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Summer 2015

MITTAG-LEFFLER THEROREM

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An Essay Submitted to
the Graduate School of
John Carroll University
in Partial Fulfillment of the Requirements
for the Degree of
Master of Science

By
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2015

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Dedication

His generosity and support overwhelms me and I would like to take this opportunity to thank Dr. David Stenson for his suggestions, and constant corrections and believing in me all through my masters program. Thanks to The Gates Millennium Scholars program for the financial support throughout, and also to my family, especially to my lovely parents, Mr. Kofi Ottie Agyeman and Mrs. Comfort Amankwah for their support. I couldn’t have done all this without the support of my fiancé, Kwaku Ayisi and all my friends.

CHAPTER 1

The Mittag-Leffler Theorem: A Brief History

Magnus Gösta Mittag-Leffler was a Swedish Mathematician who contributed immensely to the theory of functions. He was born in Stockholm, Sweden and growing up had an aptitude for higher mathematics. He continued his formal education at Uppsala and became a lecturer. His appointment as a lecturer enabled him to travel abroad for three years to continue his studies. In 1873, Magnus Gösta Mittag-Leffler left for Paris, France to study with Charles Hermite (a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions and algebra). In 1875, heeding to Hermite's advice, Mittag-Leffler left for Berlin, Germany to further his studies and learned from Mathematicians like Karl Weierstrass. Weierstrass's work in complex analysis influenced Mittag-Leffler's work for many years.

A year later and in 1877, Mittag-Leffler expanded on Weierstrass's 1876 factorization theorem and proved a similar theorem for meromorphic functions, which is now associated with Mittag-Leffler's name. A meromorphic function is one whose singularities are poles. The theorem states that given a set of poles, corresponding orders, and Laurent coefficients, it is always possible to find a function which is analytic except at those poles, with the correct orders and Laurent coefficients, up to the addition of an entire function.

The final version of Mittag-Leffler's theorem was published later in 1884 in the journal *Acta Mathematics*. After serving 45 years as the editor in chief of *Acta*, Mittag-Leffler passed away on July 7, 1927 in Stockholm. This paper focus on the understanding of the proof of Mittag-Leffler's theorem with meromorphic functions and also connects with Weierstrass theorem with entire functions. An entire function is analytic everywhere in the entire complex plane, and the most common entire function with no zeros has the form $e^{g(z)}$, where $g(z)$ is any arbitrary entire function. Note that the reciprocal of an entire function that has zeros is a meromorphic function, where the reciprocal function has a pole where the original function has a zero. To understand how a Mittag-Leffler function is created, we first need to review some of the basic ideas in complex analysis. The third chapter focus on the proof of Mittag-Leffler's theorem and Weierstrass Theorem using meromorphic functions. We end this paper with an independent proof of Weierstrass Theorem which is now known as The Weierstrass Factorization Theorem. This approach uses infinite products.

CHAPTER 2

Basic Definitions and Theorems

In order to prove the Mittag-Leffler Theorem, we first review some basic concepts, definitions, and theorems which will aid in better understanding the proof of the theorem. Most of the proofs of these theorems can be found in any undergraduate text in Complex analyses. Note that all functions unless explicitly stated otherwise are complex functions.

Definition: The **disc** with center at z_0 and radius $r > 0$ is $D(z_0, r) = \{z \in \mathbb{C}; |z - z_0| < r\}$.

Definition: The **closed disc** with center at z_0 and radius r is $\overline{D}(z_0, r) = \{z \in \mathbb{C}; |z - z_0| \leq r\}$.

Definition: The **punctured disc** with center at z_0 and radius r is $D'(z_0, r) = \{z \in \mathbb{C}; 0 < |z - z_0| < r\}$.

Definition: The **unit disc** with center at 0 and radius $r = 1$ is $D(0,1) = \{z \in \mathbb{C}; |z| < 1\}$.

Definition: The **circle** with center at z_0 and radius r is $C(z_0, r) = \{z \in \mathbb{C}; |z - z_0| = r\}$.

The following definitions will help to define a region.

Definition: A subset S of $\mathbb{C}$ is **open** if for each point $a_0 \in S$, $D(a_0, \varepsilon) \subset S$ for some $\varepsilon > 0$.

Definition: An open subset of $\mathbb{C}$ is **connected** if it cannot be expressed as a union of two nonempty disjoint open sets.

Definition: A **region** is a nonempty open connected subset of $\mathbb{C}$.

Definition: A function $f(z)$ of the complex variable z is **analytic** at a point z_0 if $\exists\ \varepsilon > 0 \ni f(z)$ is differentiable in $D(z_0, \varepsilon)$. If the function is analytic, then the function is sometimes called *regular* or *holomorphic* (H).

In addition, a function is an *entire* function if $f(z)$ is holomorphic on $\mathbb{C}$. Since the derivative of a polynomial exist everywhere, it follows that every polynomial is an entire function.

Definition: A function $f(z)$ defined on region Ω is **bounded** if there exist a real number $M < \infty$ such that $|f(z)| \le M$ for all $z \in \Omega$.

Definition: Γ is called a **simple arc** (or **smooth curve**) if there exist a parametrization $z : [a,b] \to \mathbb{C}$ of Γ. Where z is continuously differentiable and $z(t_1) \ne z(t_2)$ except possibly at the endpoints and $z'(t) \neq 0$ for any $z \in [a,b]$. If $z(a) = z(b)$, then the result is a **simple closed arc.** Also, define $[\Gamma]$ to denote the boundary points and the interior points of a simple closed arc and define (Γ) to denote only the interior points of a simple closed arc.

Definition: A complex **sequence** is a mapping $s : \underset{s(n)}{\mathbb{N}} \underset{=}{\rightarrow} \underset{s_n}{\mathbb{C}}$. The sequence $\{s_n\}_{n=1}^{\infty}$ **converges** to $s \in \mathbb{C}$ if $\forall\ \varepsilon > 0, \exists\ N = N(z, \varepsilon)$ so that $n \geq N \Rightarrow |s_n - s| < \varepsilon$.

Definition: Let Ω be a region in $\mathbb{C}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$. If for $z \in \Omega$, $\{f_n(z)\}_{n=1}^{\infty}$ converges, then define $f : \Omega \rightarrow \mathbb{C}$ by $f(z) = \lim_{n \to \infty} f_n(z)$. So we say that, $\{f_n\}$ **converges (pointwise)** to f.

THEOREM 1. A sequence of function converges pointwise; $f_n \underset{\Omega}{\rightarrow} f$ if and only if $\forall \varepsilon > 0$ and $\forall z \in \Omega$, there exist $N = N(z, \varepsilon)$ such that $n > N \Rightarrow |f_n(z) - f(z)| < \varepsilon$.

Note that pointwise convergence of a sequence of continuous functions does not guarantee that the limit is continuous.

Definition: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ where Ω is a region in $\mathbb{C}$. Then f_n **converges uniformly** on Ω to $f(z)$ if $\forall \varepsilon > 0$, there exist $N = N(\varepsilon)$ such that $\forall z \in \Omega,\ n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$.

THEOREM 2. If a sequence of functions converges uniformly to $f(z)$ on Ω and if $\{f_n\}_{n=1}^{\infty}$ are continuous on Ω, then $f(z)$ is continuous on Ω.

THEOREM 3. A subset K of $\mathbb{C}$ is **compact** if and only if K is closed and bounded.

Definition: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : \Omega \to \mathbb{C}$ where Ω is a region in $\mathbb{C}$. Then f_n **converges locally uniformly** to $f(z)$ on Ω if for all compact subsets, K of Ω, f_n converges uniformly on K.

Note that a sequence of functions is locally uniformly convergent on Ω if and only if each point of Ω has a neighborhood where the sequence converges uniformly.

THEOREM 4. Let $\{f_n\}_{n=1}^{\infty}$ converge locally uniformly to $f(z)$ on Ω. If each $\{f_n\}_{n=1}^{\infty}$ is continuous on Ω then $f(z)$ is continuous on Ω.

Definition: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : \Omega \to \mathbb{C}$ where Ω is a region in $\mathbb{C}$. If the sequence of partial sums, $s_n(z) = \sum_{n=1}^{n} f_n(z)$ converges for all z in Ω, then the sum of functions $\sum_{n=1}^{\infty} f_n$ **converges.**

Definition: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : \Omega \to \mathbb{C}$ where Ω is a region in $\mathbb{C}$. If $\sum_{n=1}^{\infty} |f_n|$ converges, then $\sum_{n=1}^{\infty} f_n$ **converges absolutely.**

THEOREM 5. If $\sum_{n=1}^{\infty} f_n$ converges absolutely, then it is convergent.

<u>Weierstrass M-Test</u>. Let Ω be a region in $\mathbb{C}$. Let $f_n : \Omega \to \mathbb{C}, \ n = 1, 2, \cdots$. If

1. $\forall n, \ \exists \ M_n \in \mathbb{R}^+ \ni \ |f_n(z)| \le M_n, \ \forall z \in \Omega$ and
2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly over Ω.

Definition: If $f(z)$ is analytic in $D'(a,r) = \{z \in \mathbb{C}; 0 < |z-a| < r\}, r > 0$ and not analytic at a, then the point $z = a$ is called an **isolated singular point** of $f(z)$, and the function $f(z)$ is said to have an **isolated singularity** at $z = a$.

Definition: Let a function $f(z)$ be analytic at a point a_0. It has a **zero** of order m (or multiplicity m) at a_0 if and only if there is a function $g(z)$, which is analytic at a_0, such that $f(z) = (z - a_0)^m g(z)$, where $g(a_0) \ne 0$.

Note that zeros and poles of functions are closely related. In the theory of complex series, a powerful tool of studying singularities of a complex function is by Laurent series. We also note that the zeros of an analytic function are *isolated* when the function is not identically equal to zero.

<u>**Power Series.**</u> Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a complex power series where $\{a_n\}\subseteq\mathbb{C}$. Let $L=\limsup\left\{\sqrt[n]{|a_n|}; n=0,1,2,\cdots\right\}$ where $0\le L\le\infty$. Let $R=\frac{1}{L}\left(\frac{1}{\infty}=0, \text{ and } \frac{1}{0}=\infty\right)$. Then

1. If $R>0$ then the series converges locally uniformly and absolutely on $D(0,R)$.
2. If $R<\infty$, the series diverges for all z outside $\bar{D}(0,R)$.

R is called the Radius of Convergence for the series.

THEOREM 6. A function $f(z)$ is analytic at a_0 if and only if $f(z)=\sum_{n=0}^{\infty} c_n (z-a_0)^n$ with positive radius of convergence.

Definition: Let $\Gamma:[a,b]\to\mathbb{C}$ be any simple arc and let $f(z)$ be a function defined on Γ. If there is a number $L\in\mathbb{C}$ such that $\forall\ \varepsilon>0, \exists\ \delta>0$ so that for any partition $a=z_0<z_1<\ldots<z_n=b$ of $[a,b]$ with $\max\left\{|z_j-z_{j-1}| : j=1,\ldots n\right\}<\delta$ and any choice of $\delta_j\in\left[z_{i-1},z_j\right]$, it follows that $\left|\sum_{j=1}^{n} f(\zeta_j)(z_j-z_{j-1})-L\right|<\varepsilon$, then L is called the **line integral** of $f(z)$ along Γ and is denoted $\int_\Gamma f(z)\,dz$.

THEOREM 7. Let Γ be any simple closed curve and let $a\in(\Gamma)$. Then $\int_\Gamma \frac{1}{z-a}dz=2\pi i$.

Cauchy-Goursat Theorem. If $f(z)$ is analytic on and within Γ, then $\int_\Gamma f(z)\,dz = 0$.

Cauchy Integral Formula. Let Γ be a simple closed curve and let $f(z)$ be analytic on and within Γ and $a \in (\Gamma)$ then $f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$.

Laurent's Theorem. Let $f(z)$ be analytic in the annular region

$\{\ z : 0 \le r < |z-a| < R \le \infty\ \}$. Then $\exists\ \{A_n\}_{n=-\infty}^{\infty} \subseteq \mathbb{C}$ such that $f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$,

$\forall\ z \in A(a;r,R)$. In fact $A_n = \frac{1}{2\pi i}\int_{C(a,\rho)} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$ for any ρ with $r < \rho < R$. The function $f(z)$ can be expanded into a series of the form

$$f(z) = \sum_{n=-\infty}^{-1} A_n (z-a)^n + \sum_{n=0}^{\infty} A_n (z-a)^n,$$

where

$\sum_{n=-\infty}^{-1} A_n (z-a)^n$ is known as the singular part of $f(z)$, and

$\sum_{n=0}^{\infty} A_n (z-a)^n$ is known as the analytic part of *f*.

Isolated singularity are classified by the singular part of their expansions around the singularity.

Definition: The classification of isolated singularities may assume three possible forms:

1. **Removable** if singular part is zero – That is $A_{-n} = 0,\ \forall n \in \mathbb{N}$.
2. **Pole of order m** if singular part is finite – If there exist $m \in \mathbb{N}, \ni A_{-m} \neq 0$ and $\forall n > m, A_{-n} = 0$.
3. **Essential** if singular part is infinite – That is $\forall\ n \in \mathbb{N},\ \exists\ m > n\ \ni A_{-m} \neq 0$.

THEOREM 8. Let $R > 0$, let $z_0 \in \mathbb{C}$, let m be a positive integer, and let $f(z)$ be analytic on $0 < |z - z_0| < R$. Then z_0 is a pole of $f(z)$ of order m if and only if $\lim_{z \to z_0} (z - z_0)^m f(z)$ exist and is not equal to zero.

Example 1.

Find the Laurent series for $f(z) = \dfrac{1}{z(z-i)}$ that converges for $0 < |z - i| < 1$.

Solution: Given $f(z) = \dfrac{1}{z(z-i)}$, we can express it as $f(z) = \dfrac{1}{z-i}\left(\dfrac{1}{z}\right) = \dfrac{1}{z-i}\left(\dfrac{1}{i+z-i}\right)$.

Thus, $f(z) = \left(\dfrac{1}{z-i}\right)\left(\dfrac{1}{i\left(1 - \dfrac{z-i}{-i}\right)}\right) = \dfrac{1}{i(z-i)}\sum_{n=0}^{\infty}\left(\dfrac{z-i}{-i}\right)^n$ for $\left|\dfrac{z-i}{-i}\right| < 1$. Which implies that $f(z) = \dfrac{1}{i(z-i)}\sum_{n=0}^{\infty}\dfrac{1}{(-i)^n}(z-i)^n$ for $|z-i| < 1$ and can be written as

$f(z)=\sum_{n=0}^{\infty}\frac{1}{i(-i)^n}(z-i)^{n-1} \Rightarrow f(z)=\sum_{n=0}^{\infty} i^{n-1}(z-i)^{n-1}$ for $|z-i|<1$. Hence, the Laurent series for $f(z)$ that converges for $0<|z-i|<1$ is

$$f(z)=\sum_{n=0}^{\infty} i^{n-1}(z-i)^{n-1}=\frac{-i}{z-i}+1+i(z-i)-(z-i)^2-i(z-i)^3+\cdots .$$

The next example will be helpful in the understanding of the order of a pole but before that, assume $f(z)$ has an isolated pole of order m at a. Then the expansion of $f(z)$ is

$$f(z)=\frac{A_{-m}}{(z-a)^m}+\frac{A_{-m+1}}{(z-a)^{m-1}}+\cdots+\frac{A_{-2}}{(z-a)^2}+\frac{A_{-1}}{(z-a)}+\sum_{n=0}^{\infty} A_n(z-a)^n$$ and so

$g_1(z)=(z-a)f(z) = \frac{A_{-m}}{(z-a)^{m-1}}+\cdots+\frac{A_{-z}}{z-a}+A_{-1}+\sum_{n=0}^{\infty} A_n(z-a)^{n+1}$. If $g_1(z)$ exists and is not zero, then $m=1$, and $g_1(z)=A_{-1}\neq 0$. Similarly for $g_m(z)=(z-a)^m f(z)$, which can be expanded as

$g_m(z)=A_{-m}+A_{-m+1}(z-a)+\cdots+A_{-1}(z-a)^{m-1}+\sum_{n=0}^{\infty} A_n(z-a)^{n+m}$. Therefore

$g_m(a)=A_{-m}$ and $A_{-1}=\frac{g^{m-1}(a)}{(m-1)!}$.

Example 2.

Find the order of the pole at 0 for the function, $f(z)=\frac{1}{z^3(z-1)}$.

Solution:

Let $g_1(z) = zf(z) = \frac{1}{z^2(z-1)}$, $g_2(z) = z^2 f(z) = \frac{1}{z(z-1)}$, $g_3(z) = z^3 f(z) = \frac{1}{z-1}$.

Note that $g_1(0)$, and $g_2(0)$ does not exist, but $g_3(0) = -1 \neq 0$. So by theorem 8, there is a pole of order 3 at zero. Hence, $g_3''(z) = \frac{2}{(z-1)^3}$ and so $A_{-1} = \frac{g_3''(0)}{2!} = \frac{-2}{2} = -1$.

Proposition 1. Let $f(z)$ be analytic on $D(a;r)$. Assume $f(z)$ has a zero of order m at a but $f(z) \neq 0$ for $z \in D'(a,r)$ and $r > 0$. Let ρ be such that $0 < \rho < r$; then

$$\frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f'(z)}{f(z)} dz = m.$$

Proof. Let $f(z) = (z-a)^m g(z)$ where $g(z)$ is analytic on $D(a;r)$ and $g(a) \neq 0$. Then $f'(z) = (z-a)^m g'(z) + m(z-a)^{m-1} g(z)$ and $\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{m}{z-a}$.

So $\frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{C(a,\rho)} \left(\frac{g'(z)}{g(z)} + \frac{m}{z-a} \right) dz = \frac{1}{2\pi i} \left(0 + m \int_{C(a,\rho)} \frac{1}{z-a} dz \right)$ and

by the Cauchy Integral Formula $\int_{C(a,\rho)} \frac{1}{z-a} dz = 2\pi i$. Therefore

$$\frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}(m 2\pi i) = m.$$

□

Lemma 1. Let $f(z)$ be analytic at a and n be a positive integer. If $f_1(z) = \frac{f(z)}{(z-a)^n}$ has neither a zero nor a pole at a, then $f(z)$ has a zero of order n at a.

Proof. If $f(a) \neq 0$, then $f_1(z)$ has a pole at $z = a$ which is not true by assumption. Thus $f(z)$ must have a zero at a. Let k be the order of the zero at a. So $f(z) = (z-a)^k g(z)$ and $f_1(z) = (z-a)^{k-n} g(z)$. Since $f_1(z)$ has no zero at $a, k \leq n$ and since $f_1(z)$ has no pole at $a, k \geq n$. Thus $k = n$.

□

CHAPTER 3
The Mittag-Leffler Theorem

Let us recall the definition of meromorphic functions and accumulation points.

Definition: A function is **meromorphic** in a region Ω if the function is analytic in Ω except for poles. More specifically, a function $f(z)$ is meromorphic in Ω if, for every point $a \in \Omega$, there exists a disc $D(a,\delta) \subset \Omega$ such that either $f(z)$ is analytic in the disc or else $f(z)$ is analytic in the punctured disc $D'(a,\delta) \subset \Omega$ and the isolated singularity is a pole.

Definition: The point a_0 is an **accumulation point** of $A \subseteq \mathbb{C}$ if for every $\varepsilon > 0$, $D'(a_0,\varepsilon) \cap A \neq \varnothing$.

<u>The Mittag-Leffler Theorem</u>. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of distinct points with no accumulation points. Let $P_1(x), P_2(x), \ldots, P_n(x), \ldots$ be a sequence of polynomials with constant terms equal to zero. Then there exists a function $f(z)$, which is analytic in $\mathbb{C}$ except at $a_1, a_2, \ldots, a_n, \ldots$ where a_n is a pole and the Laurent expansion of $f(z)$ around a_n has singular part $P_n\left(\frac{1}{z-a_n}\right)$.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in the complex plane with no accumulation points, thus $\forall\, R > 0$, $D(0,R)$ can contain only finitely many terms of the sequence. This sequence may be renamed so that $|a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots$. Temporarily assume that $a_1 \neq 0$, thus all $a_n \neq 0$. Then choose a convergent series of positive (real) numbers $\{c_n\}_{n=1}^{\infty}$ so that $\sum_{n=1}^{\infty} c_n$ converges and use the c_n's as a degree of closeness in the following calculations. Let $\{P_n(z)\}_{n=1}^{\infty}$ be the sequence of polynomials with constant terms equal to zero. For $n = 1, 2, \ldots,\quad P_n\left(\frac{1}{z-a_n}\right)$ has an isolated pole (the order being the order of the polynomial) at $a_n \neq 0$. Thus $P_n\left(\frac{1}{z-a_n}\right)$ is analytic in $D(0,|a_n|)$ and can be expanded into a Taylor Series,

$$P_n\left(\frac{1}{z-a_n}\right) = b_0^{(n)} + b_1^{(n)}z + b_2^{(n)}z^2 + \cdots + b_k^{(n)}z^k + \cdots \quad , \qquad \text{for } z \in D(0,|a_n|).$$

This series is absolutely and uniformly convergent on $\overline{D}\left(0, \left|\frac{a_n}{2}\right|\right)$. Then choose s_n sufficiently large so that the partial sum $Q_n(z) = \sum_{j=0}^{s_n} b_j^n\ z^j$ satisfies

$$\left| P_n\left(\frac{1}{z-a_n}\right) - Q_n(z) \right| < c_n \text{ for } z \in \overline{D}\left(0, \left|\frac{a_n}{2}\right|\right).$$

Let $h(z) = \sum_{j=1}^{\infty} \left\{ P_j\left(\frac{1}{z-a_j}\right) - Q_j(z) \right\}$. Given any $R > 0$, there are only finitely many a_n in $D(0,R)$ where $h(z)$ has poles. Break up the series into two parts so that

$$h(z) = \sum_{|a_j| \le 2R} \left\{ P_j\left(\frac{1}{z-a_j}\right) - Q_j(z) \right\} + \sum_{|a_j| > 2R} \left\{ P_j\left(\frac{1}{z-a_j}\right) - Q_j(z) \right\}.$$ Define

$$h_1(z) = \sum_{|a_j| \le 2R} \left\{ P_j\left(\frac{1}{z-a_j}\right) - Q_j(z) \right\} \text{ and } h_2(z) = +\sum_{|a_j| > 2R} \left\{ P_j\left(\frac{1}{z-a_j}\right) - Q_j(z) \right\}.$$

Then $h_2(z)$ has no singularities in $D(0,R)$ and by the Weierstrass M-test, the series is absolutely and uniformly convergent on $D(0,R)$. So $h_2(z)$ is an analytic function on

$D(0,R)$. On the other hand, $h_1(z)$ is a finite sum of the form

$$h_1(z)=\left\{P_1\left(\frac{1}{z-a_1}\right)-Q_1(z)\right\}+\left\{P_2\left(\frac{1}{z-a_2}\right)-Q_2(z)\right\}+\cdots+\left\{P_k\left(\frac{1}{z-a_k}\right)-Q_k(z)\right\}$$

where $a_1,a_2,\ldots,a_k\in D(0,2R)$ which is a rational function with a_j's as poles for $j=1,2,\ldots,k$. For each such a_j, $h_1(z)=P_j\left(\frac{1}{z-a_j}\right)+f_j(z)$ where $f_j(z)$ is analytic in a disc around a_j. Hence for each j, $h_1(z)$ has the form of a Laurent series around a_j with singular part $P_j\left(\frac{1}{z-a_j}\right)$. Thus, for any $R>0$, $h(z)$ has the desired properties for $a_1,a_2,\ldots,a_k\in\bar{D}(0,2R)$. Since $R>0$ was arbitrary, R may be increased (to infinity) and the series for $h(z)$ is a desired function for all a_n (when $a_1\neq 0$). If $a_1=0$, use the previous argument on $\{a_n\}_{n=2}^{\infty}$ to construct $h(z)$ satisfying the desired properties for $\{a_n\}_{n=2}^{\infty}$ then add $P_1\left(\frac{1}{z}\right)$ to $h(z)$ and the resulting function is as desired.

□

Weierstrass' theorem was proven first before Mittag-Leffler's theorem but we will use Mittag-Leffler's theorem to prove the Weierstrass theorem before we give an independent proof of the Weierstrass factorization theorem. The next theorem will be helpful understanding the Mittag-Leffler's theorem.

<u>Weierstrass Theorem</u>. Given a sequence of complex numbers $a_1,a_2,a_3,\ldots$ with no accumulation point and given a sequence of natural numbers $k_1,k_2,k_3,\ldots$. There exist an entire function $f(z)$ in the complex plane such that $a_i,\ i\in\{1,2,\ldots\}$ is a zero of $f(z)$ with multiplicity $k_i,\ i\in\{1,2,\ldots\}$ and there are no other zeros.

Proof.

As in the theorem of Mittag-Leffler, let $g(z)$ be a meromorphic function with poles $\{a_j\}_{j=1}^{\infty}$ and singular part at a_j as $\frac{k_j}{z-a_j}$. Pick $\zeta \in \mathbb{C} \setminus \{a_j\}_{j=1}^{\infty} = \Omega$. Then for any $\mathrm{z} \in \mathbb{C} \setminus \{a_j\}_{j=1}^{\infty}$, define $\mathrm{G}(z) = \int_{\zeta}^{z} g(t) dt$, where the path of integration is contained in Ω. $\mathrm{G}(z)$ is not single-valued but all values differ by an integer multiple of $2\pi i$ by the Cauchy Integral formula. Then $h(z) = e^{G(z)}$ is single valued and, since $\mathrm{G}(z)$ is analytic at each $\mathrm{z} \in \Omega$, so is $h(z)$. It needs to be shown that the poles of $h(z)$ can be removed so that each a_j is a zero of order k_j for $h(z)$. By Lemma 1, this is accomplished by showing that $H_j(z) = \frac{h(z)}{(z-a_j)^{k_j}}$ has no zero or pole at a_j for $j = 1, 2, \ldots$. For each j,

$H_j(z)$ is analytic on Ω and $\frac{H_j'(z)}{H_j(z)} = \frac{\frac{h'(z)}{(z-a_j)^{k_j}} - \frac{k_j h(z)}{(z-a_j)^{k_j+1}}}{\frac{h(z)}{(z-a_j)^{k_j}}} = \frac{h'(z)}{h(z)} - \frac{k_j}{z-a_j}$. But

$h'(z) = h(z) \cdot G'(z)$ and $G'(z) = \left[\int_{\zeta}^{z} g(t) dt\right]' = g(z)$. Hence $h'(z) = h(z) \cdot g(z)$ and $\frac{H_j'(z)}{H_j(z)} = g(z) - \frac{k_j}{z-a_j}$. But $\mathrm{g}(z)$ has a simple pole at a_j with singular part $\frac{k_j}{z-a_j}$ so, subtracting $\frac{k_j}{z-a_j}$ yields a function that can be made analytic at a_j. Thus $H_j(z)$ has neither a pole nor a root at a_j. Thus $H(z)$ is an analytic function with roots only at the a_j's of orders k_j respectively.

□

CHAPTER 4

Independent Proof of Weierstrass Theorem

One of the principal activities in complex analysis is to construct holomorphic or meromorphic functions with certain prescribed behavior. Infinite products are more useful in this sense than infinite sums. In order to give an independent proof of the Weierstrass theorem, we first review some advanced concepts, and definitions which will aid in better understanding the proof of the theorem.

Definition: An infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to **converge** if

1. Only a finite number $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ of the a_n's are equal to -1;
2. If $N_0 > 0$ is so large that $a_n \neq -1$ for $n > N_0$, then $\lim_{N\to+\infty} \prod_{n=N_0+1}^{N}(1+a_n)$ exists and is nonzero.

If $\prod_{n=1}^{\infty}(1+a_n)$ converges, then we define its **value** to be

$$\left[\prod_{n=1}^{N_0}(1+a_n)\right]\cdot \lim_{N\to+\infty}\prod_{N_0+1}^{N}(1+a_n).$$

Definition: Let $p \in \mathbb{N}$ and define $E_0(z) = (1-z)$ and $E_p(z) = (1-z)e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)}$ for $1 \le p \in \mathbb{Z}$. Then $E_p(z)$ is holomorphic on all of $\mathbb{C}$ and is called a **Weierstrass elementary factor.**

Weierstrass Theorem. Let U be an open subset of $\mathbb{C}$. Let $\Omega = \{a_n\}_{n=1}^{\infty}$ be a sequence (possibly finite with finite repetition permitted) in U with no accumulation point in U.

Then there exists an analytic function, $f(z)$ on U whose zeros are precisely the a_n's and the multiplicity of the zeros are the number of times the zero is repeated in the given sequence.

The proof of the Weierstrass (Factorization) theorem will be proved after we've proven some lemma's to support our claim. Note that, in the theorem of Weierstrass, the sequence may repeat.

Lemma 2. If $0 \le x < 1$ then $1 + x \le e^x \le 1 + 2x$.

Proof. Assume that x is a real number with $0 \le x < 1$, then we show that $1 + x \le e^x \le 1 + 2x$. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and note that

$\sum_{n=2}^{\infty} \frac{1}{n!} < \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = 1,$ since $n! > 2^{n-1}$ for $j > 2$. Then $1 + x \le 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

and $e^x = 1 + x + x\left(\sum_{n=2}^{\infty} \frac{x^{n-1}}{n!}\right) \le 1 + x + x\sum_{n=2}^{\infty} \frac{1}{n!} \le 1 + x + x(1) = 1 + 2x$. We then have that

$1 + x \le e^x \le 1 + 2x$.

□

Lemma 3. If $a_n \in \mathbb{C}, |a_n| < 1$, then the partial product P_N for $\prod_{n=1}^{\infty} (1 + |a_n|)$ satisfies

$e^{\left(\frac{1}{2}\sum_{n=1}^{N} |a_n|\right)} \le P_N \le e^{\left(\sum_{n=1}^{N} |a_n|\right)}$.

Proof. By the first part of Lemma 2, $1+|a_n| \le e^{|a_n|}$ so $P_N = \prod_{n=1}^{N}(1+|a_n|) \le \prod_{n=1}^{N} e^{|a_n|} = e^{\sum_{n=1}^{N}|a_n|}$.

By the second part of Lemma 2, $1+|a_n| = 1+2\left|\frac{a_n}{2}\right| \ge e^{\left|\frac{a_n}{2}\right|}$ so

$$P_N = \prod_{n=1}^{N}(1+|a_n|) \ge \prod_{n=1}^{N} e^{\frac{1}{2}|a_n|} = e^{\sum_{n=1}^{N}\frac{1}{2}|a_n|} = e^{\frac{1}{2}\sum_{n=1}^{N}|a_n|}.$$

□

Lemma 4. If $\prod_{n=1}^{\infty}(1+|a_n|)$ converges, then $\sum_{n=1}^{\infty}|a_n|$ converges.

Proof. By Lemma 3, $P_N = \prod_{n=1}^{k}(1+|a_n|) \ge e^{\frac{1}{2}\sum_{n=1}^{N}|a_n|}$. Since $\{P_n\}$ converges by assumption, it is bounded which bounds the partial sums of $\frac{1}{2}\sum_{n=1}^{\infty}|a_n|$. Thus the series $\sum_{n=1}^{\infty}|a_n|$ must converge.

□

Lemma 5. Let $a_n \in \mathbb{C}$, $P_N = \prod_{n=1}^{N}(1+a_n)$, and $\tilde{P}_N = \prod_{n=1}^{N}(1+|a_n|)$. Then $|P_N - 1| \le \tilde{P}_N - 1$.

Proof. Suppose $a_n \in \mathbb{C}$, and write $\mathrm{P}_n = (1+a_1)(1+a_2)\ldots(1+a_N)$ and

$\tilde{P}_N = 1+|a_1|+|a_2|+\cdots|a_N|+|a_1||a_2|+\cdots+|a_1||a_2|\cdots|a_N|$. Then

$$\begin{aligned}|P_N - 1| &= |a_1 + a_2 + \cdots + a_N + a_1a_2 + a_1a_3 + \cdots + a_1a_2\cdots a_N| \\ &\le |a_1|+|a_2|+\cdots+|a_N|+|a_1a_2|+|a_1a_3|+\ldots+|a_1||a_2|\cdots|a_N| \\ &= \tilde{P}_N - 1.\end{aligned}$$

□

Lemma 6. If the infinite product $\prod_{n=1}^{\infty}\left(1+|a_n|\right)$ converges, then so does $\prod_{n=1}^{\infty}\left(1+a_n\right)$.

Proof. Since the partial products $P_N = \prod_{n=1}^{\infty}\left(1+|a_n|\right)$ converge, Lemma 4 implies that $\sum_{n=1}^{\infty}|a_n|$ also converges. Thus $\lim_{n\to\infty}|a_n| = 0$. Therefore there exists $N_0 \in \mathbb{N}$ so that $k > N_0$ implies $a_n \neq -1$. For $k > N_0$, let $Q_k = \prod_{n=N_0+1}^{k}\left(1+a_n\right)$ and $\tilde{Q}_k = \prod_{n=N_0+1}^{k}\left(1+|a_n|\right)$.

For $M > N > N_0$,

$$\begin{aligned}|Q_M - Q_N| = \left|Q_N\left(\frac{Q_M}{Q_N}-1\right)\right| = |Q_N|\left|\frac{Q_M}{Q_N}-1\right| = |Q_N|\left|\prod_{k=N+1}^{M}\left(1+a_k\right)-1\right| \\ \leq \left|\tilde{Q}_N\right|\left|\prod_{k=N+1}^{M}\left(1+|a_k|\right)-1\right| \\ = \left|\tilde{Q}_M - \tilde{Q}_N\right| .\end{aligned}$$

The convergence of $\tilde{Q}_k$ implies the convergence of Q_k. It remains to show that the limit of Q_k is not zero. Since $\lim_{n\to\infty}\prod_{k=N_0+1}^{n}\left(1+|a_k|\right) = L > 0$, and $\lim_{n\to\infty}\left(1+|a_n|\right) = 1$, for large M, N, $\prod_{k=M}^{N}\left(1+|a_k|\right)$ is close to 1. Hence choose M sufficiently large so that $N > M$ implies $\frac{1}{2} < \prod_{k=M}^{N}\left(1+|a_k|\right) < \frac{3}{2}$ so $-\frac{1}{2} < -1+\prod_{k=M}^{N}\left(1+|a_k|\right) < \frac{1}{2}$ and $\left|-1+\prod_{k=M}^{N}\left(1+|a_k|\right)\right| < \frac{1}{2}$. Then, by Lemma 5, $\left|-1+\prod_{k=M}^{N}\left(1+a_k\right)\right| < \frac{1}{2}$ so $-\frac{1}{2} < -1+\prod_{k=M}^{N}\left(1+a_k\right) < \frac{1}{2}$ and $\frac{1}{2} < \prod_{k=M}^{N}\left(1+a_k\right) < \frac{3}{2}$.

So $\lim_{N\to\infty}|Q_N| = \lim_{N\to 0}\left|\prod_{k=N_0+1}^{M-1}\left(1+a_k\right)\right|\left|\prod_{k=M}^{N}\left(1+a_k\right)\right| \geq \frac{1}{2}\left|\prod_{k=N_0+1}^{M-1}\left(1+a_k\right)\right| > 0$.

□

Lemma 7. Let $U \subseteq \mathbb{C}$ be open. Suppose that $\{f_n\}_{n=1}^{\infty} : U \to \mathbb{C}$ are holomorphic and that $\sum_{n=1}^{\infty} |f_n|$ is locally uniformly convergent. Then the sequence of partial products $F_N(z) = \prod_{n=1}^{N} (1 + f_n(z))$ converges uniformly on compact sets. In particular, the limit of these partial products defines a holomorphic function on U. Furthermore the function vanishes at a point $z_0 \in U$ if and only if $f_n(z_0) = -1$ for some n and the multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of the functions $1 + f_n$ at z_0.

Proof. Let K be a compact subset of U. Then $\sum_{n=1}^{\infty} |f_n(z)|$ converges uniformly to $F(z)$ on K and $F(z)$ is analytic on K so $F(z)$ is bounded on K by some constant C. Let $P_N(z)$ be the Nth-partial product of $\prod_{n=1}^{\infty} (1 + |f_n(z)|)$. By Lemma 3, the partial products are bounded on K by e^C. Let $0 < \varepsilon < 1$ and choose L sufficiently large so that if $M \geq N \geq L$, $\sum_{n=N}^{M} |f_n(z)| < \varepsilon$ for all $z \in K$. Then for $M \geq N \geq L$, Lemma 2 yields,

$$\begin{aligned}
|P_M(z) - P_N(z)| = \left| P_N(z) \left(\frac{P_M(z)}{P_N(z)} - 1 \right) \right| &= \left| P_N(z) \left(\prod_{n=N+1}^{M} (1 + |f_n(z)|) - 1 \right) \right| \\
&\leq |P_N(z)| \left| \prod_{n=N+1}^{M} (1 + |f_n(z)|) - 1 \right| \\
&\leq \prod_{n=1}^{N} (1 + |f_n(z)|) \cdot \left(e^{\left(\sum_{n=N+1}^{M} |f_n(z)| \right)} - 1 \right) \\
&\leq e^C \left(e^{\varepsilon} - 1 \right).
\end{aligned}$$

Since ε was arbitrary chosen between 0 and 1, $e^{\varepsilon}-1$ may be made arbitrary close to zero. Hence $\{P_N(z)\}$ is an infinite Cauchy sequence and, because K is compact, $\{P_N(z)\}$ converges uniformly on K and by Lemma 6, we are done.

□

Lemma 8. If $|z|\le 1$, then $|1-E_p(z)|\le |z|^{p+1}$.

Proof. Suppose $p=0$, then $|1-E_0(z)|=|1-(1-z)|=|z|$. Now assume $p\ge 1$, then the first derivative of Weierstrass elementary factors is,

$$E_p'(z)=(-1)e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)}+e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)}\left(1+z+z^2+\cdots+z^{p-1}\right)(1-z)=e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)}\left(-1+1-z^p\right)$$
$$=-z^p\left(e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}\right).$$

Expanding $e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)}$ into a Taylor series yields

$E_p'(z)=-z^p\left(1+\sum_{n=1}^{\infty}\alpha_n z^n\right)=-z^p+\sum_{n=p+1}^{\infty}\left(-\alpha_{n-p}\right)z^n$ where all α_n's are positive real values. But if the Taylor series for $E_p(z)$ is $1+\sum_{n=1}^{\infty}b_n z^n$, then $E_p'(z)=\sum_{n=1}^{\infty}nb_n z^{n-1}$.

Hence $0=b_1=b_2=\cdots=b_p$ and all $b_n<0$ for $n<p$. Thus $E_p(z)=1+\sum_{n=p+1}^{\infty}b_n z^n$ and

$$0 = E_p(1) = 1 + \sum_{n=p+1}^{\infty} b_n \text{ so } \sum_{n=p+1}^{\infty} b_n = -1 \text{ and } \sum_{n=p+1}^{\infty} |b_n| = 1. \text{ Hence}$$

$$\begin{aligned}
|E_p(z)-1| = \left|\sum_{n=p+1}^{\infty} b_n z^n\right| = \left|z^{p+1}\sum_{n=p+1}^{\infty} b_n z^{n-p-1}\right| = |z|^{p+1}\left|\sum_{n=p+1}^{\infty} |b_n| z^{n-p-1}\right| &\le |z|^{p+1}\left|\sum_{n=p+1}^{\infty} |b_n| z^{n-p-1}\right| \\
&\le |z|^{p+1}\sum_{n=p+1}^{\infty} |b_n||z|^{n-p-1} \\
&\le |z|^{p+1}\sum_{n=p+1}^{\infty} b_n \\
&= |z|^{p+1}.
\end{aligned}$$

□

We are now ready to proof Weierstrass (factorization) theorem.

Proof of Weierstrass. Let U be an open subset of $\mathbb{C}$. Let $\Omega = \{a_n\}_{n=1}^{\infty}$ be a sequence (possibly finite with finite repetitions permitted) in U with no accumulation point in U. If the sequence is finite, that is $\{a_1, a_2, \ldots, a_N\}$. Then $f(z) = \prod_{n=1}^{N}(z - a_n)$ is the desired function for any U. Otherwise consider $U \subseteq \mathbb{C} = \mathbb{C} \cup \{\infty\}$ and transform U to U by the Mobius transformation $T_p = \dfrac{1}{z-p}$ where $p \in U \setminus \Omega$. Then $\infty \in U$ and the accumulation points are all in the finite part of the plane. Then

1. $U \neq \mathbb{C}$
2. $\mathbb{C} \setminus U$ is compact (It is a closed subset of $\mathbb{C}$)
3. $\{a_n\}_{n=1}^{\infty} \cup \{\infty\} \subset \tilde{U}$
4. $\{a_n\}_{n=1}^{\infty} \cap \{\infty\} = \varnothing$

Henceforth refer to U as U. The accumulation points of $\{a_n\}$ are in the boundary of U, so any compact subset of U contains only finitely many a_n's. Since $\mathbb{C}\setminus U$ is compact, $\forall n,\ \exists\ \hat{a}_n \in \mathbb{C}\setminus U$ such that $d_n = |a_n - \hat{a}_n|$ is minimal. Note that $\lim_{n\to\infty} d_n = 0$ since the boundary maps to the boundary of the image. Let K be a compact subset of U. Then $d(K, \mathbb{C}\setminus U) > 0$, and there exist $\delta > 0$ so that for all $z \in K, w \in \mathbb{C}\setminus U,\ |z-w| \geq \delta > 0$. So $|z - \hat{a}_n| \geq \delta > 0$ for all $z \in K$ and all a_n. So there exist $n_0 : n > n_0$ which implies that $d_n < \frac{1}{2}|z - \hat{a}_n|$. Hence $\frac{d_n}{|z-\hat{a}_n|} < \frac{1}{2} \Rightarrow \left|\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right| < \frac{1}{2}$ for all $z \in K$ and $n > n_0$. Lemma 8 implies $\left|1 - E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right)\right| \leq \left|\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right|^{n+1} < \left(\frac{1}{2}\right)^{n+1}$. Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}$ converges, $\sum_{n=1}^{\infty}\left|1 - E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right)\right|$ converges uniformly on K. The negative of the series, $\sum_{n=1}^{\infty}\left(E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right) - 1\right)$ also converge uniformly on K. Hence Lemma 7 ensures that $\prod_{n=1}^{\infty} E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right) = \prod_{n=1}^{\infty}\left(1 + \left(E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right) - 1\right)\right)$ converges uniformly on K. Since K was an arbitrary compact subset of U, $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - \hat{a}_n}{z - \hat{a}_n}\right)$ is the desired function on U. To return to the original U and $\{a_n\}_{n=1}^{\infty}$ use the composition $f \circ T_p$.

□

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