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Stability pockets of a periodically forced oscillator in a model for seasonality

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Abstract

A periodically forced oscillator in a model for seasonality shows chains of stability pockets in the parameter plane. The frequency of the oscillator and the length of the photoperiod in the Zeitgeber are the two parameters. The present study is intended as a theoretical complement to the numerical study of Schmal et al. (2015) of stability pockets (or Arnol'd onions in their terminology). We construct the Poincaré map of the forced oscillator and show that the Arnol'd tongues are taken into chains of stability pockets by a map with a number of folds. This number is related to the rational point $(\frac{p}{q}, 0)$ on the frequency axis from which a chain of p pockets emanates. Stability pockets are already observed in an article by van der Pol and Strutt in 1928, see van der Pol and Strutt (1928) and later explained by Broer and Levi in 1995, see Broer and Levi (1995).

Keywords: Forced oscillator; Circadian clock; Entrainment; Resonance; Poincaré map; Stability pocket

1. Introduction

The numerical study by Schmal et al. [10] of a model for seasonal effects on the circadian clock of an organism (located in the supra chiasmatic nucleus), shows stability pockets and chains thereof in the parameter plane. As in [10] the seasonal effect is in particular the variation of the length of the daylight time interval, called photoperiod. This alternation of light and dark acts as a forcing, called Zeitgeber, on the circadian clock. The present study aims to complement the results of [10] by giving a theoretical background and moreover a geometrical explanation for the observed phenomena. We also indicate what is to be expected when the system is perturbed. The

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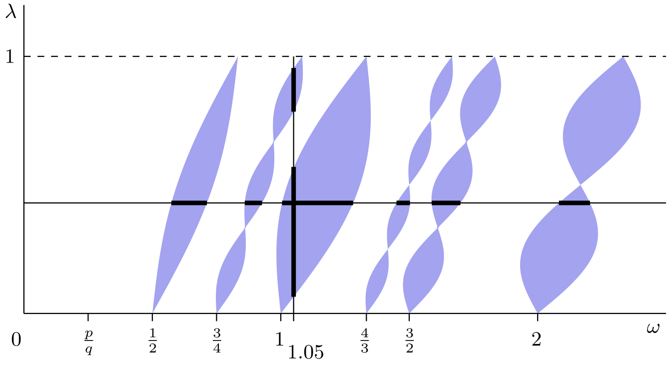

Fig. 1. Stability pockets in the parameter plane of the oscillator forced with a periodic block function.

setting of the problem is bifurcations of parameter dependent dynamical systems, in particular periodically forced oscillators.

1.1. Periodically forced oscillators

We start with a general periodically forced oscillator. Such an oscillator shows periodic dynamics (also called synchronization or entrainment) if the frequency of the forcing is close enough to a rational multiple of the frequency of the oscillator. The latter is the frequency of the isolated oscillator, that is in absence of forcing. On the other hand, if the frequency ratio is not close enough to a rational number, the dynamics is quasi periodic for a large number of irrational frequency ratio's. Now we consider the periodically forced oscillator as a system of two asymmetrically coupled oscillators. The first corresponds to the circadian clock and the second corresponds to the Zeitgeber, thus the first is forced by the second. The latter has fixed dynamics. In particular its frequency is fixed and we set it equal to one by a scaling of time. Thus the frequency of the first oscillator becomes a parameter of the system. Another important parameter is the coupling strength. In the parameter plane of coupling strength versus frequency there are regions, called Arnol'd tongues (see [1]) or resonance regions, where the first oscillator synchronizes with the second, that is there are one or more stable (and unstable) periodic solutions. These tongues are wedge shaped and their vertices lie at rational points on the frequency axis. So for a fixed value of the coupling strength there is a frequency interval where synchronization occurs. On the boundary of this interval the periodic solutions disappear in saddle–node bifurcations. If the coupling strength goes to zero the interval shrinks to a (rational) point on the frequency axis. If the coupling strength increases there may be many other bifurcations, for example period doubling bifurcations, see [6].

In certain examples of periodically forced oscillators the tongues close again in a second vertex, forming a so called *stability pocket* (called *Arnol'd onion* by Schmal et al. [10]), see Fig. 1. This phenomenon occurs for example in Hill's equation and has already been observed by van der Pol & Strutt [13], although they do not mention it explicitly as such. Much later Hill's equation has been reanalyzed by Broer & Levi [3] and in a more general setting by Broer & Simó [5], using methods not available to van der Pol & Strutt. Then the term *instability pocket* was introduced. However, it depends on the point of view whether a pocket is called a stability or an instability pocket, see Remark 3.4. Although the context of the Hill equation differs from ours, the mechanism by which the pockets are formed is the same. In both systems a general underlying system has a wedge shaped resonance region. The map that takes the parameter space of

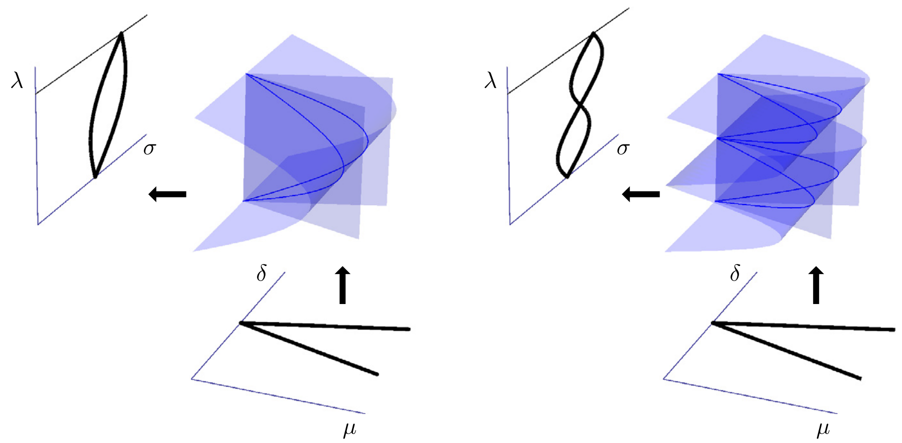

Fig. 2. Stability pockets and folds of the map $(\sigma, \lambda) \mapsto (\delta, \mu)$. Here only the folding part is shown. Left: the pocket for the (1, 1)-tongue; right: the chain of pockets for the (2, 1)-tongue. See Section 3 for the meaning of σ, λ, δ and μ.

the specific example to the parameter space of the general system, has one or more folds, thus generating pockets, see Fig. 2.

1.2. *Informal statement of results*

Our model consists of two asymmetrically coupled oscillators. The first corresponds to the circadian clock and the second to the Zeitgeber. Time has been scaled so that the frequency of the Zeitgeber is equal to one. In particular the Zeitgeber is modeled by a periodic block function where the length of the block corresponds to the photoperiod λ a parameter of the model, see Fig. 3. There are several interpretations of this model. If $\lambda = 1$ there is constant light, which can be interpreted as summer at the poles, if $\lambda = 0$ there is constant darkness: winter at the poles. But, for example $\lambda = \frac{1}{2}$, corresponds to spring or autumn at mid latitudes but this could also be interpreted as any day of the year at the equator. In the biological interpretation of the model the range of admissible values of λ depends on the latitude. Another parameter of the system is the frequency ω of the circadian clock. The parameter plane of the model, in which we find the stability pockets, is the (ω, λ) plane.

Our main result is that at each rational point $(\frac{p}{q}, 0)$ on the frequency axis a chain of p stability pockets emanates. We do this by showing that there is a map with precisely p folds taking the Arnol'd tongues to stability pockets near each of these rational points, see Figs. 2 and 1. The origin of the folds is that by varying λ from 0 to 1 the forcing is in fact constant at the endpoints $\lambda = 0$ and $\lambda = 1$. Thus the Fourier coefficients of the Zeitgeber are parameterizations of curves in the complex plane with both endpoints at zero. Since the coupling strength of the Zeitgeber and the oscillator is proportional to the absolute value of the Fourier coefficients we get folding. In case the Zeitgeber is a periodic block function, Fourier coefficients with a higher frequency number parameterize a curve which goes through zero multiple times causing chains of stability pockets. If we perturb the block function, the chains will in general open up to form a single pocket, see Section 3 and Fig. 4. Apart from folding, this map also ‘skews’.

The stability pockets, being images of the resonance tongues, are again resonance regions (or regions of entrainment). Taking a constant value for the photoperiod we find ranges of entrainment on the frequency axis, the horizontal line in Fig. 1 intersection the pockets. Taking

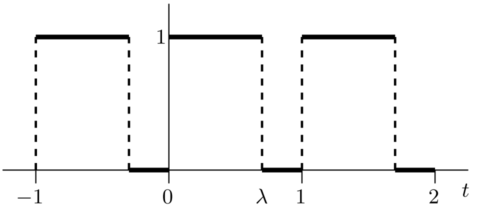

Fig. 3. A periodic block function modeling daylight. The length of the photoperiod is λ.

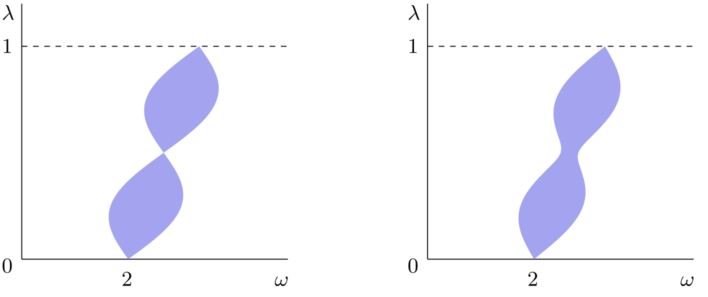

Fig. 4. The image of the (2, 1)-tongue for g_λ and for a small perturbation of g_λ.

a constant value for the frequency of the circadian clock we find ranges of entrainment for the photoperiod, the vertical line in Fig. 1. Also this line intersects several pockets, due to the skewness. It shows for example that an organism might not be able to entrain to the Zeitgeber when the photoperiod becomes long, but instead has for example a 3 : 4 resonance with the Zeitgeber, see Fig. 1. The latter is mainly caused by the 'skewing'. A similar phenomenon may occur for a slightly larger frequency of the circadian clock for short photoperiods. For more biological interpretation and examples refer to [10].

1.3. Outline

In Section 2 we introduce a system of asymmetrically coupled phase oscillators and the Poincaré map by which we analyze the model. The main results are stated in Section 3 where we first specify the details of the model. The statements of Section 3 are proved in Section 4. A generalization of the model is briefly discussed in Section 3.

2. Setting: two asymmetrically coupled phase oscillators

The model in Schmal et al. [10] consists of a differential equation for a specific two dimensional oscillator, in fact the normal form of the Hopf bifurcation, with periodic forcing. Here we take a slightly different approach. Our oscillator will be a general phase oscillator asymmetrically coupled to a second phase oscillator with constant frequency, the forcing oscillator or Zeitgeber. First we study a general coupling and approximate the Poincaré map. Then we use these results to study a model in which the forcing is a periodic block function where the photoperiod is a parameter.

We could have started with a general oscillator in two dimensions with an external periodic forcing. But then the technicalities of bringing the system in manageable form and keeping it so when the external forcing is applied would obscure the phenomenon we want to study. Our primary goal is the way the dynamics in the phase direction of the oscillator changes when the external force is applied: we are at the moment less interested in the change of the shape and position of the periodic orbit. Therefore we start with a phase oscillator from the very beginning.

Let ψ and φ be phase angles so $\psi, \varphi \in S^1$ where ψ is interpreted as the phase angle of the oscillator and φ as the phase angle of the external forcing. Furthermore, let f_μ be a function on $\mathbb{T}^2$, smoothly depending on parameters $\mu \in \mathbb{R}^m$ for some m. Moreover we assume that f_μ has a Fourier series in ψ and φ with rapidly decreasing coefficients. For more about regularity of f_μ see remarks after the main theorem. The following asymmetrically coupled system describes a periodically forced phase oscillator.

$$\begin{cases} \dot{\psi} = \omega + \mu f_\mu(\psi, \varphi) \\ \dot{\varphi} = 1. \end{cases} \tag{1}$$

The system depends on small parameters $\mu, \delta, \ldots \in \mathbb{R}^m$. The frequency of the oscillator is ω for $\mu = 0$. In that case the dynamics of the system is quasi-periodic if ω is irrational, but if ω is rational, the dynamics is periodic. Thus for $\mu = 0$ periodic dynamics is exceptional and quasi periodic dynamics is typical. This changes dramatically if $\mu \neq 0$, then periodic dynamics becomes typical. Here *typical* dynamics is used in the sense that it occurs for a set of parameter values having finite measure. The rationals have zero measure in the set of reals, but the union of resonance regions if $\mu \neq 0$ has finite measure and for small μ it grows with μ. The fate of quasi periodic dynamics is more subtle, see [7] for a systematic and comprehensive account. Our interest is in the periodic dynamics. Thus it seems natural to take a closer look near rational values of ω, therefore we set $\omega = \frac{p}{q} + \delta$, with p and q relative prime integers and δ is a small parameter. It is most efficient to study this system via a Poincaré map. A good candidate is the map that scores ψ at consecutive crossings of $\varphi = 0$ (sometimes called the stroboscopic map). Since all Poincaré maps are equivalent, results will not depend on this choice. For convenience we switch to the lift of this system to $\mathbb{R}^2$, then the differential equation becomes

$$\begin{cases} \dot{x} = \frac{p}{q} + \delta + \mu f_\mu(x, y) \\ \dot{y} = 1. \end{cases} \tag{2}$$

Now we define the (lift of the) Poincaré map as follows.

Definition 2.1. Let Φ_t be the flow of Eq. (2), then the Poincaré map P is implicitly defined by $(P(x), 1) = \Phi_1(x, 0)$.

The map P in this definition is the lift of a circle map, the Poincaré map on the circle. In actual computations it is easier to use the lift, therefore we only use P. Now since P is the lift of a circle map it must be of the form $P(x) = x + \omega + \mu h_\mu(x)$ where h_μ is a 1-periodic function. Note that for $h_\mu(x) = \sin(2\pi x)$ we obtain the Arnol'd circle map, see [1], which already explains most phenomena occurring in the forced oscillator. Unfortunately there is no easy connection between h_μ and f_μ in Eq. (2). But by a so called normal form transformation we obtain a vectorfield approximation of the Poincaré map P, see Section 4. The approximating vectorfield is on the right-hand side of the following equation

$$\dot{u} = \delta + \sum \tilde{f}_{k,l,m}\, \mu^m\, \mathrm{e}^{2\pi i k u} \tag{3}$$

where $\tilde{f}_{k,l,m}$ depends in a complicated way on the Taylor–Fourier coefficients of f_μ and the sum runs over all k and l with $pk + ql = 0$ and $m \in \{1, \ldots, n\}$. The time-1 flow of this vector field approximates the Poincaré map P. In fact, to be more precise, the time-q flow approximates $P^q - p$. We will elaborate on this in Section 4.1.

The main property of the Poincaré map P we use is that a stationary or a periodic point of P corresponds to a periodic solution of Eq. (1) and vice versa. Thus the existence of periodic solutions of Eq. (1) can be read off from the existence of stationary or periodic points of P. The vectorfield approximation does not cover the full dynamics of the original system. But hyperbolic stationary points of (3) correspond to stationary points of the Poincaré map P. Moreover by persistence of saddle–node bifurcations we recover the well-known picture of resonance tongues for P in the (ω, μ)-plane from Eq. (3). For each pair (p, q) a tongue emanates from the ω-axis at $\omega = \frac{p}{q}$. The tongue at $(p, q) = (1, 1)$ is called the *main tongue*, the others are just labeled by the pair (p, q). Indeed we may rewrite the differential equation as follows

$$\dot{u} = \delta + \mu \tilde{f}_\mu(u) \tag{4}$$

where $\tilde{f}_\mu$ is a 1-periodic function. Then at least near $(0, 0)$ stationary points exist in a region in the (δ, μ)-plane bounded by curves of saddle–node bifurcations of the form $\mu = \gamma_1 \delta$ and $\mu = \gamma_2 \delta$, for some constants γ_1 and γ_2. In the simplest case where $\tilde{f}_\mu(u) = \sin(2\pi u)$ we have $\mu = \pm\delta$.

In the next section we take a specific form for Eq. (2) containing parameters σ and λ and we study the map $(\sigma, \lambda) \mapsto (\delta, \mu)$. The inverse image of this map takes the tongues of the general equation for two asymmetrically coupled oscillators to the resonance regions of model with λ dependent forcing.

3. Main results: stability pockets

In a simple model of forcing depending on photoperiod we take a phase oscillator with a particular 1-periodic forcing. The general form will be the following.

$$\begin{cases} \dot{\psi} = \omega + \eta f(\psi) + \varepsilon g_\lambda(\varphi) \\ \dot{\varphi} = 1. \end{cases} \tag{5}$$

To ascertain that our approximation methods in Section 4.1 work we require that f and g_λ are functions on S^1 with rapidly decreasing Fourier coefficients. The function f describes the non-linearity of the oscillator and g_λ determines the external periodic forcing. As before we take $\omega = \frac{p}{q} + \sigma$, with positive integers p and q. The parameters σ, η and ε are small, but not necessarily of the same order.

In the forcing we use a function as in [10], namely one that depends on a non-small parameter λ, which determines the fraction of the period the forcing is 'on', see Fig. 3. A simple example being a block function with a block of length $\lambda \in [0, 1]$

$$\tilde{g}_\lambda(t) = \begin{cases} 1, & 0 \le t < \lambda \\ 0, & \lambda \le t < 1 \end{cases} \tag{6}$$

with $\tilde{g}_\lambda(t + 1) = \tilde{g}_\lambda(t)$ for all t. This function is not continuous and its Fourier coefficients are not rapidly decreasing. Therefore we replace $\tilde{g}_\lambda$ by another function, namely the convolution $\phi * \tilde{g}_\lambda$ which does have rapidly decreasing Fourier coefficients. In Section 4.3 we will show that the results in the sequel do not depend on the choice of ϕ.

Definition 3.1. The periodic forcing g_λ in Eq. (5) will be $g_\lambda = \phi * \tilde{g}_\lambda$ where ϕ is the function $\phi(x) = \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha x^2)$.

We use the methods of the previous section to find a vectorfield approximation of the Poincaré map for the oscillator. The following theorem gives a more precise statement. For a proof see Section 4.2 where also the meaning of *degree* will be clarified.

Theorem 3.2. *Consider Eq. (5) with conditions of Section 4.1 on the functions f and g_λ. Let P be the lift of the Poincaré map of (5), then the vectorfield approximation up to degree two of P is*

$$\dot{u} = \sigma - \frac{q}{p}\Big(\eta^2 \sum_k f_k f_{-k} + \varepsilon\eta \sum_m f_{mq} g_{-mp}(\lambda) e^{2\pi i m q u}\Big) \tag{7}$$

where $u = x - \frac{p}{q} y$. The Fourier coefficients of f and g_λ are f_k and $g_k(\lambda)$ respectively, where

$$g_k(\lambda) = \exp\left(-\frac{\pi^2 k^2}{\alpha}\right) \frac{i}{2\pi k}\big(\exp(-2\pi i k \lambda) - 1\big).$$

We wish to find a region in the parameter space of the oscillator, that is in the (σ, λ)-plane, where stationary points of Eq. (7) exist. Since this equation is a special form of Eq. (3) we consider the map $(\sigma, \lambda) \mapsto (\delta, \mu)$. The result is formulated in the following theorem.

Theorem 3.3. *Let p and q be relative prime as mentioned before. Then the leading term of the map $(\sigma, \lambda) \mapsto (\delta, \mu)$ is implicitly given by:*

$$(\delta, \mu) = (\sigma - \eta^2 c_1 - \varepsilon\eta\lambda c_2,\ \varepsilon\eta |\sin(\pi p \lambda)| c_3)$$

where c_1, c_2 and c_3 are constants. This implies that the (p, q)-tongue has p stability pockets.

The map implicitly defined in the theorem takes the main tongue of a general periodically forced oscillator into a so called *stability pocket* in the (σ, λ)-plane of the oscillator. Other tongues, depending on p, are mapped to a chain of stability pockets. In Fig. 2 a graphical representation of this result is shown. The graph of he map $(\sigma, \lambda) \mapsto (\delta, \mu)$ is a two dimensional surface in a four dimensional space. Since this is hard to draw we only show the folding direction.

Remark 3.4.

(i) Theorems 3.2 and 3.3 together support the numerical results in [10], there is an excellent qualitative agreement with their findings and the propositions. There is no reason to doubt that with some more effort this agreement can also be made quantitative.

(ii) In [10] the stability pocket is called "Arnol'd onion". Here the term *stability pocket* is used to connect with the existing literature on Hill's equation, see [3]. In Hill's equation the zero solution becomes unstable in the resonance tongues. In case such a tongue closes, a "pocket" is formed which is then called an instability pocket. In our case we are interested in synchronization, that is in the existence of stable periodic solutions. Since these exist in resonance tongues, a tongue closing and forming a "pocket" is now called a stability pocket.

(iii) In the special case $f(\psi) = \sin(2\pi\psi)$ the map becomes simpler, namely $(\delta, \mu) = (\sigma - \eta^2 c_1 - \varepsilon\eta\lambda c_2, \varepsilon\eta\sqrt{2}|\sin(\pi p \lambda)|)$. In fact the map $(\delta, \mu) = (\sigma - \lambda, |\sin(\pi p \lambda)|)$ has essentially the same properties. This map is used to draw the graphs in Fig. 2.

(iv) In Figs. 1 and 2 the implicit assumption is that the fold projects in the 'lower part' of the tongue, that is in a region in the parameter plane where the map is a diffeomorphism. If the fold projects in a higher part of the tongue where the map is no longer be a diffeomorphism and the tongues may start to overlap, which will then be reflected in overlapping stability pockets.

(v) We have to find the map $(\sigma, \lambda) \mapsto (\delta, \mu)$ for each tongue. The reason is that no single parameterization of all normal forms exists: for each resonance $p : q$ we have to compute a new normal form.

(vi) In a more accurate model of seasonality, the parameter λ will no longer be a constant, but vary in time. Since this variation is slow with respect to the frequency of the forcing we may consider a model based on a slow–fast system. From this perspective here we are studying the fast limit of such a system.

(vii) To be able to prove Theorem 3.2 we need that the vectorfield in Eq. (2) has a Fourier–Taylor series in x, y and μ. In the proof of this normal form theorem we get Fourier series as coefficients. These must exist, therefore we impose the condition that the Fourier coefficients of the vectorfield are rapidly decreasing. Intuitively the periodic block function in the vector field of Schmal et al. [10] is a simple approximation of day and night. However due to the discontinuities it has poor regularity, in particular its Fourier coefficients are not rapidly decreasing. Fortunately the discontinuities have little or nothing to do with the phenomenon we wish to explain. Therefore we consider a convolution with the block function, which approximates the original function as closely as needed and moreover satisfies the conditions of the normal form theorem. The latter are probably stronger than necessary. But for our purposes we do not need the most general conditions.

(viii) Related to the previous but from a more general point of view, the regularity of the vectorfield in Eq. (5) greatly influences the accuracy of the vectorfield approximation from the normal form theorem. In case of a C^∞ vectorfield we can invoke the Borel theorem to get at least a formal normal form from which we obtain for example results about existence of stationary points and even bifurcations thereof. In case of (real) analytic vectorfields we get a normal form which differs an exponentially small amount (in terms of parameters) from the original vectorfield. This for example puts bounds on chaotic regions. For an overview see [2] and for an early account on exponentially small estimates see [8].

(ix) The occurrence of resonance tongues and (in)stability pockets is not limited to periodically forced oscillators, see [4] for such phenomena in quasi periodically forced systems. Like in the present case, averaging is a key procedure in obtaining estimates, see for example [11].

Proof of Theorem 3.3. Compare Eq. (4)

$$\dot{u} = \delta + \mu \tilde{f}_\mu(u)$$

to Eq. (7)

$$\dot{u} = \sigma - \frac{q}{p}\Big(\eta^2 \sum_k f_k f_{-k} + \varepsilon\eta \sum_m f_{mq} g_{-mp}(\lambda) \mathrm{e}^{2\pi i mqu}\Big)$$

to determine the map $(\delta, \mu) \mapsto (\sigma, \lambda)$. We immediately see that the first component of the map is $\delta = \sigma - \eta^2 c_1 - \varepsilon\eta\lambda c_2$ with $c_1 = \frac{q}{p} \sum_k f_k f_{-k}$ and $c_2 = \frac{q}{p} f_0$. Recall that $g_0 = \lambda$. The second component is determined by $\sum_m f_{mq} g_{-mp} \exp(2\pi i mqu)$, a periodic function of u with Fourier coefficients $f_{mq} g_{-mp}$. The λ dependent factor of the modulus of these coefficients is $|\sin(\pi mp\lambda)|$, see Lemma 4.5 in Section 4.3. The latter is a periodic function of λ with $p + 1$

zeros for all m on $[0, 1]$. Or, put differently, with at least p folds for all m. Thus the number of stability pockets is determined by the first (and largest) non-zero Fourier coefficients, f_1 and f_{-1}, and thus by the map $(\delta, \mu) = (\sigma - \eta^2 c_1 - \varepsilon\eta\lambda c_2, \varepsilon\eta|\sin(\pi p\lambda)|)$. Therefore the (p, q)-tongues have a chain of p stability pockets. To describe their shape in detail one would need the remaining Fourier coefficients. □

Let us now look at a generalization of the system in the following sense. We still consider a periodically forced oscillator but now the forcing is no longer a block function for all $\lambda \in [0, 1]$. To be more precise, let h_λ be a periodic function with $h_0 = 0$, $h_1 = 1$ and $h_\lambda - g_\lambda$, C^0 small. Then the chains of stability pockets open up, see the image of the (2, 1)-tongue in Fig. 4.

Theorem 3.5. *Let h_λ be a generic perturbation of the forcing in the oscillator of Eq. (5). Then each chain emanating from $(\omega, \lambda) = (\frac{p}{q}, 0)$ opens up to form a single stability pocket.*

Proof. Just like the Fourier coefficients of g_λ, those of h_λ can be regarded as parameterizations of a closed curve in $\mathbb{C}$ with endpoints at zero, the parameter being λ. Now for $k \neq 0$ the λ dependent factor of $g_k(\lambda)$ is $\big(\exp(-2\pi ik\lambda) - 1\big)$, see Section 4.3, which passes through zero for $\lambda = \frac{n}{k}$ for $n \in \{0, 1, \dots, k\}$. When we perturb g_λ to h_λ subject to the condition that $h_0 = 0$ and $h_1 = 1$, the Fourier coefficients again parameterize closed curves in $\mathbb{C}$ with endpoints at zero, but in general not passing through zero for $\lambda \in (0, 1)$. Thus we will still have stability pockets for the more general system, but the chains of stability pockets will open up. □

4. Proofs

This section contains the proofs of the theorems in the previous sections. We first give a short review of approximating a Poincaré map and then apply this to our coupled oscillators. In a last section we discuss how to deal with a non-smooth forcing.

4.1. A vectorfield approximation of the Poincaré map

In this section we consider vectorfields on the phase space $\mathbb{R}^2$ as lifts of vectorfields on the two torus. Our aim is to construct a vectorfield approximation of the Poincaré map. Where the latter is the lift of the Poincaré map on the two torus. We will work in the context of vectorfields with smooth dependence on parameters and rapidly decreasing Fourier coefficients.

First we define a parameter dependent differential equation on $\mathbb{R}^2$. Let f_μ be a function $\mathbb{R}^2 \to \mathbb{R}$, 1-periodic in both arguments and assume f has a Taylor–Fourier expansion

$$f_\mu(x, y) = \sum_{k,l,m} f_{k,l,m}\, \mu^m \mathrm{e}^{2\pi i(kx+ly)}$$

with coefficients $f_{k,l,m}$. The indices k and l run over $\mathbb{Z}$ and m runs over $\mathbb{N}$. Now consider the differential equation

$$\begin{cases} \dot{x} = \frac{p}{q} + \delta + \mu f_\mu(x, y) \\ \dot{y} = 1 \end{cases} \tag{8}$$

depending on parameters δ and μ. The first will be interpreted as a detuning and the second as the strength of the non-linearity. The Poincaré map P of this system is defined as $(P(x), 1) = \Phi_1(x, 0)$, where Φ_t is the flow over time t of Eq. (8).

Computing the vectorfield approximation. By a sequence of near identity transformations we put system (8) in a form such that the y dependence in the first component becomes trivial, at least up to a certain order. Here we use perturbation theory so we have to be precise about the size of various components. The grading we use will come from the parameters only. The reason is that the vectorfield is periodic in x and y therefore we set degree(x) = degree$(y) = 0$. Furthermore we set degree(μ) = degree$(\delta) = 1$. With these definitions the function f_μ has a formal Fourier–Taylor expansion in x, y and μ

$$\mu f_\mu(x, y) = \sum_{k,l,m} f_{k,l,m}\, \mu^m e^{2\pi i(kx+ly)}$$

with Fourier–Taylor coefficients $f_{k,l,m}$. Finally we define the vectorfield

$$\begin{aligned} X &= X_0 + X_1 + \cdots + X_n + \cdots \\ X_0 &= \tfrac{p}{q}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ X_m &= \sum_{k,l} f_{k,l,m}\, \mu^m e^{2\pi i(kx+ly)} \frac{\partial}{\partial x} \end{aligned} . \tag{9}$$

Now we apply a standard normalization procedure (which amounts to averaging in this case) to obtain a normal form of the vectorfield X. For an overview see [2,9] and references therein, also see Remark 3.4. The result is formulated in the next proposition.

Proposition 4.1. *Let X be a parameter dependent vectorfield as in (9). Assume that X is C^∞ in the parameter μ and that the Fourier coefficients $f_{k,l,m}$ are rapidly decreasing in k and l. By a sequence of n near-identity transformations the vectorfield X is transformed into $\tilde{X}$ with*

$$\tilde{X} = X_0 + \tilde{X}_1 + \cdots + \tilde{X}_n + \mathcal{O}(|\mu|^{n+1})$$
$$\tilde{X}_m = \sum_{pk+ql=0} \tilde{f}_{k,l,m}\, \mu^m e^{2\pi i(kx+ly)} \frac{\partial}{\partial x}.$$

The new coefficients $\tilde{f}_{k,l,m}$ depend on the original coefficients in a complicated way. One exception being $\tilde{X}_1$ for which $\tilde{f}_{k,l,m} = f_{k,l,m}$. The transformed vectorfield up to order n only contains resonant terms, that is in general $\tilde{f}_{k,l,m} \neq 0$ only if the index (k, l, m) satisfies the resonance condition $pk + ql = 0$.

Proof. This will only be a sketch of the proof, for more details see the references above. The key idea of the proof is that every C^∞ near identity coordinate transformation can be approximated as closely as needed by the flow of a C^∞ vectorfield [12]. Since we work in the context of C^∞ vectorfields on the two torus (or lifts thereof) the transformation has to respect this property. The flow of a C^∞ vectorfield on the two torus is such a transformation. Since these vectorfields form a Lie algebra we can use rather standard normal form theory. Our aim is to get rid of the time (y) dependence without transforming time, therefore we apply an asymmetric transformation generated by the vectorfield $Y = g(x, y)\frac{\partial}{\partial x}$ so that the new x depends on y but not the other way around. The procedure is inductive by degree. Let $\mathcal{X}$ be the set of C^∞ vectorfields on the two torus depending on one or more small parameters μ. We define a grading by the degree of μ: $\mathcal{X}_k \subset \mathcal{X}$ is the set of vectorfields of degree k or higher in μ. In the normal form procedure we frequently use the commutator $[\cdot, \cdot]$ of vectorfields, then for $X_m \in \mathcal{X}_m$ we have $[X_m, X_n] \in \mathcal{X}_{m+n}$. Furthermore if $Y_m \in \mathcal{X}_m$ is a fixed vectorfield and $\mathrm{ad}Y_m : X \mapsto [Y_m, X]$ then $(\mathrm{ad}Y_m)^k(X_n) \in \mathcal{X}_{km+n}$. Because of these relations we may normalize for increasing degree.

Now suppose we have normalized up to degree $n-1$, then take Y of degree n. The transformation acts on the vectorfield X as $\exp(\operatorname{ad} Y)$, namely

$$\exp(\operatorname{ad} Y)(X) = \mathrm{e}^{\operatorname{ad}Y}(X) = X_0 + X_1 + \cdots + X_{n-1} + X_n + \operatorname{ad}Y(X_0) + \text{h.o.t.}$$

where $\operatorname{ad}Y(x) = [Y, X]$. As usual in normal form theory we try to solve $X_n + \operatorname{ad}Y(X_0) = 0$ for g. Let $g_{k,l}$ be the Fourier coefficients of g then on the level of coefficients the equation becomes $qf_{k,l} - 2\pi i(pk+ql)g_{k,l} = 0$. Thus we set $g_{k,l} = \frac{qf_{k,l}}{2\pi i(pk+ql)}$ provided that $pk+ql \neq 0$. This implies that in the normal form the so called *resonant terms* with coefficient $f_{k,l}$ satisfying the resonance condition $pk+ql = 0$ are retained. $\square$

Assuming we have normalized the vectorfield up to sufficiently high order, we only keep terms up to order n by truncation. As a last step we again change coordinates: $u = x - \frac{p}{q}y$ and $v = y$. Then the new vectorfield Z becomes

$$Z = \Big(\delta + \sum_{j=1}^{n} \sum_{|m|=j,\, pk+ql=0} \tilde{f}_{k,l,m}\, \mu^m \mathrm{e}^{2\pi i k u}\Big)\frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

Indeed, if $pk+ql = 0$ then changing to coordinates u and v we get $kx + ly = k(u + \frac{p}{q}y) + ly = ku + \frac{pk+ql}{q}y = ku$.

Interpretation of vectorfield Z. Now let $\tilde{X}$ be the truncated normal form of X and Z be the vectorfield defined above. The corresponding flows are denoted by respectively $\tilde{\Phi}$, Φ and Ψ. Let P be the Poincaré map of Φ defined as $(P(x), 1) = \Phi_1(x, 0)$. In a similar way we define $\tilde{P}$, then this $\tilde{P}$ is an approximation up to order n of P. Furthermore let ψ be the flow of the first component of Z. Then we have $\Psi_s(u, 0) = (\psi_s(u), s)$ but also $\Psi_s(u, 0) = (\Pi_x \tilde{\Phi}_s(u, 0) - s\frac{p}{q}, s)$, where Π_x is the projection $\Pi_x(x, y) = x$. This means that $\tilde{P^q}(u) = \Pi_x \tilde{\Phi}_q(u, 0) = \psi_q(u) + p$. Thus $\tilde{P^q} - p$ is equal to the time q flow of the first component of Z. Therefore we call Z a vectorfield approximation of P.

Using the vectorfield approximation. The vectorfield Z approximates the Poincaré map of the vectorfield X in (8) but there is no equivalence. Therefore we have to be careful when drawing conclusions about the dynamics of (8) from analysis of the vectorfield Z. Here we will be mainly interested in stationary points of Z. A first observation is that hyperbolic stationary points of Z correspond to relative equilibria of $\tilde{X}$ and thus to periodic orbits of X (provided that the difference between X and $\tilde{X}$ is small enough). Stationary points of Z satisfy equation

$$0 = \delta + \sum_{j=1}^{n} \sum_{|m|=j,\, pk+ql=0} \tilde{f}_{k,l,m}\, \mu^m \mathrm{e}^{2\pi i k u}. \tag{10}$$

Given p and q, the right hand side is a $\frac{2\pi}{q}$ periodic function. Therefore solutions, if they exist, come in q pairs. Now suppose solutions exist, then upon varying parameters μ they may disappear in tangencies. Assuming for simplicity a single parameter μ, we find a wedge shaped region in parameter space defined by $\gamma_1\mu < \delta < \gamma_2\mu$ where solutions exist. This inequality only holds near $(\delta, \mu) = (0, 0)$ and the constants γ_1 and γ_2 depend on the Taylor–Fourier coefficients $\tilde{f}_{k,l,m}$. Dynamically speaking the boundaries of the wedge are curves of saddle–node bifurcations. These form the familiar stability tongues or Arnol'd tongues. Since saddle–node bifurcations in one parameter families persist under small perturbations, the stability tongues of vectorfield Z approximate those of the Poincaré map P.

For each combination of p and q (positive and relative prime) we find a tongue. The one for $(p, q) = (1, 1)$ is called the main tongue. Since we assumed that the Taylor–Fourier coefficients rapidly decrease for increasing k and l then the leading terms in Eq. (10) are

$$0 = \delta + \gamma\mu \sin(2\pi u + \chi) \tag{11}$$

where γ and χ are determined by $\tilde{f}_{1,-1,1}$ and $\tilde{f}_{-1,1,1}$. This little calculation at least shows that the tip of the tongue is a straight cone.

Similarly the tongues for $(p, q) = (1, q)$ are determined by an equation whose leading terms are as in Eq. (11), but now γ and χ are determined by $\tilde{f}_{q,-1,1}$ and $\tilde{f}_{-q,1,1}$. Again this shows that the tip of the tongue is a straight cone, but the angle of the cone decreases with increasing q.

Remark 4.2. We collect some remarks about the vectorfield approximation.

(i) The transformation $u = x - \frac{p}{q}y$ and $v = y$ is not just useful but originates from the following. By construction the truncated vectorfield $\tilde{X} = X_0 + \tilde{X}_1 + \cdots + \tilde{X}_n$ commutes with X_0 implying that their flows $\tilde{\Phi}$ and Φ^0 also commute. In other words the flow of X_0 generates a symmetry group of $\tilde{X}$. By switching to 'co-moving' coordinates, thus by the transformation Φ^0_t, the vectorfield $\tilde{X}$ transforms to $\tilde{X} - X_0$. Therefore stationary points of $\tilde{X} - X_0$ correspond to periodic orbits of $\tilde{X}$. Because of this property such stationary points are called *relative equilibria*.

(ii) In the normal form of the oscillator we will only need terms up to degree two. Then we have to compute a few higher order (h.o.t.) terms. Let X and Y be as in the proof, but now Y is of degree 1 then the terms we need are

$$\exp(\mathrm{ad}\ Y)(X) = X_0 + X_1 + \mathrm{ad}Y(X_0) + \mathrm{ad}Y(X_1) + \frac{1}{2}(\mathrm{ad}Y)^2(X_0).$$

(iii) The coefficients of the vectorfield are Fourier series. To ensure that the normal form makes sense these series must exist. Although not apparent from the discussion above, the coefficients are sums, products and convolutions of Fourier coefficients of the original vectorfield in Eq. (8). Therefore we impose the rather strong condition that the Fourier coefficients in (8) are rapidly decreasing.

4.2. Approximation of Poincaré map of the oscillator

We apply Proposition 4.1 to the differential equation of the oscillator. The starting point is the general form of the lift of Eq. (5)

$$\begin{cases} \dot{x} = \omega + \eta f(x) + \varepsilon g_\lambda(y) \\ \dot{y} = 1. \end{cases}$$

We normalize this vectorfield to obtain an approximate Poincaré map. We assume that f and g both have a Fourier series with coefficients f_k and g_k. Then the vectorfield is

$$\begin{aligned} X &= X_0 + X_1 + X_2 \\ X_0 &= \frac{p}{q}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ X_1 &= (\eta f(x) + \varepsilon g(y))\frac{\partial}{\partial x} = \sum_{k,l} f_{k,l} \mathrm{e}^{2\pi i(kx+ly)} \frac{\partial}{\partial x} \\ X_2 &= \delta \frac{\partial}{\partial x} \end{aligned}$$

where $f_{k,0} = \eta f_k$ and $f_{0,l} = \varepsilon g_l$. For both $k \neq 0$ and $l \neq 0$ we set $f_{k,l} = 0$. In view of the resonance condition $pk + ql = 0$ for a term $\exp(2\pi i(kx + ly))\frac{\partial}{\partial x}$, see Proposition 4.1, there will be no resonant terms of degree 1, therefore we set degree$(\delta) = 2$ and degree(ε) = degree$(\eta) = 1$. Note that $\omega = \frac{p}{q} + \delta$.

The proof of Theorem 3.2 consists of determining the vectorfield of degree 1 and computing terms up to degree 2.

Proof of Theorem 3.2. Following the proof of Proposition 4.1 we take a vectorfield Y of degree 1 and we try to solve $X_1 + \mathrm{ad}Y(X_0) = 0$ for the Fourier coefficients of Y. The degree is determined by the parameters δ, ε and η. Let $Y = a(x, y)\frac{\partial}{\partial x}$ and a has Fourier coefficients $a_{k,l}$. Then on the level of Fourier coefficients we get $a_{k,0} = \eta \frac{q}{p}\frac{1}{2\pi ik} f_k$ if $k \neq 0$ and $a_{0,l} = \varepsilon \frac{1}{2\pi il} g_l$ if $l \neq 0$. Since we have solved equation $X_1 + \mathrm{ad}Y(X_0) = 0$ we have $\mathrm{ad}Y(X_1) + \frac{1}{2}(\mathrm{ad}Y)^2(X_0) = -\frac{1}{2}\mathrm{ad}Y(X_1)$. Thus we obtain up to degree two

$$\begin{aligned}\tilde{X} &= \exp(\mathrm{ad}\,Y)(X) = X_0 + X_1 + \mathrm{ad}Y(X_0) + \mathrm{ad}Y(X_1) + \frac{1}{2}(\mathrm{ad}Y)^2(X_0) + \text{h.o.t.}\\ &= X_0 + \frac{1}{2}\mathrm{ad}Y(X_1) + \text{h.o.t.}\end{aligned}$$

As a shorthand write $X_1 = f\frac{\partial}{\partial x}$, $Y = a\frac{\partial}{\partial x}$ and $\mathrm{ad}Y(X_1) = h\frac{\partial}{\partial x}$ then $h = a\frac{\partial f}{\partial x} - f\frac{\partial a}{\partial x}$. From this expression we select the resonant terms, that is the $h_{k,l}$ satisfying $pk + ql = 0$, then we are left with

$$\begin{aligned}h(x, y) &= -2\eta^2\frac{q}{p}\sum_k f_k f_{-k} - 2\varepsilon\eta\frac{q}{p}\sum_{pk+ql=0} f_k g_l \mathrm{e}^{2\pi i(kx+ly)}\\ &= \sigma - \frac{q}{p}\Big(\eta^2\sum_k f_k f_{-k} + \varepsilon\eta\sum_m f_{mq} g_{-mp}(\lambda)\mathrm{e}^{2\pi im(qx-py)}\Big).\end{aligned}$$

Setting $u = x - \frac{p}{q}y$ and $v = y$ the vectorfield approximation of the Poincaré map becomes

$$\begin{cases}\dot{u} = \delta - \eta^2\dfrac{q}{p}\displaystyle\sum_k f_k f_{-k} - \varepsilon\eta\sum_m f_{mq} g_{-mp}(\lambda)\mathrm{e}^{2\pi imqu}\\ \dot{v} = 1.\end{cases} \qquad \square$$

With this result we find the stability tongues of the vectorfield approximation. If for example $(p, q) = (1, 1)$ then the tongue boundaries follow from solving equation

$$0 = \delta - \eta^2\sum_k f_k f_{-k} - \varepsilon\eta\sum_k f_{-k} g_k \mathrm{e}^{2\pi iku}$$

for u, see Section 4.1.

4.3. The Fourier coefficients of the forcing

The forcing g_λ of the oscillator, see Eq. (6), is a piecewise constant function and therefore not even C^∞, in particular its Fourier coefficients are not rapidly decreasing as required in Proposition 4.1. In order to get a suitable approximation of g_λ we use convolution with a so called Schwartz function. For this approximation we find the Fourier coefficients. We are in particular interested in the dependence on the parameter λ. The proofs of the following proposition and lemmas are found by straightforward arguments and computations.

Proposition 4.3. *Let ϕ be a normalized Schwartz function and let $\hat{\phi}$ be its Fourier transform. Then the Fourier coefficients of the convolution $\phi * g_\lambda$ are $\hat{\phi}(k) \cdot g_k$ where $g_0 = 0$ and $g_k = \frac{i}{2\pi k}\big(e^{-2\pi i k\lambda} - 1\big)$ if $k \neq 0$. In particular we have $|g_k| = \frac{1}{2\pi k}|\sin(\pi k\lambda)|$.*

Let ϕ be a normalized Schwartz function, that is

(i) for all positive integers m and n we have $\sup_{\mathbb{R}} |x^m (\frac{d}{dx})^n \phi(x)| < \infty$,
(ii) $\int_{\mathbb{R}} \phi(x)\, dx = 1$.

The familiar Gauss function $\phi(x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2}$ is an example of such a function. For all $\alpha > 0$ the function $\phi_\alpha(x) = \alpha\phi(\alpha x)$ is again a normalized Schwartz function and the larger α, the closer $\phi_\alpha * g_\lambda$ is to g_λ.

Lemma 4.4. *Let g be a possibly non-smooth 1-periodic function with a Fourier series such that $g(x) = \sum_k g_k \exp(2\pi k x)$ and let ϕ be a normalized Schwartz function with Fourier transform $\hat{\phi}$. Then $\phi * g$ is a smooth 1-periodic function which has a Fourier series with rapidly decreasing coefficients $\hat{\phi}(k) \cdot g_k$.*

This lemma shows that the results in Theorem 3.3 do not depend on the choice of the Schwartz function ϕ since the zeros of $\hat{\phi}(k) \cdot g_k(\lambda)$, as a function of λ, are exactly those of $g_k(\lambda)$.

By an elementary calculation we immediately find the Fourier coefficients of g_λ and $\phi * g_\lambda$.

Lemma 4.5. *The Fourier coefficients g_k of the function g_λ as defined in Eq. (6), are $g_0 = \lambda$ and $g_k = \frac{i}{2\pi k}\big(e^{-2\pi i k\lambda} - 1\big)$ if $k \neq 0$. The Fourier transform of ϕ_α is $\hat{\phi}_\alpha(y) = \exp(-\frac{\pi^2 y^2}{\alpha})$.*

Our main interest is in the λ dependence of the Fourier coefficients. So far we found $\hat{\phi}(k) \cdot g_k = \exp(-\frac{\pi^2 k^2}{\alpha}) \frac{i}{2\pi k}\big(\exp(-2\pi i k\lambda) - 1\big)$ if $k \neq 0$, from which we infer that

$$|\hat{\phi}(k) \cdot g_k| = \mathrm{e}^{-\frac{\pi^2 k^2}{\alpha}} \frac{1}{2\pi k} \sqrt{2(1 - \cos(2\pi k\lambda))} = \mathrm{e}^{-\frac{\pi^2 k^2}{\alpha}} \frac{1}{\pi k} |\sin(\pi k\lambda)|.$$

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