Summer 2017

An Introduction to Topology for the High School Student

Nathaniel Ferron
John Carroll University, nferron18@jcu.edu

An Introduction to Topology for the High School Student

An Essay Submitted to the
Office of Graduate Studies
College of Arts & Sciences of
John Carroll University
in Partial Fulfillment of the Requirements
for the Degree of
Master of Arts

By
Nathaniel Ferron
2017

The creative project of Nathaniel Ferron is hereby accepted:

______________________________________ ____________________

Advisor – Dr. Douglas Norris Date

I hereby certify this is the original document

______________________________________ ____________________

Author – Nathaniel Ferron Date

Table of Contents

Introduction

When asking a current high school Geometry student about the material and content that is presented, they will normally reply with a simple response such as "shapes and proofs". I believe this is how most of the world views geometry. Large populations of high school students are unaware of the diversity and applications of different geometries. As a student in high school, I do not recall discussing the type of geometry that we were learning was considered Euclidean Geometry. It was not until my undergraduate program that I was introduced to the idea that there are different types of geometries. While looking through textbooks that are aligned to state standards for Ohio, there is little to no reference of Euclidian Geometry. I do understand the practicality of Euclidean Geometry in its application to how we view the world. I also understand for teachers there are state testing and standards that are required to be followed closely. These requirements create situations that can be difficult to deviate from. But I hope it can be agreed that all students would benefit from a deeper understanding of the diversity of geometry. The topics that will be discussed may not be appropriate for every student in every school, but some students will benefit from being challenged visually and by abstract concepts. Strong visual-spatial learners may consider some abstract and advanced concepts in Geometry more appealing while developing a deeper understanding of the subject. The following unit will take a look into the basic concepts of topology. Students will be forced to think abstractly and visualize the figures and concepts discussed through a variety of notes, assignments, and activities. Students will be introduced to the concept of topology, important figures in topology, and the properties that these figures possess. These figures are known as the sphere, torus, Klein bottle, Möbius strip, and projective plane.

Structure of Unit and Classroom

Before analyzing individual lesson plans of the unit presented in this paper, I want to provide an understanding of how I structured the worksheets, assignments, and the class in general. The worksheets for this unit were originally created in SMART Notebook, a program that utilizes a SMART Board. Having a SMART Board is not a requirement for this unit to be successful but it is important to know that this unit was designed with the intention to take advantage of the various functions of the SMART Board. These functions include easily drawing notes and assignments during class instruction, manipulating shapes or figures on the SMART Board, and allowing for an easier approach to guided notes. When going though each worksheet that was created there will be several places where red font is used. The red font is what the student might correctly state each question from one of the worksheets. This means that for the assignments, review, exit slips, and assessments the red font is essentially the answer key. When distributing these materials to students, it would be essential for the red font to not be present. For guided notes, students are expected to copy the red font how it is presented. The use of guided notes is very useful with my students as many of them struggle with taking classroom notes. Guided notes better allow for students to follow along and correctly complete their notes. I have learned through my years of teaching that students prefer guided notes in my classes and it does allow for more time in class by shortening the amount of material that the students are required to write.

Although the activities of each day in the unit will be formally discussed later, I want to provide a brief overview of how I structure my classroom. The unit was planned for a class that has 50 minutes of instructional time. Each day I begin with a bellwork problem that either is a review of previous material or a problem that will enhance the learning objective for that current lesson. As students complete the bellwork, I walk around the room and observe their progress, allowing me to assist students as needed. The solution to the bellwork problem will be discussed with the class by me or a student giving the solution. Homework and classwork assignments are collected or checked the day after they are assigned. The solutions to the problems are displayed or verbally given to the students. I encourage students to ask questions if they have incorrect solutions or did not fully understand what the question was asking. I believe that it is essential for students to discuss their misconceptions or mistakes in order to have a better understanding of what is happening in the solution. Discussing the solutions to problems as a class can lead to students communicating with each other how they saw the problem and even lead to discussing different ways of arriving at the same solution. As a result of the discussion, students develop the ability to discuss mathematical concepts. The homework or classwork is also used as a formative assessment when collected, giving me the opportunity to evaluate students understanding of the learning objectives assigned.

Standards and Content

The content that will be covered in this unit will be as follows:

- Definition of Topology
- Determine if mathematical figures are geometrically or topologically equivalent
- Creating a torus and its properties
- Creating a Mobius strip and its properties
- Creating a Klein bottle and its properties
- Identify figures that are orientable/non-orientable
- Creating a projective plane and its properties
- Global vs local properties
- Extrinsic vs intrinsic properties
- Open vs closed manifolds
- Homogenous figures

Students should be previously introduced to figures of the plane such as triangles, quadrilaterals, and circles as well as three dimensional figures like cubes, prisms and spheres. An appropriate situation in which these objectives could be implemented would be at the end of a Geometry class, a Math 4 class, or a Senior Math topics class. For Ohio schools there are End of Course Exams that conclude at the beginning of May, leaving approximately a month of time where this unit could be utilized. As previously stated, there are not any specific standards from Ohio's Learning Standards in Mathematics that align with the above topics. The above content however can be connected to the

Standards for Mathematical Practice. These standards were created for mathematical educators to implement with their students. By implementing these standards, we create situations that improve how students think mathematically. This applies to all levels of mathematics. The Standards for Mathematical Practice are as follows:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with Mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

The learning objectives will greatly involve the first three Standards for Mathematical Practice. Students will have to consistently make sense of what the problem is asking them to do or understand. When solving the problems, students might come to find their solutions are incomplete or incorrect, for which they will need to persevere to solve them. Many of these concepts are abstract, even for students who typically excel in mathematics. Students will be required to reason with these abstract concepts in order to fully comprehend the concepts in the unit. This unit will also encourage students on multiple occasions to create arguments and explain their reasoning. I personally believe that this is an extremely important mathematical skill for students to acquire, as many students believe in just using a formula or algorithm to solve problems. Several students struggle when posed with the question of "Why?" or when they are required to explain their reasoning. Expanding a student's ability related to these standards will develop and enhance how they think mathematically which should be the goal of a mathematics teacher.

<u>Unit Overview/Lesson Plans</u>

Within each lesson plan there will be an objective, procedures/strategies used, and materials required for that day. The procedures, strategies, and activities will have estimated times for each activity that may differ for classes depending on how students comprehend the content of each lesson. Each day proceeds in a sequential order designed to extend from previous lessons. Therefore if students are struggling, re-teaching or reinforcement of learning objectives could be required. Following these lesson plans, the worksheets and materials to be used for that specific lesson will be inserted directly after. These worksheets all contain the red font that was previously discussed. When reading the lesson plan chart, TSW will stand for "the student will".

Day 1

Objectives:	**Procedures/Strategies:**
• TSW determine if mathematical figures have the same topology or geometry	• TSW complete bellwork (5 mins) • TSW takes guided notes on objective (35 mins) • TSW complete Exit Slip: (10 mins)-collected at end of class
Materials: • Guided Notes: Topology vs Geometry • Exit Slip: Topology vs Geometry	**Assessment Strategies (Formative/Summative):** • The exit slip will be used as a formative assessment

The bellwork question for Day 1 requires students to describe what it means for two figures to be congruent. This question will lead students into the direction of what it means for figures to be geometrically equivalent. The guided notes will take students through the definitions of topology and geometry. Students will then investigate which figures, that are geometrically or topologically equivalent, look like while explaining the reasoning for their answers. Students will also be introduced to how two dimensional figures can be neither geometrically or topologically equivalent, in which the answer for this relationship is “neither”. The notes will then take students through the same procedure, only this time using three dimensional figures. I recommend during this section to have the class provide answers as you go through each pair of figures, allowing students to have discussions and defend their answers. If figures are topologically equivalent, students should discuss how one could take a figure and create a second figure using topological properties. At the conclusion of the guided notes, students will complete an exit slip that requires students to state if two figures have the same topology, geometry, or neither. The figure pairs will be both two dimensional and three dimensional. This should be collected and used as a formative assessment and could also be used to pair or group specific students for tomorrow’s assignment. Groups may be required due to limited resources at the teacher’s disposal as discussed in the next lesson plane. The exit slip will determine a student’s understanding of the learning objective.

Notes Name:_________________________

Topology vs. Geometry Date:___________ Period:______

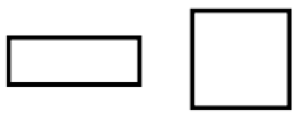
Definitions:

Topology: Two mathematical figures have the same topology if one figure can be transformed into the other figure by twisting and stretching, not tearing.

Geometry: Two mathematical figures have the same geometry if the figures have the same shape and congruent measures.

Example: Look at the following pairs of shapes and determine if they have the same geometry, topology, or have neither the same geometry/topology. Explain your reasoning

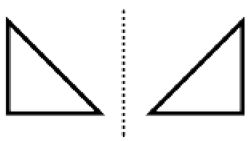
1.)



Topology

Reason (s) Why:
The figure on the left appears to be a rectangle while the right figure appears to be a square. Rectangles have opposite sides congruent and a square has all sides congruent, so their measures are not congruent. You could stretch the figure on the right to get the figure on the left.

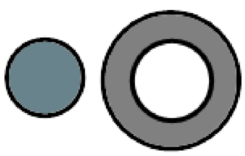
2.)



Geometry

Reason (s) Why:
The two figures have the same shape and measure. The dotted line shown is a line of reflection, which shows the figures are congruent.

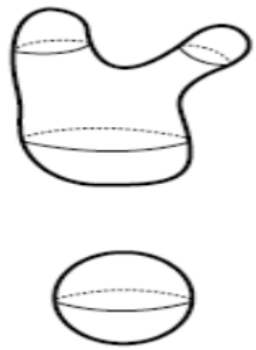
3.)



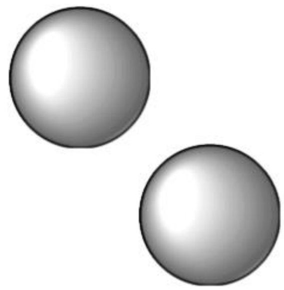
Neither

Reason (s) Why:
The figure on the right has a hole in it, so it is not the same as the figure on the left. That hole could not be created without tearing, so the figures do not have the same topology.

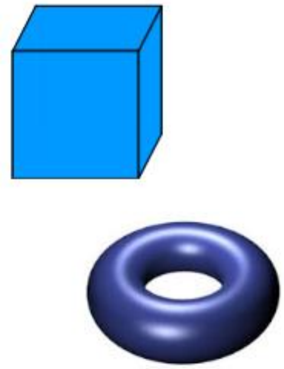
The previous examples were a two dimensional look at geometry vs topology, but we can also look at three dimensional examples.



Topology

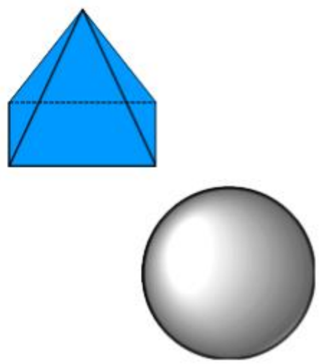


Geometry



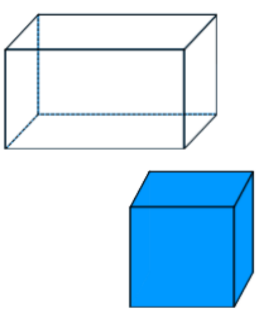
Neither

Topology

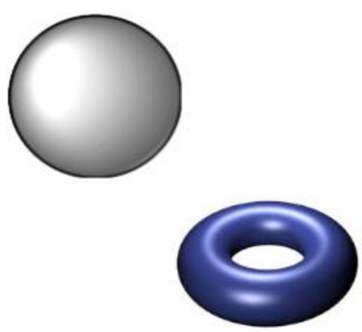


Topology

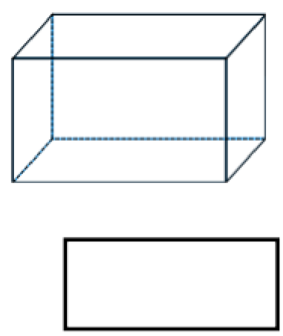
Neither



Topology



Neither



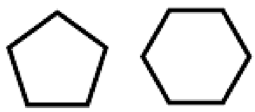
Neither

Exit Slip Name:________________________

Topology vs. Geometry Date:___________ Period:______

State if the following shapes have the same geometry, topology, or neither.

1)

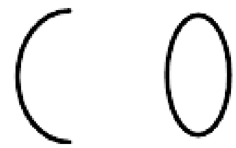


Topology

2)

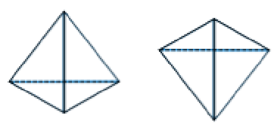
Geometry

3)



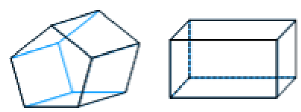
Neither

4)



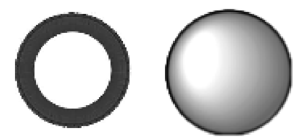
Geometry

5)



Topology

6)



Geometry

Day 2

Objectives: • TSW determine if mathematical figures have the same topology or geometry	**Procedures/Strategies:** • TSW complete bellwork (5 mins) • TSW will complete Assignment 1: Topology vs Geometry (45 mins)
Materials: • Assignment 1: Topology vs Geometry • Pipe Cleaners (1 per student) • Play-Doh (1 container per student)	**Assessment Strategies (Formative/Summative)**: • The assignment can be used as a formative assessment

The bellwork question for Day 2 is to take different pairs of figures and determine if they have the same geometry or topology, similar to the exercises from yesterday's exit slip and guided notes. These figures should be both two and three dimensional pairs. For the assignment, students will be using the pipe cleaners and Play-Doh to create figures that have the same topology or geometry. If resources are limited, pair or group students based on the outcome of the previous exit slip. I recommend placing students who answered fewer than two correctly with students who answered at least 5 correctly from the exit slip. These groupings should allow for groups to work at a consistent pace. During the assignment, students will take the pipe cleaner and try to create various letters of the alphabet. Their goal is to determine if the methods they used to form the letter are geometrical, topological, or neither. A key point for this section would be the second question, which has the students trying to create the letter I where on the worksheet the I is only a vertical strike. The answer for this on the document is geometrically, but it could be argued that the pipe cleaner is not perfectly straight when the student starts. I would make sure to state to the students that each time we will assume that we start with a straight or linear pipe cleaner, even if it appears to be curved. The second part of the activity involves students using Play-Doh to create three dimensional figures. As it was with the pipe cleaner, students should be instructed to assume they start with a perfect sphere each time before forming the desired figure. The questions that follow are designed to help students understand what physically can be done to a figure and still have it be topologically the same. Once students have finished with the pipe cleaner and Play-Doh, it is recommended that those materials are returned to the teacher. On the back of the assignment, students are given a set of figures in which they are to match all topologically equivalent figure groups together. The final question asks students to demonstrate their ability to create their own figures that have the same geometry or topology to a given figure. This question will require students to physically draw these figures, which may be difficult for some students to accomplish. The skill of drawing figures will be practiced throughout the unit.

Assignment 1 Name:________________________

Topology vs. Geometry Date:___________ Period:______

For each of the following take the given pipe cleaner and try to create the given example. You may not cut or tear the pipe cleaner. Then claim if the shapes have the same geometry, topology, or neither.

1. S — Topology

2. I — Geometry

3. L — Topology

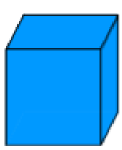
4. O — Neither

Look at the number four and explain why the correct answer is neither (not geometrically or topologically equivalent).

To create the “o” or circle, one would have to actually glue the ends together which is not allowed in topology. Geometrically they have different properties, therefore the answer is neither.

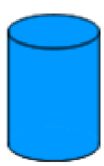
We will complete the same exercise for the following shapes using Play-Doh. For each shape, start with a ball or sphere (you can easily create this by “rolling” it between your hands. Then state if the shapes have the same geometry, topology, or neither.

5.



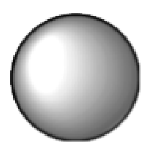
Topology

6.



Topology

7.



Geometry

8.

Topology

9.

Neither

10.

Topology

11.

Neither

12.

Neither

For your answer to question 7, did you have to assume anything?

That the Play-Doh was formed in a way that had the same measures as the Sphere.

Consider the shapes in questions 9 and 11. If you started your Play-Doh as the coffee cup, could you create a doughnut using topological properties? Explain.

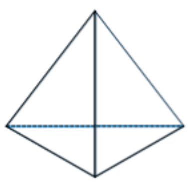
Yes because the two shapes are topologically equivalent. You flatten out the “cup” part of the coffee cup into the handle which would form a doughnut shape. The key is both shapes have one “hole” which means we can mold/form the two shapes into each other.

Consider the shapes in questions 11 and 12. If you started your Play-Doh as the doughnut, could you create a fidget spinner using topological properties? Explain.

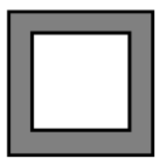
No- The doughnut shape has one hole while the spinner has three holes. If you were to go from the doughnut to the spinner you would need to rip and tear two times in order to create the three holes, making the doughnut and the spinner not topologically equivalent.

State which groups of the following shapes would be topologically equivalent.

A.



B.

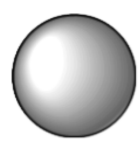


C.

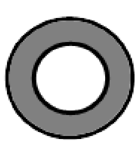
D.

E.

F.



G.



H.

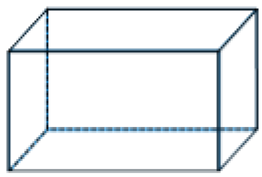
I.

Answer: A and F, B and G, H and I, C and E

One of the shapes on the previous page does not match up with any of the others. State which shape it is and then explain your reasoning why.

Shape D: Shape D has two holes, all of the other shapes have either 0, 1, or 3 holes. This means we could not create a shape D from the others without tearing or gluing together.

Take each of the following shapes and draw something geometrically equivalent, topologically equivalent, and a shape that is neither. **YOU MUST CREATE A SHAPE NOT PREVIOUSLY SEEN ON THIS ASSIGNMENT.**



Geometrically:	Answers will vary	Answers will vary
Topologically:	Answers will vary	Answers will vary
Neither:	Answers will vary	Answers will vary

Day 3

Objectives:	**Procedures/Strategies:**
• TSW determine what a two torus is (or flat torus) • TSW evaluate how a figure moves on the surface of a torus	• TSW complete bellwork (5 mins) • TSW review Assignment 1 (5 mins) • TSW complete guided notes: Torus (35 mins) • TSW complete exit slip (5 mins)
Materials: • Guided Notes: Torus • Projector connected to computer with internet, or school issued student computer devices (like a Chromebook)	**Assessment Strategies (Formative/Summative):** • Exit Slip

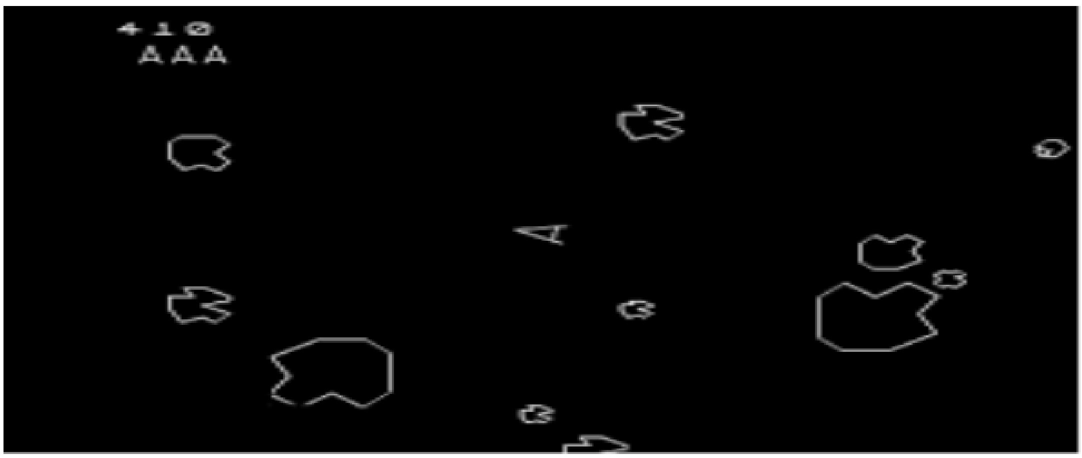
The bellwork for Day 3 is to recreate the same question at the end of assignment one and have the students create a figure having the same geometry and topology as a given figure. After bellwork, instruct students to get out assignment one and discuss any problems or questions that the students have. After that period of discussion, collect the assignment and use it as a formative assessment to determine the students' progress of mastering the learning goal from yesterday's lesson. At the beginning of the guided notes, there is a link for a YouTube clip and a picture of an old game known as Asteroids. Depending on the technology and internet access available at the school, one could simply show the clip or allow students a brief moment of time to play the actual game. If you choose to play the game there are free online simulators, but many gaming websites are blocked on school internet access. The clip and questions are designed to introduce students to the concept of a torus. Relating the concept of a torus to the video game Asteroids may help groups of students grasp the concept of what it means for the edges to be glued together. Watching the clip of the game will demonstrate how the spaceship and asteroids move throughout the torus space. The questions that follow on the back of the guided notes are designed for students to investigate what happens as a figure, a pentagon, moves throughout the surface of the torus. Students may be curious about the arrows that are marked on the rectangle in these problems. This will be clarified in the next section of the notes, which gives a visual definition of a flat two dimensional torus and explains how these arrows represent the gluing of the figure's edges. The last set of questions lead students towards what a three dimensional representation of a torus is, which they will have to draw on their exit slip. A process for creating a three dimensional torus using our flat torus will be done later in the unit. The exit slip, attached at the bottom of the guided notes, poses the question of what happens when throwing a ball in a torus. It may help to inform students that the ball is not affected by gravity and will always stay on a constant path. This exit slip can be done on a separate sheet of paper if it is desired to be used as a formative assessment.

Notes Name:______________________

Torus Date:__________ Period:______

Let's watch the following clip of a classic game called Asteroids.

https://www.youtube.com/watch?v=5rjjtJ2GMN8

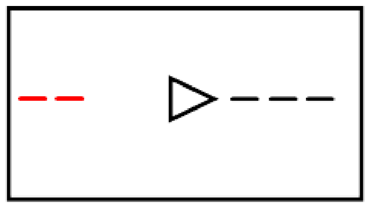


After watching the clip and using the picture answer the questions below:

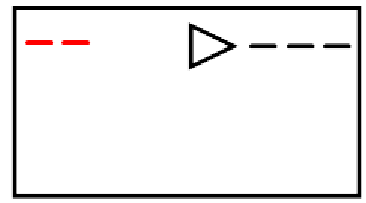
1. What shape appears to be the space for the game? Rectangle
2. What happened when an object same into contact with one of the sides of the shape? Objects would appear on the opposite side they exited. For example, if the ship went through the top side, it would reappear coming through the bottom side and vice versa.
3. If you were given a piece of paper (which is like our above mentioned shape), how could we recreate what happens to the objects in the game Asteroids? We could glue the top and bottom edges together. This would create a way for an object to continue on its path. NOTE: This would change the shape that we started with into something different. We started with a rectangle and if we physically glue the top and bottom together we would now have a shape that looks like a tube or cylinder.

Take the following images and the information above and complete the path of the "laser" of the ship in the Asteroid game. Assume the laser path is 5 dashes.

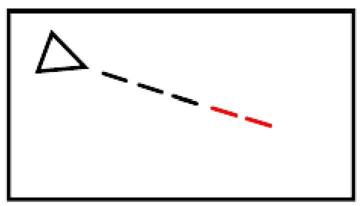
a)



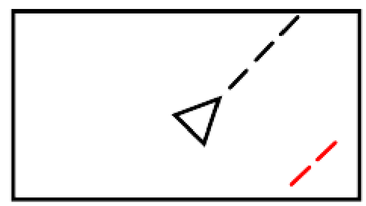
b)



c)

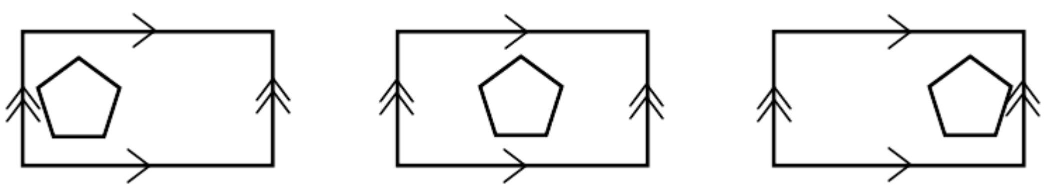


d)



What would happen if the laser continued? The ship would destroy itself.

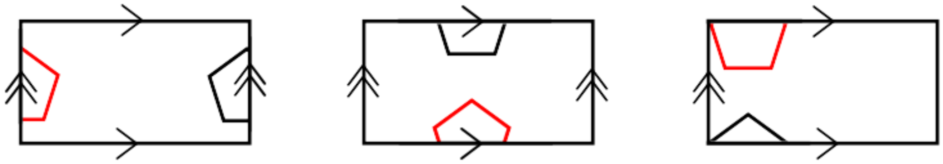
In the definition of topology, we mentioned gluing as an action that would make shapes topologically different.   Consider the right edge of the rectangles below "glued" to the left edge. As well as the bottom edge being glued to the top edge. **NOTE:** This gluing is represented by the arrows on the rectangle.



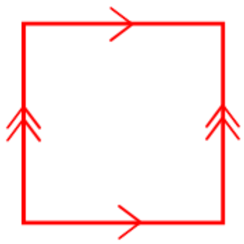
How is the pentagon moving inside the rectangles? Horizontally

Can the pentagon continue this path or will it run out of room? Yes since the edges are glued.

Complete the pictures given below (part of the pentagon is missing)



We will now consider the following two dimensional marked torus:



NOTE: The arrows represent the gluing of the shape. You must think of the gluing "matching" up the arrows.

What if instead of a two dimensional torus we had a three dimensional torus (or three torus). Consider the following questions.

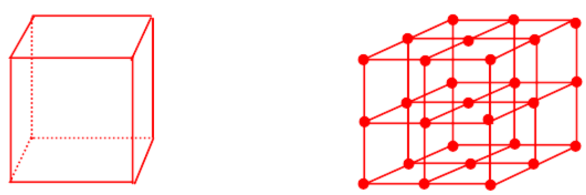
What shape did we initially use for a two dimensional torus? Square/Rectangle

What is the three dimensional equivalent to that shape? A Cube/Prism

How would the shape be glued together (be specific)? By the faces: left with right, top with bottom, and front with back.

For your exit slip, draw a representation of a three torus (explain how the gluing would work in complete sentences).

Student answers: Using a "cube" you could glue the front face to the back, the left face to the right, and the top face to the bottom. When throwing a ball at a face, it will come out from the opposite side. For example if the ball is thrown at the front face, it would come out the back face. When reviewing answers it may help to show the "jungle gym" version of a torus and what happens as someone moves throughout it.



Day 4

Objectives:	**Procedures/Strategies:**
• TSW determine what a two torus is (or flat torus) • TSW demonstrate how a figure moves on the surface of a torus	• TSW entrance activity (supplement to bellwork) 10 mins • TSW complete Assignment 2: Torus (40 mins)
Materials: • Entrance Activity • Assignment 2: Torus	**Assessment Strategies (Formative/Summative)**: • Assignment 2: Torus

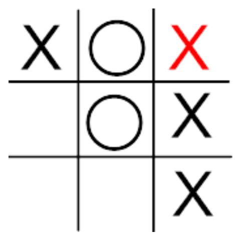
The bellwork for the day will be replaced with an entrance activity, Tic Tac Toe torus. This worksheet starts with reminding students of the familiar game of Tic Tac Toe. If students do not know how to play the game then more time should be spent with them explaining the rules, as this information will be vital later for the assignment. The next part of the activity involves students playing Tic Tac Toe torus. Tic Tac Toe torus involves having the board glued together like a torus. Students will answer questions that take them through how a Tic Tac Toe torus board works. Some students may struggle with the concept at first. Showing them winning moves on the first torus board may help them better understand how the game works now that the board has become a torus. When students have completed the assignment, verify their answers before they start Assignment 2, making sure that they understand how the new gluing of the board works. Assignment 2 starts off with students playing Tic Tac Toe torus against a partner. Encourage students to discuss their strategies with each other and how the original game has changed. Students will then determine the winning move for the "X" player for a game that has already been started. For students who struggled with the entrance activity or still do not fully understand the Tic Tac Toe torus game, it is recommended that students draw the "big" boards on a separate sheet of paper as seen on the entrance activity. This will allow students to have a better chance at seeing possible or winning moves. The next section in the assignment will have students moving a variety of figures through a torus. Students will see the original image and then have to draw where the missing part of the figure is placed in the torus. A follow up question will ask students to create their own original figure and place it in the torus so that it goes through a glued wall. The last question will ask students to further their previous investigation of a three dimensional torus. Students will now investigate what would happen if they walked through a face, instead of throwing a ball. The second part of this question will require a little more thought then the first. Students will have to abstractly think about what they would see as they look through a specific face only with their head, essentially "peaking" though a face. It may be helpful for students to discuss this concept in groups to get a better understanding of what the question is asking.

Entrance Activity Name:________________________

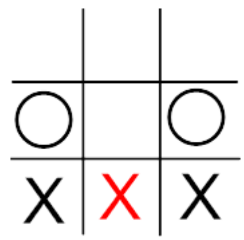
Tic Tac Toe (Torus) Date:___________ Period:______

Consider the game Tic Tac Toe, where the goal is to get three in a row! Place the winning move for the X player in each of the following games.

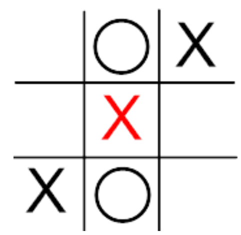
1)



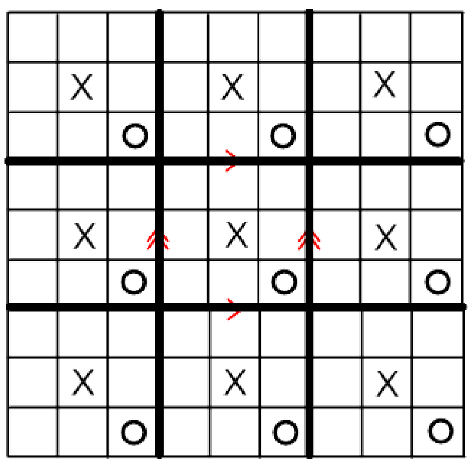
2)



3)



Consider the Tic Tac Toe board below and answer the following questions.



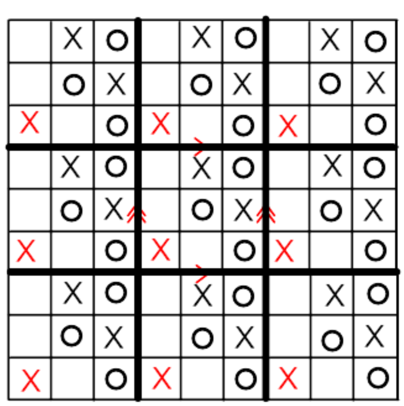
4) What do the symbols on the middle of the board mean? The board is like a torus, opposite edges are glued.

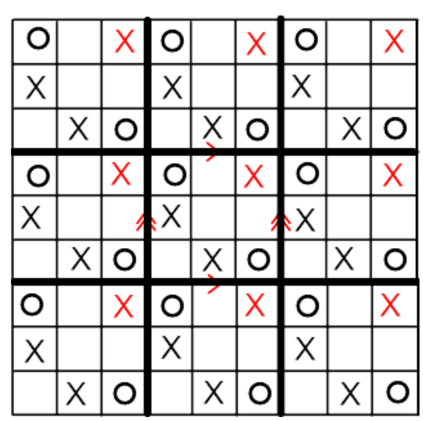
5) What happens if an "X" mark gets placed in the middle row, middle column of the middle board? An "X" will appear in the same spot on all boards.

6) What happens if an "O" make gets placed in the bottom row, right column of the middle board? An "O" will appear in the same spot on all boards.

7) Will this happen for all possible positions for the Tic Tac Toe board? Yes because of the properties of the torus.

Find the winning move for player "X" in the following games (will only take one move):





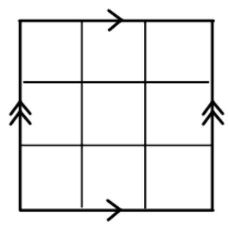
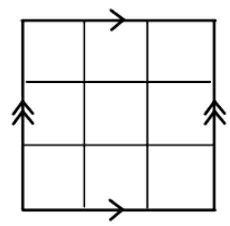
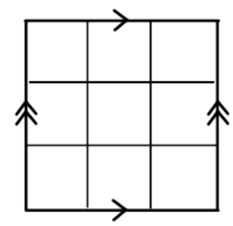
Assignment 2

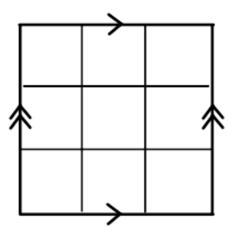
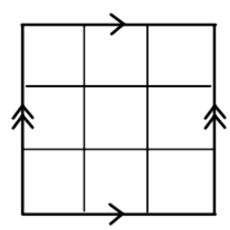
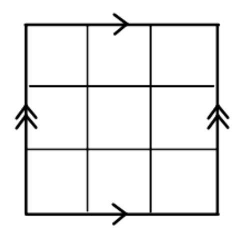
Name:________________________

Torus

Date:___________ Period:______

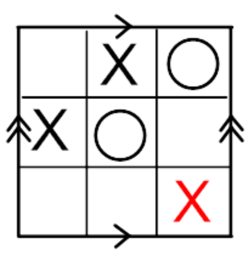
Partner with another student and play some torus Tic Tac Toe. If needed, extend the sides of the boards to help with how the torus "adds" to the board.

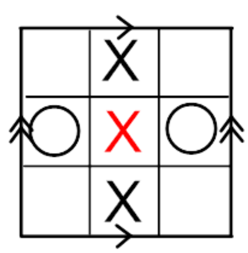




Find the winning move for the "X" player in the following Tic Tac Toe torus game. All boards are a torus and in each board there is a winning move.

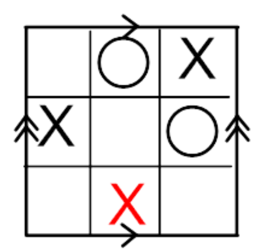
1.



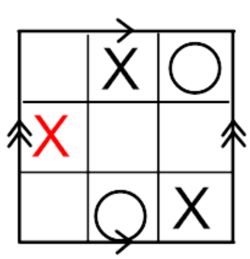
2.



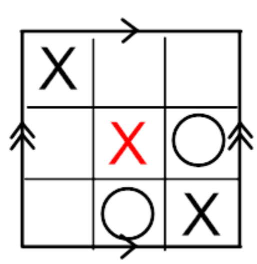
3.



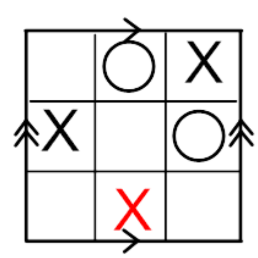
4.



5.

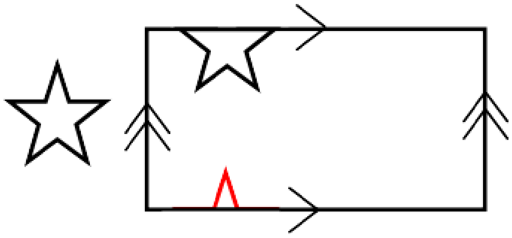


6.

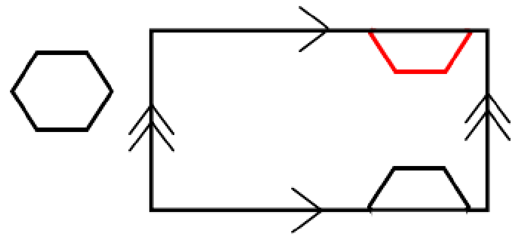


For each of the following you will be given a mathematical figure and a torus with part of that figure in it. You need to draw the missing part of the figure in the correct location in the torus.

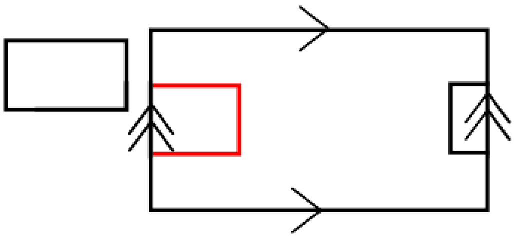
7.



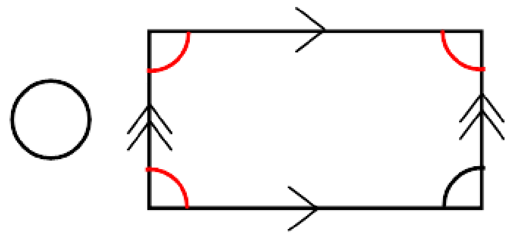
8.



9.



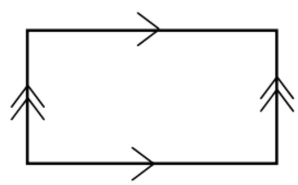
10.



11. Create your own math figure as the original image. Then place the image in a corner so that your figure has to be "finished" in multiple places (as in question 10).

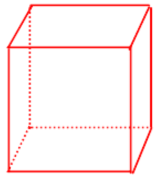
Original Figure:

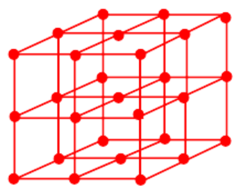
Answers will vary; suggest a square for struggling students.



12. Draw both representations of a three dimensional torus. Explain what you would see will standing inside the figure.

Images should include both:





Students should discuss how standing in a three torus would show them standing in each of the "tori" in the image on the left. The created in a torus is replicated through all the other tori.

Day 5

Objectives:	**Procedures/Strategies:**
• TSW define orientability • TSW define and create a Möbius strip • TSW define and understand the properties of a Klein Bottle	• TSW complete bellwork (5mins) • TSW review Assignment 2: Torus (5 mins) • TSW complete Orientability activity (15 mins) • TSW complete guided notes: Orientability (25 mins)
Materials:	**Assessment Strategies (Formative/Summative)**:
• Orientability Activity –"Band" Sheet and Instructions/Questions • Guided Notes: Orientability	• Orientability Activity

The bellwork question for today is for students to move an object through a torus (as previously discussed). After bellwork the solutions for Assignment 2 should be discussed as a class and students' questions should be clarified. The next activity will take students through what it means for a shape to be orientable or non-orientable. Students will draw their own figure that needs to be verified in case the created figure prevents the student from determining if anything has happened to its orientation. Once the figure is created, the students will copy it onto one side of the paper strip and then trace the image onto the other side. It is important that the students do not simply redraw the figure on the back side of the paper. The figure needs to be traced so that they can see the switching of the orientation. The activity will then direct students into what their figure does when the band is half twisted or fully twisted. This will be the first time for students to see the orientation of a figure changing and also interact with a Möbius strip. If students finish quickly, have them create new figures and experiment with multiple twists of their strip. Once the activity is completed, students will then complete the guided notes. Due to time restrictions, this part of the lesson may not be completely finished. Time is built into the next lesson that allows the class to finish the rest of the guided notes. The guided notes will start off with defining orientability, the Mobius strip, and the Klein bottle. It is important that when describing the Klein bottle that students understand the difference in the gluing of it's edges. One of the pairs of edges will have a gluing that is orientation reversing, making the Klein bottle non orientable. It should be noted to students that the gluing arrows will be in opposite directions to represent the reversing of orientation. Students will also see a comparison to how the Klein bottle and torus are represented. The next section will take a pentagon and move it throughout the surface of the Klein bottle. Questions that follow discuss how the orientation of the pentagon is reversed and how the orientation could be reversed back. This should conclude the amount of time that is remaining in the class for the guided notes.

<u>Orientable Activity</u>

Name:________________________

Date:___________ Period:______

Using the given strip of paper and pen/marker:

1. Create a figure and place it in the box below. **NOTE**: Your figure needs to have the ability to be seen as "upside down" and "right side up". For instance:

Figures that would **not** work — Figures that will work

Draw your figure here: and then verify with the teacher that it is correct.

2. Draw that figure in each square of your strip. Try to keep each figure congruent.

3. Flip your strip over and trace out each image from the other side, do not just simply re draw it. NOTE: Depending on your figure it may seem like you created a reflection.

4. Take your completed strip and form it into a band. What do you notice about the starting and ending images? Draw the starting and ending figures:

The starting and ending images are identical.
NOTE: Student images will vary based on starting image.

<u>Start</u>

<u>End</u>

5. Take the band end and give it a half twist. What do you notice about the starting and ending images? Draw the starting and ending figures:

The ending figure is an upside down version of the starting figure.

<u>Start</u>

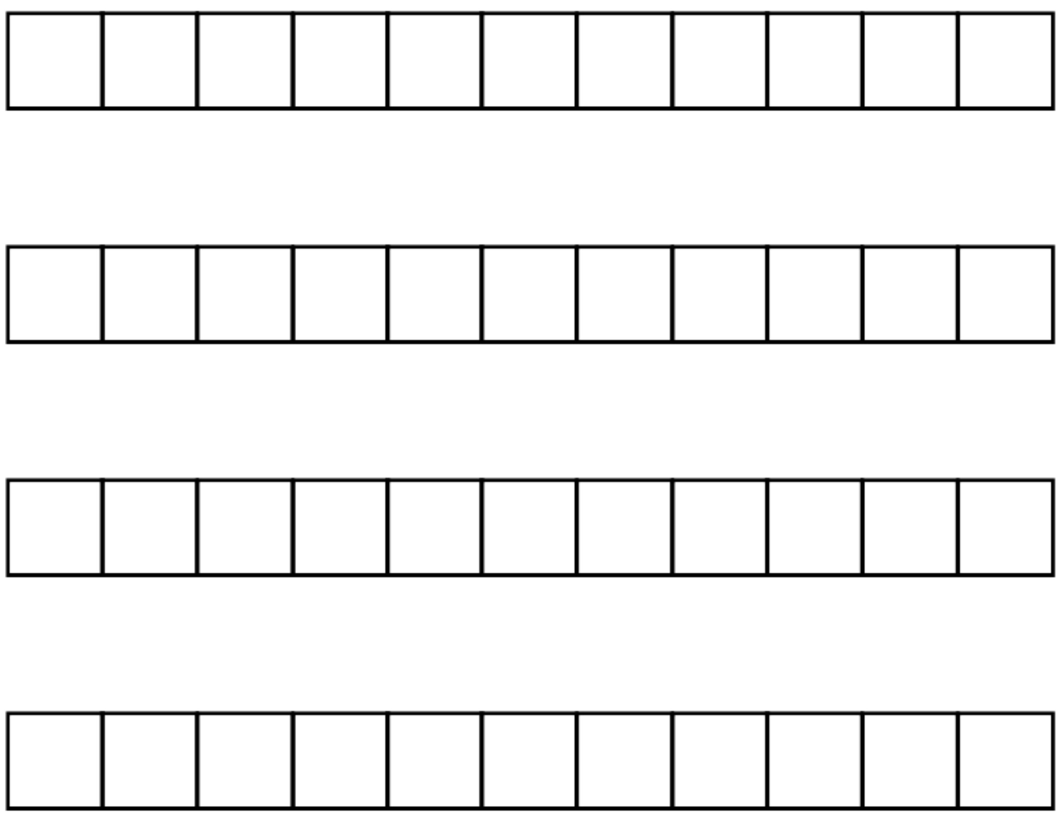
<u>End</u>

6. Take the band end and give it a full twist. What do you notice about the starting and ending images? Draw the starting and ending figures:

	Start	**End**
The ending figure is now again the same as the starting figure.		

7. Take other strips and try to find other shapes, letters, and figures that have similar or different results.

Orientable Activity Slips:



Notes Name:_________________________

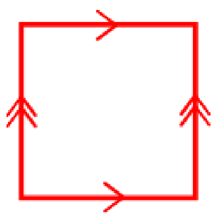
Orientability, Möbius Strip, Klein Bottle Date:___________ Period:______

Orientability: A surface is orientable if a figure making all possible global trips on the surface does not change its orientation at any time.

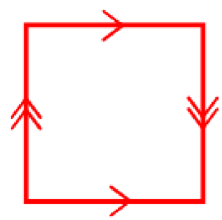
Möbius Strip: A figure created by taking a rectangular plane and gluing with a 180° or half twist. Note: We use a rectangular shape as a model, but the strip itself has no thickness.

Klein Bottle: A figure in two dimensions represented by a rectangle where one pair of opposite sides are reverse glued.

Torus:



Klein Bottle:



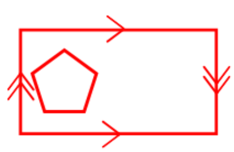
Note: The other pair of sides could be glued together.

What is the difference between the torus and the Klein bottle? One of the pairs of glued sides has the arrows going in the opposite direction (DOES NOT MATTER WHICH PAIR).

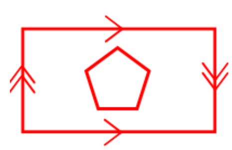
What happens when figures go "through a gluing"? As a figure passes through the gluing it's orientation is reversed, like what happened when traveling on a Möbius Strip.

Consider the pentagon moving through a Klein bottle as we did with the torus previously:

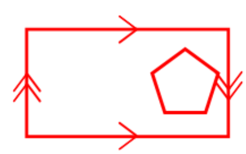
1.



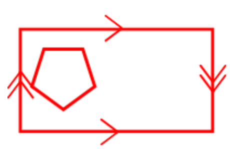
2.



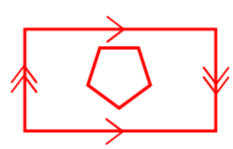
3.



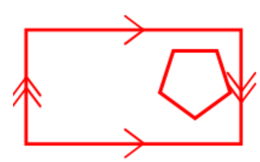
4.



5.



6.



Would this happen if the pentagon moved up and down instead of left and right? No, since gluing marks are not reversed, meaning the orientation is preserved.

What would happen if an upside down pentagon went through the right side again? It would come out with the orientation that it started with, or look like the first picture.

Day 6

Objectives:	**Procedures/Strategies:**
• TSW define orientability • TSW define and create a Möbius strip • TSW define and understand the properties of a Klein bottle	• TSW complete bellwork/entrance activity (5mins) • TSW complete guided notes (15 mins) • TSW complete Assignment 3: Orientability (30 mins)
Materials: • Entrance Activity • Guided Notes: Orientability • Assignment 3: Orientability	**Assessment Strategies (Formative/Summative)**: • Assignment 3: Orientability • Entrance Slip

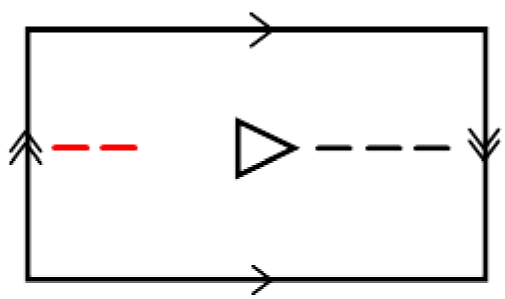
Students will begin class by completing an entrance slip instead of their bellwork. The entrance slip takes the asteroid questions from the torus notes and changes the "game board" to a Klein bottle. These questions are a review of where the students left off in the notes the previous day. The portion of the guided notes that needs to be finished will demonstrate to the students how to take a flat torus and Klein bottle and turn them into a three dimensional representation. A flat torus can be formed into a doughnut shape and a flat Klein bottle can be formed into a figure that has its neck glued into itself (as seen in the notes). This process can be very abstract for students, so it might be necessary to clarify explanations and ensure students have an understanding of each step while forming the figures. Following the conclusion of the guided notes, students will then complete Assignment 3. This assignment will begin with a section on Klein bottle Tic Tac Toe, first introducing students to a big board version. Klein bottle Tic Tac Toe can be more challenging then torus Tic Tac Toe. Students may need some assistance in comprehending how the boards surrounding the "original board" change based on the gluing of the Klein bottle. As suggested in the torus Tic Tac Toe, some students may need to draw the surrounding boards in order to see winning moves. Students will then play Klein bottle Tic Tac Toe with a partner and also determine the winning move from a preset board. The next section will take various figures partially shown on a Klein bottle that need to be completed. Students will have to interpret how the gluing changes the orientation of the given figure. The final problem of the assignment asks the students to find Möbius strips within the Klein bottle. One way to help students who struggle with this is to remind them of what property the Klein bottle and Möbius strip have in common. This property is that figures will switch their orientation while traveling across both figures (or that the Klein bottle and Möbius strip are nonorientable). For a student to find a Möbius strip, they need to create a strip on the Klein bottle that includes the orientation reversing edges as the start and end of the band. When reviewing the assignment in the following lesson, ensure that students understand how to create the Möbius strip on a Klein bottle, as there are multiple ways of doing this.

Entrance Slip Name:__________________________

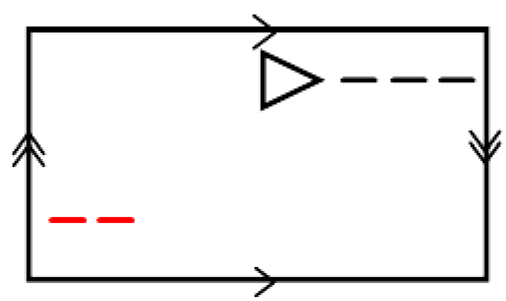
Orientability Date:___________ Period:______

What if the "game board" we used for Asteroids was a Klein bottle instead of a tours? How would the new board change the "laser" being shot from your ship? Complete the laser path below. Remember each shot contains 5 dashes.

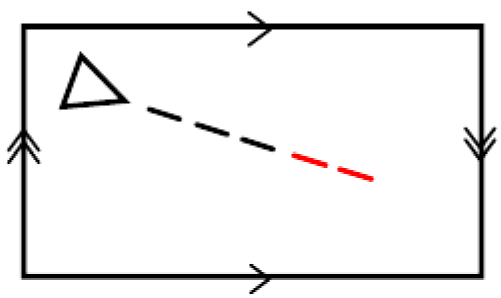
1.



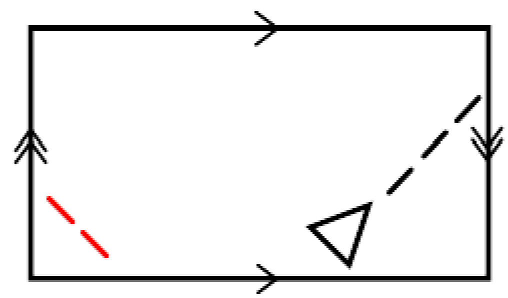
2.



3.



4.

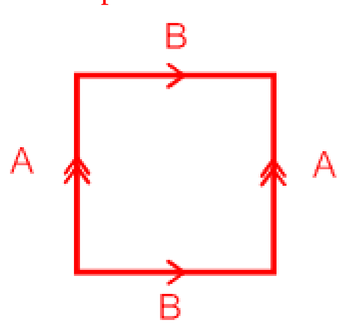


Continuation of Day 5 Guided Notes

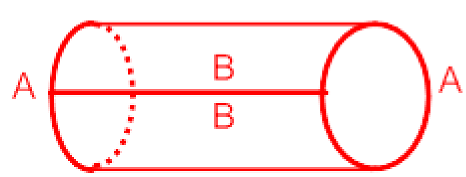
Lets see how we can take our two dimensional torus and Klein bottle and visualize them in a three dimensional space.

To visualize a torus:

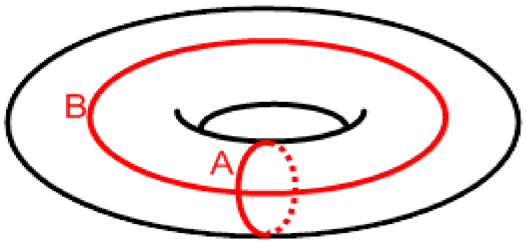
1. Draw our representation of a 2-torus.



2. Form it into a cylinder by matching the "B" edges.



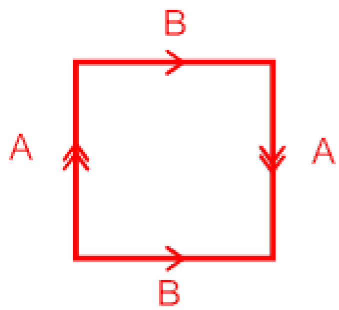
3. Stretch the cylinder and attach the "A" edges together to form the doughnut shape:



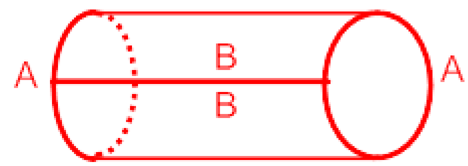
NOTE: Black space is still drawn by student but used black to show contrast for the doughnut shape

To visualize a Klein bottle:

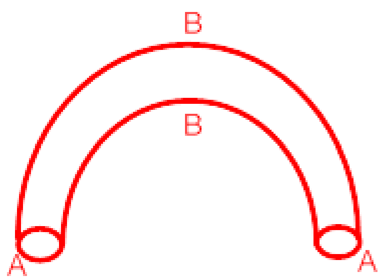
1. Draw our representation of a Klein Bottle.



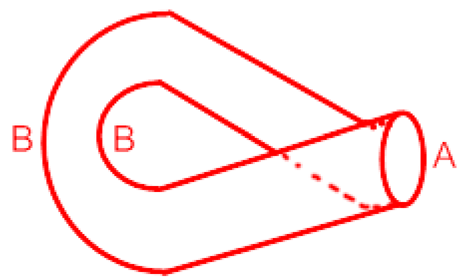
2. Form it into a cylinder by matching the "B" edges.



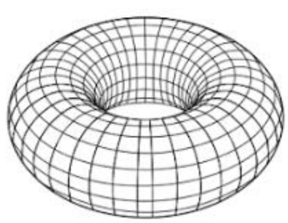
3. Stretch the cylinder out and start to bend.

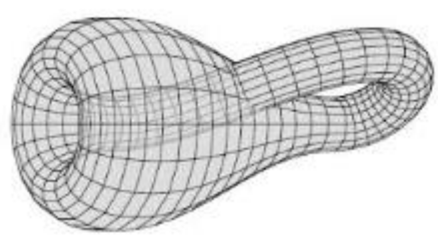


4. Put one end through the cylinder and connect it to the other end.



Since our ability of creating these figures may not be perfect, we can look at these images that computers can generate to give us a better understanding.

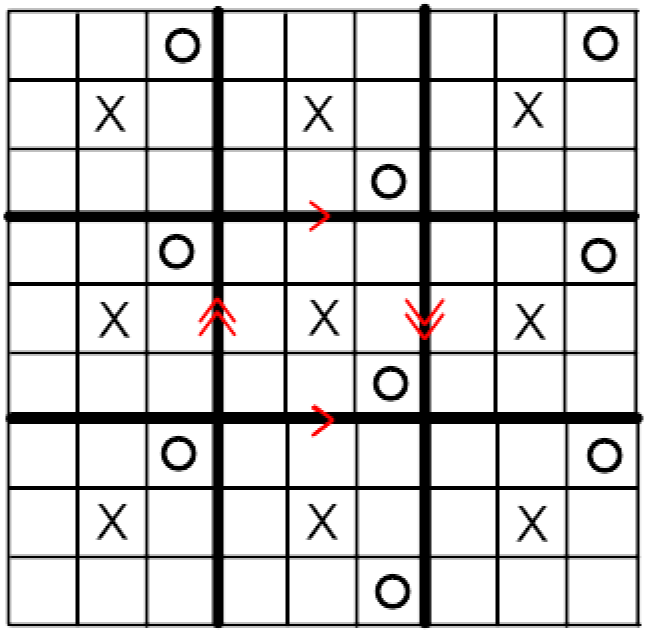




Assignment 3 Name:_________________________

Orientability, Möbius Strip, Klein Bottle Date:___________ Period:______

Directions: Answer the following questions about the given Tic Tac Toe board.



1. What do the symbols on the middle board mean?
The board is like a Klein bottle.

2. What happens if an "X" mark gets placed in the middle row, middle column of the middle board?
An "X" will appear in the middle row, middle column on all the other boards.

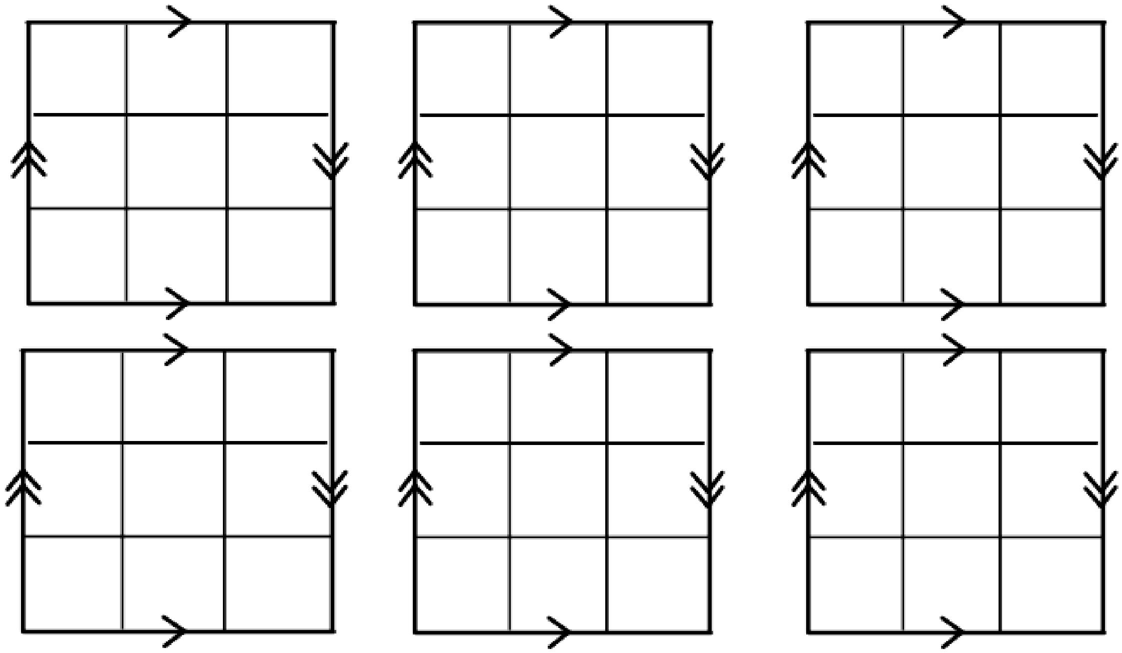
3. What happens if an "O" make gets placed in the bottom row, right column of the middle board?

An "O" will be switched to the top row, right column on the boards that are left and right of the current board. Boards in the same middle column will have the "O" place in the original position of bottom row/right column.

4. Why did the X keep the same position everywhere but the O switched from the top row to the bottom row?

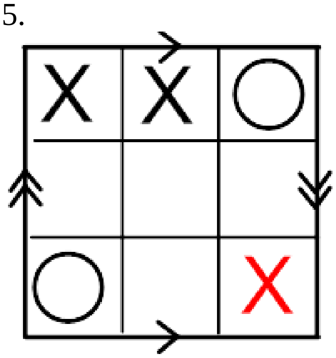
The gluing of the above Klein bottle board switches orientation from when moving left to right. This means that anything in the top row will switch to the bottom row for adjacent boards and vice-versa. The middle row does not change since the position is directly in the middle. NOTE: Make sure to inform students that the position of the X does not change moving from board to board when it is in the middle row, middle column but the orientation of the figure does. Consider the letter T instead of the letter X. When moving to boards in other columns, the T will appear upside down in the middle position, middle row. The letter X will not show this orientation switch.

Partner with another student and play some Klein bottle Tic Tac Toe. If needed, extend the sides of the boards to help with how the Klein bottle switches the board. You may use your own paper if desired.

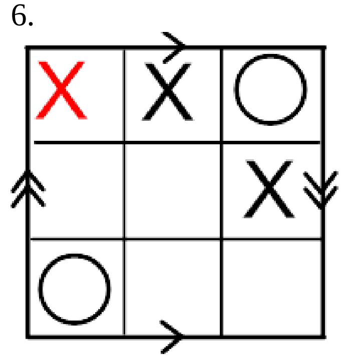


Find the winning move for the "X" player in the following Tic Tac Toe Klein bottle game. All boards are a Klein bottle and in each board there is a winning move.

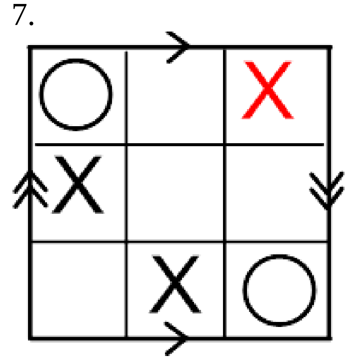
5.



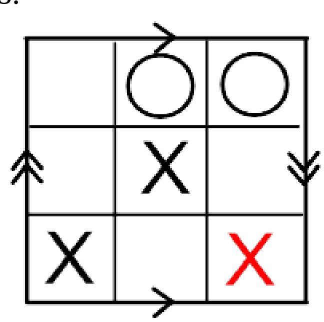
6.



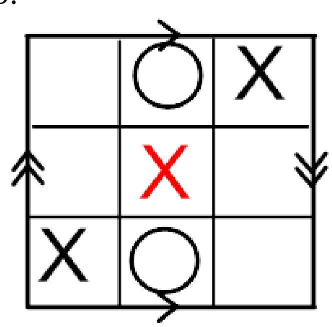
7.



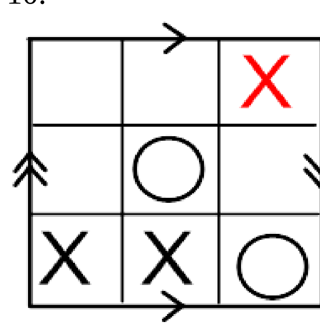
8.



9.

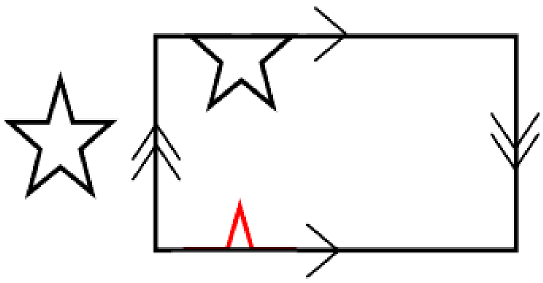


10.

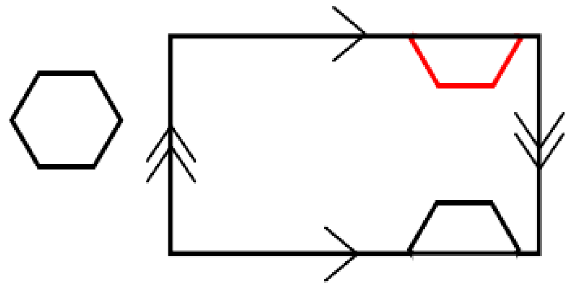


For each of the following you will be given a mathematical figure and a Klein bottle with par of the figure in it. You need to draw the missing part of the figure in the correct location of the Klein bottle.

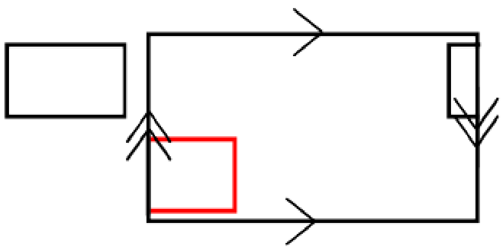
11.



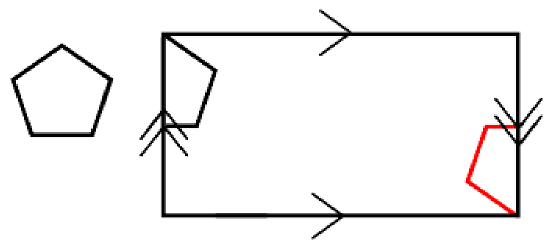
12.



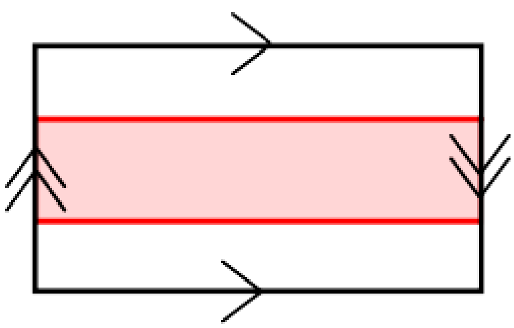
13.

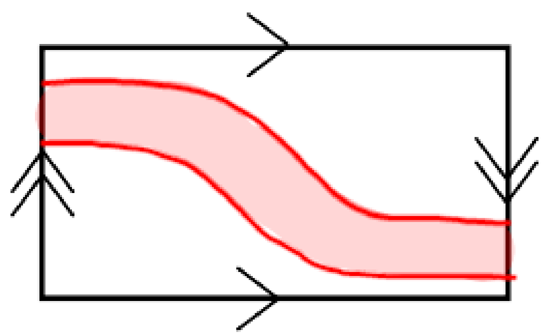


14.



Using the two blank Klein bottle models below, draw a different Möbius strip on each model. Then explain how you know each is a Möbius strip.





The key for this to work is that then ends of the strip align with the orientation reversing part of the Klein bottle. Student answers could vary, but the previous stated property must be present as shown above.

Day 7

Objectives:	**Procedures/Strategies:**
• TSW identify a projective plane and its properties	• TSW complete bellwork (5mins) • TSW review Assignment 3 (5mins) • TSW complete guided notes (30 mins) • TSW complete exit slip (10 mins
Materials: • Guided Notes: Projective Plane • Exit Slip	**Assessment Strategies (Formative/Summative)**: • Exit Slip

The bellwork for the lesson is for students to draw a flat torus, a flat Klein bottle, and explain what the makes the two figures different. After bellwork, Assignment 3 should be reviewed as a class and any questions or concerns should be covered. Students will then complete the guided notes worksheet on the projective plane. Students will be given the definition of a projective plane. The next part of the guided notes will have the students draw the flat versions of the torus and Klein bottle. It is recommended that the students should try to draw the topological projective plane based on the given definition. Next, the students will take the topological projective plane and create an analogous three dimensional projective space. This was done for the torus and Klein bottle, but the topological projective plane is even more abstract. For students to better understand the process, explain when the topological projective plane is made into the circle topologically by having all of the opposite points on the circle glued together. One could describe this as any two points on the end of a diameter of the circle are glued together. From the gluing of the opposite points, the circle can be formed into a hemisphere that is the geometric projective plane, where the rim has the opposite points glued together. When a figure crosses through the rim, its orientation will be switched, just as it would be in the topological projective plane. It is important for the student to understand how figures move about the geometric projective plane. These concepts will be used in the exit slip, as students will have to answer questions that implement critical thinking skills about figures moving on the projective plane. Students will be required to think of distances on the geometric projective plane and demonstrate an understanding of how all opposite points are glued together. This exit slip may require more attention from the teacher and grouping of students is suggested. Groupings of students will promote communication about the content and also increase their chances of arriving at the correct solutions.

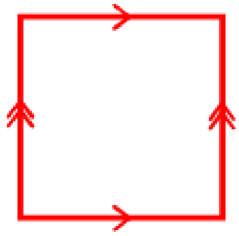
Notes Name:________________________

Projective Plane Date:___________ Period:______

Projective Plane: A projective plane can be represented by a rectangle where both pairs of opposite sides change the orientation as a figure travels from one side to another.

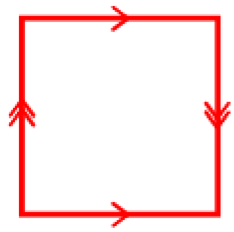
To better understand this definition, we can look at the previous examples of gluing planes:

No pairs switching



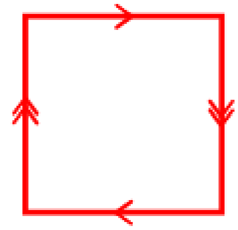
Torus

One pair switching



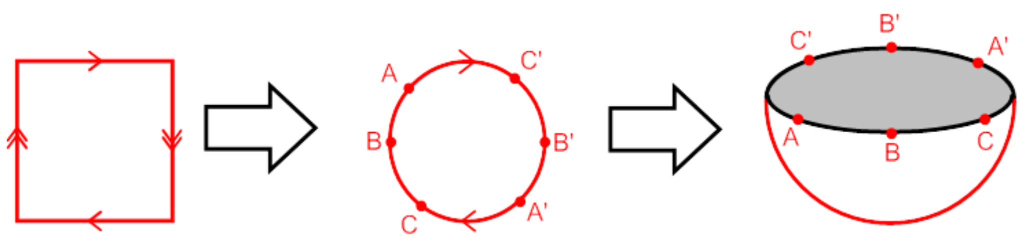
Klein bottle

Both pairs switching



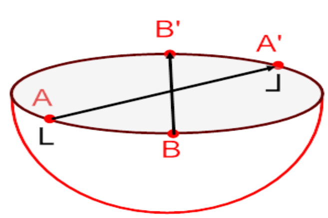
Projective plane

We have represented a torus and Klein bottle in three dimensional space, so we can also try and do the same with the projective plane.



This is a representation where all opposite points are glued on the circular boundary together: A to A', B to B', and C to C' etc. This can then be represented in three dimensions by the hemisphere, where the rim or edge is where our opposite points are glued.

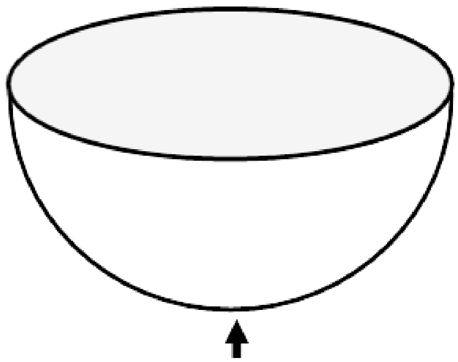
What does a figure look like after it crosses through the rim? The figure will have it's right and left sides switched, or its orientation will change.



Exit Slip Name

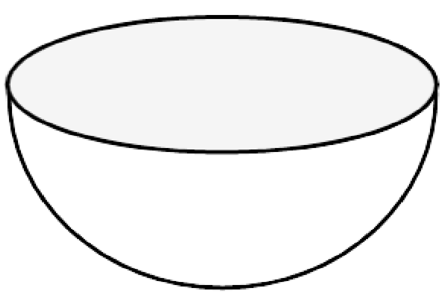
Projective Plane Date:___________ Period:______

Think of a figure living on a projective plane at the pole (in the case for the projective plane pictured below think of the south pole). If a figure left his house on a journey in a straight path which brought him back home, at what point would the figure be farthest from it's home? Explain.



No matter which direction the figure travels, it will be farthest from the house at the rim of the hemi-sphere. As the figure crosses the rim, it will come out on the "opposite side" and the figure will then be getting close to the house. Since the rim is where it switches from getting farther from the house to getting closer to the house, it must be the farthest point from the house.

Two sibling "figures" are ready to move out of their parent's house. They can move anywhere on the projective plane and because they fight all the time they want to be as far away as possible from each other. Where could they build their houses so that they are as far apart as possible? Explain your answer with sentences and draw a picture on the projective plane.



Some student answers will vary but the important property for this to work is the siblings need to be 90 degrees apart from each other. For instance one answer could be one sibling anywhere on the rim while the other is on the pole. Another answer would be to have the siblings 90 degrees apart on the rim (since on the rim the opposite sides are glued making points 180 degrees away from the same point).

Day 8

<table>
<tr><td><u>Objectives</u>:
TSW identify a surface and its properties
TSW distinguish between properties of figures
TSW identify homogenous figures</td><td><u>Procedures/Strategies:</u>
TSW complete bellwork (10mins)
TSW complete guided notes (40 mins)</td></tr>
<tr><td><u>Materials</u>:
Guided Notes: Figure Properties
Blank Paper and String
Beach ball already inflated</td><td><u>Assessment Strategies (Formative/Summative)</u>:
Assignment 4 (Tomorrow)</td></tr>
</table>

The bellwork for today is for the students to draw the three dimensional representation of our three defined figures: the torus, Klein bottle, and projective plane. Students do not need to describe how the figures come from their flat versions, although this could be an enrichment question for those who finish quickly. Extra time was added to this bellwork to allow students to display their answers and discuss how to draw each figure. Following the bellwork, students will complete the guided notes where they will first be introduced to a sphere as a defined figure. The sphere now gives us four defined figures that we will refer to throughout the guided notes. Next we will define extrinsic and intrinsic properties followed by an activity that demonstrates these properties. Students are to take a blank piece of paper and draw a segment of any length, but it is recommended that they keep it somewhere near the center. Once the students create the segment, they are to curve the paper into a cylinder and then compare planes and cylinders. For the intrinsic part, direct students to discuss how we could test if the plane and the cylinder have the same intrinsic properties. Students should arrive at the conclusion of testing the length of the segments that were created. To test their length, measure the segment while on the plane with the string and then do the same to the cylinder segment. The outcome will show the segments have the same length, making them intrinsically the same. Extrinsically the figures are different, due to the curvature of the cylinder and flatness of the plane while looking at the figures from three space. The notes will then look at local and global properties, which are defined and followed by examples. The examples allow students to determine if figures have similar or different local and global properties. It is important to note that these properties are thought of as “local geometry” and “global topology”. This may assist students in understanding the difference between local and global properties. The next property to be discussed is to determine if a surface is considered homogenous. The activity to simulate this property uses a beach ball that is already inflated. The students should compare the beach ball to a sphere and then discuss if the ball has the same local geometry at all points. After arriving at the conclusion that the sphere is homogenous, the students should discuss how the ball could be changed in order for it to be nonhomogeneous. Students should be directed to deflate the ball, causing the surface to be nonhomogeneous to the sphere which can be easily demonstrated. After the activity, make sure to discuss the four figures that we have defined and how they are all homogenous.

Notes Name:________________________

Figure Properties Date:___________ Period:______

We have now discussed three specific figures: Torus, Klein bottle, and Projective plane

A fourth figure that we will discuss is something we are familiar with: Sphere

IMPORTANT: When referring to the sphere we are only concerned with the surface. We are concerned about what it is like to be on the actual surface.

Drawing a Sphere:

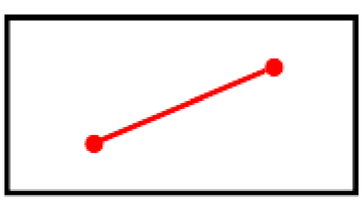
Note: Draw this figure for the students. To draw the sphere first draw a circle, then the solid arc, and then the dotted arc. Use this space to draw multiple spheres or even use different strategies.

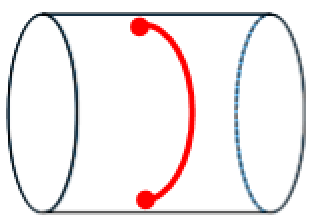
Now that we have seen our four figures, we can discuss some properties/definitions.

Intrinsic Property: Intrinsic properties are perceived from being on the surface. If you are on a plane, then you perception would be of a two dimensional figure.

Extrinsic Property: Extrinsic properties are perceived from looking from outside the surface. If you are off of the plane, then your perception would be like a 3-D figure looking at a 2-D figure.

Example: Take the paper given to you and draw a line segment on it. Then roll it into a cylinder (making sure the segment is completely seen). Draw what you see.





Answers may vary for students

What does the above activity show about the intrinsic geometry for the paper and cylinder? The segment on the cylinder is the same intrinsically as on the plane paper. We know this by taking a string at both endpoints of the segment on the paper and measuring the segment on the cylinder to find they have the same measure. Therefore the cylinder and the paper are intrinsically the same.

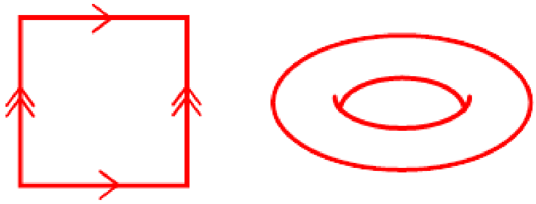
What does the above show about the extrinsic geometry for the paper and cylinder? The paper no longer looks the same after we turned it into a cylinder. Before it was flat like a plane but not it has extrinsic curvature. Looking at it from our viewpoint we can see this difference, therefore the pare and cylinder are not extrinsically the same.

Local Property: Local properties deal with small regions on the figure.

Global Property: Global properties deal with the entire figure, not just a portion.

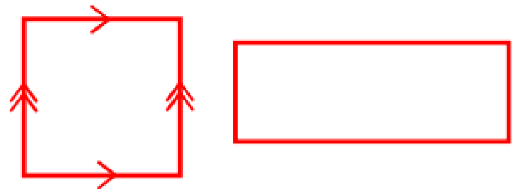
Example: Identify if the following pairs of figures have equivalent local geometries and/or global topological properties

Flat Torus and Doughnut Surface



Local Geometry: Different due to curvature on doughnut.
Global Topology: Equivalent since we can create the doughnut from the flat torus.

Flat Torus and a Plane



Local Geometry: Equivalent since both are flat surfaces.
Global Topology: Different by considering how a figure moves different globally on each surface.

NOTE: We will think of the geometry of a figure for local properties and the topology of a figure for global properties.

Homogeneous: A figure is homogenous if the local geometry is the same at all points.

Nonhomogeneous: A figure is nonhomogeneous if the local geometry is not the same everywhere.

Use the demonstration with the beach ball to answer following questions.

When the beach ball was fully inflated, was it homogeneous or nonhomogeneous?
Homogeneous

Which surface previously discussed does the beach ball model? The sphere

What happened to the beach ball as it was deflated? Why? It became nonhomogeneous because the local geometry is now different for some parts of the surface.

What are the four homogeneous surfaces that we have discussed so far?
The sphere, torus, Klein bottle, and projective plane (talk these out with students, have them think what the four could be and ask them to justify their answer).

Closed Manifold: Finite, there is no bound to the surface.

Open Manifold: Infinite, the surface spreads without bound.

Day 9

Objectives:	**Procedures/Strategies:**
• TSW identify a surface and its properties • TSW TSW distinguish between intrinsic/extrinsic properties, local/global properties, and open/closed figures • TSW identify homogenous figures	• TSW complete bellwork (5mins) • Vocab Flash Cards (10 mins) • TSW complete Assignment 4: Surface Properties (35 mins)
Materials: • Assignment 4: Figure Properties • Rubber bands/Scissors • Index Cards	**Assessment Strategies (Formative/Summative):** • Assignment 4

Bellwork for today will be a review of the Tic Tac Toe torus and Klein bottle games that have been previously played. Give students a Tic Tac Toe board for a torus and Klein bottle in which they need to determine the wining move for the “X” player. Following bellwork, distribute index cards to students and have them copy the 8 definitions that were introduced yesterday: intrinsic, extrinsic, local, global, homogeneous, nonhomogeneous, closed, and open. One side of the index card should state the word and the other side should have the definition. Students may elect to include examples of images with the definitions to assist in reinforcing the properties. These vocabulary cards will be used as a reference for students during today’s assignment and also give the students a study reference for their upcoming summative assessment. Have each student show that they have completed their vocabulary cards and then proceed to give them Assignment 4, a pair of rubber bands, and scissors. The rubber band and scissors will be used for the beginning part of the assignment. This resource should be helpful to students who are struggling to visualize what happens intrinsically and extrinsically to the rubber band as it is cut and twisted. Students will then answer a series of questions on determining if specific figures are open or closed. The students will be asked how to make a figure closed that was previously identified as open. The back side of the assignment will give students a description of various figures and ask them to determine if they are homogeneous or nonhomogeneous figures. Students will have to explain their reasoning and demonstrate their understanding of what it means for a figure to be homogeneous. The final section of the assignment gives students pairs of figures and asks them to determine if the figures have the same local or global properties. Students will again have to state their reasoning and demonstrate understanding. Encourage students to use complete sentences when explaining their reasoning. When reviewing the assignment the following day, make sure that students explain their reasoning or give students the reasoning behind the explanations. This will give students a chance to confirm that their reasoning was correct or show that they did not fully understand the solution.

Assignment 4 Name:________________________

Figure Properties Date:___________ Period:______

Extrinsic vs Intrinsic

You are given two rubber bands. One of the you leave alone, while the other you cut, give a full twist, and then glue back together. Answer the questions that follow.

1. Do the rubber bands have the same intrinsic topology? Why? Yes, an object on the surface would not be able to tell the difference between the two bands since the full twist would not change the orientation of the figures on the band.

2. Do the rubber bands have the same extrinsic topology? Why? No, we can clearly see a difference between the two bands. Their topology has changed from cutting and then gluing a twist into the second band.

3. How could we form the second band to make the intrinsic **and** extrinsic topologies different? We can only give the second band a half twist. This would cause the orientation to change for a figure that is traveling on the band, like a Möbius strip.

Closed vs Open

For the following figures, state if they are closed or open.

4. A circle
Closed

5. A line
Open

6. A two holed doughnut
Closed

7. A sphere
Closed

8. A plane
Open

9. An infinitely long cylinder
Open

10. A flat torus
Closed

11. Klein Bottle
Closed

12. Projective Plane
Closed

13. What could we do to number 5 to change your answer? We can change it to a segment, since a line is infinitely long it is open. A segment is finite.

14. What could we do to number 9 to change your answer? We can change it to just a bounded cylinder, which would make it finite.

Homogeneous vs Nonhomogeneous

Take each of the following example and explain why they are homogenous or nonhomogeneous.

15. A piece of paper. Homogeneous, the geometry is consistent throughout the surface, like a plane or flat torus.

16. A piece of paper crumpled into a ball. Nonhomogeneous, the geometry is different throughout each part of the paper.

17. A basketball. Homogeneous, the geometry is consistent throughout the surface, like a sphere.

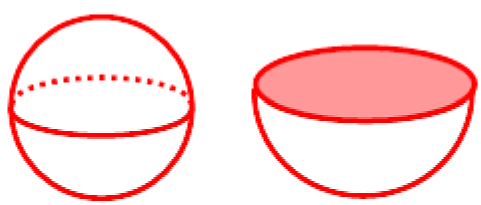
18. A basketball with the air let out (deflated). Nonhomogeneous, the geometry is different throughout points of the deflated basketball.

19. A solid hemisphere (like a ball of Play-Doh that is cut in half). Nonhomogeneous, the geometry is different on the "flat part" of the hemisphere when compared to the curved part.

Local Geometry vs Global Topology

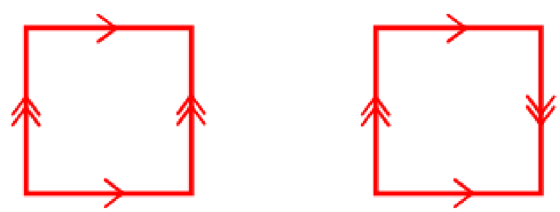
Draw each of the following figures and then determine if the local geometries and/or global topologies are equivalent. Explain your reasoning for each answer.

Solid Sphere and Solid Hemisphere



Local Geometry: Different due to the flat space on the hemisphere.
Global Topology: Equivalent since we can create a semicircle topologically from the sphere by pulling out the flat part.

Klein Bottle and Torus (Flat Versions)



Local Geometry: Equivalent since both are flat surfaces.
Global Topology: Different since the orientability between the two figures is different.

Day 10

Objectives:	**Procedures/Strategies:**
• TSW review previous learning objectives for the unit	• TSW complete bellwork (5mins) • TSW review Assignment 4 (5mins) • TSW complete review activity (10-15mins) • TSW complete review assignment (35 mins)
Materials:	**Assessment Strategies (Formative/Summative)**:
• Review Assignment • Review Activity materials (explained below)	• Review Activity/Assignment

The bellwork question will be another review for the torus and Klein bottle. Give the students a figure that is placed on a flat torus and flat Klein bottle. Make sure that the figure is going through an edge so that the students have to draw the missing portion of the figure in the torus or Klein bottle. Review Assignment 4 as previously stated in Day 8. The review activity problems are given below in a worksheet format. This review is intended for students to see the problem and then inform the teacher of their selected response. These problems could be shown using technology such as Plickers, Kahoot, Turning Point, or other student response systems. All students should be given adequate time to respond and then the correct solution should be discussed. At the end of the activity students should complete the review assignment for the remainder of the class.

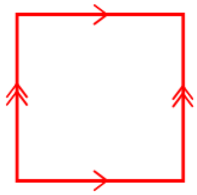
Day 11

Objectives:	**Procedures/Strategies:**
• TSW review previous learning objectives for the unit	• TSW complete summative assessment • TSW complete missing assignments or challenge puzzles
Materials:	**Assessment Strategies (Formative/Summative)**:
• Summative Assessment	• Summative Assessment

There will be no bellwork today. Students will be given the assessment at the beginning of the class and the remainder of the period to complete it. After the assessment students have the above mentioned options to stay occupied. The test is composed of by 3 learning objectives, and each question is labeled (by parenthesis above the question number) with the objective it corresponds to and the point total for the question. Students are expected to show all work and use complete sentences when required.

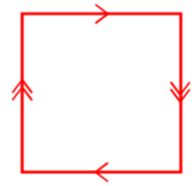
Review Activity Problems

1. The following is an example of what?



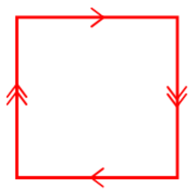
A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

2. The following is an example of what?



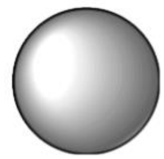
A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

3. The following is an example of what?



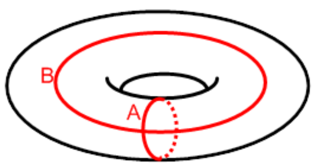
A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

4. The following is an example of what?



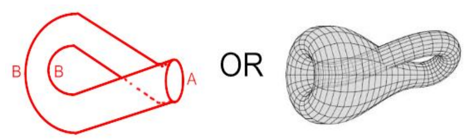
A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

5. The following is an example of what?



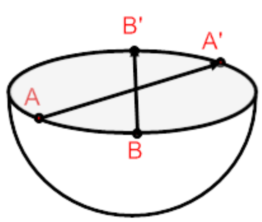
A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

6. The following is an example of what?



A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

7. The following is an example of what?



A. Sphere
B. Torus
C. Klein Bottle
D. Projective Plane

8. The following mathematical figures might share which property?

Sphere
Torus
Klein Bottle
Projective Plane

A. Congruent
B. Orientable
C. Nonorientable
D. Homogeneous

9. Of the following mathematical figures, which one is orientable?

A. Möbius Strip
B. Torus
C. Klein Bottle
D. Projective Plane

10. Which of the following actions prevents figures from being topologically equivalent?

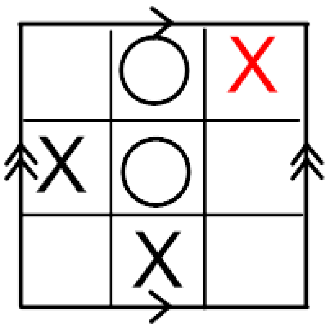
A. Bending
B. Stretching
C. Tearing
D. Twisting

Assessment Review Name:____________________________

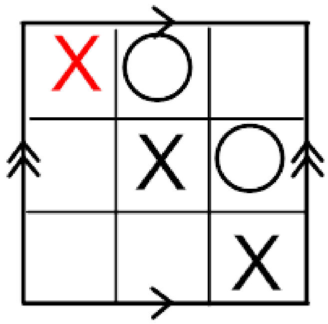
Unit: Topology Date:____________ Period:_______

Find the winning move for the “X” player in the following Tic Tac Toe torus game. All boards are a torus and in each board there is a winning move.

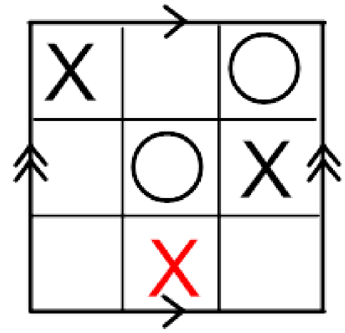
1.



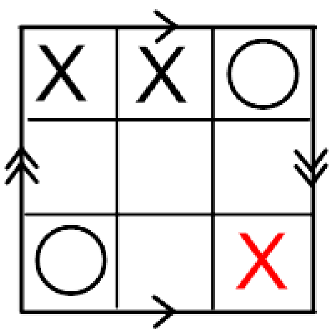
2.



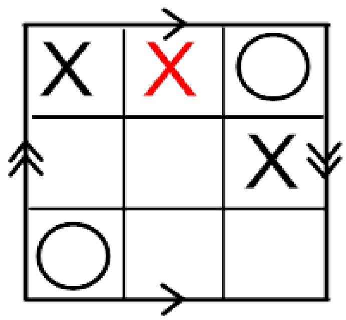
3.



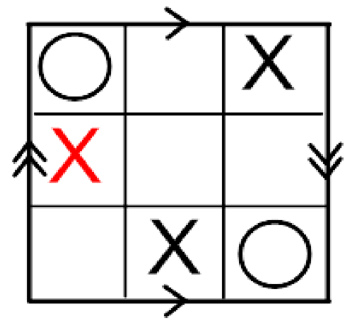
4.



5.

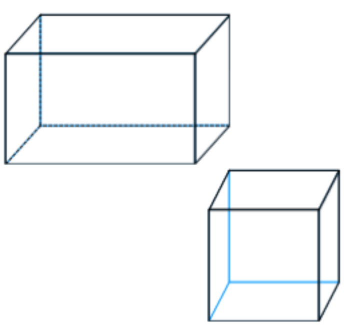


6.



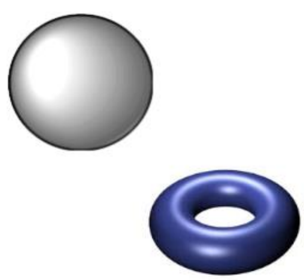
Look at the following pairs of figures and determine if they have the same geometry, topology, or have neither the same topology/geometry.

7.



Topology

8.



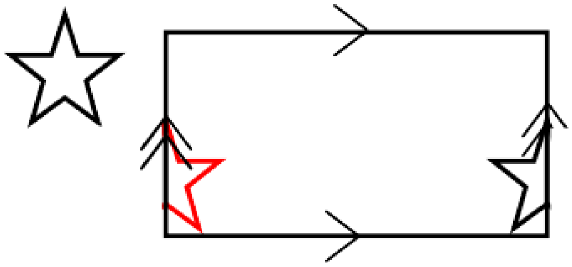
Neither

9.

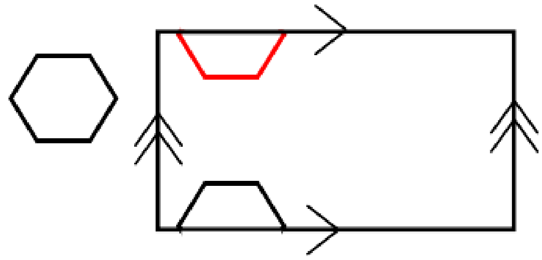
Topology

For each of the following you will be given a mathematical figure and a torus with part of that figure in it. Draw the missing part of the figure in the correct location of the torus.

10.

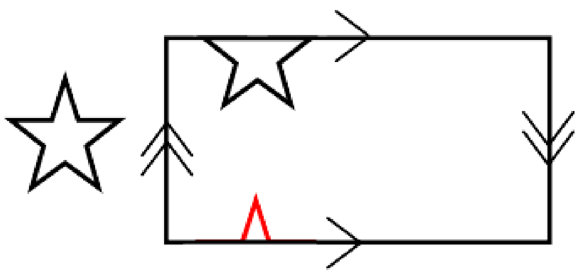


11.

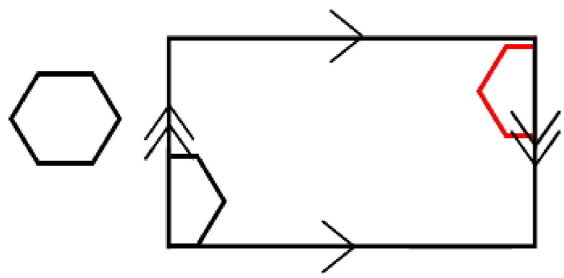


For each of the following you will be given a mathematical figure and a Klein bottle with part of that figure in it. Draw the missing part of the figure in the correct location on the Klein bottle.

12.

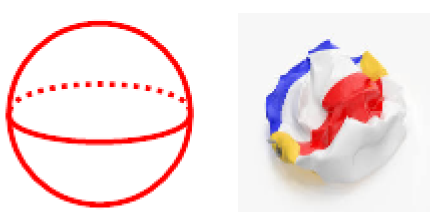


13.



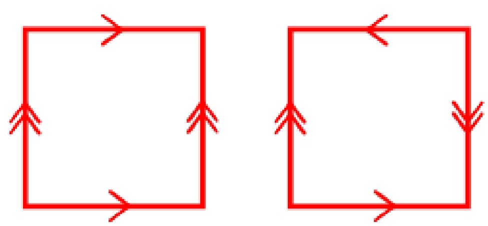
Draw each of the following figures and then determine if the local geometries and/or global topologies are equivalent. Explain your reasoning for each answer.

14. Sphere and Deflated Beach Ball



Local Geometry: Different due to the nonhomogeneous geometry of the beach ball.
Global Topology: Equivalent since we can blow air into the beach ball to make a sphere.

15. Torus and Projective Plane



Local Geometry: Different due to the projective plane being curved and the torus being flat.
Global Topology: Different since the orientability between the two figures is different.

Assessment: Topology Name:___________________________

Date:____________ Period:______

I can...

A. Distinguish between Topology and Geometry

B. Identify Geometric/Topological properties in mathematical figures

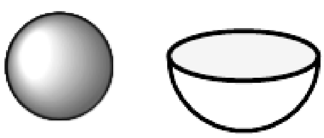
C. I can use properties of a torus, Klein bottle, and orientability to determine relationships amongst mathematical figures.

Score for each learning objective		
A	**B**	**C**

Directions: Make sure to completely answer each question and use complete sentences when necessary. You may draw pictures or diagrams to help explain your answer.

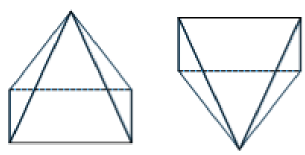
For the following, state if the following figures are geometrically or topologically equivalent. If they are neither, then state that.

(1a)
1.



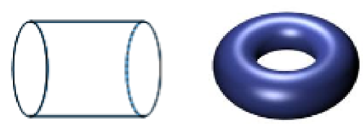
Topology

(1a)
2.



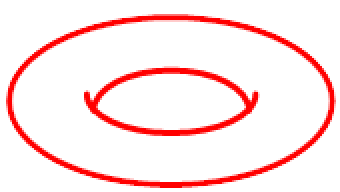
Geometry

(1a)
3.



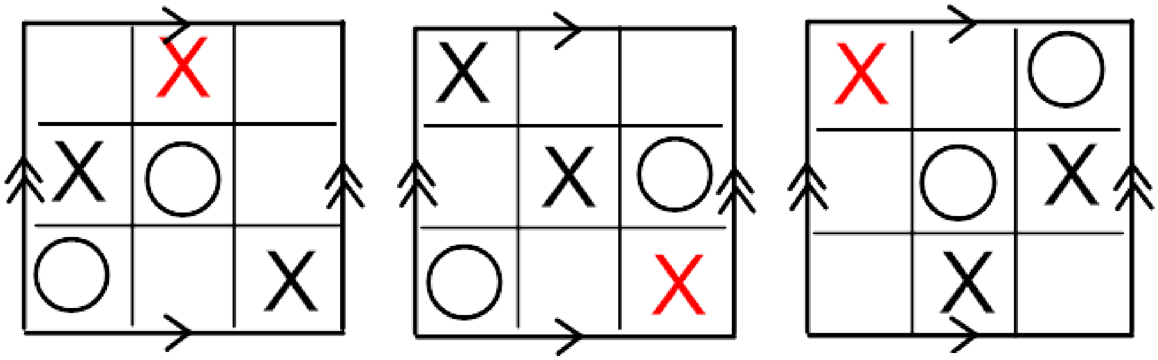
Neither

(2a)
4. Draw a figure that is topologically equivalent and a figure that is not topologically equivalent to the following figure.

Figure	**Topologically**	**Not Topologically**
	 Answers may vary but should be similar.	Answers will vary , look for figures without a "hole" or several "holes.

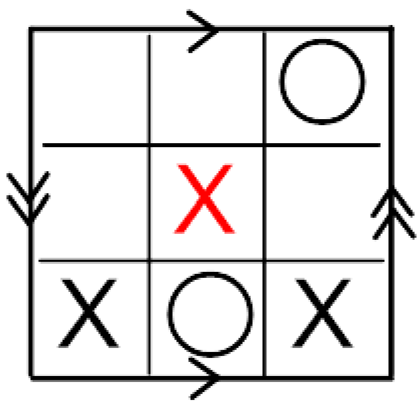
(3c)

5. Determine the winning move for the “X” player in Tic Tac Toe torus game



(3c)

6. Determine the winning move for the X player in the following Tic Tac Toe Klein bottle game. Explain how you know this is the winning move (a picture may help).



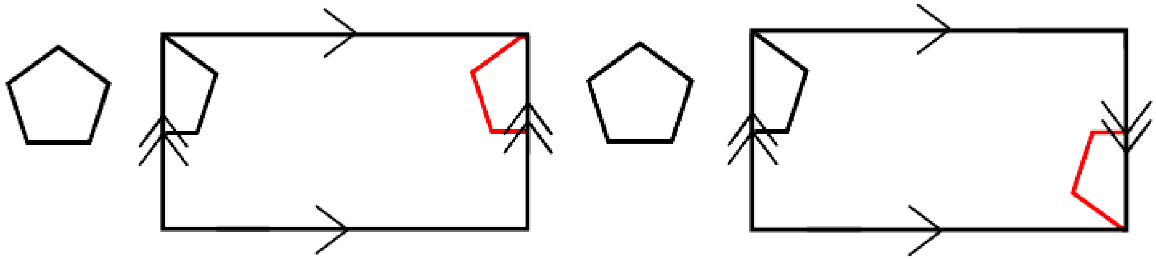
Student may draw the chart with the extension of the board to show how the gluing moves the “X” to create a diagonal win. If students do not draw the chart, they need to describe how the gluing of the Klein bottle creates the X as a winning move.

(4b)

7. Give an example of two figures which have equivalent global topologies but different local geometries. How do you know your figures meet these requirements? You may draw pictures if you choose so.

Student answers will vary, but examples include a sphere and deflated sphere, flat torus and a doughnut shape, solid sphere and solid hemisphere. Students should discuss what makes the shapes topologically the same and how their local geometries differ.

(3c)
8. You are given a mathematical figure in a torus. Complete what the rest of the figure will look like in the torus. Then take the same figure and show what happens in a Klein bottle. Explain why there is a difference for the figure in the torus compared to the Klein bottle.



The difference is due to the Klein bottle being nonorientable. As it passes through the edge the orientation of the pentagon is switched.

(3c)
9. Explain how you could make a Möbius strip. Describe what happens to a figure as it travels across the surface of the surface of the Möbius strip (refer to the properties of the Möbius strip).
A Möbius strip can be created by taking a band, performing a half twist on the band, and then gluing the two ends together. A figure traveling on a Möbius strip will have its orientation switch due to then half twist in the band (since it is nonorientable).

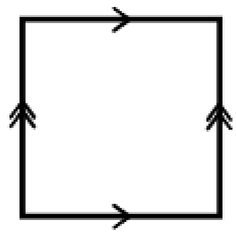
(2a)
10. Describe why the following figures are topologically different.

The first figure has one "hole", whereas the second shape has two "holes". For the shapes to be topologically equivalent, you need to tear the figure with one hole at least once and then glue it back together in order to create a shape with two holes. Therefore they are topologically different.

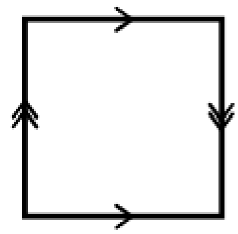
(4b)

11. Give the correct name for each of the four figures below,

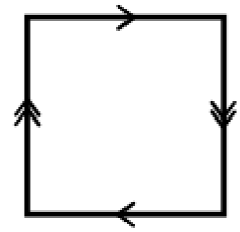
Sphere



Torus



Klein Bottle



Projective Plane

(4b)

12. Use the above names to fill in the chart below.

	Orientable	**Nonorientable**
Curved Local Geometry		
Flat Local Geometry		

(2b)

13. State which geometric property all of the four above figures have in common? Explain what the property means about the figures?

Homogeneous: Their local geometry is the same for the entire figure.

Conclusion

Topology is a subject that can drive our students to think in an entirely different way mathematically. Even though this unit is just an introduction to topology, it still is sufficient enough to promote critical and abstract thinking. One of the main goals in a mathematics classroom should be to enhance a student's ability in think critically and to solve problems. Although these skills are essential to mathematical reasoning, they are also essential to student learning in all disciplines. Improving those skills will improve students' reasoning in all classes. Within this unit, students are consistently engaged in activities, assignments, and problems that force them to practice and utilize these skills. Students also are given several opportunities to enhance their communication skills with respect to mathematical content. Students will consistently have conversation with not only the teacher but with other students as well. Topology also introduces students to an area of mathematics they did not know existed. It will deepen various skills and concepts students previously used and create a new foundation for abstract thinking. Through exercises, students are pushed beyond their previous visual-spatial perception skills. Even though this unit is not included in Ohio's Learning Standards in Mathematics, it does promote growth in fundamental mathematical practices that can be achieved by students.

Reference

1. Weeks, J. R. (2002). *The Shape of Space*(2nd ed.). New York, NY: Marcel Dekker.