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WHO CAME UP WITH THIS? MATHEMATICS HISTORY IN THE CLASSROOM

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The essay of Katie S. Schneck is hereby accepted:

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I certify that this is the original document

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Table of Contents

1. INTRODUCTION

Ever since I was a child, I knew I wanted to be a teacher. I did not, however, always want to be a math teacher. My original goal was to teach history at the high school level. I eventually changed my mind and ended up becoming a math teacher. I am happy I switched content areas, but I still have that love and appreciation of history.

Mathematics appealed to me for many reasons. I am a very logical person by nature, so I appreciated the logical foundations and processes present in mathematics. Mathematics always seemed so neat and tidy. Throughout my college coursework, and especially here at John Carroll, I have come to realize that is not the case. Mathematics as we know it is the result of centuries of struggle and hard work from countless men and women. Many of the things that we take for granted today were not always so obvious or the way things were done. An excellent example of this is our decimal number system. Many of us, students included, don't realize that our decimal number system wasn't widespread in the West until the 13th century. In addition to that, many early civilizations didn't even think zero was a number and had no symbol for it!

I like to tell people that I became a math teacher because there is always a right answer. My coursework has taught me that even that is not correct. My previous statement implies that mathematics is a very rigid discipline with no room for creativity and discovery. While there are axioms in place and logic must be followed, mathematics is an ever-growing field and has great potential for creativity and different approaches. It is often the case that there are different paths to the same results. Sometimes those different paths lead to some incredible, new results. There is a great deal of beauty to be found in mathematics, and I had not truly appreciated before studying its history.

High school students are also lacking in this appreciation. The closest students usually come to asking about the historical context of a lesson is to ask, "What evil person came up with this?" or something along those lines. The truth is, it was probably more than one "evil person." The concepts that seem so boring and awful to the current high school students were once revolutionary and exciting. The concepts could have been a mathematician's life works, and not just another day in math class.

The history of mathematics is rarely included in a high school math classroom. The recently adopted Common Core Mathematics Standards make no mention of incorporating history into the math curriculum. Textbooks tend to address math history by placing an occasional picture and small textbox on the side of page, never as the focus of a lesson. Math history is treated as pieces of trivia used to fill time, and not to deepen students' appreciation and understanding. In the current educational atmosphere, this is understandable, considering the increase in testing and the decrease in instructional time. However, it is important to make the effort. Including history in the math classroom is important because it offers many benefits. It humanizes the concepts, helps students make connections to the world around them, and it helps to deepen student understanding.

Mathematics is a subject that many students struggle with. This causes them to become frustrated and want to give up. What they don't realize is that many mathematicians felt the same way. Mathematicians could work for years, or even decades, on the same problem before reaching a breakthrough. Some may have thought they had solved the problem, only to find out after the fact that they made an error. Mathematics was an intense and very personal thing for many mathematicians. The drama between Fior, Tartaglia and Cardano, which is described in the "Cubic Clash" lesson, shows how mathematicians don't always have all the answers, and just how personally invested they were in their work. If students are shown that mathematicians struggled with the same concepts they have difficulty with, it will help them to see that mathematics is a very human endeavor. They will begin to see that mistakes are okay, even necessary. They will understand that you can learn more from mistakes than you ever could if you got it right the first time.

Learning about mathematical history will also help students to make connections with the world they live in and see how mathematics is influenced by society. Many students view math as a tool of torture, not something that is a result of the world around them. A favorite question from students is, "When will I use this in real life?" Well, most mathematics developed from real-life situations. In the "Chances Are" lesson, students will see how the popular activity of gambling in the 17th century led to a whole new branch of mathematics that is very applicable today. Probability and odds are discussed daily, and students will now have a better understanding of where it all started. Students will also learn how a 19th century woman was the first to see and predict many of the amazing things modern-day technology can do. Almost every student has a smart phone, and the codes that play their music, display the graphics for their favorite game, and allow them to read their texts all started with Ada Lovelace. These few examples will help students to realize that math is a result of the world around them, not a separate entity.

Including a historical aspect to mathematics in the class room will also assist students in developing a deeper understanding of the concepts. By putting a problem into a meaningful context, it becomes a much richer task. Students are better able to visualize the problem, develop a plan, and troubleshoot. In “Eratosthenes and the Circumference of the Earth,” students use many geometric concepts, such as arc length and right triangle trigonometry, but because they are used in a concrete, historical way, the tasks become more than just tasks. “Approximating Pi, Archimedes Style,” has student perform constructions that would otherwise be tedious and frustrating. If students know why they are doing or learning something, they become much more motivated and willing to put forth effort.

For these reasons, mathematics history has a valuable place in the classroom. What follows are five individual lessons that can be used independently in various places in the high school math curriculum. The lessons are based on some of my favorite stories from the history of mathematics. They are designed to introduce students to the history behind a concept, and then to actually apply that concept. Students will learn how the concept was developed, and then put it to work. The combination of historical information and practical application will help students to deepen their understanding on multiple levels.

The lessons are ready to be implemented in the classroom. For the teacher’s reference, standards are included. These standards are taken from the Common Core Standards for Mathematics, for reference purposes.

2. LESSON #1 ERATOSTHENES AND THE CIRCUMFERENCE OF EARTH

Teacher's Materials

This lesson is intended for a high school geometry class. It introduces students to Eratosthenes and how he calculated the circumference of the earth, and directs students to apply his method to other situations, such as calculating the circumference of Mars. The lesson requires that students have knowledge of theorems resulting from a parallel line being cut by a transversal, right triangle trigonometry, formulas for arc length and circumference, and proportions.

Objectives

- Students will be able to determine the circumference of a planet using Eratosthenes' method.
- Students will be able to understand and explain how Eratosthenes calculated the circumference of the earth.

Standards

- HSG.SRT.C.8 – Use trigonometric ratios to solve right triangles in applied problems.
- HSG.MG.A.1 – Use geometric shapes, their measures, and their properties to describe objects

Materials

- *Eratosthenes and the Circumference of the Earth* handout
- *Work Page* handout
- *Eratosthenes on Mars* handout
- Scientific calculator

Classtime Needed

One 45-minute period

Lesson Plan Outline

1. Read the first paragraph of the Eratosthenes and the Circumference of the Earth handout
2. Work through Eratosthenes calculations with the class.
    a. Students should follow along on the Work Page
3. Students can begin working on the Eratosthenes on Mars worksheet with any time left. What they do not finish can be assigned for homework
4. Optional Extension – An Alternative Method (Found on the Eratosthenes and Circumference of the Earth handout)

Notes

When working through the calculations, it is up to the teacher's discretion as to whether to give students the first handout as well as the work page. Some may choose to only hand out the work page, and then distribute the main handout afterwards in order to prevent students from reading ahead or just copying down answers. Regardless, students should receive the first handout at some point.

The first handout contains a section called "An Alternative Method." This utilizes similar triangles and achieves comparable results. This can be shown to students at any time. Keep in mind that the Mars worksheet is set up using the original method. Students can choose to solve it using similar triangles if they would like to.

For your reference, the actual circumference of Mars is 13,263 miles. Students should get a circumference of 13,250 miles, if they rounded as was shown in the example with Earth's circumference. You can use this method to create a problem where students find the circumference of any planet. If you create an imaginary planet, you do not have to worry too much about the measurements of the distance between the two points, the height of an object, and its shadow. If you want to create the problem for a planet with a known circumference, you must start with the radius and work backwards to make sure it works out to an accurate measurement. That is how I created the problem for Mars.

Another possible extension is to recreate the whole process. It can be adapted using any two locations so long as one is significantly north of the other. Measurements would need to be taken at local noon, when the sun is at its highest point. This is NOT necessarily exactly 12 p.m. There will need to be some adjustments made since neither of the measurements would be taken at a time when there is no shadow. This would be a great project to pair with a sister school or a relative who lives far away.

Name ________________________________ Date ___________ Period _____

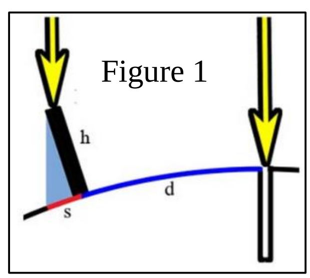
Eratosthenes and the Circumference of the Earth

Eratosthenes was a Greek mathematician, geographer and astronomer who lived in ancient times. You may have heard of the Sieve of Eratosthenes, which was one of the first known methods for systematically determining prime numbers. One of his most famous results is from 240 B.C. when he produced a fairly accurate calculation for the circumference of the Earth, which is very impressive considering he did it with only his knowledge of geometry and a shadow! Let's take a look at how he did it.

<u>It All Started With A Well…</u>

In the ancient Egyptian city of Syene, near modern-day Aswan, there was a well. On most days, you couldn't see the sun in the well. However, on the summer solstice, the sun was directly overhead, which meant you could see the sun in the well. Eratosthenes knew that in the city of Alexandria, about 500 miles to north, there was never a day when the sun was directly overhead. The reason for the differences in shadows is due to the curvature of the earth, and Eratosthenes realized he could use this information to calculate the circumference of the earth as well!

Eratosthenes' method initially uses the Alternate Interior Angle Theorem, which states that alternate interior angles of parallel lines are congruent. The parallel lines in this case are the sun's rays. Now, technically, the rays are not exactly parallel, but since the sun is much bigger than the earth (over 1 million earths could fit inside the sun!), they are parallel enough for Eratosthenes' purposes.



Eratosthenes took a measurement of the length of the shadow that an object cast in Alexandria on the same day, at the same time as the sun was directly overheard in Syene, thus casting no shadows. Figure 1 illustrates the rays from the sun hitting the well in Syene and an object in Alexandria.[1]

[1] All figures in this lesson can be found at http://www.coopertoons.com/education/eratosthenes/eratosthenes_earth.html.

The figure is not to scale! The *s* represents the length of the shadow, *h* the height of the object measured, and *d* the distance between the two cities.

Let's take a look at a different diagram to get a better understanding of the situation. The two parallel lines in Figure 2, also not to scale, represent the sun's rays. The point A represents Alexandria, and C represents Syene. Line BC represents the ray that goes down the well in Syene, and continues to the center of the earth. Line DE is the ray that creates the shadow in Alexandria. These parallel lines create $\angle EDA$ and $\angle ABC$, which are alternate interior angles, and therefore congruent.

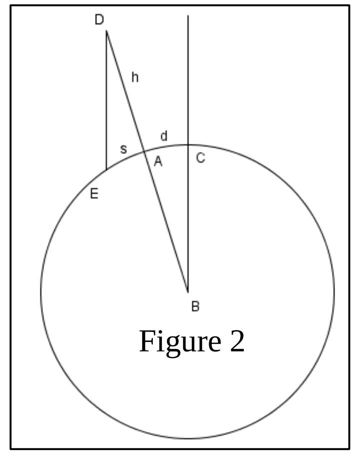
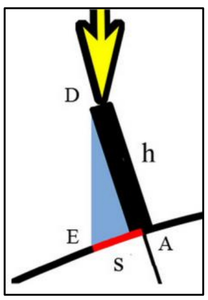


Figure 2

Figure 3



The next step in the process is to find the measure of those angles. To find the angle measure, let's zoom in on the shadow cast in Alexandria. If you look closely at Figure 3, you can see a right triangle (not exactly a right triangle because of the curvature of the earth, but close enough to get a good estimate). Remember that *h* and *s* are the height of an object and the length of its shadow. We do not know what object Eratosthenes used in Alexandria, as those records are lost, but perhaps he used an obelisk, such as Cleopatra's Needle, which stood 68 feet tall. On the summer solstice, its shadow would have been 8.6 feet long.

Using our knowledge of right triangle trigonometry, we know that

$$\tan(\angle EDA) = \frac{s}{h}.$$

Since we want to find an angle measure, we are actually going to use inverse trig, so

$$\angle EDA = \tan^{-1}\left(\frac{s}{h}\right).$$

Substituting in our assumed values for *h* and *s*, we get

$$\angle EDA = \tan^{-1}\left(\frac{8.6}{68}\right),$$

which simplifies to $\angle EDA = \tan^{-1}(0.1265)$,

finally giving us $\angle EDA = 7.2°$,

which is also the angle that the sun's rays hit Alexandria on the summer solstice.

We previously determined that $\angle EDA \cong \angle ABC$, so we further know that $\angle ABC = 7.2°$ as well. The distance between Alexandria and Syene is an arc with $\angle ABC$ as the central angle. The arc is a portion of the earth's total circumference, and we need to find out what proportion of it. To do so, divide 7.2° by 360°, the number of degrees in a full circle. The result is 2%, or $\frac{1}{50}$, of the earth's total circumference.

The distance from Alexandria to Syene is 2% or $\frac{1}{50}$ of the earth's circumference. Based on information from caravans, Eratosthenes knew the distance between the two cities to be 5,000 stadia, an ancient measure equal to 500 miles. Now, to find the circumference, all he had to do was set up a proportion using either the percent or fraction to determine the whole circumference. One possible equation to solve is $\frac{1}{50}c = 500$, where c is the circumference. Solving the equation gives a circumference of 25,000 miles, which is remarkably accurate! The circumference has been calculated by NASA as 24,901 miles, so Eratosthenes number was only off by about 0.4%. As a comparison, centuries later, Christopher Columbus tried to calculate the circumference and came up with 15,000 miles, which is off by about 40%! Math may not have been his strong suit...

<u>An Alternative Method</u>

The circumeference can also be calculated by using similar triangles and the ratios of their sides. Using Figure 2 again, we can "see" similar triangles in ΔEDA and ΔABC, since we have already shown

$$\angle EDA = \angle ABC ,$$

and we also know $\angle EAD = \angle BAC$,

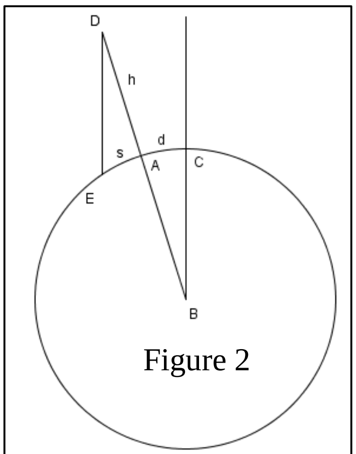


Figure 2

since they are vertical angles. (Technically, these are not triangles, since the earth is curved, but the curvature is not great enough to cause a huge error. We will still get a good estimate.) Since the sides of similar triangles are proportional, we can come up with the following ratio:

$$\frac{s}{h} = \frac{d}{r}.$$

This can be solved for $$r = \frac{dh}{s},$$

giving us the radius of the earth. We know the circumference is

$$C = 2\pi r,$$

so by substitution, we can find the circumference with the formula

$$C = \frac{2\pi dh}{s} = \frac{2\pi \cdot 500 \cdot 68}{8.6} = 24{,}840 \text{ miles},$$

which is only off by 0.25%.

Name ______________________________________ Date ______________ Period ______

Eratosthenes and the Circumference of the Earth – Work Page

As we read through the story of Eratosthenes, complete your work on this paper. The figures are not to scale!

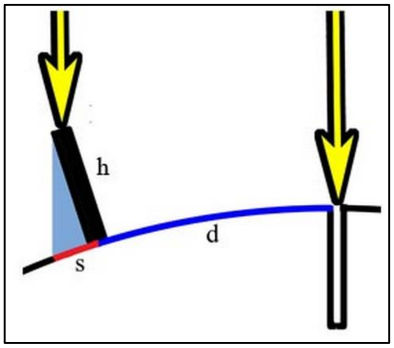


Figure 1

1. Label the cities of Alexandria and Syene on the diagrams.
2. Mark the parallel lines that represent the sun's rays.
3. Mark the angles that are congruent because of the Alternate Interior Angles Theorem. Write them below

   ____________= ___________

4. The next step is to find the measure of these angles
   a. Name the "right triangle" we will use

      ____

   b. What angle measure are we solving for?

      ____

   c. Fill in the values of h = __________

      and s =__________

   d. Which trig ratio will we use? ______

   e. Solve the trig equation for the angle. Show your work below.

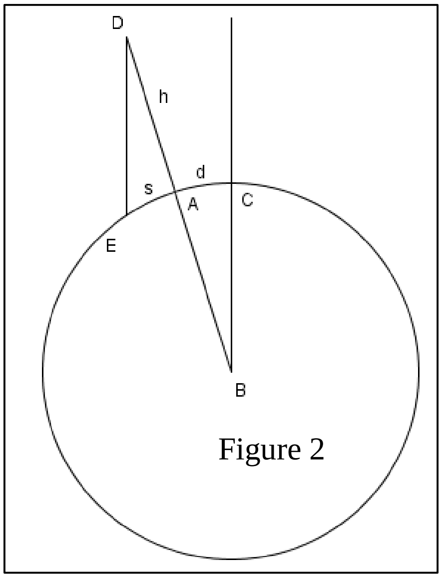


Figure 2

   f. Write the angle measures you determined below.

      ________________________ and ____________________

5. Now we need to consider arc length.

   a. What arc length are we considering? ___________________

   b. What is the central angle associated with the arc length? _________

   c. Determine what fraction or percent of the total circumference that the arc covers. Show your work below.

      Fraction/percent of total circumference = _________

6. Now we will put all of the information together to find the circumference.

   a. What is the actual arc length, or distance between the two cities? ________

   b. Set up an equation relating the fraction/percent of the arc length, the circumference and the actual arc length.

      ____

   c. Solve the equation. Show your work below.

      The circumference is _________________________

Name ______________________________________ Date _____________ Period ______

Eratosthenes on Mars

Now that you've seen how Eratosthenes calculated the circumference of the earth, you can apply his method to any planet, assuming we can get there to take some measurements! We are going to take a trip to Mars to determine its circumference!

Once we get to Mars, we set up Base Camp. We also set up a Research Lab 530 miles to the north. We notice that, on a particular day, the sun is directly over Base Camp, meaning there are no shadows on the ground. On that same day, a 42-foot radio tower at the Research Lab casts a 10.8 ft shadow. Use the information provided and Eratosthenes' method to determine the circumference of Mars.

1. Draw a figure or two to help model the problem. Use your handout to help guide you.

2. Label the Base Camp and Research Lab on the figure.
3. Mark the parallel lines that represent the sun's rays.
4. Mark the angles that are congruent because of the Alternate Interior Angles Theorem. Write them below

   ____________= ___________
5. The next step is to find the measure of these angles
   a. Name the "right triangle" we will use _____________

   b. What angle measure are we solving for? _________

   c. Fill in the values of *h* = __________ and *s* =____________

   d. Which trig ratio will we use? ______
   e. Solve the trig equation for the angle. Show your work below.

   f. Write the angle measures you determined: ______________

   and ______________

6. Now we need to consider arc length.
    a. What arc length are we considering? ___________________

    b. What is the central angle associated with the arc length? _________
    c. Determine what fraction or percent of the total circumference that the arc covers. Show your work below.

       Fraction/percent of total circumference = _________
7. Now we will put all of the information together to find the circumference.
    a. What is the actual arc length, or distance between the camp and lab? _____________
    b. Set up an equation relating the fraction/percent of the arc length, the circumference and the actual arc length.

       ____
    c. Solve the equation. Show your work below.

       The circumference of Mars is ________________________________

8. Now that we know the circumference of Mars, we can use the formula for circumference of a circle to determine the radius. Use your answer from #7c to determine the radius of Mars.

       Radius of Mars _______________________________
9. Recall the Eratosthenes determined the circumference of the earth to be 25,000. Using this measurement, and the formula for circumference of a circle, calculate the radius of the earth.

       Radius of the earth ______________________

10. Eratosthenes used Euclidean geometry to calculate the earth's circumference with a surprising degree of accuracy, considering when he did it and the tools he had at his disposal. However, since he used Euclidean geometry on a sphere, there are some measurements and assumptions that are not exactly accurate. Since the earth is such a large scale object, these small errors do not greatly impact his results. What are some of these inconsistencies or falsehoods that Eratosthenes used in his measurements? Name as many as you can find.

3. LESSON #2 APPROXIMATING PI, ARCHIMEDES STYLE

Teacher's Materials

This lesson is intended for a high school geometry class. In this lesson, students will learn how Archimedes approximated pi, and will perform a similar exploration to come up with their own approximation of pi using a dynamic geometry program. It can easily be adapted to appropriately challenge and support both high-achieving students and students who struggle. Students will be using Geogebra, so they should have some prior familiarity with it. Students will also need to have studied polygons, including the Pythagorean theorem, right-triangle trigonometry, and circles. Knowledge of perimeter and area is necessary as well.

Objectives

- Students will be able to understand and explain how Archimedes approximated pi.
- Students will be able approximate pi using a dynamic geometry software by inscribing and circumscribing circles on polygons

Standards

- 8.NS.A.2 – Use rational approximations of irrational numbers.
- HSG.C.A.3 – Construct the inscribed and circumscribed circles of a triangle and quadrilateral

Materials

- *Approximating Pi, Archimedes Style* Handout
- *Archimedes Revisited – Approximating Pi Using Geogebra* Handout
- *How He Actually Did It – Without Geometry* Handout
- Computer with projector
- Computer lab or Chromebooks (each student needs access to Geogebra)

Classtime Needed

Two 45-minute periods or one 90-minute period

Lesson Plan Outline

1. Read Approximating Pi, Archimedes Style handout
2. Show Geogebra Demonstration on screen - http://tube.geogebra.org/m/15088
3. Archimedes Revisited – Approximating Pi Using Geogebra Activity in computer lab
4. Discussion Questions
5. How He Actually Did It – Without Geometry handout

Notes

Archimedes was more than just a mathematician. He was a physicist, inventor, astronomer and engineer, among other things. I have only mentioned a few of his achievements here, but he has many other accomplishments to his name. A possible extension activity would be to have students research what else he accomplished during his lifetime.

Archimedes' approximation of pi anticipated modern calculus with its use of infinitesimals. Parallels can be made between his method and the use of limits, giving students a taste of calculus.

The Geogebra demonstration listed in step 2 of the lesson plan shows what happens when polygons with greater number of sides are inscribed and circumscribed around a circle. This is great tool to show students that the more sides there are, the closer the polygons become to the circle. This results in better and better approximations of pi. The demonstration allows you to move the slider to see how the polygons change, as well as the calculations.

Another interesting extension activity would be to have students research and present the history of the calculation of pi. There are several different ways it has been done, and the age of computers has led to an astronomical increase in the number of digits we know.

In step 6 of the construction process, students need to divide the perimeter of the polygon by the diameter of the circumscribed and inscribed circles. Geogebra does not have a function to determine the diameter of a circle, so students will need to recognize that the diameter is simply two times the radius. A sample input for the formula bar in this step would be = perimeter[polygon]/(2*circumference[circle]), where students substitute the names of their polygon and circles in place of those words.

This activity does not have to be completed with Geogebra. It can be adapted for any interactive geometry program, such as Geometer's Sketchpad. You will need to work through the activity to rewrite the steps, as each program uses different commands and has different procedures. You should still be able to achieve the same results, as long as you adjust the activity to work within the constraints of the program that you have at your disposal.

The method used in the Geogebra activity is different from Archimedes' method in that students will be inscribing and circumscribing circles on a polygon, not polygons on a circle as he did. The activity uses perimeter and circumference, but can be modified to use area as the basis of the ratio. The resulting ratio needed would then be $\frac{a}{r^2}$ instead of $\frac{p}{d}$ or $\frac{p}{2r}$.

The last handout in this lesson describes how Archimedes actually worked through this problem in his day. He did not actually use the geometric figures as students will do in this lesson. He used an arithmetic approach that found the perimeters after the number of sides had been doubled using the harmonic and geometric means. This is a great example of how there are multiple paths to the same answer. Students will hopefully then realize that if they are stuck on a certain problem, a new approach will possibly be the key to solving it. Also, if there are musically inclined students in the class, the connection between the harmonic mean and music can be mentioned, and students can be encouraged to look into that on their own.

Name ______________________________ Date ____________ Period _____

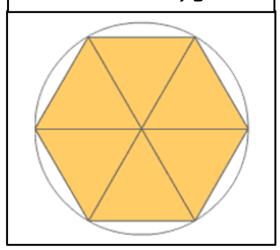
Approximating Pi, Archimedes Style

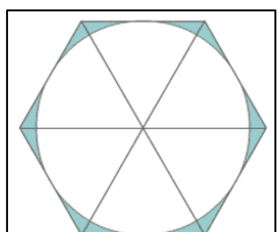
We have all heard of the famous mathematical constant pi, or π. It pops up in math classes on a regular basis, such as with circles, trigonometry, and many other areas. We know it is an irrational number, and that it approximately equals 3.14, but not exactly. We use pi a lot, but where did it come from?

Pi is the ratio of the circumference of a circle to its diameter. Mathematicians have been attempting to approximate pi for centuries. The ancient Babylonians had a rough estimate for pi, and the Rhind Papyrus showed that the Egyptians also used an approximation for pi. Chinese mathematicians came up with their own approximations as well. Some say that the Bible contains an approximation as well, even though it is not very accurate!

One of the earliest attempts to calculate pi was done by Archimedes of Syracuse, a Greek mathematician who lived in the 3^{rd} century B.C. He is considered the greatest mathematician of antiquity, and one of the greatest ever. He is well known for many of his achievements. He created many war machines to help defend his city, and his Heat Ray was featured on several episodes of *Mythbusters*. There is a legend that he was asked to determine if a crown that was supposed to be pure gold had been made with some silver to save money, and he was struggling to come up with a solution. He thought of a solution while he was taking a bath, and was so excited he is said to have run through the streets naked shouting “Eureka!”

Inscribed Polygon





Circumscribed Polygon

In addition to his work in mechanics, among other things, Archimedes came up with an ingenious method for approximating pi. He achieved this by beginning with a circle, and then inscribing and circumscribing regular polygons, as shown in the figures to the right.[2] He then compared the perimeters of the circumscribed and inscribed polygons to the circumference of the circle. The circumscribed polygon’s perimeter is greater than the circumference, while the inscribed polygon’s is less. He was then able to come up with an upper and a lower bound for pi. He repeated this several times, with polygons with more and more sides. He finally used polygons with 96 sides, and was able to determine that pi was greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$, or between 3.1408 and 3.1429. This is very close to the known value of 3.14159… and he did this without calculators or computers!.

[2] These figures can be found at http://www.pbs.org/wgbh/nova/physics/approximating-pi.html.

Ever since the days of Archimedes, mathematicians have been on a quest to calculate pi to more and more digits. Once the age of computers dawned, progress has grown exponentially. The greatest number of digits of pi calculated to date is 12.1 trillion digits, achieved in 2013 by A.J. Yee and S. Kondo. While this accomplishment is impressive, it is not actually necessary. For most earthly calculations, ten digits will suffice. To calculate the circumference of the observable universe to within an atom's length, you would only need 39 digits of pi. Still, the feat is impressive!

There is one last story about Archimedes you should know. He was killed in 212 B.C. during the Second Punic War by the invading Roman Army. The Roman General Marcellus had ordered that Archimedes not be harmed, since he was a valuable asset, but his soldiers did not follow orders. When the soldiers approached Archimedes, he was working on some geometric figures he had drawn on the ground. His last words are said to be "Don't disturb my circles!" right before he was killed. Now that is dedication!

Name ______________________________________ Date ______________ Period ______

Archimedes Revisited – Approximating Pi Using Geogebra

Now that you know a little bit about how Archimedes came up with his approximation of pi, it's your turn! You are going to use a method similar to what Archimedes did to create an upper and a lower bound for pi. You will start by constructing a regular polygon with a specified number of sides, and then inscribing and circumscribing a circle around the polygon. You will then compare the perimeter of the polygon with the circumferences of the circle. If c_i is the circumference of the inscribed circle, c_c the circumference of the circumscribed circle, and the perimeter is p, then we know the following to be true:

$$p < c_c \text{ and } p > c_i$$

Remember that pi is the ratio of circumference to diameter, so if we divide each inequality by the diameter of the particular circle, we then find that these are true:

$$\frac{p}{d_c} < \pi \text{ and } \frac{p}{d_i} > \pi \text{, which can be combined as } \boxed{\frac{p}{d_c} < \pi < \frac{p}{d_i}}$$

This is an approximation of pi! You will be using Geogebra to construct and calculate the upper and lower bounds of pi several times by using polygons with more and more sides. Follow the steps below, and you'll be walking in Archimedes' steps thousands of years later!

<u>Approximating Pi Using Geogebra</u>

1. Turn Axes off
2. Draw a Regular Polygon with 4 sides
3. Construct the circumscribed circle by using the "Draw Circle Through 3 Points" tool, choosing three of the vertices of the polygon. Rename the circle to indicate it is the circumscribed or outer circle.
4. Find the center of the circumscribed circle using either the "Midpoint or Center" tool or using the formula bar. This will also be the center of the inscribed circle. Rename the point "Center."
5. Construct the inscribed circle using the "Circle with Center Through Point" tool, using the center of the outer circle, and dragging the edge of the circle to one of the sides of the polygon, NOT a vertex! Rename the circle to indicate it is the inscribed or inner circle.

6. In the formula bar, calculate the ratio of the perimeter of the polygon to the diameter of both the inscribed and circumscribed circle. The results will be your upper and lower bounds for pi!
   (Hint: there is no tool or function to calculate the diameter of a circle in Geogebra. Is there another way you could represent the diameter?)
7. Record your results in the table below. Repeat the construction for polygons with 5, 6, 10, and 20 sides to get better and better approximations.
8. Just for fun, do the construction one more time, choosing a polygon with at least 20 sides.

Number of Sides	$\frac{p}{d_c}$	$\frac{p}{d_i}$	Upper and Lower Bounds of Pi
4			________ < π < ________ .
5			________ < π < ________ .
6			________ < π < ________ .
10			________ < π < ________ .
20			________ < π < ________ .
_______ sides			________ < π < ________ .

Questions

1. The first ten digits of pi are 3.141592653… How many digits matched in your best approximation?

2. We used circumference to compare the circles to the polygon and approximate pi.
   We know that $\pi = \frac{Circumference}{diameter}$ because of the formula for circumference.
   Could we have used area to approximate pi? What ratio would we have to have used to get the upper and lower bounds?

3. While the approach we used is based on the same logic as what Archimedes did so many years ago, there is something different in our process. What did we do differently from Archimedes?

Name ___________________________________Date _____________ Period ______

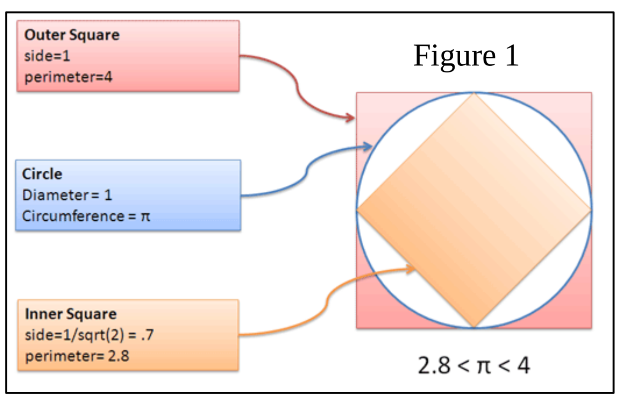
How He Actually Did It – Without Geometry!

Now that you've tried your hand at approximating pi, let's take a closer look at how Archimedes actually did it himself. We already know he calculated the area and perimeter of polygons inscribed and circumscribed on a circle, but we didn't really discuss just how he did this. Archimedes lived over 2,000 years ago, so clearly he didn't use Geogebra. While he could have done this geometrically, Archimedes devised a method using means that looks a little different than what we did.

There were three means that Archimedes used. The geometric mean was a way for mathematicians to calculate the area of a rectangle in terms of a square. The harmonic mean is so named because of its connection to music theory, but it also has a geometric application. You are familiar with the arithmetic mean of two numbers, which is the average. The harmonic mean of two numbers is the reciprocal of the arithmetic mean of their reciprocals.

Archimedes started with a circle of diameter 1 unit. Since pi is the ratio of circumference to diameter, this meant the circumference of this particular circle would be pi. He didn't know what the circumference was, so he started with a shape whose perimeter he did know. Archimedes originally inscribed and circumscribed regular hexagons, but in this case we will start with squares, since it is a bit easier to work with.

The circumscribed, or outer square, has a side length equal to the diameter, which is 1, so its perimeter is 4. The inscribed, or inner square, takes a bit more work to find the perimeter. Drawing the diagonals of the square divides it into four isosceles right triangles with legs of length ½. Solving for the hypotenuse, which is also the side of the square, using the Pythagorean Theorem, and we get a side length of $1/\sqrt{2}$, which is approximately 0.7. Now we can determine that the inner perimeter is 2.8. Archimedes knew the circumference of the circle had to be between these two values, so he got his first rough estimate of pi. In this case, we know that pi must be greater than 2.8, but less than 4. Figure 1 illustrates this process.[3]



We know that this isn't a very good estimate, and Archimedes did too. That is why he repeated the process using polygons with more and more sides. Specifically, he doubled

[3] This figure is found at http://betterexplained.com/articles/prehistoric-calculus-discovering-pi/.

the number of sides each time he did it. When the polygons on the inside and outside have more sides, they are closer and closer to the circle, so their perimeters are closer and closer to pi! Since Archimedes started with a hexagon, the next shape he used was a dodecagon. In our case, our next shape would be an octagon.

These octagons are being inscribed and circumscribed on the same circle with a diameter of 1. The side lengths, and thus perimeters, of octagons are more difficult to calculate. This is where Archimedes used his method utilizing means. He knew that perimeter of the circumscribed octagon would be the harmonic mean of the perimeters of inscribed and circumscribed squares. Furthermore, he knew the perimeter of the inscribed octagon would be the geometric mean of the perimeter of the inscribed square and circumscribed octagon. The harmonic mean of two numbers a and b is $H(a,b)=\frac{2ab}{a+b}$ and the geometric mean is $G(a,b)=\sqrt{a\cdot b}$.

In general, let p_n be the perimeter of an inscribed polygon with n sides and P_n be the perimeter of a circumscribed polygon with n sides. The perimeter of the circumscribed polygon with $2n$ sides, or double the sides, is given by

$$P_{2n}=\frac{2P_n p_n}{P_n+p_n}.$$

The perimeter of the inscribed polygon with $2n$ sides is given by

$$p_{2n}=\sqrt{p_n\cdot P_{2n}}\ .$$

We could use this process to determine the perimeters of the inscribed and circumscribed octagons, and then do the same thing for 16-sided polygons, 32-sided polygons, etc, indefinitely. This would result in the following sequence of numbers that finds upper and lower bounds of pi:

$$P_4, p_4, P_8, p_8, P_{16}, p_{16},...$$

We can use a graphing calculator to calculate this sequence. Start by storing the value of P_4, which is 4, in A, and the value of p_4, which is $4/\sqrt{2}$, in B. Next, determine P_8 by entering the formula for the harmonic mean of A and B, and store that value in A. Your calculator entry will be $(2AB)/(A+B)\rightarrow A$. Now we can calculate p_8 by finding the geometric mean of our new A value and B, storing the result in B. The calculator entry will be $\sqrt{AB} \to B$. By doing this we find that $P_8 = 3.31$ and $p_8 = 3.06$, which are our new upper and lower bounds for pi. At this point, to find P_{16}, you just have to find the harmonic mean of A and B again. Since the calculations are stored in the calculator's memory, just hit "2nd" "Enter" twice, and the calculator will recall the formula, giving us $P_{16} = 3.18$. Do the same to find that $p_{16} = 3.12$. So by doubling the number of sides just twice, we already have approximated pi to be between 3.12 and 3.14. Carry this process out again to find P_{32} and p_{32}.

P_{32} = ______________ p_{32} = ______________

This iterative process will yield better and better approximations of pi. Archimedes did this process several times, ending with 96-sided polygons. At the time, decimals hadn't been discovered, so he was working only with fractions and square roots. Chances are, if he would have known about decimals, he probably would have continued his calculations even further. Still, his approximation isn't too shabby for not even having a calculator!

4. LESSON #3 CHANCES ARE… THE BIRTH OF PROBABILITY

Teacher's Materials

This lesson is intended for a high school classroom as an introduction to probability or as a supplemental activity. It can be used as early as an Algebra 1 classroom. Students need to have an understanding of proportions and percents in order to complete this activity, as well as a basic understanding of probability as the ratio of the number of favorable events to the total number of events.

Objectives

- Students will be able to determine how to fairly divide the prize pot of a game between two players when interrupted before completion.
- Students will appreciate and understand the historical importance of Pascal and Fermat's correspondence.

Standards

- HSS.MD.B.5 – Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.
- HSS.MD.B.5.A – Find the expected payoff for a game of chance.

Materials

- *Chances Are…* Handouts (1 per student)
- *The Birth of Probability Practice* Worksheet (1 per student)
- Coins (1 per pair of students)
- Calculator (optional)

Classtime Needed

Two 45-minute periods or one 90-minute period

Lesson Plan Outline

I. Part 1 - Coin Flip Game in Partners

II. Part 2 – Student Discussions to Determine a Fair Division of the Prize

III. Part 3 – Discussion and Historical Background of Problem of Points

IV. Part 4 – Demonstration of Pascal and Fermat's method of determining expectation

   a. Example 1 – Game to 3 points, score of 2-1 when interrupted

   b. Example 2 – Game to 10 points, score of 7-8 when interrupted

V. Worksheet for Individual Practice in class or for homework

Notes

Part 1

Group students in pairs, handing out a *Chances Are*… handout to each students and a coin to each pair of students. Explain the game and have students begin. Monitor their progress and create some type of interruption before students finish their game. Some suggestions are to have a phone go off, a visitor come to the room, or you can simply ask students to stop where they are. You do not want students to actually get to 10 points.

Part 2

Tell students we can't continue the game. The prize money has to be handed out now, and they need to come up with some fair way of distributing the money between the two players. If students are stuck, or not having meaningful conversations, the following suggestions may serve as prompts for their discussions:

- Who would be most likely to win if the game was finished?
- What is the most important information in determining who should get more money?
- Would their method work for other scores/point values?
- Would both players be satisfied that the division was fair?

Monitor the discussions and make note of groups that have taken different approaches to help facilitate the full-class discussion.

If any pairs finished their game, assign them a score to work with.

Part 3

After the small-group discussion, have several of the groups share their methods. Groups can be chosen ahead of time based on observations during Part 2, or at random. List all the different methods on the board. Once they are listed, discuss the pros and cons of each. Some questions to consider:

- Which methods look at the number of points scored already? Which methods look at the number of points needed to win? Do these produce different results?
- Would these methods produce different results if used on a bigger scale, say 100 points to win?
- Does the difference of the two scores matter for these methods?
- Does how far along the game is matter for each method? Does interrupting the game early on produce different results than interrupting it towards the end?

Students can offer their input on which method they think is the most fair, as well as any concerns they have with other methods. A vote can be taken to see which one the class as a whole would choose.

After the discussion, have students read through Part 3 of their handout out loud. Ask if there are any questions at this point in time.

Part 4

The explanation of the Pascal/Fermat method starts with a relatively simple problem of a game to 3 points with a score of 2-1. The method for determining the expected value is based on finding the number of coin tosses needed to guarantee a winner, then using Pascal’s triangle. There is another method of calculating the expected values that can be used instead. This is explained below.

This method illustrates a game where it requires 3 points to win, with a $64 prize. Before moving on, it should be understood that a tied score, such as 2-2, 1-1, or 0-0 means each player should receive 50% of the prize, or $32, since no player has an advantage. A game is interrupted with a score of 2-1. What would happen if the game continued? If A wins the round, the score would be 3-1, with A winning the whole prize of $64. If B wins the round, the score would be tied 2-2, meaning each player would receive $32. Since A winning and B winning are equally likely, meaning a 50% probability, Pascal determined the expected value of the winnings of Player A as 0.50×64 + 0.50×32 = 48. He did this by multiplying the probability of winning by the amount of prize money received. So if the game is interrupted with a score of 2-1, Player A should receive $48, or 75% of the prize.

What if the score was 2-0? Again, let's look at what could happen in the next round. If A wins, the score would be 3-0, and A would win all $64. If B wins, the score would be 2-1, which we just figured out has an expected value of $48. Therefore, the expected value can be calculated as 0.50×64 + 0.50×48 = 56. If a game ends with a score of 2-0, then Player A should receive $56, or 87.5% of the prize.

Using this same method, you can fill in the rest of Figure 1 to the right.[4] This chart is based on a $64 payout, but the percentages can be determined and applied to any other prize amount. This method can become quickly cumbersome since certain expected values must be worked out before determining other expected values, but this is still valuable for more advanced students to be able to see and understand.

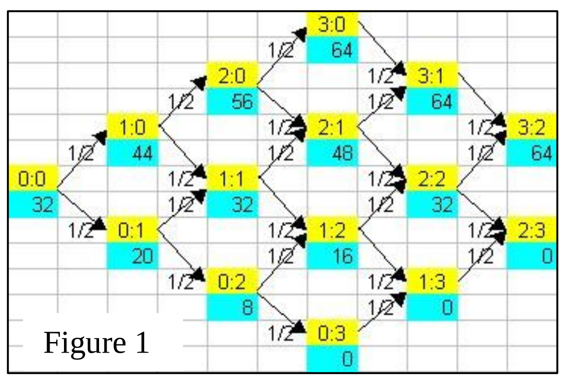


Figure 1

Practice Worksheet

The first page of the practice sheet offers students the opportunity to practice calculating payouts for different games being interrupted at different times. The second page

[4] Chart can be found at http://www.eprisner.de/MAT109/FermatPascal.html.

describes two previous attempts to solve the Problem of Points, and asks students to analyze what problems may arise from these solutions.

Either before completing the practice sheet or afterwards, a discussion of the timeline of these events would help students to realize how long it took these ideas that we take for granted now to develop. It took over a century! Pacioli's early attempt was in the late 15^{th} century, Tartaglia's attempt came in the mid-16^{th} century, and Pascal and Fermat finally solved it in the mid-17^{th} century. This helps illustrate that math is a constantly evolving science, and hopefully students will appreciate that more have a greater appreciation for this after this lesson.

Name ____ Date ______________ Period ______

Chances Are… The Birth of Probability

Part 1: You and a partner are going to play a game of flipping a coin. There is a $100 prize for the winning player (not real money, don't get too excited!). Player 1 will be heads, Player 2 will be tails. For every coin toss landing on heads, Player 1 gets a point. For every tails, Player 2 gets a point. The first player to 10 points wins the entire $100 pot. Record your results below.

PLAYER 1 POINTS	PLAYER 2 POINTS

Part 2: The game has been cut short! Since neither player has yet scored 10 points, it is unclear how the prize money should be divided up fairly. Discuss with your partner to determine a fair method for dividing up the prize money. Try to come up with a method that both players agree is fair!

Part 3: The game that you just played and discussed is actually a famous problem called "The Problem of Points," or "Division of Stakes" problem. The problem asks how to fairly divide up a prize pot if a game is ended before either player earns the required number of points. This problem was the topic of much discussion in the 17^{th} century, and led to the development of modern probability.

Humans have participated in games and gambling for almost as long as mankind has been around. It wasn't until the 16^{th} and 17^{th} centuries when people began to realize that outcomes could be predicted with some accuracy. Gamblers and underwriters alike began to look for reliable guidelines to increase the possibility of profit and gain when gambling.

The earliest example of an attempt to define probability belongs to Gerolamo Cardano, who wrote a manuscript in 1550 in which he discussed the outcomes of rolls of the dice, the problem of points, as well as a rudimentary definition of probability. Although he wrote the manuscript in 1550, it wasn't found until 1576 and only published in 1663, so he is not usually credited as one of the founders of the field of probability. That honor goes to two men, Blaise Pascal and Pierre de Fermat.

In 1654, when Fermat was a judge living in Toulouse, France, and Pascal was a former child prodigy living in Paris, the two struck up a correspondence that would change the study of probability forever. A frequent gambler named Chevalier de Méré posed the "Problem of Points" to Pascal in hopes that he could get a satisfactory answer. Pascal developed a solution, and wrote to Fermat for confirmation.

Pascal's solution focused on how many points were still needed to win, which is unlike all the other approaches that were being taken at the time. He calculated the probability that each player would win if the game continued, and divided the prize accordingly. We will take a closer look at his method in the next section.

Fermat and Pascal wrote back and forth several times to further develop Pascal's ideas. Their letters represent the beginning of modern probability theory, including the first mention of the concept of "expected value." This correspondence cemented their places as the fathers of probability theory.

Part 4: How Did They Do It?

Example 1: To begin, we will look at a simplified version of the game we played, where we only play to 3 points instead of 10. One thing we must establish is that each coin flip is equally likely to land on heads as tails. Let's see what happens when Player A is ahead 2-1 when the game is interrupted, using Pascal and Fermat's method.

1. How many points does Player A need to win? ___________ Player B? _________

2. Could the game be over after 1 coin flip? _____ Can you GUARANTEE it will be over after 1 coin flip? ______

   Explain your answer. ____

3. How many coin flips would you need to GUARANTEE the game will be over? Explain your answer. ____

   ____

4. List all the possible outcomes for three coin flips needed to guarantee a winner. For example, one outcome is three heads, so Player A would get the one point they need after the first flip, and win the game.

   AAA

5. How many possible outcomes where there for three flips? _______

6. How many of those outcomes would result in Player A winning? ________ Player B winning? _________

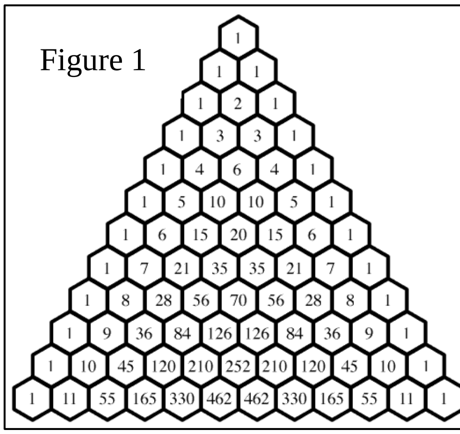
7. To determine the proportion of the prize that Player A should get, Pascal and Fermat calculated the number of times that Player A could be expected to win, divided by the total number of possibilities. In this case, what is the proportion of the possible outcomes did Player A win? _________

   What is this proportion as a percent? ____________

8. Thus, if a game to 3 points is interrupted with Player A winning 2-1, Player A should receive______% of the prize. If the prize is $100, Player A should receive $_____ and Player B should receive $_____.

This method can be used for any number of points and any score along the way. It is, however, quite tedious to list out all of the possible outcomes. Pascal and Fermat agreed, and in their correspondence came up with a method utilizing Pascal's Triangle. Pascal's Triangle, as shown in Figure 1, is created by starting with 1's along each outside diagonal, and then creating each entry by adding the two entries above it.[5] Don't be fooled though, Pascal was not the first mathematician to use this triangle. It is believed that the triangle was first used in China, but Pascal gets the glory.

Figure 1



Here is how Pascal and Fermat used the triangle to solve the Points Problem. Once a game is interrupted, determine how many flips are necessary to guarantee a winner. This

[5] Triangle image can be found at http://mathforum.org/workshops/usi/pascal/mo.pascal.html.

is done by adding the number of points Player A needs with the number of points Player B needs. In our previous example of a 3-point game with Player a winning 2-1, we needed 3 flips to ensure a winner. Next, we look at that row of the triangle, in this case the third row. The third row has the numbers 1 2 1. Since Player A needs 1 point to win and Player B needs 2 points, the first 2 numbers give the number of ways Player A can win, while the last number give the number of ways Player B can win. So Player A can win 1 + 2 = 3 ways, while Player B can win 1 way. Add the entire row to determine the total number of outcomes, which is 1 + 2 + 1 = 4 outcomes.

You may be concerned that the number of ways to win and the number of outcomes don't match what we did earlier. Remember, we are dealing with the *probability* of winning, so let's write the proportions to see if they match up.

Probability of Player A winning __________ As a percent _______

Probability of Player B winning __________ As a percent _______

As you can see, the percentages, and thus, the probabilities are the same! Let's try Pascal and Fermat's method on another example.

Example 2

Consider a game to 10 points that is stopped with Player A losing 7:8. The prize is $64.

1. How many flips are necessary to guarantee a winner? ___________

2. Which row of Pascal's Triangle should you look at? ____________ Write it

   here: ____

3. The first _____ numbers in the row represent the ways Player A can win.

   The last _____numbers in the row represent the ways Player B can win.

4. The total number of outcomes is __________.

5. What is the probability of Player A winning? ____________ As a Percent _______

   What is the probability of Player B winning? ____________ As a Percent _______

6. How much money should Player A receive? ______

   How much should Player B receive?_______

Name ______________________________ Date ____________Period ______

The Birth of Probability – Practice

Consider a game of coin flips, where heads and tails are equally likely outcomes. The winner of the game is the first to score 20 points. The prize is $50. Pascal scores a point when the coin lands on heads, Fermat scores when the coin lands on tails. Their game is interrupted when Pascal is winning with a score of 19-17.

1. How many flips are necessary to guarantee a winner?___________

2. Which row of Pascal's triangle should you use? _________ Write it below.

   ____

3. The first _____ numbers in the row represent the ways Pascal can win.

   The last _____numbers in the row represent the ways Fermat can win.

4. The total number of outcomes is __________.

5. What is the probability of Pascal winning? _______ As a Percent _______

   What is the probability of Fermat winning? _______ As a Percent _______

6. How much money should Pascal receive? ________

   How much should Fermat receive?________

Ms. Schneck and Mrs. Hoslar playing a coin flip game to 100 points. They play during their lunch period until the bell rings. The score when the bell rings is 94-95, with Ms. Schneck losing at this point.

7. What is the probability of Ms. Schneck winning? ________ As a Percent _______

   What is the probability of Mrs. Hoslar winning? _________ As a Percent _______

8. If the prize is $200, how much money should Ms. Schneck receive? ___________

9. If the prize is $1,000,000, how much money should Mrs. Hoslar receive? _______

There were many failed attempts to solve the Points Problem before Pascal and Fermat came up with a satisfactory solution. Below, consider some other solutions and explain what was not satisfactory or fair about them.

10. In a 1494 textbook he wrote, Luca Pacioli discussed the points problem and gave a solution where the prize was divided up based on the percentage of points won up to that point. For instance, in a 10-point game, if Player A was winning 7-3, they would get 70% of the prize, since they had score 70% of the points so far. If Player A was losing 0-1, they would get 0% of the prize. The number of points needed to win did not matter.

11. In the 16th century, a mathematician named Tartaglia wasn't happy with Pacioli's results. He didn't think it was fair to award the whole prize to one person if they were winning 1-0, since the game was a long way from over. He came up with another method by looking at the division of the prize on the ratio of the size of the lead to the length of the game. For instance, in a 100-point game, a score of 57-47 would pay out the same as a score of 99-89.

5. LESSON #4 CLASH OF THE CUBICS

Teacher's Materials

This lesson is intended for a high school Algebra classroom. Students need to be able to solve quadratic equations using various methods, including the quadratic formula. This lesson would serve as a good pre-cursor to introducing complex numbers, as students will likely create equations that have complex solutions. The lesson can also be used as an introduction to irrational answers in a earlier class, as students will also create equations with non-perfect squares. In addition, this lesson can lead to a meaningful discussion of the discriminant, as students may notice that the term under the radical is what affects the type of solution they get.

Objectives

- Students will be able to solve quadratic equations using a variety of methods
- Students will be able to analyze the quadratic formula and identify what parts of it give more "difficult" solutions.

Standards

- HSN.CN.C.7 – Solve quadratic equations with real coefficients that have complex solutions.
- HSA.REI.B.4 – Solve quadratic equations in one variable.
- HSA.REI.B.4.B – Solve quadratic equations by inspection, taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions

Materials

- *Clash of the Cubics* handout
- *I Challenge You to a Duel* handout
- *Clash of the Cubics – A Challenge for You* handout
- *Tartaglia's Poem* handout

Classtime Needed

Two 45-minutes periods or one 90-minute period

Lesson Plan Outline

1. Read through Clash of the Cubics worksheet as a class
2. Quickly review methods for solving quadratic equations
    a. Graphing
    b. Factoring
    c. Square roots
    d. Completing the square
    e. Quadratic formula
3. I Challenge You to a Duel – students work in pairs
4. Class discussion about results and what kind of problems they solved
5. Divide the class into two groups and have the groups try to stump each other
6. Tartaglia's poem
7. Challenge Worksheet for homework or extra credit

Notes

The tale of the Tartaglia and Fior can be presented in many ways. If the class is theatrically inclined, an interesting approach to sharing the story is to have all or some of the students read it the night before, then act it out for the class the next day. Another option would be to have students make posters or some other type of media presentation to share the story. These could possibly be used as an extension assignment as well.

During the review of quadratic equations, lead the student discussion by asking some guiding questions. Some questions might be as follows: Are some methods are easier than others? Can every equation be solved using every method? Can you tell by looking at an equation whether it will be easy to solve or more difficult? Students should begin to think about how the different parts of the equation and the quadratic formula can affect the answers and difficulty level.

When creating equations for the duel, students may need some guidance. I would suggest preparing some equations for students who lose theirs or don't complete them. Some types of equations to include would be one with complex solutions, irrational solutions, a GCF, or equations with a b or c value of zero. When groups are solving their duel problems, stress that they are to get only exact answers – no decimal approximations!

After the duel, students can present some problems that stumped them. The most likely equations to stump them will be equations with complex solutions. A parallel can be drawn to how Fior's method only worked for certain types of equations, and now we have a similar situation. We will have to use a new approach in order to solve these problems, which lead into complex numbers. An introduction to complex numbers can begin the next day in class. The discussion can also focus on the discriminant, since students may have figured out that the discriminant controls what type of solution a quadratic has.

This lesson can be used in Algebra 1 as an introduction to irrational numbers too, as students will likely get irrational solutions. A potential drawback is that they will also get solutions with a negative square root, which is not generally addressed in Algebra 1.

After the discussion, the class can split into two groups and use what they discussed to try to come up with some equations that are really stumpers. The two groups can face off in a re-creation of the University of Bologna contest between Tartaglia and Fior.

Tartaglia's poem for Cardano is included as an interest piece. Students will see how mathematics was written at this time, and hopefully gain a greater appreciation for how notation and symbolism has simplified the process since then!

The challenge sheet contains some of the problems that Fior gave to Tartaglia in the duel. Students are to try to write the equation that he is describing. This is a good exercise in translating from words to mathematical notation, and it will again help student gain an appreciation for how difficult it was to do math back then, and what a great achievement it is that these men could understand it all. It also has students attempt to work through the cubic formula, which is clearly more complicated than the quadratic formula. Again, students can gain appreciation for how impressive these achievements are, especially considering these mathematicians did all of this without the assistive technology we have grown accustomed to and dependent on. Students will not be able to work it all the way out because it will have complex numbers, but they can make some headway with it. Maybe then the quadratic formula won't seem so bad!

Name ____________________________________Date _____________Period ______

Clash of the Cubics – A Tale of Challenges, Duels and Theft

Mathematics is often thought of as a boring, tiresome subject, and mathematicians are portrayed as dull old men. This could not be farther from the truth! Mathematics is a dynamic field of study, and mathematicians are very creative and competitive. Mathematicians are not all dull and in fact are not all women. The field is growing in diversity every day. This is the story of one such mathematician, and how one type of equation consumed his life and ruined his reputation.

Leading up to the 16th century, mathematicians were able solve linear and quadratic equations, which you also know how to do. The next logical step was to try to solve cubic equations, which have an x^3 term, and no higher exponents. Some mathematicians had claimed that it was impossible to come up with a general solution, or one that would work for every cubic equation. Some mathematicians weren't satisfied with that and continued to work on finding a solution.

A mathematician named Scipione del Ferro found a solution that would work for one type of cubic equation, $x^3 + px = q$. He did not share his discovery, but on his deathbed he taught his student Antonio Fior how to solve them. Fior was not the best of students, but he had the secret weapon that no one else had, so he wanted to make use of it.

During this time, it was common for academics to demonstrate their methods and results by challenging one another, with the hopes of bolstering their reputations. These challenges could be intense and included referees and very strict rules. Being very confident in his method, Fior decided to challenge a top mathematician of the time, Niccolo Tartaglia, to a mathematical duel. According to the rules of the duel, each mathematician was to give the other 30 cubic equations, and who ever solved all 30 in the shortest time would win the duel.

Tartaglia can from a poor Italian family and despite this found he had a great talent for mathematics. He was very ambitious and taught himself enough mathematics to become one of the leading mathematicians of his time. When Fior challenged him, he did not know how to solve cubics, but he set to work and eventually worked out a solution. Tartaglia's solution was even better than Fior's because it worked for more types of cubics, not just equations of the form $x^3 + px = q$. His method relied on the square roots of negative numbers, which Fior and others considered to be impossible.

When it came time for the contest, which took place at Bologna University in 1535, Fior presented Tartaglia with 30 equations of the type that he knew how to solve. Tartaglia gave him 30 problems, but included many different types of cubics. Thanks to his superior method, Tartaglia was able to solve the problems he received from Fior in less than two hours, whereas Fior struggled with his. Tartaglia was the clear winner, firmly establishing his reputation.

After the duel, Tartaglia did not reveal his methods, despite great interest. Another famous mathematician of the time, Gerolamo Cardano, desperately wanted to know his secret. After repeated requests, Tartaglia finally shared his method with Cardano with the agreement that Cardano would not publish it. Even when he did share it, he made Cardano work for it by disguising his solution in a poem. Cardano and his student, Lodovico Ferrari, used Tartaglia's and Fior's methods and built upon them, eventually discovering a general solution to the cubic. Ferrari even realized that he could use a similar process to solve quartics, or equations with an x^4 term. All of these solutions, for both cubics and quartics, involved complex numbers, something that was not readily accepted at the time. However, a mathematician named Rafael Bombelli explored how to formally manipulate the square root of negative numbers in order to arrive at the known real solutions of cubic equations. (You may wonder whether there is a general solution for quintic equations, those with an x^5 term. It is actually impossible to have a general solution to a quintic equation, as proven by Abel and Galois.)

Cardano did originally agree to not publish Tartaglia's solution, but after 6 years, Tartaglia still hadn't published his results, so Cardano included his method, as well as Fior's method, in a book called *Ars Magna*. Cardano did give credit to Tartaglia in the introduction of the book, but Tartaglia was still furious about what he considered to be intellectual theft. Tartaglia spent years accusing Cardano of theft and trying to draw him into a debate, but Cardano didn’t get sucked in. Tartaglia kept this up until 1548 when he was challenged by Ferrari, Cardano’s student, to another mathematical duel. By this point, since he and Cardano had continued to work with cubics and quartics, Ferrari had a much better understanding of these equations than Tartgalia. Tartaglia quickly realized that he was clearly the inferior mathematician in the contest, not Ferrari, and he fled town in the middle of the night, thus losing the contest. As a result, he ruined his reputation and his career, all over cubic equations!

Name ____________________________________ Date _____________ Period ______

I Challenge You To A Duel!

We have learned many methods for solving quadratic equations so far. List them below

1.
2.
3.
4.
5.

We are going to have a mathematical duel right here in class! You and a partner are going to work together to come up with FIVE quadratic equations of the form $\boxed{ax^2 + bx + c = 0}$ where $a \neq 0$. If you would like to use your notes and textbook, you may. List your five equations below

1.
2.
3.
4.
5.

Next you will be challenged by another pair of students. You and your partner are to work together to try to solve their equations. You may use whatever method you choose, but you are to give only exact answers, no decimal approximations!

Who were you challenged by? ____

Write and solve the equations you were given below. Show your work!

1.	2.	3.

4.	5.

Name ____________________________________ Date _____________ Period ______

Clash of the Cubics – A Challenge for YOU!

Back in the time of Tartaglia and Fior, mathematics looked a little different. We are used to opening up a textbook and seeing pages and pages of equations to solve. Back then, the symbolism and notation that we are used to just didn't exist. Listed below are a few of the problems that Fior gave to Tartaglia. Can you write the equation that he is describing? You do NOT have to solve.

1. Find me a number such that when its cube root is added to it, the result is six.

2. Find me two numbers in double proportion (x and $2x$) such that when the square of the larger number is multiplied by the smaller, and this product is added to the two original numbers, the result is forty.

3. Find me a number such that when it is cubed, and the said number is added to the cube, the result is five.

4. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is this profit?

5. There is a tree, 12 braccia high, which was broken into two parts at such a point that the height of the part which was left standing was the cube root of the length of the part that was cut away. What was the height of the part that was left standing?

Reading these problems should make you happy you didn't go to school back then, right?? Flip the page over for another challenge!

BONUS

After reading all about how Fior, Tartaglia and Cardano came up with ways to solve cubic equations, you may be curious to know how they did it. When a general solution was finally published by Cardano, he had taken what Fior and Tartaglia started and came up with a cubic formula. The quadratic formula isn’t terribly complicated to remember or use, but the cubic formula is a little more involved than that!

The formula for solving a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ is

$$x = \left\{ q + \left[q^2 + \left(r - p^2 \right)^3 \right]^{1/2} \right\}^{1/3} + \left\{ q - \left[q^2 + \left(r - p^2 \right)^3 \right]^{1/2} \right\}^{1/3} + p$$

where $p = \dfrac{-b}{3a}$, $q = p^3 + \dfrac{bc - 3ad}{6a^2}$, and $r = \dfrac{c}{3a}$

Your challenge is to begin to solve a cubic using the cubic formula. In order to receive any bonus points, you must have at least determined the value of *p*, *q* and *r*. After that, you should plug those values into the formula and simplify as much as you can. The further you go, the more points you can earn. (NOTE: You will NOT be able to completely solve the equation, as it involves complex numbers. More to come on that in later days!)

6. Begin to solve the following cubic equation: $x^3 + 4x^2 - 7x - 10 = 0$

Tartaglia's Poem

When Tartaglia finally shared his solution with Cardano, he wasn't going to just give it to him. Being the brilliant mind that he was, he decided to encode the solution in a poem. Below is a translation of the poem he sent Cardano, along with the mathematical translation beside each line. The poem rhymed and was much more artistic in its original Italian, but it is still possible to enjoy the skill it takes to write the poem. The poem can be found at http://www.maa.org/publications/periodicals/convergence/how-tartaglia-solved-the-cubic-equation-tartaglias-poem.

01) *When the cube with the cose beside it*	<x^3+px >
02) *Equates itself to some other whole number,*	<$=q$>
03) *Find two other, of which it is the difference.*	<$u-v=q$>
04) *Hereafter you will consider this customarily*	
05) *That their product always will be equal*	<$uv=$>
06) *To the third of the cube of the cose net.*	<3/3, instead of $(p/3)^3$>
07) *Its general remainder then*	
08) *Of their cube sides, well subtracted,*	< $\sqrt[3]{u}-\sqrt[3]{v}$ >
09) *Will be the value of your principal unknown.*	<$=x$>
10) *In the second of these acts,*	
11) *When the cube remains solo ,*	<$x^3=px+q$>
12) *You will observe these other arrangements:*	
13) *Of the number*	<q>
you will quickly make two such parts,	<$q=u+v$>
14) *That one times the other will produce straightforward*	<$uv=$>
15) *The third of the cube of the cose in a multitude,*	<$p^3/3$, instead of $(p/3)^3$>
16) *Of which then, per common precept,*	
17) *You will take the cube sides joined together.*	< $\sqrt[3]{u}+\sqrt[3]{v}$ >
18) *And this sum will be your concept.*	<$=x$>
19) *The third then of these our calculations*	<$x^3+q=px$>
20) *Solves itself with the second, if you look well after,*	
21) *That by nature they are quasi conjoined.*	
22) *I found these, & not with slow steps,*	
23) *In thousand five hundred, four and thirty*	
24) *With very firm and strong foundations*	
25) *In the city girded around by the sea.*	(Katscher)

Would you have been able to figure out the solution? It should make you appreciate that I don't give you formulas in this way! Although it gives me an idea…

6. LESSON #5 LADIES FIRST – THE FIRST COMPUTER PROGRAMMER

Teacher's Materials

This lesson can be used for any high school math class as a supplemental lesson. It introduces students to Ada Lovelace, who wrote the first computer program. Students are also given the chance to see how a computer program is written by modeling a cup-stacking activity. This lesson is a great way for students to begin to understand what computer science is all about, and hopefully spark their interest to pursue it further.

Objectives

- Students will learn about Ada Lovelace and appreciate her contributions to the field of computer science.
- Students will be able to translate actions into a simple program and be able to trouble-shoot if there are problems with the code.

Materials

- *Ladies First – The First Computer Programmer* Handout
- *Cups – An Unplugged Programming Activity* Handout
- A stack of at least six cups, one per group
- Cup Stack Cards, one set per group
- Blank Index Cards, one set per group

Classtime Needed

One 45-minute period

Lesson Plan Outline

1. Read Ladies First Handout
2. Cups – An Unplugged Programming Activity
3. Discussion of results and code blocks

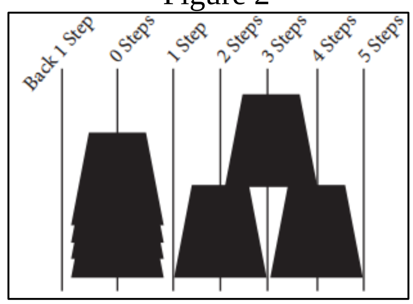
Notes

Before reading the Ada Lovelace biography, ask students what they think computer programming is and who wrote the first computer program. Many students may be surprised to find that computer programming has its roots in the 19th century and that a woman wrote the first program.

Figure 1

Before beginning the Cups activity, write the six programming symbols on the board for quick reference. It is also suggested that the class work through an example of the simplest one, shown in Figure 1, together.[6] This will help students to understand how the activity should work. Have the class instruct you as to what to do first. Do ONLY what they tell you to. Do not put the cup down until they tell you to. A student can be recording the steps on the board, and then as a class translate it to code.

Figure 2



During the explanation of the activity, stress to students that a movement forward is just half the width of a cup. Students will want to move the cup a whole length for one movement. Figure 2 to the right will help to illustrate this.

For the activity, students should be grouped in threes or fours. Each group should receive a set of Cup Stack Cards, a stack of cups, index cards and handouts. Set aside a corner of the room for the "Robot Room." This is where the robots can take their cups while the rest of their group is programming. Here they can practice the different commands with the cups.

Groups should start with the simpler Cup Stack cards, then work their way to the more complex ones. While they are programming, they should NOT have the cups. The cups should be with the robots. This is to ensure that students are able to think abstractly and write the program without having to physically move the cups.

[6] All figures from this lesson can be found at http://csedweek.org/files/CSEDrobotics.pdf.

Once the program is written, the robot returns to the group and begins to execute the program. There should be no talking while the robot is at work, just like if a computer were running the program. It is possible the program may have an error. If this is the case, then the robot should leave and the rest of the group can work together to debug the program. Once they've fixed it, then the robot can return and run the program again.

The activity can continue until either everyone has gone or time runs out. Groups should be attempting harder and harder stacks as the activity progresses.

After the activity, students may want to see how an actual computer program works. I have created a simple program using Scratch, which is accessible at https://scratch.mit.edu/projects/70523344/. Blocks of the code are shown and examined on the second page of the activity sheet. Be sure to let them see the program run so that they can relate the code to the program. All of the "wait" blocks in the code are there simply so that students can see the movements, otherwise the cup would move right to the final spot without showing the individual commands. The commands in Scratch are not the same as the ones we used, so some adjustments have to be made. The screen is set on a coordinate plane, so movements are based on the coordinates of the object. To pick a cup up, the y-value must be changed by a positive number. To put the cup down, the y-value has to be changed by a negative number. Sideways movement is based on the x-coordinate.

Scratch is a one of many beginning programming languages that is very user-friendly and able to be picked up quite quickly. Encourage students to start a free account and see what they can come up with!

Name ________________________________ Date ____________Period ______

Ladies First – The First Computer Programmer

Computer programming and technology are fields that seem to be dominated by men. It may be surprising to hear that the first computer programmer was actually a woman! Ada Lovelace, who was born Augusta Ada Byron in 1815, was the daughter of famed poet Lord Byron. Her parents' marriage was a miserable one, and her father left only weeks after her birth. She never saw him again.

Her mother hoped to prevent her daughter from developing the same temperamental and moody personality and poetic tendencies as her father, so she hired tutors to teach her daughter mathematics and science. While Ada excelled at mathematics, her mother's efforts were not entirely successful. Ada was very imaginative in her approach to science and mathematics, even designing a flying machine when she was 13 years old. Her creativity continued to impact her work for the rest of her career, and helped her to make a lasting name for herself.

Ada's social circle included many of the brightest minds of the time, including scientist Michael Faraday, author Charles Dickens, and Mary Somerville, one of the first women admitted to the Royal Astronomical Society. Somerville introduced her to Charles Babbage, who had drawn up plans for a Difference Engine to perform mathematical calculations. Ada was enthralled by Babbage's ideas, and they struck up a considerable correspondence discussing mathematics, logic, and many other topics.

Babbage eventually began creating plans for an Analytical Engine. He had trouble finding support in England, since his first machine was not complete, but he was more fortunate overseas. An Italian mathematician, Louis Menebrea, wrote an article describing the engine, and Ada was asked to translate the article to English. Babbage asked her to include her thoughts in the translation, since she understood the machine so well. She worked feverishly on the task, and her notes ended up being three times longer than the actual article!

In her notes, she described plans for how the machine could perform several tasks, such as handling letters and symbols in addition to numbers, calculate Bernoulli numbers, and she described loops for repeated actions. Her codes are considered to be the first computer program. Due to her forward-thinking intuition, she was also able to predict many of the things that computers would eventually be able to do, such as compose music and produce graphics. She anticipated many of the things we consider relatively new in computing by over a century!

As is so often the case, her work was not widely recognized during her lifetime. Her work was rediscovered in the mid-20th century, and became part of the inspiration for Alan Turing's work on early modern computers in the 1940's. She received many awards posthumously, and even had a programming language named after her by the U.S. Department of Defense.

Ada Lovelace's writings laid the groundwork for modern-day computer programming. She described the blocks of code that are so essential to our computers, phones, televisions, and the multitude of other devices we use on a daily basis. If anyone ever says you do something like a girl, you can be sure to take it as a compliment!

Name ___________________________________ Date _____________Period ______

Cups – An Unplugged Programming Activity

Every piece of technology we use on a daily basis operates using computers, from iPads to smartphones. Contrary to popular belief, computers are not actually that smart. They will only do exactly what they are told to do. These directions come in the form of codes. Codes take certain actions and turn them into symbolic language that a computer can understand. Codes allow computers to perform all kinds of amazing tasks, just like Ada Lovelace predicted.

In this activity, you and your group will practice writing some simple codes to operate a robotic arm. Each group will receive a card with an arrangement of cups stacked in a particular way. It is your job to write the directions for how the robotic arm can move in order to stack the cups to match the picture. There are only six movements that can be used by the robotic arm. They are:

↑ - Pick cup up
↓- Put cup down
→ - Move cup ½ width forward
← - Move cup ½ width backwards
↻ - Turn cup 90° clockwise on its side
↺ - Turn cup 90° counterclockwise on its side

Steps

1. Choose one member of your group to be the robotic arm. Send them to the "Robot Room" while the rest of the group writes a program. The robot takes the cups with them!
2. Choose a card from the Cup Stacks pile.
3. Write a code on one of the blank cards provided using only the six symbols above. When the robot follows these directions, they should create the stack on the original card.

4. Once the code is written, call the robot back to group. The robot then follows the movements on the card.
5. Be on the lookout for incorrect movements. If there are any problems with the code, work together to try to de-bug it, and then ask the robot to re-run it.
6. Choose another classmate to be the robot and try it again!

<u>Rules</u>

1. ONLY the six symbols above can be used in the code.
2. Cups stay with the robot at all times, not with the programmers!
3. Once the robot returns, there should be no talking!

Work through the activity until time is called or your group has programmed every card. If you finish early, try the extension activity below!

It’s your turn to get creative! Sketch a possible stack of cups below. Write the code for it, and see if your robot can create the stack.

Sketch:

Code:

Now that you've tried your hand at writing a program, you may wonder how this can be done on a computer. There are many easy-to-learn programming languages that a beginner can use to really understand how a computer program works. A few of these are Scratch, Starnova Logo, and Snap!, which all utilize blocks to help students see which functions and commands can be used together. Take a look at a simple Scratch program that models this activity, which is found at https://scratch.mit.edu/projects/70523344/ . The initial screen of the program is shown in Figure 1.

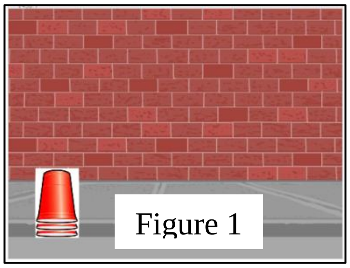


Figure 1

Now that you've seen the program, let's take a look at some of the code. The code for the first cup in the stack is shown in Figure 2. In the activity, you were able to tell the robot to move half a cup width. In Scratch, the programming window is set on a coordinate plane, so movements have to be based on coordinates. The "wait" commands are added in the code to allow students time to see the movements.

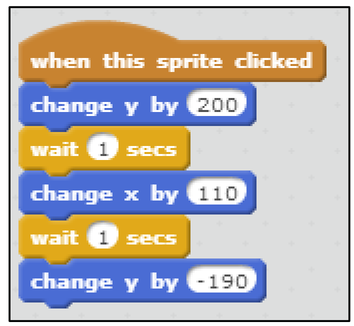


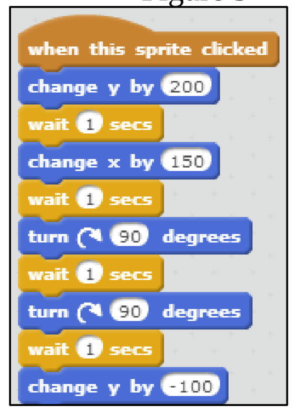
Figure 2

What is the first movement in this code block?

The second movement?

The third movement?

Figure 3

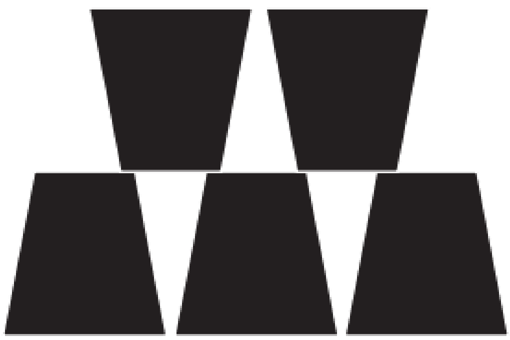


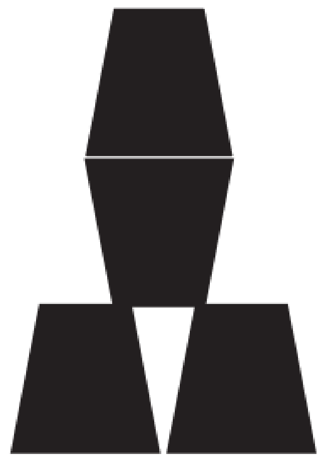
The code block for the middle cup is similar. The third cup has a little different code, shown in Figure 3. Again, the wait time is there so that the movements can be seen by the viewer. There are five movements in this code block. Can you translate this code into the symbols you used earlier?

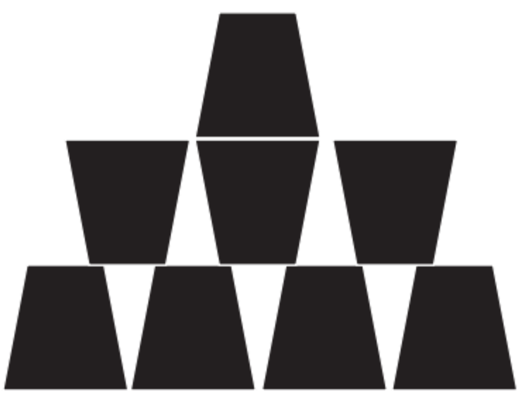
Code:

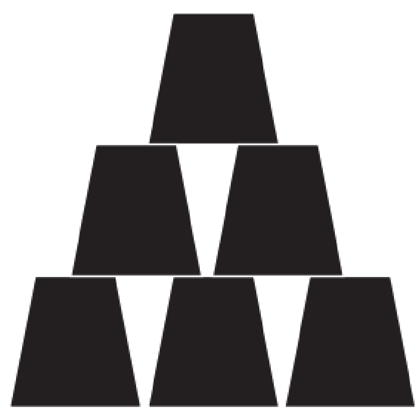
Scratch is a programming community where you can see other people's projects, borrow from them, and learn how to use it quite easily. It is available for free at scratch.mit.edu, so if you are interested check it out! While Ada Lovelace didn't use Scratch or any of the other programming languages out there, her ideas formed the basis for these languages. It is impressive that her work is still influential over a century later!

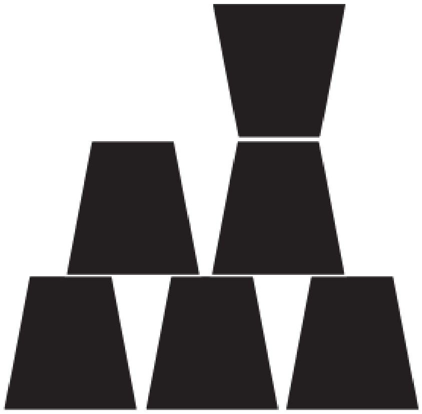
Cup Stack Cards

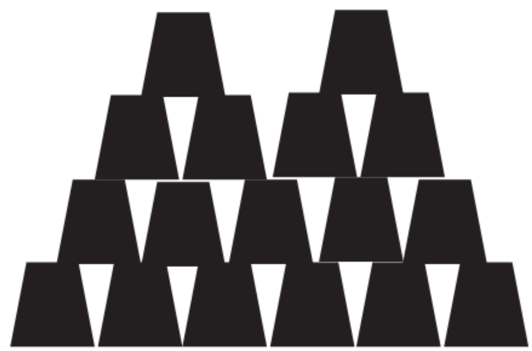












7. CONCLUSION

> *"In classrooms, we often treat mathematics as if we were learning on an island. We travel to that island once a day for mathematics and delve into a study that is pure, clean, and logically solid and has clear lines and no dirty corners. Students think that mathematics is closed, dead, emotionless, all discovered… By including (the history of mathematics), we can rescue students from the island of mathematics and relocate them on the mainland of life that contains mathematics that is open, alive, full of emotion, and always interesting." Bidwell, Mathematics Teacher Sept. 1993 p. 461*

This collection of lessons is a starting point for rescuing students from the island Bidwell describes. This is by no means an exhaustive collection. While I have not had the opportunity to test these lessons in the classroom yet, I am excited to try them and share them with my colleagues.

I have always had an interest in all things historical, but in researching for these lessons I have found a renewed interest in the history of mathematics. I was entertained and impressed by the stories I read. It is fascinating how much these mathematicians accomplished with the few resources they had at their disposal. To learn the stories behind the formulas gave me a much greater appreciation for the work of these mathematicians. These lessons will help students to see that the history of math is useful, especially in the three ways mentioned previously. Mathematics History is beneficial in that it humanizes the mathematical process, helps students to make connections to the history and culture of the world around them, and it helps to deepen understanding of the concepts by enriching the tasks.

The first benefit of humanizing mathematics makes it much more interesting and motivating for students. Some of the characters in the story of mathematics were very eccentric and intriguing. Some worked in tough conditions, overcoming class and gender barriers. Sometimes the history just shows that these men and women were just that: regular men and women. These lessons offered a peek into the lives of the mathematicians involved in them. For them, mathematics was life, something worth fighting for and even dying for, in the case of Archimedes. Students were shown some of the struggles mathematicians had. A common misconception students have is that mathematicians were geniuses that could just think up things like formulas and proofs in little to no time. While many of them were geniuses, they worked on problems for years. They made mistakes, were proven wrong, and had to start over, as is seen in the Fior/Tartaglia/Cardano battle. It is important for students to understand this, because they often think only geniuses are able to understand these topics when, in reality, everyone is capable of learning mathematics, the effort just needs to be put in. Students may not be able to relate to sheer intelligence of some mathematicians, but they can absolutely relate to the struggles they endured, and that is perhaps the biggest benefit from studying the history of mathematics.

One of the biggest complaints from students is they feel math doesn't apply to the real world, when in fact, mathematics informs much of the world we know, and helping students to realize this is another benefit of including mathematics history. Learning about the foundations of programming will help students to appreciate just how much programming affects their life. Students were shown how probability was born from one of the most enduring human past times – gambling. These are just two examples of how mathematics developed from the world around it. Mathematics can also be used to learn much about the culture it developed in. Looking at the artwork of different civilizations can show you a lot about their knowledge of geometry and patterns. Studying the origins of the words they used for numbers can help to see what their culture was like. There is much to be learned about a culture if one digs just a little deeper into their art, everyday objects, and even numbers. Hopefully students now have more of a desire to do so.

A third benefit that comes from including mathematics history is that it assists students in developing a deeper and more enduring understanding of the concepts being taught. Including historical information in lessons and problems immediately makes the task a richer one. Students can better see why they should take certain steps and what they should do next. They can understand the process in a much more concrete way, as opposed to an abstract checklist they must follow for some reason. Placing content in a historical scenario will also increase retention, since students will be creating more connections during the learning process.

There are many more benefits to introducing students to the history of mathematics. The history is a rich one, with much to offer anyone willing to learn. Learning about the people behind the mathematics has reintroduced me to my love of history, and I want my students to feel the same way. I want to change how students perceive mathematics. I want them to be curious about the people behind the formulas. I want them to question where certain ideas, concepts, or even words originated. I want them to appreciate the beauty in mathematical solutions. I want them to be in awe of how mathematicians that lived centuries ago were able to see and discover certain ideas with very little technology within their reach. I want students to discover that the history of mathematics is also important, and is often the key to truly understanding many concepts. I want them to understand that struggling with mathematics is okay, and even common among the best mathematical minds of their time. Mathematics is deeply woven into human history, and knowing something of the history will help students to master the mathematics as well as to spark their interest. By changing how students view mathematics, we can begin to change and improve mathematics' reputation from a method of torture to a necessary key to world around us. These lessons are a good starting point, but there is so much more out there that can help to engage students in the mathematics classroom like never before.

8. REFERENCE PAGE

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[2] Aczel, Amir. “The Origin of the Number Zero.” *Smithsonian Magazine* (December 2104) <http://www.smithsonianmag.com/history/origin-number-zero-180953392/?no-ist>. [Accessed 19 July 2015].

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