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GENERAL FLIPS AND THE CD-INDEX

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ABSTRACT OF DISSERTATION

Daniel J Wells

The Graduate School
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2010

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ABSTRACT OF DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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at the University of Kentucky

By
Daniel J Wells
Lexington, Kentucky

Director: Dr. Carl W Lee, Professor of Mathematics
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GENERAL FLIPS AND THE **CD**-INDEX

We generalize bistellar operations (often called flips) on simplicial manifolds to a notion of general flips on PL-spheres. We provide methods for computing the **cd**-index of these general flips, which is the change in the **cd**-index of any sphere to which the flip is applied. We provide formulas and relations among flips in certain classes, paying special attention to the classic case of bistellar flips. We also consider questions of “flip-connecticity“, that is, we show that any two polytopes in certain classes can be connected via a sequence of flips in an appropriate class.

KEYWORDS: combinatorial geometry, polytopes, bistellar flips, **cd**-index, PL-spheres

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Chapter 1 History

1.1 Introduction

Bistellar flips on simplicial complexes have been studied and used in a number of settings, both theoretical and practical. In this dissertation we will consider a non-simplicial generalization of bistellar flips and study the **cd**-index of this class of operations.

In Chapter 1 we present some basic definitions and a brief historical summary of major results regarding bistellar flips, shellings, and face numbers and flag numbers. These topics provide the motivation for this research. References are given for further research into these topics.

In Chapter 2 we provide an overview of ideas and results of this dissertation with some discussion of the broader context. This is meant to serve as a brief introduction and so we leave formal definitions and technicalities for later chapters.

Chapter 3 defines general flips and studies their **cd**-indices. General flips are a generalization of bistellar operations, often called *flips*, on non-simplicial complexes. The **cd**-index of a flip records how the flip affects the **cd**-index of a PL-sphere, such as the boundary complex of a polytope, that it is applied to. We present formulae which can be used to compute the **cd**-index of any flip, as well as formulae for several special classes of general flips. We pay special attention to the **cd**-index of the classic bistellar flips. An enumeration result is given for flips that are "almost bistellar".

Chapter 4 turns to the issue of flip connectivity. We present some results showing any pair of polytopes, or PL-spheres in certain classes, can be connected with a sequence of general flips belonging to certain classes. We also address the question of monotone flips and sequences of flips that are monotone with respect to the **cd**-index.

1.2 Bistellar operations

Bistellar operations, often called *flips*, have been defined in a number of settings. Regardless of the setting, bistellar flips are a certain class of minimal local changes to a simplicial structure. In the (combinatorial) topological setting, the simplicial structure is a pure abstract simplicial complex. In geometric settings, the simplicial structures may be restricted to the boundaries of simplicial polytopes. As an applied tool, bistellar flips have been employed in computational geometry and in the modeling of surfaces and other manifolds. They can be also applied to several algebraic settings, including the study of toric varieties. Bistellar flips can also be generalized to general oriented matroids that may not be realizable as real point configurations. See [30] for a more detailed discussion of bistellar flips in these and other settings.

We will consider the topological and geometric settings here and summarize some major results.

1.2.1 Bistellar operations and simplicial complexes

A (finite) simplicial complex is a non-empty collection of subsets of a finite vertex set that is closed under inclusion. The elements of the simplicial complex are called *faces* or *simplices*. The dimension of a simplex is one less than the cardinality of its vertex set. We call a complex *pure* if all maximal faces have the same dimension. We will consider only pure complexes except as otherwise noted. The dimension of a pure complex is the dimension of its maximal faces. Note that for both pure and non-pure complexes, a complex can be determined by a list of its maximal faces, called *facets*. Faces of dimension zero are called *vertices*; those of dimension one are *edges*. Faces that are one dimension below facets are called *subfacets*. We write $F \prec G$ if F is a facet of G, and $E \prec\!\!\prec G$ if E is a subfacet of G. Note that the empty set is considered to be a face of dimension -1.

Each abstract simplicial complex has an associated topological space equipped with a regular cell decomposition where each abstract simplex corresponds to a geometric simplex of the same dimension and the geometric simplices are identified along their common faces. We will use the same terminology for both the abstract and topological complexes, making a distinction only where there is danger of confusion. We will use *closure* to mean the closure under inclusion of the abstract complex C and denote it by $\overline{C}$.

There are several notions that are useful in discussing simplicial complexes and bistellar flips. The *star* of a face σ in a simplicial complex C is

$$\mathrm{st}(\sigma) = \{\tau \in C : \sigma \subseteq \tau\},$$

that is, the set of all faces containing σ. The star is generally not a simplicial complex, since it is not necessarily closed under inclusion. The closed star $\overline{\mathrm{st}}(\sigma)$ is closure under inclusion of the star $\mathrm{st}(\sigma)$, and thus is a valid complex.

The *link* of a face σ of C is

$$\mathrm{lk}(\sigma) = \overline{\mathrm{st}}(\sigma) \setminus \mathrm{st}(\sigma).$$

That is, the $\mathrm{lk}(\sigma)$ consists of faces of C disjoint from σ to which σ may be added to obtain an face of C.

If the link of each vertex of a pure simplicial complex is a PL-sphere or PL-ball, then we call the complex a *combinatorial manifold (with boundary)*.

The *boundary* of a simplicial manifold C is the closure of the set of subfacets of C that are contained in only one facet. We denote this set by ∂C. The boundary of the boundary of a simplicial manifold is the empty set. Note that this means that, strictly speaking, $\partial\partial C$ is not a simplicial complex.

The *free join* of two complexes C and K with disjoint vertex sets is

$$C * K = \{\sigma \cup \tau : \sigma \in C, \tau \in K\}.$$

Intuitively, the free join is obtained by embedding C and K in disjoint, non-parallel affine subspaces and connecting every pair of simplices, one from each complex, to form new simplices. The free join of simplicial complexes corresponds to the Cartesian product of their face posets. The boundary operation interacts with the free join in a type of Leibnitz' rule. For disjoint complexes C and K,

$$\partial(C * K) = (\partial C * K) \cup (C * \partial K)\,.$$

Observe that for any face σ in a pure simplicial complex,

$$\overline{\mathrm{st}}(\sigma) = \sigma * \mathrm{lk}(\sigma).$$

We can now succinctly define bistellar flips in the setting of abstract simplicial complexes.

Definition 1.2.1. Two simplicial complexes C and C' are said to be related by a *bistellar flip* if there are simplices $\sigma \in C \setminus C'$ and $\tau \in C' \setminus C$ such that

1. $C \setminus \mathrm{st}\,\sigma = C' \setminus \mathrm{st}\,\tau$,
2. $\mathrm{lk}\,\sigma = \partial\tau$ in C,
3. $\mathrm{lk}\,\tau = \partial\sigma$ in C'.

The combinatorial type of the bistellar flip is determined by the dimensions of the simplices σ and τ, or equivalently, by the dimensions of C and σ. The two simplices σ and τ will always have complementary dimensions so that $\dim(\sigma)+\dim(\tau) = \dim(C)$. Thus there are $d + 1$ combinatorial types of (directed) bistellar flips in dimension d.

Figures 1.1 and 1.2 show all of the bistellar flips possible in two and three dimensions, respectively.

1.2.2 Bistellar operations and polytopes

The "bistellar" in *bistellar flip* is there because a bistellar flip can be achieved by a stellar subdivision followed by an appropriate inverse stellar subdivision. In their 1974 paper ([14]), Ewald and Shephard investigate geometric stellar equivalences of boundary complexes of polytopes. In the proof of the main theorem they construct a sequence of pairs of stellar and inverse stellar operations that amount to bistellar flips. However, Ewald and Shephard did not use the term bistellar.

Ewald and Shephard define boundary complexes K and K' of convex polytopes to be *geometrically stellar equivalent* if there is a sequence of complexes

$$K = K_0, K_1, K_2, \ldots, K_k, K_{k+1} = K'$$

such that for each pair of consecutive complexes, one can be obtained from the other by a single geometric stellar subdivision. A *geometric stellar subdivision* is an operation isomorphic to the change that occurs to the boundary complex by adding a single new vertex just beyond one face of a convex polytope but beneath all other faces (see [15] for technical definitions of beneath and beyond) and taking the convex hull. Note that this definition allows one to pick a new embedding of the polytope between each stellar subdivision or inverse stellar subdivision. Ewald and Shephard prove the following theorem.

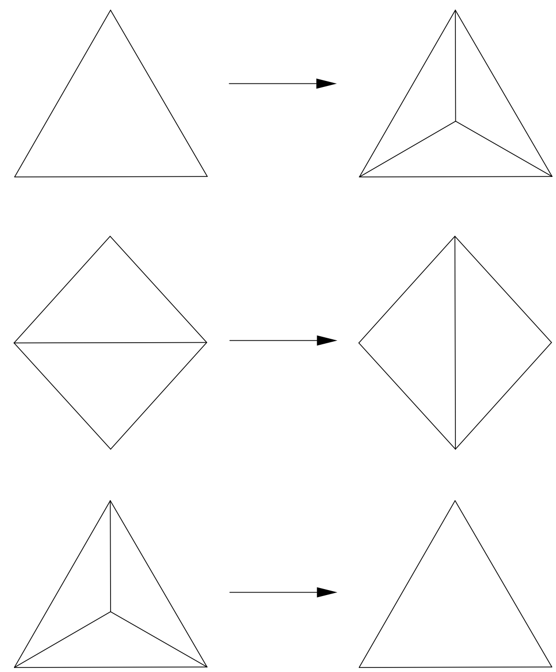

Figure 1.1: Two dimensional bistellar flips

Theorem 1.2.2 (Ewald–Shephard 1974, [14])**.** *The boundary complex of any d-dimensional convex polytope is geometrically stellar equivalent to the boundary complex of any other d-dimensional convex polytope.*

Note in particular that this theorem does not specify that the polytopes must be simplicial. However, the proof deals only with simplicial polytopes, since every polytope is geometrically stellar equivalent to its complete barycentric subdivision, which is simplicial. The core idea of the proof is to superimpose scaled copies of the two simplicial polytopes in "strong general position" with respect to each other and then watch the changes to the boundary complex of the convex hull of their union as the initially small polytope grows larger and the initially large polytope shrinks. They show that when the combinatorial type of the boundary complex changes, there is a single stellar subdivision of the old complex isomorphic to a single stellar subdivision of the new complex. From our perspective, the interesting thing to note is that the natural changes in the proof are in fact *bistellar* changes, not just stellar ones.

In 1978 Ewald improved upon the previous result and gave an upper bound on the number of stellar and inverse stellar operations needed to obtain (the boundary of) a simplicial polytope from (the boundary of) a simplex.

Theorem 1.2.3 (Ewald 1978, [13])**.** *If P is a simplicial d-polytope and P' is a d-simplex, there is a chain of k stellar operations or their inverses from P to P',*

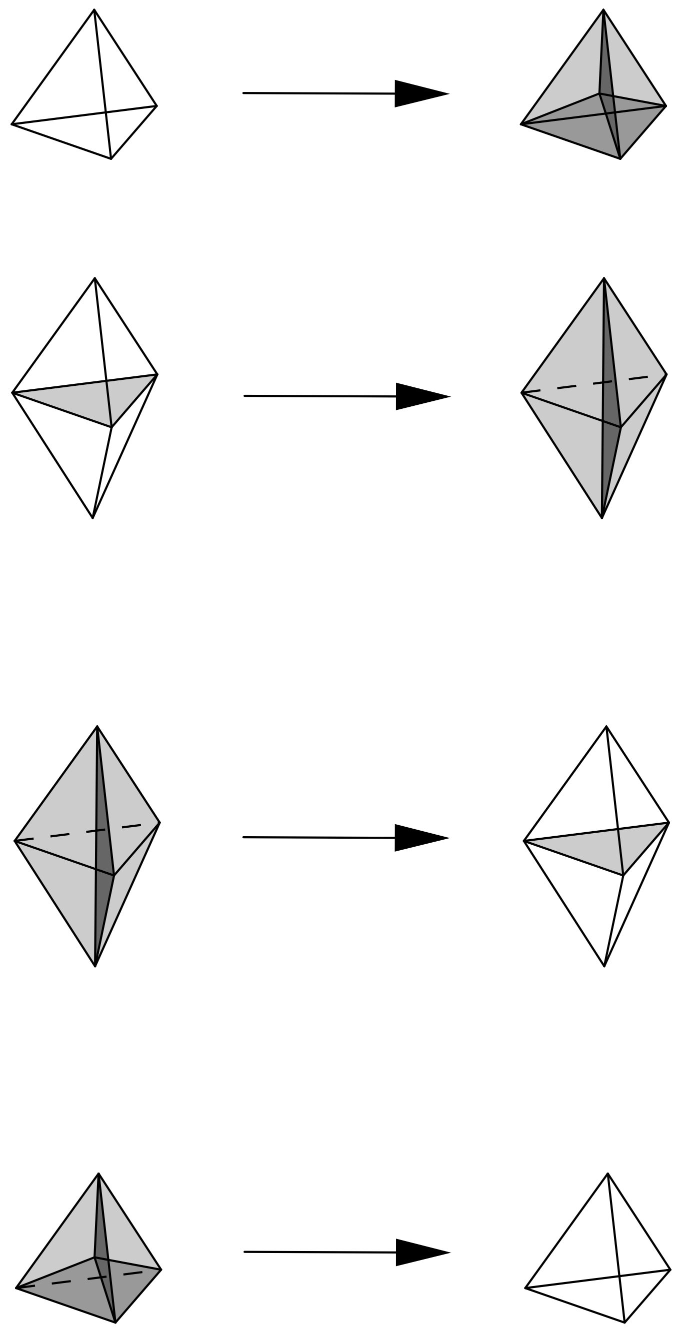

Figure 1.2: Three dimensional bistellar flips

where $k \leq 2(f_{d-1}(P) - v_{\max}(P)) + d - 1 - f_0(P)$. Here $v_{\max}(P)$ is the largest number of facets which contain a common vertex of P.

Also in 1978, Pachner published a paper ([23]) that specifically looks at the *bistellar* equivalence of simplicial polytopes. He shows that for $d \leq 4$, all combinatorial (simplicial) d-spheres are bistellar equivalent to the boundary of a $(d+1)$-simplex. By relating all simplicial polytopes to the appropriate stacked polytope, Pachner also proves the following theorem.

Theorem 1.2.4 (Pachner 1978, [23])**.** *If P and Q are two simplicial polytopes in the same dimension with the same number of vertices, then the boundary complex of one can be obtained from the boundary complex of the other by a sequence of bistellar flips that preserves the number of vertices.*

Pachner later refines the results on bistellar equivalence of simplicial polytopes in a 1981 paper [24]. He shows that the minimal number of bistellar flips required to obtain a simplicial polytope with a given h-vector is at least $h_{\lfloor \frac{d}{2} \rfloor} - 1$. Pachner shows that this bound is tight for $d \leq 5$, and conjectured that it is tight for $d \geq 6$ as well. Lee [17] later showed that this is indeed the case.

1.2.3 Bistellar flips and PL-manifolds

A *PL-manifold* is a topological manifold equipped with a piecewise-linear structure. For our purposes, it can be assumed that this structure is determined by a particular cellular structure on the manifold.

The following results demonstrate that PL-manifolds behave in a manner generally consistent with intuition.

Theorem 1.2.5.

1. *The union of two d-dimensional PL-balls attached along their common boundary is a d-dimensional PL-sphere.*
2. *The closure of the complement of a d-dimensional PL-ball embedded in a d-dimensional PL-sphere is a d-dimensional PL-ball.*
3. *The union of two d-dimensional PL-balls, attached along a $(d-1)$-dimensional PL-ball on their boundary is a d-dimensional PL-ball.*
4. *The free join of two PL-balls is a PL-ball.*
5. *The free join of a PL-ball with a PL-sphere is a PL-ball.*
6. *The free join of two PL-spheres is a PL-sphere.*

See section 4.7 of [7], [16], and [29] for these and other fundamental results in PL-topology.

In a 1987 paper [25], Pachner studies bistellar flips in a broader setting of simplicial PL-manifolds. A *PL-manifold* is a topological manifold (with boundary) equipped with an atlas having charts related by piecewise linear functions. A *simplicial PL-manifold* is a particular triangulation of a PL-manifold.

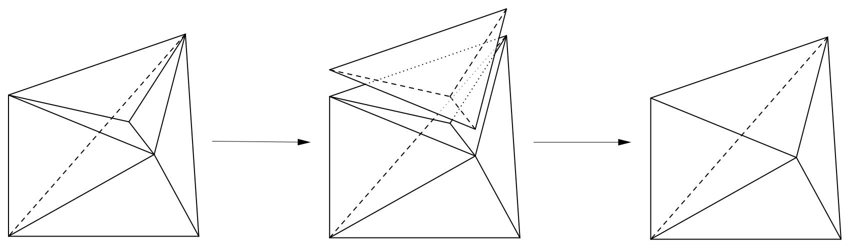

Figure 1.3: A shelling operation

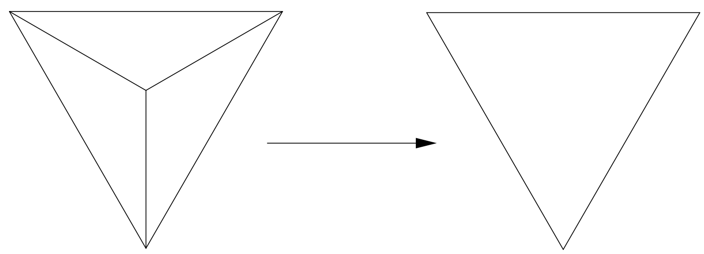

Figure 1.4: A bistellar flip

Theorem 1.2.6 (Pachner 1987, [25])**.** *Two compact (boundary-less) simplicial PL-manifolds are PL-homeomorphic if and only if they are bistellar equivalent.*

Theorem 1.2.7 (Pachner 1987, [25])**.** *Every simplicial PL-sphere can be obtained from the boundary of a simplex by a finite sequence of bistellar flips.*

This latter result is strengthened in the same paper to show that every simplicial PL-sphere is the boundary complex of some shellable ball.

The ideas of shelling moves and bistellar flips are closely related. Pachner refers to bistellar flips as "the inner equivalent to shellings". Shelling and inverse shelling operations consist of gluing on or removing facets which meet the boundary of a manifold, with certain technical restrictions. Note that the last step of shelling a sphere, which changes the complex from a ball to a sphere, is not considered a shelling move in this context. Pachner's definition of shelling and inverse shelling moves ensures that these moves do not change the topology of the manifold. While shellings deal with single facets along the boundary of a manifold, bistellar flips deal with multiple facets in the interior of a manifold. Further, shelling and inverse shelling operations induce bistellar flips on the boundary complex. Figures 1.3 and 1.4 show a shelling operation and the corresponding bistellar flip.

Together, shelling operations and bistellar flips provide a combinatorial description of PL-homeomorphism for PL-manifolds.

Let us recall here what it means for a cell complex to be shellable.

Definition 1.2.8. A pure cell complex S is *shellable* if there is a *shelling order* of its facets $F_1, F_2, F_3, \ldots, F_s$ such that

1. $\overline{F_1}$ is shellable;
2. $\forall 2 \leq k \leq s, \overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$ is shellable of one less dimension;
3. $\overline{F_k}$ can be shelled starting with the facets of $\overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$.

Any ordering of a finite collection of vertices is a shelling order.

We say that the facets of $\overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$ (subfacets of S) are *buried* by F_k. These are faces that cease to be part of the boundary when F_k is shelled on. A complex $\bigcup_{i=1}^{k} \overline{F_i}$, where $1 \leq k \leq s$, is called an *initial shelling segment* of S. Likewise, the complex $\bigcup_{i=k}^{s} \overline{F_i}$ is called a *final shelling segment* of S.

We will employ a technique of Stanley's called *S-shelling* which was used in [35]. At each step of the shelling process, we "cap" the complex by adding a single new facet glued to the boundary of the ball to form a sphere. In general the *semi-suspension* of a PL-ball is this PL-sphere obtained by "capping" the ball with a single new facet glued to the boundary of the ball.

In 1971, Bruggesser and Mani [8] showed that all convex polytopes are shellable. Consider a polytope P with a line passing through its interior so that the line does not intersect more than one of the supporting hyperplanes of the facets of P in one place. Consider a point starting on the line in the interior of P and moving outwards to infinity, and then returning from infinity to the interior along the other half of the line. The point will pass through the supporting hyperplanes of the facets one by one. This order on the facets induces a shelling order and is called a *line shelling*. There is a subtle difference between S-shellablility and the standard definition of shellability, however line shellings of polytopes satisfy both definitions.

Theorem 1.2.9 (Pachner 1990, [26])**.** *Two PL-manifolds with boundary are PL-homeomorphic if and only if they are related by a sequence of shelling, inverse shelling, and bistellar operations.*

Theorem 1.2.10 (Pachner 1990, [26])**.** *Two PL-balls, or two PL-manifolds with spherical boundaries are PL-homeomorphic if and only if they are related by a sequence of shelling and inverse shelling operations.*

Theorem 1.2.11 (Casali 1995, [9])**.** *Two PL-manifolds that coincide along their boundaries are PL-homeomorphic if and only if they are bistellar equivalent.*

1.3 Face numbers and flag numbers

1.3.1 Face numbers

The study of the face numbers began with Euler and his famous polyhedral relation. In modern terminology, it is stated as follows.

Theorem 1.3.1 (Euler's Polyhedral Relation)**.** *If P is a three dimensional polytope, then*

$$f_0(P) - f_1(P) + f_2(P) = 2.$$

The f_i here are the *face numbers* or f-vector, which is simply a record of the number of faces in each dimension. More precisely, $f = (f_{-1}, f_0, f_1, \dots)$ where

$$f_i = f_i(C) = \text{number of } i\text{-faces of } C.$$

Trying to understand and characterize the f-vectors of polytopes has been a major goal in the study of polytopes. A generalization of Euler's Polyhedral Relation was discovered, with many attempts at proofs, between Schläfli's discovery in 1852 [31], arguably the first, and Poincaré's 1899 proof [27, 28], now recognized as the first complete proof. Other earlier attempts relied on the then unproven assumption that the boundary complexes of polytopes were always shellable.

Theorem 1.3.2 (Euler–Poincaré–Schläfli Relation)**.** *If P is a convex d-polytope with f_i i-faces, then*

$$\sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d.$$

More on this relation and its history can be found in Chapter 8 of [15].

The next great step forward in the quest to characterize the f-vectors of polytopes came with Steinitz's 1906 publication of a complete characterization for three dimensions.

Theorem 1.3.3 (Steinitz, [36])**.** *The f-vectors of convex 3-polytopes are exactly those integer vectors $(f_0, f_0 + f_2 - 2, f_2)$ such that*

$$\frac{1}{2} f_0 + 2 \leq f_2 \leq 2 f_0 - 4.$$

The situation in four dimensions has proven more complicated, and no analogous result has yet been shown.

A new set of relations were conjectured for simplicial polytopes by Dehn [10] in 1905 and proven in 1927 by Sommerville [32]. This relation is especially simple and beautiful when expressed in terms of the h-vector.

If we let $F(t) = \sum_{j=0}^{d} f_{j-1} t^j$ be the generating function for the f-vector, then the h-vector of a simplicial polytope can be defined as the coefficients in the related generating function $(1-t)^d F(\frac{t}{1-t}) = \sum_i h_i t^i$. Explicitly,

$$h_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{d-i}{d-j} f_{j-1}.$$

Now we can state the Dehn–Sommerville Relations.

Theorem 1.3.4 (Dehn–Sommerville Relations)**.** *If P is a simplicial d-polytope with h-vector $(h_0, h_1, h_2, \ldots, h_d)$, then for $0 \leq k < \frac{d}{2}$*

$$h_k = h_{d-k}.$$

The next flurry of activity in the area occurred in the 1970s. In 1970 Barnette [1] proved the Lower Bound Conjecture for simplicial polytopes, providing a tight lower bound for the number of faces in all dimensions for d-dimensional polytopes with a given number of vertices. This was followed a few months later by McMullen's proof [20] of the Upper Bound Conjecture. The Upper Bound Theorem asserts that the cyclic polytope has the maximum number of faces in all dimensions of any polytope in the same dimension with the same number of vertices. McMullen's proof is notable not only for the result, but for introducing the idea of h-vectors of simplicial polytopes. Our h_i is equivalent to McMullen's g_{i-1}^{d+1}, and this g^{d+1} vector was one of a sequence of invariants.

The Upper Bound Theorem can be stated (for simplicial polytopes, which implies the general result) as $h_k \leq \binom{n-d+k-1}{k}$ for all simplicial d-polytopes with n vertices. McMullen, with Walkup, then proposed, and proved for some special cases, the Generalized Lower Bound Conjecture which asserts that the h-vector of simplicial polytopes is unimodal [22].

The culmination of this work was the publication of proofs in 1981 of the sufficiency by Billera and Lee [6], and necessity by Stanley [33], of a characterization conjectured by McMullen in 1971. The result, now known as the g-Theorem, provides a complete characterization of the h-vectors, and therefore f-vectors, of simplicial polytopes.

1.3.2 Flag vectors

For simplicial d-polytopes the g-vector ($g_i = h_i - h_{i-1}$, $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$; $g_0 = h_0 = 1$) is the final say in combinatorial invariants. It is, in a sense, a "minimal encoding" of the f-vector, and the f-vector carries a lot of combinatorial information. However, for non-simplicial polytopes in dimensions three and above, there is more combinatorial information to be had. The h-vector has been extended in several different useful ways, but there is no single obviously "correct" definition.

In 1985, Bayer and Billera [3] gave a generalization of the Dehn-Sommerville relations for polytopes, spheres, and Eulerian posets. This generalization is given in terms of the flag h-vector, which they called the *extended h-vector*. For an introduction to posets (partially ordered sets) see [**?**].

The flag f-vector of a graded poset of rank d having a minimal and a maximal element is a vector indexed by subsets of ranks. For each subset $S = \{s_1, s_2, s_3, \ldots, s_n\} \subseteq \{0, \ldots, d-1\}$, f_S is the number of chains of poset elements $x_1 < x_2 < x_3 < \cdots < x_n$, where the rank of x_i is $s_i + 1$. The flag f-vector of a polytope is given by considering the poset to be the faces ordered by inclusion. The rank of a face is its dimension plus one.

Example 1.3.5. Consider a hexagonal prism. We compute the flag f-vector:

S	f_S
$\varnothing$	1
$\{0\}$	12
$\{1\}$	18
$\{2\}$	8
$\{0,1\}$	$18 \cdot 2 = 36$
$\{0,2\}$	$6 \cdot 4 + 2 \cdot 6 = 36$
$\{1,2\}$	$6 \cdot 4 + 2 \cdot 6 = 36$
$\{0,1,2\}$	$12 \cdot 3 \cdot 2 = 72$

$\Diamond$

The flag h-vector is defined in terms of the flag f-vector with an inclusion-exclusion formula analogous to the formula for the simplicial h-vector.

$$h_S = \sum_{T \subseteq S} (-1)^{|S \smallsetminus T|} f_T.$$

The flag f- or h-vector of a polytope is defined to be the flag f- or h-vector of its face lattice.

Example (1.3.5 Continued)**.** From the flag f-vector we compute the flag h-vector:

S	f_S	h_S
$\varnothing$	1	1
$\{0\}$	12	$12 - 1 = 11$
$\{1\}$	18	$18 - 1 = 17$
$\{2\}$	8	$8 - 1 = 7$
$\{0,1\}$	36	$36 - 18 - 12 + 1 = 7$
$\{0,2\}$	36	$36 - 8 - 12 + 1 = 17$
$\{1,2\}$	36	$36 - 8 - 18 + 1 = 11$
$\{0,1,2\}$	72	$72 - 36 - 36 - 36 + 8 + 18 + 12 - 1 = 1$

$\Diamond$

Observe that the flag h-vector satisfies the symmetry relation $h_S = h_{\bar{S}}$, where $\bar{S}$ is the set complement.

Although the flag f-vector has 2^d entries, the affine span of the flag numbers of d-polytopes has far fewer dimensions than this. Here is the theorem which is given in the previously mentioned paper by Bayer and Billera.

Theorem 1.3.6 (Bayer–Billera 1985, [3])**.** *For $d \geq 1$,*

$$\begin{aligned}\dim \operatorname{aff}\{\{f_S(P)\}_{S \subseteq \{1,\dots,d\}} : P \textit{ is an Eulerian poset of rank } d+1\}& \\ = \dim \operatorname{aff}\{\{f_S(P)\}_{S \subseteq \{0,\dots,d-1\}} : P \textit{ is a } d\textit{-polytope}\}& \\ = F_d - 1,&\end{aligned}$$

where F_d is the dth Fibonacci number.

1.3.3 The cd-index

The **ab**-index of a d-polytope, or more generally, of a graded poset of rank $d+1$, is a polynomial in the non-commuting variables **a** and **b** defined by the flag h-vector. For each set $S \subseteq \{0,\ldots,d-1\}$, define the index monomial of S to be $u_S = u_0 u_1 \cdots u_{d-1}$ where

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S \ (i \text{ is } \mathbf{a}\text{bsent}), \\ \mathbf{b} & \text{if } i \in S \ (i \ \mathbf{b}\text{e there}). \end{cases}$$

Then the **ab**-index is

$$\Phi = \sum_{S \subseteq \{0,\ldots,d-1\}} h_S u_S.$$

Fine observed that this polynomial can be written in terms of $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{ab}+\mathbf{ba})$. Bayer and Klapper [4] confirmed this and proved that the **ab**-index can indeed be written in terms of $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$ (with integer coefficients) exactly when the poset in question satisfies the Generalized Dehn–Sommerville equations. This new index is called the **cd**-*index* and we will denote the **cd**-index of a poset or complex P by $\Psi(P)$.

Example (1.3.5 Continued)**.** From the flag h-vector we compute the **ab**- and **cd**-indices:

u_S	h_S	$(\mathbf{a}+\mathbf{b})^3$	$(\mathbf{ab}+\mathbf{ba})(\mathbf{a}+\mathbf{b})$	$(\mathbf{a}+\mathbf{b})(\mathbf{ab}+\mathbf{ba})$
aaa	1	1		
baa	11	1	10	
aba	17	1	10	6
aab	7	1		6
bba	7	1		6
bab	17	1	10	6
abb	11	1	10	
bbb	1	1		

The **ab**-index is $\Phi = 1\mathbf{aaa}+11\mathbf{baa}+17\mathbf{aba}+7\mathbf{aab}+7\mathbf{bba}+17\mathbf{bab}+11\mathbf{abb}+1\mathbf{bbb}$. This can be rewritten $\Psi = \mathbf{c}^3 + 10\mathbf{dc} + 6\mathbf{cd}$ in terms of **c** and **d**. $\Diamond$

Lemma 1.3.7. *Suppose S is a PL-sphere of dimension $d-1$ with F-vector $(f_0, f_1, \ldots)$. Then*

$$\Psi(S) = \begin{cases} \mathbf{c} & \text{if } d=1 \\ \mathbf{c}^2 + (f_0-2)\mathbf{d} & \text{if } d=2 \\ \mathbf{c}^3 + (f_0-2)\mathbf{dc} + (f_2-2)\mathbf{cd} & \text{if } d=3 \ . \\ \mathbf{c}^4 + (f_0-2)\mathbf{dc}^2 + (f_1-f_0)\mathbf{cdc} + (f_3-2)\mathbf{c}^2\mathbf{d} & \\ \quad +(f_{03}-2f_0-2f_3+4)\mathbf{d}^2 & \text{if } d=4 \end{cases}$$

It should be noted that the **cd**-index has a Fibonacci number of terms, which is one more than the affine span of the possible flag f-vectors of polytopes. The difference of one is accounted for by the fact that the coefficient of $\mathbf{c}^d$ is always one for any sphere. See Theorem 1.3.6.

The **cd**-index of polytopes, spheres, and more general posets have been topics of much research in the years since Bayer and Klapper's introductory 1991 paper. Stanley showed in 1993 that the **cd**-index is non-negative for all S-shellable regular CW-spheres [35]. Bayer characterized in 2001 the signs of the coefficients of Eulerian posets in [2]. There are **cd**-words with coefficients that are always non-negative and others which can have positive or negative coefficients. Further, there are no upper bounds for the non-negative coefficients nor are there lower bounds for the possibly negative coefficients.

Ehrenborg and Readdy prove in [12] that the **ab**-index induce a coalgebra homomorphism from a Newtonian coalgebra of graded posets to the Newtoninan coalgebra of **ab**-polynomials. Further, when one restricts to Eulerian posets and **cd**-polynomials the **cd**-index is still a coalgebra homomorphism.

In [5] Billera and Ehrenborg prove that the **cd**-index of polytopes satisfies certain monotonicity properties. They use this monotonicity to prove that the d-simplex provides a term-wise lower bound for the **cd**-index of d-polytopes and further that the cyclic polytope $C(n,d)$ provides a term-wise upper bound for d-polytopes with n vertices. Recall that these polytopes also serve as the lower and upper bounds for the f-vectors of simplicial polytopes.

Chapter 2 Overview

2.1 General flips

Bistellar flips have an intimate relationship with shelling moves. If a facet is shelled onto or off of a shellable simplicial ball, the boundary of that ball changes by a bistellar move. In this sense, bistellar moves are to spheres what shelling moves are to balls. Shelling and inverse shelling moves have also been defined for (simplicial) PL-manifolds with boundary. See Section 1.2.3 for more background on this. In particular, this allows one to define equivalence classes of PL-manifolds with boundary, including those that are not shellable. These shelling moves, which deal with the facets along the boundary, along with bistellar flips, which keep the boundary fixed, form a complete characterization of the PL-homeomorphism classes of PL-manifolds with boundary. So, in this view, bistellar moves are to the interior of a manifold what shelling is to the boundary.

Shelling is well-defined for complexes that are not simplicial. Thus we are inspired by this idea of shelling moves on non-simplicial complexes to define something analogous to bistellar flips that can be applied to non-simplicial manifolds. Although we will save a formal definition of general flips until Chapter 3, we will now give an intuitive picture of these general flips and how they relate to bistellar flips.

Consider a shellable ball B along with a shelling order of its facets $F_1, F_2, F_3, \dots, F_s$. We will use S_i to denote the boundary of the union of the first i facets. Thus

$$S_i = \partial(F_1 \cup F_2 \cup \cdots \cup F_i).$$

When a new facet F_{i+1} is shelled onto the ball, some of its faces are glued to the existing faces of the boundary S_i. Those faces that are "buried" are now interior faces of $F_1 \cup F_2 \cup \cdots \cup F_i \cup F_{i+1}$ and are thus not faces of S_{i+1}. The other faces of F_{i+1} are now part of the new boundary complex S_{i+1}. If F_{i+1} is a simplex, then S_i and S_{i+1} are related by a bistellar flip. The same change can be considered when F_{i+1} is something other than a simplex.

In Figure 2.1, the two rightmost spheres are related to the leftmost by general flips. The lower flip is the bistellar flip $\langle \Sigma_3 \rangle_2$. The upper flip is a non-simplicial polytopal flip. The middle figures show the related balls being glued onto the shaded facets. Figure 2.2 shows diagrams of the same two flips.

This, roughly, is our notion of a general flip. A general flip replaces a collection of facets of a sphere or other manifold (played by S_i above) that happen to match part of a PL-sphere (F_i above) with the remaining faces of that sphere.

Bistellar flips, stellar subdivisions, and inverse stellar subdivisions are all examples of general flips. (See Lemma 4.0.2.) However this class also includes many other operations. We will consider general, shellable, polytopal, simplicial, and bistellar flips. Each of these types is strictly contained in the previous.

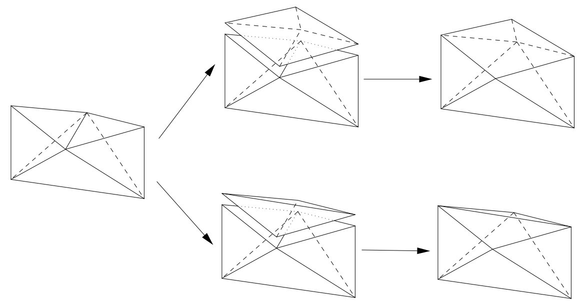

Figure 2.1: A general flip and a bistellar flip realized as the result of "gluing"

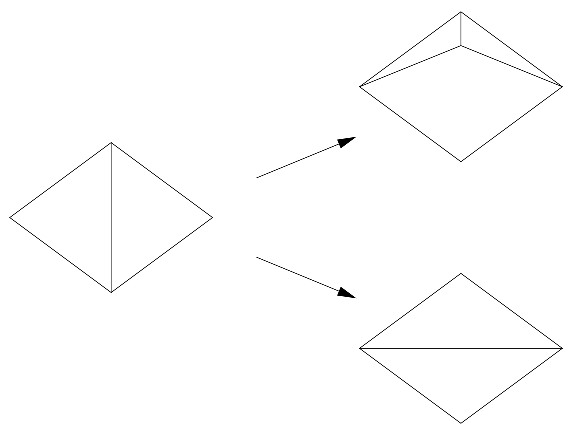

Figure 2.2: A general flip and a bistellar flip

2.2 The cd-index of flips

A flip is defined by a PL-sphere S with its facets partitioned into two subcomplexes, S^+ and S^-, which are each PL-balls. We refer to S^+ as the top or new patch and S^- as the bottom or old patch. The flip is denoted

$$\left\langle S^-, S^+ \right\rangle .$$

The **cd**-index of the flip is defined to be the difference between the **cd**-indices of the semi-suspensions of the two patches,

$$\Psi \left\langle S^-, S^+ \right\rangle = \Psi(\tilde{\mathcal{S}}(S^+)) - \Psi(\tilde{\mathcal{S}}(S^-)).$$

This can also be expressed as

$$\Psi \left\langle A, B \right\rangle = \Psi(A \cup B) - 2\Psi(\tilde{\mathcal{S}}(A)) + \Psi(A \cap B) \cdot \mathbf{c},$$

where $\tilde{\mathcal{S}}(A)$ denotes the semi-suspension of A. (See Proposition 3.2.3.)

If a flip is defined by the initial and final segments of a shelling order for a shellable sphere, then the **cd**-index can also be calculated from the **cd**-index of the defining sphere, its facets, and certain flips in two dimensions lower. By induction, it is enough to know the **cd**-indices of all faces of the defining sphere.

Corollary (3.2.9)**.** *For any shellable sphere S with facets partitioned into initial and final shelling segments S^- and S^+, there exists a flip $\langle E^-, E^+\rangle$ for each interior subfacet E of S^- determined by the shelling order on S such that*

$$\Psi\left\langle S^-, S^+\right\rangle = \Psi(S) - \sum_{F \prec S^-} \Psi(F)\cdot \mathbf{c} - \sum_{E \prec\!\!\prec int\, S^-} \Psi\left\langle E^-, E^+\right\rangle \cdot (2\mathbf{d} - \mathbf{c}^2)$$

where the first summation is over all facets of S^- and the second is over all interior subfacets of S^-.

2.3 The cd-index of bistellar flips

Bistellar flips are precisely those flips we call polytopal flips of complexity 0. We use Σ_d to denote the boundary complex of a d-simplex, and the special notation $\langle\Sigma_d\rangle_k$ to denote the $d-1$ dimensional bistellar flip that replaces k simplices with $d-k+1$ simplices. We show that the **cd**-index of bistellar flips can be computed recursively in terms of other, mostly lower dimensional flips.

Proposition (3.3.1)**.** *The following recurrence holds:*

$$\begin{aligned}
\Psi\left\langle\Sigma_d\right\rangle_k &= \Psi\left\langle\Sigma_d\right\rangle_0 - k\Psi\left\langle\Sigma_{d-1}\right\rangle_0 \mathbf{c} - \sum_{i=0}^{k-2}(k-i-1)\Psi\left\langle\Sigma_{d-2}\right\rangle_i (2\mathbf{d}-\mathbf{c}^2),\\
\Psi\left\langle\Sigma_d\right\rangle_0 &= \Psi(\Sigma_d).
\end{aligned}$$

We use $\langle\Sigma_d\rangle_0$ to denote the degenerate flip $\langle\varnothing, \Sigma_d\rangle$ which replaces the empty set with (the boundary complex of) a d-simplex. While it makes sense to consider such operations flips, they must often be treated as a special case.

We will also show that the **cd**-index of bistellar flips can be computed directly from the shelling components of the **cd**-index. Stanley defines these **cd**-polynomials in terms of the changes to the **cd**-index of a simplex as it is S-shelled. The shelling components form a basis for the **cd**-indices of simplicial polytopes that corresponds to the simplicial h-vector. Using that fact, it is straightforward to write

$$\Psi\left\langle\Sigma_d\right\rangle_k = \sum_{i=k}^{d-k} \check{\Phi}_i^d.$$

Further, we show the **cd**-index of a bistellar flip is, essentially, the difference of shelling components.

Proposition (3.3.4)**.**

$$\Psi\left\langle\Sigma_d\right\rangle_k \mathbf{c} = \check{\Phi}_k^{d+1} - \check{\Phi}_{d-k+1}^{d+1}.$$

This result leads us to give a nicer recursion for bistellar flips involving the derivation G.

Proposition (3.3.5)**.**

$$\Psi \left\langle \Sigma_d \right\rangle_k = \frac{1}{2} G \left[\Psi \left\langle \Sigma_{d-1} \right\rangle_k + \Psi \left\langle \Sigma_{d-1} \right\rangle_{k-1} + \Psi \left\langle \Sigma_{d-2} \right\rangle_{k-1} \mathbf{c} \right].$$

It is possible to give explicit formulae for the individual coefficients of the **cd**-index for any simplicial sphere in terms of its simplicial h-vector. We do this, and can thus prove our observation that the **cd**-indices of bistellar flips "begin" with the **cd**-index of a simplex.

Proposition (3.3.8)**.** *For $k \leq \lfloor d/2 \rfloor$,*

$$\Psi \left\langle \Sigma_d \right\rangle_k = \Psi(\Sigma_{k-1}) \cdot \mathbf{d}\mathbf{c}^{d-k-1} + \Omega,$$

where the terms of Ω have at most $d-k-2$ final $\mathbf{c}$*s.*

2.4 The cd-index of simplicial flips of complexity 1

The polytopal flips of complexity 1 are those whose defining d-polytopes have $d+2$ vertices, that is, one more than a simplex of the same dimension. These types of small polytopes have a lot of structure and this translates to small flips as well, especially when the polytope is also simplicial.

Every simplicial d-polytope with $d+2$ vertices has a boundary complex $\Sigma_m * \Sigma_n$, where $m+n = d+2$. We show that connected shellable collections of facets on such a polytope (and thus flips defined by them) are indexed by Young diagrams that fit in an $m \times n$ rectangle.

We also show that each such flip can be achieved as a sequence of exactly m bistellar flips, where $m \leq n$. This gives the following formula for the **cd**-index of these small simplicial flips.

Proposition (3.4.8)**.** *If λ is a partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ with $0 \leq \lambda_i \leq n$ for all i, and F_λ is the flip defined by the corresponding $m \times n$ Young diagram matrix for λ, then*

$$\Psi(F_\lambda) = \sum_{i=1}^{m} \Psi \left\langle \Sigma_{m+n-2} \right\rangle_{\lambda_i + i - 1}.$$

2.5 Semi-simplicial flips

General flips lack some of the structure of bistellar flips. One feature lost is that bistellar flips nicely "factor". That is, we can write a bistellar flip whose defining polytope (a d-simplex) is $\Delta_{k-1} * \Delta_{d-k}$ as

$$\left\langle \Sigma_d \right\rangle_k = \left\langle \partial\Delta_{k-1} * \Delta_{d-k}, \Delta_{k-1} * \partial\Delta_{d-k} \right\rangle,$$

where $*$ denotes the free join.

Analogous to this property we define a *semi-simplicial* flip to be a general flip of form $\langle P * \partial\Delta_k, \partial P * \Delta_k \rangle$ where P is any polytope. We show that these flips have **cd**-indices that look very much like the **cd**-index of bistellar flips "mixed" with $\Psi(P)$. The "mixing" is done by the *mixing operator* M, a bilinear operator M on **cd**-polynomials such that

$$\Psi(P * Q) = M(\Psi(P), \Psi(Q)).$$

Proposition (3.5.1)**.**

$$\begin{aligned}\Psi \langle P * \partial\Delta_k, \partial P * \Delta_k \rangle = M\left(\Psi(P), \Psi(\Sigma_k)\right) - (k+1) M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c} \\ - M\left(\Psi(P), \sum_{i=0}^{k-1} (k-i)\Psi \langle \Sigma_{k-2} \rangle_i\right)(2\mathbf{d} - \mathbf{c}^2).\end{aligned}$$

2.6 Flip connectivity

Any two simplicial d-polytopes can be connected by a sequence of bistellar flips. Any two d-polytopes can be connected by a sequence of stellar subdivisions, followed by bistellar flips, followed by inverse stellar subdivisions. In the context of polytopes, both bistellar flips and stellar subdivisions and their inverses are polytopal flips. Therefore we know that any two polytopes can be connected by polytopal flips. However, there are infinitely many polytopal flips in each dimension.

When the defining polytope has q vertices more than a simplex of the same dimension, we say the flip has *complexity* q. Thus bistellar flips have complexity 0.

Theorem (4.1.1)**.** *Any d-polytope can be obtained from a d-simplex by a sequence of polytopal flips with complexity bounded above by k, where k is the smallest integer such that no facet has more than $k + d$ vertices. Further, this can be done so that each intermediate step yields a polytope.*

Two simplicial polytopes with the same number of vertices can be connected by bistellar flips that preserve the number of vertices. We extend this result to non-simplicial polytopes.

Proposition (4.1.6)**.** *If P and Q are two polytopes in the same dimension with the same number of vertices, then the boundary complex of one can be obtained from the boundary complex of the other by a sequence of polytopal flips that preserve the number of vertices.*

It is not possible in general to preserve the number of vertices and bound the complexity. We demonstrate this with an example and then give some rough bounds on the number of extra vertices that may be necessary to connect the simplex to a given non-simplicial polytope.

2.7 (Non-)Monotonicity

Bistellar flips are monotonic in the sense that their **cd**-indices have all non-negative or all non-positive coefficients. General flips are not, in general, monotonic. We discuss some classes of low dimensional or low complexity flips that are monotonic and give some examples of non-monotonic flips.

2.8 Semi-simplicial flips

For low dimensional polytopes, $d \leq 5$, semi-simplicial flips are sufficient to connect all polytopes. We describe how to use these flips to connect any non-simplicial polytope to a simplicial polytope in the process of proving the following more general result.

Theorem (4.3.1)**.** *If P is the boundary complex of a $(d-5)$-simplicial d-dimensional polytope with facets having at most $d+q$ vertices, then P can be obtained from Σ_d by a sequence of semi-simplicial flips with complexity at most q.*

Chapter 3 The cd-index of flips

3.1 Shelling and flipping

The classical bistellar operations, or *flips* can be motivated by the changes to the boundary of a shellable simplicial ball as it is being shelled.

A bistellar flip is a local change to a (pure) d-dimensional simplicial complex that replaces a collection of k pairwise adjacent facets with $d-k+1$ pairwise adjacent facets. In the shelling picture, the old patch of facets is that part of the boundary complex that is being buried by the new simplex being shelled on, and the new patch is the exposed part of the new simplex.

Bistellar flips can be defined without reference to this shelling picture, as we did in Chapter 1. This definition naturally applies to simplicial manifolds in general. See, for example [23].

We now generalize the notion of a flip to non-simplicial complexes. Let us revisit the shelling picture. Consider a shellable ball that is not necessarily simplicial. At each shelling step, there is a patch of facets that is buried and a new patch to replace it. Again, we can define these general flips without reference to shellings.

Definition 3.1.1. We say two d-dimensional PL-manifolds C and C' are related by a d-dimensional *general flip* if there exist a collection A of facets of C and a collection B of facets of C' with the following properties:

1. $C \smallsetminus \operatorname{int} \bigcup A = C' \smallsetminus \operatorname{int} \bigcup B$,
2. $\bigcup A$ and $\bigcup B$ are PL-balls,
3. $\partial \left(\bigcup A\right) = \partial \left(\bigcup B\right)$ is a PL-sphere.

The flip is denoted $\langle A, B\rangle$, and we write $C' = \langle A, B\rangle\, C$.

Although technically excluded by this definition, it will occasionally be convenient to include the degenerate flip-like operations $\langle \varnothing, S\rangle$ and $\langle S, \varnothing\rangle$ which “create” and “destroy”, respectively, an entire PL-sphere. We will generally include creation and destruction flips in notions for classes of flips, but the reader should keep in mind that these are not general flips. In particular, a creation or destruction flip necessarily changes the topology of any PL-manifold to which it is applied. These operations are to flips what the first and last steps of a shelling order are to shelling operations.

We should note here that requiring PL-balls and PL-spheres ensures certain very important properties hold that do not hold for more general classes of balls and spheres. In particular, the following facts are vital for flips to preserve the topology of the manifolds to which they are applied.

Theorem 3.1.2 (1.2.5). *1. The union of two d-dimensional PL-balls attached along their common boundary is a d-dimensional PL-sphere.*

2. *The closure of the complement of a d-dimensional PL-ball embedded in a d-dimensional PL-sphere is a d-dimensional PL-ball.*

See section 4.7 of [7], [16], and [29] for these and other fundamental results in PL-topology.

The first fact means that in our definition of flip, $A \cup B$ is always a PL-sphere. The second ensures us that we can go the other direction. If we begin with a PL-sphere and a PL-ball A embedded in it, we can take the closure of the complement of A as B.

Further, we can state that if a PL-manifold C is related to a PL-sphere by general flips, that C is also a PL-sphere.

The definition of general flip is very broad, but we will primarily concern ourselves with general flips satisfying some condition on $A \cup B$ stronger than being a PL-sphere. We will say that a flip $\langle A, B\rangle$ is a *shellable flip* if $A \cup B$ is a shellable sphere and A and B are initial and final segments of some shelling order. We will call a flip $\langle A, B\rangle$ a *polytopal flip* if it is a shellable flip and $A \cup B$ is realizable as a polytope. If the flip is polytopal and $A \cup B$ is a simplicial polytope, then we will call it a *simplicial flip*. We can consider the classic bistellar flips to be simplicial flips $\langle A, B\rangle$ where $A \cup B$ is a simplex.

We will also be interested in discriminating among flips according to their *complexity*.

Definition 3.1.3. The *complexity* of a d-dimensional flip $\langle A, B\rangle$ is

$$f_0(A \cup B) - d - 2.$$

This number is the number of "extra" vertices in the sphere $A \cup B$ when compared to a simplex of appropriate dimension. Thus bistellar flips have complexity 0, since they are precisely the general flips where $A \cup B$ is the boundary of a simplex. A flip that subdivides a square into four triangles has complexity 1, since the associated sphere $A \cup B$ is a square-based pyramid, which has one more vertex than a tetrahedron.

3.2 A general formula for $\Psi \langle A, B\rangle$.

Given a flip $\langle A, B\rangle$, we can consider the two complexes $\tilde{\mathcal{S}}(A)$ and $\tilde{\mathcal{S}}(B)$, which are the semi-suspensions of A and B, respectively. Recall that the semi-suspension of a ball is the sphere obtained by adding a single new facet attached to the boundary of the ball. This is sometimes referred to as "capping" the ball. Refer to [35] for a precise definition of the semi-suspension of near-Eulerian posets. So now we have two PL-spheres. Since they are (finite) PL-manifolds, the cell structure on them is a finite regular CW cell structure. The face posets of regular CW-complexes are Eulerian [34], and Eulerian posets have **cd**-indices [4]. So we can make this definition.

Definition 3.2.1. The **cd**-index of a flip $\langle A, B\rangle$ is

$$\Psi \langle A, B\rangle = \Psi(\tilde{\mathcal{S}}(B)) - \Psi(\tilde{\mathcal{S}}(A)).$$

To justify that this is a good definition for the **cd**-index of a flip, we will prove this theorem.

Theorem 3.2.2. *If S and S' are PL-spheres that are related by the general flip $\langle A, B\rangle$ so that $S' = \langle A, B\rangle S$, then*

$$\Psi(S') - \Psi(S) = \Psi \langle A, B\rangle .$$

Proof. To show that the differences $\Psi(\tilde{\mathcal{S}}(B)) - \Psi(\tilde{\mathcal{S}}(A))$ and $\Psi(S') - \Psi(S)$ are the same we will show the differences in the flag f-vectors are the same. This is sufficient since the coefficients of the **cd**-index are linear combinations of the entries of the flag f-vector. We will do this by arguing that the only chains of faces that contribute to the difference are those that are contained entirely in A or B.

Consider a chain of faces of S that contains no face of A. This is also a chain in S', since S and S' agree outside of A and B. Thus such a chain contributes nothing to the difference.

Consider next a chain that contains both faces of A and faces of $S \setminus A$. Let F be the maximal dimensional face in this chain. The face F cannot be a face of A, or else the entire chain would be in A. Let E be the highest dimensional face of this chain that is also a face of A. Since E is a face of A also contained in F, E must lie on the boundary of A. Therefore all lower dimensional faces in the chain are also in ∂A. Since $\partial A = \partial B$, this chain of faces also appears in S'. Thus this chain does not contribute to the difference.

Swapping the roles of S and S', the same argument shows that chains in S' not entirely in B do not contribute to the difference. Thus the only chains that contribute are those contained entirely in A or B.

The difference $\Psi(\tilde{\mathcal{S}}(B)) - \Psi(\tilde{\mathcal{S}}(A))$ consists of those chains contained entirely in A or B, along with chains that contain the "caps" of A or B. By a parallel argument, those chains not contained entirely in A or B do not contribute to this difference. Thus the two differences are equal. □

3.2.1 The general formula

The remainder of this chapter is devoted to various formulae for computing the **cd**-index of flips of various classes. We begin with a completely general formula.

Proposition 3.2.3. *For any PL-balls A and B with $\partial(A) = \partial(B)$,*

$$\Psi \langle A, B\rangle = \Psi(A \cup B) - 2\Psi(\tilde{\mathcal{S}}(A)) + \Psi(A \cap B) \cdot \mathbf{c}. \tag{3.1}$$

Equivalently, for any PL-sphere S and a full dimensional PL-ball A embedded in S,

$$\Psi \left\langle A, \overline{S \smallsetminus A}\right\rangle = \Psi(S) - 2\Psi(\tilde{\mathcal{S}}(A)) + \Psi(\partial A) \cdot \mathbf{c}.$$

Proof. First we note that if $\mathcal{S}(S)$ is the (full) suspension of a PL-sphere S, then $\Psi(\mathcal{S}(S)) = \Psi(S) \cdot \mathbf{c}$. This fact appears as Lemma 1.1 in [35]. Whenever we refer to the suspension of a d-dimensional PL-sphere S, we mean the Pl-sphere obtained by adding two

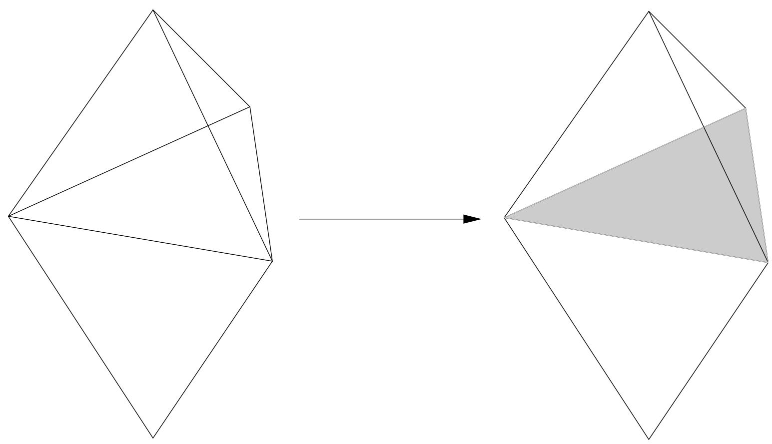

Figure 3.1: The flip $\left\langle \overline{\mathrm{conv}(\Sigma_2 * \Sigma_1)}, \Delta_2 * \Sigma_1 \right\rangle$

identical $d+1$-faces to S, each of which have all of S as their boundary. This is topologically equivalent, but not combinatorially equivalent, to the usual topological notion of suspension which is often realized instead as a a free join with Σ_0.

Now $\Psi(A \cup B) + \Psi(\mathcal{S}(A \cap B)) = \Psi(\tilde{\mathcal{S}}(A)) + \Psi(\tilde{\mathcal{S}}(B))$. The complex A together with one of the maximal faces of $\mathcal{S}(A \cap B)$ forms $\tilde{\mathcal{S}}(A)$. This leaves $\mathrm{int}(B)$ and close of the other maximal face of $\mathcal{S}(()A \cap B)$, which together form $\mathcal{S}(B)$.

Hence, the right side of Equation (3.1) is equal to the difference $\Psi(\tilde{\mathcal{S}}(B)) - \Psi(\tilde{\mathcal{S}}(A))$. □

Note also that Proposition 3.2.3 is a direct consequence of Lemma 3.3 in [5] which states that

$$\Psi(A \cup B) = \Psi(\tilde{\mathcal{S}}(A)) + \Psi(\tilde{\mathcal{S}}(B)) - \Psi(A \cap B)\mathbf{c}.$$

Example 3.2.4. Consider the simple (non-polytopal) flip

$$\left\langle \overline{\mathrm{conv}(\Sigma_2 * \Sigma_1)}, \Delta_2 * \Sigma_1 \right\rangle.$$

We use Δ_k to denote the solid k-dimensional simplex, so $\Sigma_k = \partial\Delta_k$. This flip, shown in Figure 3.1, replaces a single triangular bipyramid ($A = \overline{\mathrm{conv}(\Sigma_2 * \Sigma_1)}$) with two tetrahedra ($B = \Delta_2 * \Sigma_1$).

To calculate the **cd**-indices for this example, we will find f_0, f_1, f_3, and f_{03} by direct counting and apply Lemma 1.3.7.

The necessary flag f-vector entires for $A \cup B$ are $f_0 = 5, f_1 = 9, f_3 = 3$, and $f_{03} = 13$. So

$$\begin{aligned}\Psi(A \cup B) &= \mathbf{c}^4 + (f_0 - 2)\mathbf{dc}^2 + (f_1 - f_0)\mathbf{cdc} + (f_3 - 2)\mathbf{c}^2\mathbf{d} + (f_{03} - 2f_0 - 2f_3 + 4)\mathbf{d}^2 \\ &= \mathbf{c}^4 + 3\mathbf{dc}^2 + 4\mathbf{cdc} + \mathbf{c}^2\mathbf{d} + \mathbf{d}^2.\end{aligned}$$

Now we consider $\tilde{\mathcal{S}}(A)$. Since A consists of a single facet, the semi-suspension of A is the suspension of ∂A. Thus

$$\Psi(\tilde{\mathcal{S}}(A)) = \Psi(\partial A)\mathbf{c} = (\mathbf{c}^3 + 3\mathbf{dc} + 4\mathbf{cd})\mathbf{c}.$$

Therefore,

$$\begin{aligned}\Psi\left\langle \mathrm{conv}(\Sigma_2 * \Sigma_1), \Delta_2 * \Sigma_1\right\rangle &= \mathbf{c}^4 + 3\mathbf{dc}^2 + 4\mathbf{cdc} + \mathbf{c}^2\mathbf{d} + \mathbf{d}^2 \\ &\quad - 2(\mathbf{c}^3 + 3\mathbf{dc} + 4\mathbf{cd})\mathbf{c} \\ &\quad + (\mathbf{c}^3 + 3\mathbf{dc} + 4\mathbf{cd})\mathbf{c} \\ &= \mathbf{c}^2\mathbf{d} + \mathbf{d}^2.\end{aligned}$$

$\Diamond$

Now we turn to flips $\langle S^-, S^+\rangle$, where $S^- \cup S^+$ is a shellable sphere. Let S be a PL-sphere with an shelling order $F_1, F_2, \ldots, F_k, \ldots, F_s$ and let

$$S_k^- = S^- = F_1 \cup F_2 \cup \cdots \cup F_k$$

and

$$S_k^+ = S^+ = F_{k+1} \cup F_{k+2} \cup \cdots \cup F_s.$$

This defines a flip $\langle S^-, S^+\rangle$.

We will now show that shelling orders of spheres are reversible.

Lemma 3.2.5. *If S is a d-dimensional spherical cell complex with shelling order $F_1, F_2, \ldots, F_s$, then $F_s, F_{s-1}, \ldots, F_1$ is also a shelling order.*

Proof. We proceed by induction. Observe that a 0-sphere consists of two disjoint vertices, and is trivially shellable in either order. Suppose that shelling orders of spheres in dimension less than d are reversible. Let S be a d-dimensional spherical cell complex with shelling order $F_1, F_2, F_3, \ldots, F_s$. Thus we have

1. $\overline{F_1}$ is shellable;
2. $\forall 2 \leq k \leq s, \overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$ is shellable of one less dimension;
3. $\overline{F_k}$ can be shelled starting with the facets of $\overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$.

We must show that the reverse order $F_s, F_{s-1}, F_{s-2}, \ldots, F_k, \ldots, F_1$ satisfies

1. F_s is shellable;
2. $\forall s-1 \geq k \geq 1, F_k \cap \bigcup_{i=k+1}^{s} F_i$ is shellable of one less dimension; and
3. F_k can be shelled starting with $F_k \cap \bigcup_{i=k+1}^{s} F_i$.

The fact that $\overline{F_s}$ is shellable is immediate from the fact that S is shellable. Since S is a sphere, it is a manifold with no boundary. Thus every subfacet of S is contained in exactly two facets. In particular, each facet of $\overline{F_k}$ is contained in some $\overline{F_i}$ for exactly one $i \neq k$. Therefore $\overline{F_k \setminus \bigcup_{i=1}^{k-1} F_i} = \overline{F_k} \cap \bigcup_{i=k+1}^{s} \overline{F_i}$. The facet $\overline{F_k}$ can be shelled starting with $\overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$, and, since ∂F_k is a $d-1$ dimensional sphere, that this shelling order is reversible by induction. The reverse order starts with $\overline{F_k} \cap \bigcup_{i=k+1}^{s} \overline{F_i}$. $\square$

Theorem 3.2.6. *If S is a sphere with shelling order $F_1, F_2, \ldots, F_k, \ldots, F_s$, $S_k^- = F_1 \cup F_2 \cup \cdots \cup F_k$ is the initial shelling segment, and $S_k^+ = F_{k+1} \cup F_{k+2} \cup \cdots \cup F_s$ is the final segment, then*

$$\Psi \left\langle S_k^-, S_k^+ \right\rangle = \Psi(S) - \sum_{i=1}^{k} \Psi(F_i)\mathbf{c} - \sum_{i=1}^{k} \Psi(R_{F_i})(2\mathbf{d} - \mathbf{c}^2). \tag{3.2}$$

Proof. For any shellable sphere S with a shelling order $F_1, F_2, \ldots, F_s$, we can write

$$\Psi(S) = \sum_{i=1}^{s} \Psi_S(F_i),$$

where

$$\Psi_S(F_i) = \Psi(\tilde{\mathcal{S}}(F_1 \cup F_2 \cup \cdots \cup F_i)) - \Psi(\tilde{\mathcal{S}}(F_1 \cup F_2 \cup \cdots \cup F_{i-1}))$$

is the contribution made by F_i to the **cd**-index of S.

It is important to note that this contribution depends on the particular shelling order chosen. We have two related shelling orders for S, the given shelling order and its reverse. We will use $\overrightarrow{\Psi}_S(F)$ and $\overleftarrow{\Psi}_S(F)$ to distinguish between the given forward order and the reverse order, respectively.

Thus we can now write

$$\Psi \left\langle S_k^-, S_k^+ \right\rangle = \sum_{i=k+1}^{s} \overleftarrow{\Psi}_S(F_i) - \sum_{i=1}^{k} \overrightarrow{\Psi}_S(F_i).$$

This will ultimately be a useful thing to do because we have a recursive formula for the contribution of a facet.

Now consider applying the "same" trick to the **cd**-index of a flip, rather than just a sphere. We can decompose the **cd**-index of a sphere by considering how the **cd**-index changes as we shell the sphere up from nothing. With a flip, we have two different but related spheres so we need a slightly different trick. We will begin with the rather trivial flip $\left\langle S_0^-, S_0^+ \right\rangle = \langle \varnothing, S \rangle$ that creates the whole sphere S out nothing. We then incrementally get closer to our desired flip by simultaneously shelling S^- up from nothing and shelling S^+ down from all of S. That is, we consider the sequence of flips $\left\langle S_0^-, S_0^+ \right\rangle, \left\langle S_1^-, S_1^+ \right\rangle, \ldots, \left\langle S_k^-, S_k^+ \right\rangle$ and define the contribution to $\Psi \left\langle S^-, S^+ \right\rangle$ by F_i to be

$$\Psi_{\langle S^-, S^+ \rangle}(F_i) = \Psi \left\langle S_i^-, S_i^+ \right\rangle - \Psi \left\langle S_{i-1}^-, S_{i-1}^+ \right\rangle.$$

We can then write

$$\Psi \left\langle S_k^-, S_k^+ \right\rangle = \Psi(S) + \sum_{i=1}^{k} \Psi_{\langle S^-, S^+ \rangle}(F_i).$$

Playing with our notation a bit reveals that

$$\begin{aligned}
-\Psi_{\langle S^-,S^+\rangle}(F_j) &= \Psi\left\langle S^-_{j-1}, S^+_{j-1}\right\rangle - \Psi\left\langle S^-_{j}, S^+_{j}\right\rangle \\
&= \sum_{i=1}^{j} \overrightarrow{\Psi}_S(F_i) - \sum_{i=j+1}^{s} \overleftarrow{\Psi}_S(F_i) \\
&\quad - \sum_{i=1}^{j-1} \overrightarrow{\Psi}_S(F_i) + \sum_{i=j}^{s} \overleftarrow{\Psi}_S(F_i) \\
&= \overrightarrow{\Psi}_S(F_j) + \overleftarrow{\Psi}_S(F_j). \qquad \text{(A)}
\end{aligned}$$

We now need to make use of two identities in addition to the result above. The first is the aforementioned formula for the contribution of a facet. This formula is derived by Lee [19] by analyzing Stanley's S-shelling argument for the non-negativity of the **cd**-index [35].

$$\overrightarrow{\Psi}_S(F) = \Psi(R_F)\mathbf{d} + \sum_{i=\ell+1}^{r} \overrightarrow{\Psi}_F(E_i)\mathbf{c}. \qquad \text{(B)}$$

where F is a facet of S with shelling order $E_1, E_2, \ldots, E_\ell, \ldots, E_r$ that is compatible with the shelling order of S. Specifically, we must have that $E_1 \cup \cdots \cup E_\ell$ is the intersection of F with the union of the previous facets and $R_F = \partial(E_1 \cup \cdots \cup E_\ell)$ is the boundary of that intersection.

The second identity is derived from the observation that for any sphere S, the union of the semi-suspensions of S^- and S^+ is equal to the union of S and the (full) suspension of $R_S = \partial S^- = \partial S^+ = S^- \cap S^+$. So we have

$$\sum_{i=1}^{k} \overrightarrow{\Psi}_S(F_i) + \sum_{i=k+1}^{s} \overleftarrow{\Psi}_S(F_i) = \Psi(S) + \Psi(R_S)\mathbf{c}.$$

$$\left(\Psi(S) - \sum_{i=k+1}^{s} \overrightarrow{\Psi}_S(F_i)\right) + \left(\Psi(S) - \sum_{i=1}^{k} \overleftarrow{\Psi}_S(F_i)\right) = \Psi(S) + \Psi(R_S)\mathbf{c}.$$

This implies

$$\Psi(S) - \Psi(R_S)\mathbf{c} = \sum_{i=k+1}^{s} \overrightarrow{\Psi}_S(F_i) + \sum_{i=1}^{k} \overleftarrow{\Psi}_S(F_i). \qquad \text{(C)}$$

Note that the use of R_S matches the use of R_F above.

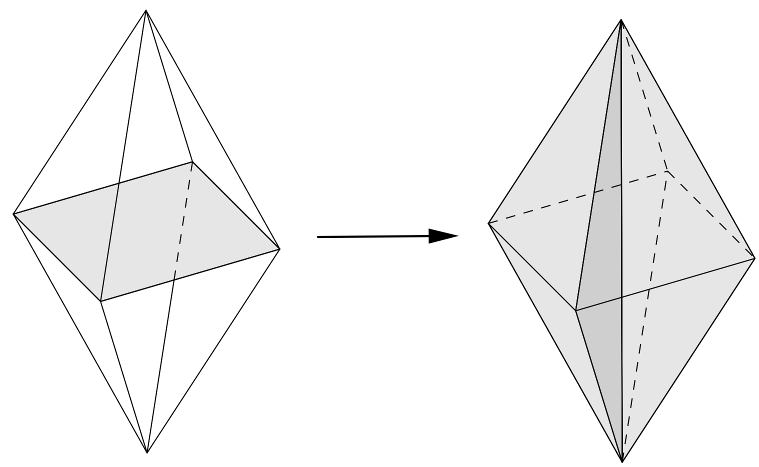

Figure 3.2: The flip $\langle Q * \partial\Delta_1, \partial Q * \Delta_1\rangle$, where Q is a square

Equations (A), (B), and (C) together imply

$$\begin{aligned}
-\Psi_{\langle S^-, S^+\rangle}(F_i) &= \overrightarrow{\Psi}_S(F_i) + \overleftarrow{\Psi}_S(F_i) \\
&= \Psi(R_{F_i})\mathbf{d} + \sum_{j=\ell+1}^{r} \overrightarrow{\Psi}_{F_i}(E_j)\mathbf{c} \\
&\quad + \Psi(R_{F_i})\mathbf{d} + \sum_{j=1}^{\ell} \overleftarrow{\Psi}_{F_i}(E_j)\mathbf{c} \\
&= 2\Psi(R_{F_i})\mathbf{d} + \left(\Psi(F_i) - \Psi(R_{F_i})\mathbf{c}\right)\mathbf{c} \\
&= \Psi(F_i)\mathbf{c} + \Psi(R_{F_i})(2\mathbf{d} - \mathbf{c}^2)
\end{aligned}$$

□

Example 3.2.7. Consider the polytopal flip

$$\langle Q * \partial\Delta_1, \partial Q * \Delta_1\rangle,$$

where Q is a solid square and Δ_1 is a line segment. This flip, shown in Figure 3.2, replaces two square-based pyramids sharing a base with four tetrahedra aranged around a common edge.

We will compute $\Psi\langle Q * \partial\Delta_1, \partial Q * \Delta_1\rangle$ via Theorem 3.2.6. As in Example 3.2.4, we will compute the **cd**-indices of spheres of dimension at most 3 (boundaries of 4-dimensional balls) by directly counting chains of faces.

We need to compute the **cd**-indices of the union $(Q * \partial\Delta_1) \cup (\partial Q * \Delta_1) = S$, the two facets of $Q * \partial\Delta_1$ (F_1, F_2), and the boundary of the collection of subfacets buried by each facet of $Q * \partial\Delta_1$ (R_{F_1}, R_{F_2}).

Observe that $S = (Q * \partial\Delta_1) \cup (\partial Q * \Delta_1) = Q * \Delta_1$. Counting we see $f_0(S) = 6$, $f_1(S) = 13$, $f_3(S) = 6$, and $f_{03}(S) = 26$. Thus

$$\begin{aligned}\Psi(S) &= \mathbf{c}^4 + (6-2)\mathbf{dc}^2 + (13-6)\mathbf{cdc} + (6-2)\mathbf{c}^2\mathbf{d} + (26 - 2\cdot 6 - 2\cdot 6 + 4)\mathbf{d}^2 \\ &= \mathbf{c}^4 + 4\mathbf{dc}^2 + 7\mathbf{cdc} + 4\mathbf{c}^2\mathbf{d} + 6\mathbf{d}^2.\end{aligned}$$

Both of the facets of $Q * \partial\Delta_1$ are square-based pyramids which have 5 vertices and 5 facets. Thus

$$\Psi(F_1) = \Psi(F_2) = \mathbf{c}^3 + 3\mathbf{dc} + 3\mathbf{cd}.$$

The first facet in any shelling order buries nothing, so $R_{F_1} = \varnothing$ and

$$\Psi(R_{F_1}) = 0.$$

The second facet buries a single square face, so $R_{F_2} = \partial Q$, and

$$\Psi(R_{F_2}) = \mathbf{c}^2 + 2\mathbf{d}.$$

We thus have

$$\begin{aligned}\Psi\left\langle Q * \partial\Delta_1, \partial Q * \Delta_1\right\rangle &= \Psi(S) + \Psi(F_1)\mathbf{c} + \Psi(F_2)\mathbf{c} \\ &\quad + \Psi(R_{F_1})(2\mathbf{d} - \mathbf{c}^2) + \Psi(R_{F_2})(2d - \mathbf{c}^2) \\ &= \quad \mathbf{c}^4 + 4\mathbf{dc}^2 \quad + 7\mathbf{cdc} \quad + 4\mathbf{c}^2\mathbf{d} \quad + 6\mathbf{d}^2 \\ &\quad -(\mathbf{c}^3 + 3\mathbf{dc} \quad + 3\mathbf{cd})\mathbf{c} \\ &\quad -(\mathbf{c}^3 + 3\mathbf{dc} \quad + 3\mathbf{cd})\mathbf{c} \\ &\quad \qquad\qquad\qquad\qquad -2(\mathbf{c}^2 \quad + 2\mathbf{d})\mathbf{d} \\ &\quad +(\mathbf{c}^2 + 2\mathbf{d})\mathbf{c}^2 \\ &= \qquad\qquad\qquad \mathbf{cdc} \quad + 2\mathbf{c}^2\mathbf{d} \quad + 2\mathbf{d}^2.\end{aligned}$$

◊

3.2.2 Further unpacking of the recursive formula

It is possible to use the equation in Theorem 3.2.6 to write the **cd**-index of a flip recursively in terms of (mostly) lower dimensional flips.

The first term and first summation are already the **cd**-indices of flips. Specifically, $\Psi(S) = \Psi\left\langle \varnothing, S\right\rangle$ and $\Psi(F_i) = \Psi\left\langle \varnothing, F_i\right\rangle$. The first is a flip in the same dimension as $\langle S^-, S^+\rangle$, and the second is a flip in one dimension lower.

To deal with the R_{F_i} term, we recall that $R_{F_i} = \partial(E_1 \cup E_2 \cup \cdots \cup E_\ell)$ and so it is in fact the boundary of a ball that is already equipped with a shelling. Thus we have a natural sequence of flips defined by this shelling that builds up R_{F_i}. Employing this idea we can write

$$\Psi(R_{F_i}) = \sum_{j=1}^{\ell} \Psi\left\langle (E_j)^-, (E_j)^+\right\rangle. \tag{3.3}$$

Note that the shelling sequence $(E_j)_{j=1}^{\ell}$ and the ℓ here all depend on F_i.

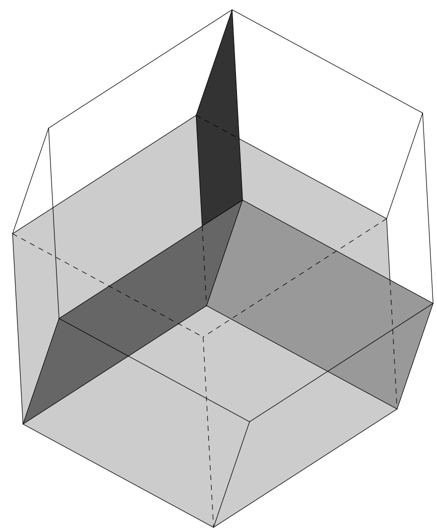

Figure 3.3: Four cubes meeting at a common vertex

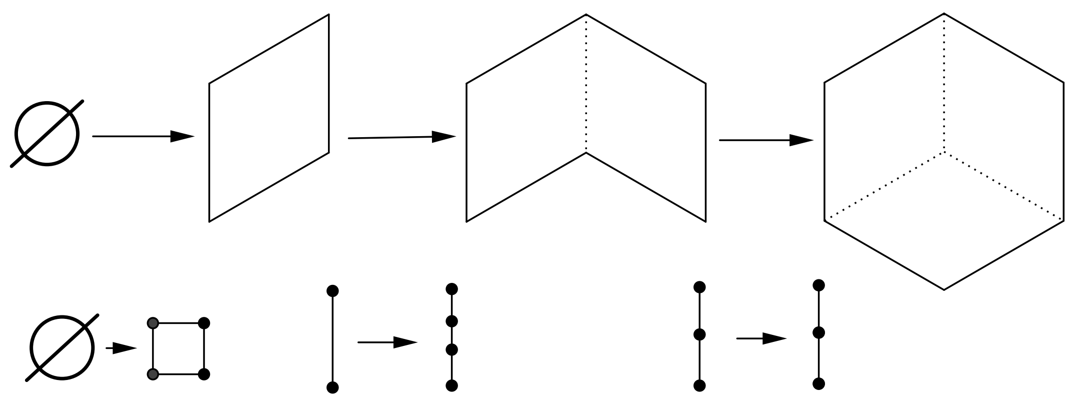

Figure 3.4: R_{F_4} built up by a sequence of flips

Example 3.2.8. Consider F_1, F_2, F_3, and F_4 to be the four facets that meet at the vertex of a 4-dimensional cube, as in Figure 3.3. The sphere R_{F_4} is a hexagon and is the boundary of the three squares (E_1, E_2, E_3) buried by F_4. Consider the shelling order of (E_1, E_2, E_3) induced by the shelling order on $\{F_i\}$.

The first square E_1 does not bury anything, so the associated flip is $\langle \varnothing, E_1 \rangle$. The square E_2 buries one edge, so the associated flip is $\langle C_1, C_3 \rangle$, where C_i denotes the 1-dimensional complex consisting of a chain of i segments. The final square E_3 in R_{F_4} buries two edges, so the associated flip is $\langle C_2, C_2 \rangle$. See Figure 3.4.

$\Diamond$

Note that we use $F \prec S$ to mean F is a facet of S, and $E \prec\!\!\prec S$ to mean E is a facet of a facet, that is, a subfacet, of S.

Corollary 3.2.9. *For any shellable sphere S partitioned into initial and final shelling segments S^- and S^+, there exists a flip $\langle E^-, E^+\rangle$ for each interior subfacet E of S^- determined by the shelling order on S such that*

$$\Psi\left\langle S^-, S^+\right\rangle = \Psi\left\langle \varnothing, S\right\rangle - \sum_{F \prec S^-} \Psi\left\langle \varnothing, F\right\rangle \mathbf{c} - \sum_{E \prec\!\!\prec int\, S^-} \Psi\left\langle E^-, E^+\right\rangle (2\mathbf{d} - \mathbf{c}^2). \quad (3.4)$$

Proof. This summation in (3.3) is over all subfacets buried by the facet F_i in the shelling order for S. But in the equation (3.2) of Theorem 3.2.6 we then sum this expression over all of the facets in S^-. Since each subfacet is buried by at most one facet, we are in fact counting each subfacet in the interior of S^-. So we can now write the **cd**-index of a shellable flip in terms of one trivial flip in the same dimension, trivial flips in one dimension lower, and flips in two dimensions lower. □

Although this corollary is written without explicit reference to a specific shelling order, one must choose a shelling order for S^- and a set of compatible shelling orders for the facets of S^- in order for the flips $\langle E^-, E^+\rangle$ to be defined. Any such set of shelling orders will give the same result, although the individual flips do depend on the choice of shelling orders.

3.3 Bistellar flips

It is natural to seek first to understand the **cd**-index of the classic bistellar flips, so we now turn our attention to flips of complexity 0. We have several ways to attack this question computing the **cd**-index of bistellar flips. First, we can apply our general formula to the bistellar case to derive a recursive formula for the **cd**-index of bistellar flips in terms of other bistellar flips. Alternatively, we can make use of the fact that the **cd**-index of a simplicial polytope, and thus that of a simplicial flip, is determined entirely by the simplicial h-vector. It is already known that the g-vector of bistellar flips is either all zeros, or has exactly one component that is ± 1 (see [17]). Consequently, the h-vector is also very nice. We can employ this information in two directions. We can use a result of Stanley's that gives a basis for the **cd**-indices of simplicial Eulerian posets [35]. This shows how a given component of the h-vector contributes to the **cd**-index. We can also look at how a given coefficient of a given **cd**-word is determined by the whole h-vector.

One should note that although the **cd**-index of a simplicial flip is determined entirely by the f-vector, simplicial flips may be applied to non-simplicial spheres. It is in this setting that one might be interested in the **cd**-index of a bistellar or other simplicial flip rather than just the f-, g-, or h-vectors.

3.3.1 Recursive formula

A bistellar flip is a polytopal flip with complexity 0. The union of the old and new patches is the boundary of a simplex. We can specify a particular bistellar flip by giving the dimension of this simplex and the number of facets in each part. Of course, any two of these values determines the third. It is standard to use the number of old

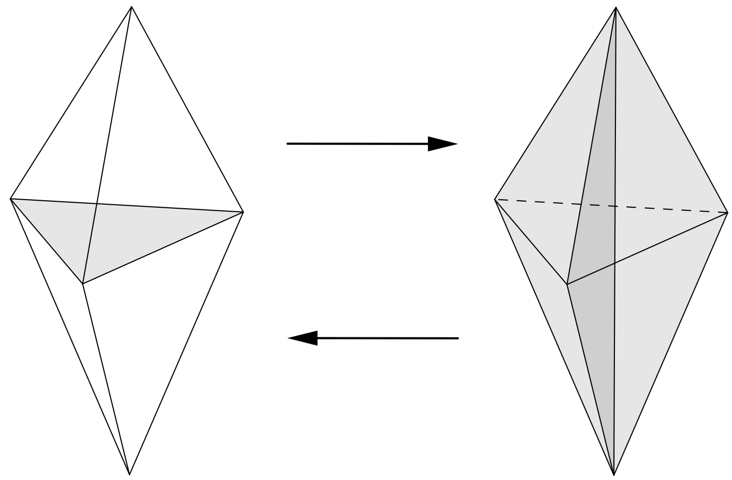

Figure 3.5: $\langle \Sigma_4 \rangle_2$ and $\langle \Sigma_4 \rangle_3$

and new facets to index bistellar flips. However, we prefer to use the dimension and the number of old facets. We will denote by $\langle \Sigma_d \rangle_k$ the $d-1$ dimensional flip that replaces k pairwise adjacent $(d-1)$-simplices with $d-k+1$ pairwise adjacent $(d-1)$-simplices. The union of the two patches is the boundary of a d-simplex, denoted by Σ_d. Note that flips $\langle \Sigma_d \rangle_k$ and $\langle \Sigma_d \rangle_{d-k+1}$ are the inverses of each other, and thus their **cd**-indices are the negatives of each other. See Figure 3.5. Also observe that $\langle \Sigma_d \rangle_0 = \langle \varnothing, \Sigma_d \rangle$. Thus $\Psi \langle \Sigma_d \rangle_0 = \Psi \Sigma_d$.

Proposition 3.3.1. *The following recurrence holds:*

$$\begin{aligned} \Psi \langle \Sigma_d \rangle_k &= \Psi \langle \Sigma_d \rangle_0 - k \Psi \langle \Sigma_{d-1} \rangle_0 \mathbf{c} - \sum_{i=0}^{k-2} (k-i-1) \Psi \langle \Sigma_{d-2} \rangle_i (2\mathbf{d} - \mathbf{c}^2); \\ \Psi \langle \Sigma_d \rangle_0 &= \Psi(\Sigma_d). \end{aligned}$$

Proof. The recursion follows from Corollary 3.2.9. Each facet of Σ_d is a copy of Σ_{d-1}, so the sum in the second term in the right hand side of equation (3.4) collapses to $k\Psi(\Sigma_{d-1})$.

To obtain the last term we express the rightmost sum of equation (3.4) over interior subfacets as an explicit double sum. For each of the k original facets we sum over the subfacets that it buries. We wish to compute

$$\sum_{F \prec \Sigma_d^-} \sum_{E \prec F^-} \Psi \left\langle E^-, E^+ \right\rangle .$$

Since the subfacets of Σ_d are $d-2$ dimensional simplices, the flips $\langle E^-, E^+ \rangle$ are $d-3$ dimensional bistellar flips. In particular, $\langle E^-, E^+ \rangle = \langle \Sigma_{d-2} \rangle_i$ where i is the number of $(d-2)$-dimensional facets of F shelled on before E, since each facet is adjacent to all others. Thus

$$\sum_{E \prec F^-} \Psi \left\langle E^-, E^+ \right\rangle = \sum_{i=0}^{|F^-|} \Psi \langle \Sigma_{d-2} \rangle_i .$$

Now we ask what $|F^-|$ (the number of facets in F^-) is. Again this is just the number of facets of Σ_d shelled on before F. Thus our double summation is now

$$\sum_{F \prec \Sigma_d^-} \sum_{E \prec F^-} \Psi \left\langle E^-, E^+ \right\rangle = \sum_{j=0}^{k} \sum_{i=0}^{j} \Psi \left\langle \Sigma_{d-2} \right\rangle_i .$$

Reversing the order of summation gives us the result we want.

Note that although there are two variables in the recursion, the set of initial values $k = 0$ is sufficient since k is bounded above by $d + 1$. □

Example 3.3.2. We compute $\Psi \left\langle \Sigma_5 \right\rangle_3$ using Proposition 3.3.1.

$$\begin{aligned} \Psi \left\langle \Sigma_5 \right\rangle_3 = {} & \Psi\Sigma_5 - 3\Psi\Sigma_4\mathbf{c} \\ & - (3-0-1)\Psi\Sigma_3(2\mathbf{d} - \mathbf{c}^2) - (3-1-1)\Psi \left\langle \Sigma_3 \right\rangle_1 (2\mathbf{d} - \mathbf{c}^2). \end{aligned}$$

Apply Proposition 3.3.1 again to $\Psi \left\langle \Sigma_3 \right\rangle_1$.

$$\Psi \left\langle \Sigma_3 \right\rangle_1 = \Psi\Sigma_3 - \Psi\Sigma_2\mathbf{c}.$$

So,

$$\begin{aligned} \Psi \left\langle \Sigma_5 \right\rangle_3 &= \Psi\Sigma_5 - 3\Psi\Sigma_4\mathbf{c} - 2\Psi\Sigma_3(2\mathbf{d} - \mathbf{c}^2) - (\Psi\Sigma_3 - \Psi\Sigma_2\mathbf{c})(2\mathbf{d} - \mathbf{c}^2) \\ &= \Psi\Sigma_5 - 3\Psi\Sigma_4\mathbf{c} - 3\Psi\Sigma_3(2\mathbf{d} - \mathbf{c}^2) + \Psi\Sigma_2\mathbf{c}(2\mathbf{d} - \mathbf{c}^2) \\ &= \mathbf{c}^5 + 4\mathbf{dc}^3 + 9\mathbf{cdc}^2 + 9\mathbf{c}^2\mathbf{dc} + 12\mathbf{d}^2\mathbf{c} + 4\mathbf{c}^3\mathbf{d} + 10\mathbf{dcd} + 12\mathbf{cd}^2 \\ &\quad - 3\left(\mathbf{c}^4 + 3\mathbf{dc}^2 + 5\mathbf{cdc} + 3\mathbf{c}^2\mathbf{d} + 4\mathbf{d}^2\right)\mathbf{c} \\ &\quad - 3\left(\mathbf{c}^3 + 2\mathbf{dc} + 2\mathbf{cd}\right)(2\mathbf{d} - \mathbf{c}^2) \\ &\quad + \left(\mathbf{c}^2 + \mathbf{d}\right)\mathbf{c}(2\mathbf{d} - \mathbf{c}^2) \\ &= 0. \end{aligned}$$

This computation agrees with the fact that $\left\langle \Sigma_5 \right\rangle_3$ does not change the number of faces in the patch, but only changes their orientation. In each even dimension ($\left\langle \Sigma_5 \right\rangle_3$ is a 4-dimensional flip) there is such a flip that is an analogue of the familiar $\left\langle \Sigma_3 \right\rangle_2$ that "flips" the diagonal of a triangulated quadrilateral. ♢

3.3.2 Explicit formula

When S-shelling a simplicial sphere, the change to the h-vector (and thus the **cd**-index) that occurs at each step is the same as occurs during some particular shelling step in shelling a simplex. Stanley denotes the change to the **cd**-index when shelling on the i^{th} facet (counting from zero) of Σ_d by $\check{\Phi}_i^d$. That is,

$$\check{\Phi}_i^d = \Psi\left(\Lambda_{i+1}^d\right) - \Psi\left(\Lambda_i^d\right),$$

where Λ_i^d is the semi-suspension of the complex consisting of the first i facets of Σ_d. These form a basis for the space of **cd**-indices of simplicial Eulerian posets so that for any such poset P,

$$\Psi(P) = \sum_{i=0}^{d-1} h_i(P)\check{\Phi}_i^d.$$

See [35]. We call these polynomials $\check{\Phi}_i^d$, introduced by Stanley, the ith*shelling components* of the **cd**-index.

Note that the semi-suspension of d facets of Σ_d is equivalent to all of Σ_d. Furthermore, Σ_d is already a sphere, so the semi-suspension is undefined. Thus we simply take $\Lambda_d^d = \Lambda_{d+1}^d = \Sigma_d$. This means that $\check{\Phi}_d^d = 0$.

We know that the h-vector of $\langle \Sigma_d \rangle_k$, when $k \leq d-k$, is given by

$$h_i = \begin{cases} 1 & \text{if } k \leq i \leq d-k, \\ 0 & \text{otherwise.} \end{cases}$$

So therefore we know that

$$\Psi \langle \Sigma_d \rangle_k = \sum_{i=k}^{d-k} \check{\Phi}_i^d.$$

Of course this does little good unless we understand what these basis polynomials $\check{\Phi}_i^d$ are. Ehrenborg and Readdy [12] shed some light on them by applying coalgebraic techniques. In particular, they show that the shelling components can be computed recursively by means of a derivation on **cd**-polynomials. Define G to be a derivation on **cd**-polynomials such that $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$.

Theorem 3.3.3 (Ehrenborg–Ready)**.** *The following recursion holds for* $\check{\Phi}_i^d$*:*

$$G\left(\check{\Phi}_i^d\right) = \check{\Phi}_{i+1}^{d+1}.$$

As a base for this recursion we observe that $\check{\Phi}_0^d = \Psi(\Sigma_{d-1})\mathbf{c}$. This is the **cd**-index of the suspension of Σ_{d-1}.

3.3.3 Another recursion

First we consider the difference of shelling components of the **cd**-index.

Proposition 3.3.4.

$$\Psi \langle \Sigma_d \rangle_k \mathbf{c} = \check{\Phi}_k^{d+1} - \check{\Phi}_{d-k+1}^{d+1}.$$

Proof. Consider the quantity $\check{\Phi}_k^{d+1} - \check{\Phi}_{d-k+1}^{d+1}$. If we write this in terms of the Λ complexes we have

$$\Psi\left(\Lambda_{k+1}^{d+1}\right) - \Psi\left(\Lambda_k^{d+1}\right) - \Psi\left(\Lambda_{d-k+2}^{d+1}\right) + \Psi\left(\Lambda_{d-k+1}^{d+1}\right).$$

Now observe that we can take the two positive complexes, pull off the caps, and glue them together to form a complete $(d+1)$-simplex. The two caps then fit together to form the suspension of the shared boundary of the two complexes. We can do likewise with the negative complexes. Since the **cd**-index of the suspension of a sphere is the **cd**-index of the sphere times $\mathbf{c}$, we can rewrite our quantity (with a small abuse of notation) as

$$\Psi\left(\Sigma_{d+1}\right) + \Psi\left(\partial\Lambda_{k+1}^{d+1}\right)\mathbf{c} - \Psi\left(\Sigma_{d+1}\right) - \Psi\left(\partial\Lambda_k^{d+1}\right)\mathbf{c} = \left[\Psi\left(\partial\Lambda_{k+1}^{d+1}\right) - \Psi\left(\partial\Lambda_k^{d+1}\right)\right]\mathbf{c}.$$

This quantity is measuring how the **cd**-index of the boundary changes when we shell on a simplex. This is precisely the **cd**-index of a bistellar flip. The new simplex in Λ^{d+1}_{k+1} intersects all k of the simplices of Λ^{d+1}_{k}, so in particular we have this is the **cd**-index of the bistellar flip $\langle\Sigma_d\rangle_k$.

$\square$

Proposition 3.3.5. $\Psi\langle\Sigma_d\rangle_k = \frac{1}{2}G\left[\Psi\langle\Sigma_{d-1}\rangle_k + \Psi\langle\Sigma_{d-1}\rangle_{k-1} + \Psi\langle\Sigma_{d-2}\rangle_{k-1}\mathbf{c}\right].$

Proof. By applying the derivation G to $\Psi\langle\Sigma_d\rangle_k$ we see

$$\begin{aligned} G\left(\Psi\langle\Sigma_d\rangle_k\right) &= \sum_{i=k}^{d-k} G\left(\check{\Phi}^d_i\right) \\ &= \sum_{i=k}^{d-k} \check{\Phi}^{d+1}_{i+1} \\ &= \Psi\langle\Sigma_{d+1}\rangle_{k+1} + \check{\Phi}^{d+1}_{d-k+1} \\ &= \Psi\langle\Sigma_{d+1}\rangle_{k} - \check{\Phi}^{d+1}_{k}. \end{aligned}$$

Shifting indices on the last two lines, we can solve for $\Psi\langle\Sigma_d\rangle_k$ to get

$$\begin{aligned} \Psi\langle\Sigma_d\rangle_k &= G\Psi\langle\Sigma_{d-1}\rangle_k + \check{\Phi}^d_k && (3.5) \\ &= G\Psi\langle\Sigma_{d-1}\rangle_{k-1} - \check{\Phi}^d_{d-k}. && (3.6) \end{aligned}$$

If we add these together, we can apply the G recursion on $\check{\Phi}$ and Proposition 3.3.4.

$$\begin{aligned} 2\Psi\langle\Sigma_d\rangle_k &= G\Psi\langle\Sigma_{d-1}\rangle_k + G\Psi\langle\Sigma_{d-1}\rangle_{k-1} + \check{\Phi}^d_k - \check{\Phi}^d_{d-k} \\ &= G\left[\Psi\langle\Sigma_{d-1}\rangle_k + \Psi\langle\Sigma_{d-1}\rangle_{k-1} + \check{\Phi}^{d-1}_{k-1} - \check{\Phi}^{d-1}_{d-k-1}\right] \\ &= G\left[\Psi\langle\Sigma_{d-1}\rangle_k + \Psi\langle\Sigma_{d-1}\rangle_{k-1} + \Psi\langle\Sigma_{d-2}\rangle_{k-1}\mathbf{c}\right] \end{aligned}$$

$\square$

Example 3.3.6. We use Proposition 3.3.5 to calculate $\Psi\langle\Sigma_5\rangle_2$ from the **cd**-index of lower dimensional bistellar flips.

$$\begin{aligned} \Psi\langle\Sigma_5\rangle_2 &= \frac{1}{2}G\left[\Psi\langle\Sigma_4\rangle_2 + \Psi\langle\Sigma_4\rangle_1 + \Psi\langle\Sigma_3\rangle_1\mathbf{c}\right] \\ &= \frac{1}{2}G\left[\left(\mathbf{cdc} + \mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2\right) + \left(\mathbf{dc}^2 + 3\mathbf{cdc} + 3\mathbf{c}^2\mathbf{d} + 4\mathbf{d}^2\right) + \left(\mathbf{dc}^2 + 2\mathbf{cdc}\right)\right] \\ &= G\left[\mathbf{cdc}^2 + 3\mathbf{cdc} + 2\mathbf{c}^2\mathbf{d} + 3\mathbf{d}^2\right] \\ &= \left(\mathbf{cdc}^2 + \mathbf{d}^2\mathbf{c} + \mathbf{dcd}\right) + 3\left(\mathbf{d}^2\mathbf{c} + \mathbf{c}^2\mathbf{dc} + \mathbf{cd}^2\right) \\ &\quad + 2\left(\mathbf{dcd} + \mathbf{cd}^2 + \mathbf{c}^3\mathbf{d}\right) + 3\left(\mathbf{cd}^2 + \mathbf{dcd}\right) \\ &= \mathbf{cdc}^2 + 3\mathbf{c}^2\mathbf{dc} + 4\mathbf{d}^2\mathbf{c} + 2\mathbf{d}^2\mathbf{c} + 6\mathbf{dcd} + 8\mathbf{cd}^2. \end{aligned}$$

$\diamond$

3.3.4 Term-wise explicit formula

Looking at tables of coefficients of the **cd**-indices of bistellar flips in various dimensions, we make two observations. First, looking in reverse lexicographic order, all the zeros occur as an initial string. Secondly, the first non-zero coefficients, again in reverse lexicographic order, are precisely the coefficients of **cd**-indices of simplices.

Verifying these observations is our motivation for the following proposition. We want to compute the coefficient of a particular **cd**-word in the **cd**-index of a simplicial flip.

Let $[w]$ denote the coefficient of the **cd**-word w in a given **cd**-polynomial and let u_T be the **cd**-word corresponding to the subset $T \subseteq \{1, \ldots, e-1\}$ in the following manner:

Write the characteristic 01-word for the set T. That is, the digit in position i is 1 if $i \in T$ and 0 if $i \notin T$. Append a 0 at the right end of the word. Now for each 1, replace the 1 and the digit to its right with **d** and then replace each remaining 0 with **c**. We will use the notation $\breve{S}$ to indicate an arbitrary choice of one of the two sets that has S as its "difference set". The difference set $\widehat{S}$ of $S \subseteq \{0, \ldots, d-1\}$ is the set

$$\{i \in \{1, \ldots, e-1\} : (i \in S \text{ and } i-1 \notin S) \text{ or } (i \notin S \text{ and } i-1 \in S)\}.$$

Proposition 3.3.7. *If $h = (h_0, h_1, \ldots, h_d)$ is a simplicial h-vector satisfying the Dehn-Sommerville relations $h_i = h_{d-i}$, then the corresponding* **cd***-index is given by*

$$[u_T] = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \sum_{R \subseteq \breve{S}} (-1)^{|\breve{S} \setminus R|} \left(\prod_{i=1}^{|R|-1} \binom{r_{i+1}+1}{r_i+1} \right) \left(\sum_{i=0}^{r_{|R|}+1} \binom{e-i}{e-r_{|R|}-1} h_i \right),$$

Proof. The basic flow of the conversion is thus: simplicial h-vector $\to$ simplicial f-vector $\to$ flag f-vector $\to$ flag h-vector "=" **ab**-index $\to$ **cd**-index.

Given a simplicial h-vector, one forms the f-vector with the well know relation

$$f_i = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i.$$

The entires f_T of the flag f--vector is determined by the simplicial f-vector entry for the maximum element of T. There are $f_{t_{|T|}}$ faces of dimension $t_{|T|}$ and each is a simplex. The number of T-chains passing though a given $t_{|T|}$ is the product of binomial coefficients. Each binomial coefficient counts the number of ways to choose one t_i-face contained in a given t_{i+1}-simplex.

$$f_T = \left[\prod_{i=1}^{|T|-1} \binom{t_{i+1}+1}{t_i+1} \right] f_{t_{|T|}}.$$

From the we flag f-vector, we obtain the flag h-vector according to the definition.

$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

Now we need to express the coefficients of the **cd**-index in terms of the coefficients of the **ab**-index (the flag h-vector). To make this clear will we need some notation. For any set $S \subseteq \{0, \ldots, d-1\}$, we can form the difference set $\widehat{S} = \{i \in \{1, \ldots, d-1\} : |\{i, i-1\} \cap S| = 1\}$. Further given a set $T \in \{1, \ldots, d-1\}$ we can form two sets S such that $\widehat{T} = S$. We will denote an arbitrary choice of one of these two sets by $\breve{S}$. We will write $\widehat{h}_T$ to mean $h_{\breve{T}}$.

We need to know which **cd**-words contribute to the coefficient of a given **ab**-word. That is, given an **ab**-word v, which **cd**-words have v in their **ab**-expansions?

Observe that we can view a **cd**-word as a description of the set of **ab**-words that appear in its expansion. To create an **ab**-word that will appear in the expansion, we choose **ab** or **ba** for each **d** and we choose **a** or **b** for each **c**. It is even more profitable to think of the **d**s as marking pairs of positions at which the letters in the **ab**-word must be **d**ifferent. The **c**s can then still be seen as marking single places where an arbitrary letter is needed.

So a **cd**-word contributes to h_S corresponding to an **ab**-word w if the w has differences in at least the places mandated by the **d**s. It is convenient to assign a set $T \subseteq \{1, \ldots, d-1\}$ to each **cd**-word. u_T is the **cd**-word corresponding to T in the following manner:

Write a 01-word by replacing **d** with 10 and **c** with 0 and then remove the final 0. Now T is the set with this characteristic 01-word. That is, the digit in position i is 1 if $i \in T$ and 0 if $i \notin T$.

The **cd**-words contributing to h_S are precisely those of the form u_T where $T \subseteq \widehat{S}$. So we could now write a large and highly redundant system of linear equations. For each $S \subseteq \{0, \ldots, d-1\}$,

$$\sum_{T \subseteq \widehat{S}} [u_T] = h_S.$$

To make this system easy to solve, we will choose a subset of Fibonacci many of these constraints and thus remove all redundancy. There is one constraint for each h_S, so we need to choose Fibonacci many subsets of $\{0, \ldots, d-1\}$. Observe that there are the correct Fibonacci number of subsets of $\{1, \ldots, d-1\}$ that have no consecutive elements. The **ab**-words with these as their difference sets are precisely those that have a unique **cd**-word with maximal **d**s that contributes to it. Our special subset of constraints is for each $S \subset \{1, \ldots, d-1\}$ with no consecutive elements,

$$\sum_{T \subseteq S} [u_T] = h_{\breve{S}}.$$

The associated matrix for this system is triangular, and so we can apply back substitution to get

$$[u_T] = \sum_{S \subset T} (-1)^{|T \setminus S|} h_{\breve{S}}.$$

Note that since T must necessarily have no consecutive entries, the sum is already restricted to subsets S that contain no consecutive elements.

Now we can simply nest our string of transformations to obtain the result. □

Proposition 3.3.8. *For $k \leq \lfloor d/2 \rfloor$,*

$$\Psi \left\langle \Sigma_d \right\rangle_k = \Psi(\Sigma_{k-1}) \cdot \mathbf{dc}^{d-k-1} + \Omega,$$

where the terms of Ω have at most $d-k-2$ final $\mathbf{c}$*s.*

Proof. We show that

1. $[v\mathbf{c}^{d-k}]\Psi \left\langle \Sigma_d \right\rangle_k = 0$.
2. $[w\mathbf{dc}^{d-k-1}]\Psi \left\langle \Sigma_d \right\rangle_k = [w]\Psi(\Sigma_{k-1})$.

We know that

$$h_i \left\langle \Sigma_d \right\rangle_k = \begin{cases} 1 & \text{if } k \leq i \leq d-k, \\ 0 & \text{if } i < k \text{ or } i > d-k. \end{cases}$$

So we can apply the formula

$$[u_T]\Delta_k \Psi(\Sigma_d) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \sum_{R \subseteq \breve{S}} (-1)^{|\breve{S} \setminus R|} \left(\prod_{i=1}^{|R|-1} \binom{r_{i+1}+1}{r_i+1} \right) \left(\sum_{i=0}^{r_{|R|}+1} \binom{d-i}{d-r_{|R|}-1} h_i \left\langle \Sigma_d \right\rangle_k \right).$$

1. Now we observe that for $u = v\mathbf{c}^{d-k}$, the corresponding difference set T has maximum element $t_{|T|} \leq k-1$. To see this apply the algorithm for converting a $\mathbf{cd}$-word to the difference set:

   a) Replace each $\mathbf{d}$ with 10.

   b) Replace each $\mathbf{c}$ with 0.

   c) Remove the final 0.

   d) Convert the characteristic 01-word to a set.

   Since u ends in c^{d-k}, we know we have at least $d-k-1$ final 0s in the characteristic 01-word. If the last letter of v is $\mathbf{d}$, then this becomes 10 giving one more final 0. If the last letter of v is $\mathbf{c}$, we still have one more final 0. Thus the last 1 is no further right than the $(k-1)^{\text{st}}$ position.

   Now consider $\breve{T}$. We will choose the set with the smaller maximum element, so the corresponding $\mathbf{ab}$-word will end in $\mathbf{a}$. To compute the $\mathbf{ab}$-word corresponding to $\breve{T}$ from the 01-word representing T:

   a) Write $\mathbf{a}$.

   b) Reading writing right to left, write the same letter as the previous spot for a 0 and the opposite letter for a 1.

So the rightmost **b** (corresponding to the maximum element of $\breve{T}$) is no further right than $(k-2)$nd position (counting from zero this time, since the **ab**-word represents as subset of $\{0,\ldots,k-1\}$) because we have at least $d-k$ final **a**s. Furthermore, for any subset $S \subseteq T$, the maximum element of $\breve{S}$ will be at most $k-2$, and thus the maximum element of any subset R of $\breve{S}$ is also bounded by $k-2$.

But looking at our formula, we see that every term has a factor of $h_i \langle \Sigma_d \rangle_k$ where $i \leq r_{|R|}+1$, where $R \subseteq \breve{S}$ and $S \subset T$. Thus $i \leq r_{|R|}+1 \leq k-2+1 = k-1 < k$ and so $h_i \langle \Sigma_e \rangle_k = 0$. Therefore $[v\mathbf{c}^{d-k}]\Psi \langle \Sigma_d \rangle_k = 0$.

2. Now we turn our attention to **cd**-words of the form $w\mathbf{dc}^{d-k-1}$. The corresponding difference set T has maximum element k, and so $\breve{T}$ (again taking the set with the smaller maximum element) is $k-1$. As before our formula involves h_i only up to $i = \max \breve{T}+1 = k$. So the only nonzero terms we have are when $\max R = \max \breve{S} = \max \breve{T} = k-1$. Let $A = T \setminus \{k-1\}$. We now rewrite the formula in terms of A since every set that contributes a nonzero term must contain $k-1$.

$$[w\mathbf{dc}^{d-k-1}]\Psi \langle \Sigma_d \rangle_k = \sum_{S\subseteq A}(-1)^{|A\setminus S|}\sum_{R\subseteq \breve{S}}(-1)^{|\breve{S}\setminus R|}\left(\prod_{i=1}^{|R|-1}\binom{r_{i+1}+1}{r_i+1}\right)\binom{(k-1)+1}{r_{|R|}+1}.$$

Observe that $w = u_A$. Since we have

$$[w]\Psi(\Sigma_{k-1}) = \sum_{S\subseteq A}(-1)^{|A\setminus S|}\sum_{R\subseteq \breve{S}}(-1)^{|\breve{S}\setminus R|} \left(\prod_{i=1}^{|R|-1}\binom{r_{i+1}+1}{r_i+1}\right)\left(\sum_{i=0}^{r_{|R|}+1}\binom{d-i}{d-r_{|R|}-1}h_i(\Sigma_{k-1})\right),$$

we need only show

$$\sum_{i=0}^{r_{|R|}+1}\binom{d-i}{d-r_{|R|}-1}h_i(\Sigma_{k-1}) = \binom{(k-1)+1}{r_{|R|}+1}.$$

The left-hand side is the general formula for the $r_{|R|}$ term of the f-vector in terms of the h-vector, and right-hand side is precisely the number of $r_{|R|}$-faces of a $(k-1)$-simplex.

Thus we have $[w\mathbf{dc}^{d-k-1}]\Psi \langle \Sigma_d \rangle_k = [w]\Psi(\Sigma_{k-1})$.

□

Example 3.3.9. Suppose $u = \mathbf{dcdc}^5$. We compute the difference set corresponding to this **cd**-word.

1. **dcdccccc**
2. 10**c**10**ccccc**
3. 1001000000
4. 100100000
5. $\{1, 4\} = T$

We now compute $\breve{T}$. We have $T = \{1, 4\}$, which corresponds to 100100000.

1. **a**
2. $\mathbf{abbbaaaaaa} = \mathbf{ab}^3\mathbf{a}^6$

The letter changes at the sixth and ninth positions from the right. This corresponds to $\breve{T} = \{1, 2, 3\}$. $\Diamond$

3.4 Flips of complexity 1

We now turn our attention to simplicial flips that involve one more vertex than bistellar flips. These are the simplicial flips of complexity 1. Recall that this means that the union of the old and new facets is a simplicial d-polytope with $d+2$ vertices.

Small polytopes, that is, d-polytopes with $d + 2$ vertices, are almost simplices and have structure that is almost as nice as a simplex. Each facet of a simplex is determined by the single vertex it does not contain. In a small polytope, each facet is determined by the two vertices not contained in the facet. This structure is easily understood by looking at the Gale diagram.

The *Gale transform* of a collection of n (possibly repeated) points in $\mathbb{R}^d$ is a collection of n (possibly repeated) points in $\mathbb{R}^{n-d-1}$. This is a duality relation in the sense that a set of points S is the same as the the Gale transform of the gale transform of S. When it is the combinatorial structure of the convex hull of the points that is of interest, one usually uses a Gale diagram instead of a Gale transform. If P is a d-dimensional polytope with vertex set V having n vertices, then a configuration G of n points in $\mathbb{R}^{n-d-1}$ is a *Gale diagram* of V (or P) if the convex hull of the Gale transform of G is combinatorially equivalent to P. Thus, rather than a unique dual, the Gale diagram is any representative from an equivalence class of Gale transforms that corresponds to the class of combinatorially equivalent polytopes. We can choose the Gale diagram so that all the points lie on a unit sphere.

Suppose G is a Gale diagram of a polytope P and we use the same label set V for the points in both the polytope and its gale diagram. Then a set $S \subseteq V$ is the label set of the vertices of a face of P (we will simply say S *is* a face of P) if and only if the compliment $S^c = V \setminus S$ "captures the origin" in G. We say that a set of points *catptures the origin* if the origin is contained in the relative interior of the convex hull

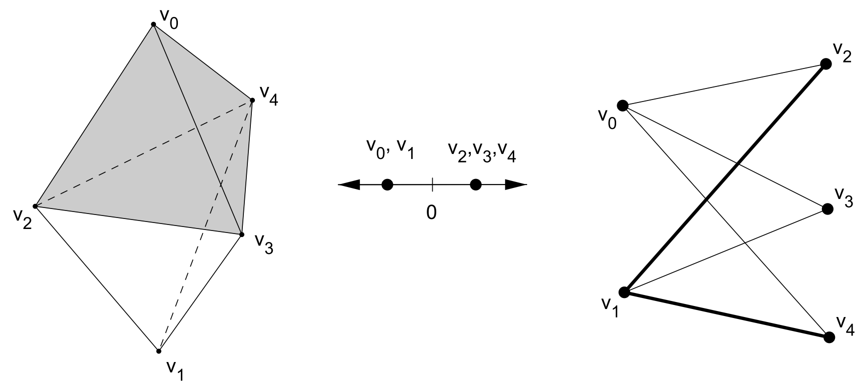

Figure 3.6: A patch of facets of $P_{2,3}$

of the set. When S is a face of P, we say S^c is a coface of S. A maximal face is a facet, so a minmal set that captures the origin in the Gale diagram is a *cofacet*.

The Gale diagram of a small d-polytope has dimension $(d+2)-d-1=1$. So the points are naturally divided into three classes: the negative points, the positive points, and the points at the origin. In the case of simplicial polytopes, there are no points at the origin. This gives a partition of the vertices into two sets. This is in fact the Radon partition, and further gives the only two minimal non-faces of the polytope. A cofacet in the Gale diagram is a pair of points that capture the origin. This happens exactly when the pair consists of one point from each set. Thus the facets are determined by such pairs. See Chapter 6 of [15] for more on polytopes with few vertices and their Gale diagrams.

What can a Gale diagram tell us about a small simplicial flip? We first must know what a shellable subset of the facets looks like in the Gale diagram. Let us denote by $P_{m,n}$ the small simplicial polytope with a Radon partition with sizes m and n. Rather than the usual Gale diagram of $P_{m,n}$, we will consider the complete bipartite graph $K_{m,n}$. The vertices of the graph correspond to the vertices of the polytope. Each edge corresponds to the facet that contains all vertices except the endpoints. A subset of facets of $P_{m,n}$ then corresponds to a subset of the edges of $K_{m,n}$. Figure 3.6 shows some facets of $P_{2,3}$ highlighted, the Gale diagram of $P_{2,3}$, and $K_{2,3}$ with the corresponding edges highlighted.

A simplicial complex is shellable if and only if there is an ordering on the facets such that at each shelling step there is a unique minimal new face contained in the new facet. Translating this we get that a subset of edges of $K_{m,n}$ corresponds to a shellable complex if and only if there is an ordering of the edges such that at each step there is a *unique maximal new superset* of the new edge. By *unique maximal new superset* of the new edge, we mean a set of vertices of the graph that contains the endpoints of the new edge and does not contain both endpoints of any other edge such that every other such superset has fewer vertices.

Definition 3.4.1. The *edge diameter* of a graph is the maximum over pairs of edges of the number of interior vertices in a minimal path containing that pair of edges.

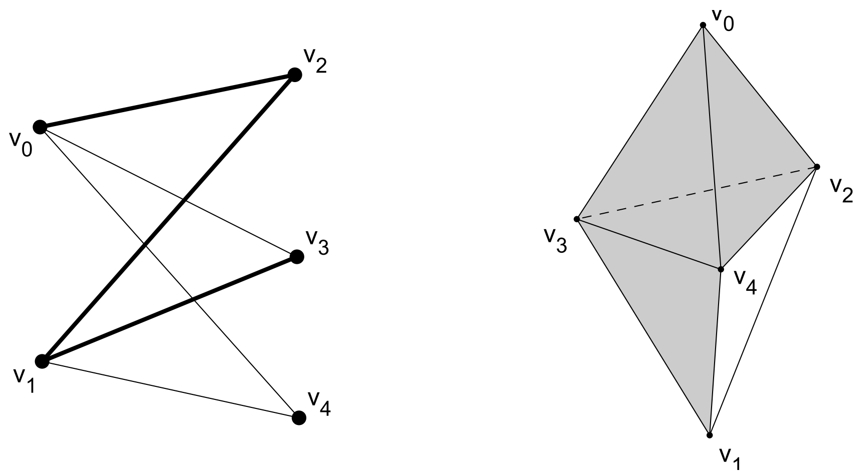

Figure 3.7: A shellable subset of $P_{2,3}$

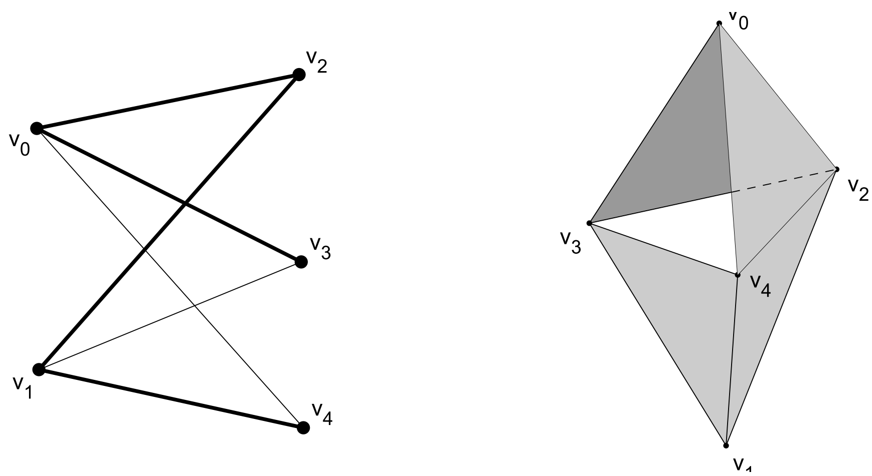

Figure 3.8: A non-shellable subset of $P_{2,3}$

Compare this to the usual diameter, which is maximum over pairs of vertices of the number of edges in a minimal path containing that pair of vertices.

Figure 3.7 shows a shellable collection of facets of $P_{2,3}$ and the corresponding subset of $K_{2,3}$. Figure 3.8 shows a non-shellable collection and the corresponding subset of edges.

Proposition 3.4.2. *A subgraph $G \subseteq K_{m,n}$ has an ordering on the edges such that when adding the edges one at a time there is a unique maximal superset of the new edge containing now other edges at each step if and only if there is an ordering so that the edge diameter of G is at most 2 at each step.*

Proof. We first show that the diameter condition is necessary. We assume throughout that the edges of G are connected, since a disconnected graph corresponds to a set of facets that is not shellable.

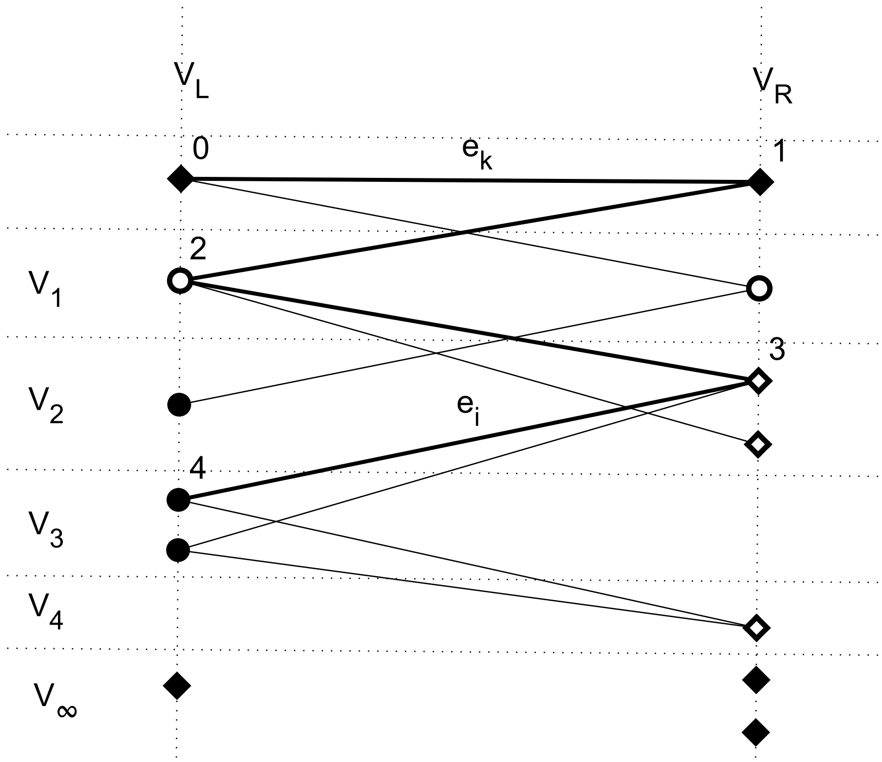

Figure 3.9: Graph with large edge diameter

Suppose that $G \subseteq K_{m,n}$ has edge diameter at least 3 and let $G = (e_1, e_2, e_3, \ldots, e_k)$ be any ordering of the edges of G. There must exist a pair of edges e_i, e_j that are exactly three vertices apart along the minimal path between them. Without loss of generality we may assume that $e_j = e_k$. We will label the vertices so that we have $e_k = \{0, 1\}$, $e_i = \{3, 4\}$ and so that the three vertices between e_k and e_i on some shortest path are 1, 2, and 3.

We partition the vertices into $V(K_{m,n}) = e_k \sqcup V_1 \sqcup V_2 \sqcup \cdots \sqcup V_l \sqcup V_\infty$ according to their distance from e_k. We also have a natural bipartition $V_i = V_i^L \sqcup V_i^R$ (*L*eft and *R*ight subsets) induced by the bipartition V^L, V^R on the whole graph where vertex 0 is in V^L. We know that V_1, V_2, and V_3 are non-empty since $2 \in V_1^L, 3 \in V_2^R$, and $4 \in V_3^L$. If S is a unique maximal new superset of e_k, then $\{0, 1\} \subset S$, $V_1 \cap S = \varnothing$, and $V_\infty \subset S$. However, we have a choice for the remaining vertices. We can safely add any of the remaining vertices from V^L or from V^R without including both endpoints of any other edge, since there are no edges between vertices on the same side. So to be maximal we may include either all of the remaining vertices on the left, or all the remaining ones or the right. Thus S cannot be maximal. See Figure 3.9 for an example. The diamond shaped vertices form one maximal new superset of e_k, and the black solid vertices form another.

To show that the condition is sufficient, we suppose instead that G has edge diameter at most two. Then we can repeat the above construction, except this time

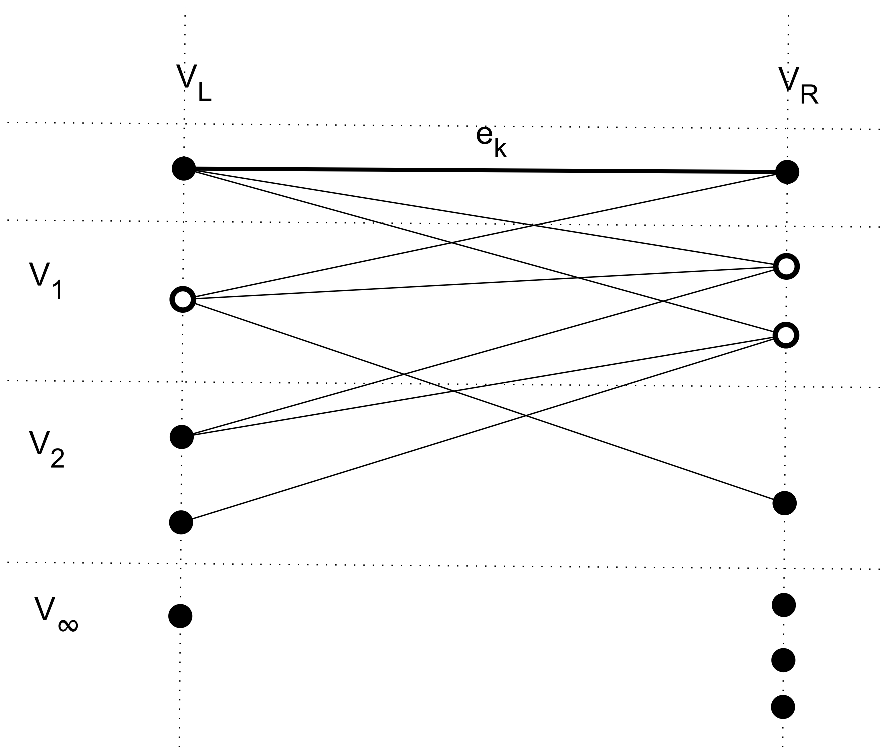

Figure 3.10: Graph with small edge diameter

we know that there is no minimal chain of edges with more than two interior vertices. This means that the partition is $V = e_k \sqcup V_1 \sqcup V_2 \sqcup V_\infty$. Further, there can be no edges between vertices in V_2 since such an edge would be three vertices from e_k, or between edges in V_∞ since they would be infinitely far away from e_k. So a new superset can contain all of V_2. Thus the set $S = e_k \sqcup V_2 \sqcup V_\infty = V \setminus V_1$ is the unique maximal new superset of e_k. See Figure 3.10. The solid vertices form the unique new superset of e_k.

□

We can improve this characterization by looking at the bipartite incidence matrix of the subgraph. For a subgraph $G \subseteq K_{m,n}$, define an $m \times n$ incidence matrix $M(G)$ with rows and columns corresponding to the vertices of $K_{m,n}$. Thus each entry in $M(G)$ corresponds to an edge in $K_{m,n}$, where $M_{i,j} = 1$ if $\{i, j\} \in E(G)$ and $M_{i,j} = 0$ otherwise.

If G has edge diameter two, then every pair of edges either share an endpoint or have endpoints connected by a third edge. Edges that share an endpoint correspond to entries in M that are in the same row or the same column. So a 0/1-matrix corresponds to a bipartite graph with edge diameter at most two if for every pair of non-zero entries that do not share a row or column, there is a third non-zero entry that shares a row with one and a column with the other.

Since we care only about the combinatorial structure of the collection of facets of $P_{m,n}$, we do not care about the labels on the vertices of the polytope, the graph, or the matrix. So two matrices are equivalent if they differ only by a permutation of the rows and columns.

Proposition 3.4.3. *Let M be an $m \times n$ $0/1$-matrix such that for each pair of entires $M_{i,j} = M_{k,l} = 1$, $i = k$, $j = l$, $M_{i,l} = 1$, or $M_{k,j} = 1$. Then M is equivalent to a matrix in which every entry above or to the left of a nonzero entry is also nonzero.*

Proof. Let M be a matrix with the property in question. Sort the rows left to right and then the columns top to bottom in order of decreasing number of positive entries. Suppose that we have $M_{i,j} = 0, M_{i,l} = M_{k,j} = M_{k,l} = 1$, where $i < k$ and $j < l$. Since row i must have at least as many positive entries as row k, there must be a column c such that $M_{i,c} = 1$ and $M_{k,c} = 0$.

$$\begin{array}{cccccc} & j & & l & & c \\ i & 0 & \cdots & 1 & \cdots & 1 \\ & \vdots & & \vdots & & \vdots \\ k & 1 & \cdots & 1 & \cdots & 0 \end{array}$$

But then $M_{i,c} = M_{k,j} = 1$ and $M_{k,c} = M_{i,j} = 0$, which violates the property in question. □

Observe that an array of boxes corresponding to the non-zero entires of a 0/1-matrix M as in Proposition 3.4.3 in which every entry above or to the left of a nonzero entry is also nonzero is precisely what is called a Young diagram. Thus we will call such a matrix a *Young diagram matrix.* The combinatorial difference between a Young diagram matrix and a Young diagram is that extracting the diagram from the matrix loses the information of dimensions of the matrix.

Corollary 3.4.4. *Since every Young diagram can be built up from the empty diagram with a Young diagram at each step, the edge diameter condition alone, without providing a shelling order, is enough to guarantee shellability.*

Corollary 3.4.5. *A collection of facets of a small simplicial polytope is shellable if and only if for every pair of facets, either thy are adjacent or each is adjacent to a common third facet. Here two facets are adjacent if they intersect in a subfacet.*

Corollary 3.4.6. *Since shellable patches on small simplicial polytopes are indexed by Young diagram matrices, there $\binom{m+n}{m}$ small simplicial flips defined on $P_{m,n}$ unless $m = n$.*

There is an additional type of automorphism of $P_{m,m}$ (swapping the two groups of vertices) that is not accounted for by swapping rows and columns. Thus if $M' = M^T$, then the flips corresponding to to those matrices will be combinatorially the same, but are counted separately by the binomial coefficient above. Since $P_{m,m}$ is in dimension $2(m-1)$, this means that there are more small simplicial flips in odd dimensions than in even dimensions.

Proposition 3.4.7. *Every small simplicial flip defined on $P_{m,n}$ is achievable by a sequence of exactly m bistellar flips. Further, this is the shortest such sequence.*

Proof. Consider a given flip $\langle P^-_{m,n}, P^+_{m,n}\rangle$ defined on $P_{m,n}$ where $m \leq n$. Let V be the labeled point set consisting of the vertices of $P_{m,n}$ labeled 1 to $m+n$, and one additional point labeled 0 positioned such that exactly the facets of $P^+_{m,n}$ are visible from 0. The convex hull of V is a polytope on which the flip $\langle P^-_{m,n}, P^+_{m,n}\rangle$ may be performed.

Recall that the dimension of a Gale diagram of n points in $\mathbb{R}^d$ is $n-d-1$. So in this case, the dimension of the Gale diagram is $(m+n+1)-(m+n-2)-1 = 2$. No face of $\text{conv}(V)$ consists of all but one point of V, so there are no points at the origin. Thus the standard Gale diagram G of V consists of points labeled $0, 1, \ldots, m+n$ arranged on the unit circle in $\mathbb{R}^2$.

Consider the Gale diagram G' obtained from G by moving the point 0 to its antipodal point. Let $i, j \in V$ be different from 0. If $\{0, i, j\}$ captures the origin in G, then 0 is on the same side of the line ij as the origin. Thus in G' the point labeled 0 is on the opposite side of the line ij from the origin and thus $\{0, i, j\}$ does not capture the origin. Likewise those sets $\{0, i, j\}$ that do not capture the origin in G do capture it in G'. The sets $\{0, i, j\}$ that capture the origin in G correspond to facets of $\text{conv}(V)$ that do not contain 0, which are precisely the facets of $P^-_{m,n}$, and those that capture the origin in G' correspond to the facets of $P^+_{m,n}$.

So applying the flip $\langle P^-_{m,n}, P^+_{m,n}\rangle$ to $\text{conv}(V)$, which has Gale diagram G, yields a new polytope with Gale diagram G'.

We want to achieve this same change by a sequence of bistellar flips. First note that in a Gale diagram consisting of points on the unit circle and the origin, we can slide points on the circle around and as long as the point does not cross a diameter that has a point on the opposite end the combinatorial type of the corresponding polytope does not change. See [15, pp. 109–111] for this fact along with a description of 2-dimensional Gale diagrams.

So what change occurs when we do slide one point (i) past the antipode (or tail) of another point (j)? Suppose $\{i, j, k\}$ is a cofacet before the move. Such a set is a cofacet exactly when each of the three diameters divides the other two points. In particular, i and k are on opposite sides of the diameter through j before the move. Therefore after i crosses the tail of j, i and k are on the same side, and thus the points $\{i, j, k\}$ is no longer a cofacet. This means that one effect of moving i past the tail of j is that every old facet containing neither i nor j ceases to be a facet.

What about sets $\{i, j, k\}$ that do not capture the origin before the move? We are assuming that there are no points between i and the tail of j, nor between j and the tail of i. If k were between the tails of i and j, then $\{i, j, k\}$ would be a cofacet, so k must be between i and j. Thus when i moves past that tail of j, k will be between the tails of i and j, and thus $\{i, j, k\}$ will become a cofacet. So for every $k \neq i, j$, $\{i, j, k\}$ is a cofacet after the move if and only if it was not a cofacet before the move.

Consider the case where we move $i = 0$ past some other point j. Looking at the corresponding polytope what we see is that before the move, we have one a simplicial facet for each point (other than 0 and j) in the Gale diagram on the same side of the diameter through j as 0. After the move we lose all of these and replace them with a simplicial facet for each of the other points in the Gale diagram. All of the cofacets involved contain 0 and j, thus none of the facets contain 0 or j. Further, since each pair of involved facets before the move (and each pair of involved facets after the move) differs by one vertex, they share a subfacet. We can thus describe the effect of moving 0 past a tail in the Gale diagram as replacing some number of pairwise adjacent simplices with another number of pairwise adjacent simplices where the total number of simplices is $m+n+1-2 = (m+n-2)+1$. Thus this change is precisely a bistellar flip. See section 5 of [21] for another depiction of these operations on Gale diagrams of d-polytopes with $d+3$ vertices.

Further, every bistellar flip induces this sort on change in the Gale diagram. Every bistellar flip in dimension $d = m+n-2$ involves exactly $d+1 = m+n-1$ vertices. Thus there are exactly two vertices not contained in any of the faces involved in the flip. So there must be two vertices that are contained in every cofacet involved. But since the polytope is simplicial, every cofacet contains exactly 3 points, so there are exactly 2 points that are contained in all involved cofacets. If there are more than the two points changing their relationship to each other, than this condition of exactly two common points cannot be fulfilled.

Observe that since the diameter through 0 divides the m points from the n points, the point 0 must pass at least $\min\{m, n\}$ tails. Thus a flip defined by $P_{m,n}$ can be achieved by a sequence of m bistellar flips. Further, if $m \leq n$, this is the shortest possible sequence.

□

Figure 3.11 shows the polytope obtained by adding a point to $P_{2,3}$ and the corresponding Gale diagram before and after moving the new point, labeled 0, to its antipode. This flip shown, shown in Figure 3.12, corresponds to the Young diagram matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The corresponding partition is $(1, 1)$.

Proposition 3.4.8. *If λ is a partition with exactly m parts, each between zero and n, and F^n_λ is the flip defined by the corresponding $m \times n$ Young diagram matrix for λ, then*

$$\Psi(F^n_\lambda) = \sum_{i=1}^{m} \Psi \left\langle \Sigma_{m+n-2} \right\rangle_{\lambda_i + i - 1}.$$

Proof. To find an expression for the **cd**-index of small flips in terms of bistellar flips, we need only understand which m bistellar flips produce a given small flip. For this we return to the Gale diagram in the previous proof. We have G the standard Gale

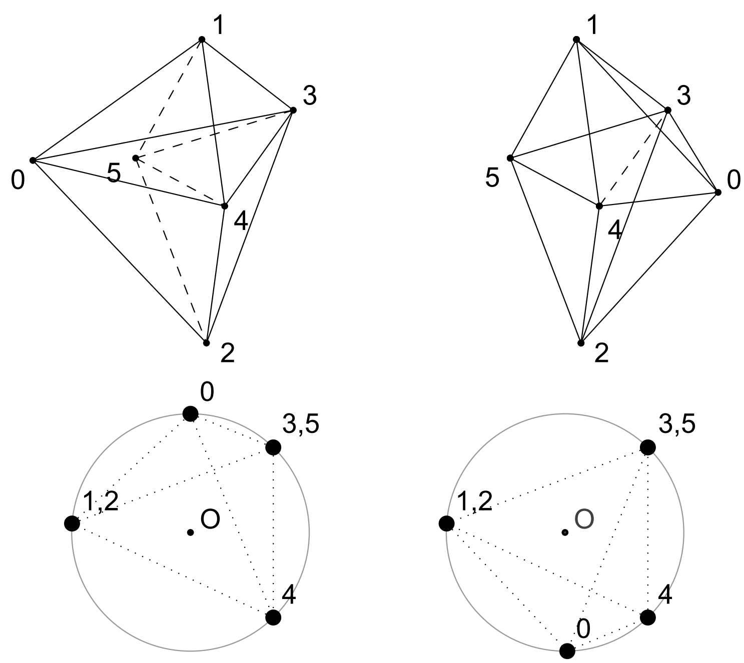

Figure 3.11: Polytope and Gale diagram before (left) and after (right) applying $F^3_{(1,1)}$

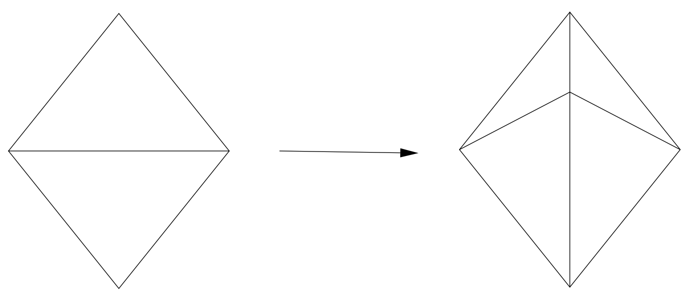

Figure 3.12: The small simplicial flip $F^3_{(1,1)}$

diagram of $P_{m,n}$ with one extra vertex labeled 0 so that the facets in $P^+_{m,n}$ are not part of the convex hull, but those of $P^-_{m,n}$ are. We will assume that 0 is at the bottom of the circle and will pass vertices 1 through m in order. Denote by G_i the intermediate stage achieved after moving 0 past i tails. Thus $G = G_0$, and $G' = G_m$.

We need to determine which bistellar flip occurs as 0 passes the tail of the point i. So in the context of the Gale diagram, we need to know how many cofacets contain both 0 and i in G_{i-1}. This is the number of vertices on the opposite side of the edge $\{0, i\}$. In G_{i-1}, the point 0 is just below the diameter through i, so we need to know the number of points above this diameter. A point j with $1 \leq j \leq m$ will be above this diameter if and only if $j \leq m$. Now consider a point $k > m$. If m is above the diameter at i, then $\{0, i, k\}$ captures the origin in G_0, since 0 starts below the diameter. Thus the (i, k) entry in the matrix corresponding to this flip is a 1.

Suppose that we have any point ℓ such that the (i,ℓ) entry is a one. Then $\{0,i,\ell\}$ captures the origin in G_0 and therefore ℓ is above the diameter at i, since 0 is below it.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be the partition corresponding to the flip in question. Then number of points above the diameter at i in G, and thus the number of facets buried as 0 passes that diameter, is $\lambda_i + i - 1$. This lets us state the result

$$\Psi(F^n_\lambda) = \sum_{i=1}^{m} \Psi \left\langle \Sigma_{m+n-2} \right\rangle_{\lambda_i+i-1}.$$

□

We can also apply the general formula of equation (3.4) to obtain another formula in terms of the bistellar flips.

Proposition 3.4.9. *If M is an $m \times n$ Young matrix, then*

$$\begin{aligned}\Delta\Psi(M) = &\sum_{k=0}^{m-1} \Psi \left\langle \Sigma_{m+n-2} \right\rangle_k \\ &- |M|\Psi(\Sigma_{m+n-3})\mathbf{c} \\ &- \sum_{i=1}^{m}\sum_{j=1}^{n} M_{i,j} \sum_{k=0}^{i+j-3} \Psi \left\langle \Sigma_{m+n-4} \right\rangle_k (2\mathbf{d} - \mathbf{c}^2).\end{aligned}$$

where $|M|$ is the sum of the entries of M (the number of positive entries).

Proof. This follows from Corollary (3.2.9) and Proposition 3.4.9 using the Young matrix interpretation of patches on small polytopes. The first term is $\Psi(P_{m,n})$ written in terms of bistellar flips as in the previous proposition. Since we are working with a simplicial polytope, we need only count the facets in $P^-_{m,n}$ for the second term. For the triple summation we use the shelling order on $P^-_{m,n}$ given by reading all the positive entries of M left to right, top to bottom. So the number of adjacent facets shelled on before facet corresponding to $M_{i,j}$ is just the number of entries above in the same column $(i-1)$ and to the left in the same row $(j-1)$. The intersections with these previous facets are the lower subfacets. So the lower facet corresponding to $M_{i,j}$ will contribute $i+j-2$ terms to the sum. Since all faces are simplices, the flips defined by these lower subfacets are bistellar flips and each lower facet will contribute one bistellar flip of each type from 0 to $i+j-3$. □

3.5 Semi-simplicial flips

Bistellar flips have additional structure that is not captured in the definition of simplicial flips. One aspect that is lost is the fact that a bistellar flip $\left\langle \Sigma_d \right\rangle_k$ "factors" as $\langle \partial\Delta_{k-1} * \Delta_{d-k}, \Delta_{k-1} * \partial\Delta_{d-k}\rangle$, where $*$ denotes the free join. Further the polytope formed by the union of the old and new patches is $\Delta_{k-1} * \Delta_{d-k}$. Here we use Δ_d to denote the simplicial complex consisting of the entire simplex, including the d-face.

Although we already have used Σ_d to denote $\partial\Delta_d$, we will use the latter wherever we want to emphasize the difference between a solid simplex and its boundary.

We can recapture this extra structure by considering generals flips that still have this presentation except with one of the simplices replaced with a generic polytope P. We call such a flip $\langle P * \partial\Delta_k, \partial P * \Delta_k\rangle$ *semi-simplicial*. Note that the complexity of the flip depends only on P and not on k. Observe also that $(P * \partial\Delta_k) \cup (\partial P * \Delta_k) = \partial\,(P * \Delta_k)$ is the boundary of a polytope, and that the two patches are each precisely the facets containing P and Δ_k, respectively. Thus the patches are initial and final segments of some shelling sequence. So semi-simplicial flips are polytopal flips.

To compute the **cd**-index of this flip, we will need to know how the **cd**-index interacts with the free join. From [12] and more satisfyingly in [11] we learn that

$$\Psi(P * Q) = M(\Psi(P), \Psi(Q))$$

where M, called the mixing operator, is a bilinear operator on **cd**-polynomials. This can be computed recursively in the ring of **cd**-polynomials. We refer the reader to [11] for this recursion.

Proposition 3.5.1.

$$\begin{aligned}\Psi\left\langle P * \partial\Delta_k, \partial P * \Delta_k\right\rangle = M\left(\Psi(P), \Psi(\Sigma_k)\right) &- (k+1)M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c}\\ &- M\left(\Psi(P), \sum_{i=0}^{k-1}(k-i)\Psi\left\langle \Sigma_{k-2}\right\rangle_i\right)(2\mathbf{d}-\mathbf{c}^2).\end{aligned}$$

Proof. We begin with the general recursive formula for the **cd**-index of a shellable flip.

$$\Psi\left\langle P^-, P^+\right\rangle = \Psi(P) - \sum_{F \prec P^-}\Psi(F)\mathbf{c} - \sum_{E \prec\!\!\prec \mathring{P}^-}\Psi\left\langle E^-, E^+\right\rangle(2\mathbf{d}-\mathbf{c}^2).$$

Recall that this formula presupposes a shelling order so that P^- consists of an initial segment of that order. Let us now apply this to the flip $\langle P * \partial\Delta_k, \partial P * \Delta_k\rangle$.

$$\begin{aligned}\Psi\left\langle P * \partial\Delta_k, \partial P * \Delta_k\right\rangle &= \Psi(P * \partial\Delta_k)\\ &- \sum_{F \prec (P*\partial\Delta_k)}\Psi(F)\mathbf{c} - \sum_{E \prec\!\!\prec \operatorname{relint}(P*\partial\Delta_k)}\Psi\left\langle E^-, E^+\right\rangle(2\mathbf{d}-\mathbf{c}^2).\end{aligned}$$

For the first term, we simply apply the mixing operator. For the second term, we must recognize that the facets of $P * \partial\Delta_k$ are $P * \sigma$ where σ is a facet of Δ_k. So we can write

$$\begin{aligned}\Psi\left\langle P * \partial\Delta_k, \partial P * \Delta_k\right\rangle &= M\left(\Psi(P), \Psi(\Sigma_k)\right)\\ &- (k+1)M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c}\\ &- \sum_{E \prec\!\!\prec \operatorname{relint}(P*\partial\Delta_k)}\Psi\left\langle E^-, E^+\right\rangle(2\mathbf{d}-\mathbf{c}^2).\end{aligned}$$

Digesting the final summation requires somewhat more thought. The subfacets of $P * \partial\Delta_k$ are of one of two forms. They may be $F * \sigma$ where $F \prec P$ and $\sigma \prec \Delta_k$ or $P * \tau$ where $\tau \prec\!\!\prec \Delta_k$. However, the sum is only over interior subfacets, so we must identify and exclude any subfacets on the boundary. Observe that $\partial(P * \partial\Delta_k) = (\partial P * \partial\Delta_k) \cup (P * \partial\partial\Delta_k) = \partial P * \partial\Delta_k$. Thus the boundary subfacets are precisely those of the form $F * \sigma$. So we can write

$$\begin{aligned}\Psi \langle P * \partial\Delta_k, \partial P * \Delta_k\rangle = M\left(\Psi(P), \Psi(\Sigma_k)\right) - (k+1)M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c} \\ - \sum_{\sigma \prec \Delta_k} \sum_{\tau \prec \sigma} \Psi\left\langle (P*\tau)^-, (P*\tau)^+\right\rangle (2\mathbf{d} - \mathbf{c}^2).\end{aligned}$$

Recall now that flips in the last summation are defined by a compatible shelling order order on $P * \sigma$. But a shelling order in $P * \sigma$ is defined solely by an order on σ. Since σ is a simplex, the order is arbitrary. Let $\partial\sigma = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_i \cup \cdots \cup \tau_k$, where $\tau_1, \ldots, \tau_k$ is a shelling order of σ. Then

$$\begin{aligned}(P * \tau_i)^- &= (P * \tau_i) \cap \left[(P * \tau_1) \cup (P * \tau_2) \cup \cdots \cup (P * \tau_{i-1})\right] \\ &= P * \left[\tau_i \cap (\tau_1 \cup \tau_2 \cup \cdots \cup \tau_{i-1})\right]\end{aligned}$$

and

$$\begin{aligned}(P * \tau_i)^+ &= (P * \tau_i) \cap \left[(P * \tau_{i+1}) \cup (P * \tau_{i+2}) \cup \cdots \cup (P * \tau_k)\right] \\ &= P * \left[\tau_i \cap (\tau_{i+1} \cup \tau_{i+2} \cup \cdots \cup \tau_k)\right].\end{aligned}$$

Now since the **cd**-index of a flip is actually a linear combination of the **cd**-indices of the old and new complexes, we can factor the bilinear operator M out of the flip, that is,

$$\Psi\left\langle (P*\tau)^-, (P*\tau)^+\right\rangle = M\left(\Psi(P), \Psi\left\langle \tau^-, \tau^+\right\rangle\right).$$

So we can now write

$$\begin{aligned}\Psi \langle P * \partial\Delta_k, \partial P * \Delta_k\rangle = M\left(\Psi(P), \Psi(\Sigma_k)\right) - (k+1)M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c} \\ - M\left(\Psi(P), \sum_{\sigma \prec \Delta_k} \sum_{\tau \prec \sigma} \Psi\left\langle \tau^-, \tau^+\right\rangle\right)(2\mathbf{d} - \mathbf{c}^2).\end{aligned}$$

Now the τ flips are bistellar flips and just as with the bistellar flips formula, we can simplify the double summation to a single summation to give a final equation.

$$\begin{aligned}\Psi \langle P * \partial\Delta_k, \partial P * \Delta_k\rangle = M\left(\Psi(P), \Psi(\Sigma_k)\right) - (k+1)M\left(\Psi(P), \Psi(\Sigma_{k-1})\right)\mathbf{c} \\ - M\left(\Psi(P), \sum_{i=0}^{k-1}(k-i)\Psi\left\langle \Sigma_{k-2}\right\rangle_i\right)(2\mathbf{d} - \mathbf{c}^2).\end{aligned}$$

□

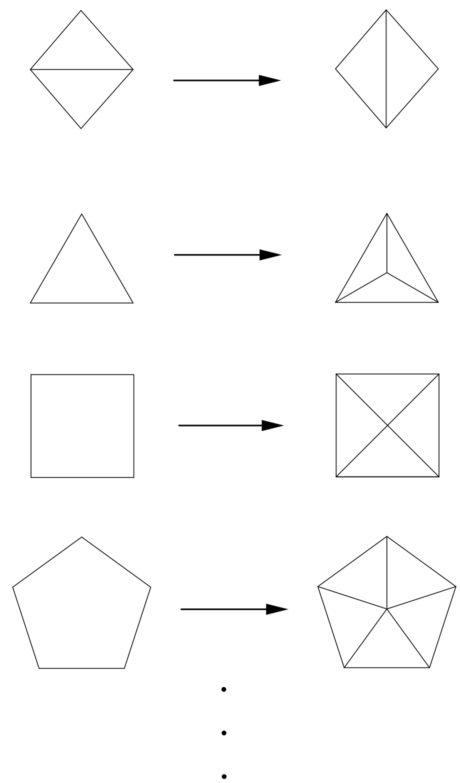

Figure 3.13: 2-dimensional semi-simplicial flips

3.5.1 Low dimensional semi-simplicial flips

In low dimensions ($d = 2, 3$), we can describe all of the semi-simplicial flips and give explicit formulae for their **cd**-indices.

By definition, a d-dimensional semi-simplicial flip is entirely determined by a polytope P in dimension at most d.

In two dimensions, the semi-simplicial flips are $\langle \Sigma_3 \rangle_2$ and $\pm \langle \text{n-gon}, \partial(\text{n-gon}) * \Delta_0 \rangle$. The latter class consists of the flips that subdivide a polygon and their inverses that cut off a vertex. In the positive direction, these flips add one vertex and $(n-1)$ facets. Thus they have the **cd**-index

$$\Psi \langle (\text{n-gon}), \partial(\text{n-gon}) * \Delta_0 \rangle = \mathbf{dc} + (n-1)\mathbf{cd}.$$

Figure 3.13 shows all 2-dimensional semi-simplicial flips with complexity at most 2.

In three dimensions we have two classes of semi-simplicial flips. The first consists those of the form $\pm \langle (\text{n-gon}) * \partial\Delta_1, \partial\text{n-gon}) * \Delta_1 \rangle$. These (in the positive direction) replace two pyramids over an n-gon with n tetrahedra. We can compute the **cd**-index of these flips by direct counting.

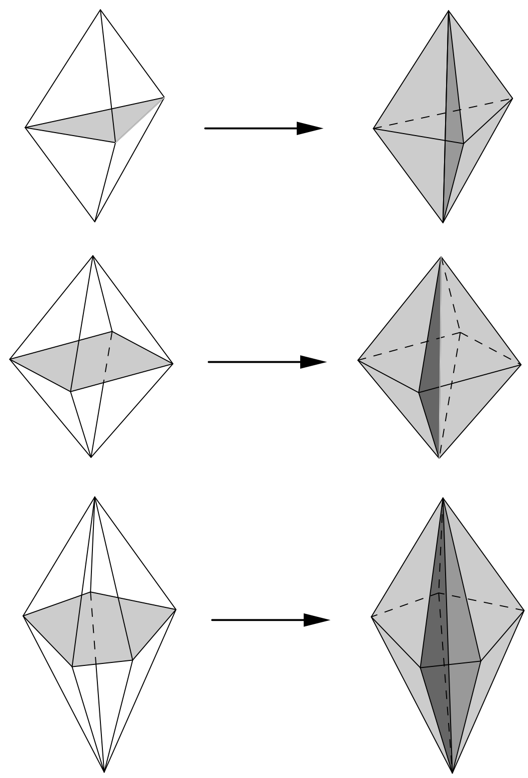

Figure 3.14: 3-dimensional semi-simplicial flips

$$
\begin{aligned}
[\mathbf{c}^4]\Psi\left\langle A,B\right\rangle &= f_{-1}(B) - f_{-1}(A) = 0\\
[\mathbf{dc}^2]\Psi\left\langle A,B\right\rangle &= (f_0(B)-2) - (f_0(A)-2) = 0\\
[\mathbf{cdc}]\Psi\left\langle A,B\right\rangle &= (f_2(B)-f_3(B)) - (f_2(A)-f_3(A)) = (n-1)-(n-2)\\
[\mathbf{c}^2\mathbf{d}]\Psi\left\langle A,B\right\rangle &= (f_3(B)-2) - (f_3(B)-2) = n-2\\
[\mathbf{d}^2]\Psi\left\langle A,B\right\rangle &= (f_{03}(B)-2f_0(B)-2f_3(B)+4) - (f_{03}(B)-2f_0(B)-2f_3(B)+4)\\
&= (4n-2(n+2)-2n+4) - (2n+2-n(n+2)-2\cdot 2+4)
\end{aligned}
$$

Thus, these have **cd**-index

$$\langle(\text{n-gon}) * \partial\Delta_1, \partial(\text{n-gon}) * \Delta_1\rangle = \mathbf{cdc} + (n-2)\mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2.$$

Figure 3.14 shows these flips for $n = 3, 4, 5$.

The other class consists of stellar subdivisions of 3-faces and their inverses. Let P be a 3-polytope with f-vector $f = (f_0, f_1, f_2)$. Again we count:

	A	B
f_{-1}	1	1
f_0	f_0	f_0+1
f_2	f_2	f_2+f_1
f_3	1	f_2
f_{03}	f_0	$2f_1+f_2$

$$\begin{aligned}
[\mathbf{c}^4]\Psi\langle A,B\rangle &= f_{-1}(B)-f_{-1}(A)=0\\
[\mathbf{dc}^2]\Psi\langle A,B\rangle &= (f_0(B)-2)-(f_0(A)-2)=1\\
[\mathbf{cdc}]\Psi\langle A,B\rangle &= (f_2(B)-f_3(B))-(f_2(A)-f_3(A))\\
&= (f_2+f_1-f_2)-(f_2-1)\\
[\mathbf{c}^2\mathbf{d}]\Psi\langle A,B\rangle &= (f_3(B)-2)-(f_3(B)-2)=f_2-1\\
[\mathbf{d}^2]\Psi\langle A,B\rangle &= (f_{03}(B)-2f_0(B)-2f_3(B)+4)-(f_{03}(B)-2f_0(B)-2f_3(B)+4)\\
&= (2f_1+f_2-2(f_0+1)-2f_2)-(f_0-2f_0-2)
\end{aligned}$$

So we have

$$\langle P,\partial P*\Delta_0\rangle = \mathbf{dc}^2+(f_1-f_2+1)\mathbf{cdc}+(f_2-1)\mathbf{c}^2\mathbf{d}+(2f_1-f_2-f_0)\mathbf{d}^2.$$

Applying Euler's relation, we can simplify this to

$$\langle P,\partial P*\Delta_0\rangle = \mathbf{dc}^2+(f_0-1)\mathbf{cdc}+(f_2-1)\mathbf{c}^2\mathbf{d}+(f_0+f_2-4)\mathbf{d}^2.$$

Chapter 4 Flip connectivity and monotonicity

In Chapter 1 we saw a number of results showing that any two members of some certain class of simplicial complexes could be connected by a sequence of operations. Those results show that in a very strong sense, bistellar equivalence and PL-homeomorphism are the same thing for simplicial PL-manifolds with the same (or no) boundary. Thus in this context, one can demonstrate homeomorphism of $(d-1)$-dimensional complexes by constructing a sequence of simple combinatorial operations chosen from a pool of just $d+1$ such operations.

The 1974 result of Ewald and Shephard is not restricted to simplicial polytopes, and thus gives us our first result concerning the general flip equivalence of polytopes. Let us take a second look at this theorem.

Theorem (1.2.2) (Ewald–Shephard 1974, [14]). *The boundary complex of any d-dimensional convex polytope is geometrically stellar equivalent to the boundary complex of any other d-dimensional convex polytope.*

Recall that the boundary complexes of two polytopes are said to be geometrically stellar equivalent if they are related by a sequence of geometric stellar subdivisions. We claim these geometric stellar subdivisions are a special type of general flip.

Lemma 4.0.2. *Geometric stellar subdivisions and their inverses are polytopal flips.*

Proof. A geometric stellar subdivision is accomplished by adding a new vertex close enough to the barycenter of a face F so that it is beyond all facets containing F but beneath all other facets of the polytope, and then taking the convex hull. Both the old and new patches consist of all facets of a polytope containing a given face, and are thus shellable. We need only argue that their union is realizable as a convex polytope.

If F is a facet, then the union is $\mathrm{Pyr}(F)$. Otherwise, consider a Gale diagram G of the convex hull of the vertices of $\overline{\mathrm{st}}(F)$. There is a Gale diagram G_1 for $\overline{\mathrm{st}}(F)$ along with the new point p introduced by the stellar subdivision such that projecting onto the hyperplane with the point corresponding to p as its normal yields the diagram G, but with an extra point corresponding to p at the origin. For example, if we take G to be the leftmost diagram in Figure 4.1, then center diagram is G_1. Note that points in G_1 project back to their positions in G.

All of the new faces created by the stellar subdivision contain p. Thus the corresponding cofaces in G_1 are those that do not contain p.

Consider now a third Gale diagram G_2 obtained from G_1 by moving p to its antipode. This is the rightmost diagram in Figure 4.1. If C was a cofacet in G_1 containing p, then C is not a coface in G_2. Since C is a cofacet in G_1, the convex hull of $C \setminus \{p\}$ intersects the vertical axis (the axis through p) on the negative side. But in G_2, the point p is on also on the negative half. The convex hull of such a set C is shown in Figure 4.1. Likewise, if C' is a set of points such that the relative interior

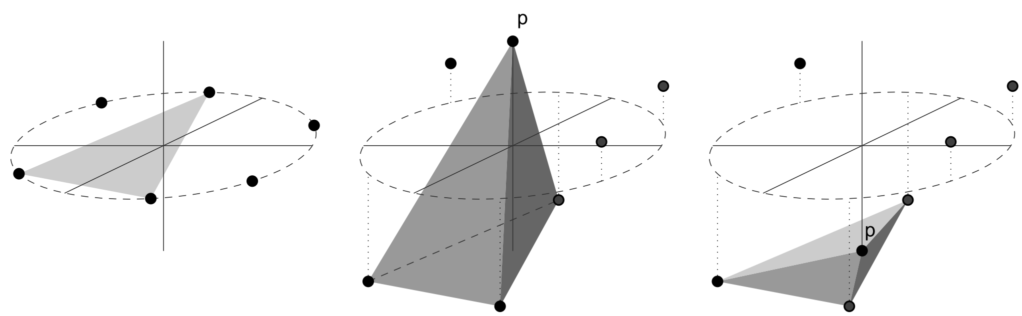

Figure 4.1: G, G_1, and G_2

of its convex hull intersects the positive half of the axis, then $C' \cup \{p\}$ is not a coface in G_1, but is a coface in G_2. A set C intersects the vertical axis in its relative interior exactly when it is a coface in G.

Thus the facets of the polytope associated with G_2 correspond exactly to the facets of $\operatorname{conv}(\overline{\operatorname{st}}\, F \cup \{p\})$ that contain p, along with the facets of $\operatorname{conv}(\overline{\operatorname{st}}\, F)$ that are not also facets of $\operatorname{conv}(\overline{\operatorname{st}}\, F \cup \{p\})$. This is the polytope we wanted. □

This lemma allows us to view the Ewald–Shephard result as a result about general flips: "Any two d-polytopes can be connected by a sequence of polytopal flips."

Corollary 4.3.2 will state that, for $d \leq 6$, all d-polytopes can be built up with semi-simplicial flips of bounded complexity. Thus in particular we can use the formulae in section 3.5.1 to give a description of the **cd**-indices of 4-dimensional polytopes.

Corollary 4.0.3. *If P is a 4-dimensional polytope, then*

$$\begin{aligned}\Psi(P) = \mathbf{c}^4 + 3\mathbf{dc}^2 + 5\mathbf{cdc} + 3\mathbf{c}^2\mathbf{d} + 4\mathbf{d}^2 \\ + \sum_{n\geq 3} k_n \left(\mathbf{cdc} + (n-2)\mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2\right) \\ + \sum_{Q} \ell_Q \left(\mathbf{dc}^2 + (f_0(Q)-1)\mathbf{cdc} + (f_2(Q)-1)\mathbf{c}^2\mathbf{d} + (f_0(Q) + f_2 9Q) - 4)\mathbf{d}^2\right),\end{aligned}$$

where the last summation is over 3-dimensional polytopes Q, and k_n and ℓ_Q are integers.

By reversing the algorithm described in Section 4.3 we can be somewhat more specific. Any 4-polytope can be built from Σ_4 by a sequence of bistellar flips that monotonically increases the number of vertices, followed by a sequence of semi-simplicial flips that are all applied in the negative direction.

Corollary 4.0.4. *If P is a 4-dimensional polytope, then*

$$\begin{aligned}\Psi(P) = {} & \mathbf{c}^4 + 3\mathbf{dc}^2 + 5\mathbf{cdc} + 3\mathbf{c}^2\mathbf{d} + 4\mathbf{d}^2 \\ & + k_3\left(\mathbf{dc}^2 + 3\mathbf{cdc} + 3\mathbf{c}^2\mathbf{d} + 4\mathbf{d}^2\right) \\ & + \ell_{\Sigma_3}\left(\mathbf{cdc} + \mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2\right) \\ & - \sum_{n>3} k_n \left(\mathbf{cdc} + (n-2)\mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2\right) \\ & - \sum_{Q \neq \Sigma_3} \ell_Q \left(\mathbf{dc}^2 + (f_0(Q)-1)\mathbf{cdc} + (f_2(Q)-1)\mathbf{c}^2\mathbf{d} + (f_0(Q)+f_2 9Q)-4)\mathbf{d}^2\right),\end{aligned}$$

where the last summation is over 3-dimensional polytopes Q other than the simplex, ℓ_{Σ_3} is an integer, and k_n and ℓ_Q are non-negative integers.

4.1 Bounding complexity and preserving vertex numbers

When we pass from bistellar flips on the boundary complexes of simplicial polytopes to polytopal flips on polytopes we gain a lot of complexity. Instead of using just $d+1$ types of flips to connect d-polytopes, we must now use an unbounded number of flips. In Ewald and Shephard's method, our pool of possible flips consists of one for each combinatorial type of polytope in each dimension less than d.

We can, however, simplify things somewhat by restricting ourselves to polytopes with facets that do not have too many vertices. This gives us a finite, albeit large, set of facet types to deal with. We will see that polytopes with "small" facets can, in fact, be built by "small" flips. This will reduce our pool of possible flips to a finite, if still unwieldy, collection.

Recall that we defined the *complexity* of a flip $\langle A, B\rangle$ to be $f_0(A \cup B) - d - 1$, where $A \cup B$ is the boundary complex of a d-polytope. We will now show that flips with bounded complexity are sufficient to build polytopes with facets with bounded vertex numbers.

Theorem 4.1.1. *If P is the boundary complex of a d-dimensional polytope with facets having at most $d+k$ vertices, then P can be obtained from Σ_d by a sequence of polytopal flips with complexity at most k. Further, this can be done so that each intermediate complex can be realized as the boundary complex of a convex polytope.*

Proof. Let $\bar{P}$ be a d-polytope with facets having at most $d+k$ vertices, and let P denote the boundary complex. Let Q denote the boundary complex of $\mathrm{Pyr}(\bar{P})$. There is a line shelling of Q that begins with the facet $\mathrm{Pyr}(F)$ for a chosen facet F of P and adds the base $\bar{P}$ last. This gives a shelling order for $Q \setminus \{\bar{P}\}$ (Q without the facet $\bar{P}$) that begins with $\mathrm{Pyr}(F)$.

Observe that $Q \setminus \{\bar{P}\}$ is a shellable ball with boundary complex P. Each shelling step induces a polytopal flip on the boundary complex. Thus the shelling order yields a sequences of flips connecting P to $\mathrm{Pyr}(F)$ for a chosen facet F of P.

If P has a simplex as a facet, then we can choose this simplex to be F to connect P to a simplex and we are done. Otherwise we can repeat the process on $\mathrm{Pyr}(F)$, selecting the initial facet F_1 of $\mathrm{Pyr}(F)$ to be different from F. This connects P to $\mathrm{Pyr}(F)$ to $\mathrm{Pyr}(F_1)$, where $\mathrm{Pyr}(F_1) = \mathrm{Pyr}^2(E)$ for some facet E of F.

Since $f_0(\mathrm{Pyr}(F)) < f_0(P)$ whenever P is not a pyramid with base F, iterating this process will eventually terminate. The final iteration with have a simplex as the initial facet. Together, the sequences of flips induced by the several shelling sequences connect P to Σ_d.

Now we must show that each flip has complexity at most k. Each flip employed is defined by a shelling step that adds a facet of the form $\mathrm{Pyr}^{d-\ell-1}(E)$, where E is some ℓ-face of P. Each facet of P has at most $d+k$ vertices, and so each ℓ-dimensional face of P has at most $k+\ell+1$ vertices. Now observe that the pyramid operation adds one vertex and increases the dimension by one. Thus each $\mathrm{Pyr}^{d-\ell-1}(E)$ has at most $k+\ell+1+(d-\ell-1) = d+k$ vertices, and the flip has complexity at most k.

It remains only to show that each intermediate complex can be realized as the boundary complex of a polytope. Each of the intermediate complexes is the boundary complex of an initial shelling segment of a line shelling of the polytope $\mathrm{Pyr}(\bar{P})$. Every initial segment of a line shelling is the collection of facets either visible or not visible from a point outside the polytope. The complex in question is the boundary of such an initial segment and thus is a *shadow boundary*. Every shadow boundary can be realized convexly. In particular a shadow boundary can be directly realized as the boundary of the shadow (projection onto a hyperplane). $\square$

Example 4.1.2. If we take P to be a cube, we can apply use a shelling of the pyramid over a cube to connect the cube to a square-based pyramid. Then we repeat on the square-based pyramid to connect to a simplex. Running the sequence in reverse gives a sequence of polytopal flips, with complexity at most 1, that builds a cube from simplex. This process is shown in full in Figure 4.2. Here are the **cd**-indices of the flips shown, following the directions of the arrows.

$$2\mathbf{dc} + 3\mathbf{cd}$$

$$0$$

$$0$$

$$-\mathbf{dc} - 2\mathbf{cd}$$

$$2\mathbf{dc} + 3\mathbf{cd}$$

$$\mathbf{dc} + \mathbf{cd}$$

$$\mathbf{dc} + \mathbf{cd}$$

$$\mathbf{dc} - \mathbf{cd}$$

$$-\mathbf{dc} - 3\mathbf{cd}$$

$\lozenge$

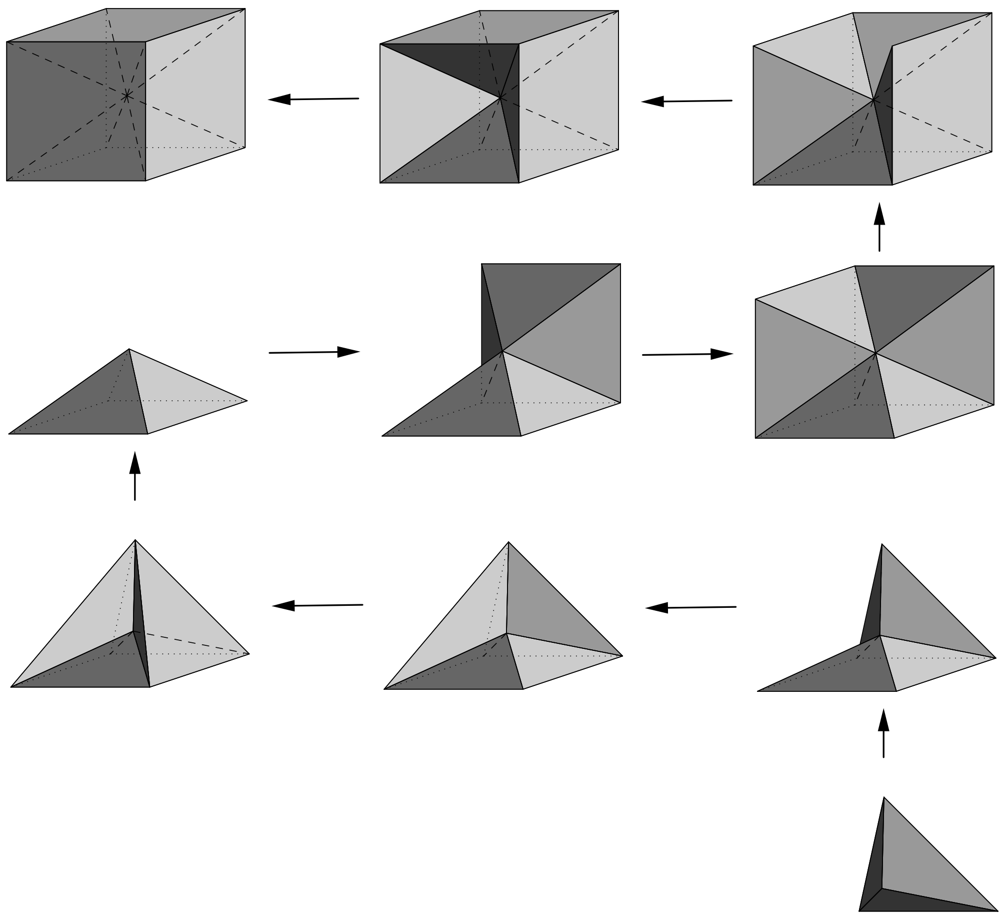

Figure 4.2: Building a cube with polytopal flips

We saw in Theorem 1.2.4 of Chapter 1 that Pachner showed that any two simplicial d-polytopes with the same number of vertices can be connected by a sequence of flips that preserves the number of vertices. The analogous statement is true for non-simplicial polytopes and polytopal flips, but not if we enforce the same bound on complexity as in the previous theorem. The cube, even in 3-dimensions, provides proof that, if we bound the complexity, then it might be necessary to change the number of vertices.

Counterexample 4.1.3. *No general flip admissible on the boundary complex of the d-cube preserving f_0 has complexity less than* $3 \cdot 2^{d-2} - d$.

Proof. Every flip of the form $\langle F, G \rangle$, where F is a single facet, increases the number of vertices. However, any collection of two or more facets of the d-cube must have at least $3 \cdot 2^{d-2}$ vertices. Thus a flip that replaces such a collection must have a complexity of at least $3 \cdot 2^{d-2} - d$. $\square$

Note in particular that beginning with $d = 3$, the complexity of any admissible flip preserving the number of vertices is greater than the bound from Theorem 4.1.1.

This example also shows that when building a polytope from a simplex using flips of bounded complexity, it is sometimes necessary to add extra vertices that are later removed. The algorithm in Theorem 4.1.1 provides some upper bounds for the maximum required number of extra vertices in such a construction.

Proposition 4.1.4. *A d-polytope P with facets having at most $d + k$ vertices can be obtained from a simplex by a sequence of polytopal flips of complexity at most k such that each intermediate complex has at most $f_0(P) + n$ vertices, where n is the codimension of the maximal simplex that is a face of P.*

Proof. The algorithm described in the proof of Theorem 4.1.1 consists of some number of iterations. Each iteration involves exactly one extra vertex, specifically the apex of the pyramid. This vertex is removed from the boundary complex when the final facet is shelled on in each iteration. So we need only determine a bound on the number of required iterations. The process must terminate with a simplex. The initial polytope of the nth iteration, including the final simplex, is of the form $\mathrm{Pyr}^n(E)$ where E is a codimension n face of P. So the number of iterations is minimized when the codimension of E, a simplex, is minimized. □

Without further information about P we have a more general bound.

Corollary 4.1.5. *A d-polytope P with facets having at most $d + k$ vertices can be obtained from a simplex by a sequence of polytopal flips of complexity at most k such that each intermediate complex has at most $f_0(P) + d - 1$ vertices.*

However, these bounds are not sharp. A 3-cube can be built from the simplex using flips of complexity at most 1 in such a way that only one extra vertex is needed. The algorithm described above requires two.

Thus to extend Theorem 1.2.4, we will have to ignore the complexity of the flips we use.

Theorem 4.1.6. *If P and Q are two polytopes in the same dimension with the same number of vertices, then the boundary complex of one can be obtained from the boundary complex of the other by a sequence of polytopal flips that preserve the number of vertices.*

Proof. Suppose P is the boundary complex of a polytope with n vertices. It is enough to show that there is a sequence of polytopal flips, each resulting in a convex polytope with n vertices, that connects P to some simplicial polytope. Then given two polytopes P and Q with n vertices, we can connect them to simplicial polytopes P' and Q', respectively, and apply Theorem 1.2.4.

The operation of *pulling* a vertex can be defined as moving a vertex slightly out from its original location in the boundary complex of a polytope so that its new location is beyond those faces that contained it and beneath all other faces. Thus we can consider this to be a special case of a stellar subdivision where the face being "subdivided" is a vertex. Since stellar subdivisions and their inverses are polytopal flips, so are *pulling* operations and their inverses.

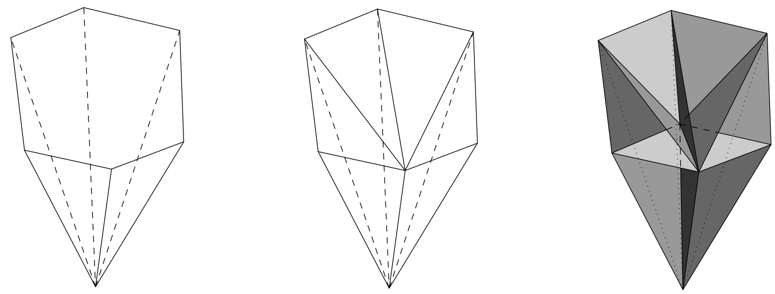

Figure 4.3: P_3, $T(P_3)$, and P_4

It is known (see [18], for example) that pulling each vertex of a polytope in turn, in any order, results in a simplicial polytope. This is sometimes called a *pulling triangulation* of the boundary. $\square$

4.2 (Non-)Monotonicity

It would be of great interest to show that all polytopes, or all polytopes in a certain class, can be obtained from a simplex by a monotonic sequence of flips. By a *monotonic sequence of flips*, we mean a sequence of flips so that all the coefficients of the **cd**-indices of all the flips are non-negative. This would give a new geometric proof of the non-negativity of the **cd**-index for polytopes. A monotonic sequence of bistellar flips to construct any simplicial polytope would give a geometric demonstration that the g-vector is non-negative, and possibly a new proof of the g-theorem.

Unfortunately, we do not have such monotonic flip sequences. In fact, general flips are not even individually monotonic.

Proposition 4.2.1. *In each dimension $d \geq 2$, there is a flip of complexity 3 that is non-monotonic.*

Proof. We will construct a family of polytopes P_d with $d+4$ vertices so that in each case there is a vertex v contained in fewer than half the facets but that also shares a facet with every other vertex. Since the collection of facets containing a given vertex is shellable, we can then define a flip $\left\langle \overline{\mathrm{st}}(v), P \smallsetminus \mathrm{st}(v) \right\rangle$ that decreases f_0 and increases f_{d-1}. The change in f_0 is precisely the coefficient of $\mathbf{c}^{d-2}\mathbf{d}$ in the **cd**-index of the flip, while the change in f_{d-1} is the coefficient of $\mathbf{d}\mathbf{c}^{d-2}$.

Let H be a hexagon with vertices labeled v_0 through v_5. Define $P_3 = \mathrm{Pyr}(H)$. If P is a pyramid over a non-simplex simplicial polytope, let $T(P)$ denote the result of pulling v_0 to triangulate the base of the pyramid. For each $d > 3$, define $P_d = \mathrm{Pyr}(T(P_{d-1}))$. Figure 4.3 shows P_3, $T(P_3)$, and P_4.

Observe that since v_1 is adjacent to v_0, pulling v_0 never introduces any new edges containing v_1. Taking a pyramid introduces exactly one new edge containing v_1. Thus in each simplicial polytope $T(P_{d-1})$, the vertex v_1 is a simple vertex. Thus v_1 is contained in exactly $d-1$ facets of $T(P_{d-1})$ and d facets of $P_d = \mathrm{Pyr}(T(P_{d-1}))$.

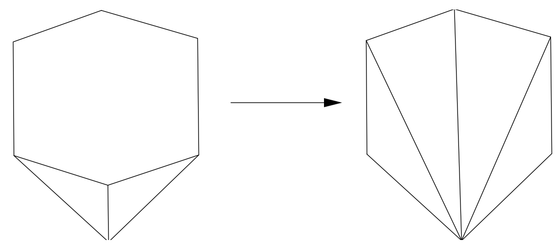

Figure 4.4: A non-monotonic polytopal flip

Further observe that as a vertex in the base of a pyramid, v_1 shares a facet with every other vertex. If we denote the set of facets containing v_1 by A, and the collection of remaining facets by B, then $f_0(A) = f_0(B) - 1$. The flip we are interested in is $\langle A, B \rangle$. Figure 4.4 shows the flip in dimension 2.

Since $f_{d-1}(A) = d$, we now must show that $f_{d-1}(P_d) > 2d$. We proceed by induction on d. Suppose that $f_{d-2}(P_{d-1}) > 2(d-1)$.

$$
\begin{aligned}
f_{d-1}(P_d) &= f_{d-1}(\mathrm{Pyr}(T(P_{d-1}))) \\
&= f_{d-2}(T(P_{d-1})) + 1 \\
&\geq f_{d-2}(P_{d-1}) + 2 \\
&> 2(d-1) + 2 \\
&= 2d
\end{aligned}
$$

As a base case, observe that $f_2(P_3) = f_2(\mathrm{Pyr}(H)) = 7$. □

Example 4.2.2. We compute the **cd**-index of the flip $\langle A, B \rangle$ shown in Figure 4.4. Recall from Lemma 1.3.7 that $\Psi = \mathbf{c}^3 + (f_0 - 2)\mathbf{dc} + (f_2 - 2)\mathbf{cd}$ for 3-dimensional PL-spheres. Thus the **cd**-index of this flip is

$$
\begin{aligned}
\Psi \langle A, B \rangle &= (f_0(B) - f_0(A))\mathbf{dc} + (f_2(B) - f_2(A))\mathbf{cd} \\
&= (6-7)\mathbf{dc} + (4-3)\mathbf{cd} \\
&= \mathbf{cd} - \mathbf{dc}.
\end{aligned}
$$

Bistellar flips are monotonic, but being simplicial is not, in general, enough to guarantee monotonicity. Even at complexity 1, there are non-monotonic simplicial flips. The first such counterexamples are 7-dimensional and are defined by collections of facets of $P_{7,3}$ and $P_{6,4}$.

Counterexample 4.2.3.

$$\begin{aligned}\Psi(F^7_{(6,5,0)}) &= -\Psi(F^7_{(7,2,1)})\\ &= \Psi\left\langle\Sigma_8\right\rangle_{(6+0)} + \Psi\left\langle\Sigma_8\right\rangle_{(5+1)} + \Psi\left\langle\Sigma_8\right\rangle_{(0+2)}\\ &= \mathbf{cdc}^5 + 4\mathbf{c}^2\mathbf{dc}^4 + 5\mathbf{d}^2\mathbf{c}^4 + 5\mathbf{c}^3\mathbf{dc}^3 + 13\mathbf{dcdc}^3 + 16\mathbf{cd}^2\mathbf{c}^3\\ &\quad + 2\mathbf{dc}^2\mathbf{dc}^2 + 7\mathbf{cdcdc}^2 + 7\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 + 10\mathbf{d}^3\mathbf{c}^2\\ &\quad - 3\mathbf{c}^5\mathbf{dc} - 14\mathbf{dc}^3\mathbf{dc} - 36\mathbf{cdc}^2\mathbf{dc} - 42\mathbf{c}^2\mathbf{dcdc} - 56\mathbf{d}^2\mathbf{cdc}\\ &\quad - 22\mathbf{c}^3\mathbf{d}^2\mathbf{c} - 56\mathbf{dcd}^2\mathbf{c} - 68\mathbf{cd}^3\mathbf{c} - \mathbf{c}^6\mathbf{d} - 6\mathbf{dc}^4\mathbf{d} - 22\mathbf{cdc}^3\mathbf{d}\\ &\quad - 42\mathbf{c}^2\mathbf{dc}^2\mathbf{d} - 54\mathbf{d}^2\mathbf{c}^2\mathbf{d} - 44\mathbf{c}^3\mathbf{dcd} - 110\mathbf{dcdcd} - 132\mathbf{cd}^2\mathbf{cd}\\ &\quad - 20\mathbf{c}^4\mathbf{d}^2 - 76\mathbf{dc}^2\mathbf{d}^2 - 154\mathbf{cdcd}^2 - 114\mathbf{c}^2\mathbf{d}^3 - 156\mathbf{d}^4\end{aligned}$$

Counterexample 4.2.4.

$$\begin{aligned}\Psi(F^6_{(5,4,3,0)}) &= -\Psi(F^6_{(6,3,2,1)})\\ &= \Psi\left\langle\Sigma_8\right\rangle_{(5+0)} + \Psi\left\langle\Sigma_8\right\rangle_{(4+1)} + \Psi\left\langle\Sigma_8\right\rangle_{(3+2)} + \Psi\left\langle\Sigma_8\right\rangle_{(0+3)}\\ &= \mathbf{c}^2\mathbf{dc}^4 + \mathbf{d}^2\mathbf{c}^4 + 2\mathbf{c}^3\mathbf{dc}^3 + 5\mathbf{dcdc}^3 + 6\mathbf{cd}^2\mathbf{c}^3\\ &\quad + \mathbf{c}^4\mathbf{dc}^2 + 4\mathbf{dc}^2\mathbf{dc}^2 + 9\mathbf{cdcdc}^2 + 7\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 + 10\mathbf{d}^3\mathbf{c}^2\\ &\quad - 2\mathbf{c}^2\mathbf{dc}^2\mathbf{d} - 2\mathbf{d}^2\mathbf{c}^2\mathbf{d} - 4\mathbf{c}^3\mathbf{dcd} - 10\mathbf{dcdcd} - 12\mathbf{cd}^2\mathbf{cd}\\ &\quad - 2\mathbf{c}^4\mathbf{d}^2 - 8\mathbf{dc}^2\mathbf{d}^2 - 18\mathbf{cdcd}^2 - 14\mathbf{c}^2\mathbf{d}^3 - 20\mathbf{d}^4\end{aligned}$$

With this evidence of poorly behaved flips, it is interesting to note any classes of flips that are monotonic. Pulling flips and other stellar subdivisions, like bistellar flips, induce monotonic changes in the **cd**-index. This is shown in [5]. This is somewhat encouraging to note, since we have seen that, together, bistellar flips and inverse pulling flips are sufficient to construct all polytopes. However, we do not yet have an algorithm to give a monotonic sequence of such flips. Further, one cannot expect a monotonic flip sequence to be found using the strategy of connecting a non-simplicial polytope to a simplicial one and then to a simplex. Among other conditions, such a monotonic sequence must be monotonic in number of vertices and facets, but a non-simplicial polytope will have fewer facets than a simplicial polytope obtained from it via pulling flips.

The formulae we derived in Section 3.5.1 show that for $d \leq 3$, d-dimensional semi-simplicial flips are monotonic. Again, this looks encouraging. We will show next that semi-simplicial flips are sufficient to construct all 3- and 4-dimensional polytopes from a simplex. However, the construction we give allows the flips to applied in both the positive and negative directions.

4.3 Semi-simplicial flips

Recall that in Section 3.5 we defined the *semi-simplicity* flips to be those general flips with the form

$$\left\langle E * \partial\Delta_k, \partial E * \Delta_k\right\rangle,$$

(and their inverses) where E is some convex polytope and $k \geq 0$. If P is a low dimensional polytope, then we can use these flips to connect the boundary complex of P to the boundary complex of a simplicial polytope while very naturally bounding the complexity of the flips.

This class of flips is smaller than the class of flips needed in the proof of Theorem 4.1.1. There is just one pair of d-dimensional semi-simplicial flips for each polytope with dimension at most d. Compare this to the flips of Theorem 4.1.1, which are indexed by initial shelling sequences of polytopes with dimension at most d.

Theorem 4.3.1. *If P is the boundary complex of a $(d-4)$-simplicial d-dimensional polytope with facets having at most $d+q$ vertices, then P can be obtained from Σ_d by a sequence of semi-simplicial flips with complexity at most q.*

Corollary 4.3.2. *If P is the boundary complex of a d-dimensional polytope with facets having at most $d+q$ vertices for some $d \leq 5$, then P can be obtained from Σ_d by a sequence of semi-simplicial flips with complexity at most q.*

We will prove Theorem 4.3.1 by describing an algorithm to transform a $(d-5)$-simplicial d-polytope into a fully simplicial d-polytope using semi-simplicial flips. The bound on the complexity comes basically for free because the complexity of a semi-simplicial flip, $\langle E * \partial\Delta_k, \partial E * \Delta_k \rangle$, is determined by the number of extra vertices of E. Thus the complexity of all semi-simplicial flips is naturally bounded.

The general strategy of the algorithm is this. Beginning with the facets, and working our way down in dimension, we will apply semi-simplicial flips to "poorly behaved" faces. These flips will remove the poorly behaved faces and thus ensure that any remaining essential non-simpliciality is in lower dimensions. Ultimately, all higher dimensional faces will be free joins of simplices with lower dimensional faces that are, by assumption, simplices themselves.

We now present this algorithm in detail as a sequence of propositions building up to Theorem 4.3.1. As already mentioned, the complexity of these flips is naturally bounded, so we will make no further mention of it.

Definition 4.3.3. A polytope is *k-simplicial* if all its k-faces are simplices.

Proposition 4.3.4. *If P is the boundary complex of a $(d-2)$-simplicial d-dimensional polytope, then P can be connected to a simplicial polytope via a sequence of semi-simplicial flips.*

Proof. By assumption, only the facets of P are not simplices. For each non-simplex facet F, apply the semi-simplicial flip $\langle F, \partial F * \Delta_0 \rangle$. Note that we may consider F to be $F * \partial\Delta_0$. The facets of $\partial F * \Delta_0$ are each the free join of a facet of F with Δ_0, that is, pyramids over subfacets of P. Since all of the subfacets of P are simplices, the new facets created by the flip are all simplices. Figure 4.5 shows, on the right, the result of applying these flips to the 3-dimensional cube on the left.

Note that these semi-simplicial flips always result in the boundary complex of a polytope. The flips here are stellar subdivisions that can be realized geometrically

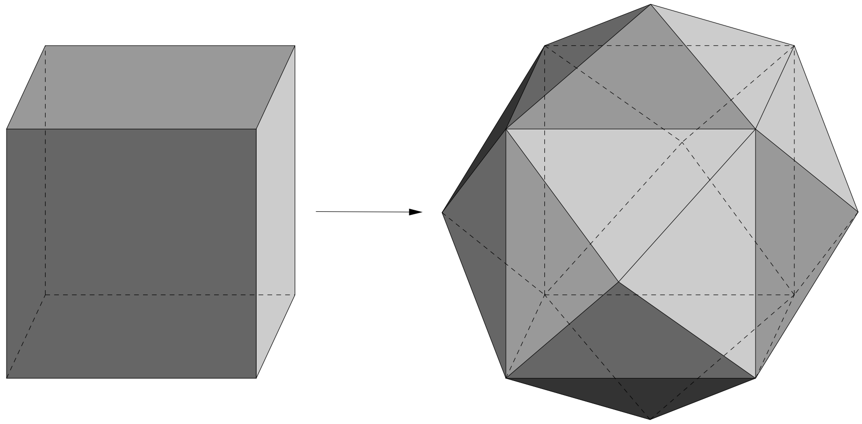

Figure 4.5: Subdividing the faces of a cube

by introducing a new vertex beyond, but sufficiently close to, the barycenter of the facet. □

Proposition 4.3.5. *If P is the boundary complex of a $(d-3)$-simplicial d-dimensional polytope, then P can be connected to a simplicial polytope via a sequence of semi-simplicial flips.*

Proof. By applying the method of the previous proposition, we can transform P into a complex P' with the property that every facet is the free join of a subfacet with Δ_0.

We can deal with each non-simplex subfacet E of P' by applying the semi-simplicial flip $\langle E * \partial\Delta_1, \partial E * \Delta_1\rangle$. Since it is a subfacet, we know that E is always contained in exactly two facets. This flip will remove the subfacet E, and each of the new facets will be the free join of a facet of E (a codimension 3 face of P') with a line segment δ_1. In dimension 3, these are the (non-bistellar) flips shown in Figure 3.14. Since P, and therefore P', is $(d-3)$-simplicial, those codimension 3 faces are simplices, and so the result of making these flips is a simplicial complex.

We can ensure that these flips are admissible, that is, that they do not result in two faces intersecting in something other than a common, possibly empty, face, by making some preparatory semi-simplicial flips. For each subfacet E of P' that is not already a pyramid over a codimension 3 face, apply a stellar subdivision on each of the two facets containing E. Denote the two new vertices by u and v. With that done, we can now geometrically realize the flip by lifting u and v just beyond E. When we consider the convex hull of the new configuration, we see that the edge $\{u, v\}$ has been introduced, and the E is no longer a face. This two step process is shown in Figure 4.6 in the trivial (and unnecessary) case where E is an edge.

□

Proposition 4.3.6. *If P is the boundary complex of a $(d-4)$-simplicial d-dimensional polytope, then P can be connected to a simplicial polytope via a sequence of semi-simplicial flips.*

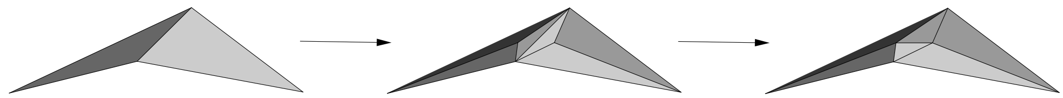

Figure 4.6: Using semi-simplicial flips to remove a subfacet

Proof. By applying the methods of the previous two propositions, we can obtain a complex P' such that all facets and subfacets of P' are free joins of codimension 3 faces with simplices.

Since we must now deal with faces in codimension 3, we can no longer assume that we can directly apply a semi-simplicial flip. Let E be a face of P' in codimension 3. If E is contained in exactly three facets of P', then $\overline{\mathrm{st}}(E) = E * \partial\Delta_2$. Apply the semi-simplicial flip $\langle E * \partial\Delta_2, \partial E * \Delta_2\rangle$. This removes the face E and all new facets are free joins of codimension 4 faces (facets of E) with simplices.

If E is contained in more than 3 facets, then we must reduce that number. To do this we will perform some preparatory flips on subfacets containing E that reduce the number of facets containing E.

Consider the structure of $\overline{\mathrm{st}}(E)$. The face figure of E is a polygon, since $\mathrm{codim}(E) = 3$. Each facet containing E has form $E * \Delta_1$. Thus $\overline{\mathrm{st}}(E)$ is $E * S$, where S is the boundary of an n-gon, for some $n > 3$.

Consider a particular subfacet F of P' that contains E. We know that $F = E * v_i$ for some vertex v_i of S. The two facets containing F are $G_1 = E * \mathrm{conv}\{v_{i-1}, v_i\}$ and $G_2 = E * \mathrm{conv}\{v_i, v_{i+1}\}$ where v_{i-1} and v_{i+1} are the two vertices of S adjacent to v_i. The flip $\langle F * \partial\Delta_1, \partial F * \Delta_1\rangle$ is admissible provided the vertices v_{i-1} and v_{i+1} do not appear in any common face. We can guarantee that situation if we first isolate $\overline{\mathrm{st}}(E)$ as follows. If P is the boundary complex of a $(d-4)$-simplicial d-dimensional polytope, then P can be connected to a simplicial polytope via a sequence of semi-simplicial flips.

Denote the subfacets containing the face E by $F_1, F_2, \ldots, F_n$, the facets containing E by $G_1, G_2, , \ldots, G_n$, and the vertices of S by $v_1, v_2, \ldots, v_n$ so that $F_i = E * v_i$, and $G_i = E * \{v_i, v_{i+1}\}$ where the indices are considered modulo n.

Subdivide each facet G_i by introducing a new vertex u_i. The removes all the G_i. The face E is now contained in $2n$ facets, each of the form $E * \mathrm{conv}\{u_i, v_i\}$ or $E * \mathrm{conv}\{u_i, v_{i+1}\}$. Note in particular that there are no facets that contain both u_i and u_{i+1} for any i.

Apply the flip

$$\langle F_i * \{u_{i-1}, u_i\}, \partial F_i * \mathrm{conv}\{u_{i-1}, u_i\}\rangle$$

for each i. We now have a complex where the face E is contained in subfacets $E * u_i$ and facets $E * \mathrm{conv}\{u_i, u_{i+1}\}$.The face E is still contained in n facets, but we now know that u_i and u_j are contained in a common facet only if i and j differ by one.

We can now apply the flip

$$\langle (E * u_i) * \{u_{i-1}, u_{i+1}\}, \partial(E * u_i) * \mathrm{conv}\{u_{i-1}, u_{i+1}\}\rangle\,,$$

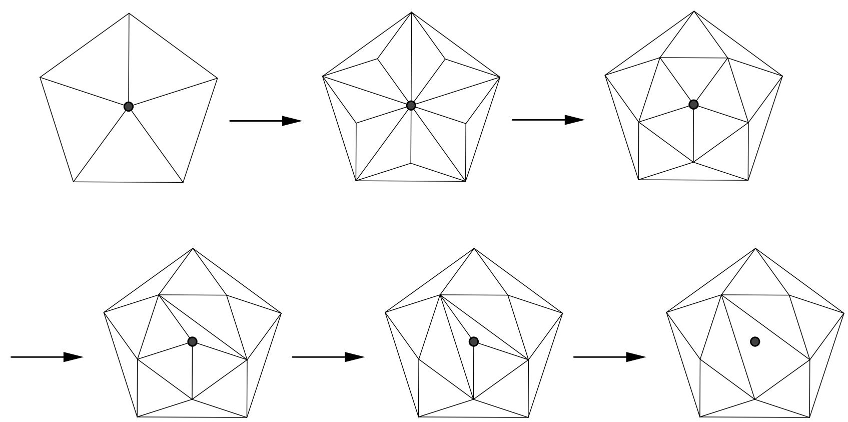

Figure 4.7: Isolating and removing a vertex

which reduces the number of facets containing E by one.

Renumbering the vertices in the link of E, we can apply more flips of this form until E is contained in exactly 3 facets.

We now need only to apply the flip $\langle E * \partial\Delta_2, \partial E * \Delta_2\rangle$. This will be admissible provided that the copy of Δ_2 we are creating does not already exist. The vertices of the Δ_2 in question are three of the u_i. None of the flips we have performed created any facets containing more than two of these, so the flip is admissible. $\square$

Figure 4.7 shows an analogous process applied in a 2-dimensional setting. The highlighted vertex plays the roll of the non-simplicial face E.

In particular, Theorem 4.3.1 implies that all d-polytopes are connected by semi-simplicial flips of bounded complexity for $d \leq 5$.

Appendices

Shelling components of the cd-index

	$\mathbf{c}$
$\check{\Phi}^1_0$	1
$\check{\Phi}^1_1$	0

	$\mathbf{c}^2$	$\mathbf{d}$
$\check{\Phi}^2_0$	1	0
$\check{\Phi}^2_1$	0	1
$\check{\Phi}^2_2$	0	0

	$\mathbf{c}^3$	$\mathbf{dc}$	$\mathbf{cd}$
$\check{\Phi}^3_0$	1	1	0
$\check{\Phi}^3_1$	0	1	1
$\check{\Phi}^3_2$	0	0	1
$\check{\Phi}^3_3$	0	0	0

	$\mathbf{c}^4$	$\mathbf{dc}^2$	$\mathbf{cdc}$	$\mathbf{c}^2\mathbf{d}$	$\mathbf{d}^2$
$\check{\Phi}^4_0$	1	2	2	0	0
$\check{\Phi}^4_1$	0	1	2	1	1
$\check{\Phi}^4_2$	0	0	1	1	2
$\check{\Phi}^4_3$	0	0	0	1	1
$\check{\Phi}^4_4$	0	0	0	0	0

	$\mathbf{c}^5$	$\mathbf{dc}^3$	$\mathbf{cdc}^2$	$\mathbf{c}^2\mathbf{dc}$	$\mathbf{d}^2\mathbf{c}$	$\mathbf{c}^3\mathbf{d}$	$\mathbf{dcd}$	$\mathbf{cd}^2$
$\check{\Phi}^5_0$	1	3	5	3	4	0	0	0
$\check{\Phi}^5_1$	0	1	3	3	4	1	2	2
$\check{\Phi}^5_2$	0	0	1	2	3	1	3	4
$\check{\Phi}^5_3$	0	0	0	1	1	1	3	4
$\check{\Phi}^5_4$	0	0	0	0	0	1	2	2
$\check{\Phi}^5_5$	0	0	0	0	0	0	0	0

	$\mathbf{c}^6$	$\mathbf{dc}^4$	$\mathbf{cdc}^3$	$\mathbf{c}^2\mathbf{dc}^2$	$\mathbf{d}^2\mathbf{c}^2$	$\mathbf{c}^3\mathbf{dc}$	$\mathbf{dcdc}$	$\mathbf{cd}^2\mathbf{c}$	$\mathbf{c}^4\mathbf{d}$	$\mathbf{dc}^2\mathbf{d}$	$\mathbf{cdcd}$	$\mathbf{c}^2\mathbf{d}^2$	$\mathbf{d}^3$
$\check{\Phi}^6_0$	1	4	9	9	12	4	10	12	0	0	0	0	0
$\check{\Phi}^6_1$	0	1	4	6	8	4	10	12	1	3	5	3	4
$\check{\Phi}^6_2$	0	0	1	3	4	3	8	10	1	4	8	6	8
$\check{\Phi}^6_3$	0	0	0	1	1	2	5	6	1	4	9	7	10
$\check{\Phi}^6_4$	0	0	0	0	0	1	2	2	1	4	8	6	8
$\check{\Phi}^6_5$	0	0	0	0	0	0	0	0	1	3	5	3	4
$\check{\Phi}^6_6$	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1: Coefficients of the shelling compnents for dimensions 1 through 6

The cd-index of bistellar flips

	$\mathbf{c}^2$	$\mathbf{d}$
$\langle\Sigma_2\rangle_0$	1	1
$\langle\Sigma_2\rangle_1$	0	1

	$\mathbf{c}^3$	$\mathbf{dc}$	$\mathbf{cd}$
$\langle\Sigma_3\rangle_0$	1	2	2
$\langle\Sigma_3\rangle_1$	0	1	2
$\langle\Sigma_3\rangle_2$	0	0	0

	$\mathbf{c}^4$	$\mathbf{dc}^2$	$\mathbf{cdc}$	$\mathbf{c}^2\mathbf{d}$	$\mathbf{d}^2$
$\langle\Sigma_4\rangle_0$	1	3	5	3	4
$\langle\Sigma_4\rangle_1$	0	1	3	3	4
$\langle\Sigma_4\rangle_2$	0	0	1	1	2

	$\mathbf{c}^5$	$\mathbf{dc}^3$	$\mathbf{cdc}^2$	$\mathbf{c}^2\mathbf{dc}$	$\mathbf{d}^2\mathbf{c}$	$\mathbf{c}^3\mathbf{d}$	$\mathbf{dcd}$	$\mathbf{cd}^2$
$\langle\Sigma_5\rangle_0$	1	4	9	9	12	4	10	12
$\langle\Sigma_5\rangle_1$	0	1	4	6	8	4	10	12
$\langle\Sigma_5\rangle_2$	0	0	1	3	4	2	6	8
$\langle\Sigma_5\rangle_3$	0	0	0	0	0	0	0	0

	$\mathbf{c}^6$	$\mathbf{dc}^4$	$\mathbf{cdc}^3$	$\mathbf{c}^2\mathbf{dc}^2$	$\mathbf{d}^2\mathbf{c}^2$	$\mathbf{c}^3\mathbf{dc}$	$\mathbf{dcdc}$	$\mathbf{cd}^2\mathbf{c}$	$\mathbf{c}^4\mathbf{d}$	$\mathbf{dc}^2\mathbf{d}$	$\mathbf{cdcd}$	$\mathbf{c}^2\mathbf{d}^2$	$\mathbf{d}^3$
$\langle\Sigma_6\rangle_0$	1	5	14	19	25	14	35	42	5	18	35	25	34
$\langle\Sigma_6\rangle_1$	0	1	5	10	13	10	25	30	5	18	35	25	34
$\langle\Sigma_6\rangle_2$	0	0	1	4	5	6	15	18	3	12	25	19	26
$\langle\Sigma_6\rangle_3$	0	0	0	1	1	2	5	6	1	4	9	7	10

	$\mathbf{c}^7$	$\mathbf{dc}^5$	$\mathbf{cdc}^4$	$\mathbf{c}^2\mathbf{dc}^3$	$\mathbf{d}^2\mathbf{c}^3$	$\mathbf{c}^3\mathbf{dc}^2$	$\mathbf{dcdc}^2$	$\mathbf{cd}^2\mathbf{c}^2$	$\mathbf{c}^4\mathbf{dc}$	$\mathbf{dc}^2\mathbf{dc}$	$\mathbf{cdcdc}$
$\langle\Sigma_7\rangle_0$	1	6	20	34	44	34	84	100	20	72	140
$\langle\Sigma_7\rangle_1$	0	1	6	15	19	20	49	58	15	54	105
$\langle\Sigma_7\rangle_2$	0	0	1	5	6	10	24	28	10	36	70
$\langle\Sigma_7\rangle_3$	0	0	0	1	1	4	9	10	5	18	35
$\langle\Sigma_7\rangle_4$	0	0	0	0	0	0	0	0	0	0	0

	$\mathbf{c}^2\mathbf{d}^2\mathbf{c}$	$\mathbf{d}^3\mathbf{c}$	$\mathbf{c}^5\mathbf{d}$	$\mathbf{dc}^3\mathbf{d}$	$\mathbf{cdc}^2\mathbf{d}$	$\mathbf{c}^2\mathbf{dcd}$	$\mathbf{d}^2\mathbf{cd}$	$\mathbf{c}^3\mathbf{d}^2$	$\mathbf{dcd}^2$	$\mathbf{cd}^3$
$\langle\Sigma_7\rangle_0$	100	136	6	28	72	84	112	44	112	136
$\langle\Sigma_7\rangle_1$	75	102	6	28	72	84	112	44	112	136
$\langle\Sigma_7\rangle_2$	50	68	4	20	54	66	88	36	92	112
$\langle\Sigma_7\rangle_3$	25	34	2	10	28	36	48	20	52	64
$\langle\Sigma_7\rangle_4$	0	0	0	0	0	0	0	0	0	0

Table 2: Coefficients of the **cd**-indices of bistellar flips in dimensions 1 through 6

	$\langle\Sigma_{14}\rangle_0$	$\langle\Sigma_{14}\rangle_1$	$\langle\Sigma_{14}\rangle_2$	$\langle\Sigma_{14}\rangle_3$	$\langle\Sigma_{14}\rangle_4$	$\langle\Sigma_{14}\rangle_5$	$\langle\Sigma_{14}\rangle_6$	$\langle\Sigma_{14}\rangle_7$
$\mathbf{c}^{14}$	**1**	0	0	0	0	0	0	0
$\mathbf{dc}^{12}$	13	**1**	0	0	0	0	0	0
$\mathbf{cdc}^{11}$	90	13	**1**	0	0	0	0	0
$\mathbf{c}^2\mathbf{dc}^{10}$	363	78	12	**1**	0	0	0	0
$\mathbf{d}^2\mathbf{c}^{10}$	429	89	13	**1**	0	0	0	0
$\mathbf{c}^3\mathbf{dc}^9$	1000	286	66	11	**1**	0	0	0
$\mathbf{dcdc}^9$	2275	637	143	23	**2**	0	0	0
$\mathbf{cd}^2\mathbf{c}^9$	2550	702	154	24	**2**	0	0	0
$\mathbf{c}^4\mathbf{dc}^8$	2001	715	220	55	10	**1**	0	0
$\mathbf{dc}^2\mathbf{dc}^8$	6708	2364	715	175	31	**3**	0	0
$\mathbf{cdcdc}^8$	12285	4277	1274	306	53	**5**	0	0
$\mathbf{c}^2\mathbf{d}^2\mathbf{c}^8$	8283	2847	834	196	33	**3**	0	0
$\mathbf{d}^3\mathbf{c}^8$	11154	3826	1118	262	44	**4**	0	0
$\mathbf{c}^5\mathbf{dc}^7$	3002	1287	495	165	45	9	**1**	0
$\mathbf{dc}^3\mathbf{dc}^7$	13286	5642	2145	705	189	37	**4**	0
$\mathbf{cdc}^2\mathbf{dc}^7$	32580	13715	5159	1674	442	85	**9**	0
$\mathbf{c}^2\mathbf{dcdc}^7$	36036	15015	5577	1782	462	87	**9**	0
$\mathbf{d}^2\mathbf{cdc}^7$	48048	20020	7436	2376	616	116	**12**	0
$\mathbf{c}^3\mathbf{d}^2\mathbf{c}^7$	18020	7436	2728	858	218	40	**4**	0
$\mathbf{dcd}^2\mathbf{c}^7$	45500	18746	6864	2154	546	100	**10**	0
$\mathbf{cd}^3\mathbf{c}^7$	54960	22620	8272	2592	656	120	**12**	0
$\mathbf{c}^6\mathbf{dc}^6$	3431	1716	792	330	120	36	8	**1**
$\mathbf{dc}^4\mathbf{dc}^6$	18863	9371	4290	1770	636	188	41	**5**
$\mathbf{cdc}^3\mathbf{dc}^6$	57330	28301	12857	5256	1868	545	117	**14**
$\mathbf{c}^2\mathbf{dc}^2\mathbf{dc}^6$	85503	41886	18852	7621	2673	768	162	**19**
$\mathbf{d}^2\mathbf{c}^2\mathbf{dc}^6$	111969	54877	24713	9997	3509	1009	213	**25**
$\mathbf{c}^3\mathbf{dcdc}^6$	70070	34034	15158	6050	2090	590	122	**14**
$\mathbf{dcdcdc}^6$	175175	85085	37895	15125	5225	1475	305	**35**
$\mathbf{cd}^2\mathbf{cdc}^6$	210210	102102	45474	18150	6270	1770	366	**42**
$\mathbf{c}^4\mathbf{d}^2\mathbf{c}^6$	28173	13585	5995	2365	805	223	45	**5**
$\mathbf{dc}^2\mathbf{d}^2\mathbf{c}^6$	102414	49326	21736	8560	2908	804	162	**18**
$\mathbf{cdcd}^2\mathbf{c}^6$	200655	96551	42497	16713	5669	1565	315	**35**
$\mathbf{c}^2\mathbf{d}^3\mathbf{c}^6$	144309	69381	30507	11983	4059	1119	225	**25**
$\mathbf{d}^4\mathbf{c}^6$	196482	94450	41522	16306	5522	1522	306	**34**

Table 3: Selected coefficients of the **cd**-indices of 13-dimensional bistellar flips

The cd-index of complexity 1 simplicial flips

	$\mathbf{c}^2$	$\mathbf{d}$
$F^2_{(0,0)}$	1	2
$F^2_{(1,0)}$	0	2
$F^2_{(1,1)}$	0	0

	$\mathbf{c}^3$	$\mathbf{dc}$	$\mathbf{cd}$
$F^3_{(0,0)}$	1	3	4
$F^3_{(1,0)}$	0	2	4
$F^3_{(1,1)}$	0	1	2
$F^3_{(2,0)}$	0	1	2
$F^3_{(2,1)}$	0	0	0
$F^3_{(3,0)}$	0	0	0

	$\mathbf{c}^4$	$\mathbf{dc}^2$	$\mathbf{cdc}$	$\mathbf{c}^2\mathbf{d}$	$\mathbf{d}^2$
$F^3_{(0,0,0)}$	1	4	9	7	10
$F^3_{(1,0,0)}$	0	2	7	7	10
$F^3_{(1,1,0)}$	0	1	5	5	8
$F^3_{(1,1,1)}$	0	1	3	3	4
$F^3_{(2,1,0)}$	0	0	3	3	6
$F^3_{(2,1,1)}$	0	0	1	1	2
$F^3_{(2,2,0)}$	0	0	1	1	2
$F^4_{(0,0)}$	1	4	8	6	8
$F^4_{(1,0)}$	0	2	6	6	8
$F^4_{(1,1)}$	0	1	4	4	6
$F^4_{(2,0)}$	0	1	4	4	6
$F^4_{(2,1)}$	0	0	2	2	4
$F^4_{(2,2)}$	0	0	0	0	0
$F^4_{(3,0)}$	0	1	2	2	2
$F^4_{(3,1)}$	0	0	0	0	0
$F^4_{(4,0)}$	0	0	0	0	0

Table 4: Coefficients of the **cd**-indices of small simplicial flips in dimensions 1 through 3.

	$\mathbf{c}^5$	$\mathbf{dc}^3$	$\mathbf{cdc}^2$	$\mathbf{c}^2\mathbf{dc}$	$\mathbf{d}^2\mathbf{c}$	$\mathbf{c}^3\mathbf{d}$	$\mathbf{dcd}$	$\mathbf{cd}^2$
$F^4_{(0,0,0)}$	1	5	14	18	24	10	26	32
$F^4_{(1,0,0)}$	0	2	9	15	20	10	26	32
$F^4_{(1,1,0)}$	0	1	6	12	16	8	22	28
$F^4_{(1,1,1)}$	0	1	5	9	12	6	16	20
$F^4_{(2,0,0)}$	0	1	6	12	16	8	22	28
$F^4_{(2,1,0)}$	0	0	3	9	12	6	18	24
$F^4_{(2,1,1)}$	0	0	2	6	8	4	12	16
$F^4_{(2,2,0)}$	0	0	2	6	8	4	12	16
$F^4_{(2,2,1)}$	0	0	1	3	4	2	6	8
$F^4_{(2,2,2)}$	0	0	0	0	0	0	0	0
$F^4_{(3,0,0)}$	0	1	5	9	12	6	16	20
$F^4_{(3,1,0)}$	0	0	2	6	8	4	12	16
$F^4_{(3,1,1)}$	0	0	1	3	4	2	6	8
$F^4_{(3,2,0)}$	0	0	1	3	4	2	6	8
$F^4_{(3,2,1)}$	0	0	0	0	0	0	0	0
$F^4_{(3,3,0)}$	0	0	0	0	0	0	0	0
$F^4_{(4,0,0)}$	0	1	4	6	8	4	10	12
$F^4_{(4,1,0)}$	0	0	1	3	4	2	6	8
$F^4_{(4,2,0)}$	0	0	0	0	0	0	0	0
$F^5_{(0,0)}$	1	5	13	15	20	8	20	24
$F^5_{(1,0)}$	0	2	8	12	16	8	20	24
$F^5_{(1,1)}$	0	1	5	9	12	6	16	20
$F^5_{(2,0)}$	0	1	5	9	12	6	16	20
$F^5_{(2,1)}$	0	0	2	6	8	4	12	16
$F^5_{(2,2)}$	0	0	1	3	4	2	6	8
$F^5_{(3,0)}$	0	1	4	6	8	4	10	12
$F^5_{(3,1)}$	0	0	1	3	4	2	6	8
$F^5_{(3,2)}$	0	0	0	0	0	0	0	0
$F^5_{(4,0)}$	0	1	3	3	4	2	4	4
$F^5_{(4,1)}$	0	0	0	0	0	0	0	0
$F^5_{(5,0)}$	0	0	0	0	0	0	0	0

Table 5: Coefficients of the **cd**-indices of small simplicial flips in dimension 4.

	$\mathbf{c}^6$	$\mathbf{dc}^4$	$\mathbf{cdc}^3$	$\mathbf{c}^2\mathbf{dc}^2$	$\mathbf{d}^2\mathbf{c}^2$	$\mathbf{c}^3\mathbf{dc}$	$\mathbf{dcdc}$	$\mathbf{cd}^2\mathbf{c}$	$\mathbf{c}^4\mathbf{d}$	$\mathbf{dc}^2\mathbf{d}$	$\mathbf{cdcd}$	$\mathbf{c}^2\mathbf{d}^2$	$\mathbf{d}^3$
$F^4_{(0,0,0,0)}$	1	6	20	34	44	32	80	96	14	52	104	76	104
$F^4_{(1,0,0,0)}$	0	2	11	25	32	28	70	84	14	52	104	76	104
$F^4_{(1,1,0,0)}$	0	1	7	19	24	24	60	72	12	46	94	70	96
$F^4_{(1,1,1,0)}$	0	1	6	16	20	20	50	60	10	38	78	58	80
$F^4_{(1,1,1,1)}$	0	1	6	14	18	16	40	48	8	30	60	44	60
$F^4_{(2,1,0,0)}$	0	0	3	13	16	20	50	60	10	40	84	64	88
$F^4_{(2,1,1,0)}$	0	0	2	10	12	16	40	48	8	32	68	52	72
$F^4_{(2,1,1,1)}$	0	0	2	8	10	12	30	36	6	24	50	38	52
$F^4_{(2,2,0,0)}$	0	0	2	10	12	16	40	48	8	32	68	52	72
$F^4_{(2,2,1,0)}$	0	0	1	7	8	12	30	36	6	24	52	40	56
$F^4_{(2,2,2,0)}$	0	0	1	5	6	8	20	24	4	16	34	26	36
$F^4_{(2,2,2,1)}$	0	0	1	3	4	4	10	12	2	8	16	12	16
$F^4_{(2,2,2,2)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^4_{(3,1,1,0)}$	0	0	1	7	8	12	30	36	6	24	52	40	56
$F^4_{(3,1,1,1)}$	0	0	1	5	6	8	20	24	4	16	34	26	36
$F^4_{(3,2,1,1)}$	0	0	0	2	2	4	10	12	2	8	18	14	20
$F^4_{(3,2,2,1)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^4_{(3,3,1,1)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^4_{(4,1,1,1)}$	0	0	1	3	4	4	10	12	2	8	16	12	16

Table 6: Coefficients of the **cd**-indices of small simplicial flips in dimension 5 determined by the small simplicial polytope $P_{4,4}$.

	$\mathbf{c}^6$	$\mathbf{dc}^4$	$\mathbf{cdc}^3$	$\mathbf{c}^2\mathbf{dc}^2$	$\mathbf{d}^2\mathbf{c}^2$	$\mathbf{c}^3\mathbf{dc}$	$\mathbf{dcdc}$	$\mathbf{cd}^2\mathbf{c}$	$\mathbf{c}^4\mathbf{d}$	$\mathbf{dc}^2\mathbf{d}$	$\mathbf{cdcd}$	$\mathbf{c}^2\mathbf{d}^2$	$\mathbf{d}^3$
$F^5_{(0,0,0)}$	1	6	20	33	43	30	75	90	13	48	95	69	94
$F^5_{(1,0,0)}$	0	2	11	24	31	26	65	78	13	48	95	69	94
$F^5_{(1,1,0)}$	0	1	7	18	23	22	55	66	11	42	85	63	86
$F^5_{(1,1,1)}$	0	1	6	15	19	18	45	54	9	34	69	51	70
$F^5_{(2,0,0)}$	0	1	7	18	23	22	55	66	11	42	85	63	86
$F^5_{(2,1,0)}$	0	0	3	12	15	18	45	54	9	36	75	57	78
$F^5_{(2,1,1)}$	0	0	2	9	11	14	35	42	7	28	59	45	62
$F^5_{(2,2,0)}$	0	0	2	9	11	14	35	42	7	28	59	45	62
$F^5_{(2,2,1)}$	0	0	1	6	7	10	25	30	5	20	43	33	46
$F^5_{(2,2,2)}$	0	0	1	4	5	6	15	18	3	12	25	19	26
$F^5_{(3,0,0)}$	0	1	6	15	19	18	45	54	9	34	69	51	70
$F^5_{(3,1,0)}$	0	0	2	9	11	14	35	42	7	28	59	45	62
$F^5_{(3,1,1)}$	0	0	1	6	7	10	25	30	5	20	43	33	46
$F^5_{(3,2,0)}$	0	0	1	6	7	10	25	30	5	20	43	33	46
$F^5_{(3,2,1)}$	0	0	0	3	3	6	15	18	3	12	27	21	30
$F^5_{(3,3,0)}$	0	0	1	4	5	6	15	18	3	12	25	19	26
$F^5_{(3,3,1)}$	0	0	0	1	1	2	5	6	1	4	9	7	10
$F^5_{(4,0,0)}$	0	1	6	13	17	14	35	42	7	26	51	37	50
$F^5_{(4,1,0)}$	0	0	2	7	9	10	25	30	5	20	41	31	42
$F^5_{(4,1,1)}$	0	0	1	4	5	6	15	18	3	12	25	19	26
$F^5_{(4,2,0)}$	0	0	1	4	5	6	15	18	3	12	25	19	26
$F^5_{(4,2,1)}$	0	0	0	1	1	2	5	6	1	4	9	7	10
$F^5_{(4,3,0)}$	0	0	1	2	3	2	5	6	1	4	7	5	6
$F^5_{(5,0,0)}$	0	1	5	10	13	10	25	30	5	18	35	25	34
$F^5_{(5,1,0)}$	0	0	1	4	5	6	15	18	3	12	25	19	26
$F^5_{(5,1,1)}$	0	0	0	1	1	2	5	6	1	4	9	7	10
$F^5_{(5,2,0)}$	0	0	0	1	1	2	5	6	1	4	9	7	10

Table 7: Coefficients of the **cd**-indices of small simplicial flips in dimension 5 determined by the small simplicial polytope $P_{5,3}$.

	$\mathbf{c}^6$	$\mathbf{dc}^4$	$\mathbf{cdc}^3$	$\mathbf{c}^2\mathbf{dc}^2$	$\mathbf{d}^2\mathbf{c}^2$	$\mathbf{c}^3\mathbf{dc}$	$\mathbf{dcdc}$	$\mathbf{cd}^2\mathbf{c}$	$\mathbf{c}^4\mathbf{d}$	$\mathbf{dc}^2\mathbf{d}$	$\mathbf{cdcd}$	$\mathbf{c}^2\mathbf{d}^2$	$\mathbf{d}^3$
$F^6_{(0,0)}$	1	6	19	29	38	24	60	72	10	36	70	50	68
$F^6_{(1,0)}$	0	2	10	20	26	20	50	60	10	36	70	50	68
$F^6_{(1,1)}$	0	1	6	14	18	16	40	48	8	30	60	44	60
$F^6_{(2,0)}$	0	1	6	14	18	16	40	48	8	30	60	44	60
$F^6_{(2,1)}$	0	0	2	8	10	12	30	36	6	24	50	38	52
$F^6_{(2,2)}$	0	0	1	5	6	8	20	24	4	16	34	26	36
$F^6_{(3,0)}$	0	1	5	11	14	12	30	36	6	22	44	32	44
$F^6_{(3,1)}$	0	0	1	5	6	8	20	24	4	16	34	26	36
$F^6_{(3,2)}$	0	0	0	2	2	4	10	12	2	8	18	14	20
$F^6_{(3,3)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^6_{(4,0)}$	0	1	5	9	12	8	20	24	4	14	26	18	24
$F^6_{(4,1)}$	0	0	1	3	4	4	10	12	2	8	16	12	16
$F^6_{(4,2)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^6_{(5,0)}$	0	1	4	6	8	4	10	12	2	6	10	6	8
$F^6_{(5,1)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$F^6_{(6,0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8: Coefficients of the **cd**-indices of small simplicial flips in dimension 5 determined by the small simplicial polytope $P_{6,2}$.

The cd-indices of 3-dimensional semi-simplicial flips

Complexity	$\mathbf{c}^4$	$\mathbf{dc}^2$	$\mathbf{cdc}$	$\mathbf{c}^2\mathbf{d}$	$\mathbf{d}^2$
0	1	3	5	3	4
0	0	1	1	2	0
0	0	1	3	3	4
1	0	1	2	2	0
1	0	1	4	4	6
2	0	1	3	2	0
2	0	1	5	4	7
2	0	1	5	6	9
2	0	1	5	7	10
3	0	1	4	2	0
3	0	1	6	5	9
3	0	1	6	6	10
3	0	1	6	7	11
3	0	1	6	8	12
3	0	1	6	9	13
4	0	1	5	2	0
4	0	1	7	5	10
4	0	1	7	6	11
4	0	1	7	7	12
4	0	1	7	8	13
4	0	1	7	9	14
4	0	1	7	10	15
4	0	1	7	11	16
5	0	1	6	2	0
5	0	1	8	6	12
5	0	1	8	7	13
5	0	1	8	8	14
5	0	1	8	9	15
5	0	1	8	10	16
5	0	1	8	11	17
5	0	1	8	12	18
5	0	1	8	13	19

Coefficients of all possible **cd**-indices of semi-simplicial flips in dimension 3 with complexity at most 5

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