag{2.38}$$

for $k$ satisfying both $k \geq 0$ and $k \geq \sup_{\Omega_r} u(x) - \delta$ if $\overline{\Omega}_r \cap D \neq \emptyset$ and for only $k \geq \sup_{\Omega_r} u(x) - \delta$ otherwise, where $A_{k,r} = \{x \in \Omega_r : u(x) > k\}$. Here $\Omega_{r-\sigma r}$ is concentric to $\Omega_r$ and $r \leq r_0$ for some positive $r_0$.

We note that the only difference in this definition is that the regions $\Omega_r$ are required to be concentric with the domain $\Omega_R$. With this definition, we state a corollary.

**Corollary 2.4.5.** *If $|x-y| \geq 2\rho$, then $G^\rho(x) \in \widetilde{\text{ß}}_m(\overline{\Omega}_R(x), M_R, \gamma, \delta, 0)$ where $\delta > 0$ is arbitrary, $\gamma = \gamma(n, \theta)$, $R = \frac{|x-y|}{4}$, and $M_R = \sup_{\overline{\Omega}_R(x)} G^\rho$.*

*Proof.* First, we note by Theorems 2.3.1 and 2.4.2 that $\|G^\rho\|_{L^\infty(\Omega_R)} \leq M$ for some $M$. Thus, since $LG^\rho = f = 0$ in $\Omega_R$, from (2.9), we have

$$\int_{A_{k,r}} a_{ij} u_{x_i} \phi_{x_j} \, dx = 0$$

where $\eta$ and $\phi$ are defined the same way as in Proposition 2.3.2. Then the same proof leads to the result. □

Now consider an analog of Lemma 2.3.7 for $G^\rho$:

**Corollary 2.4.6.** *Let $|x-y| \geq 2\rho$, $R = \frac{|x-y|}{4}$, and fix $\Omega_R(x) = B_R(x) \subset \Omega$. There exists a positive integer $s = s(n, \theta)$ so that for any $B_{4r} \subset B_R(x)$, concentric with $B_R(x)$, at least one of the following inequalities hold for $G^\rho$:*

1. $osc(G^\rho, B_r) \leq 2^s r$
2. $osc(G^\rho, B_r) \leq \left(1 - \frac{1}{2^{s-1}}\right) osc(G^\rho, B_{4r})$

*Proof.* The proof is almost the same as the proof of Lemma 2.3.7. The dependence on $s$ is different. One of the conditions on $s$ was (2.15). But, from Corollary 2.4.5, since $G^\rho(x) \in \widetilde{\mathfrak{B}}_m(\overline{\Omega}_R(x), M_R, \gamma, \delta, 0)$ for any $\delta > 0$, we may choose $\delta = M_R$ to omit this condition. Then, $s$ no longer depends on the bound $M_R$. Also, since $q = \infty$, $r^{1-\frac{n}{q}}$ becomes $r$. The rest of the proof goes without change. □

We also have an analog for Lemma 2.3.10.

**Corollary 2.4.7.** *Let $\Omega$ be a Lipschitz domain and assume the boundary conditions (2.2) and (2.3). Let $|x-y| \geq 2\rho$, $R = \frac{|x-y|}{4}$, and $R_0 = \min\{r_0, R\}$. For $r \leq R_0/16C$ and any $x \in \partial\Omega$, there exists a positive integer $s = s(n, \theta, C)$ so that for any $\Omega_{16Cr} \subset \Omega_{R_0}(x)$, concentric with $\Omega_{R_0}(x)$, at least one of the following inequalities hold for $G^\rho$:*

1. $osc(G^\rho, \Omega_r) \leq 2^s r$
2. $osc(G^\rho, \Omega_r) \leq \left(1 - \frac{1}{2^{s-1}}\right) osc(G^\rho, \Omega_{16Cr})$

*Here, $C$ depends on the Lipschitz constant $M$.*

*Proof.* Replace $u$ with $G^\rho$ in the proof of Lemma 2.3.10, with the only difference being that by Lemma 2.4.1, $k = M_{16Cr} - \frac{\omega_{16Cr}}{2^t}$ is always positive. □

We now state a theorem for Hölder continuity for $G^\rho$:

**Theorem 2.4.8.** *Let $\Omega$ be a Lipschitz domain and assume the boundary conditions (2.2) and (2.3). Let $|x-y| \geq 2\rho$, $R = \frac{|x-y|}{4}$, and $R_0 = \min\{r_0, R\}$. Then $G^\rho$ belonging to $\widetilde{\mathfrak{B}}_2(\overline{\Omega}_{R_0}, M_{R_0}, \gamma, M_{R_0}, 0)$ satisfies a Hölder condition in $\overline{\Omega}_{R_0}$. Moreover, there exists $\alpha$ such that $G^\rho$ satisfies the estimate*

$$|G^\rho(z) - G^\rho(z')| \leq C\left(\frac{|z-z'|}{R_0}\right)^\alpha \left(1 + \sup_{\Omega_{R_0}} |G^\rho(x)|\right), \qquad z, z' \in \Omega_{R_0} \tag{2.39}$$

*where $C$ and $\alpha$ both depend on $n$, $\theta$, $|\Omega|$, and the Lipschitz constant.*

*Proof.* The proof follows immediately from applying Corollaries 2.4.5, 2.4.6, and 2.4.7, and Theorem 2.4.2 with Lemma 2.3.6. □

Following an argument from Grüter and Widman, we will now show that there exists a Green function $G(\cdot, y)$ such that $G(\cdot, y) \in W_D^{1,s}(\Omega)$ for any $s \in [1, \frac{n}{n-1})$. Furthermore, this function $G(\cdot, y)$ is also in $W_D^{1,2}(\Omega \backslash B_r(y))$ for any $r > 0$. Then from the Hölder estimate (2.39), we also get a continuous extension of $G(\cdot, y)$ onto $\partial\Omega$.

For the next theorem, define weak $L^p$ for $p > 1$ as

$$L_p^*(\Omega) = \{f : f \text{ is measurable and } \|f\|_{L_p^*(\Omega)} < \infty\}$$

where

$$\|f\|_{L_p^*(\Omega)} = \sup_{t>0} t\, |\{x \in \Omega : |f(x)| > t\}|^{\frac{1}{p}}$$

**Theorem 2.4.9.** *Let $s \in [1, \frac{n}{n-1})$. There exists a sequence $G^{\rho_k}(\cdot, y)$ and a Green function $G(\cdot, y) \in W_D^{1,s}(\Omega) \cap W_D^{1,2}(\Omega \backslash B_r(y))$ such that $G^{\rho_k}(\cdot, y) \rightharpoonup G(\cdot, y)$ as $k \to \infty$.*

*Proof.* We will start by showing a weak $L^{\frac{n}{n-1}}(\Omega)$ estimate for $\nabla G^\rho$. That is,

$$\|\nabla G^\rho\|_{L^*_{\frac{n}{n-1}}(\Omega)} \leq C(n,L) \tag{2.40}$$

Define a cutoff function $\eta \in C^\infty(\mathbb{R}^n)$ to be so that $\eta = 1$ outside $B_{2r}(y)$, $\eta = 0$ in $B_r(y)$, and $|\nabla\eta| \leq \frac{C}{r}$. Then inserting the test function $\eta^2 G^\rho$ in the weak formulation for $G^\rho$ (2.29), we have

$$\int_\Omega a_{ij} G^\rho_{x_i}(\eta^2 G^\rho_{x_j} + 2\eta\eta_{x_j} G^\rho)\ dx = \fint_{B_\rho(y)} \eta^2 G^\rho\ dx$$

Using the ellipticity condition, we then obtain

$$\int_\Omega \theta\eta^2|\nabla G^\rho|^2\ dx \leq \fint_{B_\rho(y)} \eta^2 G^\rho\ dx + C\int_\Omega \eta|\nabla\eta||G^\rho||\nabla G^\rho|\ dx$$

Then if $r \geq 2\rho$, we have

$$\begin{aligned}
\int_{\Omega\setminus B_r(y)} \theta|\nabla G^\rho|^2\ dx &\leq C\int_{B_{2r}(y)\setminus B_r(y)} |\nabla\eta||G^\rho||\nabla G^\rho|\ dx \\
&\leq C\int_{B_{2r}(y)\setminus B_r(y)} \frac{|\nabla\eta|^2|G^\rho|^2}{\varepsilon}\ dx + \int_{B_{2r}(y)\setminus B_r(y)} \varepsilon|\nabla G^\rho|^2\ dx \\
&\leq \frac{C}{r^2\varepsilon}\int_{B_{2r}(y)\setminus B_r(y)} |G^\rho|^2\ dx + \int_{\Omega\setminus B_r(y)} \varepsilon|\nabla G^\rho|^2\ dx
\end{aligned}$$

where $\varepsilon > 0$. Then, choosing $\varepsilon = \frac{\theta}{2}$ and using the estimate from Theorem 2.4.4, we obtain

$$\begin{aligned}
\int_{\Omega\setminus B_r(y)} |\nabla G^\rho|^2\ dx &\leq \frac{C}{r^2}\int_{B_{2r}(y)\setminus B_r(y)} |G^\rho|^2\ dx \\
&= \frac{C}{r^2}\int_r^{2r}\int_{|x-y|=s} |x-y|^{4-2n}\ d\sigma(x)\ ds \\
&= \frac{C}{r^2}\int_r^{2r} s^{3-n}\ ds
\end{aligned}$$

so that

$$\int_{\Omega\setminus B_r(y)} |\nabla G^\rho|^2\ dx \leq Cr^{2-n} \tag{2.41}$$

If $r \le 2\rho$, we note that

$$\begin{aligned}
\theta \int_\Omega |\nabla G^\rho|^2 \, dx &\le \int_\Omega a_{ij} G^\rho_{x_i} G^\rho_{x_j} \, dx \\
&= \fint_{B_\rho(y)} G^\rho \, dx \\
&\le C\rho^{-n} \left( \int_{B_\rho(y)} (G^\rho)^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{2n}} \rho^{\frac{n(n+2)}{2n}} \\
&\le C\rho^{\frac{2-n}{2}} \left( \int_\Omega |\nabla G^\rho|^2 \, dx \right)^{\frac{1}{2}}
\end{aligned}$$

so that (2.41) holds for all $r > 0$. Next, defining $\Omega_t := \{x \in \Omega : |\nabla G^\rho(x)| > t\}$ and setting $r = t^{-\frac{1}{n-1}}$, then by Chebyshev's inequality and (2.41), we have

$$t^2 |\Omega_t \cap (\Omega \backslash B_r(y))| \le C t^{\frac{n-2}{n-1}}$$

which is equivalent to

$$|\Omega_t \cap (\Omega \backslash B_r(y))| \le C t^{\frac{-n}{n-1}} \tag{2.42}$$

Also,

$$|\Omega_t \cap B_r(y)| \le C r^n = C t^{\frac{-n}{n-1}}$$

Combining this with (2.42) gives the weak $L^{\frac{n}{n-1}}(\Omega)$ estimate for $\nabla G^\rho$, (2.40).

Next, we claim that

$$\|f\|_{L^{p-\varepsilon}(\Omega)} \le |\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}} \left( \frac{p-\varepsilon}{\varepsilon} \right)^{\frac{1}{p}} \|f\|_{L^*_p(\Omega)} \tag{2.43}$$

for $0 < \varepsilon \le p-1$. This is true since

$$\begin{aligned}
\|f\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega)} &= (p-\varepsilon) \int_0^\infty \alpha^{p-\varepsilon-1} |\{|f| > \alpha\}| \, d\alpha \\
&= (p-\varepsilon) \left( \int_0^A + \int_A^\infty \right) \alpha^{p-\varepsilon-1} |\{|f| > \alpha\}| \, d\alpha \\
&= I + II
\end{aligned}$$

We have

$$I \le (p-\varepsilon)|\Omega| \int_0^A \alpha^{p-\varepsilon-1} \, d\alpha = |\Omega| A^{p-\varepsilon} \tag{2.44}$$

and

$$
\begin{aligned}
II &= (p-\varepsilon)\int_A^\infty \alpha^{p-\varepsilon-1}|\{|f|>\alpha\}|\,d\alpha \\
&\le (p-\varepsilon)\|f\|^p_{L^*_p(\Omega)}\int_A^\infty \alpha^{-\varepsilon-1}\,d\alpha \\
&= (p-\varepsilon)\|f\|^p_{L^*_p(\Omega)}\frac{A^{-\varepsilon}}{\varepsilon}
\end{aligned} \tag{2.45}
$$

Choosing $A = \left(\frac{(p-\varepsilon)\|f\|^p_{L^*_p(\Omega)}}{|\Omega|\varepsilon}\right)^{\frac{1}{p}}$, we obtain from (2.44) and (2.45) that

$$
\|f\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega)} \le |\Omega|^{\frac{\varepsilon}{p}}\left(\frac{p-\varepsilon}{\varepsilon}\right)^{\frac{p-\varepsilon}{p}}\|f\|^{p-\varepsilon}_{L^*_p(\Omega)} \tag{2.46}
$$

and hence, (2.43). So, we may use (2.40) and (2.43) with $p = \frac{n}{n-1}$ and $\varepsilon = p - s$ to obtain

$$
\|\nabla G^\rho\|_{L^s(\Omega)} \le C\|\nabla G^\rho\|_{L^*_{\frac{n}{n-1}}(\Omega)} \le C(n, L, s, |\Omega|) \tag{2.47}
$$

where $s \in [1, \frac{n}{n-1})$.

Next, define $s_k = \frac{n}{n-1} - \frac{1}{k}$ and choose a sequence $\rho_{l_1}$ which tends to 0 as $l_1 \to \infty$. Then from the estimate (2.47) and (2.41), the sequence $\{G^{\rho_{l_1}}\}$ is bounded in $W^{1,s_1}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$. So, by weak compactness, there exists a subsequence $\{G^{\rho_{l_1 l_2}}\}$ and a function $G(\cdot, y) \in W^{1,s_1}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$ such that $G^{\rho_{l_1 l_2}}(\cdot, y) \rightharpoonup G(\cdot, y)$ in $W^{1,s_1}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$ as $l_2 \to \infty$. Similarly, the sequence $\{G^{\rho_{l_1 l_2}}\}$ is bounded in $W^{1,s_2}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$. So, there exists a subsequence $\{G^{\rho_{l_1 l_2 l_3}}\}$ such that $G^{\rho_{l_1 l_2 l_3}}(\cdot, y) \rightharpoonup G(\cdot, y)$ in $W^{1,s_2}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$ as $l_3 \to \infty$. Using an inductive argument, we see that for each $k$, there exists a subsequence $\{G^{\rho_{l_1 \cdots l_{k+1}}}\}$ such that $G^{\rho_{l_1 \cdots l_{k+1}}}(\cdot, y) \rightharpoonup G(\cdot, y)$ in $W^{1,s_k}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$ as $l_{k+1} \to \infty$. So if we define the sequence $G^{\rho_k} = G^{\rho_{l_1 \cdots l_{k-1} k}}$, then given any $s \in [1, \frac{n}{n-1})$, we have that $\{G^{\rho_k}\}$ converges weakly to $G(\cdot, y)$ in $W^{1,s}_D(\Omega) \cap W^{1,2}_D(\Omega\backslash B_r(y))$. □

**Theorem 2.4.10.** *Given any $s \in [1, \frac{n}{n-1})$, the function $G(\cdot, y)$ solves the mixed problem (2.1) with $f = \delta_y$ ($\delta_y$ being the Dirac−$\delta$ measure at $y$), $f_D = 0$, and $f_N = 0$*

*in the sense that*

$$\int_\Omega a_{ij}(x) G_{x_i}(x,y)\phi_{x_j}(x)\ dx = \phi(y) \qquad \text{for any } \phi \in W_D^{1,s'}(\Omega)\cap C(\Omega)$$

*where $s'$ is the Hölder conjugate of $s$.*

*Proof.* Consider the sequence $\{G^{\rho_k}\}$ from the proof of Theorem 2.4.9. Then from the weak formulation for $G^{\rho_k}(\cdot,y)$ (2.29), we have

$$\int_\Omega a_{ij}(x) G_{x_i}^{\rho_k}(x,y)\phi_{x_j}(x)\ dx = \fint_{B_{\rho_k}(y)} \phi(x)\ dx$$

The right side converges to $\phi(y)$ as $k\to\infty$ since $\phi$ is continuous. Also, from Theorem 2.4.9, since

$$\begin{aligned}\langle A,\varphi\rangle &= \int_\Omega a_{ij}(x)\varphi_{x_i}(x)\phi_{x_j}(x)\ dx\\ &\le C\|\nabla\varphi\|_{L^s(\Omega)}\|\nabla\phi\|_{L^{s'}(\Omega)}\end{aligned}$$

is a bounded linear functional on $W_D^{1,s}(\Omega)$, we have

$$\int_\Omega a_{ij}(x)(G_{x_i}^{\rho_k}(x,y) - G_{x_i}(x,y))\phi_{x_j}(x)\ dx \to 0, \qquad \text{as } k\to\infty$$

thus, giving the result. □

We note that Theorem 2.4.8 implies that $G^{\rho_k}$ extends continuously to $\partial\Omega$. Also, from the pointwise bound 2.4.4, we have a uniform bound for the Hölder norm of each $G^{\rho_k}$ on compact sets of $\overline{\Omega}\backslash\{y\}$. Hence, from Rudin [33, p. 158], for each $y$, we may find a subsequence $\rho_k$ tending to 0 such that $G^{\rho_k}(\cdot,y)$ converges uniformly to $G(\cdot,y)$ on compact subsets of $\overline{\Omega}\backslash\{y\}$. This implies that $G(\cdot,y)$ is Hölder continuous in $\overline{\Omega}\backslash\{y\}$. Furthermore, in light of Theorem 2.4.4, we have

$$G(x,y) \le C|x-y|^{2-n}, \qquad x\ne y \tag{2.48}$$

We have the following representation theorem for solutions to the mixed problem with zero Dirichlet data.

**Theorem 2.4.11.** *Given any $s \in [1, \frac{n}{n-1})$, if $u$ is a weak solution to the mixed problem (2.1) with $f_D = 0$, $f_N \in W_D^{-1/2,2}(\partial\Omega)$, and $f \in L^{s'}(\Omega)$, then*

$$u(y) = \int_\Omega f(x)G(x,y)\ dx + \langle f_N, G(\cdot, y)\rangle_N. \tag{2.49}$$

*Moreover, this function $G$ is unique.*

*Proof.* From the above discussion, there is a sequence $\{G^{\rho_k}\}$ from the proof of Theorem 2.4.9 which also converges uniformly on compact subsets of $\overline{\Omega}\backslash\{y\}$. Since $u \in W_D^{1,2}(\Omega)$ is an acceptable test function in the weak formulation for $G^{\rho_k}(\cdot, y)$ (2.29), we have

$$\int_\Omega a_{ij}(x)G^{\rho_k}_{x_i}(x,y)u_{x_j}(x)\ dx = \fint_{B_{\rho_k}(y)} u(x)\ dx.$$

Also, from the weak formulation for $u$ (2.4),

$$\int_\Omega a_{ij}(x)u_{x_i}(x)G^{\rho_k}_{x_j}(x,y)\ dx = \int_\Omega f(x)G^{\rho_k}(x,y)\ dx + \langle f_N, G^{\rho_k}(\cdot, y)\rangle_N$$

Thus, from the symmetry condition (2.30), we have

$$\fint_{B_{\rho_k}(y)} u(x)\ dx = \int_\Omega f(x)G^{\rho_k}(x,y)\ dx + \langle f_N, G^{\rho_k}(\cdot, y)\rangle_N.$$

The left side converges to $u(y)$ as $k$ tends to $\infty$ by Lebesgue's differentiation theorem. Also, Theorem 2.4.9 implies

$$\int_\Omega f(x)G^{\rho_k}(x,y)\ dx \to \int_\Omega f(x)G(x,y)\ dx, \quad k \to \infty$$

and the uniform convergence of $\{G^{\rho_k}\}$ implies

$$\langle f_N, G^{\rho_k}(\cdot, y)\rangle_N \to \langle f_N, G(\cdot, y)\rangle_N, \quad k \to \infty$$

To show uniqueness, we adopt the definition of weak solution taken from Littman, Stampacchia, and Weinberger [26]. For a measure $\mu$ of bounded variation on $\Omega$, we say that $w \in L^1(\Omega)$ is a very weak solution of the mixed problem $Lw = \mu$ with zero Neumann data and zero Dirichlet data if

$$\int_\Omega w\psi\ dx = \int_\Omega \phi\ d\mu \tag{2.50}$$

for every $\phi \in C(\overline{\Omega})$ satisfying the mixed problem (2.1) with $f = \psi \in C(\overline{\Omega})$, $f_D = 0$, and $f_N = 0$.

If $\psi \in \left(W^{1,2}_D(\Omega)\right)'$, then the Lax-Milgram theorem gives the existence of a unique weak solution $\phi \in W^{1,2}_D(\Omega)$ to the mixed problem (2.1) with $f = \psi$, $f_D = 0$, and $f_N = 0$. Furthermore, from Corollary 2.3.11, if $\psi \in C(\overline{\Omega})$, then $\phi \in C(\overline{\Omega})$. So from (2.49), given any $\psi \in C(\overline{\Omega})$, we have

$$\phi(y) = \int_\Omega \psi(x)G(x,y)\ dx \tag{2.51}$$

We also know that there exists a unique function $\widetilde{G}(\cdot,y) \in W^{1,2}_D(\Omega\backslash B_r(y)) \cap W^{1,1}_D(\Omega)$ which is a very weak solution to

$$\begin{cases} L\widetilde{G} = \delta_y & \text{in } \Omega \\ \widetilde{G} = 0 & \text{on } D \\ \frac{\partial \widetilde{G}}{\partial \nu} = 0 & \text{on } N \end{cases}$$

where $\delta_y$ is the Dirac-$\delta$ measure at $y$. That is,

$$\phi(y) = \int_\Omega \psi_(x)\widetilde{G}_(x,y)\ dx \tag{2.52}$$

So, from (2.51) and (2.52), we have

$$\int_\Omega \psi(x)(G(x,y) - \widetilde{G}(x,y))\ dx = 0, \quad \text{for any } \psi \in C(\overline{\Omega}) \tag{2.53}$$

This implies $G = \widetilde{G}$, thus giving uniqueness of the Green function. $\square$

**Future Work**

We close this chapter with a list of questions.

- Can we study the fundamental solution for the Lamé system?
- Can we study the fundamental solution for general elliptic systems?

- Can we study the fundamental solution for the Robin problem?

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- Max Steckler Fellowship, awarded by the Mathematics Department of the University of Kentucky (2009, 2010).
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Mathematics Publications

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